CEMC Math Circles - Grade 11/12

November 11 - 17, 2020
Mathematical Games - Solutions

Here are the solutions to this weeks problems. There is a description of each game, followed by a discussion about the winning strategy.

Warm-up Activity: 21 Flags (described here)

The rules of **21 Flags** are the following:
- There are 21 flags.
- Two players take turns removing either 1, 2, or 3 flags.
- The player who takes the last flag wins.

Solution. In this game, you always want to go first! This is because if you go first, no matter what your opponent does, you can arrange that the number of flags remaining is a multiple of 4. For instance, if there are 4 flags left and it's your opponents turn, *no matter what they do*, you can win! Similarly, if there are 8 flags left and it's your opponents turn, you can make sure that on their next turn, there are 4 flags left (and so on). Since there are 21 flags to start, if you go first you will always be able to win.

This game illustrates a concept we'll see a few more times: the idea of *preserving* some kind of condition. Most often (though not in this game), this takes the form of *parity*, i.e. whether something is even or odd. □

A Rook on a Chessboard:

A rook is placed on the bottom right square of an 8×8 chessboard. On each players turn, they move it any number of spaces to the left, or any number of spaces up (never to the right or down). The player who moves the rook to the top left square wins.

Solution. In this game, you always want to go second. No matter what your opponent does, move back to a diagonal square. Since you can't move the rook to a new diagonal square without first moving off, you will eventually win! □

The Calendar Game:

Players take turns writing down dates. The first player must begin by writing down January 1. After this, the next player takes the previous date and may increase either the month or the day arbitrarily, but *not both*. For example, the second player could choose January 12, or May 1, but not Feburary 2. The player who writes down December 31 wins.

Solution. This game is secretly the same as **The Calendar Game**! Working backwards, notice that if you write down November 30, you will win. Indeed, no matter what your opponent chooses, you will be able to write down December 31. Similarly, if you write down October 29, no matter what your opponent chooses, you will be able to write down either November 30, or December 31. Continuing in this fashion, the same logic applies to September 28, August 27, and so on. In particular, if you go first and write down January 20, you will be able to win!

To see that this is the same as **The Calendar Game**, write the dates and months down on a grid: the winning positions are just the diagonal squares! □

The Left Handed Queen:

A queen is placed near the bottom right square of an 8×8 chessboard. On each players turn, they can move it any number of spaces to the left, diagonally up and to the left, or up. The player who moves the queen to the top left square wins.

Solution. Working backwards, we can label the chessboard with winning and losing positions. To start, we can label all squares where we can get to the top left corner in a single move; these are winning positions. Next, if a square *only* has moves to winning positions, it must be a losing position! Similarly, any square with a move to a losing position is a winning position, and so on. Continuing in this way, we can fill in the whole chessboard. In particular, the starting position given is a winning position: if you go first, you can always win.

If we add labels to the chessboard starting at zero, then the (top) losing positions have coordinates:

$$\{(1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), (12, 21), ...\}$$

Can you figure out what this sequence is? □

Easy Nim:

The game begins with two piles of stones. One has 5 and the other has 7. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone loses.

Solution. **Easy Nim** is much easier than **Nim**. This is just another example of a diagonal game! The winning strategy here is to go first, and make the number of stones in both piles equal. Once you get to two piles of two stones, *no matter what your opponent does*, you will be able to force them to take the last stone. □

> **Nim:** (can be played online here)
>
> The game begins with five piles of stones. There are $1, 2, 3, 4$, and 5 stones in each pile (to save space, we might write this as $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$). On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone loses.

Solution. This game is by far the hardest on this list! If there are only two piles (**Easy Nim**) then this is just a diagonal game like **A Rook on a Chessboard**. In other words, $m \oplus n$ is a winning position exactly when $m \neq n$ (if this is the case, your move should be to make them equal). However, if there are three or more piles, you might have to draw out all the possible moves to understand the winning and losing positions. For instance, $1 \oplus 2 \oplus 3$ is a losing position, and $1 \oplus 2 \oplus 3 \oplus 4$ is a winning position. However, there's a very nice (but slightly complicated) solution that works for *any* number of piles and stones.

