WUOLAH

Tema-1-Ejercicios-RECURRENCIA-Re...

CarlosGarSil98

Fundamentos de análisis de algoritmos

1º Grado en Ingeniería Informática

Escuela Técnica Superior de Ingeniería
Universidad de Huelva

KEEP
CALM
AND
ESTUDIA
UN POQUITO

1. Calcular la eficiencia del siguiente algoritmo:

```
int ejemplo1(int n)
{
    if (n <= 1)
          return 1;
    else
          return (ejemplo1(n - 1) + ejemplo1(n - 1));
}
```

T_{if} = condición + devuelve = 1 + 1 = 2 ⟶ cuando $n \leq 1$

T_{else} = condición + devuelve + llamada + $T(n-1)$ + resta + suma + llamada + $T(n-1)$ + resta

$T_{else} = 7 + 2T(n-1)$ ⟶ cuando $n > 1$

$$T(n) \begin{cases} 2 & \text{si } n \leq 1 \\ 2T(n-1) + 7 & \text{si } n > 1 \end{cases}$$

$T(n) = 2T(n-1) + 7$; $T(n) - 2T(n-1) = 7$ No Homogénea

$T(n) - 2T(n-1) = 1^n \cdot 7 \cdot n^0$; $(x-2)(x-1)^{0+1} = 0$ ⟶ Raíces $\begin{cases} r_1 = 2 & \text{simple} \\ r_2 = 1 & \text{simple} \end{cases}$

$T(n) = C_1 \cdot r_1^n \cdot n^0 + C_2 \cdot r_2^n \cdot n^0 = C_2 \cdot 1^n + C_1 \cdot 2^n = C_2 + C_1 \cdot 2^n$

$T(1) = 2 \longrightarrow C_1 \cdot 2^n + C_2 = 2$; $C_2 = 2 - 2C_1$ Cogemos $T(1)$ por ser caso base

$T(2) = 2T(2-1) + 7 = 2T(1) + 7 = 2(2) + 7 = 4 + 7 = 11$

$T(2) = C_1 \cdot 2^2 + C_2 = C_1 \cdot 2^2 + 2 - 2C_1 = 11$; $4C_1 - 2C_1 = 11 - 2$; $2C_1 = 9$; $C_1 = 9/2$

$C_2 = 2 - 2C_1 = 2 - 2 \cdot 9/2 = 2 - 9 = -7$

$$T(n) = \frac{9}{2} \cdot 2^n - 7$$

2. Calcular la eficiencia del siguiente algoritmo:

```
int ejemplo2(int n)
{
   if (n == 1)
         return n;
   else
         return (ejemplo2(n/2) + 1);
}
```

T_{if} = condicion + devuelve = $1+1=2$ $\longrightarrow$ cuando $n=1$

T_{else} = condicion + devuelve + llamada + $T(n/2)$ + division + suma $\longrightarrow$ Cuando $n \neq 1$

$$T(n) = \begin{cases} 2 & \text{si } n = 1 \\ T(n/2) + 5 & \text{si } n \neq 1 \end{cases}$$

$T(n) = T(n/2) + 5$; $T(n) - T(n/2) = 5$ No Homogénea

cambio base $n = 2^k$ $T(2^k) - T(2^{k-1}) = 5$ cambio base $2^k = t_k$

$t_k - t_{k-1} = 1^k \cdot 5 \cdot k^0$ cambio base $t_k = x$ $(x-1)(x-b)^{d+1} = 0$; $(x-1)(x-1) = 0$

Raiz doble = 1

$T(n) = T(2^k) = c_1 \cdot r_1^k \cdot k^0 + c_2 \cdot r_1^k \cdot k^1 = c_1 \cdot r_1^k + c_2 \cdot r_1^k \cdot k$

tenemos 2^k y queremos $n \rightarrow 2^k = n$; $\log_2 n = k$ realizamos dicho cambio de base

$T(n) = c_1 \cdot r_1^{\log_2 n} + c_2 \cdot r_1^{\log_2 n} \cdot \log_2 n$; $T(n) = c_1 + c_2 \cdot \log_2 n$; $T(1) = 2 = c_1 + c_2 \log_2 n$

$c_1 = 2 - c_2 \cdot 0$; $c_1 = 2$

$T(2) = T(2/2) + 5 = T(1) + 5 = 2 + 5 = 7$; $c_1 + c_2 \log_2 2 = 7$; $2 + c_2 \cdot 1 = 7$; $c_2 = 7 - 2 = 5$

