# Evaluation of an Agent-Based Economic model

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## Abstract

We will use the the work of Riccetii et al. [27] as the basis for our project. Building an agent-based model that produces business cycle fluctuation and can produce periods of weakened performance through feedback mechanisms. I will then attempt to analyse how these crashes are created. Ultimately attempting to answer the unknown of economics with the help of agent-based models.

**Motivation**

A model is a way of representing a part of the real world in a logical and objective way. By gathering empirical evidence (information acquired by experimentation or observation), we can verify the degree of homomorphism between the model and the system it seeks to represent. It is the economists job to use models to study the interaction and behaviour of economic agents (corporations, countries, families, etc. ) and how economies work. As one of the founders of Economics Alfred Marshall said:

> "Economics is, on the one side, a science of wealth; and, on the other, that part of the social science of man's action in society, which deals with his efforts to satisfy his wants." [24]

And yet today, economics is still widely regarded as a social science that studies how agent's decisions and behaviour effect the exchange and consumption of resources. There is much dispute as to what we should assume about how agents act. The majority of economists view of the economy embraces the neoclassical approach, in which we assume agents will behave rationally, agents have full information of their environment and are always consistent to some optimal criteria [8]. Yet models with this approach have failed to describe economic crises, but instead set out the phenomena they explain as equilibria problems and apply "shocks" (sudden increases or decreases in model variables) as a way to represent an economic crisis. Which completely avoids any investigation as to how or why such crises emerge.

For the neoclassical view, the behaviour of agents arise from a constrained optimization problem, this is known as rational choice theory. Therefore agents behavior is not defined explicitly and instead arises from the defined optimization problem. In order to decide how to act, agents solve the problem in order to find his/her best choice, and behave such that he or she knows what to expect from taking their economic decision.

This has been challenged through Herbert Simon and others, by studying cognitive psychology of economic behaviour [32] [21]. Showing that humans have bounded rationality because of cognitive limitations, both in the amount of available information and in the ability of processing it in the correct way. This has sparked a new economic approach, in which the economy is viewed as a complex system of heterogeneous agents with bounded rationality and limited information. Instead of each agent finding his/her optimal decision by solving a defined optimization problem, agents have adaptive expectations, adjusting to their environment with the limited information that they have.

New ways of modelling have been created in the scientific community, thanks to the coming of the computer age. One of which are Agent-Based Models (ABMs), which seek to represent systems of objects whose properties change as a result of interactions. These type of models have been applied to many fields of study, including epidemics, behavioral studies, network theory, etc. ABMs are comprised of independent autonomous agents that make decisions following systematic rules, the dynamics of such models are driven by the successive interactions of agents. The main use for modellers using ABMs is to repeatedly run the model in order to understand systems that are otherwise difficult to explain or replicate, rather than predict future outcomes. This is because much more effort has been placed on trying to achieve models that are representative of reality, rather than understanding what it will be like. [12] Since economies can be viewed as a complex adaptive system and given the limitations of neoclassical economic methods, ABMs are a good method for economic study. Applying ABMs in economics is known as Agent-based Computational Economics (ACE) and is defined as 'The computational study of economic processes modelled as dynamic systems of interacting agents' [35].

For this project we have chosen to build an agent-based model as we believe they sit closer to real-world economies, getting rid of the rational agent assumption, modelling agent behaviour adaptatively and fundamentally having a network of interacting entities which show large-scale effects arise from micro-level interactions [28] [35].

**Why credit networks?**

Economic depressions have captivated the attention of economists. If we are to avoid crises our understanding of them is vital. Irving Fisher investigated the Great Depression of the 1930s and concluded that large debts weakened the economy prior to the crisis [13]. Fisher's debt deflation theory is still relevant today as shown by Mervyn King on the UK recession of 1990s [22]. And after the financial crisis of 2008, it is becoming more and more apparent that the performance of credit markets (where institutions issue debt to investors) plays a major role in financial distress [7].

Although stocks and shares typically appear in the news, it is the credit market that shows signs of distress before the equity market. More recently Bernanke [5] showed that credit-market frictions amplify shocks in the economy, emphasising their importance in models if we are to acknowledge the evidence found in real-world economic systems. Riccetti et al. [27] and Gatti [14] take inspiration from Bernanke [5] and build a financial accelerator agent-based model to understand feedback mechanisms in credit markets [27] [14].

Despite the interesting dynamics of the Riccetti model [27], it was subject to no empirical validation. This is common throughout the agent-based model literature and is due to a lack of standardisation (as explored in [36]). Only work on the network (relationships between agents) was validated with Japanese banking data by Bargigli [4]. We take inspiration from the credit market ABM published by Riccetti et al. and use it to build a simplified model in this project, analysing the dynamics that arise and attempting to perform more validation wherever we can.

**Real world context**

We will now explain the terminology that is used throughout this report. In the real world, markets allow for the exchange of resources, sellers offer goods or services in exchange for money or other goods from buyers. There are different types of markets depending on what is being exchanged or what type of economic agent is using it. Companies and goverments exchange money through a market known as the credit market (also called the bond market), this credit is usually given to firms in the form of loans and banks are repaid over a given time period with some interest rate. By company we mean an entity whose goal is to generate profits through selling goods or services, we also call these firms. A persons or corporations net worth is defined as the value of assets they own, minus their liabilities. Therefore we could also say that companies will often want to expand their net worth, since having more debt (liability) than they can afford is generally not desired. A companies net worth is usually a good measure of financial health.

The mechanics of markets have been heavily studied in economics. Alfred Marshall popularized the conversation of how supply and demand influence prices in markets. It should be noted that both finance and economics study markets, although each studies these for a different purpose. The study of the management and flow of money is known as finance, while economics deals studies markets as systems of interaction. There is some overlap between economics and finance, but in this project we move away from the financial literature as we´re not concerned with analysing prices, interest rates, the flow of credit or risk. Agent-based models have been applied in finance, in fact these were the first I encountered. Lettau [23] presents an easily solvable problem, where agents can decide how much of an asset to buy, or they can hold onto their money. If they purchase any amount of an asset they will receive a payment (dividend) of random size at the next period. This problem is easy to solve since the purchase price of the asset is already known, therefore if we know the likely size of the dividend then the amount to purchase can be known to avoid losing money. Agents learn through a genetic algorithm, this is where candidate agents are produced and by keeping the best, removing the worst and adding random mutations, agents evolve overtime and will seem to "learn" an optimal solution, as they do in Lettau´s model.

Other early ABM used in finance include studies on currency exchange rates, like the one by Arifovic [3]. Interestingly agents here fail to find a optimum solution to a problem with multiple equilibria, which was a similar result when proposing the problem to people. We also note the work of Streit and Borenstein which apply ABMs with fuzzy logic to study interest rates [34]. Fuzzy logic was popularized by Lofti Zadeh when he introduced mathematical objects called Fuzzy Sets [37]. These sets introduce a way of defining ambiguity, which act similarly to the more traditional ways of treating uncertainty with probability theory. Another paper encountered using Fuzzy theory was that of Gupta *et al.* [18] which doesn´t use an agent model but caught my attention. They solve asset portfolio optimization (strategies to maximise profit by finding the best portfolio of investments), using fuzzy programming which is an interesting new area.

