# Modal Analysis

## VU 325.100

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2020S

# Free Oscillation Eigenvalue Problem

# Linear, Single DoF Oscillator

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{1}$$

- inhomogeneous, linear, second order ODE with constant coefficients
- homogeneous solution describes transient response
- particulate solution describes steady-state response

# Complex Representation

A physical signal can be represented by the real part of it's complex representation

$$x(t) = \Re\{\hat{x}e^{\hat{s}t}\} \quad \text{with } \hat{x}, \hat{s} \in \mathbb{C} \tag{2}$$

- complex representation is often more convenient to work with
- amplitude and phase encoded in $\hat{x} = a + bj$ with $a, b \in \mathbb{R}$
- damping and frequency encoded in $\hat{s}$

Derive the relations for amplitude, phase, damping and frequency for the real valued signal $x(t) = x_0 e^{\sigma t} \cos(\varphi + \omega t)$.

# Multi Degree of Freedom System

A system of $n$ coupled, linear, second-order ODEs may be written in matrix form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}(t) \tag{3}$$

with the vector of degrees of freedom $\boldsymbol{x} = [x_1, \ldots, x_n]^T$,
$n \times n$ mass, damping and stiffness matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$, and
time dependent forcing vector $\boldsymbol{f}(t) = [f_1(t), \ldots, f_n(t)]^T$.

- can be obtained in various ways
- can be recast into first order system using $\boldsymbol{y} = [\boldsymbol{x}, \dot{\boldsymbol{x}}]^T$

# Free, un-damped Oscillations

- free oscillations occur without excitation, thus $\boldsymbol{f} = \boldsymbol{0}$
- un-damped implies $\boldsymbol{C} = \boldsymbol{0}$

Inserting into the general system (3) simplifies to

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0} \tag{4}$$

We can insert the general solution of the from

$$\boldsymbol{x}(t) = \Re\{\hat{\boldsymbol{x}} e^{j\omega t}\} \tag{5}$$

to obtain the un-damped, free-oscillation eigenvalue problem

$$[\boldsymbol{K} - \omega^2 \boldsymbol{M}]\hat{\boldsymbol{x}} = 0 \tag{6}$$

# Types of Eigenvalue Problems

| | |
|---|---|
| standard | $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ |
| generalised | $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{B}\boldsymbol{x}$ |
| quadratic | $[\boldsymbol{K} + \lambda \boldsymbol{C} + \lambda^2 \boldsymbol{M}]\boldsymbol{x} = \boldsymbol{0}$ |
| polynomial | $[\boldsymbol{A}_0 + \lambda \boldsymbol{A}_1 + \cdots + \lambda^n \boldsymbol{A}_n]\boldsymbol{x} = \boldsymbol{0}$ |
| nonlinear | $\boldsymbol{A}(\lambda)\boldsymbol{x} = \boldsymbol{0}$ |

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# Conversion to Standard Form

A generalised EV problem can transformed to a standard one by pre-multiplying $\boldsymbol{B}^{-1}$ or $\boldsymbol{A}^{-1}$ giving

$$\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{or} \quad \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{x}$$

A quadratic EV problem my can be converted to a generalised one by $\boldsymbol{y} = \lambda\boldsymbol{x}$ and

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \lambda \begin{bmatrix} -\boldsymbol{C} & -\boldsymbol{M} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \tag{7}$$

The transformation in (7) is not the only possible one. Note also the similarity to the transformation of higher order ODEs.

# Solving an Eigenvalue Problem

We consider the standard eigenvalue problem

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \quad \text{or equivalently} \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0} \tag{8}$$

which has a non-trivial solutions ($x_i \neq 0$) if

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \tag{9}$$

- The aim is to compute the roots $\lambda_i$ of the characteristic equation (9) of the eigenvalue problem (8).
- The roots are called *Eigenvalues*.
- For each eigenvalue one can compute a corresponding eigenvector $\boldsymbol{x}_i$.

The eigenvectors are not unique as $(\boldsymbol{A} - \lambda_i \boldsymbol{I})$ is singular. Unique eigenvectors are obtained by setting constraints e.g. on their length.

# Eigenvalue Solvers

**By hand** find the roots of the characteristic polynomial.

**Vector iteration** computes a single eigenvalue/vector iteratively. Also: inverse vector iteration, Rayleigh quotient Iteration, ...

