Static Program Analysis

Assignment Project Exam Help

Part 5 – widening and narrowing

https://powcoder.com

Add WeChat powcoder

http://cs.au.dk/~amoeller/spa/

Anders Møller & Michael I. Schwartzbach

Computer Science, Aarhus University

Interval analysis

- Compute upper and lower bounds for integers
- Possible applications:
  - array bounds checking
  - integer representation
  - …
- Lattice of intervals:

$$Intervals = lift(\{\ [l, h]\ \mid l,h \in N \wedge l \leq h\ \})$$

where

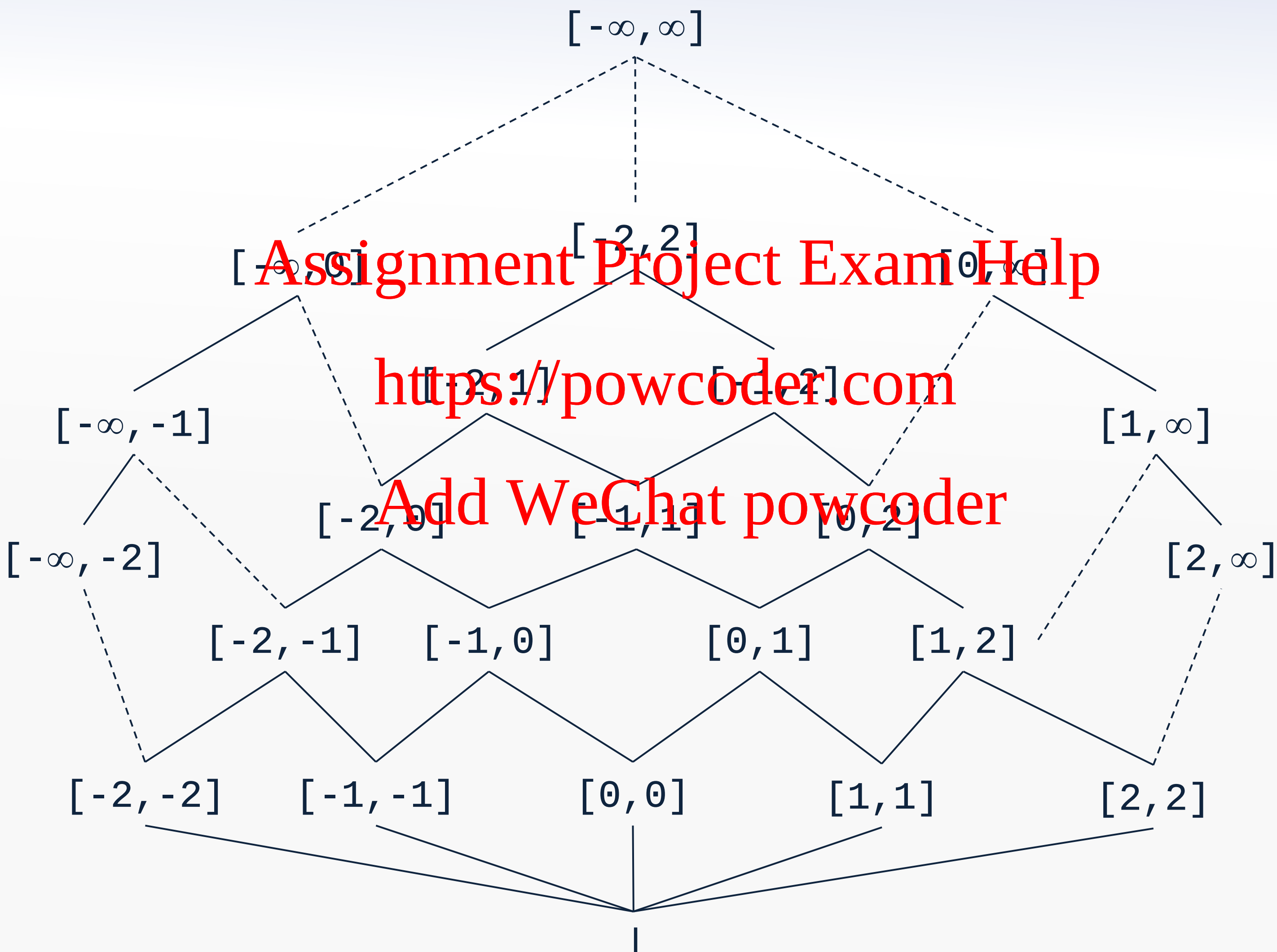
$$N = \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$$

