# LECTURE #1: Vectors in 2-/3-dimensional space

§1.1, §1.2

- This course will be mostly concerned with curves and surfaces in 2/3-dimensions.
- To make sense of this, we need the concept of a vector!

  "Vectors have magnitude <u>and</u> direction".

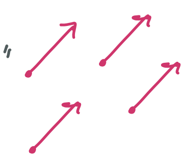
<u>DEF</u> $\mathbb{R}^n$ denotes the n-dimensional Cartesian plane. Points in $\mathbb{R}^n$ are described by tuples of the form $(x_1, x_2, \dots, x_n)$.

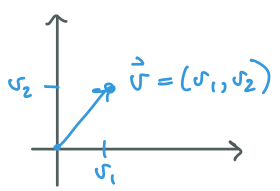
→ In particular, $\mathbb{R}^3$ is the set of 3-tuples of the form $(x, y, z)$.

<u>DEF</u> A <u>vector</u> $\vec{v} \in \mathbb{R}^3$ is a tuple $\vec{v} = (v_1, v_2, v_3)$ of real numbers.

→ We think of $\vec{v}$ as an "arrow" from the origin to $(v_1, v_2, v_3)$.

<u>IMPORTANT POINT</u>: Although they look just like points, the "point" is that we can move (translate) vectors wherever we need.






the coordinates of $\vec{v}$ describe the direction

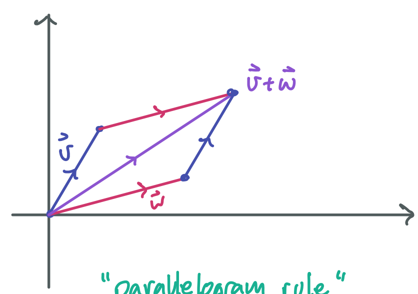
We can add and scale vectors:

- sum of $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is:

  $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

- scalar product of $v = (v_1, v_2, v_3)$ and $\lambda \in \mathbb{R}$ is

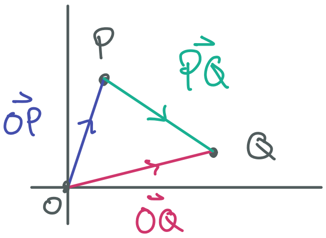
  $\lambda v = (\lambda v_1, \lambda v_2, \lambda v_3)$




"parallelogram rule"

<u>EX</u> The zero vector is $\vec{0} = (0,0,0)$. Given a point $P=(p_1,p_2,p_3)$, we write $\vec{OP} = (p_1,p_2,p_3)$. If $Q=(q_1,q_2,q_3)$, we write:

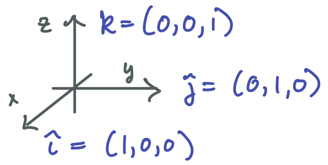
$$\vec{PQ} = \vec{OQ} - \vec{OP} = (q_1-p_1,\, q_2-p_2,\, q_3-p_3)$$




$\star$ $\vec{PQ}$ is the "arrow" between P & Q.

We'll use this formula a lot!

<u>EX</u> $\mathbb{R}^3$ comes with three "standard" vectors:




Any vector can be written as

$$(a,b,c) = a\hat{\imath} + b\hat{j} + c\hat{k}.$$

( $\{\hat{\imath},\hat{j},\hat{k}\}$ is called a basis )

<u>RMK</u> Although we represent points and vectors the same way, we think of them differently, as convenient. What's the difference?

Vectors satisfy various properties: (Check!)

(1) $\vec{0} + \vec{v} = \vec{v}$ (additive identity)
(2) $1\vec{v} = \vec{v}$ (multiplicative identity)
(3) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ (commutativity)
(4) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (associativity)
(5) $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$ (multiplicative associativity)
(6) $(\lambda+\mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$ (additive distributivity)
(7) $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ (multiplicative distributivity)

**EX** If $P=(p_1,p_2,p_3)$ and $Q=(q_1,q_2,q_3)$ are distinct points, there is a unique line $\ell$ through $P$ and $Q$.

→ If $X=(x,y,z)$ is any point on $\ell$, $\vec{PX}$ is parallel to $\vec{PQ}$, so $\vec{PX}=t\vec{PQ}$ for some $t\in\mathbb{R}$. Thus:

$$(x-p_1,\,y-p_2,\,z-p_3) = t(q_1-p_1,\,q_2-p_2,\,q_3-p_3) = t\vec{v}$$

where $\vec{v}=(q_1-p_1,\,q_2-p_2,\,q_3-p_3)$ (the direction vector)

→ We can write this as $(x,y,z)=(p_1,p_2,p_3)+t\vec{v}$. or

$$\begin{cases} x=p_1+tv_1 \\ y=p_2+tv_2 \\ z=p_3+tv_3 \end{cases}$$ (called the parametric form)

