Polyform tiling

Introduction

A **polyomino** (word derived from **domino**) is a geometric plane figure made of the union of finitely many edge-connected squares from the regular square lattice. (Some authors require also that a polyomino be simply connected, i.e. that it have no enclosed holes, but that is not the definition used here.) Polyominoes are common objects in recreational mathematics. An *n*-omino is a polyomino with *n* squares; the name is commonly written with a Greek prefix.

Problems with polyominoes commonly involve arranging polyominoes to fill some region, or the whole plane, without gaps or overlaps, subject to some constraints. Sets of the twelve pentominoes in wood or plastic are widely available; traditional problems are to fill a 3 by 20, 4 by 15, 5 by 12 or 6 by 10 rectangle with them; the complete sets of solutions to these problems were determined by computer in the 1960s.

Polyiamonds (word derived from **diamond**) are shapes analogous to polyominoes, but made up of equilateral triangles taken from the regular tiling by such triangles. **Polyhexes** (singular variously either **polyhex** or **polyhexe**) are shapes analogous to polyominoes, but made up of regular hexagons taken from the tiling by the regular hexagon. Both polyiamonds and polyhexes are widely used in recreational mathematics, similarly to polyominoes. **Polykites** are shapes analogous to polyominoes, but made up of kites taken from the [3.4.6.4] Laves tiling, and are among less-widely-used types of polyforms.

Problems of filling rectangles with copies of a single polyomino have been much studied; problems of filling other regions, and of filling regions with polyhexes or polyiamonds, have been studied less. This page concerns only tiling the whole plane with copies of a single polyomino, polyhex, polyiamond or polykite.

It is unclear what results on plane polyomino, polyhex and polyiamond tiling were found first by whom, when. Tiling the polyominoes of orders 1 through 6 is easy, and John Conway determined the tiling properties of the heptominoes with the aid of Conway's criterion. The list of non-tiling octominoes was published by Martin Gardner from correspondence from David Bird. Results through the hexahexes and the 9-iamonds were also published by Gardner from his correspondence with Bird and others. A paper with some results for 9-ominoes is listed below. Other people may have obtained results for higher orders, probably through computer or computer-aided search, but results in this area have not generally been formally published. Even in 1983, when the properties of hexominoes had long been known (the fact that they all tile was published by Martin Gardner in Scientific American, July 1965, page 102), it was possible for a paper to be published showing that they all tiled, as if a new result. (F. W. Barnes, Every hexomino tiles the plane, J. Combin. Inform. System Sci. 8 (1983), no. 2, 113–115; MR 86f:05051.)

From time to time I play with problems in recreational mathematics, sometimes by computer; since the summer of 1996 I have, on and off, written and run various programs to determine tiling properties of polyominoes, polyhexes and polyiamonds. Some of the results here may have been computed by others before being presented here, but the only other substantial computations on such matters I am aware of in recent years are those of Glenn Rhoads. Computations with my programs whose results are shown here were first run on single computers for the lower orders, 1996–2002; then for 20-ominoes through 23-ominoes, 16-hexes through 19-hexes and 22-iamonds through 28-iamonds in parallel at DPMMS, Cambridge during summer 2003; then for higher orders by Paul Church on SHARCNET in 2007. Computations for polykites were run in 2023.

There is a hierarchy of preferred types of tiling in use here, based on that of Grünbaum and Shephard in *Tilings and Patterns*. The most preferred type is a tiling by translation (in which the centroids of the tiles will form a lattice; if a shape tiles with translates only then it has a tiling in which the centroids of the tiles form a lattice; this is a non-trivial theorem, and some papers on this subject are listed below). The existence of such a tiling is determined by a condition on the boundary of the shape, similar to the well known Conway criterion. Then come the tilings by 180° rotation, as found by Conway's criterion. (Many people would look for these first and ignore translation tilings). We now need some terminology: a plane tiling is said to be isohedral if the symmetry group of the tiling acts transitively on the tiles; and n-isohedral if the tiles fall into n orbits under the action of the symmetry group of the tiling. A shape is said to be anisohedral if it tiles, but not isohedrally. So next most preferred are the isohedral tilings; then come the anisohedral tiles, which are probably the most interesting. (Part of Hilbert's 18th problem supposed that there were no anisohedral tiles in two dimensions (the first was found by Heesch in 1935) and asked for one in three dimensions.) After this are left over the non-tiling shapes. A refinement to the hierarchy used is, for anisohedral shapes, to minimise the number of orbits of tiles; a shape is said to be n-anisohedral, or to have isohedral number n, if it admits a n-isohedral tiling but no m-isohedral tiling for any $m < n$.

