LECTURE #2: Cross products and determinants

§1.3

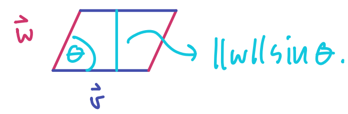
- It's very useful to be able to find vectors orthogonal to a given vector.

DEF The cross product of $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is the vector $\vec{v} \times \vec{w}$ characterized by:

(1) $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

(2) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$ (the area of parallelogram spanned by $\vec{v}$ & $\vec{w}$)

(3) $\vec{v}, \vec{w}$, and $\|\vec{v} \times \vec{w}\|$ are a "right handed" set of vectors, ie look like:



"Right hand rule"

NOTE This is only sensible in $\mathbb{R}^3$ (why?)

EX

$$\left\{ \begin{array}{lll} \hat{\imath} \times \hat{\jmath} = \hat{k} & \hat{\jmath} \times \hat{k} = \hat{\imath} & \hat{k} \times \hat{\imath} = \hat{\jmath} \\ \hat{\jmath} \times \hat{\imath} = -\hat{k} & \hat{k} \times \hat{\jmath} = -\hat{\imath} & \hat{\imath} \times \hat{k} = -\hat{\jmath} \end{array} \right\}$$

MORE PROPERTIES

(1) $\vec{v} \times \vec{w} = 0$ if and only if $\vec{v}$ and $\vec{w}$ are parallel or one of $\vec{v}$ or $\vec{w}$ are zero.

(2) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ [anti-commutative]

(3) $(\lambda \vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda \vec{w})$ [scalar associativity]

(4) $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$ [additive distributivity]

(*) $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ [not associative!]

$(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0} \times \hat{\jmath} = \vec{0}$ but $\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{\jmath}$.

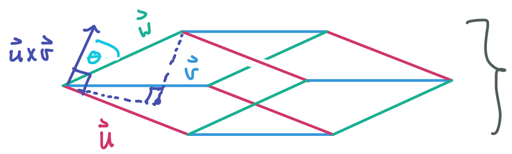
• Another useful and related operation is the following.

DEF The triple scalar product of $\vec{u}, \vec{v}$ and $\vec{w}$ is the (number) $(\vec{u} \times \vec{v}) \cdot \vec{w}$ (in that order), also sometimes denoted $[\vec{u}, \vec{v}, \vec{w}]$.

PROP $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is the (signed) volume of the parallelpiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

PF On one hand, $(\vec{u} \times \vec{v}) \cdot \vec{w} = \|\vec{u} \times \vec{v}\| \, \|\vec{w}\| \cos\theta$ where θ is the angle between $\vec{u} \times \vec{v}$ & $\vec{w}$.

On the other hand, we have:



Volume is $\underbrace{\|\vec{u} \times \vec{v}\|}_{\text{area of base}} \cdot \underbrace{\|\vec{w}\| \cos\theta}_{\text{height}}$.

NOTE This volume is signed; ie $(\hat{\imath} \times \hat{\jmath}) \cdot \hat{k} = 1$ but $(\hat{\jmath} \times \hat{\imath}) \cdot \hat{k} = -1$.

COR We have the following property of the triple product:

$$\vec{u} \times \vec{v} \cdot \vec{w} = \vec{w} \times \vec{u} \cdot \vec{v} = \vec{v} \times \vec{w} \cdot \vec{u}.$$

These are all the (signed) volume of the same parallelpiped!

• While Properties (1) - (3) of the cross product are straightforward, Property (4) takes more work. First, we need a fact.

FACT $\vec{u} = \vec{v}$ if and only if $\vec{u} \cdot \vec{x} = \vec{v} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^3$

PF If $\vec{u} = \vec{v}$, then we certainly have $\vec{u} \cdot \vec{x} = \vec{v} \cdot \vec{x}$ for any $\vec{x}$.