Although these are interesting models we believe ABMs are better suited as a tool for economics than finance. This is because we focus more on the interactions and the aggregates that arise from this, rather than trying to solve equilibria problems involving the maximisation of profits. Agent-based Computational Economics (ACE) is closely linked with the work at the Santa Fe Institute founded in 1984, which pursues the understanding of natural, artificial and social systems. This institute sparked the new way of viewing economic problems with three books, all titled 'The economy as an evolving complex system'. This new approach, called the Santa Fe approach identified models having cognitive foundations, structural foundations, no global controller and exhibiting continual adaptation, perpetual novelty and out-of-equilibrium dynamics. These characteristics are what helped the Santa Fe approach develop a close relationship with ABMs.

ABMs are also closely related to the literature of simulations, computer programs that simulate aggregate effects of a policy, by applying such policies to individuals and calculating the overall results. Simulations however are more policy oriented whilst ABM are more focused on general equilibrium feedbaks through interactions.

**ABM structure**

ABMs are also closely related to the literature of simulations, computer programs that simulate aggregate effects of a policy, by applying such policies to individuals and calculating the overall results. Simulations however are more policy oriented whilst ABM are more focused on general equilibrium feedbaks through interactions. Yet ABM and simulations share the fact that they try to represent some system changing over time. At their core they consist of a well-defined set of functions which relate input to outputs, these functions set out the rules that shape our model. In fact, for simulations and ABM these functions describe a recursive system, that is we we take the output of the function is dependent on its past outputs. Hence ABM are Markov chain, that the probability of the model entering a certain state only depends on the current state and not on previous states [20]. Markov cains are a powerful tool for analysing stochastic systems, as the systems change over time we express each change in state with a transition matrix. Unfortunately for ABM the number of possible states can be very large and so transition matrices become unfeasible.

As defined in the book "Agent-Based Models in Economics: A toolkit" [28], for an ABM, at each time $t$, we describe an agent $i$, $i \in 1... n$, by some state variables $\mathbf{x_{i,t}}$. The evolution of the agents state is specified by a an equation

$$\mathbf{x_{i,t-1}} = \mathbf{f}_i(\mathbf{x}_{i,t}, \mathbf{x}_{-i,t}, \boldsymbol{\theta}_i, \boldsymbol{\xi}_{i,t}) \tag{1}$$

where $\xi_{\mathbf{i,t}}$ are stochastic terms, and $\theta_i \in \Theta$ is a vector of parameters from some parameter space $\Theta$. The subscipt of $i$´s indicate that the function $\mathbf{f}$, parameters $\boldsymbol{\theta}$ and stochastic variables $\boldsymbol{\xi}$ depend on the specific agent $i$ evolving, and may also depend on the state $\mathbf{x}_{-i}$ of all other agents. The above equation expresses that each individual agent $i$ evolves over time according to the function $\mathbf{f}_i$ which depends on stochastic variables, fixed variables called parameters and the state of other agents. It is important to recognise the use of these subscripts, it expresses the individuality of each agent.

We also define the state of the system as a whole as a matrix of all individual states $\mathbf{X}_t = (\mathbf{x}_{i,t})$. Transforming our Equation 1, to consider all individual states, we have the following

$$\mathbf{X}_{t+1} = \mathbf{F}(\mathbf{X}_t, \boldsymbol{\theta}, \boldsymbol{\Xi}_t) \tag{2}$$

where $\boldsymbol{\Xi}_t$ is a matrix of all stochastic variables at time $t$, and $\boldsymbol{\theta}$ here expresses all parameters.

Notice that presenting the problem this way is completely different to optimization problems as is done in neoclassical economics. Often we are interested in some statistic $y$ of our economy, because these statistics depend on the state of the system, finding them often mean running the simulation (applying iterations of Equations 2 & 1). Finding the relationship between states and some statistics of interest is difficult due to the random terms $\boldsymbol{\Xi}$ and possible errors in measurement. The sequence of random numbers for one simulation is given by the computers random number generator. For a given number called a seed, this generator outputs a specific sequence of random numbers. This gives us the option to replicate results by keeping our seed value and other parameters constant, or by changing the seed we can ensure a different sequence of random numbers is produced. Keeping the seed constant may be useful, as it allows us to see the effect of changes in $\mathbf{x}_{i,t}$, $\mathbf{x}_{-1,t}$ and $\boldsymbol{\theta}_i$ on a copy of the simulation. But the true usefulness of the seed is that by changing the seed we ensure that the simulation evolves through a different path by producing a different sequence of random numbers. By running many simulations, each with a different seed, we can obtain a distribution of observations for some statistic of interest, this is known as Monte Carlo analysis.

**Choosing a framework**

The Santa Fe Institute, which sparked a growing interest in ABMs, created a simulation platform called SWARM. It consisted of libraries to build agent-based simulations, implementing a concept of time, an event scheduler and tools for monitoring experiments. SWARM is now a unmaintained project but it's impact on the community led to the creation of another set of tools called Netlogo. Being more modern Netlogo provides an easier experience, moving away from traditional programming (as is the case with SWARM) and providing it's own programming language and integrated development environment (IDE). Having such free and open-source tools is great to see, we choose not to use Netlogo in particular and instead build our own model with the Python programming language. There are a few reasons for this decision. A programming language designed for ABM usage may seem useful, it is primarily focused on being used by school teachers and non-programmers. It's language is also limited, all Netlogo code must be within one file (experimental feature allows this though). Accessing file reads and writes from Netlogo is not well designed, which we would need if we are to store results and perform Monte Carlo analysis. As a programmer, using a more flexible language should be easier.

We need a framework that will make it easy to individually model firms, banks, institutions, etc. Also since real-world systems are composed of entities with limited information and computational capabilities, it is important for our model to have encapsulation for agents, hiding data and methods of particular agents from others. And so for this reason and that of abstraction we choose an object-oriented paradigm. This gives us a few programming languages to choose from, but we choose Python as our language. It's dynamically typed and memory management will lift the workload of programming, allowing us to focus more on the dynamics of our model. It's packaging system also comes with libraries for performing data analysis and visualisation such as numpy and matplotlib which will be useful for this project.

## The Model

### Model Overview

Choosing the work of [27] as our main inspiration, we use their model as our starting point for this project.

Our modelled economy is composed of two markets: credit and goods. Firms make produce which then sells on the goods market, firms also buy credit from banks on the credit market to keep production going. We assume that firms sell all their product output, each firm selling at a stochastic price, following a normal distribution. The amount of credit that a firm can borrow depends on its previous performace, by letting firms agents follow an adaptive behavioral rule for leverage. This model produces a network-based financial accelerator: an cascade of firm bankruptcies can be produced from a small shock.