and intervals are ordered by inclusion:

$$[l_1, h_1] \sqsubseteq [l_2, h_2] \text{ iff } l_2 \leq l_1 \wedge h_1 \leq h_2$$

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

The interval lattice



Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

bottom element here interpreted as “not an integer”

Interval analysis lattice

- The total lattice for a program point is

  $$Vars \rightarrow Intervals$$

  that provides bounds for each (integer) variable

Assignment Project Exam Help

- If using the worklist solver that initializes the worklist with only the *entry* node, use the lattice *lift*(*Vars* → *Intervals*)

https://powcoder.com

  - bottom value of *lift*(*Vars* → *Intervals*) represents "unreachable program point"

Add WeChat powcoder

  - bottom value of *Vars* → *Intervals* represents "maybe reachable, but all variables are non-integers"

- This lattice has *infinite height*, since the chain

  $$[0,0] \sqsubseteq [0,1] \sqsubseteq [0,2] \sqsubseteq [0,3] \sqsubseteq [0,4] \ldots$$

  occurs in *Intervals*

Interval constraints

- For assignments:

$$[\![\, x = E \,]\!] = JOIN(\mathrm{v})[x \to eval(JOIN(\mathrm{v}), E)]$$

Assignment Project Exam Help

https://powcoder.com

- For all other nodes:

Add WeChat powcoder

$$[\![\mathrm{v}]\!] = JOIN(\mathrm{v})$$

where $JOIN(\mathrm{v}) = \bigsqcup_{\mathrm{w} \in pred(\mathrm{v})} [\![\mathrm{w}]\!]$

Evaluating intervals

- The *eval* function is an *abstract evaluation*:
  - $eval(\sigma, x) = \sigma(x)$
  - $eval(\sigma, intconst) = [intconst, intconst]$
  - $eval(\sigma, E_1 \text{ op } E_2) = \overline{\text{op}}(eval(\sigma, E_1), eval(\sigma, E_2))$
- Abstract operators:
  - $\overline{\text{op}}([l_1, h_1], [l_2, h_2]) =$

$$[\min_{x \in [l_1, h_1], y \in [l_2, h_2]} x \text{ op } y, \max_{x \in [l_1, h_1], y \in [l_2, h_2]} x \text{ op } y]$$

not trivial to implement!

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

Fixed-point problems

- The lattice has infinite height,
  so the fixed-point algorithm does not work ☹

Assignment Project Exam Help

- The sequence of approximants

  $f^i(\perp)$ for $i = 0, 1, \ldots$

  is not guaranteed to converge

https://powcoder.com

Add WeChat powcoder

- (Exercise: give an example of a program where this happens)

- Restricting to 32 bit integers is not a practical solution
- *Widening* gives a useful solution…

Does the least fixed point exist?

- The lattice has infinite height, so Kleene's fixed-point theorem does not apply ☹

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

- **Tarski's fixed-point theorem:**

In a complete lattice L, every monotone function $f\colon L \to L$ has a unique least fixed point given by

$$\mathrm{lfp}(f) = \sqcap\{\, x \in L \mid f(x) \sqsubseteq x \,\}$$

(Proof?)

