Assignment 2 - Triangulation and Linear Programming

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We give the algorithm as follows.

- Given a simple polygon P a with n vertices.
- Perform triangulation on P.
- Construct the dual graph G of P.
- Find a root node R of G whose degree is 1.
- Perform $LabelNumberOfChildNodes(G, R)$, which is described in Algorithm 1.
- Select the node v_i whose label is n where $n = max(label_i, 0 < i < n \text{ and } label_i \leq \lfloor 2n/3 \rfloor)$
- Find the diagonal line corresponding to the edge between v_i and its parent.

Correctness

The first case is when we can pick the node with the exact label, that is $\lfloor 2n/3 \rfloor$. In this case, one polygon that the algorithm returns has exactly $\lfloor 2n/3 \rfloor + 2$ vertices. The other polygon has exactly $\lfloor n/3 \rfloor + 2$ vertices. Hence the algorithm is correct.

The second case, shown in Figure 1, is when we cannot find the exact label, so we have to find the closest node v^*, whose label is the maximum one that is less than $\lfloor 2n/3 \rfloor$. In this case, there is 2 branches starting from the parent of v^*. Hence, the label of the parent of v^* is the sum of the labels of its 2 children,

Algorithm 1 LabelNumberOfChildNodes

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Require: a dual graph $G$ and a node $v_i$
  Label $v_i$ as $Visited$
  if $degree(v_i) > 1$ then
      $NumNodes = 0$
      for Each neighbor $v_j$ of $v_i$ do
          if $v_j$ is not $Visited$ then
              $NumNodes = NumNodes + LabelNumberOfChildNodes(G, v_j)$
          end if
      end for
      $label_i = 1 + NumNodes$
      Return $label_i$
  else
      $label_i = 1$
      Return $label_i$
  end if
```

and it is greater than $\lfloor 2n/3 \rfloor$. Therefore, if v^* is the maximum value between the 2 children, then the label of v^* is greater than $\lfloor n/3 \rfloor$. That is,

$$n/3 \leq label(v^*) < 2n/3$$

So if we cut by the edge between v^* and its parent, neither of the 2 polygons has more than $\lfloor 2n/3 \rfloor + 2$ vertices. Then the algorithm is correct.

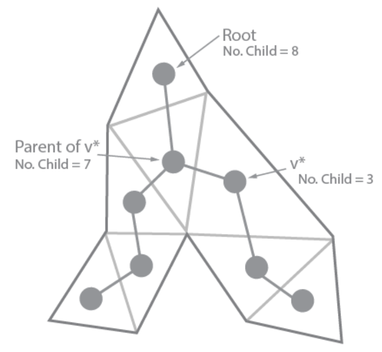

Figure 1: A polygon whose the tree of its dual graph has 2 branches

Running Time

- Performing triangulation on P takes $O(n \log n)$ (by splitting into y-monotone polygons and triangulate each one)

- Constructing the dual graph G of P takes $O(n)$
- Finding a root node R of G whose degree is 1 takes $O(n)$
- Performing $LabelNumberOfChildNodes(G, R)$ takes $O(n)$ because we traverse each node only once
- Selecting the appropriate node v_i take $O(n)$
- Finding the diagonal line corresponding to the edge between v_i and its parent takes constant time

Thus, the algorithm performs in $O(n \log n)$

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a

This classification problem will be solved by a randomized incremental algorithm, by first shuffling the points randomly and initially choosing two points of different classes (P_1 and P_2). The line that passes through both of them will be our initial separating line l.

As the question is whether there is a line l with the points of P_1 above or on it, and the points of P_2 below or on it, we can assume that the correct position of points from P_1 is above l, and the correct position of points from P_2 is below l.

For each of the next points, we check whether it is in the correct position or not. If not, we modify the line l. The detailed algorithms is as follows.

Algorithm:

$FindSeparatingLine(S)$:

1. Given: Set S of m points of class P_1 and n point of class P_2
2. Output: Separating line (it contains 2 points of different classes) if any or "Not Found"
3. Initialize empty array Z
4. S $\leftarrow$ RandomPermutation(S)
5. Initialize l our initial separating line to be the line between 2 different random points of P_1 and P_2
6. For every other point $p \in S$:
   (a) insert p to Z

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    (b) If $p$ is in the proper position
        i. continue
    (c) Else
        i. $l \leftarrow$ create a new line between $p$ and the point of different class
           in $l$
        ii. loop on every point in the other class, connect $p$ to the new point
            if that point is not in the correct position
        iii. loop on every point in $Z$, if there is a point that not in the proper
             position, return "Not Found"
7. return $l$
```

Correctness

We prove the algorithm by induction.

Based on our selection, the classifier l always contains 1 point from P_1 and 1 point from P_2.