### Model Details

#### Capital Structure

We assume that the level of production of a firm $Y_{i,t}$ is proportional to its financial capital (the value of credit it owns).

$$Y_{i,t} = \phi K_{i,t}^{\beta} \tag{3}$$

where $\phi > 1$ and $0 < \beta < 1$ and $K_{i,t}$ is the total capital of firm $i$ at step $t$, composed as the sum of the firms net worth $A_{i,t}$ and its debt $B_{i,t}$: $K_{i,t} = A_{i,t} + B_{i,t}$. The fact that $\beta < 1$ derives a risk-based aggregate supply curve as per [17]

Firms predict expected price in a simple way: future price is equal to the price in the previous step, that is $p_{i,t}^{e} = p_{i,t-1}$. For those that dont know, leverage is the ratio of asset value (in this case debt size) to the cash needed to purchase it. Firms should prefer to borrow more credit for less, i.ethey prefer higher leverage. As spoken in the previous chapter it is important for us to focus on leverage rather than asset prices, since results from real world show that leverage cycles are indictive of financial crisis [15]. We follow the theory of dynamic trade-off theory by letting firms have a target leverage which follows an adaptive behavioral rule as follows:

$$\begin{cases} L^*_{i,t} = L_{i,t-1}(1 + adj * u) \;\; if \;\; p_{i,t} > R_{z,i,t} \\ L^*_{i,t} = L_{i,t-1}(1 - adj * u) \;\; if \;\; p_{i,t} \leq R_{z,i,t} \end{cases} \tag{4}$$

Where interest rate paid for firm $i$ is $R_{z,i,t}$ given from bank $z$ at time $t$. The $adj$ variable is a parameter that sets the maximum leverage change between the two periods and is multiplied by a random variable drawn from a uniform distribution $u \sim U(0, 1)$

The amount of debt that a firm borrows from banks is a function of net worth and target leverage:

$$B^*_{i,t} = A_{i,t} L^*_{i,t} \tag{5}$$

With this capital structure, more debt is obtained with increased profits and leverage, because the firms net worth $A_{i,t}$ is the accumulation of its profits $Pr_{i,t}$

$$A_{i,t+1} = A_{i,t} + Pr_{i,t} = \sum_t Pr_{i,t} = \sum_t [p_{i,t} Y_{i,t} - R_{i,t} B_{i,t}] \tag{6}$$

We can see that the firms profits are simply its output $Y_{i,t}$ all sold at price $p_{i,t}$, minus the debt $B_{i,t}$ which is paid at interest rate $R_{i,t}$. Therefore increasing debt is obtained in times of growth for the firm. Also because of the adaptive leverage, firms will invest more if it gets a cheap interest rate.

We make the simplifying assumtion that firms only take out one debt at a time which is paid in the following period. This is one of the only differences between our baseline model and [27] which uses at most two debts.

The net worth of banks $A_{z,t}$ is modelled by:

$$A_{z,t+1} = A_{z,t} + Pr_{z,t} = \sum_t Pr_{i,t}$$

where bank profits are defined as:

$$Pr_{i,t} = \sum_i [R_{i,z,t} B_{i,z,t}] - r^{CB}_t D_{i,z} - c \cdot A_{z,t} - bad_{z,t}$$

the bank deposits $D_{z,t}$ is simply the outstanding credit minus the net worth $D_{z,t} = B_{z,t} - A_{z,t}$, and $bad_{z,t}$ are the non-performing loans of bank $z$ (i.e. the bad dedt). The central bank interest rate is represented by $r^{CB}_t$, and $c$ is the cost of running the bank.

We need to obtain a bank behviour where financially strong banks will lend money more favourably, as we assume banks want to increase their market share. They should also be more willing to give credit to firms that are performing well, a risk premium. Therefore we adopt the following rule for interest rates:

$$R_{i,z,t} = rCB_t + \underbrace{\gamma A_{z,t-1}^{-\gamma}}_{\text{bank-specific}} + \underbrace{\gamma \frac{L_{i,t}}{1 + \frac{A_{i,t-1}}{\max\{A_{j,t-1} j=1,\ldots,N\}}}}_{\text{firm-specific (risk premium)}} \tag{7}$$

The bank specific term decreases as the bank becomes more financially strong. For the firm-specific term we see that for firms with little net worth the denominator is close to 1 and stronger firms are charged a lower interest as the denominator is larger for theseȧ smaller risk premium while weaker firms are charged more. The presence of this risk premium in the interest rate forms the network financial accelerator.

With all the above considered it is important to note that if a firms sale price and size of production isn´t enough to generate funds to pay off the debt they owe, then they go bankrupt. We now see in the following section how this influences other agents.

**Firm-bank network**

The credit network can be thought of as nodes (which represent firms or banks) and links connecting these nodes (which represent when a certain firm/bank is borrowing/lending with another). We store this credit network as an adjacency matrix since firms only attain loans from a single bank. These network links are initially random, but over time firms will choose out of $\chi$ banks to borrow from the one offering the lowest interest rate. Hence interest rates influence both the profits of firms and the topology of the credit network. As mentioned in [27] and [14] a single firm will effect others through its bank. When a firm goes bankrupt its bank now carries the bad debt and increases its interest rates, this causes further financial fragility through other firms now charged higher interest rates or through banks that lose customers. However, if a firm switches partner it itself may survive bankrupcy through lower interest. These dynamics are the financial accelerator.

A firm or bank goes bankrupt (defaults) when their net worth is negative, $A_t < 0$. This will occur then $Pr_t < A_t$, i.e. when firms or banks fail to produce enough profits. Defaulted firms and banks are replaced by new agents with a random net worth following $U \sim (0, 2)$ and the firm-bank partners of new agents is also uniformly selected.

This network structure evolves endogenously, showing the emergence of business fluctuations and how the bankruptcy of a firm affects banks (and other firms) which may also go bankrupt with a consequent credit crunch. Having large effects arise from micro behviour of individual agents is key for agent-based models as mentioned in [30].

**Timeline of events**

The events taken during a single step in our model is shown in Figure 1, the bank-specific interest from Equation 7 is calculated, firms then use this term to search for a better bank. In the following order firms calculate their target leverage, how much debt they can borrow, their capital, how much then produce they can output and at what price. Then knowing how financially strong they are, the interest rate for each firm is set completely with Equation 7. After this is done firms will sell the goods (indirectly) by updating their profits and net worth. Banks

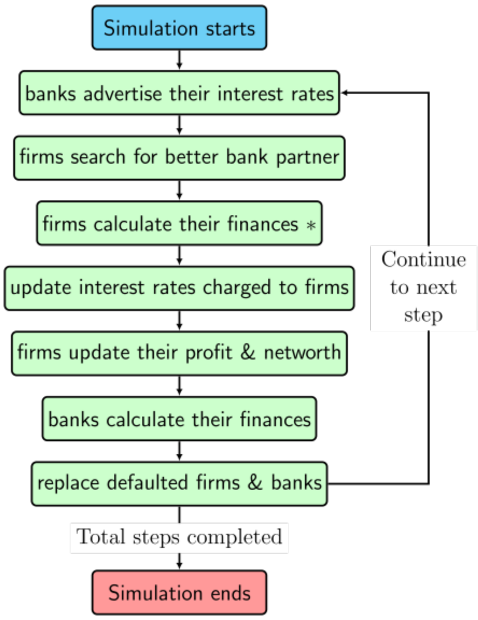



Figure 1: Timeline of events in baseline model

then update their deposits, debt, profit and then net worth. Then finally defaulted firms and banks are replaced with the methods described in the previous section.