$T(n) = 2 + 5 \cdot \log_2 n$

3. Calcular la eficiencia del algoritmo de las Torres de Hanoi por expansión de la recurrencia.

```
Hanoi(origen,destino,pivote,discos):
    si discos=1
        moveruno(origen,destino)
    en otro caso
        Hanoi(origen,pivote,destino,discos-1)
        moveruno(origen,destino)
        Hanoi(pivote,destino,origen,discos-1)
```

Hanoi (origen, destino, pivote, disco) ⟶ disco nos dice el tamaño, el número de discos, por tanto, disco = n, quedando de la siguiente forma

T_{si} = condición + llamada (moveruno)

T_{sino} = condición + llamada (Hanoi) + $T(n-1)$ + resta + llamada (moveruno) + + llamada (Hanoi) + $T(n-1)$ + resta

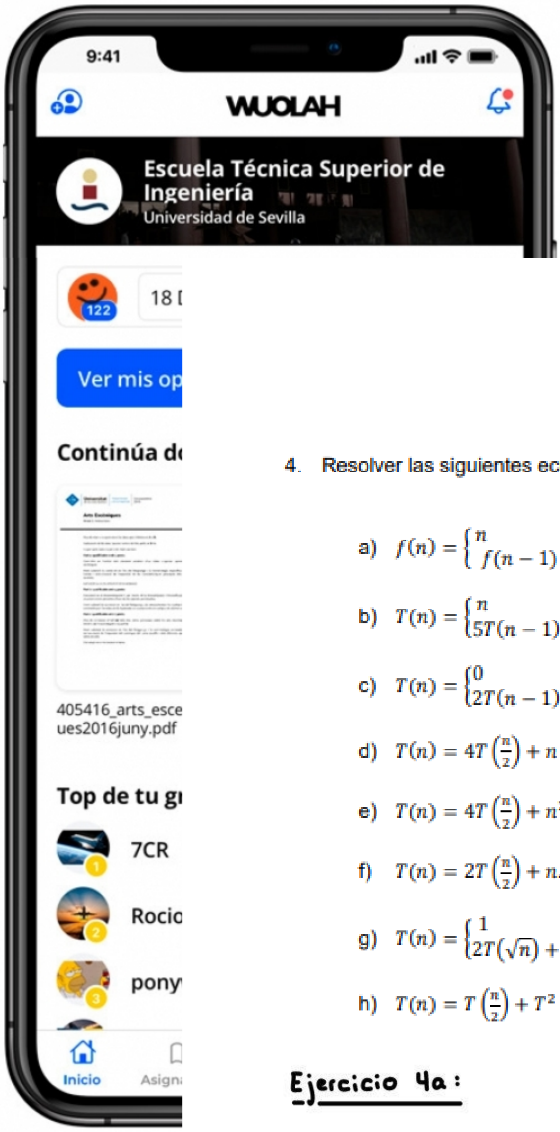
$$T(n) \begin{cases} T_{si} & \text{si } n=1 \\ T_{sino} & \text{si } n\neq 1 \end{cases} \longrightarrow T(n) \begin{cases} 2 & \text{si } n=1 \\ 2T(n-1)+6 & \text{si } n\neq 1 \end{cases}$$

$$T(n) = 2T(n-1)+6 = 2(2T(n-2)+6)+6 = 2^2\cdot T(n-2)+6+12 = 2^2(2T(n-3)+6)+6+12 =$$
$$= 2^3\cdot T(n-3)+6+12+6\cdot 2^2 = 2^n\cdot T(1)+6+6\cdot 2^1+6\cdot 2^2+\dots 6\cdot 2^{n-1};$$

$$T(n) = 2^n + 6\cdot 2^{n-1} - 6$$

$T(1) = 2$
$T(2) = 2\cdot T(2-1)+6 = 2\cdot 2+6 = 10$
$T(3) = 2\cdot T(3-1)+6 = 2\cdot 10+6 = 26$
$T(4) = 2\cdot T(4-1)+6 = 2\cdot 26+6 = 58$

$T(1) = 2$
$T(2) = 2^2+6\cdot 2^{2-1}-6 = 10$
$T(3) = 2^3+6\cdot 2^{3-1}-6 = 26$
$T(4) = 2^4+6\cdot 2^{4-1}-6 = 58$



