Modal Analysis

VU 325.100

Johann WASSERMANN   Florian TOTH

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Modal Subspace & Model Reduction

Oscillation Modes

For a system of equations of motion

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f} \tag{1}$$

we can formulate and solve an eigenvalue problem to obtain eigenvalues λ_i and corresponding mode shapes $\boldsymbol{v}_i$.

A system with n degrees of freedom has n eigenvalues and modes.

Modal Coordinates

We introduce modal coordinates q_i to describe the original DoF vector $\boldsymbol{x}$ as a weighted sum of mode shapes $\boldsymbol{v}_i$ by

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{v}_1 q_1 + \boldsymbol{v}_2 q_2 + \cdots + \boldsymbol{v}_n q_n = \sum_{i=1}^{n} \boldsymbol{v}_i q_i \\ &= \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \boldsymbol{V}\boldsymbol{q} \end{aligned} \tag{2}$$

We introduce the modes as a new basis, the *modal basis*.

Transformation to Modal Coordinates

We use $\boldsymbol{x} = \boldsymbol{V}\boldsymbol{q}$ and pre multiply the equation of motion (1) with $\boldsymbol{V}^T$ to obtain

$$\underbrace{\boldsymbol{V}^T\boldsymbol{M}\boldsymbol{V}}_{\underline{\boldsymbol{M}}}\ddot{\boldsymbol{q}} + \underbrace{\boldsymbol{V}^T\boldsymbol{C}\boldsymbol{V}}_{\underline{\boldsymbol{C}}}\dot{\boldsymbol{q}} + \underbrace{\boldsymbol{V}^T\boldsymbol{K}\boldsymbol{V}}_{\underline{\boldsymbol{K}}}\boldsymbol{q} = \underbrace{\boldsymbol{V}^T\boldsymbol{f}}_{\underline{\boldsymbol{f}}} \tag{3}$$

where the modal system matrices $\underline{\boldsymbol{M}}$, $\underline{\boldsymbol{C}}$, and $\underline{\boldsymbol{K}}$ are diagonal.

For mass-normalised modes we obtain n un-coupled single DoF oscillators of the form

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = \underline{f}_i \tag{4}$$

The n physical DoFs $\boldsymbol{x}$ and the n modal DoFs $\boldsymbol{q}$ are equivalent.

$\underline{\boldsymbol{C}}$ might not be diagonal if un-damped modes are used.

Model Order Reduction

Use a sub-set of $m < n$ modes to approximate the physical DoFs

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \approx \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ \vdots \\ v_{n1} \end{bmatrix} \eta_1 + \cdots + \begin{bmatrix} v_{1m} \\ v_{2m} \\ v_{3m} \\ \vdots \\ v_{nm} \end{bmatrix} \eta_m = \begin{bmatrix} v_{11} & \cdots & v_{1m} \\ v_{21} & \cdots & v_{2m} \\ v_{21} & \cdots & v_{3m} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nm} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}$$

$$\boldsymbol{x} \approx \boldsymbol{v}_1 \eta_1 + \cdots + \boldsymbol{v}_m \eta_m = \sum_{i=1}^{m} \boldsymbol{v}_i \eta_i = \boldsymbol{V} \boldsymbol{\eta} \tag{5}$$

Usually only a few modes $m \ll n$ are required to achieve a good approximation of the global system dynamics.

Reduced System

1. Define a set of m modes to use: $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m]$
2. Compute the reduced system matrices and forcing vector

$$\underbrace{\underline{\boldsymbol{M}}}_{m\times m} = \underbrace{\boldsymbol{V}^T}_{m\times n} \underbrace{\boldsymbol{M}}_{n\times n} \underbrace{\boldsymbol{V}}_{n\times m} \qquad \underline{\boldsymbol{C}} = \boldsymbol{V}^T\boldsymbol{C}\boldsymbol{V} \qquad \underline{\boldsymbol{K}} = \boldsymbol{V}^T\boldsymbol{K}\boldsymbol{V} \tag{6}$$

$$\underline{\boldsymbol{f}} = \boldsymbol{V}^T\boldsymbol{f} \tag{7}$$

3. Solve the reduced system for the modal coordinates $\boldsymbol{\eta}$

$$\underline{\boldsymbol{M}}\ddot{\eta} + \underline{\boldsymbol{C}}\dot{\eta} + \underline{\boldsymbol{K}}\eta = \underline{\boldsymbol{f}} \qquad \left(\underline{\boldsymbol{K}} + j\omega\underline{\boldsymbol{C}} - \omega^2\underline{\boldsymbol{M}}\right)\hat{\eta} = \hat{\underline{\boldsymbol{f}}} \tag{8}$$

4. Use the coordinate transformation to obtain the full solution

$$\boldsymbol{x} = \boldsymbol{V}\eta \tag{9}$$

When to use Sub-Space Formulations?