**QR-algorithm** computes all eigenvalues iteratively using QR-decompositions.

**Subspace iterations** extract a sub-set of eigenvalues at one end of the spectrum. Important algorithms: Lanczos, Arnoldi.

**Contour-integral** based algorithms calculate all eigenvalues inside a given contour in the complex plane.

Eigenvalue solvers available in Matlab, numpy/scipy typically use combinations of above techniques. They often use publicly available libraries, e.g. LAPACK[a], ARPACK[b], or FEAST[c].

[a] `http://www.netlib.org/lapack`
[b] `http://www.caam.rice.edu/software/ARPACK/`
[c] `http://www.ecs.umass.edu/~polizzi/feast/index.htm`

# Vector Iteration

The basic vector iteration to iteratively solve a standard EV problem (8) is

$$\boldsymbol{y}_k = \boldsymbol{A}\boldsymbol{x}_k \quad \text{and} \quad \boldsymbol{x}_{k+1} = \frac{\boldsymbol{y}_k}{\|\boldsymbol{y}_k\|}$$

where $\|\boldsymbol{y}_k\|$ converges towards the largest eigenvalue and $\boldsymbol{x}_k$ towards the corresponding eigenvector.

- only yields largest (in magnitude) eigenvalue
- converges baldy if $\lambda_n/\lambda_{n-1} \approx 1$, i.e. the second largest EV is almost the same size as the largest

Vector iteration is also known as *power iteration*, *power method*, or *von Mises iteration*.

# Inverse Vector Iteration

$\lambda$ is an eigenvalue of $\boldsymbol{A}$ $\quad\Leftrightarrow\quad$ $\mu = 1/\lambda$ is an eigenvalue of $\boldsymbol{A}^{-1}$

Do vector iteration with $\boldsymbol{A}^{-1}$ to obtain the eigenvalue $\mu$ with largest magnitude, which gives by $\lambda = 1/\mu$ the eigenvalue with smallest magnitude of $\boldsymbol{A}$. The corresponding eigenvector is equal.

The iteration rule is, thus

$$\boldsymbol{y}_k = \boldsymbol{A}^{-1}\boldsymbol{x}_k \quad \text{and} \quad \boldsymbol{x}_{k+1} = \frac{\boldsymbol{y}_k}{\|\boldsymbol{y}_k\|},$$

where $\|\boldsymbol{y}_k\|$ converges towards $\mu$.

In each iteration we solve the same linear system $\boldsymbol{A}\boldsymbol{y}_k = \boldsymbol{x}_k$ with different right hand side, which can be done efficiently by pre-computing the LU factorisation of the system matrix $\boldsymbol{A}$.

Has the same limitations as vector iteration.

# Shifted (inverse) Vector Iteration

- $\lambda$ is an eigenvalue of $\boldsymbol{A}$
- $\lambda - \sigma$ is an eigenvalue of $\boldsymbol{A} - \sigma \boldsymbol{I}$
- $\mu = \frac{1}{\lambda - \sigma}$ is an eigenvalue of the inverse $(\boldsymbol{A} - \sigma \boldsymbol{I})^{-1} = \boldsymbol{B}$
- a suitable shift point $\sigma \approx \lambda_i$ makes $\mu_i$ the largest EV of $\boldsymbol{B}$

One can do vector iteration with $\boldsymbol{B}$ to compute $\mu$ and the corresponding eigenvector, which is also an eigenvector of $\boldsymbol{A}$.

If the shift point exactly equals an eigenvalue of $\boldsymbol{A}$, i.e. $\sigma = \lambda_i$ then $(\boldsymbol{A} - \sigma \boldsymbol{I})$ is singular (= not invertible) by definition $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{x} = 0$. Thus, inverse vector iteration will not converge.

Choose shift points *close to* eigenvalues, i.e. $\sigma = \lambda_i + \delta$ with $\delta > 0$.