One thing we'll need is the idea of a *binary number*. It's basically the same thing as a decimal number, but we're only allowed to use ones and zeros. Instead of expanding the digits of a number by powers of ten, we use powers of two (we usually include a subscript of 2 to indicate that this is a binary number). For example:

$$101101_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 32 + 8 + 4 + 1 = 45$$

To work backwards, write your decimal number as a sum of single powers of two (think about why you can always do this!). For example:

$$61 = 32 + 16 + 8 + 4 + 1 = 111101_2$$

What does this have to do with **Nim**? Let's write the number of stones in each pile in binary. For example, think about the piles $13 \oplus 4 \oplus 7$ and $11 \oplus 3 \oplus 9 \oplus 1$. In fact, these are examples of a winning and losing position. If we add up the number of ones in each column, we get the following:

$$
\begin{array}{rl}
13: & 1101 \\
4: & 100 \\
7: & \underline{111} \\
& 1312
\end{array}
\qquad\qquad
\begin{array}{rl}
11: & 1011 \\
3: & 11 \\
9: & 1001 \\
1: & \underline{1} \\
& 2024
\end{array}
$$

Notice that in the second case ($11 \oplus 3 \oplus 9 \oplus 1$) every column had an even number of ones in it. In the first case ($13 \oplus 4 \oplus 7$), there are some columns with an odd number of ones in them. In fact, this is how we can distinguish winning and losing positions! If every column has an even number of ones in it, then it is a losing position, and if there is a column with an odd number of ones in it, then it is a winning position. We just have to figure out why this works.

To see that we can always go from a winning position to a losing position, think about our example above. Consider the leftmost column that has an odd number of ones in it, and pick one of the

piles that has a one in it. You can change the numbers in this row so that all columns will have an even number of ones in them! For instance, we can change the 13 pile from 1101 to 0011 (or in other words, from 13 to 3).

$$
\begin{array}{rl}
3: & 0011 \\
4: & 100 \\
7: & \underline{111} \\
& 0222
\end{array}
\qquad\qquad
\begin{array}{rl}
7: & 0111 \\
3: & 11 \\
9: & 1001 \\
1: & \underline{1} \\
& 1024
\end{array}
$$

To see that any move from a losing position ends in a winning position, think about what happens if you take any number of stones away from some pile. In particular, you must change *at least one* digit in some row, but no other rows! In other words, there will be a column with an odd number of ones in it. For example, think about what happens if we change the pile of 11 to a pile of 7 (or any other number). If you try this strategy for **Easy Nim**, you will see that this is the same diagonal strategy as before.

Once again, parity was important! Can you use this to check whether $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$ is a winning or losing position? $\qquad\qquad\square$
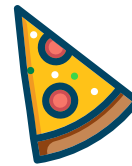
Disclaimer: There are no solutions provided for the challenge problems. If you're interested in one of them or you think you have a solution, get in touch!

CEMC Math Circles - Grade 11/12

November 18 - 24, 2020
Partitioning Puzzles - Solutions

Here are the solutions to this weeks problems. There is a description of each problem, followed by a discussion about the solution.

Problem 1. Once a day, you break a large 4×8 rectangular chocolate bar into its smaller 1×1 pieces, to distribute to your class of 32 students. After some time, you notice that *no matter how* you break up the chocolate bar, it always requires the *same* number of breaks. How many does it take? Why?

Remarks: You can only break the chocolate along its grid lines, and you can only break a *single* piece at a time (the way one might actually break up a chocolate bar). If you can do this problem, consider a chocolate bar of size $m \times n$. Now how many does it take?

Solution. It will always take 31 breaks to break the chocolate bar into its 1×1 squares. You can think about all of the possible ways to break it up (there are many), but working backwards is much easier. Each time you make a break, you will have *exactly one* more piece (maybe not of size 1×1) than you did before. Working backwards, this means you will always make 31 breaks (one less than the total number of squares). If the chocolate bar had size $m \times n$, you would always perform $mn - 1$ breaks.

