OPTIMAL CONTROL

Heat transfer

Optimal control problem - Control problem

Let the heat equation in the cross section of the steel plate be given by

$$
\begin{aligned}
\rho c y_t - \operatorname{div}(k \nabla y) &= 0 && \Omega \times (0, T) \\
-k y_\nu &= g(t)(y - w) && \Gamma_0 \times (0, T) \\
y_\nu &= 0 && \partial\Omega \setminus \Gamma_0 \times (0, T) \\
y(0) &= y_0 && \Omega
\end{aligned}
\tag{1}
$$

Where $y(x, t)$ is the temperature for $x \in \Omega$, $t \in (0, T)$. We aim to minimize the following cost function

$$
J(y, w) = \frac{1}{2} \int_\Omega (y(T) - y_d)^2 \, dx + \frac{\gamma}{2} \int_0^T \int_{\Gamma_0} w(t)^2 \, ds \, dt \tag{2}
$$

Optimality conditions - Lagrangian

Weak form of the state equation (1)

$$\int_\Omega \rho c y_t v \; dx + \int_\Omega k \nabla y \nabla v \; dx + \int_{\Gamma_0} g(t) y v \; ds = \int_{\Gamma_0} g(t) w v \; ds \qquad \forall v \in H^1(\Omega) \tag{3}$$

Yields the Lagrangian

$$\begin{aligned} L(y,p,w) = {} & \frac{1}{2}\int_\Omega (y(T) - y_d)^2 \; dx + \frac{\gamma}{2}\int_0^T \int_{\Gamma_0} w(t)^2 \; ds \; dt \\ & - \int_0^T \int_\Omega \rho c y_t p \; dx \; dt - \int_0^T \int_\Omega k \nabla y \nabla p \; dx \; dt \\ & - \int_0^T \int_{\Gamma_0} g(t)(y - w) p \; ds \; dt \end{aligned}$$

Optimality conditions - Adjoint equation

From $L_y(\bar{y}, \bar{p}, \bar{w})h = 0$ we find the adjoint equation

$$\begin{aligned} -\rho c p_t - \operatorname{div}(k\nabla p) &= 0 && \text{in } \Omega \times (0, T) \\ -k\partial_\nu p &= pg(t) && \text{on } \Gamma_0 \times (0, T) \\ -k\partial_\nu p &= 0 && \text{on } \Gamma_1 \times (0, T) \\ \rho c p(T) &= y(T) - y_d && \text{in } \Omega \end{aligned} \tag{4}$$

where χ is the characteristic function.

Optimality conditions - Frame Title

From the variational inequality of we can extract the gradient. As

$$
\begin{aligned}
L_w(\bar{y}, \bar{p}, \bar{w})(w - \bar{w}) &\geq 0, \\
\gamma \int_0^T \int_{\Gamma_0} \bar{w}(w - \bar{w}) \, ds \, dt + \int_0^T \int_{\Gamma_0} g(w - \bar{w})\bar{p} \, ds \, dt &\geq 0, \\
\int_0^T \int_{\Gamma_0} \underbrace{(\gamma w + gp)}_{\nabla f} h \, ds \, dt &\geq 0,
\end{aligned}
$$

the gradient of our system is given by

$$\nabla f = \gamma w + gp|_{\Gamma_0}.$$

Optimality conditions - Reduced functional

The reduced functional is then defined by

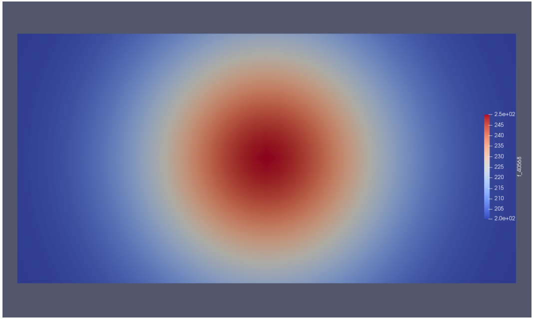
$$f(w) := J(Sw, w) = \frac{1}{2}\int_{\Omega}(S(w)(T) - y_d)^2 \, dx + \frac{\gamma}{2}\int_0^T \int_{\Gamma_0} w(t)^2 \, ds \, dt \tag{5}$$

