Assignment 5 - Delaunay Triangulation and Voronoi Diagram

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Let denote LB and UB as a lower bound and upper bound of of maximum number that an arbitrary line can intersect with a triangulation T of n points. We claim that :

$$\text{LB} = \text{No. Triangles in T} + 1$$
$$\text{UB} = \text{No. Triangles in T} + 1$$

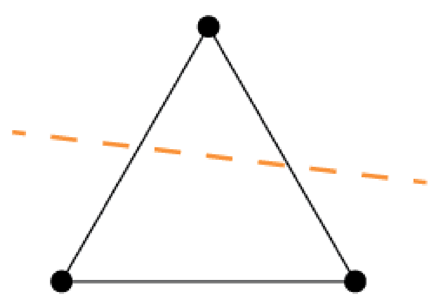

Figure 1: Smallest Triangulation

The LB is illustrated by the example from Figure 1 ,the smallest triangulation having 1 triangle, that the maximum number of intersection is 2. For UB, we have observed that for 1 triangle the number of intersection is 2, which is the number of triangle plus 1. Let's assume that for k triangles, the max number of intersection is $k+1$. Next, if we add one more triangle, one of its edges will be shared with the one of the old triangles, thus it can generate only 1 more intersection (we already know that there are at most 2 intersections with each triangle). Thus, the max number of intersection is $k+2$. Therefore, we can conclude that for a triangulation T the UB of intersection between an arbitrary and T equals to the number of triangles in T plus 1 as we claim.

(According to the lecture, the number of triangle in a triangulation of n points is $2n-2-h$ where h is the number of vertices on the convex boundary).

In naive approach of finding average intersection, traversing n^2 pairs and checking with all edges $O(n)$ takes $O(n^3)$ running time. To improve the efficiency, the data structure using duality, proven in Exercise Week 4 Question 3, is used. Such that we can query the number of intersection between an arbitrary line and edges in $O(\log n)$ expected time, instead of $O(n)$ time. Hence, the total complexity is reduced to $O(n^2 \log n)$.

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a

In this section, we have to prove that the edge set of the EMST of P contains the edges of a nearest neighbor graph.

Assume that there is an edge between the points a and b of the nearest neighbor graph which is not in the EMST, then we could replace the edge containing a by ab in the tree and that will decrease the total edge length of the EMST, because the current edge containing a is longer than ab. So the EMST is not EMST. This contradiction proves the statement.

b

In this section, we have to prove that the set of edges of the Gabiel graph of P contains and EMST of P.

Assume that the edge set of the Gabiel graph does not contain an EMST of P, which means there is an edge pq in the EMST that is not in the Gabiel graph.

This implies there is a point k inside the circle taking pq as the diameter. Hence, pk and qk will become smaller than pq. Thus, pq is not a valid edge of the EMST. This contradiction proves the statement.

c

Suppose that the Delaunay graph of P does not contain the Gabriel graph of P, which means there are a pair of points p, q that are not in the same triangle, but the circle c taking pq as the diameter is empty. We will prove that at least a triangle in the triangulation is illegal.

Indeed, there is several triangles between p and q because pq is not a triangle edge. Let q_1, q_2 be 2 points in one of these triangles, which contains p. We know that the circle c is empty, so both q_1 and q_2 are outside of c. Thus, the circumcircle of $\triangle q_1 q q_2$ is bigger than c, which implies that its diameter is larger than pq. Then, this circumcircle contains p. So $\triangle q_1 q q_2$ is not a valid triangle.

Similarly, we can prove that the triangle containing p is also invalid. Thus, the graph is not the Delaunay graph anymore.

By contradiction, we have just proven that the Delaunay graph of P contains the Gabiel graph of P.

d

Firstly, we have to prove that if pq is an edge in the Gabriel graph, then the Delaunay edge between p and q intersects its dual Voronoi edge.

Let v be the third point in the Delaunay triangle containing p and q, then v is not in the region of the circle taking pq as the diameter, due to the Gabriel graph definition.

We know that the common intersection of the 3 Voronoi edges is the circumcenter of $\triangle pvq$. In this case, $\angle pvq \leq \pi/2$ because v is outside of the circle, so the circumcenter is inside the triangle region. Thus, the Voronoi edge between p and q intersects pq.

Secondly, we have to prove that if the Delaunay edge between p and q intersects its dual Voronoi edge, then pq is an edge in the Gabriel graph.

Let v be the third point in the Delaunay triangle containing p and q, then the common intersection s is the circumcenter of $\triangle pvq$. s is thus equidistant from p, q, v.

Because the Delaunay edge between p and q intersects its dual Voronoi edge, s is in the half-plane defined by pq and containing v.

We know that $sv = sq = sp = r$ where r is the radius of the circle taking pq as the diameter. Therefore, v is not in the circle inner region.

This completes the proof.