Урматы

Vlad Leonov

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Содержание

Часть I

Задание 1

Классификация уравнений 2-го порядка, характеристики

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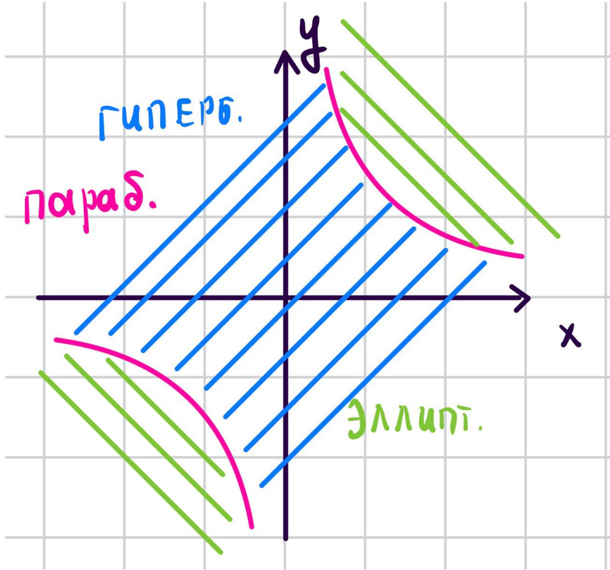
a) $yu_{xx} + 2u_{xy} + xu_{yy} - u_y = 5x$;



$$\begin{pmatrix} y & 1 \\ 1 & x \end{pmatrix}$$

$\Delta_1 = y, \Delta_2 = xy - 1$

Эллиптический, если

$$\begin{cases} y > 0 \\ xy - 1 > 0 \end{cases} \quad \text{либо} \begin{cases} y < 0 \\ xy - 1 > 0 \end{cases}$$

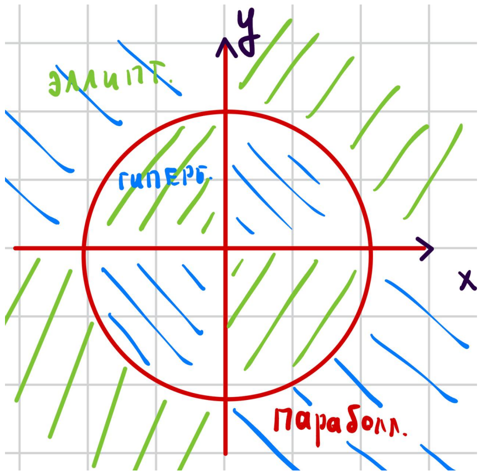
Гиперболический, если

$$\begin{cases} y < 0 \\ xy - 1 < 0 \end{cases} \quad \text{либо} \begin{cases} y > 0 \\ xy - 1 < 0 \end{cases}$$

Параболлический, если

$xy - 1 = 0$

б) $(x^2+y^2-1)u_{xx}+xyu_{yy}-u_x=0.$



$$\begin{pmatrix} x^2+y^2-1 & 0 \\ 0 & xy \end{pmatrix}$$

Эллиптический, если

$xy(x^2+y^2-1)>0$

Гиперболический, если

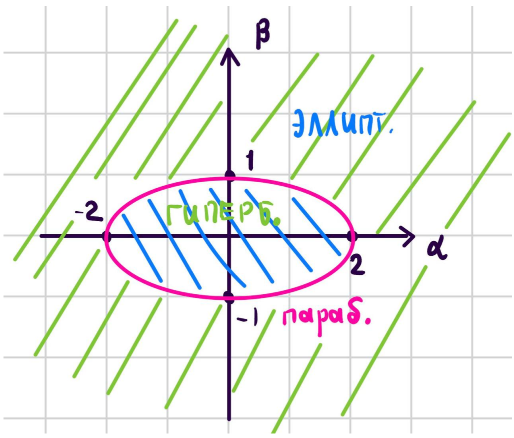
$xy(x^2+y^2-1)<0$

Параболлический, если

$xy(x^2+y^2-1)=0$

2

$u_{xx}+2\alpha u_{xz}+u_{yy}+4\beta u_{yz}+4u_{zz}=0.$



$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 2\beta \\ \alpha & 2\beta & 4 \end{pmatrix}$$

$\Delta_3=4-4\beta^2-\alpha^2$

Эллиптический, если

$4-4\beta^2-\alpha^2>0$

Гиперболический, если

$4-4\beta^2-\alpha^2<0$

Параболлический, если

$4-4\beta^2-\alpha^2=0$

2.1(2)