If the dataset contains only one point in P_1 and one point in P_2, then the algorithm pick the line l passing these points to be the classifier, and this is the correct one.

Assume that after processing point i, the algorithm has already had the correct solution. Let p^* be the next point, p_i is the point of the same class and p'_i is the point of the different class (P_1 or P_2) which are currently on l. When we insert p^*, the new classifier should group it to the correct group.

If p^* is in the correct order already (the point of P_1 should be above or on l, and the point of P_2 should be below or on l), then the classifier l is correct and the algorithm does not do anything.

If p^* is not in the correct order, which means it is on the "outer" space of the group of p_i, the line l should be modified to group p^* to the proper position. Our algorithm decides to rotate the line l so that it contains p^*. This is a proper position for p^* because every point can be on l, and p_i is also in that group because l was rotated to the "outer" direction. Since the old $l = p_i p'_i$ was the correct classifier, this guarantees that the new line $l = p^* p'_i$ correctly classifies the class of p^*.

After correcting the position of p^*, there can be a case that some points of the opposite class are wrong positioned, as being shown in figure 2. The algorithm has to perform a loop on the opposite class to correct those points. After this correction, all points of the opposite class are in the right place. The line l can be changed to $p^* p_i''$ where p_i'' is the point that better classifies the opposite class.

If after that, there are still points outside of their correct positions, which means there are (again) some points in the class of p^* that are in wrong places, then we know that there are no agreements in this case. Thus, there are no line classifiers available for this setting. The algorithm indeed returns false.

If all of the points are correctly positioned, then l is obviously the correct classifier. So, if the previous step is correct, then the current step is also correct. This implies that our incremental approach is correct.

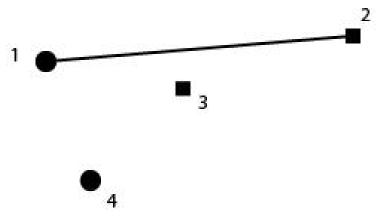

Figure 2: Example of 2-round modification. Circles: P_1, Squares: P_2. Intitially, p_3 is in the correct order, but after checking p_4, we rotate the line to p_4p_2 so that P_1 is above the line. So p_3 is not correct. Then we have to loop on P_2 to put p_3 back to the proper position. The final result is p_4p_3.

Running Time

In order to shuffle all points of P, the algorithm takes $O(n+m)$.

The 2-round check has to traverse the previous processed points for checking correctness, this process takes $O(i)$ where i is the number of points which have been processed. Hence, the expected running time is :

$$O(n+m)+\sum_{i=1}^{m+n} (Pr[\,p_i \in \text{ProperPosition}\,] * O(1) + Pr[\,p_i \notin \text{ProperPosition}\,] * O(i))$$

We know that $Pr[\,p_i \in \text{ProperPosition}\,] \leq 1$ and $Pr[\,p_i \notin \text{ProperPosition}\,]$ is never greater than the probability of selecting 2 points from i points. Thus, we have :

$$Pr[\,p_i \notin \text{ProperPosition}\,] \leq 2/i$$

Hence,

$$\begin{aligned} T(n+m) &= O(n+m) + \sum_{i=1}^{n+m} (O(1) + O(2/i)O(i)) \\ &= O(n+m) \end{aligned}$$

(b)

The worst case is when the algorithm has to determine l every time processing a new point. Then, the running time will become $O(n^2)$.

Let's consider the case that we perform shuffling on P_2 which only takes $O(n)$. The algorithm takes $O(m)$ to find the first separating line between all points in P_1 and one point of P_2. Then, for iterating the other points of P_2, the running time when the algorithm has to find a better separating line changes sligthly to $O(m+i)$, where i is the number of processed points in P_2 so far. Hence, the expected running time is :

$$\begin{aligned} T(n+m) \quad &= \quad O(n) + O(m) \\ & \quad + \sum_{i=1}^{n} Pr[\, p_i \in \text{ProperPosition}\,] * O(1) \\ & \quad + \sum_{i=1}^{n} Pr[\, p_i \notin \text{ProperPosition}\,] * O(m+i) \end{aligned}$$

Hence,

$$\begin{aligned} T(n+m) &= O(n) + O(m) + \sum_{i=1}^{n} (O(1) + O(2/i)O(m+i)) \\ &= O(m) + 3O(n) + O(m) \sum_{i=1}^{n} 1/i \\ &= O(m) + 3O(n) + O(m)O(\ln n) \\ &= O(m) + 3O(n) + O(m)O(\log n) \\ &= O(m \log n) \end{aligned}$$

We can conclude that if we shuffle only some subset of data, in this case only P_2, the algorithm would perform worse.