Other hierarchies that have been used in the literature include minimising the number of aspects (orientations) in which a tile appears, using only copies under direct isometries if possible, or minimising the size of a patch of tiles repeating under translation to tile the plane. A variation on the concept of isohedral number is the size of the smallest patch of tiles repeating isohedrally; these numbers are the same for asymmetric tiles, but may differ for some symmetric tiles, in particular where the induced symmetries of tiles in different orbits differ.

Because some investigators consider Conway's criterion first without regard to tilings by translation (at low orders, almost all of the shapes tiling by translation also tile by 180° rotation), tables are also provided showing, for those shapes tiling by translation, how many do or do not also tile by 180° rotation.

Lists of shapes and tilings (in gzipped PostScript, formatted for A4 paper) are available here; as the order increases, fewer lists are provided; higher-order lists can be provided on request, but the size of the lists increases exponentially and so it is unlikely lists in this form will be of much use, although the raw tiling data may be. The following tables show the numbers of shapes in each class, with links to the lists where appropriate.

If there are problems viewing the PostScript files, first make sure that your browser has not removed the `.gz` from the name but left the file compressed, or uncompressed the file but saved it under a name ending in `.gz`. (While I'd like to provide PDF files for wider accessibility, experiment suggests they would be about ten times the size of the gzipped PostScript, which reflects the mathematical structure of the tilings in a more compact way than PDF.) The files for polykites, added more recently, are PDF, as these size considerations have become less significant over time.

Subject to limitations of time and space, my programs can in principle resolve the tiling status of any polyomino, polyhex, polyiamond or polykite that either does not tile, or tiles the plane periodically. My programs would in principle run for ever on an aperiodic monotile. Within the ranges covered by these tables, there are no aperiodic polyominoes, polyhexes or polyiamonds, and two aperiodic polykites, and the tiling status of every shape within the ranges covered by these tables, except for the two aperiodic polykites, could be determined by my programs.

If you find these lists interesting or useful, please let me know.

Tables of tiling results

Polyominoes

Results and listings are provided here through the 25-ominoes.

n	*n*-ominoes	holes	translation	180°	isohedral	anisohedral	non-tilers
1	1	0	1	0	0	0	0
2	1	0	1	0	0	0	0
3	2	0	2	0	0	0	0
4	5	0	5	0	0	0	0
5	12	0	9	3	0	0	0
6	35	0	24	11	0	0	0
7	108	1	41	60	3	0	3
8	369	6	121	199	22	1	20
9	1285	37	213	748	80	9	198
10	4655	195	522	2181	323	44	1390
11	17073	979	783	5391	338	108	9474
12	63600	4663	2712	17193	3322	222	35488
13	238591	21474	3179	31881	3178	431	178448
14	901971	96496	8672	85942	13590	900	696371
15	3426576	425449	16621	218760	43045	1157	2721544
16	13079255	1849252	37415	430339	76881	2258	10683110
17	50107909	7946380	48558	728315	48781	1381	41334494
18	192622052	33840946	154660	2344106	551137	7429	155723774
19	742624232	143060339	185007	3096983	93592	5542	596182769
20	2870671950	601165888	573296	9344528	2190553	18306	2257379379
21	11123060678	2513617990	876633	17859116	3163376	22067	8587521496
22	43191857688	10466220315	1759730	31658109	3542450	47849	32688629235
23	168047007728	43425174374	2606543	49644736	1065943	10542	124568505590

24	654999700403	179630865835	8768743	172596719	39341178		202169	475147925759
25	2557227044764	741123699012	10774339	228795554	31694933		28977	1815832051949

The following table gives details of *k*-anisohedral polyominoes of order *n*.

n	***k* = 2**	***k* = 3**	***k* = 4**	***k* = 5**	***k* = 6**
8	1	0	0	0	0
9	8	0	1	0	0
10	41	3	0	0	0
11	89	18	1	0	0
12	214	6	2	0	0
13	406	24	0	1	0
14	874	24	1	0	1
15	1107	49	1	0	0
16	2210	46	1	0	1
17	1316	60	2	0	3
18	7380	42	7	0	0
19	5450	85	2	0	5
20	18211	86	5	0	4
21	21866	199	2	0	0
22	47702	135	9	1	2
23	10390	149	3	0	0
24	201834	324	11	0	0
25	28784	182	8	1	2

