

Tema-1-Ejercicios-RECURRENCIA-Re...



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Fundamentos de análisis de algoritmos



1º Grado en Ingeniería Informática



**Escuela Técnica Superior de Ingeniería
Universidad de Huelva**

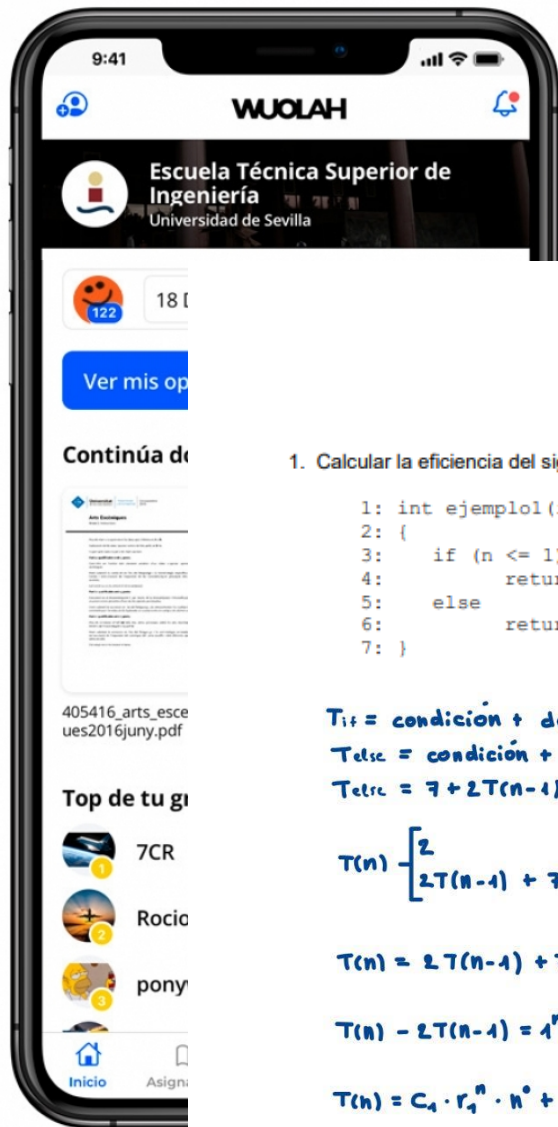


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Ya disponible para el móvil y la tablet.





**KEEP
CALM
AND
ESTUDIA
UN POQUITO**



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1. Calcular la eficiencia del siguiente algoritmo:

```
1: int ejemplo1(int n)
2: {
3:     if (n <= 1)
4:         return 1;
5:     else
6:         return (ejemplo1(n - 1) + ejemplo1(n - 1));
7: }
```

$T_f = \text{condición} + \text{devuelve} = 1 + 1 = 2 \rightarrow \text{Cuando } n \leq 1$

$T_{else} = \text{condición} + \text{devuelve} + \text{llamada} + T(n-1) + \text{resta} + \text{suma} + \text{llamada} + T(n-1) + \text{resta}$

$T_{else} = 7 + 2T(n-1) \rightarrow \text{Cuando } n > 1$

$$T(n) = \begin{cases} 2 & \text{si } n \leq 1 \\ 2T(n-1) + 7 & \text{si } n > 1 \end{cases}$$

$$T(n) = 2T(n-1) + 7; \quad T(n) - 2T(n-1) = 7 \quad \text{No Homogénea}$$

$$T(n) - 2T(n-1) = 1^n \cdot 7 \cdot n^0; \quad (x-2)(x-1)^{0+1} = 0 \rightarrow \text{Raíces} \begin{cases} r_1 = 2 & \text{simple} \\ r_2 = 1 & \text{simple} \end{cases}$$

$$T(n) = C_1 \cdot r_1^n \cdot n^0 + C_2 \cdot r_2^n \cdot n^0 = C_1 \cdot 1^n + C_2 \cdot 2^n = C_1 + C_2 \cdot 2^n$$

