Modal Analysis

VU 325.100

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Time and Frequency Domain

Single degree of freedom oscillator

The equation

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

is usually written as

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = f(t) \tag{1}$$

with the natural frequency ω and the damping ratio ζ.

Time Domain

We have an *initial value problem* to determine $\boldsymbol{x}(t)$ in

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f} \tag{2}$$

with known initial conditions and forcing

$$\dot{\boldsymbol{x}}(t_0) = \boldsymbol{v}_0 \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \qquad \boldsymbol{f} = \boldsymbol{t}(t)$$

which can be solved by forward integration in time.

Numeric Time Integration

The second order ODE system can be recast into a first order system and solved by

- Forward Euler (explicit, stability!)
- Backward Euler
- Multi-step methods (Runge-Kutter, . . .)

For structural dynamics problems one often uses specialized methods for second order ODEs

- Newmark family
- Wilson-θ
- Hilber, Hughes, Taylor (HHT) method

Most algorithms require a start value for the acceleration $\boldsymbol{a}_0$ too.

Harmonic Forcing

We assume harmonic forcing of the form

$$\boldsymbol{f}(t) = \Re\left\{\hat{\boldsymbol{f}} e^{j\omega t}\right\} \tag{3}$$

and a damped system with $\boldsymbol{C} > 0$ to obtain the *steady state* solution

$$\left(\boldsymbol{K} + j\omega\boldsymbol{C} - \omega^2\boldsymbol{M}\right)\hat{\boldsymbol{x}} = \hat{\boldsymbol{f}} \quad \text{or} \quad \boldsymbol{Z}(\omega)\hat{\boldsymbol{x}} = \hat{\boldsymbol{f}} \tag{4}$$

where we have introduced the *dynamic stiffness* matrix

$$\boldsymbol{Z}(\omega) = \boldsymbol{K} + j\omega\boldsymbol{C} - \omega^2\boldsymbol{M} \tag{5}$$

One matrix decomposition per frequency value necessary.

Transfer Function

When we consider the force at DoF j as the only input and the displacement at DoF i as an output we can write the transfer function

$$H_{ij}(\omega) = \frac{x_i(\omega)}{f_j(\omega)} \tag{6}$$

For the full system in matrix form we get

$$\begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & \dots & H_{1n}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & \dots & H_{2n}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1}(\omega) & H_{n2}(\omega) & \dots & H_{nn}(\omega) \end{bmatrix} \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_n \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix} \quad \text{or} \quad \boldsymbol{H}(\omega)\hat{\boldsymbol{f}} = \hat{\boldsymbol{x}} \tag{7}$$

with the *receptance* matrix $\boldsymbol{H}$.

Receptance and Dynamic Stiffness

- the dynamic stiffness matrix is symmetric (because it is composed of symmetric parts)
- dynamic stiffness and receptance are an inverse pair, i.e.

$$\boldsymbol{Z}^{-1} = \boldsymbol{H}$$

- as $\boldsymbol{Z}$ is sparse $\boldsymbol{H}$ is dense
- both $\boldsymbol{Z}$ and $\boldsymbol{H}$ are symmetric
- *reciprocity*: a single force input at i leads to a displacement output at j, which is the same as if the force was applied to j and the displacement measured at i

Mobility and Accelerance

The mobility $\boldsymbol{Y}(\omega)$ and the accelerance $\boldsymbol{A}(\omega)$ can be derived from the receptance $\boldsymbol{H}(\omega)$ by differentiation in the frequency domain

$$\boldsymbol{Y}(\omega) = j\omega\boldsymbol{H}(\omega)$$
$$\boldsymbol{A}(\omega) = -\omega^2\boldsymbol{H}(\omega)$$

Oscillation Modes

The solutions of the generalised eigenvalue problem

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) \boldsymbol{v} = \boldsymbol{0} \tag{8}$$

are the natural frequencies ω_i and corresponding mode shapes $\boldsymbol{v}_i$.