Work published in [26] expands on this model, adding agents of individuals and adding labour and deposit markets. Since we were more interested in business cycles, it made sense to avoid unneccessary complexities for the beginning of the project.

## Model Dynamics & Evaluation

The model published in [27] was subject to no empirical validation. This is common throughout the agent-based model literature and is due to a lack of standardisation (as explored in [36]). For the validity of a model can be defined as the degree of homomorphism between the model and the system it tries to represent [28]. Therefore performing validation on our models is crucial for our scientific findings to hold correctness, without it we cannot say our models are representative of the systems they try to represent. If ABMs are to succeed, more empirical validation must be performed in the literature.

Firstly we must characterize what output variables from the model are of interest, and it would be wise to follow the economic theory that variation in leverage is key to understanding economic cycles [17] [5] [15]. Our model presented above does have some simplifications compared to our inspiration [27], such as the interest rate setting, computation of deposits, bank costs, recovery of non-performing loans, and for the firm decision to increase or reduce the target leverage. But we will show that our reduced model still displays a financial accelerator though leverage cycles.

**First simulations**

We run the simulation with 1000 total simulation steps, 500 firms, 50 banks and with the parameters in Table 1 and plot graphs for total firm output, debt, price and bank net worth in Figure 2. Whether our choice in these parameter values is correct and how each variable effects the system will be discussed later.

| Parameter | Value | Meaning |
|---|---|---|
| $\alpha$ | 0.1 | Mean firm price |
| var | 0.2 | Firm price variance |
| $\gamma$ | 0.02 | Interest rate parameter (see Eq 7) |
| $\chi$ | 5 | Number of potential partners searched |
| $\lambda$ | 4 | Intensity of choice |
| adj | 0.1 | Maximum leverage adjustment |
| $\phi$ | 3 | Production parameter (see Eq **??**) |
| $\beta$ | 0.7 | Production parameter (see Eq **??**) |
| $r^{CB}$ | 0.02 | Central bank interest rate |
| $c$ | 0.01 | Cost of running bank |

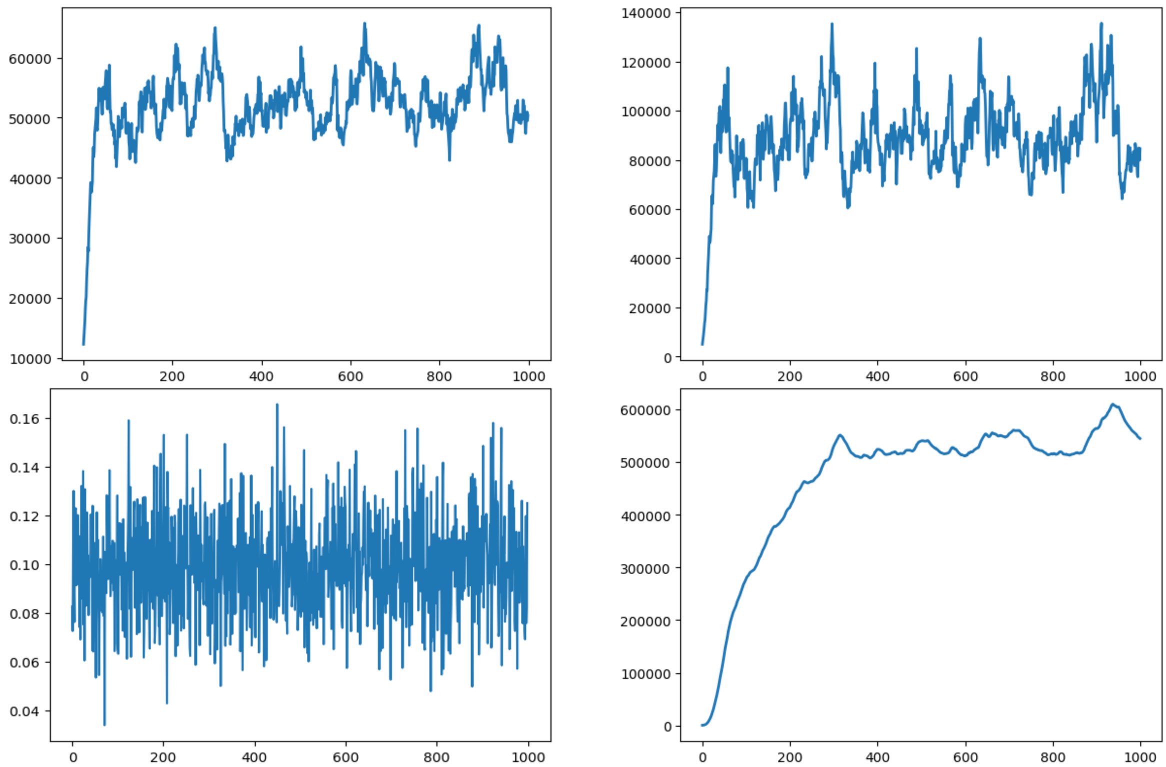
Table 1: Simulation Parameter Values

With Figure 2 we can see how output, debt, price and bank net worth change thoughout one particular run of our simulation. At the start of the simulation we can see a drastic increase in firms output and debt, shown in Figures 2a & 2b. This steep increase in output and debt shouldn´t be of concern because our model is designed to be a systematic representation of an economy, and so if the initialization steps do not break our model´s ability to express economic behaviour, then we can safely ignore the beginning of our simulations. After the early steps in this particular simulation, we can see that firm output and debt level off and start showing repetitious behaviour. There are many periods in our simulation where the aggregate firm´s output and debt climbs high then falls sharply before beginning to rise again, we shall refer to this phenomena as a cycle. Whether these cycles are analogous to economic crises requires further investigation, comparing our model to both economic theory and empirical evidence.

From Figure 2c we can see how the sum of all firm´s prices doesn´t appear to change deterministically throughout the simulation. This is because each firms price follows a normal distribution $\mathcal{N}(\alpha, \mathrm{var})$, and so the sum of prices across all firms itself follows a normal distribution, albeit with different mean and variance. Figure 2d shows a positive and more smooth increase in total bank net worth, recall that net worth is the accumulation of profits. This means that overall the majority of banks are healthy since increases in this curve indicate a positive flow of bank profits.

