Introduction

My research lies in the fields of low-dimensional and geometric topology. Specifically, I study 3- and 4-manifolds, and the interactions between them. The topology of 4-manifolds is particularly exciting due to *exotic* phenomena: $\mathbb{R}^n$ admits a unique smooth structure if $n \neq 4$, but $\mathbb{R}^4$ admits uncountably many smooth structures. I also study embeddings of knotted curves and surfaces in 3- and 4-manifolds, since they witness a wide variety of rich behaviour, and are used as building blocks to construct more complicated manifolds. Surgery operations on surfaces can even be used to change the smooth structure of a given manifold. I am particularly interested in the differences between surfaces in 4-manifolds up to varying degrees of equivalence (e.g., homotopy, isotopy, concordance) as well as how surgeries and smooth structures depend on these relationships. I specialize in constructions of "difficult to find" diffeomorphisms to show certain 4-manifolds (usually homotopy 4-spheres) are smoothly standard, and in the theory of trisections.

My main contributions include standardness theorems about Gluck twists of roll spun knots and Gompf's twisted doubles (§1), a non-orientable analogue of a classical theorem of Laudenbach-Poénaru with a variety of applications (§3), and extensions of the recently introduced theory of trisections to new settings (§3 and §4). In each section, I summarize my previous and ongoing work. I also describe its significance, and give objectives to be carried out within the next few years.

1. Gluck twists of roll spun knots

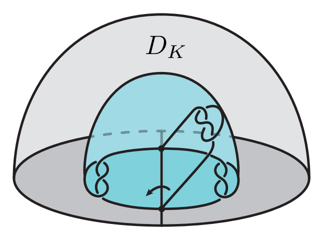
One of the most important unresolved problems in low dimensional topology is the smooth 4-dimensional Poincaré conjecture, which asks whether S^4 admits any exotic smooth structures. The most well-known source of potential counterexamples is originally due to Gluck [11], and begins with an embedded 2-sphere, $S \subset S^4$, usually called a *2-knot*. For the unit sphere $S^2 \subset \mathbb{R}^3$, let $r_\theta : S^2 \to S^2$ be the diffeomorphism that rotates S^2 by an angle of θ about the z-axis.

Definition 1.1. Let $S \subset S^4$ be a smoothly embedded 2-sphere. The *Gluck twist of S^4 along S* is the smooth homotopy 4-sphere

$$\Sigma_S = (S^4 - \mathrm{int}(N)) \cup_\tau N$$

where N is a regular neighborhood of S diffeomorphic to $S^2 \times D^2$, and τ is the automorphism of $\partial N \cong S^2 \times S^1$ sending $(x, \theta) \mapsto (r_\theta(x), \theta)$.

One can check that Σ_S is a homotopy 4-sphere, and so by a celebrated theorem of Freedman [8], is homeomorphic to S^4. However, despite its apparent simplicity, it remains an open question whether all Gluck twists are *standard*, i.e., diffeomorphic to the standard S^4. For example, if the sphere S is unknotted (isotopic to the unit sphere in the equator of S^4), then this operation does produce a standard S^4. Various families of 2-knots are known to have standard Gluck twists, many of which arise from spinning constructions. If $K \subset S^3$ is a knot (an embedded S^1), then $K\#(-K)$ bounds a natural disk $D_K \subset B^4$ (right), obtained by rotating K through the interior of B^4 (here, $-K$ is the *mirror* of K).



Definition 1.2. Suppose that $K \subset S^3$ is a knot. The *spin* of K, denoted $S(K)$, is the 2-knot obtained by doubling the pair (B^4, D_K), i.e.,

$$(S^4, S(K)) = (B^4, D_K) \cup (S^3 \times I, (K\# - K) \times I) \cup (B^4, D_K).$$

Gluck [11] proved that the Gluck twist of a spun knot is standard. Further generalizations of spun knots were later given by Litherland [25], who defined the family of deform spun 2-knots, which include twist roll spun knots. These may be interpreted as a modification of Definition

1.2, wherein one replaces the product $(S^3 \times I, (K\# - K) \times I))$ with a more complicated properly embedded annulus, or *concordance*, in $S^3 \times I$ with the correct boundary. Except in the case that K is a torus knot [25], Gluck twists of twist roll spun knots have remained notably absent from the list of smooth homotopy 4-spheres known to be standard, since usual techniques do not seem to apply in this case. In joint work, Hannah Schwartz and I partially remedied this situation.