Try this with some small chocolate bars to see it in action! $\square$

Problem 2. You begin with a cube made from 27 smaller cubes glued together (arranged just like a Rubik's cube, although this has nothing to do with this problem). You would like to cut it into its 27 component cubes with a large knife– what is the minimum number of cuts required to do so? How do you know?

Remarks: This is not a trick question. The knife only cuts in a straight line; it does not bend, and you cannot move the pieces while you are making a cut. You may re-arrange, stack, or assemble the pieces however you would like in between each cut, so that you cut more than one piece at the same time. While reading this remark, you should try to convince yourself that you need at *least* 4 cuts.
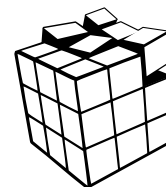
Solution. This is a slightly tricky question. First, we note that we can certainly separate the cube into 27 pieces using 6 cuts: just cut twice along 3 different faces without every moving any of the pieces. The question is: if we're allowed to re-assemble the pieces in between each cut, can we do better?

The answer is no! Think about the cube at the very centre. No matter how you re-assemble the pieces, it will take always 6 cuts to separate it from all of its neighbours, since those cuts can't be performed at the same time. Thus, we'll always need at least 6 cuts.

Perhaps surprisingly, a $4 \times 4 \times 4$ Rubik's cube can also be cut into its 64 pieces in only 6 moves (figure out how). If you thought this was easy: how many cuts does it take to cut apart an $n \times n \times n$ cube? □

> **Problem 3.** You have a single piece of pizza, in the shape of a (perfect) equilateral triangle with side length 1 unit. You and your friend would like to divide it equally: what is the *length* of the shortest curve which cuts the triangle into two pieces of equal area?

Remarks: This is not a trick question. The curve does not have to be a straight line, nor does it have to start or end at any of the corners. The *median* of the triangle (in red on the right) is one curve that works; you should convince yourself that the length of this curve is $\sqrt{3}/2$. Can you do better than this?

Solution. Of the three problems, this one is the hardest. We'll start by setting some notation: we will call the equilateral triangle T. Note that all of the interior angles of T are all equal to $60°$. In radians, this is simply $\pi/3$. If we want, we can plot this triangle in the plane with two vertices at $(0,0)$ and $(1,0)$ (like all of the pictures below).

We can do some trigonometry to figure out the length of the median (illustrated in orange on the right): it is the opposite side of a triangle with angle $60°$ and hypotenuse equal to one, so has length $\sin(60°) = \sqrt{3}/2 \approx 0.866$. Even better, the area of a triangle is equal to (base $\times$ height)/2, so using the median (the height), we find that the area of T is $(1 \cdot \sqrt{3}/2)/2 = \sqrt{3}/4 \approx 0.433$.
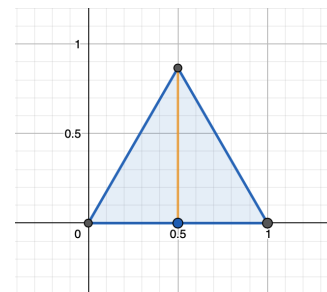
Thus, the median is one possible curve which splits T into two regions of equal area, and has length $\sqrt{3}/2$. The question is: can we do any better?

There are essentially two possibilities: the curve has endpoints on the "same side," or on "two different sides." The first case is easier, and maybe something you discounted right away without giving a rigorous justification. To deal with this, we'll use the following fact, which will appear later:

A circle is the shortest curve among all curves enclosing a fixed area.

Take a moment to parse what this means. In other words, if we want to enclose a region of some fixed area, the shortest curve which does so is a circle. This seems obvious when you think about it, but proving it rigorously is surprisingly hard. If you'd like to know how, ask me!