$4u_{xx} - 4u_{xy} - 2u_{yz} + u_y + u_z = 0$

$$\left(4\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} - 2\frac{\partial^2}{\partial y \partial z}\right)u + u_y + u_z = 0$$

$$\left(\left(\underbrace{2\frac{\partial}{\partial x} - \frac{\partial}{\partial y}}_{\frac{\partial}{\partial \xi}}\right)^2 - \left(\underbrace{\frac{\partial}{\partial y} + \frac{\partial}{\partial z}}_{\frac{\partial}{\partial \mu}}\right)^2 + \left(\underbrace{\frac{\partial}{\partial z}}_{\frac{\partial}{\partial \nu}}\right)^2\right)u + u_y + u_z = 0$$

$u_{\xi\xi} - u_{\mu\mu} + u_{\nu\nu} + u_\mu = 0$ гиперболический

$u = u(x(\xi, \mu\nu), y(\xi, \mu, \nu), z(\xi, \mu, \nu))$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\underbrace{\frac{\partial x}{\partial \xi}}_{2} + \frac{\partial u}{\partial y}\underbrace{\frac{\partial y}{\partial \xi}}_{-1} + \frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial \xi}}_{0}$$

$$\frac{\partial u}{\partial \mu} = \frac{\partial u}{\partial x}\underbrace{\frac{\partial x}{\partial \mu}}_{0} + \frac{\partial u}{\partial y}\underbrace{\frac{\partial y}{\partial \mu}}_{1} + \frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial \mu}}_{1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x}\underbrace{\frac{\partial x}{\partial \nu}}_{0} + \frac{\partial u}{\partial y}\underbrace{\frac{\partial y}{\partial \nu}}_{0} + \frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial \nu}}_{1}$$

Замена

$$\begin{cases} x = 2\xi \\ y = -\xi + \mu \\ z = \mu + \nu \end{cases}$$

Приведение к каноническому виду уравнений 1-го порядка в случае двух независимых переменных в области

2.11(6)

$u_{xy} + 2xyu_y - 2xu = 0$

$$u_y = v$$

$$v_x + 2xyv - 2xu = 0 \quad \text{Продиффиринцируем по } y$$

$$\frac{\partial}{\partial y}(v_x + 2xyv - 2xu) = 0 \;\Rightarrow\; v_{xy} + 2xyv_y = 0$$

$$v_y = \tau$$

$$\tau_x + 2xy\tau = 0 \quad \frac{d\tau}{dx} = -2xy\tau \Rightarrow \frac{d\tau}{\tau} = -2xydx$$

$$\tau = \exp(-x^2y)C(y) \qquad v = \int_0^y C(t)e^{-x^2t}\,dt + f(x)$$

$$u = \frac{1}{2x}(v_x + 2xyv)$$

$$\boxed{u = \frac{1}{2x}\left(\int_0^y C(t)(-2xt)e^{-x^2t}\,dt + f'(x) + 2xy\int_0^y C(t)e^{-x^2t}\,dt + f(x)\right)}$$

1

а) $y^3 u_{xy} - y u_{yy} - 3y^5 u_x + (2+3y^3)u_y = 0,\ y > 0,\ x \in \mathbb{R}^1;$
$u|_{y=1} = 1+3x, \quad u_y|_{y=1} = 3(4+3x),\ x \in \mathbb{R}^1.$

Ур-е характеристик

$$-y^3 dxdy - y(dx)^2 = 0 \quad dx(y^3 dy + ydx) = 0$$

$$\begin{cases} dx = 0 \\ y^3 dy + ydx = 0 \end{cases} \Leftrightarrow \begin{cases} x = C_1 \\ y(y'y^2 + 1) = 0 \end{cases} \Leftrightarrow \begin{cases} C_1 = x \\ C_2 = x + \frac{y^3}{3} \end{cases}$$

Делаем замену

$$\begin{cases} \xi = x \\ \mu = x + \frac{y^3}{3} \end{cases}$$

$$u_x = u_\xi + u_\mu$$

$$u_y = y^2 u_\mu$$

$$u_{yy} = y^4 u_{\mu\mu} + 2y u_\mu \quad u_{xy} = u_{\xi\mu} y^2 + u_{\mu\mu} y^2$$