Widening

- Introduce a *widening* function ω: L $\rightarrow$ L so that

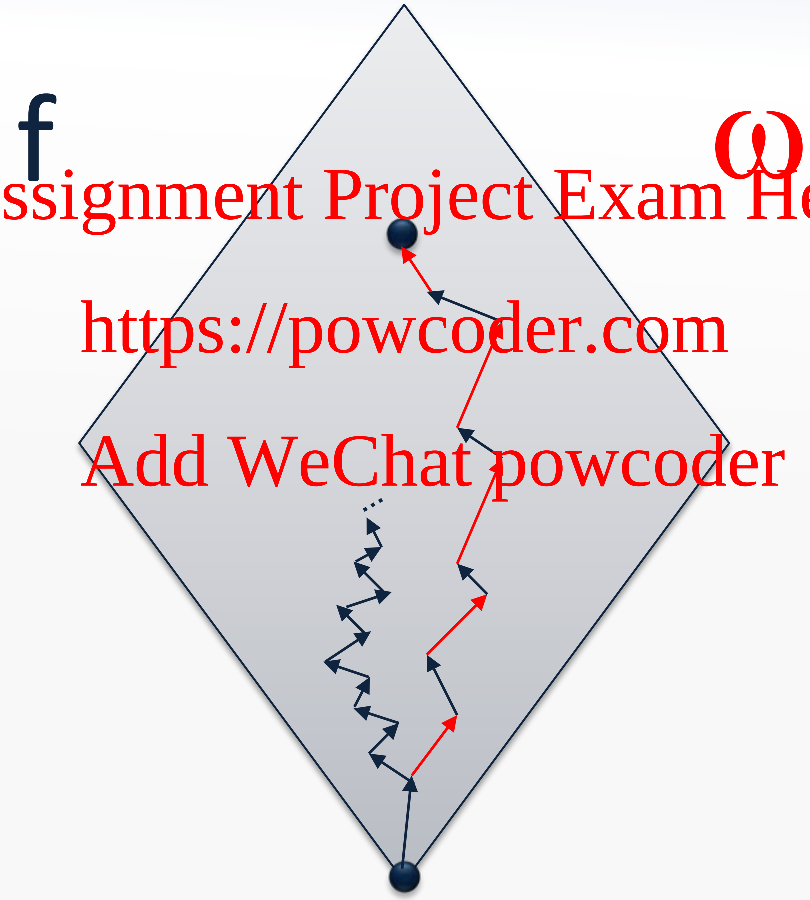
  $(\omega \circ f)^i(\perp)$ for $i = 0, 1, \ldots$

  converges on a fixed point that is a safe approximation of each $f^i(\perp)$

- i.e. the function ω coarsens the information

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

Turbo charging the iterations



Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

Simple widening for intervals

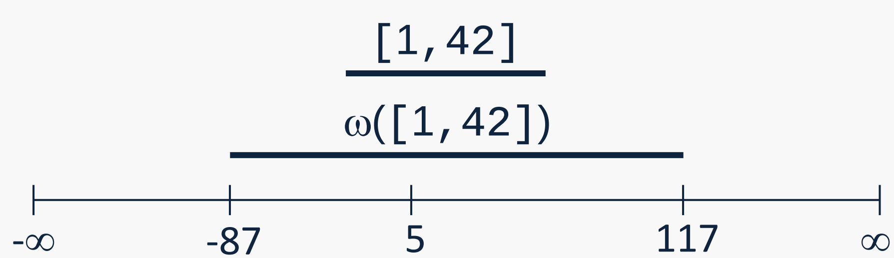
- The function ω: L $\rightarrow$ L is defined pointwise on

  L = (*Vars* $\rightarrow$ *Intervals*)^n

- Parameterized with a fixed finite set *B*
  Assignment Project Exam Help
  - must contain $-\infty$ and ∞ (to retain the $\top$ element)
  https://powcoder.com
  - typically seeded with all integer constants occurring in the given program
  Add WeChat powcoder
- Idea: Find the nearest enclosing allowed interval
- On single elements from *Intervals* :

$$\omega([a,b]) = [\ max\{i \in B \mid i \leq a\},\ min\{i \in B \mid b \leq i\}\ ]$$

$$\omega(\bot) = \bot$$



Divergence in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
   x = 7;
   x = x+1;
   y = y+1;
}
```

$[x \to \perp, y \to \perp]$
$[x \to [8,8], y \to [0,1]]$
$[x \to [8,8], y \to [0,2]]$
$[x \to [8,8], y \to [0,3]]$
…

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

Simple widening in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
   x = 7;
   x = x+1;
   y = y+1;
}
```