# Rayleigh Quotient Iteration

Inverse vector iteration with changing shift point $\sigma$, i.e.

$$\boldsymbol{y}_k = \boldsymbol{B}_k \boldsymbol{x}_k \quad \text{and} \quad \boldsymbol{x}_{k+1} = \frac{\boldsymbol{y}_k}{\|\boldsymbol{y}_k\|}$$

with shift point equal to an estimate of the Rayleigh quotient of $\boldsymbol{A}$

$$\sigma_k = \frac{\boldsymbol{x}_k^* \boldsymbol{A} \boldsymbol{x}_k}{\boldsymbol{x}_k^* \boldsymbol{x}_k} \quad \text{thus} \quad \boldsymbol{B}_k = (\boldsymbol{A} - \sigma_k \boldsymbol{I})^{-1}$$

- $\boldsymbol{x}_k$ converging towards eigenvectors
- $\|\boldsymbol{y}_k\|$ converging towards eigenvectors $\mu_k$ of $\boldsymbol{B}_k$
- desired eigenvalues $\lambda = \sigma_k + \frac{1}{\mu_k}$

A linear system has to be solved in each iteration, but convergence is much faster.

# Higher Eigenvalues

- vector iteration only yields ends of the spectrum
- shift could be used to get arbitrary EVs, but we don't know where to shift

- once some eigenvectors are known we can use vector iteration in a direction orthogonal to these eigenvectors
- converges towards next eigenvalue
- needs re-orthogonalisation in each iteration (due to numerical errors)

# Gram-Schmid Process

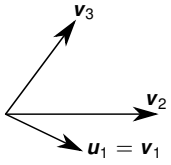
Define a projection of $\boldsymbol{v}$ onto $\boldsymbol{u}$ as

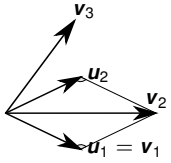
$$p(\boldsymbol{v}, \boldsymbol{u}) = \frac{\boldsymbol{u}^* \boldsymbol{v}}{\boldsymbol{u}^* \boldsymbol{u}} \boldsymbol{u} \tag{10}$$

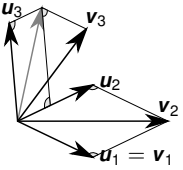
Assume a set of basis vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ then an equivalent orthogonal basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ is constructed by

$$\boldsymbol{u}_1 = \boldsymbol{v}_1 \tag{11}$$

$$\boldsymbol{u}_k = \boldsymbol{v}_k - \sum_{j=1}^{k-1} p(\boldsymbol{v}_k, \boldsymbol{u}_j) \tag{12}$$




- $[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k]$ is an orthogonal matrix
- an ortho*normalized* basis $\boldsymbol{Q} = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k]$ is obtained from $\boldsymbol{e}_k = \boldsymbol{u}_k / \|\boldsymbol{u}_k\|$

# QR-decomposition

Any $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ may be decomposed such that

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$$

where $\boldsymbol{Q}$ is unitary and $\boldsymbol{R}$ is upper triangular.

1. regard the columns of $\boldsymbol{A}$ as the original basis, i.e. $\boldsymbol{A} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$
2. Compute an orthonormal basis $\boldsymbol{Q}$, e.g. via the Gram-Schmid process[a]
3. Express the $\boldsymbol{v}_k$ via the new basis, i.e. $\boldsymbol{a}_i = \sum_{k=1}^{n} \boldsymbol{e}_k^* \boldsymbol{a}_i \boldsymbol{e}_k$, which can be written as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \ldots & \boldsymbol{e}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1^* \boldsymbol{a}_1 & \cdots & \boldsymbol{e}_1^* \boldsymbol{a}_n \\ & \ddots & \vdots \\ 0 & & \boldsymbol{e}_n^* \boldsymbol{a}_n \end{bmatrix} = \boldsymbol{Q}\boldsymbol{R}$$

where $\boldsymbol{R}$ is upper triangular due to the orthogonality used in the creation of $\boldsymbol{Q}$.

[a] Alternative methods are: Gives rotations or Housholder reflections.