The following table shows, for those shapes tiling by translation, how many do or do not also tile by 180° rotation.

n	**180° as well**	**translation only**
1	1	0
2	1	0
3	2	0
4	5	0
5	9	0
6	24	0
7	41	0
8	121	0
9	212	1
10	520	2
11	773	10
12	2577	135

13	3037	142
14	8081	591
15	13954	2667
16	32124	5291
17	41695	6863
18	118784	35876
19	150188	34819
20	411484	161812
21	604304	272329
22	1305265	454465
23	1954823	651720
24	5326890	3441853
25	7331606	3442733

Polyhexes

Results and listings are provided here through the 21-hexes.

n	***n*-hexes**	**holes**	**translation**	**180°**	**isohedral**	**anisohedral**	**non-tilers**
1	1	0	1	0	0	0	0
2	1	0	1	0	0	0	0
3	3	0	3	0	0	0	0
4	7	0	6	1	0	0	0
5	22	0	12	9	1	0	0
6	82	1	36	39	1	1	4
7	333	2	60	197	33	4	37
8	1448	13	209	721	88	36	381
9	6572	67	387	2717	611	73	2717
10	30490	404	1054	8211	1803	258	18760
11	143552	2323	1468	19836	2985	501	116439
12	683101	13517	6895	74497	21250	999	565943
13	3274826	76570	7001	132611	23564	1383	3033697
14	15796897	429320	23920	403534	100221	4835	14835067
15	76581875	2373965	49496	1127416	392655	4685	72633658
16	372868101	13004323	126732	2364402	438188	10576	356923880
17	1822236628	70641985	146046	4173316	433150	8497	1746833634
18	8934910362	381260615	638800	15604400	4762862	42156	8532601529
19	43939164263	2046521491	647502	21318428	2327445	12931	41868336466
20	216651036012	10936624026	2772935	73965916	18839643	129325	205618704167
21	1070793308942	58228136539	4162861	154211880	46717154	84555	1012359995953

The following table gives details of *k*-anisohedral polyhexes of order *n*.

n	*k* = 2	*k* = 3	*k* = 4	*k* = 5	*k* = 6	*k* = 7	*k* = 8	*k* = 9	*k* = 10
6	1	0	0	0	0	0	0	0	0
7	3	1	0	0	0	0	0	0	0
8	27	7	2	0	0	0	0	0	0
9	66	5	2	0	0	0	0	0	0
10	226	26	4	0	0	0	2	0	0
11	380	97	19	3	1	0	0	1	0
12	879	95	23	2	0	0	0	0	0
13	1097	267	14	5	0	0	0	0	0
14	4419	390	24	1	1	0	0	0	0
15	3772	864	47	2	0	0	0	0	0
16	9488	1002	82	1	1	0	0	1	1
17	6718	1718	53	6	0	0	1	1	0
18	40023	2074	56	1	2	0	0	0	0
19	12630	265	34	1	0	0	1	0	0
20	127213	2039	59	3	11	0	0	0	0
21	76836	7583	132	2	2	0	0	0	0

The following table shows, for those shapes tiling by translation, how many do or do not also tile by 180° rotation.

n	180° as well	translation only
1	1	0
2	1	0
3	3	0
4	6	0
5	12	0
6	36	0
7	59	1
8	208	1
9	373	14
10	1024	30
11	1380	88
12	5966	929
13	6012	989
14	20061	3859
15	32939	16557
16	91715	35017
17	99540	46506

18	381899	256901
19	390781	256721
20	1486746	1286189
21	1856858	2306003

Polyiamonds

Results and listings are provided here through the 30-iamonds.