Orthogonality of the Modes

We write Eq. (8) for mode i and pre-multiply with the transpose of mode $\boldsymbol{v}_j$

$$\boldsymbol{v}_j^T \left(\boldsymbol{K} - \omega_i^2 \boldsymbol{M} \right) \boldsymbol{v}_i = 0 \tag{9}$$

Similarly, we write Eq. (8) for mode j, transpose ($\boldsymbol{K}$ and $\boldsymbol{M}$ are symmetric), and post multiply with mode i

$$\boldsymbol{v}_j^T \left(\boldsymbol{K} - \omega_j^2 \boldsymbol{M} \right) \boldsymbol{v}_i = 0 \tag{10}$$

Subtracting Eq. (9) from Eq. (10) we get

$$\left(\omega_i^2 - \omega_j^2 \right) \boldsymbol{v}_j^T \boldsymbol{M} \boldsymbol{v}_i = 0 \tag{11}$$

Mass- and Stiffness Orthogonality

For distinct eigenvalues $\omega_i \neq \omega_j$ we get

$$\boldsymbol{v}_j^T \boldsymbol{M} \boldsymbol{v}_i = 0 \tag{12}$$

For distinct eigenvalues $\omega_i \neq \omega_j$, the corresponding modes $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are orthogonal with respect to $\boldsymbol{M}$.

Substituting Eq. (12) into Eq. (9) we get

$$\boldsymbol{v}_j^T \boldsymbol{K} \boldsymbol{v}_i = 0 \tag{13}$$

For distinct eigenvalues $\omega_i \neq \omega_j$, the corresponding modes $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are orthogonal with respect to $\boldsymbol{K}$.

Orthogonality for repeated Eigenvalues

We assume modes i and j have the same eigenvalue $\omega_i = \omega_j = \omega_0$, thus

$$\left(\boldsymbol{K} - \omega_0^2 \boldsymbol{M}\right) \boldsymbol{v}_i = 0$$
$$\left(\boldsymbol{K} - \omega_0^2 \boldsymbol{M}\right) \boldsymbol{v}_j = 0$$

Multiplying mode $\boldsymbol{v}_j$ by a constant c and summing the equations

$$\boldsymbol{K}\left(\boldsymbol{v}_i + c\boldsymbol{v}_j\right) - \omega_0^2 \boldsymbol{M}\left(\boldsymbol{v}_i + c\boldsymbol{v}_j\right) = 0 \tag{14}$$

Any linear combination of modes with a common eigenvalue ω_0 is also and eigenvector

$$\left(\boldsymbol{K} - \omega_0^2 \boldsymbol{M}\right)\left(\boldsymbol{v}_i + c\boldsymbol{v}_j\right) = 0$$

Diagonalizing the System

We collect the mode shape vectors $\boldsymbol{v}_i$ in the columns of the modal matrix

$$\boldsymbol{V} = \left[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\right] \tag{15}$$

Due to the orthogonality of the mode shapes we see:

The mode shapes diagonalize the system matrices

$$\boldsymbol{V}^T\boldsymbol{M}\boldsymbol{V} = \begin{bmatrix} m_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & m_n \end{bmatrix} \qquad \boldsymbol{V}^T\boldsymbol{K}\boldsymbol{V} = \begin{bmatrix} k_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & k_n \end{bmatrix}$$

and thus decouple the equations of motion.

The m_i are called the generalized masses, and the k_i are called generalised stiffnesses of the system and we have $\omega_i = \sqrt{k_i/m_i}$.

Mode Normalization

The scaling of a mode is arbitrary, different methods are common

- Norm of 1
- maximum nodal displacement is 1
- Mass-normalized, i.e. $\boldsymbol{v}_i \boldsymbol{M} \boldsymbol{v}_i = 1$

For mass-normalized modes $\boldsymbol{v}_i$ we have

$$\boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} = \boldsymbol{I} \qquad \boldsymbol{V}^T \boldsymbol{K} \boldsymbol{V} = \begin{bmatrix} \omega_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_n^2 \end{bmatrix}$$

Most eigenvalue solvers return mass-normalized modes.