**Dynamics**

(a) Aggregate Firm Output (b) Aggregate Firm Debt




(c) Aggregate Firm Price (d) Aggregate Bank Networth

Figure 2: Total firm output, debt, price and bank net worth over a single simulation run. Horizontal axis corresponds to each step of simulation as described by Figure 1

We need to be sure that the cyclic behaviour seen in Figure 2 is not a coincidence with our parameter values and random seed. ABMs are characterized by adaptive expectations, therefore consistency may or may not arise. We must therefore repeatedly run our model to gain knowledge about the dynamics of our model. This is known as Monte Carlo analysis, where we run the simulation many times to see the models behaviour and properties arise from its stochastic foundations. To perform a Monte Carlo analysis we must not only run many simulations but we must also ensure that for each simulation run our sequence of random numbers is different, otherwise we would be running the same simulation multiple times and the significance of our results would be diminished. This is done by setting a variable in our code called the random seed, this variable is used to initialize the random number generator and so by making sure our random seed variable is different for every run we can be sure each simulation is different.

We perform our Monte Carlo analysis by running and storing the results of 500 simulations with parameter values from Table 1. Using our analysis tools we created, we compare all the simulations, finding the similarities in dynamics that arise. Finding out if the results shown for one simulation in Figure 2, are a common occurance from the underlying randomness of our model. For example, for each of the simulations we store the total firm production output (as we showed in Figure 2a), and we use this to calculate the average across all 500 simulations, which we plot in Figure 3. This graph is much more flat and no cyclical behaviour appears. This is indicative that most simulations generally are without cycles or that every simulation has cycles such that when we take the average they get washed out like destructive wave interference. Either way the graph confirms stability in our model.

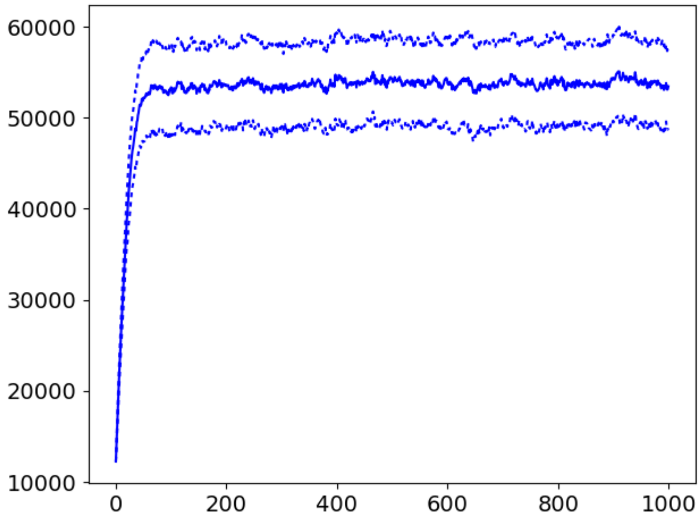



Figure 3: Median firm output (solid line) and standard deviations (dashed lines) over 200 simulations

Understanding more about these cycles and building a deterministic way of finding these cycles and their properties is crucial. Therefore a large amount of time was spent on building the software infrastructure that runs, saves simulations and analyses results.

We could detect cycles by monitoring when large declines/increases in certain variables occur, although the underlying stochasticness may cause accidental detection of cycles. To tackle this we use splines [9] (a polynomial function that is constructed from many other polynomials). The simplest spline is a combination of linear functions where successive datapoints are joined with a straight line. In our case we use cubic splines (a spline composed of cubic functions). This is so that our new splined simulation variables are twice continuiously differentiable, meaning that we can safely measure their change (first derivative) and how the change changes (second derivative). Using splines to express our noisy data into more manageble expressions, is known as spline smoothing. The naming is possibly given because this method both transforms the noise and also gives a twice continuously differentiable property. An example of how we apply splines to firm output is shown in Figure 4.

These splines provide a useful way to approximate the overall trend of variables in our simulations. Comparing the trends will give us some insight into how the variables change with respect to each other. A well known measure for this is the cross-correlation, where we take two vectors $x$ and $y$ (values of two variables for each step in the simulation), and the correlation between $x$ and $y$ is calculated as follows:

$$\sum_n x[n+k] \cdot y[n]$$

The above formula walks through each simulation step, represented by $n$, and calculates the

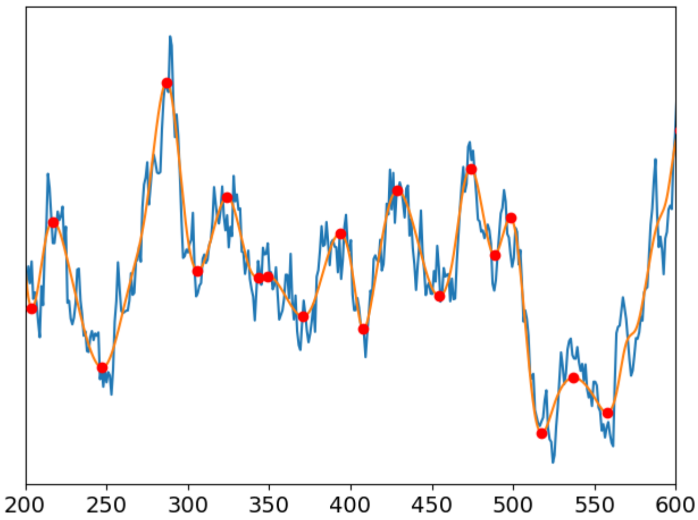



Figure 4: Raw total firm production from one simulation (blue line) and it's smoothed data (orange line) from using splines. Points of economic boom and busts (inflection points) are indicated with red dots. Horizontal axis corresponds to the number of simulation steps.

product of $x$ and $y$. The value of $k$ represents a lag value, if $k = 0$ then the above formula calculates the similarities between $x$ and $y$ at the same step through the simulation, but by changing the value of $k$ we calculate how the $x$ variable relates to values of $y$ further behind or further ahead in the simulation. A positive cross-correlation between two variables means that they both follow the same change in direction, that if one variable increases then we expect the other variable to also increase, likewise the same for decreasing. Negative cross-correlation values indicate inverse proportionality, i.e. that as one increases/decreases the other will change in the opposite direction, decreasing/increasing. It's worth remembering that we cannot assume anything about the rate at which they change together, cross-correlation shows the linear relationship between the two variables (when they increase and decrease at the same rate). Therefore if our variables are nonlinear, cross-correlation won't capture this information.

Cross-correlation plots were given in the similar model from [27], they found that across almost all 100 simulations a positive correlation between firm net worth and the subsequent leverage, highlighting that as firms' networth increases, they also increase leverage. Despite the simplifications we made with respect to their model, we find similar results across 500 simulations with parameter values from Table 1, which are shown in Figure **??**.