In particular, if we want to enclose the most area with the shortest curve that starts and ends on one side of T, it had better be a semicircle! If we use a semi circle (centered at $(1/2, 0)$), we can

figure out the required radius r fairly easily. The area enclosed by the semicircle is $\pi r^2/2$, but on the other hand this has to be $\sqrt{3}/8$ (half of the area of T). Rearranging, we find that:

$$\frac{\pi r^2}{2} = \frac{\sqrt{3}}{8} \implies r = \sqrt{\frac{\sqrt{3}}{4\pi}} \approx 0.371$$

If we plot the circle with this radius centred at $(1/2, 0)$, we get the picture on the right. Moreover, the length l of this curve is exactly half of the circumference, so we get:

$$l = \frac{\pi \cdot 2r}{2} = \pi r = \pi\sqrt{\frac{\sqrt{3}}{4\pi}} \approx 1.166$$

This is way worse than our original candidate! We conclude that if the curve we choose has its endpoints on the same side, the shortest it could be is 1.166.

What about the second case? This time we'll need a trick, since our useful fact about circles doesn't really apply anymore. Suppose that we have a curve that starts and ends on different sides of T. Then we can take six copies of T (and our curve) to form a hexagon, and a closed curve! You might worry that the endpoints won't match up; you can check that if we're careful about how we do this, they will.
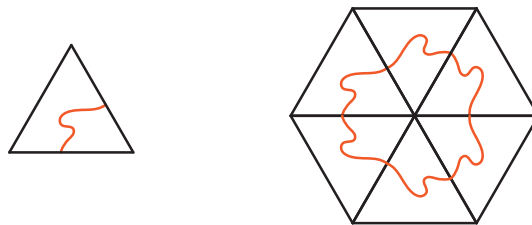
Great! Now we do have a closed curve, and we know that it must have the shortest length among all curves enclosing exactly half the area of the hexagon. Therefore, it must be a circle! Since the area of the hexagon is six times the area of T, we can figure out what the radius R must be in this case. We find that:

$$\pi R^2 = \frac{1}{2}(6 \cdot \sqrt{3}/4) \implies R = \sqrt{\frac{3\sqrt{3}}{4\pi}} \approx 0.643$$

Even better, the length L of the curve (the portion of the circle in T) is exactly 1/6 of the circumference of this circle, so we get:

$$L = \frac{1}{6} \cdot \pi(2R) = \frac{\pi}{3} \cdot \sqrt{\frac{3\sqrt{3}}{4\pi}} \approx 0.673$$
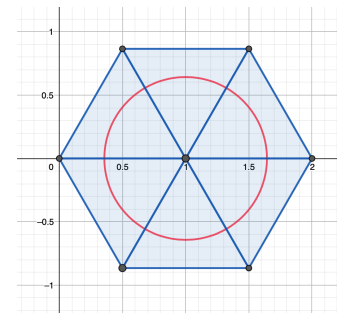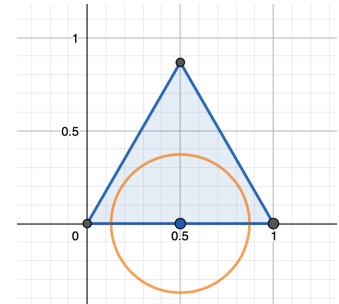
This is better than our estimate using the median! We conclude that if the curve we chose has endpoints on two different sides, this is the best we can do. In particular, this is the length of the shortest curve which cuts an equilateral triangle into two regions of equal area. □

$L \approx 0.673$

$R \approx 0.643$
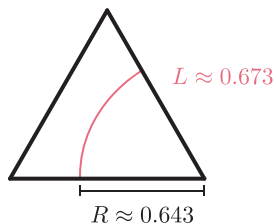
Other Remarks: If you'd like to know how to prove that circles have the least perimeter among all shapes with a fixed area, let me know! If you could do this problem or thought this solution was cool, try doing this for other regular polygons. What if T was a square? What if T was a pentagon?

CEMC Math Circles - Grade 11/12

November 25 - December 2, 2020
## Dwellings with many Doors - Solutions

Here are the solutions to this weeks problems. There is a description of each problem, followed by a discussion about the solution.