S is the solution mapping $S : L^2(P) \to L^2(Q)$ with the spaces

$$\begin{aligned} Q &:= \Omega \times (0, T) \\ P &:= \Gamma_0 \times (0, T) \\ L^2(Q) &:= L^2(0, T; L^2(\Omega)) \\ L^2(P) &:= L^2(0, T; L^2(\Gamma_0)) \end{aligned}$$

Numerical solution - Method

- Finite difference time-discretization
- Finite element space-discretization
- Projected gradient descent method



Numerical solution - Time discretization

We discretize the system in time uniformly by

$$t^n = n \cdot \Delta t, \; n = 0, \dots, N, \; \Delta t = \frac{T}{N}.$$

The time derivative is approximated using the backwards Euler approximation

$$y_t(x, t^n) = f(x, t^n, y(t^n)) \approx \frac{y^n(x) - y^{n-1}(x)}{\Delta t} = f(x, t^n, y^n).$$

Numerical solution - State equation

The discretized weak form of the state equation is given by

$$\int_{\Omega} \rho c y^{n+1} v + \Delta t \cdot k \nabla y^{n+1} \nabla v \, dx + \Delta t \int_{\Gamma_0} g(t) y^{n+1} v \, ds =$$

$$= \int_{\Omega} \rho c y^{n} v \, dx + \Delta t \int_{\Gamma_0} g(t) w v \, ds, \ \forall \, v \in H^1(\Omega),$$

$$y^0 = y_0(x).$$

Numerical solution - Adjoint equation

The adjoint equation can be transformed to a parabolic initial-value problem by the time reversal

$$\tilde{p}(x,t) = p(x, T-t).$$

The discretized weak form of the reversed adjoint equation is then given by

$$\begin{aligned}\int_\Omega \rho c \tilde{p}^{n+1} h + \Delta t \cdot k \nabla \tilde{p}^{n+1} \nabla h \, dx + \Delta t \int_{\Gamma_0} g(t) \tilde{p}^{n+1} h \, ds = \\ = \int_\Omega \rho c \tilde{p}^n h \, dx, \; \forall \; h \in H^1(\Omega), \\ \tilde{p}^0 = \frac{1}{\rho c}(y(T) - y_d).\end{aligned}$$

We regain the original adjoint equation by reversing the steps

Projected gradient method - Gradient descent step

To find the control minimizing the cost functional we use a projected gradient descent method. A step in this method is defined by

$$\begin{aligned}
\tilde{w}_{k+1} &= w_k - \alpha_k \nabla f_k, \\
w_{k+1} &= \mathcal{P}_{[w_a, w_b]}(\tilde{w}_{k+1}).
\end{aligned}$$

The projection operator $\mathcal{P}_{[w_a, w_b]}$ is defined by

$$\mathcal{P}_{[w_a, w_b]}(w)(x) = \begin{cases} w_a(x), \text{ if } w(x) < w_a(x), \\ w(x), \text{ if } w(x) \in [w_a(x), w_b(x)], \\ w_b(x), \text{ if } w(x) > w_b(x). \end{cases}$$

Projected gradient method - Calculating step length

The step length α_k is chosen using line descent with backtracking Armijo condition. We look for a step length satisfying the Armijo condition

$$f(\tilde{w}_{k+1}) \leq f(w_k) - \alpha_k \cdot c_1 ||\nabla f_k||^2, \tag{6}$$

where $0 < c_1 < 1$ is a predetermined constant. The step length α_k is determined according to the process

$\alpha_k \leftarrow \alpha_0$

while (6) not satisfied:

$$\alpha_k \leftarrow \frac{1}{2}\alpha_k$$

with α_0 predetermined according to the size of the admissible set.