# QR-algorithm

The QR-algorithm uses the QR-decomposition

$$\boldsymbol{A}_k = \boldsymbol{Q}_k \boldsymbol{R}_k \tag{13}$$

where $\boldsymbol{Q}_k$ is unitary and $\boldsymbol{R}_k$ is upper triangular in the iteration rule

$$\boldsymbol{A}_{k+1} = \boldsymbol{R}_k \boldsymbol{Q}_k = \boldsymbol{Q}_k^* \boldsymbol{A}_k \boldsymbol{Q}_k = \boldsymbol{Q}_k^* \cdots \boldsymbol{Q}_0^* \boldsymbol{A}_0 \boldsymbol{Q}_0 \cdots \boldsymbol{Q}_k. \tag{14}$$

Starting from $\boldsymbol{A}_0 = \boldsymbol{A}$ it (usually) converges to $\boldsymbol{A}_k \to \boldsymbol{R}$ in upper triangular form, i.e. computes the Schur decomposition $\boldsymbol{Q}\boldsymbol{R}\boldsymbol{Q}^* = \boldsymbol{A}$ of the matrix $\boldsymbol{A}$.

- $\boldsymbol{Q} = \prod_k \boldsymbol{Q}_k$ is unitary since all $\boldsymbol{Q}_k$ are unitary, therefore
- $\boldsymbol{A}$ and $\boldsymbol{R}$ are similar and have the same eigenvalues.
- As $\boldsymbol{R}$ is upper triangular, eigenvalues can be read from the diagonal.
- Eigenvectors can be computed by inverse vector iteration with shift.

The QR-algorithm computes all eigenvalues of a matrix simultaneously.

# Subspace Iteration

- We only want a subset $p < n$ of eigenvalues of $\boldsymbol{A} \in \mathbb{C}^{n \times n}$
- the $\lambda_p$ should be computed simultaneously

Compute an orthogonal basis $\boldsymbol{X} \in \mathbb{C}^{n \times p}$, i.e. $\boldsymbol{X}^* \boldsymbol{X} = \boldsymbol{I}$, and use it in the iteration

$$\boldsymbol{Z}_{k+1} = \boldsymbol{A}\boldsymbol{X}_k \quad \text{where} \quad \boldsymbol{X}_k \boldsymbol{R}_k = \boldsymbol{Z}_k$$

is the QR-decomposition of $\boldsymbol{Z}_k$. The largest $p$ eigenvalues appear in the diagonal of $\boldsymbol{R}_k$.

$\boldsymbol{A}\boldsymbol{X}_k = \boldsymbol{X}_k \boldsymbol{R}_k$ is an approximation of a *partial* Schur decomposition.

# Convergence Criteria

All iterative methods need a convergence criterion for termination. Usually one can use the relative norm of the update, i.e.

$$\frac{\|\lambda_{k+1} - \lambda_k\|}{\|\lambda_k\|} < \epsilon$$

If the corresponding eigenvector $\boldsymbol{x}_i$ is also calculated, one can take the norm of the residual of the eigenvalue equation, e.g.

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{x}_i = \boldsymbol{r}, \qquad \|\boldsymbol{r}\| < \epsilon$$

**Example for Python function**

```
def VectorIteration(A,x0,tolerance=1e-6):
  epsilon = 1.0
  while epsilon > tolerance:
    # compute recursion & update epsilon
  return eigenvalue, eigenvector
```

# Exercises

- Templates are available in TUWEL
- Submission of selected tasks in TUWEL
- Solutions to all tasks should be presented and discussed during workshop
- Distribute the work within your team

# Exercise Templates: getting started...

Template files are *Jupyter Notebook*s)

1. Setup your Python distribution, e.g. anaconda: `https://www.anaconda.com/download` (use Python 3)
2. Download the templates from TUWEL
3. Open *Jupyter* and load the notebook file

# Eigenvalue Solvers

Use a small $4 \times 4$ matrix with known eigenvalues to test different eigenvalue solvers.

## Tasks

Try out the different methods

1. vector iteration
2. inverse vector iteration
3. Rayleigh quotient iteration
4. higher eigenvalues using inverse vector iteration in orthogonal direction
5. QR-algorithm
6. subspace-iteration
7. numpy/matlab

## Tipps

- define functions for the different methods
- check if your implementation works for negative eigenvalues too

# Eigenvalues and Mode Shapes

Use the provided mass and stiffness matrix of an FE-model of a rectangular plate. A static problem is solved in the template example to illustrate the problem.

## Tasks

1. Visualize the structure of the system matrices (e.g. as image)
2. Visualize the geometry (see template)
3. Compute a different static problem
4. Compute and visualise natural frequencies and oscillation modes for different boundary conditions: free, east edge ($x = x_{\text{min}}$) clamped , all edges clamped

# Submission

Each template notebook contains a set of questions, which should be answered during the teamwork-phase. You must submit

- PDF file of the notebook answering the questions, and
- all files necessary to reproduce the results in the notebook.