4. Resolver las siguientes ecuaciones y dar su orden de complejidad:

a) $f(n) = \begin{cases} n & si\ n = 0\ ó\ n = 1 \\ f(n-1) + f(n-2) & en\ otro\ caso \end{cases}$

b) $T(n) = \begin{cases} n & si\ n = 0,1\ ó\ 2 \\ 5T(n-1) - 8T(n-2) + 4T(n-3) & en\ otro\ caso \end{cases}$

c) $T(n) = \begin{cases} 0 & si\ n = 0 \\ 2T(n-1) + 1 & en\ otro\ caso \end{cases}$

d) $T(n) = 4T\left(\frac{n}{2}\right) + n\,, \quad n > 1\ y\ potencia\ de\ 2$

e) $T(n) = 4T\left(\frac{n}{2}\right) + n^2\,, \quad n > 1\ y\ potencia\ de\ 2$

f) $T(n) = 2T\left(\frac{n}{2}\right) + n.\log n\,, \quad n > 1\ y\ potencia\ de\ 2$

g) $T(n) = \begin{cases} 1 & si\ n = 2 \\ 2T(\sqrt{n}) + \log\log n & con\ n \geq 4 \end{cases}$

h) $T(n) = T\left(\frac{n}{2}\right) + T^2\left(\frac{n}{4}\right) \quad$ sin caso base

Ejercicio 4a:

$f(n) = f(n-1) + f(n-2)$; $f(n) - f(n-1) - f(n-2) = 0$ Homogénea

$$x^2 - x - 1 = 0 \longrightarrow \frac{-b \pm \sqrt{b^2 - 4\cdot a\cdot c}}{2\cdot a} = \frac{1 \pm \sqrt{1 - 4\cdot 1\cdot(-1)}}{2\cdot 1} \quad r_1 = \frac{1+\sqrt5}{2} \quad r_2 = \frac{1-\sqrt5}{2}$$

$$f(n) = C_1\cdot\left(\frac{1+\sqrt5}{2}\right)^n\cdot n^0 + C_2\cdot\left(\frac{1-\sqrt5}{2}\right)^n\cdot n^0$$

$f(0) = 0 = C_1\cdot 1 + C_2\cdot 1$; $0 = C_1 + C_2$; $C_1 = -C_2$

$$f(1) = 1 = C_1\cdot\left(\frac{1+\sqrt5}{2}\right)^1 + C_2\cdot\left(\frac{1-\sqrt5}{2}\right)^1 ; \; C_2\cdot\left(\frac{1-\sqrt5}{2}\right)^1 = 1 - C_1\cdot\left(\frac{1+\sqrt5}{2}\right)^1 ;$$

$$C_2 = \frac{1 - C_1\cdot\left(\frac{1+\sqrt5}{2}\right)^1}{\left(\frac{1-\sqrt5}{2}\right)^1} = -\left(\frac{1+\sqrt5}{2}\right)^1 + C_1\cdot\left(\frac{3+\sqrt5}{2}\right)^1$$

$$C_1 = -\left(-\left(\frac{1+\sqrt5}{2}\right)^1 + C_1\cdot\left(\frac{3+\sqrt5}{2}\right)^1\right) ; \; C_1\cdot\left(\frac{5+\sqrt5}{2}\right)^1 = \left(\frac{1+\sqrt5}{2}\right)^1 ; \; C_1 = \frac{\sqrt5}{5}$$

$-C_2 = C_1$; $C_2 = -C_1 = -\frac{\sqrt5}{5}$

$$f(n) = \frac{\sqrt5}{5}\left(\frac{1+\sqrt5}{2}\right)^n - \frac{\sqrt5}{5}\left(\frac{1-\sqrt5}{2}\right)^n$$

Ejercicio 4b:

$T(n) = 5T(n-1) - 8T(n-2) + 4T(n-3)$; $T(n) - 5T(n-1) + 8T(n-2) - 4T(n-3) = 0$ Homogenea

$x^3 - 5x^2 + 8x - 4 = 0$ → simplificamos mediante método de Ruffini

$$\begin{array}{c|cccc} & 1 & -5 & 8 & -4 \\ 2 & & 2 & -6 & 4 \\ \hline & 1 & -3 & 2 & 0 \\ 2 & & 2 & -2 & \\ \hline & 1 & -1 & 0 & \\ 1 & & 1 & & \\ \hline & 1 & 0 & & \end{array}$$