n	*n*-iamonds	holes	translation	180°	isohedral	anisohedral	non-tilers
1	1	0	0	1	0	0	0
2	1	0	1	0	0	0	0
3	1	0	0	1	0	0	0
4	3	0	2	1	0	0	0
5	4	0	0	4	0	0	0
6	12	0	8	4	0	0	0
7	24	0	0	21	2	0	1
8	66	0	24	32	10	0	0
9	160	1	0	111	22	6	20
10	448	4	62	200	54	25	103
11	1186	25	0	462	52	53	594
12	3334	108	291	1162	534	47	1192
13	9235	450	0	1875	523	97	6290
14	26166	1713	539	4195	1339	281	18099
15	73983	6267	0	9410	3218	280	54808
16	211297	21988	2625	17978	9315	343	159048
17	604107	75185	0	24034	2177	345	502366
18	1736328	251590	6177	73175	29426	1367	1374593
19	5000593	828408	0	79925	15423	619	4076218
20	14448984	2692630	21923	251808	101314	2478	11378831
21	41835738	8661287	0	358096	140072	1504	32674779
22	121419260	27624040	37721	631919	110794	8292	93006494
23	353045291	87479663	0	796571	46748	1811	264720498
24	1028452717	275392248	238267	3104082	1639279	16742	748062099
25	3000800627	862593661	0	2980476	710736	3458	2134512296
26	8769216722	2690285608	299203	6088310	970251	48453	6071524897
27	25661961898	8359581585	0	10167660	3002062	5459	17289205132
28	75195166667	25893044920	1338140	23661369	8462256	95311	49268564671
29	220605519559	79978118632	0	21477012	895291	9416	140605019208
30	647943626796	246433568617	3492733	84485584	29458807	333739	401392287316

The following table gives details of *k*-anisohedral polyiamonds of order *n*.

n	***k* = 2**	***k* = 3**	***k* = 4**	***k* = 5**	***k* = 6**	***k* = 7**	***k* = 8**
9	5	0	1	0	0	0	0
10	23	1	1	0	0	0	0
11	43	8	2	0	0	0	0
12	38	8	1	0	0	0	0
13	85	9	2	0	0	0	1
14	274	7	0	0	0	0	0
15	259	13	7	1	0	0	0
16	324	16	1	1	1	0	0
17	326	16	3	0	0	0	0
18	1344	17	5	0	1	0	0
19	591	25	1	0	0	0	2
20	2348	125	4	0	1	0	0
21	1466	32	4	0	0	0	2
22	8240	43	6	0	3	0	0
23	1773	34	4	0	0	0	0
24	16608	121	13	0	0	0	0
25	3408	46	4	0	0	0	0
26	48372	62	17	0	2	0	0
27	5386	57	13	0	0	0	3
28	95189	113	8	0	1	0	0
29	9351	59	2	0	0	0	4
30	330833	2881	20	0	5	0	0

The following table shows, for those shapes tiling by translation, how many do or do not also tile by 180° rotation.

n	**180° as well**	**translation only**
2	1	0
4	2	0
6	8	0
8	24	0
10	61	1
12	269	22
14	501	38
16	2121	504
18	4709	1468
20	14978	6945
22	26675	11046

24	123081	115186
26	179707	119496
28	660366	677774
30	1479114	2013619

Polykites

Results and listings are provided here through the 24-kites. Numbers of anisohedral polykites include aperiodic polykites, but the PDF listings do not. The numbers here relate only to tilings where all tiles are aligned to the same underlying [3.4.6.4] Laves tiling (this means that the 1-kite is not, for the purposes of this table, considered to have a 180° rotation tiling, but does not affect other figures in the tables).

n	***n*-kites**	**holes**	**translation**	**180°**	**isohedral**	**anisohedral**	**non-tilers**
1	1	0	0	0	1	0	0
2	2	0	0	0	1	0	1
3	4	0	0	3	1	0	0
4	10	0	0	0	4	1	5
5	27	0	0	0	0	1	26
6	85	1	13	23	34	1	13
7	262	4	0	0	52	3	203
8	873	28	0	0	37	3	805
9	2917	141	0	356	399	0	2021
10	10011	702	0	0	0	15	9294
11	34561	3187	0	0	0	2	31372
12	120815	13967	776	3851	6305	95	95821
13	424468	59061	0	0	2316	4	363087
14	1501441	244660	0	0	2597	20	1254164
15	5334181	995780	0	27173	18769	126	4292333
16	19035075	4002302	0	0	7362	37	15025374
17	68167472	15919384	0	0	0	1	52248087
18	244928324	62810243	29732	331123	372745	3650	181380831
19	882555803	246169244	0	0	79051	12	636307496
20	3188480899	959581657	0	0	0	987	2228898255
21	11546307356	3723544997	0	1408559	982778	1983	7820369039
22	41901879156	14393894755	0	0	0	19	27507984382
23	152359984459	55461808153	0	0	0	1	96898176305
24	554991766194	213112695000	1430791	15860606	15492179	112752	341846174866