Receptance and Modal Parameters I

Taking the expression for the receptance of an un-damped system

$$\boldsymbol{H}(\omega)^{-1} = \left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) \tag{16}$$

and post- and pre-multiplying with the modal matrix and its transpose, respectively

$$\boldsymbol{V}^T \boldsymbol{H}(\omega)^{-1} \boldsymbol{V} = \begin{bmatrix} \omega_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_n^2 \end{bmatrix} - \omega^2 \boldsymbol{I} \tag{17}$$

Remembering that $\boldsymbol{V}$ is orthogonal, inversion gives a simple expression for the receptance matrix.

Invert Eq. (17) and solve for $\boldsymbol{H}$ (bring modal matrices to the rhs) ...

Receptance and Modal Parameters II

The receptance is directly related to the modal parameters of the system by

$$\boldsymbol{H}(\omega) = \boldsymbol{V} \begin{bmatrix} \ddots & & 0 \\ & \frac{1}{\omega_i^2 - \omega^2} & \\ 0 & & \ddots \end{bmatrix} \boldsymbol{V}^T \tag{18}$$

or for a single transfer function

$$H_{ij}(\omega) = \frac{V_{i1} V_{j1}}{\omega_1^2 - \omega^2} + \frac{V_{i2} V_{j2}}{\omega_2^2 - \omega^2} + \cdots + \frac{V_{in} V_{jn}}{\omega_n^2 - \omega^2} \tag{19}$$

Which term will dominate the response when the system is excited close to a natural frequency?

Types of Damping Models

Damping model can be classified in

viscous damping $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}$

proportional damping $\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K}$

structural damping $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}_r\boldsymbol{x} + j\boldsymbol{K}_i\boldsymbol{x} = \boldsymbol{f}$

Proportional damping is a useful modelling assumption that can simplify the analysis.

Many real damping mechanisms are non-proportional, e.g. visco-elasticity, wave radiation, ...

Rayleigh Damping I

The damping matrix is assumed proportional to mass and stiffness matrix

$$\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K} \tag{20}$$

Pre and post-multiplication with the (mass normalized) modal matrices gives

$$\boldsymbol{V}^T \boldsymbol{C} \boldsymbol{V} = \alpha \boldsymbol{I} + \beta \begin{bmatrix} \omega_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_n^2 \end{bmatrix} = \begin{bmatrix} \ddots & & 0 \\ & \alpha + \beta \omega_i^2 & \\ 0 & & \ddots \end{bmatrix}$$

which is again diagonal.

Rayleigh Damping II

Looking at the similarity to the single DoF oscillator,Eq. (1), we obtain the damping ratio of each mode as

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \tag{21}$$

which shows the frequency-characteristic of Rayleigh damping.

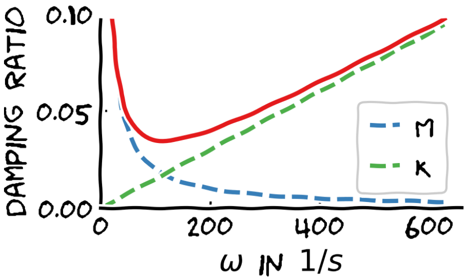


Figure: The mass-proportional term α controls the low-frequency behaviour and the stiffness-proportional term β the high-frequency behaviour.

Modal Damping

One simply considers the un-damped system and adds arbitrary modal damping ratios ζ_i in the diagonalized system.