$[x \to \bot, y \to \bot]$

$[x \to [7,\infty], y \to [0,1]]$

$[x \to [7,\infty], y \to [0,7]]$

$[x \to [7,\infty], y \to [0,\infty]]$

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

$B = \{-\infty, 0, 1, 7, \infty\}$

Correctness of simple widening

- This form of widening works when:
  - ω is an *extensive* and *monotone* function, and
  - the sub-lattice $\omega(L)$ has *finite height*

Assignment Project Exam Help

- $\omega \circ f$ is monotone and $\omega(L)$ has finite height, so $(\omega \circ f)^i(\perp)$ for i = 0, 1, … converges

https://powcoder.com

Add WeChat powcoder

- Let $f_\omega = (\omega \circ f)^k(\perp)$ where $(\omega \circ f)^k(\perp) = (\omega \circ f)^{k+1}(\perp)$
- $\mathrm{lfp}(f) \sqsubseteq f_\omega$ follows from Tarski's fixed-point theorem, i.e., f_ω is a safe approximation of lfp(f)

Narrowing

- Widening generally shoots over the target
- *Narrowing* may improve the result by applying f

Assignment Project Exam Help

- We have $f(f_\omega) \sqsubseteq f_\omega$ so applying f again may improve the result!

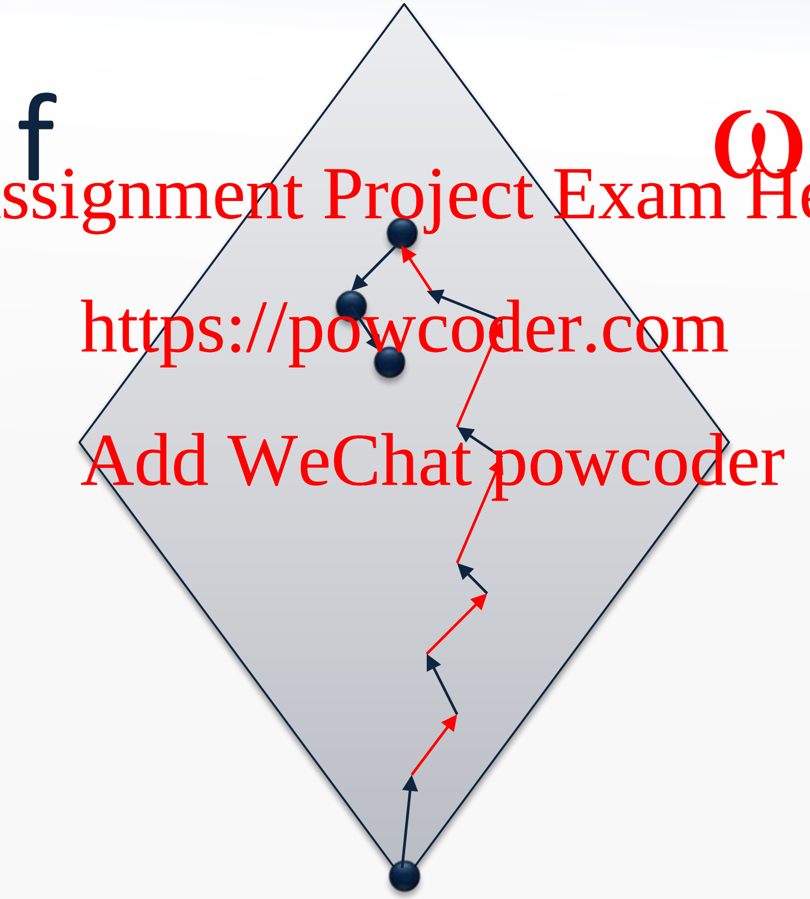
https://powcoder.com

- And we also have $lfp(f) \sqsubseteq f(f_\omega)$ so it remains safe!