The following table gives details of k-anisohedral polykites of order n.

n	***k*** **= 2**	***k*** **= 3**	***k*** **= 4**	***k*** **= 5**	***k*** **= 6**	***k*** **= ∞**

4	1	0	0	0	0	0
5	0	1	0	0	0	0
6	1	0	0	0	0	0
7	2	1	0	0	0	0
8	0	2	0	0	0	1
9	0	0	0	0	0	0
10	0	12	0	0	2	1
11	0	2	0	0	0	0
12	52	42	1	0	0	0
13	3	1	0	0	0	0
14	1	15	1	0	3	0
15	124	0	2	0	0	0
16	6	31	0	0	0	0
17	0	1	0	0	0	0
18	3624	26	0	0	0	0
19	9	3	0	0	0	0
20	0	986	0	0	1	0
21	1976	2	5	0	0	0
22	0	17	0	0	2	0
23	0	1	0	0	0	0
24	112493	250	9	0	0	0

The following table shows, for those shapes tiling by translation, how many do or do not also tile by 180° rotation.

n	180° as well	translation only
6	9	4
12	380	396
18	10668	19064
24	322587	1108204

Polymorphic tiles

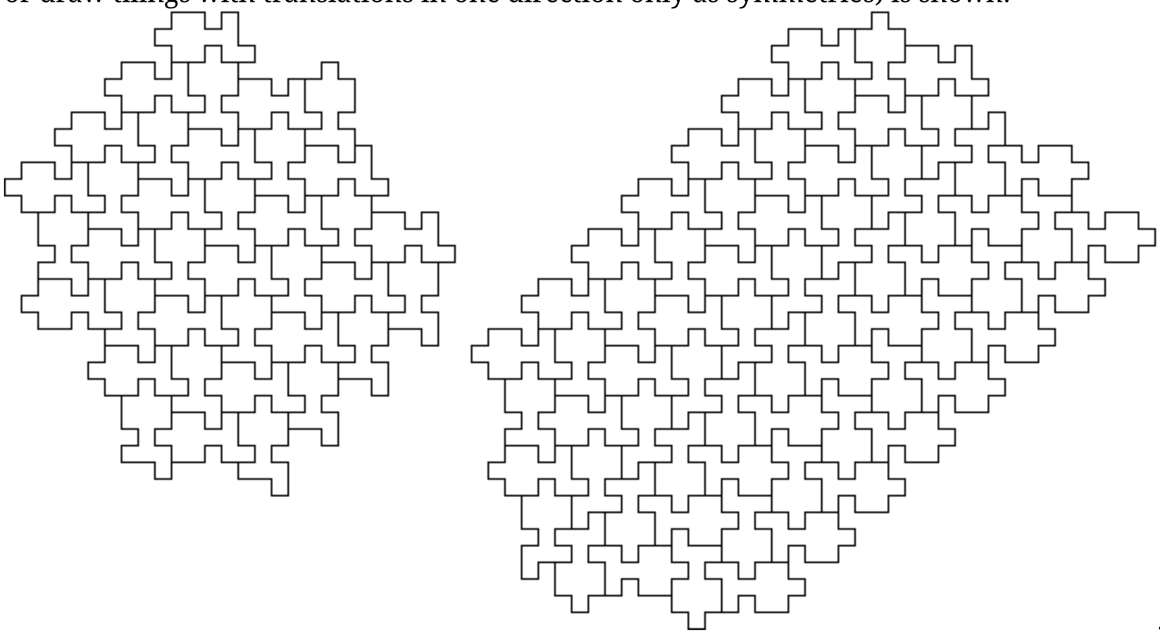
A shape is said to be *k*-morphic if it tiles the plane in exactly *k* ways. Examples for *k*-morphic polyominoes for all *k* from 0 to 10 inclusive have been presented in the literature (mainly by Fontaine and Martin), with infinite families up to *k* = 9 and a single example (up to similarity) of a 10-morphic tile.

