

Assignment 5 - Delaunay Triangulation and Voronoi Diagram

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Let denote LB and UB as a lower bound and upper bound of of maximum number that an arbitrary line can intersect with a triangulation T of n points. We claim that :

LB = No. Triangles in T + 1

UB = No. Triangles in T + 1

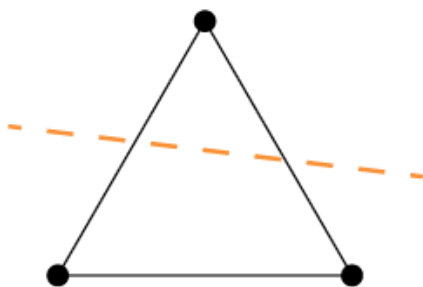


Figure 1: Smallest Triangulation

The LB is illustrated by the example from Figure 1, the smallest triangulation having 1 triangle, that the maximum number of intersection is 2. For UB, we have observed that for 1 triangle the number of intersection is 2, which is the number of triangle plus 1. Let's assume that for k triangles, the max number

of intersection is $k + 1$. Next, if we add one more triangle, one of its edges will be shared with the one of the old triangles, thus it can generate only 1 more intersection (we already know that there are at most 2 intersections with each triangle). Thus, the max number of intersection is $k + 2$. Therefore, we can conclude that for a triangulation T the UB of intersection between an arbitrary and T equals to the number of triangles in T plus 1 as we claim.

(According to the lecture, the number of triangle in a triangulation of n points is $2n - 2 - h$ where h is the number of vertices on the convex boundary).

In naive approach of finding average intersection, traversing n^2 pairs and checking with all edges $O(n)$ takes $O(n^3)$ running time. To improve the efficiency, the data structure using duality, proven in Exercise Week 4 Question 3, is used. Such that we can query the number of intersection between an arbitrary line and edges in $O(\log n)$ expected time, instead of $O(n)$ time. Hence, the total complexity is reduced to $O(n^2 \log n)$.

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a

In this section, we have to prove that the edge set of the EMST of P contains the edges of a nearest neighbor graph.

Assume that there is an edge between the points a and b of the nearest neighbor graph which is not in the EMST, then we could replace the edge containing a by ab in the tree and that will decrease the total edge length of the EMST, because the current edge containing a is longer than ab . So the EMST is not EMST. This contradiction proves the statement.

b

In this section, we have to prove that the set of edges of the Gabriel graph of P contains and EMST of P .

Assume that the edge set of the Gabriel graph does not contain an EMST of P , which means there is an edge pq in the EMST that is not in the Gabriel graph.

This implies there is a point k inside the circle taking pq as the diameter. Hence, pk and qk will become smaller than pq . Thus, pq is not a valid edge of the EMST. This contradiction proves the statement.

c

Suppose that the Delaunay graph of P does not contain the Gabriel graph of P , which means there are a pair of points p, q that are not in the same triangle, but the circle c taking pq as the diameter is empty. We will prove that at least

a triangle in the triangulation is illegal.

Indeed, there is several triangles between p and q because pq is not a triangle edge. Let q_1, q_2 be 2 points in one of these triangles, which contains p . We know that the circle c is empty, so both q_1 and q_2 are outside of c . Thus, the circumcircle of $\triangle q_1 q q_2$ is bigger than c , which implies that its diameter is larger than pq . Then, this circumcircle contains p . So $\triangle q_1 q q_2$ is not a valid triangle.

Similarly, we can prove that the triangle containing p is also invalid. Thus, the graph is not the Delaunay graph anymore.

By contradiction, we have just proven that the Delaunay graph of P contains the Gabriel graph of P .

d

Firstly, we have to prove that if pq is an edge in the Gabriel graph, then the Delaunay edge between p and q intersects its dual Voronoi edge.

Assume $\triangle pvq$ a Delaunay triangle. Due to Gabriel graph definition, v should not be in the region of the circle taking pq as the diameter.

We know that the common intersection of the 3 Voronoi edges is the circumcenter of $\triangle pvq$. In this case, $\angle pvq \leq \pi/2$ because v is outside of the circle, so the circumcenter is inside the triangle region. Thus, the Voronoi edge between p and q intersects pq .

Secondly, we have to prove that if the Delaunay edge between p and q intersects its dual Voronoi edge, then pq is an edge in the Gabriel graph.

Denote s the common intersection of bisectors pq , qv and pv . s is thus equidistant from p, q, v .

Because the Delaunay edge between p and q intersects its dual Voronoi edge, s is in the half-plane defined by pq and containing v .

We know that $sv = sq = sp = r$ where r is the radius of the circle taking pq as the diameter. Therefore, v is not in the circle inner region.

This completes the proof.