Sub-space formulations can be used to efficiently

- model global system dynamics
- describe system close to one frequency
- define constraints

Reduced systems might not be appropriate to

- capture local phenomena, e.g. stress concentrations at holes
- describe non-linear phenomena, e.g. large displacements, non-linear material, ...

Choice of the Sub-Space

- Normal modes are a very good basis for global system dynamics
- Consider the frequency range of interest
- Determine if modes are excited or not
- Importance of damping

Modal Contribution Factor

Describes the relative contribution of each mode to the total solution, e.g. by defining

$$\boldsymbol{\rho} = \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|} \tag{10}$$

the sum of all contributions equals 1, i.e. $\sum \rho_i = 1$.

For a meaningful interpretation of the modal contributions the modes must be appropriately normalized, e.g. to unit displacement for interpretation in terms of displacement contribution.

If you scale a mode by a factor s_i the corresponding modal coordinate η_i must be scaled by $1/s_i$ to obtain the modal coordinate q_i of the scaled basis:
$s_i^2(m_i\ddot{q}_i + c_i\dot{q}_i + k_i q_i) = s_i f_i$

Modal (Mass) Participation Factor

The participation factor for a mode $\boldsymbol{v}_i$ in direction for the rigid body displacement or rotation $\boldsymbol{e}_j$ is defined by

$$\Gamma_{ij} = \frac{\boldsymbol{v}_i^T \boldsymbol{M} \boldsymbol{e}_j}{\boldsymbol{v}_i^T \boldsymbol{M} \boldsymbol{v}_i} \tag{11}$$

It describes the content of rigid body motion in a particular mode.

The *modal (mass) participation factor* originates from the modelling of inertia loads, e.g. for assessing earthquake forcing which are proportional to Γ_{ij}.

Arbitrary (Modal) Basis

If one wants to approximate a deformation $\boldsymbol{x}$ by a set of arbitrary modes $[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m] = \boldsymbol{V}$ one can solve the over-determined system

$$\boldsymbol{V}\boldsymbol{\eta} = \boldsymbol{x} \tag{12}$$

in the least-squares sense, i.e. minimizing the L2-norm of the residual vector $\boldsymbol{r} = \boldsymbol{V}\boldsymbol{\eta} - \boldsymbol{x}$.

Can be used to assess *how well* a selected modal basis $\boldsymbol{V}$ can approximate the solution $\boldsymbol{x}$.

The size of the residual indicates how good the modal approximation is. The shape of the residual might indicate what additional shape is necessary to perfectly approximate the give deformation.

Restricting the Subspace

- When using a modal subsapce the motion of the system is only possible in the modal coordinates
- Adding modes adds degrees of freedom, removing modes removes degrees of freedom
- The not-used modes can be seen as constraints for the system

Linear Constraint Equations

Consider a linear constraint equation of the form

$$b_1 x_1 + \cdots + b_n x_n = \boldsymbol{b}^T \boldsymbol{x} = 0 \tag{13}$$

For k linear constraint equations we can write

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0} \tag{14}$$

where the rows of the $k \times n$ constraint matrix are the constraint vectors $\boldsymbol{b}_k^T$.

For setting $x_1 = 0$ and $x_2 = x_3$ we would use

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \end{bmatrix}$$

Enforcing the Constraints

We now introduce a new set of $n - k$ coordinates $\boldsymbol{y}$ according to

$$\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{y} \tag{15}$$

In order for all constraint to be fulfilled in the new coordinates we must have

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{Q}\boldsymbol{y} = \boldsymbol{0} \quad \forall\, \boldsymbol{y} \tag{16}$$

meaning the $n - k$ columns of the $n \times (n - k)$ matrix $\boldsymbol{Q}$ span the null-space of $\boldsymbol{B}$.

The null-space can be computed e.g. using a singular value decomposition (SVD), see template for exercise 3.

Solving the Constraint System

We now solve the system in terms of the constrained degrees of freedom $\boldsymbol{y}$ and project into the subspace (i.e. pre-multiply by $\boldsymbol{Q}^T$) obtaining

$$\boldsymbol{Q}^T\boldsymbol{M}\boldsymbol{Q}\ddot{\boldsymbol{y}} + \boldsymbol{Q}^T\boldsymbol{C}\boldsymbol{Q}\dot{\boldsymbol{y}} + \boldsymbol{Q}^T\boldsymbol{K}\boldsymbol{Q}\boldsymbol{y} = \boldsymbol{Q}^T\boldsymbol{f} \tag{17}$$

The solution for the full DoF vector is recovered from (15).

Notice the the similarity to the modal equations of motion, Eq. (4).