My programs can attempt to determine the number of tilings by a polyomino, polyiamond, polyhex or polykite, whether finite, or an uncountable infinity. However, they cannot handle all shapes within the ranges above. The following in particular cannot be handled:

- aperiodic shapes;

- shapes tiling in a countable infinity of ways (may not exist);
- shapes tiling in finitely many ways, at least one of which is not periodic in two non-parallel directions; and
- shapes tiling in an uncountable infinity of ways, at least one being periodic, but not provable to tile in an uncountable infinity of ways by finding a region or strip that forms part of a periodic tiling and can be filled in more than one way.

A dimorphic 12-omino of the third type (tilings drawn manually, since my programs cannot find or draw tilings with translations in one direction only as symmetries) is shown:



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My programs have found three new 10-morphic tiles, an 11-omino, a 16-hex and a 19-omino, and an 11-morphic tile, a 19-omino.

These tables show the numerical results obtained on polymorphic polyominoes, polyiamonds and polyhexes. Non-tiling shapes are not included. The columns headed 1 through 11 show the number of shapes of order *n* found to tile in that number of ways, the column headed ∞ shows the number of shapes found to tile in an uncountable infinity of ways, and the column headed “skipped” shows the number of shapes that tile the plane that were not successfully handled by the program (most probably because they were of the third or fourth type above, or otherwise were too complicated for the program to handle or met the heuristics to detect tiles of those types). The PDF listings do not include any of those “skipped” shapes.

Polyominoes

n	1	2	3	4	5	6	7	8	9	10	11	∞	skipped
1	0	0	0	0	0	0	0	0	0	0	0	1	0
2	0	0	0	0	0	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	0	0	0	0	0	2	0
4	0	0	0	0	0	0	0	0	0	0	0	5	0

5	1	0	0	0	0	0	0	0	0	0	0	11	0
6	0	0	0	0	0	0	0	0	0	0	0	35	0
7	6	6	3	0	0	0	0	0	0	0	0	89	0
8	20	18	4	2	0	0	1	0	0	0	0	298	0
9	193	84	14	6	1	0	0	0	0	0	0	752	0
10	749	257	41	2	2	1	0	0	0	0	0	2018	0
11	3222	809	148	31	12	3	2	0	0	1	0	2392	0
12	9026	1440	153	22	4	0	0	0	0	0	0	12803	1
13	25090	3645	435	94	25	4	3	3	0	0	0	9368	2
14	63746	5681	416	56	17	2	1	0	0	0	0	39176	9
15	180669	11842	665	85	13	1	0	1	0	0	0	86306	1
16	366557	16758	1128	142	37	6	2	0	0	0	0	162257	6
17	683157	30733	1473	264	63	9	2	1	0	0	0	111321	12
18	2192816	65557	2226	238	31	1	1	1	0	0	0	796444	17
19	2936540	72811	2130	360	96	7	2	1	1	1	1	369160	14
20	9444080	126363	3550	322	68	42	1	1	0	0	0	2552238	18
21	18299457	221446	4767	458	90	14	0	0	2	0	0	3394939	19

Polyhexes

n	1	2	3	4	5	6	7	8	9	10	∞	skipped
1	1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	0	0	0	0	3	0
4	0	0	0	0	0	0	0	0	0	0	7	0
5	0	0	0	0	0	0	0	0	0	0	22	0
6	2	0	0	0	0	0	0	0	0	0	75	0
7	28	8	4	0	1	0	0	0	0	0	252	1
8	196	70	13	4	0	0	1	0	0	0	770	0
9	1085	224	41	7	1	0	0	0	0	0	2425	5
10	3869	640	64	14	1	0	0	0	0	0	6738	0
11	13782	1840	212	34	8	1	1	0	1	0	8911	0
12	53467	5108	369	29	3	1	0	0	0	0	44661	3
13	116624	8357	619	92	20	4	4	0	0	0	38809	30
14	349511	16146	656	84	10	6	0	0	0	0	166088	9
15	1094609	35174	1231	84	8	1	0	0	0	0	443142	3
16	2260495	50854	1878	188	32	4	0	0	0	1	626405	41
17	4040091	78581	1748	269	33	3	1	0	0	0	640255	28