→ Provided $v_1,v_2,v_3\neq 0$ we can write this as

$$\frac{x-p_1}{v_1}=\frac{y-p_2}{v_2}=\frac{z-p_3}{v_3}$$ (called the implicit form)

**EX** Where do the lines $\ell_1,\ell_2$ below intersect?

$$\begin{cases} \ell_1 : (x,y,z)=(5,1,3)+t(0,2,-1) \\ \ell_2 : (x,y,z)=(-1,3,0)+t(1,1,\tfrac{1}{3}) \end{cases}$$

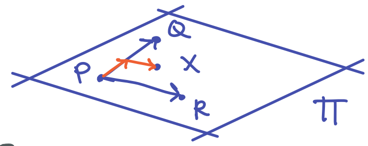
→ If $\ell_1,\ell_2$ have a point in common, then for some $t,s$ we have:

$$(5,\,1+2t,\,3-t) = (-1+s,\,3+s,\,\tfrac{1}{3}s)$$

→ First coordinate ⇒ $s=6$, second ⇒ $t=1$, third is consistent.
Thus, $\ell_1\cap\ell_2 = (5,3,2)$.

**EX** What about three points? Suppose $P=(p_1,p_2,p_3)$, $Q=(q_1,q_2,q_3)$, and $R=(r_1,r_2,r_3)$ are not collinear. Then there is a unique plane $\Pi$ passing through $P,Q$, and $R$.

→ For any point $X$ on $\Pi$, we have:




$\vec{PX} = t\vec{PQ} + s\vec{PR}$ for some $t, s \in \mathbb{R}$.

→ Just like the previous example, we can rewrite this as:

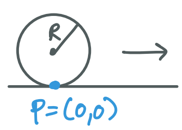
$$(x-p_1, y-p_2, z-p_3) = t\underbrace{(q_1-p_1, q_2-p_2, q_3-p_3)}_{\vec{v}} + s\underbrace{(r_1-p_1, r_2-p_2, r_3-p_3)}_{\vec{w}}$$

→ In other words,

$$(x,y,z) = (p_1,p_2,p_3) + t\vec{v} + s\vec{w}. \quad \text{(parametric form)}$$

RMK Since $P, Q, R$ are not collinear, $\vec{v}, \vec{w}$ aren't parallel → in fact, $t$ and $s$ are unique. (Check this!)

GOOD EX Consider what happens to a fixed point on a wheel as it rolls along the ground. If the wheel has radius $R$ and and rotates at 1 radian/s, and the point $P$ initially rests on the ground, give a (parametric) equation describing the position of $P$ over time.




$t=0 \Rightarrow P=(0,0)$
At time $t$, the center moves $Rt$ units to the right, (Check!)
so the center is at $(Rt, R)$.

If $(x,y)$ are the position of $P$, then we have:

$$\begin{aligned} P(t) &= \vec{OP} = \vec{OC} + \vec{CP} \quad \text{(* Note: 2D vectors here!)} \\ &= (Rt, t) + \left(R\cos\left(\tfrac{3\pi}{2}-t\right), R\sin\left(\tfrac{3\pi}{2}-t\right)\right) \\ &= (R(t-\sin t), R(1-\cos t)). \end{aligned}$$

• We've talked about adding and scaling vectors - what about multiplying them?

**DEF** The dot product of $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \in \mathbb{R}.$$

**NOTE** $\vec{v} \cdot \vec{w}$ is a scalar, not a vector! It's connected to length.

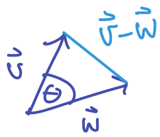
**DEF** If $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, the length (or norm) of $\vec{v}$ is:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \left(v_1^2 + v_2^2 + v_3^2\right)^{1/2} \in \mathbb{R}.$$

- If $\vec{v} \neq \vec{0}$, then $\vec{v} / \|\vec{v}\|$ has length 1, and is called a unit vector.
- The distance between $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is:

$$\|\vec{PQ}\| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}$$

- Even better, $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta$ and so,




**THM** Let $\vec{v}, \vec{w} \in \mathbb{R}^3$ be nonzero vectors, and let $0 \leq \theta \leq \pi$ be the angle between $\vec{v}$ and $\vec{w}$. Then

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = \frac{1}{2}\left(\|\vec{v}\|^2 + \|\vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2\right)$$

**PF** Use the previous identity! $\square$

Here are two extremely important identities:

(1) Cauchy-Schwartz Inequality: If $\vec{v}, \vec{w} \in \mathbb{R}^3$ then:

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$$

(*) This inequality appears in many different disguises, and is extremely useful – try to prove it!