Подставим

$$y^3(u_{\xi\mu} y^2 + u_{\mu\mu} y^2) - y(y^4 u_{\mu\mu} + 2y u_\mu) - 3y^5(u_\xi + u_\mu) + (2+3y^3)y^2 u_\mu = 0$$

$$y^5 u_{\xi\mu} - 3y^5 u_\xi = 0 \quad u_{\xi\mu} - 3u_\xi = 0$$

$$u_\xi = v$$

$$v_\mu - 3v = 0 \quad \frac{dv}{d\mu} = 3v \quad v = e^{3\mu} C(\xi)$$

$$u_\xi = e^{3\mu} C(\xi) \quad du = e^{3\mu} C(\xi) d\xi$$

$$u = \int C(\xi) e^{3\mu}\, d\xi + f(\mu) = e^{3\mu} g(\xi) + f(\mu)$$

$$u = e^{3x+y^3} g(x) + f(x + \frac{y^3}{3})$$

$$u|_{y=1} = e^{3x+1} g(x) + f(x + \frac{1}{3}) = 1 + 3x$$

$$u_y = 3y^2 e^{3x+y^3} g(x) + y^2 f'(x + \frac{y^3}{3})$$

$$u_y|_{y=1} = 3e^{3x+1} g(x) + f'(x + \frac{1}{3}) = 3(4 + 3x)$$

$$3 + 9x - 3f(x + \frac{1}{3}) = 3(4 + 3x) - f'(x + \frac{1}{3}),\ p = x + \frac{1}{3}$$

$$f'(p) = 3f(p) + 9 \qquad f(p) = e^{3p}C - 3 \qquad g(x) = \frac{4 + 3x - Ce^{3p}}{e^{3x+1}}$$

$$g(x) = (4 + 3x)e^{-3x-1} - C$$

$$u = e^{3x+y^3}((4 + 3x)e^{-3x-1} - C) + Ce^{3x+y^3} - 3 = 3x + 1$$

$$\boxed{u = e^{y^3-1}(4 + 3x) - 3}$$

б) $x^2 u_{xx} - 9y^2 u_{yy} + 3xu_x - 3yu_y = 0, \quad x > 1,\ y > 1$
$u|_{x=y} = y^{2/3}, u_x|_{x=y} = y^{-3} + y^{-1/3}, y > 1$

Ур-е характеристик

$$x^2(dy)^2 - 9y^2(dx)^2 = 0$$

$$(\frac{dy}{dx})^2 = (\frac{3y}{x})^2$$

$$\begin{cases} \frac{dy}{y} = \frac{3dx}{x} \\ \frac{dy}{y} = -\frac{3dx}{x} \end{cases} \Leftrightarrow \begin{cases} C_1 = \frac{y}{x^3} \\ C_2 = yx^3 \end{cases} \Leftrightarrow \begin{cases} \xi = \frac{y}{x^3} \\ \mu = yx^3 \end{cases}$$

$$u_x = -3u_\xi \frac{y}{x^4} + 3u_\mu yx^2,\ u_y = \frac{u_\xi}{x^3} + u_\mu x^3$$

$$u_{xx} = 9u_{\xi\xi} y^2 x^{-8} + 9u_{\mu\mu} y^2 x^4 - 18u_{\xi\mu} x^{-2} y^2 + 12u_\xi yx^{-5} + 6u_\mu yx$$

$$u_{yy} = u_{\xi\xi} x^{-6} + u_{\mu\mu} x^6 + 2u_{\xi\mu}$$

$$-36u_{\xi\mu}y^2 + u_\xi(12yx^{-3} - 9x^{-3}y - 3x^{-3}y) + u_\mu(6yx^3 + 9yx^3 - 3x^3y) = 0$$

$$-36u_{\xi\mu}y^2 + 12u_\mu yx^3 = 0$$

$$u_\mu = v$$

$$3v_\xi y = vx^3 \quad 3\frac{dv}{d\xi} = \xi^{-1}v \quad \frac{dv}{v} = \frac{d\xi}{3\xi}$$

$$v = \xi^{\frac{1}{3}}C(\mu)$$

$$u = \xi^{\frac{1}{3}}f(\mu) + g(\xi) \quad u = (x^{-3}y)^{\frac{1}{3}}f(yx^3) + g(yx^{-3})$$

$$u_x = -y^{1/3}x^{-2}f(yx^3) + 3xy^{4/3}f' - 3x^{-4}yg'$$

$$u|_{x=y} = y^{-\frac{2}{3}}f(y^4) + g(y^{-2}) = y^{2/3}$$

$$u_x|_{x=y} = -y^{-5/3}f(y^4) + 3y^{7/3}f'(y^4) - 3y^{-3}g'(y^{-2}) = y^{-3} + y^{-1/3}$$

$$\frac{2}{3}y^{-1/3} = -\frac{2}{3}y^{-5/3}f(y^4) + 4y^{7/3}f'(y^4) - 2y^{-3}g'(y^{-2})$$