Non-Proportional Viscous Damping

To decouple the system on needs to solve the quadratic eigenvalue problem

$$[\boldsymbol{K} + \lambda \boldsymbol{C} + \lambda^2 \boldsymbol{M}]\boldsymbol{x} = \boldsymbol{0} \tag{22}$$

Non-Proportional Structural Damping

For structural damping the stiffness matrix is complex valued. i.e. $\hat{\boldsymbol{K}} = \boldsymbol{K}_r + j\boldsymbol{K}_i$.
One has to solve a complex valued, generalised EV problem

$$[\hat{\boldsymbol{K}} + \lambda^2 \boldsymbol{M}]\boldsymbol{x} = \boldsymbol{0} \tag{23}$$

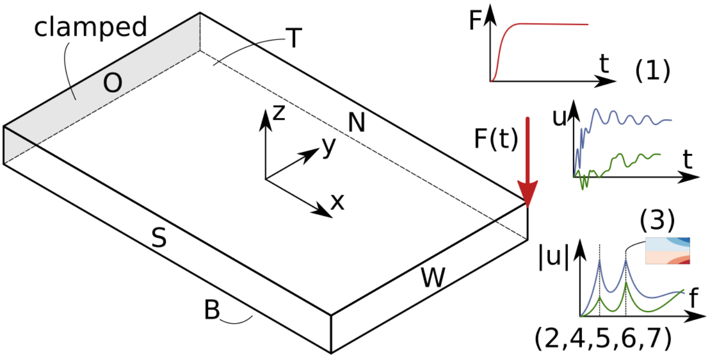
In general the eigenvectors become complex valued.

Exercises

- Templates are available in TUWEL
- Solutions should be presented during workshop
- Distribute the work within your team

Exercise 2

- Use the system matrices of the plate from last exercise
- Consider the case where it is clamped at its short edge
- Play around in time and frequency domain



Exercise 2: Tasks

1. Compute the transient response of the system for a vertical force on the corner of the plate (choosing a forcing function which will show interesting dynamics)
2. Compute the steady state response for a vertical force at the corner in the frequency range of 2-40Hz, and plot the receptances, i.e. the transfer functions for the vertical excitation at the corner with respect to the displacement of the corner and the center of the plate
3. Visualize the response at characteristic frequencies
4. Plot the average vertical response of all points of the plate surface (one layer, e.g. top or bottom, is sufficient)
5. Estimate the recentance using the time domain data
6. Compare the receptance computed by the inversion of the dynamic stiffness matrix with the one computed from the model parameters (using the first few modes)

Exercise 2: Tipps

- use a smooth step or smooth impulse for transient loading, e.g. $f(t) = 1 - e^{(t/T_s)^k}$ or $f(t) = e^{((t-t_0)/T_s)^k}$
- observe the frequency-content of the exitation signal
- compute the first few natural frequencies to get an idea of the system dynamics
- use spares matrices for the computations
- think about necessary time step size ($>$ 10 per period) and simulation duration (frequency resolution for receptance estimation)
- it might be useful to assume some slight damping ($\zeta < 0.05$)
- animate your computation results
- use the provided function for time integration

Dates

all events at Wednesday, 09:00–11:00 in BA 05

13/03/2019 overview lecture 1
20/03/2019 team meeting
27/03/2019 team meeting
03/04/2019 workshop 1 & overview lecture 2
10/04/2019 team learning
08/05/2019 workshop 2 & overview lecture 3
15/05/2019 team learning
22/05/2019 workshop 3

01/06/2019 Paper draft deadline
15/06/2019 Paper review submission deadline
01/07/2019 Paper submission deadline

1. Go through the theory and complete the exercises
2. Prepare for the workshop

Linear Algebra Rules I

inverse of product

For two square, invertible matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{C}^{n \times n}$ we have

$$(\boldsymbol{AB})^{-1} = \boldsymbol{A}^{-1}\boldsymbol{B}^{-1}.$$

inverse of orthogonal martix

An orthogonal matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ is defined as

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I},$$

thus, its inverse is equal to its transpose $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$.