$(x-2)(x-2)(x-1) = 0$

Raices = $\left[\begin{array}{l} r1 = 2 \text{ doble} \\ r2 = 1 \text{ simple} \end{array}\right.$

$T(n) = c_1 \cdot 2^n \cdot n^0 + c_2 \cdot 2^n \cdot n^1 + c_3 \cdot 1^n \cdot n^0 = c_1 \cdot 2^n + c_2 \cdot 2^n \cdot n + c_3$

$T(0) = 0 = c_1 \cdot 2^0 + c_2 \cdot 2^0 \cdot 0 + c_3 = c_1 + c_3$; $c_1 = -c_3$

$T(1) = 1 = c_1 \cdot 2^1 + c_2 \cdot 2^1 \cdot 1 + c_3 = 2c_1 + 2c_2 + c_3 = -2c_3 + 2c_2 + c_3$; $2c_2 - c_3 = 1$;

$c_3 = -1 + 2c_2$

$T(2) = 2 = c_1 \cdot 2^2 + c_2 \cdot 2^2 \cdot 2 + c_3 = 4c_1 + 8c_2 + c_3 = -4c_3 + 8c_2 + c_3$

$= -4(-1+2c_2) + 8c_2 - 1 + 2c_2 = 4 - 8c_2 + 8c_2 - 1 + 2c_2$; $2c_2 + 3 = 2$; $c_2 = -1/2$

$c_3 = -1 + 2(-1/2) = -1 - 1 = -2$

$c_1 = -(-2) = 2$

$T(n) = 2 \cdot 2^n - \frac{1}{2} \cdot 2^n \cdot n - 2$; $T(n) = 2^{n+1} - n \cdot 2^{n-1} - 2$

Ejercicio 4c:

$T(n) = 2T(n-1) + 1$; $T(n) - 2T(n-1) = 1$ No Homogénea

$f(n) = 1 = 1^n \cdot 1 \cdot n^0$ → $(x-2)(x-1)^{0+1} = 0$

Raices = $\left[\begin{array}{l} r1 = 2 \\ r2 = 1 \end{array}\right.$

$T(n) = c_1 \cdot 2^n \cdot n^0 + c_2 \cdot 1^n \cdot n^0 = c_1 \cdot 2^n \cdot c_2$

$T(0) = 0 = c_1 \cdot 2^n + c_2$; $c_2 = -c_1 \cdot 2^0$; $c_2 = -c_1$

$T(1) = 2T(1-1) + 1 = 2 \cdot 0 + 1 = 1 = c_1 \cdot 2^n + c_2$; $c_2 = 1 - 2c_1$; $c_2 = 1 + 2c_2$;

$-c_2 = 1$; $c_2 = -1$; $c_1 = -c_2 = -(-1) = 1$

$T(n) = 2^n - 1$

Ejercicio 4d:

$T(n) = 4T(n/2) + n$ $\left[\begin{array}{c}\text{cambio de base}\\ n = 2^K\end{array}\right]$ $T(2^K) = 4T(2^K/2) + 2^K$;

$T(2^K) - 4T(2^{K-1}) = 2^K$ No Homogénea

$b^K \cdot p(a) = 2^K = 2^K \cdot K^0$; $b = 0$, $a = 2 \rightarrow (x-a)^{b+1} = (x-2)^1$

$(x-4)(x-2) = 0$ Raices $\left[\begin{array}{l} r_1 = 4 \\ r_2 = 2 \end{array}\right.$

$T(2^K) = C_1 \cdot 4^K + C_2 \cdot 2^K = C_1 \cdot (2^K)^2 + C_2 \cdot 2^K$ $\left[\begin{array}{c}\text{cambio de base}\\ 2^K = n\end{array}\right]$ $T(n) = C_1 \cdot n^2 + C_2 \cdot n$

$T(n) - 4T(n/2) = n$; $C_1 \cdot n^2 + C_2 \cdot n - 4 \cdot \left(C_1 \cdot \left(\frac{n}{2}\right)^2 + C_2 \cdot \left(\frac{n}{2}\right)\right) = n$;