$$T(1) = 2 \rightarrow C_1 \cdot 2^1 + C_2 = 2; \quad C_2 = 2 - 2C_1 \quad \text{Cogemos } T(1) \text{ por ser caso base}$$

$$T(2) = 2T(2-1) + 7 = 2T(1) + 7 = 2(2) + 7 = 4 + 7 = 11$$

$$T(2) = C_1 \cdot 2^2 + C_2 = C_1 \cdot 2^2 + 2 - 2C_1 = 11; \quad 4C_1 - 2C_1 = 11 - 2; \quad 2C_1 = 9; \quad C_1 = 9/2$$

$$C_2 = 2 - 2C_1 = 2 - 2 \cdot 9/2 = 2 - 9 = -7$$

$$T(n) = \frac{9}{2} \cdot 2^n - 7$$

2. Calcular la eficiencia del siguiente algoritmo:

```

1: int ejemplo2(int n)
2: {
3:     if (n == 1)
4:         return n;
5:     else
6:         return (ejemplo2(n/2) + 1);
7: }

```

$T_{if} = \text{condicion} + \text{devuelve} = 1 + 1 = 2 \rightarrow \text{cuando } n = 1$

$T_{else} = \text{condicion} + \text{devuelve} + \text{llamada} + T(n/2) + \text{division} + \text{suma} \rightarrow \text{cuando } n \neq 1$

$$T(n) = \begin{cases} 2 & \text{si } n = 1 \\ T(n/2) + 5 & \text{si } n \neq 1 \end{cases}$$

$$T(n) = T(n/2) + 5; \quad T(n) - T(n/2) = 5 \quad \text{No Homogénea}$$

$$\text{cambio base } n = 2^k \quad T(2^k) - T(2^{k-1}) = 5 \quad \text{cambio base } 2^k = t_k$$

$$t_k - t_{k-1} = 1^k \cdot 5 \cdot k^0 \quad \text{cambio base } t_k = x \quad (x-1)(x-b)^{d+1} = 0; \quad (x-1)(x-1) = 0$$

Raíz doble = 1

$$T(n) = T(2^k) = C_1 \cdot r_1^k \cdot k^0 + C_2 \cdot r_1^k \cdot k^1 = C_1 \cdot r_1^k + C_2 \cdot r_1^k \cdot k$$

tenemos 2^k y queremos $n \rightarrow 2^k = n; \log_2 n = k$ realizamos dicho cambio de base

$$T(n) = C_1 \cdot r_1^{\log_2 n} + C_2 \cdot r_1^{\log_2 n} \cdot \log_2 n; \quad T(n) = C_1 + C_2 \cdot \log_2 n; \quad T(1) = 2 = C_1 + C_2 \log_2 1$$

$$C_1 = 2 - C_2 \cdot 0; \quad C_1 = 2$$

$$T(2) = T(2/2) + 5 = T(1) + 5 = 2 + 5 = 7; \quad C_1 + C_2 \log_2 2 = 7; \quad 2 + C_2 \cdot 1 = 7; \quad C_2 = 7 - 2 = 5$$

$$T(n) = 2 + 5 \cdot \log_2 n$$

3. Calcular la eficiencia del algoritmo de las Torres de Hanoi por expansión de la recurrencia.

```
Hanoi(origen, destino, pivote, discos):
  si discos=1
    moveruno(origen, destino)
  en otro caso
    Hanoi(origen, pivote, destino, discos-1)
    moveruno(origen, destino)
    Hanoi(pivote, destino, origen, discos-1)
```

$\text{Hanoi}(\text{origen}, \text{destino}, \text{pivote}, \text{disco}) \rightarrow$ disco nos dice el tamaño, el número de discos, por tanto, $\text{disco} = n$, quedando de la siguiente forma