Multiple Projections

Suppose one wants to solve in the constraint subspace defined by

$$\boldsymbol{x} = \boldsymbol{P}_1 \boldsymbol{y}$$

but only consider the first m modes there, i.e. solve for modal coordinates defined by

$$\boldsymbol{y} = \boldsymbol{P}_2 \eta$$

the system matrices take the form

$$\begin{aligned}\underline{\boldsymbol{K}} = \boldsymbol{P}_2^T(\boldsymbol{P}_1^T \boldsymbol{K} \boldsymbol{P}_1)\boldsymbol{P}_2 = (\boldsymbol{P}_2^T \boldsymbol{P}_1^T)\boldsymbol{K}(\boldsymbol{P}_1 \boldsymbol{P}_2) = \\ = (\boldsymbol{P}_1 \boldsymbol{P}_2)^T \boldsymbol{K}(\boldsymbol{P}_1 \boldsymbol{P}_2) = \boldsymbol{P}^T \boldsymbol{K} \boldsymbol{P}\end{aligned}$$

and we have the overall transformation

$$\boldsymbol{x} = \boldsymbol{P}\eta \quad \text{with} \quad \boldsymbol{P} = \boldsymbol{P}_1 \boldsymbol{P}_2$$

Damping Estimate by Subspace-Projection

One can approximate the solution of the quadratic EV problem

$$(\boldsymbol{K} + \lambda \boldsymbol{C} + \lambda^2 \boldsymbol{M})\boldsymbol{x} = \boldsymbol{0} \tag{18}$$

by computing $m < n$ modes $\boldsymbol{v}_i$ of the un-damped system, and subsequently solving the quadratic EV problem in the modal subspace defined by $\boldsymbol{x} = \boldsymbol{V}\boldsymbol{\eta}$, i.e.

$$(\underline{\boldsymbol{K}} + \mu \underline{\boldsymbol{C}} + \mu^2 \underline{\boldsymbol{M}})\boldsymbol{\eta} = \boldsymbol{0} \tag{19}$$

The damped natural frequency and damping ratio is then the imaginary and real part of the complex valued eigenvalue, respectively, i.e. $\mu_i = \zeta_i + j\omega_i$.

Comparing Mode Shapes

The *mode scale factor* (MSF) and the *modal assurance criterion* (MAC) describe how well two mode shapes $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ correspond. They are defined by

$$\mathrm{MSF}(\boldsymbol{v}_i, \boldsymbol{v}_j) = \frac{\boldsymbol{v}_i^* \boldsymbol{v}_j}{\boldsymbol{v}_i^* \boldsymbol{v}_i} \neq \mathrm{MSF}(\boldsymbol{v}_j, \boldsymbol{v}_i) \tag{20}$$

$$\mathrm{MAC}(\boldsymbol{v}_i, \boldsymbol{v}_j) = \frac{\left|\boldsymbol{v}_i^* \boldsymbol{v}_j\right|^2}{(\boldsymbol{v}_i^* \boldsymbol{v}_i)(\boldsymbol{v}_j^* \boldsymbol{v}_j)} = \mathrm{MAC}(\boldsymbol{v}_j, \boldsymbol{v}_i) \tag{21}$$

They are insensitive to constant phase shifts and assume a value of 1 for perfect shape correlation.

The MAC is sometimes also called *mode shape correlation coefficient* (MSCC).

Exercises

- Templates are available in TUWEL
- Solutions should be presented during workshop
- Distribute the work within your team

Exercise 3

- Use the system matrices of the plate from last exercise
- Consider the case where it is clamped at its short edge
- Use modal coordinates, dive into the modal subspace

Exercise 3: Tasks

1. Approximate a static deflection due to a vertical load on the upper-right corner $(x_{\text{max}}, y_{\text{max}})$ by selected modes.
2. Compute the free-oscillation modes and use them to compute effective system matrices.
3. Determine the modal (mass) participation factors for rigid body displacements and rotations. Which modes participate most in which directions?
4. Approximate the transient response by using a modal basis for the two given excitation signals (estimate the error with respect to the full system).
5. Compute the harmonic response of the system using a suitable modal subspace, compare Rayleigh damping and modal damping in the frequency range 0–110 Hz.
6. Evaluate the modal contribution for selected frequencies.

Exercise 3: Tipps

- Use the nullspace to enforce constraints
- The Newmark time integration can be applied to the reduced system too
- Compare the numerical effort for full and reduced system
- Use sub-critical damping i.e. $\zeta < 1$
- Normalize the modes to equal displacement to compute their contribution to the overall response

Theory Paper

- keep it as concise as possible
- read the review-guidelines in the course manual
- each student will get 1 paper to review via TUWEL

Submit your draft in TUWEL

Dates

all events at Wednesday, 09:00–11:00 in BA 05

13/03/2019 overview lecture 1
20/03/2019 team meeting
27/03/2019 team meeting
03/04/2019 workshop 1 & overview lecture 2
10/04/2019 team learning
08/05/2019 workshop 2 & overview lecture 3
15/05/2019 team learning
22/05/2019 workshop 3

01/06/2019 Paper draft deadline
15/06/2019 Paper review submission deadline
01/07/2019 Paper submission deadline

1. Go through the theory and complete the exercises
2. Prepare for the workshop
3. Start writing the theory paper