This positive cross-correlation suggests that as firms' networth increases or decreases, their leverage changes in the same way. As mentioned before we cannot make any assumtions about the rates at which they change, only that they change in the same direction. The fact that the correlations is neither -1 or 1 suggests that they aren't related linearly. Using the cross-correlation in this way allows us to see the effects of variables on others that is not so obvious from the models equations (note that networth is not in Eq 4), but it is a weak way of seeing exactly how much they effect each other, since it is likely the model has nonlinear relationships between variables.

Given that the cross-correlation between firm net worth and leverage quickly declines in Fig-

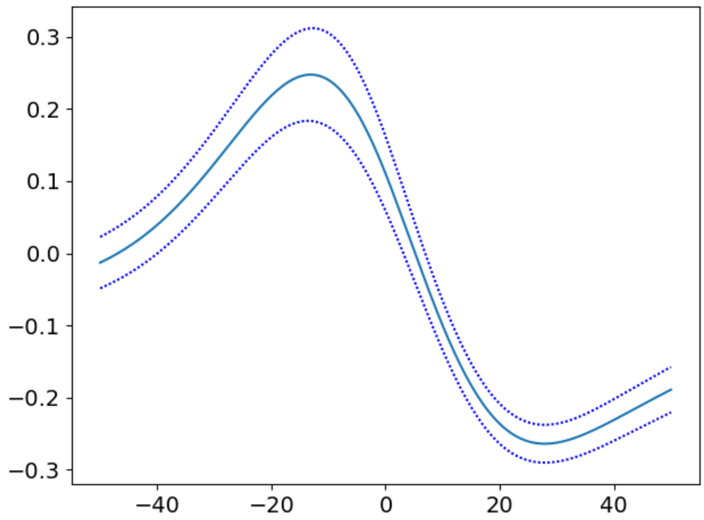



Figure 5: Average cross-correlation (solid line) between firms' networth and leverage from 500 simulations with parameter values from Table 1. Showing the amount of lag $k$ on the horizontal axis and the average cross-correlation on the vertical axis. Variance in the measurement highlighted with dashed lines.

ure 5. And given that we know a crisis occurs when most firms' net worth increases to large amounts and than declines by a large amount (similar to firm output Figure 2a). This suggests the following events; when a firm gains profits and its net worth increases (see Eq 6), the firm then increase their target leverage (positive cross-correlation with negative lag), meaning they increase borrowing and produce more goods (see Eq 5 and Eq 3). With a larger produce to sell, if firms get lucky again through the stochastic price, they will want to expand production even more. In fact with a lower risk premium, they're more likely to be offered a lower interest and hence have better profits (see Eq 7).

The decline in correlation between firms' net worth and leverage (highlighted in Figure 5), shows that the economy reaches a tipping point. As increases in firms' net worth no longer increase leverage and hence breaks the effect of the risk premium in Eq 7, therefore firms can become financially fragile despite having a large net worth. These cycles are true in the real world, leverage dramatically increased in globally from 1999 to 2006. Home-owners could get a mortgage with less than a 3% down payment, as a consequence debt ran high and Fishers debt deflation theory was true once again, causing financial fragility and starting the 2008 financial recession [15] [13]. The crashes of 1987 and 1998 also seem to be correlated to spikes in leverage [15]. These events are highlighted in the Figure 6.

**Finding crises**

Applying our smoothing spline technique (mentioned at beginning of Section 3.3) to variables from our simulation, we can find inflection points (where the variable is ∩-shaped or ∪-shaped) as a way to approximately find economic booms and crises. To locate these points we take the gradient (first derivative) of our variable under consideration and find where there is a change in sign (where our variable changes from increasing to decreasing or viceversa) [2]. Applying this technique to find crises on 500 simulations with parameter values from Table 1, the distribution of all crises found are shown in Figure 7.

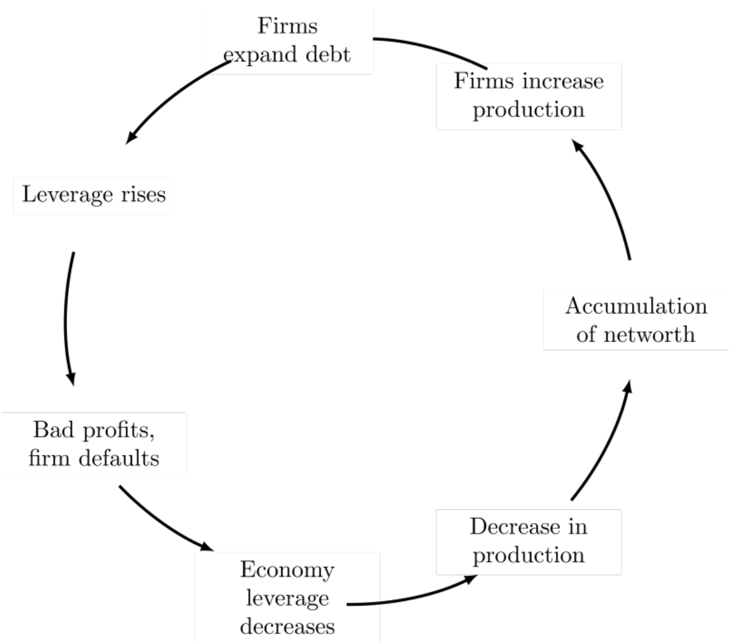



Figure 6: Economic cycle from model

The Gross Domestic Product (GDP) is the total market value of all goods and services produced in a specific time. It can serve as a measurement to indicate the size and growth of an economy. For our model GDP is simply the total firm production output (since these are the only goods we produce), which we measure at each step. We measure the crises by percent change in GDP (firm production output in our case), this is because whole numbers are not comparable across each simulation, i.e. the effects of a 500 drop in raw GDP may be small or large but it depends on the size of the economy. Our results show that across all simulations, the average crisis caused a 9.42% reduction in GDP, with standard deviation of 6.45. Since all 500 simulations run with the same parameters, these results show us how each simulation differs and our first glimpse into the stability of our model.

We must check whether the frequency and size of these cycles resemble real-world observations. This is part of our effort to perform output validation. Data was gathered from the Organisation for Economic Co-operation and Development (OECD), which is an international organisation designed to establish international standards and find solutions on social, economical and environmental issues [25]. For comparison with our results in Figure 7, we have given Table 2, produced from the OECD dataset [1], which shows the GDP loss of several countries from the first quarter in 2008 until the last quarter of 2009. From the same dataset of all OECD members during this period, we find that of countires which saw a fall in GDP, the average percent fall in GDP was 5.74% with a standard deviation of 4.84, where Belgium saw the biggest fall (20.3%). Therefore crises of such size are possible, although it should be noted that the 2008 crisis is regarded as having been most severe since the Great Depression [7] [16].