On the other hand, if $\vec{v} \cdot \vec{x} = \vec{u} \cdot \vec{x}$ for all $\vec{x}$, then in particular we have

$$\vec{v} = (\vec{v} \cdot \hat{\imath}, \vec{v} \cdot \hat{\jmath}, \vec{v} \cdot \hat{k}) = (\vec{u} \cdot \hat{\imath}, \vec{u} \cdot \hat{\jmath}, \vec{u} \cdot \hat{k}) = \vec{u}. \quad \square$$

• Now, we can prove:

PROPERTY (4) $(\vec{u}+\vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

PF For all $\vec{x}$, we have: (* use the previous corollary)

$$\begin{aligned} [(\vec{u}+\vec{v}) \times \vec{w}] \cdot \vec{x} &= (\vec{w} \times \vec{x}) \cdot (\vec{u}+\vec{v}) \\ &= (\vec{w} \times \vec{x}) \cdot \vec{u} + (\vec{w} \times \vec{x}) \cdot \vec{v} \\ &= (\vec{u} \times \vec{w}) \cdot \vec{x} + (\vec{v} \times \vec{w}) \cdot x \\ &= [(\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})] \cdot \vec{x}. \end{aligned}$$ $\square$

EX Compute $(1,0,2) \times (-1,3,0)$.

$$\begin{aligned} \rightarrow (\hat{\imath}+2\hat{k}) \times (-\hat{\imath}+3\hat{\jmath}) &= \hat{\imath} \times -\hat{\imath} + 3\hat{\imath} \times \hat{\jmath} - 2\hat{k} \times \hat{\imath} + 6\hat{k} \times \hat{\jmath} \\ &= \vec{0} + 3\hat{k} - 2\hat{\jmath} + 6(-\hat{\imath}) \\ &= (-6,-2,3) \end{aligned}$$

EX We can compute $\vec{v} \times \vec{w}$ in terms of their components. If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ then:

$$\begin{aligned} \vec{v} \times \vec{w} &= (v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}) \times (w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}) \\ &= v_1w_1(\hat{\imath} \times \hat{\imath}) + v_1w_2(\hat{\imath} \times \hat{\jmath}) + v_1w_3(\hat{\imath} \times \hat{k}) \\ &\quad + v_2w_1(\hat{\jmath} \times \hat{\imath}) + v_2w_2(\hat{\jmath} \times \hat{\jmath}) + v_2w_3(\hat{\jmath} \times \hat{k}) \\ &\quad + v_3w_1(\hat{k} \times \hat{\imath}) + v_3w_2(\hat{k} \times \hat{\jmath}) + v_3w_3(\hat{k} \times \hat{k}) \\ &= (v_2w_3 - v_3w_2)\hat{\imath} + (v_3w_1 - v_1w_3)\hat{\jmath} + (v_1w_2 - v_2w_1)\hat{k} \end{aligned}$$

• This looks complicated, but it's not an accident!

- A brief sidebar on matrices (preview for 204 and Lecture #3)

DEF A $m \times n$ matrix M is an $m \times n$ array of numbers, ie:

$$M = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{or} \quad N = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 7 & -5 \\ -1 & 2 & 6 \end{bmatrix}$$

- Matrices appear everywhere in mathematics. We'll deal with matrices a lot (just 2×2 and 3×3 today).

DEF Like vectors, we can add matrices (of the same size) and scale them. For example, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, and $\lambda \in \mathbb{R}$:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \quad \text{and} \quad \lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{bmatrix}.$$

NOTE We can actually multiply matrices (but more on this later).

Q What's a reason matrices could be useful?

- There is a rather mysterious function we need to talk about: called the determinant. It's use dates back as far as 300 BC, but to about 1800 (cf. Cauchy) in the precise modern sense.
- It is connected to solving systems of equations (more on this in MAT 204).