$C_1 \cdot n^2 + C_2 \cdot n - 4 \cdot C_1 \frac{n^2}{4} - 4C_2 \frac{n}{2} = n$; $C_1 \cdot n^2 + C_2 \cdot n - C_1 \cdot n^2 - 2C_2 \cdot n = n$;

$-C_2 \cdot n = n$; $-C_2 = 1$; $C_2 = -1 \longrightarrow T(n) = C_1 \cdot n^2 - n$ No podemos averiguar más sin condiciones iniciales

Ejercicio 4e:

$T(n) = 4T(n/2) + n^2$ $\left[\begin{array}{c}\text{cambio de base}\\ n = 2^K\end{array}\right]$ $T(2^K) = 4T(2^K/2) + (2^K)^2$;

$T(2^K) - 4T(2^{K-1}) = (2^K)^2$; No Homogénea ; $4^K = b^K \cdot p(K) = 4^K \cdot K^0$

$p(x) = (x-4)(x-4) = 0$ Raices $= r_1 = 4$ doble

$T(2^K) = C_1 \cdot 4^K + C_2 \cdot 4^K \cdot K$ $\left[\begin{array}{c}\text{cambio de base}\\ 2^K = n\end{array}\right]$ $T(n) = C_1 \cdot n^2 + C_2 \cdot n^2 \cdot \log_2(n)$

$T(n) - 4T(n/2) = n^2$; $C_1 \cdot n^2 + C_2 \cdot n^2 \cdot \log_2(n) - 4\left(C_1 \cdot \left(\frac{n}{2}\right)^2 + C_2 \cdot \left(\frac{n}{2}\right)^2 \log_2\left(\frac{n}{2}\right)\right) = n^2$

$C_2 n^2 \log_2(n) - C_2 n^2 \log_2\left(\frac{n}{2}\right) = n^2$; $C_2 n^2 \left(\log_2(n) - \log_2(n/2)\right) = n^2$

$C_2\left(\log_2(n) - \log_2(n/2)\right) = 1$; $C_2 = 1 \longrightarrow T(n) = C_1 \cdot n^2 + n^2 \cdot \log_2(n)$

Ejercicio 4f:

$\text{Log}_2 n = \text{Log}_2 2^k = K$

$T(n) = 2T(\frac{n}{2}) + n\text{Log}_2(n)$ $\left[\begin{matrix}\text{cambio de base}\\ n = 2^k\end{matrix}\right]$ $T(2^k) - 2T(2^k/2) = 2^k K$

$T(2^k) - 2T(2^{k-1}) = 2^k K$ No Homogénea $\rightarrow b^k \cdot p(k) = 2^k \cdot K \rightarrow a=2,\ b=1$

$p(x) = (x-2)(x-2)^2 = 0$ Raices $= r_1 = 2$ Triple

$T(2^k) = C_1 \cdot 2^k + C_2 \cdot 2^k \cdot K + C_3 \cdot 2^k \cdot k^2$ $\left[\begin{matrix}\text{cambio de base}\\ 2^k = n\end{matrix}\right]$

$T(n) = C_1 n + C_2 n \cdot \text{Log}_2(n) + C_3 n \text{Log}_2^2(n)$

$C_1 n + C_2 n \cdot \text{Log}_2(n) + C_3 n \text{Log}_2^2(n) - 2\left(C_1\frac{n}{2} + C_2\frac{n}{2}\cdot\text{Log}_2\left(\frac{n}{2}\right) + C_3\frac{n}{2}\text{Log}_2^2\left(\frac{n}{2}\right)\right) = n\text{Log}_2 n$

$n\text{Log}_2(n)(C_2 + C_3\text{Log}_2(n)) - n\text{Log}_2\frac{n}{2}\left(C_2 + C_3\text{Log}_2\left(\frac{n}{2}\right)\right) = n\text{Log}_2(n)$;

$n\left(\text{Log}_2(n)(C_2 + C_3\text{Log}_2(n)) - \text{Log}_2\left(\frac{n}{2}\right)\left(C_2 + C_3\text{Log}_2\left(\frac{n}{2}\right)\right)\right) = n\text{Log}_2(n)$;

$n(C_3 - C_2) + 2C_3 n\text{Log}_2(n) = n\text{Log}_2(n)$; $2C_3 = 1$; $C_3 = \frac{1}{2}$; $C_2 = \frac{1}{2}$