Polyiamonds

n	1	2	3	4	5	6	∞	skipped
1	0	0	0	0	0	0	1	0
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	0
4	0	0	0	0	0	0	3	0
5	0	0	0	0	0	0	4	0
6	2	0	0	0	0	0	10	0
7	0	0	0	0	0	0	23	0
8	2	0	0	0	0	0	64	0
9	18	8	2	0	0	0	111	0
10	62	15	3	0	0	0	261	0
11	284	64	8	1	0	0	210	0
12	491	90	12	1	0	0	1440	0
13	1623	249	30	7	0	0	586	0
14	2995	321	16	2	2	0	3018	0
15	7665	507	14	0	0	0	4722	0
16	15640	842	37	14	2	1	13725	0
17	22755	1214	34	0	1	0	2552	0
18	67185	2745	93	20	2	0	40095	5
19	85741	2846	61	6	0	0	7313	0
20	248302	5394	147	39	10	2	123623	6
21	401673	7216	107	2	0	0	90674	0

Enumerating all *k*-isohedral tilings

My programs can enumerate all *k*-isohedral tilings by a polyform (that are aligned to the corresponding underlying regular or Laves tiling). This table shows the number of isohedral tilings by *n*-ominoes for some *n* (with 2-isohedral tilings also for small *n*); more details for *k*-isohedral tilings and data for other kinds of polyform may be added later.

Polyominoes

n	1	2
1	1	0
2	6	28
3	25	82
4	74	308
5	155	545
6	578	2059
7	709	1761
8	2511	6913

9	5250
10	12527
11	14586
12	72868
13	64703
14	211554

Algorithms and data structures

The general algorithm used by my programs for determining whether polyforms tile, and how simply, is as follows.

1. List the fixed shapes of a given order, using Redelmeier's method. We define an (arbitrary) ordering for fixed tiles (first by the dimensions of the box the tile fits in then by lexicographic ordering of the bitstring indicating which cells inside that box are used), for each fixed tile generate all aspects and discard that tile unless it is of the aspect first in the ordering—i.e., we define a canonical aspect for each tile and ignore the non-canonical aspects when they get generated. Checking canonicity isn't very efficient for pure enumeration (hence why Redelmeier enumerates symmetric shapes separately and gets the enumeration of free shapes as a linear combination of the numbers with various symmetries), but it's much more efficient than checking for tiling properties 8 or 12 times and then applying linear relations to get the numbers of free shapes tiling in a given way.
2. Eliminate shapes with holes, translation tilings and 180-degree rotation tilings by considering the boundary (Conway's criterion and the translation criterion).
3. Generate and reduce a list of the possible neighbours of a tile (i.e., of those neighbours that might appear in the tiling). If shape *X* (standard position and orientation) has a putative neighbour *Y*, and position *P* is next to *X* and not contained in *Y*, but there is no possible neighbour of *X* that contains *P* and does not intersect *Y*, then *Y* is not in fact a possible neighbour of *X*. Applying this appropriately causes most shapes to be found not to tile with no subsequent backtracking search required. ("Appropriately" includes not checking all (*Y*, *P*) pairs: since this method is so efficient at showing shapes not to tile, checking all (*Y*, *P*) pairs just consumes extra time to little benefit; only *P* close to *Y* in fact need to be checked for most of the benefits.) As more information is gained, the list of possible neighbours is reduced further at subsequent stages.
4. Do a backtracking search for isohedral tilings. The programs can also list all isohedral tilings by a given shape if desired, or all *k*-isohedral tilings.
5. Do a backtracking search to find all ways of surrounding the shape (which don't include any pair of neighbours previously ruled out). If this generates too many possibilities then it jumps out to a further reduction step of determining exactly which neighbours can occur in a complete surround of a tile, but this rarely occurs.
6. Another reduction of this list based on which neighbours occur in it. Given a tile *T* and the ways of surrounding *T*, if some neighbour *U* of *T* occurs in none of these surrounds, for any isometry *I* we may eliminate all surrounds containing *IT* and *IU* as not being extensible to a tiling. This may be repeated until the lists of possible surrounds and neighbours stabilise.