Theorem 1.3 (N.-Schwartz [32])**.** *Let $S \subset S^4$ be a 2-sphere with a regular homotopy to the unknot consisting of one finger and one Whitney move. Then, the Gluck twist Σ_S is standard.*

By a theorem of Smale [35], any two smoothly embedded 2-spheres in S^4 are *regularly homotopic*, i.e., homotopic through immersions, which may be described by some finite sequence of *finger* and *Whitney* moves. We explicitly describe such a homotopy for the roll spin of a knot with unknotting number one.

Theorem 1.4 (N.-Schwartz [32])**.** *Gluck twists of m-twist n-roll spins of knots with unknotting number one are standard.*

The main tools are a recent theorem of Joseph, Klug, Ruppik, and Schwartz [18] relating the stabilization number of knotted 2-spheres in S^4 to finger and Whitney moves, and a classical theorem of Montesinos [30] on logarithmic transformations of unknotted tori in S^4.

Significance and future directions. Theorem 1.4 has the following interesting consequence. Starting with a knot K and any integer m, Gompf [12] constructs an infinite order cork: a smooth, compact, contractible 4-manifold $C_m(K)$ together with an infinite order self-diffeomorphism $f : \partial C_m(K) \to \partial C_m(K)$ such that for certain choices of K, f^n does not extend to a diffeomorphism of $C_m(K)$ for any n (by Freedman's work, f^n does extend to a homeomorphism). Since $C_m(K)$ is of Mazur type, its double is S^4, but the twisted double $\mathcal{D}_{m,n}(K) = C_m(K) \cup_{f^n} -C_m(K)$ is another homotopy 4-sphere which is not obviously standard. In fact, we show that it is diffeomorphic to a Gluck twist on a roll spun knot and so is standard in some cases, partially answering a question of Gompf and extending a result of Akbulut [1]. This illustrates another reason to study these Gluck twists in particular; they arise naturally in other important contexts.

Corollary 1.5 (N.-Schwartz [32])**.** *For all $m, n \in \mathbb{Z}$ and any knot $K \subset S^3$ with unknotting number one, Gompf's twisted double $\mathcal{D}_{m,n}(K)$ is standard.*

I am currently working on extending Theorem 1.4 to knots with higher unknotting number; to do so will require a more careful study of the subtleties of regular homotopies of 2-knots in S^4. The approach Schwartz and I take is quite different from the existing literature, and it would be interesting to see how generally it applies. In a precise sense, roll spun knots are a good testing ground to study Gluck twists in general.

Question 1.6. *To what extent can Gluck twists be understood via regular homotopies to the unknotted 2-sphere?*

The set of 2-knots satisfying Theorem 1.3 also appears to be distinct from those which are 0-concordant to the unknot, a well studied family of 2-knots with standard Gluck twists first introduced by Melvin [28]. By recent independent work of Sunukjian [36], Dai-Miller [7], and Joseph [17], there are examples of 2-knots satisfying Theorem 1.3 which are *not* 0-concordant to the unknot. In future work, I plan to investigate this relationship.

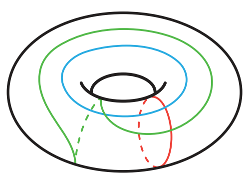
Question 1.7. *How does the condition in Theorem 1.3 relate to the notion of 0-concordance? To what extent can this condition be relaxed?*

2. Trisections of 4-manifolds

Trisections of 4-manifolds were introduced by Gay and Kirby in 2012, and have witnessed an explosion of recent research activity. Just as a Heegaard splitting is a decomposition of a 3-manifold into two handlebodies meeting along a common surface, a trisection is a decomposition of a 4-manifold into three 4-dimensional handlebodies.

Definition 2.1 ([9])**.** Let X be a smooth, oriented, connected, and closed 4-manifold. A $(g;k)$-trisection of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that:

1. $X_i \cong \natural^k S^1 \times B^3$, i.e., for each i, X_i is a 4-dimensional 1-handlebody;
2. $H_{ij} = X_i \cap X_j \cong \natural^k S^1 \times B^2$, i.e., $X_i \cap X_j$ is a 3-dimensional 1-handlebody for each $i \neq j$;
3. $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$, i.e., the triple intersection is a surface of genus g. In particular, Σ_g is a Heegaard surface for ∂X_i.