Add WeChat powcoder

- This can be iterated arbitrarily many times
  - may diverge, but safe to stop anytime

Backing up



Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

Narrowing in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
   x = 7;
   x = x+1;
   y = y+1;
}
```

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

$[x \to \perp, y \to \perp]$

$[x \to [7,\infty], y \to [0,1]]$

$[x \to [7,\infty], y \to [0,7]]$

$[x \to [7,\infty], y \to [0,\infty]]$

…

$[x \to [8,8], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

Correctness of (repeated) narrowing

Claim: $lfp(f) \sqsubseteq \ldots \sqsubseteq f^i(f_\omega) \sqsubseteq \ldots \sqsubseteq f(f_\omega) \sqsubseteq f_\omega$

- $f(f_\omega) \sqsubseteq \omega(f(f_\omega)) = (\omega \circ f)(f_\omega) = f_\omega$ since ω is extensive
  - by monotonicity of f and induction we also have, for all i:

    $f^{i+1}(f_\omega) \sqsubseteq f^i(f_\omega) \sqsubseteq f_\omega$
  - i.e. $f^{i+1}(f_\omega)$ is at least as precise as $f^i(f_\omega)$
- $f(f_\omega) \sqsubseteq f_\omega$ so $f(f(f_\omega)) \sqsubseteq f(f_\omega)$ by monotonicity of f, hence $lfp(f) \sqsubseteq f(f_\omega)$ by Tarski's fixed-point theorem
  - by induction we also have, for all i:

    $lfp(f) \sqsubseteq f^i(f_\omega)$
  - i.e. $f^i(f_\omega)$ is a safe approximation of $lfp(f)$

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

Some observations

- The simple notion of widening is a bit naive…
- Widening happens at *every* interval and at *every* node

Assignment Project Exam Help

https://powcoder.com

Add WeChat powcoder

- There's no need to widen intervals that are not "unstable"
- There's no need to widen if there are no "cycles" in the dataflow

More powerful widening

- A *widening* is a function ∇: $L \times L \rightarrow L$ that is extensive in both arguments and satisfies the following property:

  for all increasing chains $z_0 \sqsubseteq z_1 \sqsubseteq \ldots$,

  the sequence $y_0 = z_0, \ldots, y_{i+1} = y_i \nabla z_{i+1}, \ldots$ converges

  (i.e. stabilizes after a finite number of steps)

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder

- Now replace the basic fixed point solver by computing $x_0 = \bot$ and $x_{i+1} = x_i \nabla f(x_i)$ until convergence

- Theorem: $x_{k+1} = x_k$ and $\mathrm{lfp}(f) \sqsubseteq x_k$ for some k

(Proof: similar to the correctness proof for simple widening)

More powerful widening for interval analysis

Extrapolates unstable bounds to *B*:

$\bot \, \nabla \, y = y$

$x \, \nabla \, \bot = x$

Assignment Project Exam Help

$[a_1, b_1] \, \nabla \, [a_2, b_2] =$

https://powcoder.com

$[\text{if } a_1 \leq a_2 \text{ then } a_1 \text{ else } \mathit{max}\{i \in B \mid i \leq a_2\},$

$\text{if } b_2 \leq b_1 \text{ then } b_1 \text{ else } \mathit{min}\{i \in B \mid b_2 \leq i\}]$

Add WeChat powcoder

The ∇ operator on L is then defined pointwise down to individual intervals

For the small example program, we get the same result as with simple widening plus narrowing (but now without using narrowing)

Yet another improvement

- Divergence (e.g. in the interval analysis without widening) can only appear in presence of recursive dataflow constraint

Assignment Project Exam Help

https://powcoder.com

- Sufficient to "break the cycles", perform widening only at, for example, loop heads in the CFG

Add WeChat powcoder

Choosing the set *B*

- Defining the widening function based on constants occurring in the given program may not work well

```
f(x) { // "McCarthy's 91 function"
  var r;
  if (x > 100) {
    r = x - 10;
  } else {
    r = f(f(x + 11));
  }
  return r;
}
```

https://en.wikipedia.org/wiki/McCarthy_91_function

- (This example requires interprocedural and control-sensitive analysis)

Assignment Project Exam Help
https://powcoder.com
Add WeChat powcoder