7. Check for 2-isohedral and 3-isohedral tilings if there are any possible surrounds left.
8. At this point we have a collection of possible *k*-fold surrounds of a tile. We do the following to expand to (*k*+1)-fold surrounds:

   1. Check consistency: each tile within each *k*-fold surround should individually be surroundable by one of the *k*-fold surrounds, if that *k*-fold surround can be extended. Repeat until the list stabilises. (There are optimisations to speed this up when there are a great number of surrounds, and to record tiles found here to be forced.)
   2. Try to surround each tile in the first layer of each *k*-fold surround with the possible *k*-fold surrounds, in a backtracking search to extend to (*k*+1)-fold surrounds. When each is found, determine whether it contains three non-collinear tiles, in the same aspect, the translations between which yield a periodic tiling. If a periodic tiling is found, determine the smallest number of orbits (by now known to be at least 4) with which the shape tiles.

   If the shape tiles periodically, this process will terminate with a tiling. If the process does not terminate (with a periodic tiling or a finding that there are no *k*-fold surrounds for some *k*), the shape must tile (by the Extension Theorem), and must be aperiodic. In practice, the programs have internal limits on the size of coordinates of shapes allowed, and will exit if those limits are exceeded; thus any aperiodic tile, or a tile whose simplest periodic tiling is sufficiently complicated, will cause the programs to exit with an error.

A search for *k*-isohedral tilings starts with a shape in class 1, and looks for neighbours in some position. Such a neighbour is either in an already known class, or in a new class (which we may suppose is the first class not yet used). If a known class, the symmetries of the tiling may force additional tiles to be present. Repeat until all tiles of all classes are surrounded, or a contradiction is found. (In the case of symmetrical tiles, the tiles are initially assumed to have asymmetrical markings, until symmetries are forced. In fact all isohedral tilings can have asymmetrical markings placed on the tiles while staying isohedral, but this does not apply to 2-isohedral and higher tilings.) At least for isohedral tilings this is polynomial in the size of the tile (probably for *k*-isohedral tilings as well; potentially exponential in *k* though at least each *k*-isohedral tiling only gets found *k* times, once with each class in the centre, not *k*! times), although this is a different algorithm from that of Keating and Vince. Abstractly, for each *k* there is a finite set of conditions similar to Conway's criterion for determining *k*-isohedral tiling, since the number of types of *k*-isohedral tiling must be finite for each *k*. The criteria for isohedral tilings have been listed explicitly.

Bitmaps are maintained to show whether a shape in a given position and aspect is consistent with a shape in standard position and orientation. Initially only intersecting shapes (or possibly pairs of shapes enclosing a hole whose area is not a multiple of the area of the shape) are inconsistent, but as neighbours are found not to be possible in a tiling they are also marked as inconsistent.

The optimisations above reducing sets of possible neighbours are fundamentally incompatible with computing Heesch numbers, although it is possible much weaker versions could be used in that context.

The general algorithms for finding *k*-morphic tiles (which can't handle every tile) are thus:

1. Determine ways of surrounding a central tile n times, eliminating as many as possible that can't occur in a tiling.
2. If for some n we find that each way extends in at most one way (that might occur in a tiling) to surround the tile n+1 times, then we have finitely many tilings, all periodic, and we can straightforwardly list them all. (This fails to find k-morphic tiles where not all tilings are periodic in two directions, such as the 12-omino shown above.)
3. From the patches of tiles we have, find as many tilings as possible (by looking at non-collinear triples of tiles in the same aspect and seeing if the translations between them could be symmetries of a tiling extending that patch). My programs also generate isohedral, 2-isohedral and 3-isohedral tilings at appropriate points to add to this list, and other methods could also be used to find more tilings.
4. From the tilings found so far, take pairs of tilings (possibly the same tiling in different positions or orientations) and lay them on top of each other in each possible relative position and orientation. Form an infinite bipartite graph whose vertices are the tiles of the two tilings and where two vertices are joined if the corresponding tiles overlap. If this graph has a component with more than two vertices and not having translations in two non-parallel directions as symmetries, then the tile has an uncountable infinity of tilings. (Such a component corresponds to a region of a tiling, either finite or an infinite strip, that can be filled in more than one way. This method of showing a tile to have an uncountable infinity of tilings does not appear to work for all tiles with an uncountable infinity of tilings.)

Because of the cases this can't handle, the programs have heuristics to stop after surrounding too many times or finding too many possible patches without determining how many tilings a tile has.

References

The list here is deliberately very selective, giving general survey references rather than tracking results and terminology back to their original sources.

General references

The first two of these in particular have extensive bibliographies that may be used to find original sources for the older results quoted here.

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Last updated: 1 August 2023