Although at the time of writing this we are experiencing the effects of a pandemic on society and the International Monetary Fund (IMF) reports that high-income economies (similar to

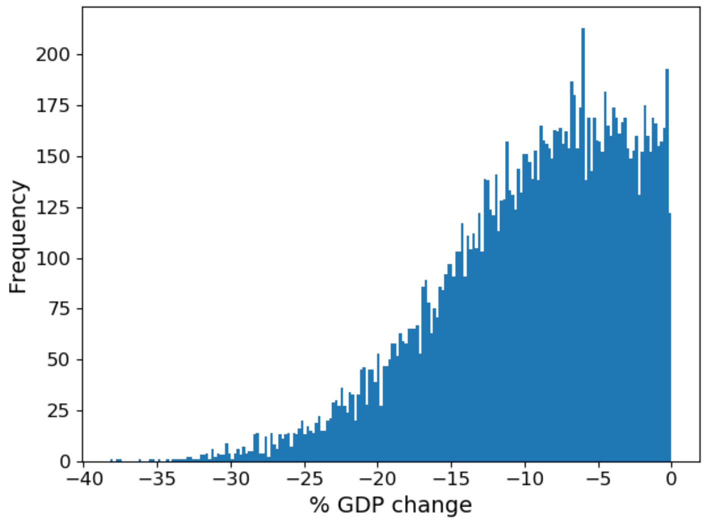



Figure 7: Distribution of crises sizes (by percent loss of GDP) from 500 simulations. An average of 28.6 crises occured per simulation, with standard deviation of 6.13.

| Country | UK | Spain | USA | Germany | Japan | Australia | Iceland | Denmark |
|---|---|---|---|---|---|---|---|---|
| GDP change (%) | -5.45 | -3.85 | -3.63 | -5.76 | -6.40 | 3.59 | -9.28 | -6.75 |

Table 2: Percentage GDP Loss from various countries from 2008 Q1 until 2009 Q4. Source: [1]

members of the OECD) will experience a change in annual GDP of $-6.1\%$, assuming COVID-19 peaks in the second quarter and recedes in the second half of 2020 [19]. And Italy has already reported it's GDP shrank 4.7% in the first quarter of this year. Comparing this to the average GDP change of $-4.29\%$ of all OECD members from 2008 until 2009, then this would place the "Great Lockdown" as the worst crisis since the Great Depression. A crisis of grand proportion perhaps, but given the limited time we have had to see such crises occur as a species, it is unclear how common crises of this magnitude are.

**Timescales realism**

Rather than comparing our individual events, we now measure the percent change in GDP at regular intervals, giving us comparable measurements to the quarterly GDP data from the OECD [1].

We must therefore determine how many simulation steps, $h^*$, are comparable to a financial quarter. In the real world, business loans are most commonly repaid on a monthly basis, but corporations acquire credit from many sources. In our simulations every firm attempts to pay its debt at each step (see Figure 1), but we should be careful with defining a financial quarter (three-month period) as three simulation steps. This is because in the real world corporations acquire funds from a range of sources, meaning that the rate at which a corporations' financial situation improves (increasing leverage) may not be similar to the rate at which it improves for a single step in the simulation.

To see how many steps $h^*$ correspond to a quarter, we measure the percent change in GDP at regular intervals of length $h$. We then score these measurements with how well they fit the data from the OECD [1]. Scores are given via the Kolmogorov-Smirnov test (calculated with the *Scipy* python library), which checks if two datasets come from the same probability distribution [33]. With this we can find which values of $h$ give intervals which appear closest to resembling real world financial quarters. So for each of the simulations ran, we check the similarities in GDP with each of the 38 OECD members in our dataset, meaning we run $500 \cdot 38 = 19000$ Kolmogorov-Smirnov tests, each giving a statistic $D$. This measurement $D$ is the largest distance between the distribution functions of our datasets. We count how many tests give $D$ less than a confidence level $D_\alpha$, calculated as follows:

$$D_\alpha = c(\alpha)\sqrt{\frac{(n+m)}{n \cdot m}} \tag{8}$$

where $c(\alpha)$ is our level of significance, found from statistical tables, and sets the confidence level of our test. The size of our datasets correspond to $n$ and $m$, the number of quarterly readings from our OECD data and the number of interval readings from each simulations. If $D < D_\alpha$ then we say the Kolmogorov-Smirnov test has passed and that the two datasets appear from the same distribution, i.e. that our GDP measurements are similar to real world data [11].

Our results would have shown how taking measurements of simulation GDP at different intervals effects the similarity between these and real world quarterly GDP measurements. Unfortunately for different values of $h$, a very small proportion of the tests passed (less than 1%). The only time where a large portion of tests passed occured when our interval size $h$ was very large, yet we discard these results since the Kolmogorov-Smirnov test yields weak results for small datasets. For simulations with parameter values from Table 1, these results show that intervals of a certain step size are not comparable to those of real financial quarters (in terms of GDP). One possibility for this is that our model does not take into account technological or population growth, which is part of the real-world and measured in our data. To tackle this we implement a basic form of growth, by adding an extra firm to our simulation with every step and adding an extra bank every twenty steps. This behviour is invoked with the '-*g*' command-line argument. Yet unfortunately, a rerun of the tests with growth simulations, our results did not improve.

By measuring our quarterly GDP as one step being equivalent to one, two, three, four, eight or fifteen months, unfortunately our results are no better, leaving us in a blind spot, with the quarterly GDP not appearing to come from the same probability distribution as the data we have from the OECD (which contains quarterly GDP from 1968 to 2019). It would have been reassuring if the Kolmogorov-Smirnov tests pointed to the number of steps, $h^*$, that correspond closely with real financial quarters. It is worth noting that the empirical data we have is our golden-standard, but it only provides us with one real measurement of an economy that arises from the structures of our society. It provides us with only a single draw from the true probability distribution which defines the realistic changes in GDP, it is therefore limiting our ability to understand what is considerded "normal" GDP change. Admittedly more data would be useful, but I think we can agree that it would be hard to rerun society and measure it's economic data again. More investigation could be done to tweak parameter values to see if we can achieve similar results to our empirical data.

Let us assume that 1 step does indeed equal 1 month. As mentioned in the previous section 28.6 crises occured per simulation on average, this means that one crisis occurs roughly every $800/28.6 \approx 28$ steps, in other words every 2.33 years. With our average crisis being a 9.43% drop in firms production output (GDP), this seems slightly unrealistic considering that such drops only rarely occur in reality, as mentioned at the end of the previous section. Such a result shouldn't make our model useless, since our statistics of interest depend on the parameter values. We can use such techniques of validation to find parameter values that create simulations with more realistic output. This is known as calibration. Similar to what we have done in this section where we calculate the same statistics in real and simulation data, and then aggregate a measure of distance between the real-world statistics and artificial statistics. Calibration based on the above technique would only allow us to find parameter values that give realistic crisis frequency and size. It is worth exploring further validation tests so that when calibration is performed we have many measures of validity.

**Distribution of Wealth**

It is generally well-known that if population income is ranked by size, the majority of the population with low income has the same accumulatibe wealth as the few of those with higher incomes. This was discovered by Vilfredo Pareto, and therefore population income is said to follow a Pareto distribution. It has been shown in [31] that a stochastic model can produce such a distribution and it is been noted in economics that stochastic analysis can help explain how such distributions in wealth arise [6].