$$-6y^{7/3}f'(y^4) + 3y^{7/3}f'(y^4) = y^{-3} \quad -3y^{7/3}f'(y^4) = y^{-3}$$

$$3\frac{df(\mu)}{dy} = -y^{-16/3} \; 3f'(\mu) = -\mu^{-4/3}$$

$$f(y^4) = (y^4)^{-1/3} + C$$

$$y^{-2/3}(y^{-4/3} + C) + g(y^{-2}) = y^{2/3} \quad g(y^{-2}) = y^{2/3} - Cy^{-2/3} - y^{-2}$$

$$f(\mu) = \mu^{-1/3} + C \quad g(\xi) = \xi^{-1/3} - C\xi^{1/3} - \xi$$

$$u = \xi^{1/3}(\mu^{-1/3} + C) + \xi^{-1/3} - C\xi^{1/3} - \xi$$

$$u = \xi^{1/3}\mu^{-1/3} + \xi^{-1/3} - \xi$$

$$\boxed{u = x^{-2} + y^{-1/3}x - yx^{-3}}$$

в) $x^2u_{xx} - xyu_{xy} - 2y^2u_{yy} + xu_x - 2yu_y = 9xy^2, \ x > 0, \ y > 0$
$u|_{x=1} = 3e^y, \quad u_x|_{x=1} = -y^2$

Ур-е хар-тик

$$x^2(dy)^2 + xydxdy - 2y^2(dx)^2 = 0$$

$$(\frac{dy}{dx})^2 + \frac{y}{x}\frac{dy}{dx} - 2(\frac{y}{x})^2 = 0$$

$$\begin{cases} \frac{dy}{dx} = -\frac{2y}{x} \\ \frac{dy}{dx} = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} C_1 = yx^2 \\ C_2 = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} \xi = yx^2 \\ \mu = \frac{y}{x} \end{cases}$$

$$u_x = 2u_\xi yx - u_\mu yx^{-2} \qquad u_y = u_\xi x^2 + u_\mu x^{-1}$$

$$u_{xx} = 4u_{\xi\xi}y^2x^2 + u_{\mu\mu}yx^{-4} - 4u_{\xi\mu}y^2x^{-1} + 2u_\xi y + 2u_\mu yx^{-3}$$

$$u_{yy} = u_{\xi\xi}x^4 + u_{\mu\mu}x^{-2} + 2u_{\xi\mu}x$$

$$u_{xy} = 2u_{\xi\xi}yx^3 - u_{\mu\mu}yx^{-3} + u_{\xi\mu}y + 2u_\xi x - u_\mu x^{-2}$$

После подстановки

$$u_{\xi\mu} = -1 \qquad u_\xi = v \qquad v_\mu = -1$$

$$v = -\mu + C(\xi) \qquad u = -\mu\xi + F(\xi) + G(\mu)$$

$$u = -y^2x + F(yx^2) + G(yx^{-1}) \qquad u_x = -y^2 + 2yxF'(yx^2) - yx^{-2}G'(yx^{-1})$$

$$u|_{x=1} = -y^2 + F(y) + G(y) = 3e^y \qquad u_x|_{x=1} = -y^2 + 2yF'(y) - yG'(y) = -y^2$$

$$\begin{cases} -y^2 + F(y) + G(y) = 3e^y \\ 2F'(y) = G'(y) \end{cases} \Rightarrow \begin{cases} -2y + F'(y) + G'(y) = 3e^y \\ 2F'(y) = G'(y) \end{cases} \Rightarrow$$

$$-2y + 3F'(y) = 3e^y \ \Rightarrow \ F(y) = e^y + \frac{y^2}{3} + C$$

$$G(y) = 3e^y + y^2 - e^y - \frac{y^2}{3} - C = 2e^y + \frac{2y^2}{3} - C$$

$$\boxed{u = -xy^2 + e^{yx^2} + \frac{y^2x^4}{3} + 2e^{yx^{-1}} + \frac{2y^2}{3x^2}}$$

г) $yu_{xx} + (x-y)u_{xy} - xu_{yy} - u_x + u_y = 0$
$u|_{y=0} = 2x^2, \quad u_y|_{y=0} = 2x, \quad 1 < x < 4$