$\mathbb{CP}^2$

Trisections also satisfy an analogue of the Reidemeister-Singer theorem: all closed 4-manifolds admit trisections, and any two trisections can be made ambiently isotopic up to a natural stabilization operation [9]. By a theorem of Laudenbach-Poénaru [24], a trisection is determined by the 3-dimensional handlebodies H_{ij}, and so a guiding principle is to try to apply 3-dimensional techniques to understand 4-manifolds. An advantage of trisections is that like Heegaard splittings, they may be described by a *trisection diagram*, by indicating each handlebody H_{ij} with a cut system of curves on the surface Σ (left: a $(1;0)$- trisection diagram for $\mathbb{CP}^2$). In fact, trisections are essentially a specialized kind of Heegaard triple, a foundational object in the theory of Heeegard-Floer homology [33].

One compelling reason to study trisections is the following. A natural invariant associated to a 4-manifold X is its *trisection genus*, that is:

$$g(X) = \min\{g : X \text{ admits a genus } g \text{ trisection}\}$$

By Haken's lemma [14], the analogous quantity for 3-manifolds is additive under connected sum, so one might ask whether the same holds true for trisections, i.e., whether $g(X\#Y) = g(X) + g(Y)$. In fact, this would be extraordinarily strong. An observation of Lambert-Cole and Meier [22] shows that this would force g to be a homeomorphism invariant, and in particular implies that $\mathbb{CP}^2, \overline{\mathbb{CP}}^2, S^2 \times S^2, \mathbb{CP}^2\#\mathbb{CP}^2, \mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and notably, S^4, all admit a unique smooth structure. Such a statement is certainly beyond reach at the present time, but it is interesting that the analogue of a 3-dimensional result could have such strong consequences. Particularly fascinating is that this meets the current progress in the production of small exotic 4-manifolds: $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ does admit exotic smooth structures [3].

Trisections can also be used to study Gluck twists (see §1). There is an analogue of a Gluck twist called a *Price twist*, performed in an neighborhood of an embedded $\mathbb{RP}^2$; if $P \subset X^4$ is a smoothly embedded $\mathbb{RP}^2$, the homotopy 4-sphere obtained by Price twisting P is denoted $\Pi_P(X)$. Using only trisection diagrams and recent work of Gay-Meier [10] and Kim-Miller [21], I gave a new proof of a theorem of Katanaga, Saeki, Teragaito, and Yamada relating these surgeries.

Theorem 2.2 (N. [31],[19])**.** *Let X^4 be a smooth, connected and closed 4-manifold, and $S \subset X$ an embedded 2-sphere. Then, $\Sigma_S(X) \cong \Pi_{S\#P_\pm}(X)$, where $P_\pm \subset X$ is any unknotted $\mathbb{RP}^2$ with Euler number ± 2. In fact, these two 4-manifolds admit stably equivalent trisections.*

Another question that arises naturally in trisection theory is whether any analogue of Waldhausen's theorem [37] holds for trisections of S^4. The simplest trisection is the $(0;0)$- trisection of S^4, and stabilizations of this trisection are called *standard*. A resolution of Question 2.3 would be an important first step towards understanding trisections of S^4, and I describe some preliminary steps towards an answer in §3 and §4.

Question 2.3. *Is every trisection of S^4 standard? Equivalently, is every trisection of S^4 a stabilization of the* $(0;0)$- *trisection?*

3. Trisections of non-orientable manifolds

In joint work, Maggie Miller and I have extended much of the theory of trisections to non-orientable 4-manifolds, first studied in this context by Rubinstein and Tillmann [34]. Many constructions in 4-manifold topology depend on an indispensable theorem of Laudenbach-Poénaru [24], and we provide a non-orientable counterpart.

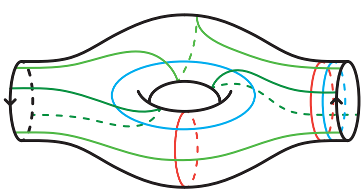
Theorem 3.1 (Miller-N. [29])**.** *Suppose that $f : \#^p S^2 \tilde{\times} S^1 \to \#^p S^2 \tilde{\times} S^1$ is a diffeomorphism. Then, f extends to a diffeomorphism of $\natural^p B^3 \tilde{\times} S^1$, i.e., there exists a diffeomorphism $F : \natural^p B^3 \tilde{\times} S^1 \to \natural^p B^3 \tilde{\times} S^1$ such that $F|_\partial = f$.*

We use primarily 3-dimensional tools, and rely heavily on Laudenbach's theorem on 2-spheres in 3-manifolds [23] as well as a theorem of Cerf [6]. Using Theorem 3.1, we are able to define trisection diagrams for non-orientable 4-manifolds (compare with §2), thus completing the theory for all closed 4-manifolds.