$T_{si} = \text{condición} + \text{llamada}(\text{moveruno})$

$T_{sino} = \text{condición} + \text{llamada}(\text{Hanoi}) + T(n-1) + \text{resta} + \text{llamada}(\text{moveruno}) + \text{llamada}(\text{Hanoi}) + T(n-1) + \text{resta}$

$$T(n) \begin{cases} T_{si} & \text{si } n=1 \\ T_{sino} & \text{si } n \neq 1 \end{cases} \rightarrow T(n) \begin{cases} 2 & \text{si } n=1 \\ 2T(n-1) + 6 & \text{si } n \neq 1 \end{cases}$$

$$T(n) = 2T(n-1) + 6 = 2(2T(n-2) + 6) + 6 = 2^2 \cdot T(n-2) + 6 + 12 = 2^2(2T(n-3) + 6) + 6 + 12 = 2^3 \cdot T(n-3) + 6 + 12 + 6 \cdot 2^2 = 2^n \cdot T(1) + 6 + 6 \cdot 2^1 + 6 \cdot 2^2 + \dots + 6 \cdot 2^{n-1};$$

$$T(n) = 2^n + 6 \cdot 2^{n-1} - 6$$

$$T(1) = 2$$

$$T(2) = 2 \cdot T(2-1) + 6 = 2 \cdot 2 + 6 = 10$$

$$T(3) = 2 \cdot T(3-1) + 6 = 2 \cdot 10 + 6 = 26$$

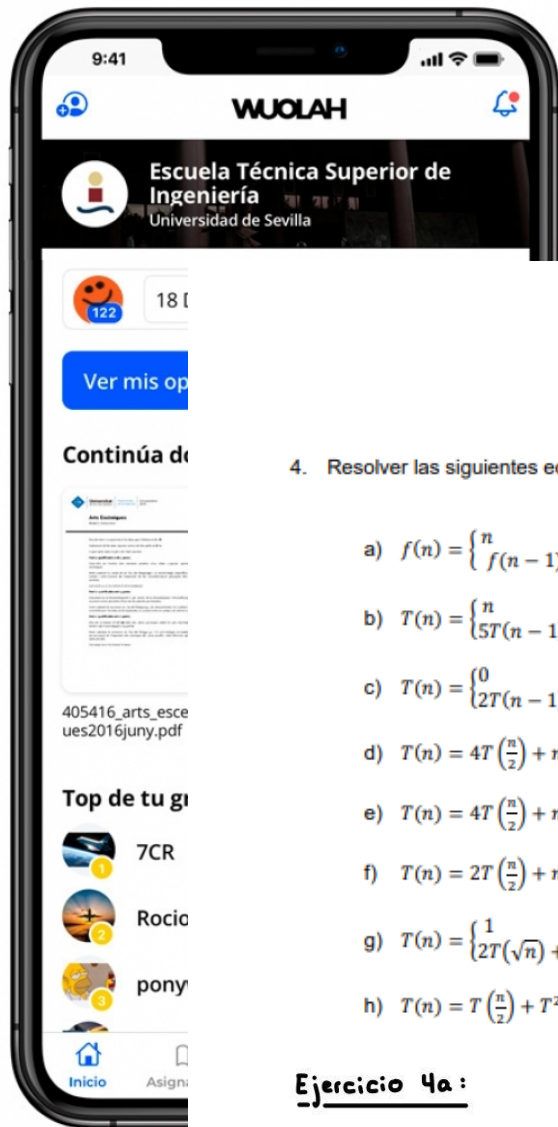
$$T(4) = 2 \cdot T(4-1) + 6 = 2 \cdot 26 + 6 = 58$$

$$T(1) = 2$$

$$T(2) = 2^2 + 6 \cdot 2^{2-1} - 6 = 10$$

$$T(3) = 2^3 + 6 \cdot 2^{3-1} - 6 = 26$$

$$T(4) = 2^4 + 6 \cdot 2^{4-1} - 6 = 58$$



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4. Resolver las siguientes ecuaciones y dar su orden de complejidad:

- $f(n) = \begin{cases} n & \text{si } n = 0 \text{ ó } n = 1 \\ f(n-1) + f(n-2) & \text{en otro caso} \end{cases}$
- $T(n) = \begin{cases} n & \text{si } n = 0, 1 \text{ ó } 2 \\ 5T(n-1) - 8T(n-2) + 4T(n-3) & \text{en otro caso} \end{cases}$
- $T(n) = \begin{cases} 0 & \text{si } n = 0 \\ 2T(n-1) + 1 & \text{en otro caso} \end{cases}$
- $T(n) = 4T\left(\frac{n}{2}\right) + n, \quad n > 1 \text{ y potencia de } 2$
- $T(n) = 4T\left(\frac{n}{2}\right) + n^2, \quad n > 1 \text{ y potencia de } 2$
- $T(n) = 2T\left(\frac{n}{2}\right) + n \cdot \log n, \quad n > 1 \text{ y potencia de } 2$
- $T(n) = \begin{cases} 1 & \text{si } n = 2 \\ 2T(\sqrt{n}) + \log \log n & \text{con } n \geq 4 \end{cases}$
- $T(n) = T\left(\frac{n}{2}\right) + T^2\left(\frac{n}{4}\right) \quad \text{sin caso base}$

Ejercicio 4a:

$$f(n) = f(n-1) + f(n-2); \quad f(n) - f(n-1) - f(n-2) = 0 \quad \text{Homogénea}$$

$$x^2 - x - 1 = 0 \rightarrow \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \begin{cases} r_1 = \frac{1 + \sqrt{5}}{2} \\ r_2 = \frac{1 - \sqrt{5}}{2} \end{cases}$$

$$f(n) = C_1 \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^n + C_2 \cdot \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

$$f(0) = 0 = C_1 \cdot 1 + C_2 \cdot 1; \quad 0 = C_1 + C_2; \quad C_1 = -C_2$$

$$f(1) = 1 = C_1 \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^1 + C_2 \cdot \left(\frac{1 - \sqrt{5}}{2}\right)^1; \quad C_2 \cdot \left(\frac{1 - \sqrt{5}}{2}\right)^1 = 1 - C_1 \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^1;$$

$$C_2 = \frac{1 - C_1 \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^1}{\left(\frac{1 - \sqrt{5}}{2}\right)^1} = -\left(\frac{1 + \sqrt{5}}{2}\right)^1 + C_1 \cdot \left(\frac{3 + \sqrt{5}}{2}\right)^1$$

$$C_1 = -\left(-\left(\frac{1 + \sqrt{5}}{2}\right)^1 + C_1 \cdot \left(\frac{3 + \sqrt{5}}{2}\right)^1\right); \quad C_1 \cdot \left(\frac{5 + \sqrt{5}}{2}\right)^1 = \left(\frac{1 + \sqrt{5}}{2}\right)^1; \quad C_1 = \frac{\sqrt{5}}{5}$$

$$-C_2 = C_1; \quad C_2 = -C_1 = -\frac{\sqrt{5}}{5}$$

$$f(n) = \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

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Ejercicio 4b:

$$T(n) = 5T(n-1) - 8T(n-2) + 4T(n-3); T(n) - 5T(n-1) + 8T(n-2) - 4T(n-3) = 0 \quad \text{Homogenea}$$

$$x^3 - 5x^2 + 8x - 4 = 0 \rightarrow \text{simplificamos mediante método de Ruffini}$$

$$\begin{array}{r|rrrr} & 1 & -5 & 8 & -4 \\ 2 & & 2 & -6 & 4 \\ \hline & 1 & -3 & 2 & 0 \\ 2 & & 2 & -2 & \\ \hline & 1 & -1 & 0 & \\ 1 & & 1 & & \\ \hline & 1 & 0 & & \end{array}$$

$$(x-2)(x-2)(x-1) = 0$$

$$\text{Raíces} = \begin{cases} r_1 = 2 & \text{doble} \\ r_2 = 1 & \text{simple} \end{cases}$$

$$T(n) = C_1 \cdot 2^n \cdot n^0 + C_2 \cdot 2^n \cdot n^1 + C_3 \cdot 1^n \cdot n^0 = C_1 \cdot 2^n + C_2 \cdot 2^n \cdot n + C_3$$

$$T(0) = 0 = C_1 \cdot 2^0 + C_2 \cdot 2^0 \cdot 0 + C_3 = C_1 + C_3; C_1 = -C_3$$