② Triangle Inequality: If $\vec{v}, \vec{w} \in \mathbb{R}^3$, then:

$$\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$$

(*) A less common version is the "reverse" triangle inequality:

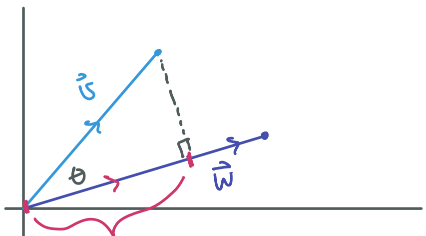
$$\big| \|\vec{v}\| - \|\vec{w}\| \big| \leq \|\vec{v} - \vec{w}\|$$ – try to prove both!

DEF Two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.

* think: "perpendicular". Why?

EX $\hat{\imath}, \hat{\jmath}, \hat{k}$ are all orthogonal.

• It's very helpful to be able to project vectors onto other vectors.




$\text{proj}_{\vec{w}} \vec{v}$ = "orthogonal projection of $\vec{v}$ onto $\vec{w}$"

• We can write: $\text{proj}_{\vec{w}} \vec{v} = \lambda \vec{w}$,

just need to find $\lambda$.

• If $\lambda \geq 0$ (ie $0 \leq \theta < \pi/2$) we can write: (Check $\lambda \leq 0$ case!)

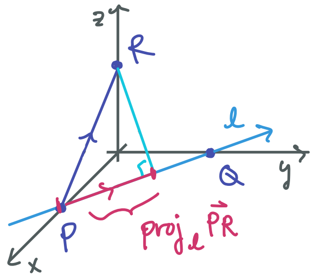
$$\lambda = |\lambda| = \frac{\|\text{proj}_{\vec{w}} \vec{v}\|}{\|\vec{w}\|} = \frac{\|\vec{v}\| \cos\theta}{\|\vec{w}\|} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}.$$

THM If $\vec{v}, \vec{w}$ are vectors in $\mathbb{R}^3$ (and $\vec{w} \neq \vec{0}$) then

$$\text{proj}_{\vec{w}} \vec{v} = \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w}.$$

HW Check that $\vec{v} - \text{proj}_{\vec{w}} \vec{v}$ and $\vec{w}$ are orthogonal.

EX What is the distance between $R = (0,0,1)$ to the line passing through the points $P = (1,0,0)$ and $Q = (0,1,0)$




* Advice: draw pictures to get intuition.

- Project $\vec{PR}$ onto $\ell$; the distance from $R$ to $\ell$ is $\|\vec{PR} - \text{proj}_\ell \vec{PR}\|$.

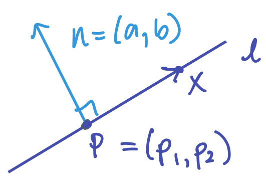
$$\|\vec{PR} - \text{proj}_{\vec{PQ}} \vec{PR}\| = \left\| (-1,0,1) - \left(\frac{(-1,0,1)\cdot(-1,1,0)}{\|(-1,1,0)\|^2}\right)(-1,1,0) \right\|$$

$$= \left\| (-1,0,1) - \tfrac{1}{2}(-1,1,0) \right\|$$

$$= \left\| \left(\tfrac{-1}{2}, \tfrac{-1}{2}, 1\right) \right\| = \sqrt{6}/2 .$$

- The same idea gives us another way to represent lines and planes.

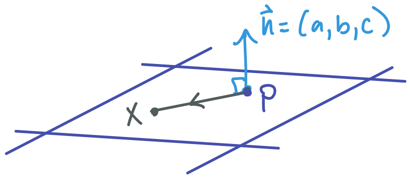
EX Here's a way to represent a line $\ell$.




Choose $\vec{n}$ normal to $\ell$. Then any point $P$ on $\ell$ satisfies $\vec{P_0P} \perp \vec{n}$, so we have an equation:

$$(x-p_1, y-p_2)\cdot(a,b) = 0$$

EX We can do this with a plane $\Pi$ in $\mathbb{R}^3$. We only need to specify a point $P_0$ on $\Pi$ and a vector $\vec{n}$ normal to $\Pi$.




Any point $X$ on $\Pi$ satisfies $\vec{P_0X} \perp \vec{n}$, so the plane is given by

$$(a,b,c)\cdot(x-p_1, y-p_2, z-p_3) = 0$$

Expanding, we get

$$ax + by + cz = (ap_1 + bp_2 + cp_3)$$

EX What's the equation of the plane passing through $P=(1,2,3)$ and normal to $\vec{n}=(1,-1,2)$? What's the distance from this plane to the origin?

- $(1,-1,2)\cdot(x-1,y-2,z-3)=0 \Rightarrow x-y+2z=5$ * draw a picture!
- The distance is the norm of the vector $\mathrm{proj}_{\vec{n}}\vec{OP}$, ie

$$\left\| \frac{(1,2,3)\cdot(1,-1,2)}{\|(1,-1,2)\|^2}(1,-1,2) \right\| = \left\| \frac{5}{6}(1,-1,2) \right\| = 5/\sqrt{6}.$$