> **Problem 1.** You live in a large house with several rooms– the floor plan is illustrated below. One day, in the midst of your infinite boredom, you wonder about the following question:
>
> Can you walk through your house in such a way that you go through each door *exactly once*? If so, how many different ways are there to do this?

Remarks: This isn't a trick question. The gaps in the walls are doors, and you can pass through each door exactly once. You can start and end wherever you would like.
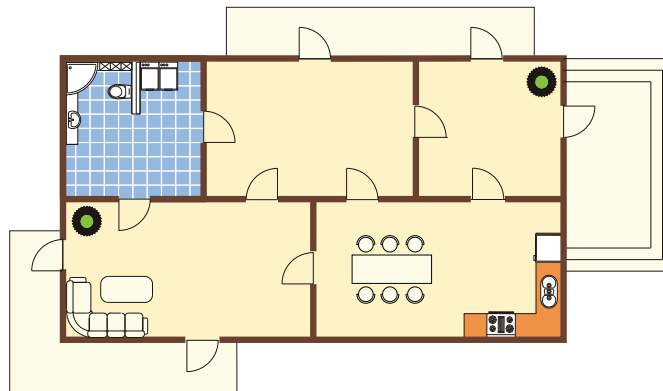

Solution. Here is a very short reason that this is impossible. There are two possibilities for such a walk: either you start and end in the same room, or you start and end in *different* rooms (we'll consider the outside of the house to be a "room"). Notice that if you start and end in the same room, this means that there must be an *even* number of doors leading in or out of each room– because no matter what, you have to leave every time you enter. Similarly, if you start and end in different rooms, this means that *exactly two* of the rooms have an odd number of doors leading in or out, and the rest must have an even number.

Looking at the schematic we see that in fact, *four* of the rooms (don't forget the outside of the house) have an odd number of doors, and so we conclude that no such walk is possible! ☐

This kind of problem was historically very interesting, and the most google-able version is called the *Bridges of Königsberg* problem. You can find lots of information on the Wikipedia page here. It was solved by Euler in 1736, and laid the foundations of topology and graph theory.

The problem is the following (it's very similar to Problem 1): can you walk through the city of Königsberg (shown below) in such a way that you cross each bridge exactly once?
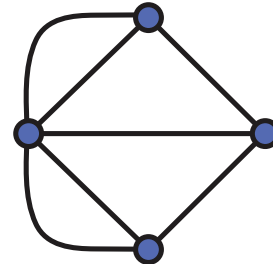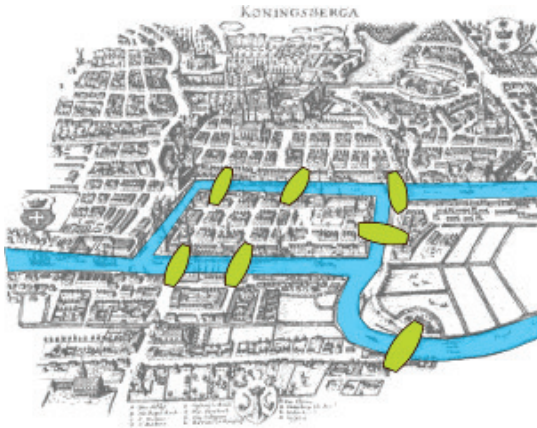
Euler pointed out that really, the drawing is irrelevant; what matters is how the different land masses are connected to each other! He drew a "graph," which consists of a point for each land mass, and and edge whenever two land masses were connected by a bridge. Now, the question really asks: can you walk through this graph in such a way that you traverse each edge exactly once?

In modern language, an **Eulerian circuit** for a graph is a walk where you start and end at the same place, and an **Eulerian path** is a walk where you start and end in different places. Euler proved (in fact, it's the same proof that we gave for Problem 1!) that a graph has an Eulerian circuit exactly when each node has an even number of edges, and that a graph has an Eulerian path exactly when there are exactly two nodes with an odd number of edges. It sounds fancy, but it's the exact same analysis that we did for Problem 1. In particular, you cannot walk through the city of Königsberg in such a way, since there are four nodes with an odd number of edges. This is a very useful trick, and we'll apply it to solve Problem 2.