$$T(1) = 1 = C_1 \cdot 2^1 + C_2 \cdot 2^1 \cdot 1 + C_3 = 2C_1 + 2C_2 + C_3 = -2C_3 + 2C_2 + C_3; 2C_2 - C_3 = 1; C_3 = -1 + 2C_2$$

$$T(2) = 2 = C_1 \cdot 2^2 + C_2 \cdot 2^2 \cdot 2 + C_3 = 4C_1 + 8C_2 + C_3 = -4C_3 + 8C_2 + C_3 = -4(-1 + 2C_2) + 8C_2 - 1 + 2C_2 = 4 - 8C_2 + 8C_2 - 1 + 2C_2; 2C_2 + 3 = 2; C_2 = -1/2$$

$$C_3 = -1 + 2(-1/2) = -1 - 1 = -2$$

$$C_1 = -(-2) = 2$$

$$T(n) = 2 \cdot 2^n - \frac{1}{2} \cdot 2^n \cdot n - 2; T(n) = 2^{n+1} - n \cdot 2^{n-1} - 2$$

Ejercicio 4c:

$$T(n) = 2T(n-1) + 1; T(n) - 2T(n-1) = 1 \quad \text{No Homogénea}$$

$$f(n) = 1 = 1^n \cdot 1 \cdot n^0 \rightarrow (x-2)(x-1)^{0+1} = 0$$

$$\text{Raíces} = \begin{cases} r_1 = 2 \\ r_2 = 1 \end{cases}$$

$$T(n) = C_1 \cdot 2^n \cdot n^0 + C_2 \cdot 1^n \cdot n^0 = C_1 \cdot 2^n + C_2$$

$$T(0) = 0 = C_1 \cdot 2^0 + C_2; C_2 = -C_1 \cdot 2^0; C_2 = -C_1$$

$$T(1) = 2T(1-1) + 1 = 2 \cdot 0 + 1 = 1 = C_1 \cdot 2^1 + C_2; C_2 = 1 - 2C_1; C_2 = 1 + 2C_1;$$

$$-C_1 = 1; C_2 = -1; C_1 = -C_2 = -(-1) = 1$$

$$T(n) = 2^n - 1$$

Ejercicio 4d:

$$T(n) = 4T(n/2) + n \left[\begin{array}{l} \text{cambio de base} \\ n = 2^k \end{array} \right] \quad T(2^k) = 4T(2^k/2) + 2^k;$$

$$T(2^k) - 4T(2^{k-1}) = 2^k \quad \text{No Homogénea}$$

$$b^k \cdot p(a) = 2^k = 2^k \cdot k^0; \quad b=0, a=2 \rightarrow (x-a)^{b+1} = (x-2)^1$$

$$(x-4)(x-2) = 0 \quad \text{Raíces} \begin{cases} r_1 = 4 \\ r_2 = 2 \end{cases}$$

$$T(2^k) = C_1 \cdot 4^k + C_2 \cdot 2^k = C_1 \cdot (2^k)^2 + C_2 \cdot 2^k \left[\begin{array}{l} \text{cambio de base} \\ 2^k = n \end{array} \right] \quad T(n) = C_1 \cdot n^2 + C_2 \cdot n$$

$$T(n) - 4T(n/2) = n; \quad C_1 \cdot n^2 + C_2 \cdot n - 4 \cdot \left(C_1 \cdot \left(\frac{n}{2}\right)^2 + C_2 \cdot \left(\frac{n}{2}\right) \right) = n;$$

$$C_1 \cdot n^2 + C_2 \cdot n - 4 \cdot C_1 \cdot \frac{n^2}{4} - 4C_2 \cdot \frac{n}{2} = n; \quad C_1 \cdot n^2 + C_2 \cdot n - C_1 \cdot n^2 - 2C_2 \cdot n = n;$$

$$-C_2 \cdot n = n; \quad -C_2 = 1; \quad C_2 = -1 \rightarrow T(n) = C_1 \cdot n^2 - n \quad \text{No podemos averiguar más sin condiciones iniciales}$$