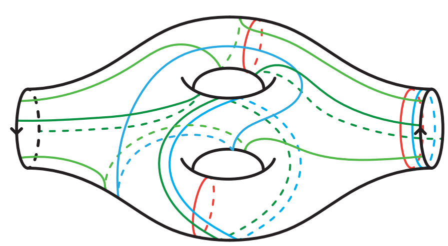
Definition 3.2. [29] A $(g;k)$- *non-orientable trisection diagram* is a tuple $(N_g;\alpha,\beta,\gamma)$ consisting of a non-orientable surface $N_g \cong \#^g(\mathbb{RP}^2\#\mathbb{RP}^2)$, together with three collections of g curves, α, β, and γ, such that:

(1) Each of α, β, and γ is a *cut system of curves* for N_g, i.e., their complement in N_g is planar, and each curve has an annular neighborhood;
(2) Each of $(N_g;\alpha,\beta)$, (N_g,β,γ), and (N_g,γ,α) describes a (non-orientable) Heegaard splitting of $\#^k S^2 \tilde{\times} S^1$.

Such a diagram describes a trisection of a 4-manifold up to handleslides among each collection of curves, and diffeomorphism of the surface (below: examples of some simple diagrams). We give an analogue of Waldhausen's theorem to characterize when collections of curves on N_g describe a Heegaard splitting of $\#^k S^2 \tilde{\times} S^1$.



$\mathbb{RP}^4$



$S^2 \tilde{\times} \mathbb{RP}^2$

Another important aspect of trisection theory is the notion of a *relative trisection*. First introduced by Gay and Kirby and studied in detail by Castro, this is a trisection of a 4-manifold X with $\partial X \neq \emptyset$; the trisection induces an open book decomposition on ∂X. One of the most foundational results is a gluing theorem of Castro [5], which guarantees that relative trisections of $(X,\partial X)$ and $(Y,\partial Y)$ inducing diffeomorphic open books on $\partial X \cong \partial Y$ can be uniquely glued along their boundary to produce a trisection of the closed manifold $X \cup_\partial -Y$. We introduce relative trisections in the non-orientable setting, and give the analogous gluing theorem. Similarly, Meier and Zupan [27] introduced *bridge trisections*, a new combinatorial way of representing knotting surfaces in 4-manifolds with respect to a fixed trisection. We also extend the theory to the non-orientable setting.

Trisections of non-orientable 4-manifolds: significance and future directions. There are many non-orientable 4-manifolds known to admit exotic smooth structures, most notably $\mathbb{RP}^4$ [4], and our extension of trisections to this setting brings the theory one step closer to understanding smooth structures via trisections.

Cappell and Shaneson [4] constructed a large family of homotopy 4-spheres as double covers of homotopy $\mathbb{RP}^4$'s (which give exotic smooth structures on $\mathbb{RP}^4$ in certain cases). One infinite family of these homotopy 4-spheres, denoted Σ_m, is now known to be standard by work of Akbulut-Kirby [2] and Gompf [13]. One can lift trisections of the Cappell-Shaneson $\mathbb{RP}^4$'s to their double covers to obtain trisections of S^4 which are at minimum, not *equivariantly* standard, which is a first step towards answering Question 2.3. This is particularly interesting because the exotic $\mathbb{RP}^4$ constructed in [4] is not even stably diffeomorphic to $\mathbb{RP}^4$, so the trisection genus invariant in §2 is unlikely to be additive in the non-orientable case. I will continue to study these trisections, and how they relate to the trisection genus invariant.

Question 3.3. *Are there non standard trisections of S^4 arising from the Cappell-Shaneson homotopy 4-spheres? Are there non-standard trisections of $\mathbb{RP}^4$?*

Together with Román Aranda, Abby Thompson, and Alex Zupan, I have also begun to investigate these homotopy spheres directly, in the hope that the perspective of trisections, or more generally, multisections (see §4) will provide fresh insight into their intricate handle decompositions.

Question 3.4. *To what extent can trisections be used to understand Cappell-Shaneson homotopy 4-spheres?*

Theorem 3.1 also has important consequences for studying non-orientable 4-manifolds via Kirby diagrams. As in the orientable case, one no longer has to specify the 3- and 4-handles.