> **Problem 2.** Your butler becomes tired of you endlessly wandering around the house, and shuts the doors to the bathroom (the room in the top left corner) Does this help matters? In other words, can you now walk through the house in such a way that you go through each remaining door exactly once? What if you insist on starting and ending at the same place?
>
> If you can do either of these walks, how many different walks of each kind are there?
>
> What is the smallest number of doors that you can close so that you can walk exactly once through each remaining door, and start and end in the same place?

Solution. We analyze the new floor plan in the same way. Now, exactly two of the rooms have an odd number of doors.

That means that there is a Eulerian path (where we start and end in different rooms– one such path is illustrated in pink) but there *cannot* be an Eulerian circuit (where we start and end in
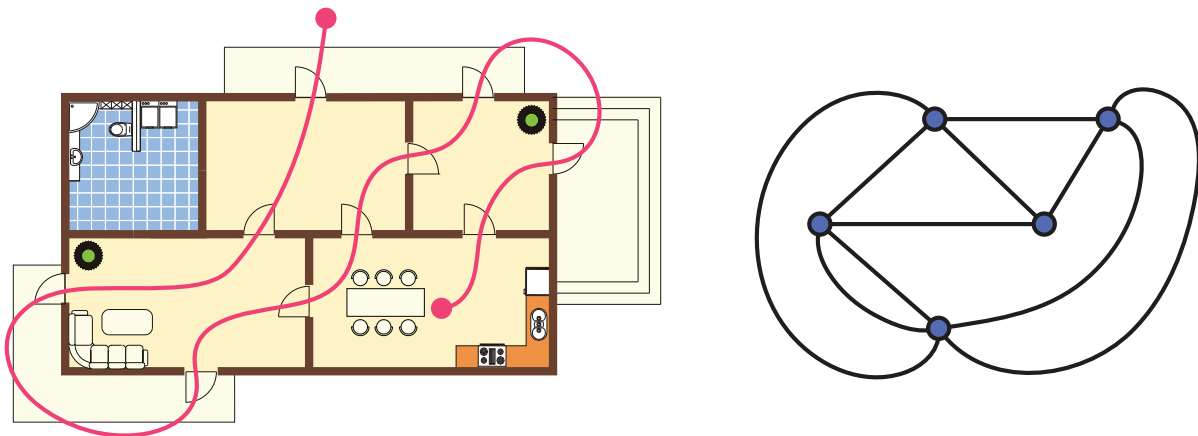
the same room). We can also draw the corresponding graph! We draw a node for each room, and connect two nodes with an edge if they share a door.

There are a lot of possible paths! If you think you have a way of counting them all, let me know. If you want to be able to find an Eulerian circuit, your butler will have to close different doors. You will need to close at least two– think about why! However, you can do it with only two. For example, if you close the doors labelled A and B, you will be able to find an Eulerian circuit.
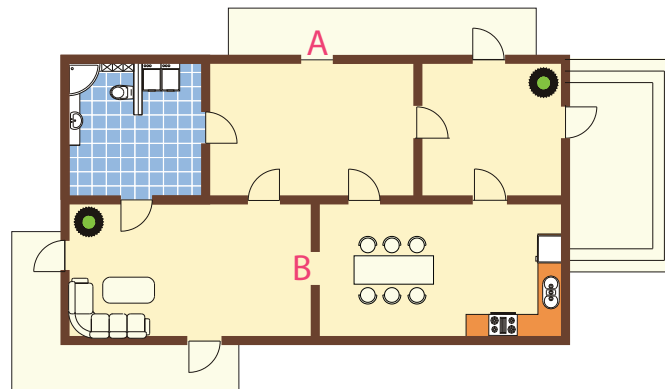
If you were able to do both of these problems and want something interesting to think about, try this challenge problem: how many ways are there to close two doors, so that the resulting house admits an Eulerian circuit?