Ejercicio 4e:

$$T(n) = 4T(n/2) + n^2 \left[\begin{array}{l} \text{cambio de base} \\ n = 2^k \end{array} \right] \quad T(2^k) = 4T(2^k/2) + (2^k)^2;$$

$$T(2^k) - 4T(2^{k-1}) = (2^k)^2; \quad \text{No Homogénea}; \quad 4^k = b^k \cdot p(k) = 4^k \cdot k^0$$

$$p(x) = (x-4)(x-4) = 0 \quad \text{raíces} = r_1 = 4 \text{ doble}$$

$$T(2^k) = C_1 \cdot 4^k + C_2 \cdot 4^k \cdot k \left[\begin{array}{l} \text{cambio de base} \\ 2^k = n \end{array} \right] \quad T(n) = C_1 \cdot n^2 + C_2 \cdot n^2 \cdot \log_2(n)$$

$$T(n) - 4T(n/2) = n^2; \quad C_1 \cdot n^2 + C_2 \cdot n^2 \cdot \log_2(n) - 4 \left(C_1 \cdot \left(\frac{n}{2}\right)^2 + C_2 \cdot \left(\frac{n}{2}\right)^2 \log_2\left(\frac{n}{2}\right) \right) = n^2$$

$$C_2 n^2 \log_2(n) - C_2 n^2 \log_2\left(\frac{n}{2}\right) = n^2; \quad C_2 n^2 (\log_2(n) - \log_2(n/2)) = n^2$$

$$C_2 (\log_2(n) - \log_2(n/2)) = 1; \quad C_2 = 1 \rightarrow T(n) = C_1 \cdot n^2 + n^2 \cdot \log_2(n)$$



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Ejercicio 4f:

$$T(n) = 2T\left(\frac{n}{2}\right) + n \log_2(n) \quad \left[\begin{array}{l} \text{cambio de base} \\ n = 2^k \end{array} \right] \quad T(2^k) - 2T(2^{k-1}) = 2^k k$$

$$T(2^k) - 2T(2^{k-1}) = 2^k k \quad \text{No Homogénea} \rightarrow b^k \cdot p(k) = 2^k \cdot k \rightarrow a=2, b=1$$

$$p(x) = (x-2)(x-2)^2 = 0 \quad \text{Raíces} = r_1 = 2 \text{ Triple}$$

$$T(2^k) = C_1 \cdot 2^k + C_2 \cdot 2^k \cdot k + C_3 \cdot 2^k \cdot k^2 \quad \left[\begin{array}{l} \text{cambio de base} \\ 2^k = n \end{array} \right]$$

$$T(n) = C_1 n + C_2 n \cdot \log_2(n) + C_3 n \log_2^2(n)$$

$$C_1 n + C_2 n \cdot \log_2(n) + C_3 n \log_2^2(n) - 2 \left(C_1 \frac{n}{2} + C_2 \frac{n}{2} \cdot \log_2\left(\frac{n}{2}\right) + C_3 \frac{n}{2} \log_2^2\left(\frac{n}{2}\right) \right) = n \log_2 n$$

$$n \log_2(n) (C_2 + C_3 \log_2(n)) - n \log_2 \frac{n}{2} \left(C_2 + C_3 \log_2\left(\frac{n}{2}\right) \right) = n \log_2(n) ;$$

$$n \left(\log_2(n) (C_2 + C_3 \log_2(n)) - \log_2\left(\frac{n}{2}\right) (C_2 + C_3 \log_2\left(\frac{n}{2}\right)) \right) = n \log_2(n) ;$$

$$n(C_3 - C_2) + 2C_3 n \log_2(n) = n \log_2(n) ; 2C_3 = 1 ; C_3 = \frac{1}{2} ; C_2 = \frac{1}{2}$$

$$T(n) = C_1 n + \frac{1}{2} n \log_2(n) + \frac{1}{2} n \log_2^2(n)$$

Ejercicio 4g:

$$T(n) = 2T(\sqrt{n}) + \log \log(n) \quad \left[\begin{array}{l} \text{cambio de base} \\ n = k^2 \end{array} \right] \quad T(k^2) - 2T(\sqrt{k^2}) = \log \log(k^2)$$

$$T(k^2) - 2T(k^{1/2}) = \log \log(k^2) \quad \text{No Homogénea}$$

$$p(x) = (x-2)(x-2) = 0 \quad \text{Raíces} = r_1 = 2 \text{ doble}$$

$$T(k^2) = C_1 \cdot 2^k \cdot k^0 + C_2 \cdot 2^k \cdot k^1 \quad \left[\begin{array}{l} \text{cambio de base} \\ 2^k = n \end{array} \right] \quad T(n) = C_1 \cdot n + C_2 \cdot n \cdot \log(n)$$

$$T(2) = 1 = C_1 n + C_2 n \log(n) = C_1 \cdot 2 + C_2 \cdot 2 \quad ; C_1 = \frac{1-2C_2}{2}$$

$$T(4) = 2T(\sqrt{4}) + \log \log(4) = 2T(2) + \log \log(4) = 2T(2) + 1 = 2 \cdot 1 + 1 = 3$$

$$T(4) = C_1 n + C_2 n \log(n) = 4C_1 + 4C_2 \cdot 2 = 4C_1 + 8C_2 = 3 ;$$

$$4\left(\frac{1-2C_2}{2}\right) + 8C_2 = 3 ; 2 - 4C_2 + 8C_2 = 3 ; 4C_2 = 1 ; C_2 = \frac{1}{4}$$

$$C_1 = \frac{1-2C_2}{2} = \frac{1-2 \cdot \frac{1}{4}}{2} = \frac{1}{4} \rightarrow T(n) = \frac{1}{4} n + \frac{1}{4} n \log(n)$$

Ejercicio 4h:

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right)^2 \left[\begin{array}{l} \text{cambio de base} \\ n = 2^k \end{array} \right] \quad T(2^k) = T(2^{k-1}) + T(2^{k-2})^2$$

Vamos a hacer uso de logaritmos ya que no es lineal:

$$\log(T(2^k)) - \log(T(2^{k-1})) - 2\log(T(2^{k-2})) = 0$$

$$u_k - u_{k-1} - 2u_{k-2} = 0 \quad \text{Homogenea}$$

$$p(x) = x^2 - x - 2 = 0 \rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-2)}}{2} = \frac{1 \pm 3}{2} = \begin{cases} r_1 = 2 \\ r_2 = -1 \end{cases}$$

$$p(x) = (x-2)(x+1) = 0$$

$$u_k = c_1 \cdot 2^k + c_2 (-1)^k = 2^k c_1 - c_2 \left[\begin{array}{l} \text{cambio de base} \\ t_k = 2^{u_k} \end{array} \right] \quad t_k = 2^{2^k c_1 - c_2}$$

$$\left[\begin{array}{l} \text{cambio de base} \\ k = \log(n) \end{array} \right] \left[\begin{array}{l} \text{cambio de base} \\ n = 2^k \end{array} \right] \quad T(n) = 2^{nc_1 - c_2} = \frac{2^{nc_1}}{2^{c_2}}$$

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right)^2 ; \left(\frac{2^{nc_1}}{2^{c_2}} \right) = \left(\frac{2^{\frac{n}{2}c_1}}{2^{c_2}} \right) + 2 \left(\frac{2^{\frac{n}{4}c_1}}{2^{c_2}} \right)^2 ;$$

$$\frac{2^{\frac{n}{2}c_1}}{2^{c_2}} = \frac{2^{\frac{n}{2}c_1}}{(2^{c_2})^2} ; 2^{\frac{n}{2}c_1} = \frac{2^{\frac{n}{2}c_1}}{2^{c_2}} ; 2^{c_2} = 1 ; c_2 = \log(1) ; c_2 = 0$$

$$T(n) = \frac{2^{nc_1}}{2^{c_2}} = \frac{2^{nc_1}}{2^0} ; T(n) = 2^{nc_1}$$