DEF The determinant of a matrix M is the unique function $\det : M_{n \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ satisfying $\det\left(\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right) = 1$ and:

$$(1) \quad \det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix} = -\det \begin{bmatrix} \vec{r}_2 \\ \vec{r}_1 \\ \vec{r}_3 \end{bmatrix}. \quad \text{(Alternating)}$$

$\star$ If $\vec{r}_1 = \vec{r}_2$, then this means $\det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix} = 0$!

(2) $\det\begin{bmatrix}\vec{r}_1+\vec{r}_1{}' \\ \vec{r}_2 \\ \vec{r}_3\end{bmatrix} = \det\begin{bmatrix}\vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3\end{bmatrix} + \det\begin{bmatrix}\vec{r}_1{}' \\ \vec{r}_2 \\ \vec{r}_3\end{bmatrix}$

FACTS We will really only use the determinant as a way to compute the cross product. We'll also adopt the notation

- For short, we'll write $|A| = \det(A)$.
- In the 2x2 case, we have: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$
- In the 3x3 case, we have a recursive formula:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

- Amazingly, we can expand along <u>any</u> row or column, ie

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{33} \end{vmatrix}$$

USEFUL FACT

- $\vec{v}\times\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ (Compare with formula)
- even better, $(\vec{u}\times\vec{v})\cdot\vec{w} = \det\begin{bmatrix}\vec{u} \\ \vec{v} \\ \vec{w}\end{bmatrix}$.

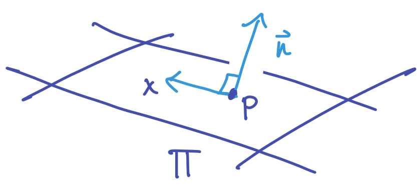
PF

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\cdot\vec{w}$$

$$= w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - v_1u_3) + w_3(u_1v_2 - v_1u_2)$$

$$= \det\begin{bmatrix}\vec{w}\\ \vec{u}\\ \vec{v}\end{bmatrix} = -\det\begin{bmatrix}\vec{u}\\ \vec{w}\\ \vec{v}\end{bmatrix} = \det\begin{bmatrix}\vec{u}\\ \vec{v}\\ \vec{w}\end{bmatrix}. \qquad \square$$

EX We can describe a plane by a normal vector and a point on the plane.



$P = (p_1, p_2, p_3)$
$\vec{n} = (n_1, n_2, n_3)$

If $x \in \pi$, then $\vec{Px} \perp \vec{n}$, so

$$(x-p_1, y-p_2, z-p_3) \cdot (n_1, n_2, n_3) = 0$$

Expanding, we get:

$$n_1 x + n_2 y + n_3 z = (n_1p_1 + n_2p_2 + n_3p_3)$$

EX If P, Q, R are three (not collinear) points on a plane, then $\vec{n} = \vec{PQ} \times \vec{PR}$ is a normal vector to the plane containing P, Q, and R.

(Draw a picture for yourself!)

EX What is the equation of the plane π through $P_0 = (1,1,1)$, $P_1 = (2,0,-1)$, and $P_2 = (3,-1,2)$?

- A normal vector $\vec{n}$ to π is

$$\vec{n} = \vec{P_0P_1} \times \vec{P_0P_2} = \det\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -2\\ 2 & -2 & 1\end{bmatrix} = -5\hat{i} - 5\hat{j} = (-5,-5,0).$$

- An equation describing π is $-5x - 5y = -10$, or $x+y = 2$.

EX What is the equation of the line ℓ given by intersecting the planes $2x-y+z=1$ and $x+y-z=2$?

- Normal vectors to each plane are $\vec{n}_1=(2,-1,1)$ and $\vec{n}_2=(1,1,-1)$.
- A direction vector for the line ℓ can be taken to be

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 3\hat{j}+3\hat{k}.$$

(in particular, $(0,1,1)$ will work).

- By inspection, $(1,1,0)$ is on both planes, so an equation for ℓ is $(x,y,z)=(1,1,0)+t(0,1,1)=(1,1+t,t)$.