$T(n) = C_1 n + \frac{1}{2} n\text{Log}_2(n) + \frac{1}{2} n\text{Log}_2^2(n)$

Ejercicio 4g:

$T(n) = 2T(\sqrt{n}) + \text{Log Log}(n)$ $\left[\begin{matrix}\text{cambio de base}\\ n = K^2\end{matrix}\right]$ $T(K^2) - 2T(\sqrt{K^2}) = \text{Log Log}(K^2)$

$T(K^2) - 2T(K^{2/2}) = \text{Log Log}(K^2)$ No Homogénea

$p(x) = (x-2)(x-2) = 0$ Raices $= r_1 = 2$ doble

$T(K^2) = C_1 \cdot 2^K \cdot K^0 + C_2 \cdot 2^K \cdot K^1$ $\left[\begin{matrix}\text{cambio de base}\\ 2^k = n\end{matrix}\right]$ $T(n) = C_1 \cdot n + C_2 \cdot n \cdot \text{Log}(n)$

$T(2) = 1 = C_1 n + C_2 n\text{Log}(n) = C_1 \cdot 2 + C_2 \cdot 2$; $C_1 = \frac{1-2C_2}{2}$

$T(4) = 2T(\sqrt{n}) + \text{Log Log}(n) = 2T(\sqrt{4}) + \text{Log Log}(4) = 2T(2) + 1 = 2 \cdot 1 + 1 = 3$

$T(4) = C_1 n + C_2 n\text{Log}(n) = 4C_1 + 4C_2 \cdot 2 = 4C_1 + 8C_2 = 3$;

$4\left(\frac{1-2C_2}{2}\right) + 8C_2 = 3$; $2 - 4C_2 + 8C_2 = 3$; $4C_2 = 1$; $C_2 = \frac{1}{4}$

$C_1 = \frac{1-2C_2}{2} = \frac{1 - 2\frac{1}{4}}{2} = \frac{1}{4} \longrightarrow T(n) = \frac{1}{4}n + \frac{1}{4}n\text{Log}(n)$

Ejercicio 4h:

$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right)^2$ $\left[\begin{array}{c}\text{cambio de base}\\ n = 2^k\end{array}\right]$ $T(2^k) = T(2^{k-1}) + T(2^{k-2})^2$

Vamos a hacer uso de logaritmos ya que no es lineal:

$\text{Log}(T(2^k)) - \text{Log}(T(2^{k-1})) - 2\,\text{Log}(T(2^{k-2})) = 0$

$u_k - u_{k-1} - 2u_{k-2} = 0$ Homogenea

$p(x) = x^2 - x - 2 = 0 \longrightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 \cdot 4 \cdot 1 \cdot (-2)}}{2} = \frac{1 \pm 3}{2} = \begin{cases} r_1 = 2 \\ r_2 = -1 \end{cases}$

$p(x) = (x-2)(x+1) = 0$

$u_k = c_1 \cdot 2^k + c_2(-1)^k = 2^k c_1 - c_2$ $\left[\begin{array}{c}\text{cambio de base}\\ t_k = 2^{u_k}\end{array}\right]$ $t_k = 2^{2^k c_1 - c_2}$

$\left[\begin{array}{c}\text{cambio de base}\\ k = \text{Log}(n)\end{array}\right]\left[\begin{array}{c}\text{cambio de base}\\ n = 2^k\end{array}\right]$ $T(n) = 2^{nc_1 - c_2} = \frac{2^{nc_1}}{2^{c_2}}$

$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right)^2$; $\left(\frac{2^{nc_1}}{2^{c_2}}\right) = \left(\frac{2^{\frac{n}{2}c_1}}{2^{c_2}}\right) + 2\left(\frac{2^{\frac{n}{4}c_1}}{2^{c_2}}\right)^2$;

$\frac{2^{\frac{n}{2}c_1}}{2^{c_2}} = \frac{2^{\frac{n}{2}c_1}}{(2^{c_2})^2}$; $2^{\frac{n}{2}c_1} = \frac{2^{\frac{n}{2}c_1}}{2^{c_2}}$; $2^{c_2} = 1$; $c_2 = \text{Log}(1)$; $c_2 = 0$

$T(n) = \frac{2^{nc_1}}{2^{c_2}} = \frac{2^{nc_1}}{2^0}$; $T(n) = 2^{nc_1}$