Corollary 3.5 (Miller-N. [29])**.** *A closed non-orientable 4-manifold is determined by its 0-, 1-, and 2-handles. Equivalently, a framed link $L \subset \#^p S^2 \tilde{\times} S^1$ which surgers to $\#^q S^2 \tilde{\times} S^1$ completely determines a closed 4-manifold.*

We also use Theorem 3.1 to study diffeomorphisms of $\#^p S^2 \tilde{\times} S^1$, and give an analogue of another theorem of Laudenbach-Poénaru.

Theorem 3.6 (Miller-N. [29])**.** *Suppose that $f : \#^p S^2 \tilde{\times} S^1 2 \to \#^p S^2 \tilde{\times} S^1$ is homotopic to the identity. Then, f is isotopic to the identity.*

When $p = 1$, this was first proved by Kim-Raymond [20], who showed (equivalently) that the diffeotopy group of $S^2 \tilde{\times} S^1$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

4. Multisections of 4-manifolds

Another natural generalization of a trisection is a *multisection*, which is the analogous decomposition with more than three pieces. Given a 4-manifold X , an n-section is a decomposition of X into 1-handlebodies $X_1, X_2, \ldots, X_n$, intersecting as in Definition 2.1. In joint work [16], Gabriel Islambouli and I introduce and study these decompositions. Multisections offer more flexibility, at the cost (or feature, depending on your perspective) more subtle behavior. In particular, the union of two (non-adjacent) handlebodies $H_{i,i+1} \cup H_{j,j+1}$ describes a Heegaard splitting of a 3-manifold which is generally not $\#^k S^1 \times S^2$, giving interesting embeddings of 3-manifolds into X. We call such a 3-manifold a *cross-section* of a multisection.

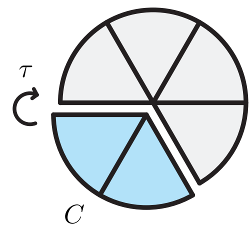
Using the exotic Mazur manifolds of [15] we immediately answer the analogue of Question 2.3 for 4-sections in the negative.

Theorem 4.1 (Islambouli-N. [16])**.** *There are infinitely many pairs of non-diffeomorphic 4-sections of S^4 with the same 3-manifold cross-sections.*

One of the present difficulties of working with trisections explicitly is that for Euler characteristic considerations, the genus of the central surface is often quite high. By contrast, this is not the case for multisections, and we witness exotic behavior in relatively low genus examples.

Theorem 4.2 (Islambouli-N. [16])**.** *The 4-manifold $E(n)_{p,q}$ has multisection genus equal to three. In particular, there are infinitely many homeomorphic but not diffeomorphic 4-manifolds admiting multisections of genus 3.*

Multisections: significance and future directions. One distinct advantage that multisections offer over trisections is compatibility with corks (see §1). By a theorem of Laudenbach-Poénaru [24], cutting and regluing a sector of a trisection can never change the ambient smooth structure. However, using a theorem of Matveyev [26], we show that every *every* cork C embedded in a 4-manifold X may be identified as two adjacent sectors of a 4-section for X. Moreover, the cork twist may be interpreted as a modification of just one of the handlebodies, which has a strong diagrammatic consequence.



Theorem 4.3. *Suppose that 4-manifolds X and X' are homeomorphic but not diffeomorphic. Then, there exists a surface, Σ, and cut systems, C_1, C_2, C_3, C_4, and C_4', for Σ such that:*

(1) $(\Sigma; C_1, C_2, C_3, C_4)$ *is a 4-section diagram for X.*
(2) $(\Sigma; C_1, C_2, C_3, C_4')$ *is a 4-section diagram for X'.*
(3) There exists a map $\tau : \Sigma \to \Sigma$ such that, $\tau(C_1) = C_1$, $\tau(C_3) = C_3$, and $\tau(C_4) = C_4'$ where τ is the restriction of a cork twist to Σ.

We explicitly describe logarithmic transformations and the Mazur cork twist in this fashion. This suggests interesting relationships between surgery and the mapping class group of the central surface Σ: 4-manifolds can effectively be described by a sequence of cut systems on a surface satisfying certain properties. I am very interested in trying to extract computable invariants for 4-manifolds in this way, and continue to investigate this in the original and more general settings.

Question 4.4. *To what extent can corks be understood using multisections? Are there computable invariants arising from the structure of a multisection?*

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