Projected gradient method - End condition

The gradient method iteration is halted when any of three conditions is met:

Maximum iteration	$k \geq \mathrm{Max.iter}$
Absolute tolerance	$f(w_k) \leq \mathrm{Abs.tol}$
Relative change	$f(w_k)/f(w_{k-1}) \geq \mathrm{Rel.tol}$

Test problem - Constructed test problem

We construct a test problem by choosing a control in the admissible set and setting the end time of this state as the desired temperature distribution. By doing this, we can be sure that a reasonable solution is attainable.

$$y_d(x) = S(w_d)(x, T)$$

For our test problem we choose the constant control

$$w_d(x, t) = 40$$

Physical constants $\rho, c, k, g(t)$ are chosen arbitrarily to display some interesting behaviour.

Test problem - Constructed test problem

Geometry:

$$\Omega = (0, L) \times (0, B)$$
$$L = 4,\ B = 2$$
$$T = 3,\ N = 30$$

Physical constants:

$$\rho = 1,\ c = 100,$$
$$k(x, t) = 50$$
$$g(t) = 15 \cdot e^{-(t-0.5T)^2}$$

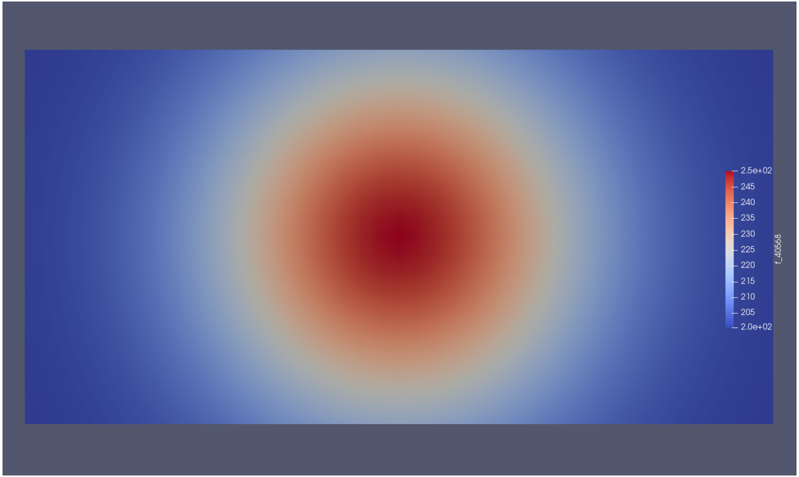
$$\Gamma_0 = \{x \in \partial\Omega : x_2 = 0 \text{ or } x_2 = B\}$$
$$\Gamma_1 = \{x \in \partial\Omega : x_1 = 0 \text{ or } x_1 = L\}$$

Control constraints: $w_a(x) = 4 \quad w_b(x) = 90$

Initial condition: $y_0(x) = 200 + 50 \cdot e^{-(x_1-0.5L)^2-(x_2-0.5B)^2}$

$$f(w) = J(w, y) = \frac{1}{2}\int_\Omega (y(x, T) - y_d(x))^2\, dx + \frac{\gamma}{2}\int_0^T \int_{\Gamma_0} w^2\, ds,\ \gamma = 10^{-5}$$

Test problem - Initial condition



Test problem - Solver parameters

Halting conditions:

$\text{Max.iter} = 20$	No more than 20 iterations
$\text{Abs.tol} = 0$	Inactive condition
$\text{Rel.tol} = 0.999$	Halt when less than 0.1% improvement

Armijo condition constants:

$$\alpha_0 = 10$$
$$c_1 = 0.5$$

$$f(\tilde{w}_{k+1}) \leq f(w_k) - \alpha_k \cdot c_1 ||\nabla f_k||^2$$

Initial control:

$$w_0(x, t) = 10 + 9 \cdot t$$

Test problem - Convergence