## Submission for Workshop 1

1. Function for vector iteration
2. Compute and plot mode shapes for rectangular plate (free-free)

Submissions are due 2 days before the workshop!

You must submit the PDF file separately (not within an archive) to allow comments on your solution via TUWEL.

# The First Team Meeting

1. introduce yourself to your team members
2. appoint chairperson and secretary
3. choose a team name, e.g. the modal hammers
4. agree on the agenda
5. work: discuss the lecture & prepare for the workshop
6. write & post the minutes of the meeting

If everyone agrees you can have more than one meeting per learning cycle.

# Tasks for Workshop

With your team

- do the suggested exercises
- summarise your findings in a short presentation (max. 15min)
- prepare for the workshop

The presentation should contain

- your main difficulty
- your main insight
- notes on the tested eigenvalue solvers
- comparison of the plate mode shapes

# Distribution of Work

- Work should be shared equally between team members
- Use your individual skills and preferences to assign tasks
- Clearly decide who does what!
- Explain what you did and discuss the results together

Every team member must be able to account for work done by others to a reasonable extent!

Example: *Susan programmed a function for vector iteration, Frank can explain what it does and how to use it.*

# Tipps for Efficient Team Learning

- Establish a common means of communication (TUWEL forum, mailing list, WhatsUp group, . . . )
- Share data with your team (file server, github repo, google docs, dropbox, . . . )
- Distribute the work between team members
- Meet twice within a learning circle
  1. discuss the lecture and distribute exercise tasks
  2. discuss your work with the team and prepare for the workshop
- prepare for team meetings

Read the Course Manual!

# Dates

all events at Wednesday, 09:00–11:00 in BA 05

04/03/2019 Introduction & course information, overview lecture 1
11/03/2019 First team meeting (attendance for team distribution)
18/03/2019 Team learning
25/03/2019 Workshop 1 (attendance), overview lecture 2
01/04/2019 Team learning
22/04/2019 Workshop 2 (attendance), overview lecture 3
29/04/2019 Team learning
06/05/2019 Workshop 3 (attendance), overview lecture 4
13/05/2019 Team learning
20/05/2019 Team learning
27/05/2019 Workshop 4 (attendance), overview lecture 5
03/06/2019 Team learning
10/06/2019 Team learning
17/06/2019 Workshop 5
24/06/2019 Final test

1. Sign up for the course in TUWEL
2. Form teams of 3-4 students: Teams will be fixed at first team meeting
3. Read the course manual

# Definitions I

Two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ are orthogonal if their inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$. For the common vector space $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$ the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is defined as $\boldsymbol{u}^T \boldsymbol{v}$ and $\boldsymbol{u}^* \boldsymbol{v}$, respectively.

The adjoint matrix $\boldsymbol{A}^*$ of $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ has elements related via $A^*_{ij} = \bar{A}_{ji}$, where the overbar denotes the complex conjugate. It is also termed *conjugate transpose* or *Hermitian transpose*.

A matrix $\boldsymbol{Q} \in \mathbb{C}^{n \times n}$ is unitary if

$$\boldsymbol{Q}^* \boldsymbol{Q} = \boldsymbol{Q} \boldsymbol{Q}^* = \boldsymbol{I},$$

i.e. its adjoint is also its transpose.

# Definitions II

A matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is similar to a matrix $\boldsymbol{C} \in \mathbb{C}^{n \times n}$, $\boldsymbol{A} \sim \boldsymbol{C}$, if and only if there is a non-singular matrix $\boldsymbol{S}$ such that

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{C}.$$

Similar matrices have *equal eigenvalues* with equal multiplicities.

If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ then there is a unitary matrix $\boldsymbol{Q} \in \mathbb{C}^{n \times n}$ such that

$$\boldsymbol{Q}^{*}\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{R},$$

is upper triangular, i.e. $R_{ij} = 0 \, \forall \, i > j$. The matrix $\boldsymbol{R}$ is called the Schur form of $\boldsymbol{A}$. The diagonal entries of $\boldsymbol{R}$ are the eigenvalues of $\boldsymbol{A}$.