INTER-UNIVERSAL TEICHMÜLLER THEORY I: CONSTRUCTION OF HODGE THEATERS

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May 2020

Abstract. The present paper is the first in a series of four papers, the goal of which is to establish an *arithmetic* version of *Teichmüller theory* for **number fields** equipped with an **elliptic curve** — which we refer to as **"inter-universal Teichmüller theory"** — by applying the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in earlier papers by the author. We begin by fixing what we call *"initial Θ-data"*, which consists of an *elliptic curve* E_F over a *number field* F, and a *prime number* $l \geq 5$, as well as some other technical data satisfying certain technical properties. This data determines various *hyperbolic orbicurves* that are related via finite étale coverings to the once-punctured elliptic curve X_F determined by E_F. These finite étale coverings admit various *symmetry properties* arising from the **additive** and **multiplicative** structures on the ring $\mathbb{F}_l = \mathbb{Z}/l\mathbb{Z}$ acting on the *l-torsion points* of the elliptic curve. We then construct *"$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters"* associated to the given Θ-data. These $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters may be thought of as *miniature models of* **conventional scheme theory** in which the **two underlying combinatorial dimensions** of a number field — which may be thought of as corresponding to the **additive** and **multiplicative** structures of a ring or, alternatively, to the **group of units** and **value group** of a local field associated to the number field — are, in some sense, **"dismantled"** or **"disentangled"** from one another. All $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters are isomorphic to one another, but may also be related to one another by means of a **"Θ-link"**, which relates certain *Frobenioid-theoretic* portions of one $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater to another in a fashion that is **not compatible** *with the respective* **conventional ring/scheme theory structures**. In particular, it is a *highly nontrivial problem to relate the ring structures* on either side of the Θ-link to one another. This will be achieved, up to certain *"relatively mild indeterminacies"*, in future papers in the series by applying the **absolute anabelian geometry** developed in earlier papers by the author. The resulting *description of an* **"alien ring structure"** [associated, say, to the *domain* of the Θ-link] in terms of a given ring structure [associated, say, to the *codomain* of the Θ-link] will be applied in the final paper of the series to obtain results in *diophantine geometry*. Finally, we discuss certain technical results concerning **profinite conjugates** *of decomposition and inertia groups in the* **tempered fundamental group** of a p-adic hyperbolic curve that will be of use in the development of the theory of the present series of papers, but are also of independent interest.

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