Since we're dealing with the wealths of firms' and banks' we need to gather empirical evidence to see how wealth is distributed amongst these in the real world. As far as I'm aware no organisational body officially monitors and publishes such evidence, although Fortune and Forbes magazines do publish such lists on their own accord. We have gathered data published by CNN called the Global 500 [10], which is a listing of the 2012 worlds largest corporations ranked by revenue and profit. This list was used due to being freely available compared to those more up to date like those of Fortune and Forbes.

We will use the Kolmogorov-Smirnov test (see previous section) to see what distribution the data follows. To do this we must first find a distribution to compare the data with, since the Kolmogorov-Smirnov test will only tell us wether two samples come from the same distribution. Therefore we use the Maximum Likelihood Estimation (MLE) which is a method that finds parameter values defining the shape of the data distribution it follows. We can then use the Kolmogorov-Smirnov test to compare the list of corporations wealth with data drawn from a probability distribution constructed with parameter values from MLE. If our Kolmogorov-Smirnov test returns a statistic $D$ less than a confidence value (see previous section), then we can assume that our empirical data does indeed follow a particular distribution.

We use a pareto distribution with a probability distribution:

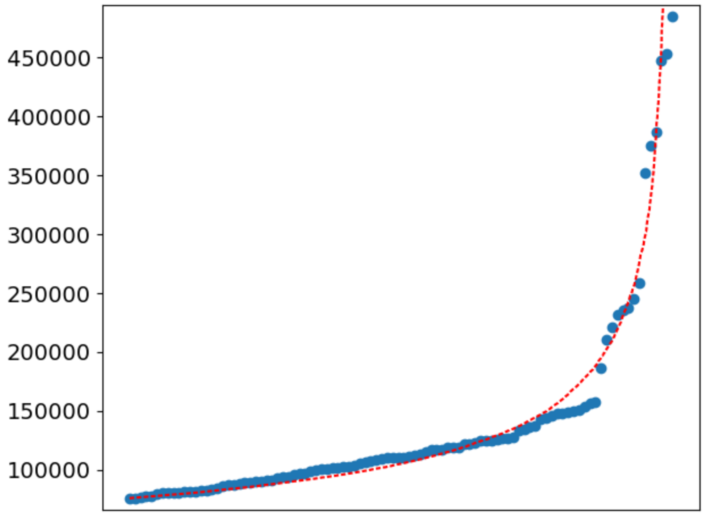



Figure 8: Red dashed line shows the pareto distribution found with Maximum Likelihood Estimation following the top 100 world's largest corporations (blue dots) ranked by millions of US Dollars in revenue (Vertical Axis). Source [10]

$$f(x,c) = (1+cx)^{-1-\frac{1}{c}} \qquad \text{defined for } \begin{cases} x \geq 0 & \text{if } c \geq 0 \\ 0 \leq x \leq \frac{-1}{c} & \text{if } c < 0 \end{cases} \tag{9}$$

Where $c$ is a parameter which defines the shape of the probability distribution [29]. When applying the above technique to the top 100 largest corporations of 2012 we achieve the distribution shown by the red dashed line in Figure 8. We then take this pareto distribution and empirical data, and perform a comparison via the Kolmogorov-Smirnov test. Setting our confidence value of $\alpha$ in Eq 8 to 0.01, meaning that there is a 99% chance that the Kolmogorov-Smirnov test is real and not discovered by chance [11]. With $D \approx 0.096$ and $D_\alpha \approx 0.23$, our test passes and we can confirm the top 100 largest corporations by revenue (in 2012) do indeed follow a pareto distribution. We also find that corporations in 2012 with positive profits also follow a pareto distribution when ranked by profit size, with $D \approx 0.060$ and $D_\alpha \approx 0.18$.

This suggests that if our model is to be realistic then it should also follow a pareto distribution. For each step of a simulation we perfrom MLE and the Kolmogorov-Smirnov test to see if firms' wealth follows a pareto distribution of the form defined by Eq 9. As usual we perform this test for 500 simulations with parameter values from Table 1. We find that for all of these simulations, more than 99.5% of steps passed the test, meaning that 99.5% of the time our simulations had all of our firms' wealth following a pareto distribution.

**Network Validation**

Work published in [4] analyses and compares empirical data with the network structure that evolves from [27]. Although our model in this project is a simplified version of [27], we can certainly learn from this analysis to draw similar conclusions with our own model.

Bargigli *et al.* [4], shows that a banks' degree (the number of firms it lends to) from real world Japanese data, follows a power law distribution. They also show this to be true for the model similar to ours in [27], although a much less skewed distribution. We run the same test on simulations with parameter values from Table 1 we too find it follows a pareto distribution (see Eq 9) in our model, which is a type of power law distribution.

The reason for the tipping point in the leverage cycle (as seen in Figure 5), can perhaps be explained by the topology of the network. Where bank defaults cause a more fractured structure, this then forces firms to switch to a new partner with interest rates they may not be able to afford, and hence more banks may succumb to more bad debt, risking other banks financial stability. Although more extensive analysis could be provided to see how failing firms and banks cause cascading effects on the rest of the network. Given how we have setup the network, as discused in Chapter **??**, firms with larger debts and hence large net worth (see Eq 5), have a much bigger effect on the network when they fail to pay their debts, than those of firms with smaller net worth. And these firms are just as able to fail as smaller ones as shown through Figure 5 and the risk premium in Eq 7 (see end of Section 3.2). It would be interesting to perform more analysis on the network topology, but unfortunately we do not have time.

## Conclusion

### Project findings

Using Python as our programming language we have built an agent-based model. We have shown that this model does indeed have economic phenomena that arise from the underlying dynamics.

Despite the simplifications made in our model compared to that of Riccetti et al [27], our agent-based model does indeed show cyclical behaviour, as seen in Figure 2a & 2b. These cycles are periods of weakend economic performance, and are validated by real world occurances (see Section 5.4).

We´ve came up with a way of detecting these crises, through the use of splines and differentiation, we´ve came up with a way to detect crises in our models. This allowed us to deterministically measure the frequency and size of crises for single simulations. We analysed how our simulations compared with that of real data from the Statistical Office of the European Union (Eurostat) and the Organisation Economic Co-operation and Development (OECD). We did this via Monte Carlo analysis (running many simulations to get distributions of statistic) and tools that we created to help analyse simulations. Results seem to show confidently that for simulations of certain parameter values, seem to have similar crises sizes and frequency. Although we did not tweak parameters to see if closer results could be achieved.

Additionally we investigated the distribution of wealth, showing that the distribution of the real world and that of our model both follow a pareto distribution very confidently.

While it would have been useful to perform some parameter calibration to see how close the model can produce an output like real data. We did come up with ways of performing validation on this model, something that is sparse in the literature. We hope that this work has shown a confincing argument into an alternative way of doing economics, as opposed to neoclassical methods of full information, rational agents.

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