Ур-е хар-тик

$$y(dy)^2 - (x-y)dxdy - x(dx)^2 = 0$$

$$\begin{cases} \frac{dy}{dx} = \frac{x}{y} \\ \frac{dy}{dx} = -1 \end{cases} \Leftrightarrow \begin{cases} y^2 = x^2 + C_1 \\ y = -x + C_2 \end{cases} \Leftrightarrow \begin{cases} \xi = y^2 - x^2 \\ \mu = y + x \end{cases}$$

$$u_x = -2xu_\xi + u_\mu \qquad u_y = 2yu_\xi + u_\mu$$

$$u_{xx} = 4x^2 u_{\xi\xi} + u_{\mu\mu} - 4xu_{\xi\mu} - 2u_\xi$$

$$u_{yy} = 4y^2 u_{\xi\xi} + u_{\mu\mu} + 4yu_{\xi\mu} + 2u_\xi$$

$$u_{xy} = -4yxu_{\xi\xi} + u_{\mu\mu} + 2(y-x)u_{\xi\mu}$$

После подстановки

$$2u_{\xi\mu}\mu^2 = 0 \quad u_\xi = v \quad v_\mu = 0$$

$$v = C(\xi) \qquad u = f(\xi) + g(\mu) \qquad u = f(y^2 - x^2) + g(y+x)$$

$$u_y = 2yf'(y^2 - x^2) + g'(y+x)$$

$$u|_{y=0} = f(-x^2) + g(x) = 2x^2 \qquad u_y|_{y=0} = g'(x) = 2x$$

$$g(x) = x^2 + C \qquad f(-x^2) + x^2 + C = 2x^2 \;\Rightarrow\; f(-x^2) = x^2 - C$$

$$u = (y+x)^2 + x^2 - y^2$$

$$\boxed{u = 2x^2 + 2xy}$$

д) $y^4 u_{yy} + y^2 u_{xy} - 2u_{xx} + 2y^3 u_y = 0$
$u|_{y=1} = x^2 + 5, \quad u_y|_{y=1} = 2x - 6,\ 1 < x < 2$

Ур-е хар-тик

$$y^4(dx)^2 - y^2 dxdy - 2(dy)^2 = 0$$

$$\begin{cases} y' = -y^2 \\ y' = \frac{1}{2}y^2 \end{cases} \Leftrightarrow \begin{cases} C_1 = x - \frac{1}{y} \\ C_2 = x + \frac{2}{y} \end{cases} \Rightarrow \begin{cases} \xi = x - \frac{1}{y} \\ \mu = x + \frac{2}{y} \end{cases} \quad u_x = u_\xi + u_\mu \qquad u_y = \frac{u_\xi}{y^2} - \frac{2}{y}$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\mu} + u_{\mu\mu} \qquad u_{yy} = \frac{u_{\xi\xi}}{y^4} + \frac{4u_{\mu\mu}}{y^4} - \frac{4u_{\xi\mu}}{y^4} - 2\frac{u_\xi}{y^3} + 4\frac{u_\mu}{y^3}$$

$$u_{xy} = \frac{u_{\xi\xi}}{y^2} - 2\frac{u_{\mu\mu}}{y^2} - \frac{u_{\xi\mu}}{y^2}$$

После подстановки

$$u_{\xi\mu} = 0$$

$$u = f(\xi) + g(\mu) \quad u = f(x - \frac{1}{y}) + g(x + \frac{2}{y})$$

$$u_y = y^{-2} f'(x - \frac{1}{y}) - 2y^{-2} g'(x + \frac{2}{y})$$

$$u|_{y=1} = f(x-1) + g(x+2) = x^2 + 5$$

$$u_y|_{y=1} = f'(x-1) - 2g'(x+2) = 2x - 6$$

$$\begin{cases} f(x-1) + g(x+2) = x^2 + 5 \\ f'(x-1) - 2g'(x+2) = 2x - 6 \end{cases} \Rightarrow \begin{cases} f(x-1) = x^2 + 5 - g(x+2) \\ 2x - g'(x+2) - 2g'(x) = 2x - 6 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(x-1) = x^2 + 5 - g(x+2) \\ g'(x+2) = 2 \end{cases} \Rightarrow \begin{cases} f(x-1) = x^2 - 2x + 1 - C \\ g(x+2) = 2x + 4 + C \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(x-1) = (x-1)^2 - C \\ g(x+2) = 2(x+2) + C \end{cases} \Rightarrow \begin{cases} f(a) = a^2 - C \\ g(b) = 2b + C \end{cases}$$

$$\boxed{u = (x - \frac{1}{y})^2 + 2(x + \frac{2}{y})}$$

2

$x^2u_{xx} - 4y^2u_{yy} + xu_x - 4yu_y = 0, \quad \frac{1}{x^2} < y < x^2,\ x > 0$
$u|_{y=\frac{1}{x^2}} = 1 + 2x^4,\ u|_{y=x^2} = 2 + x^4$

Ур-е характеристик

$$x^2(dy)^2 - 4y^2(dx)^2$$

$$\begin{cases} \frac{dx}{x} = \frac{dy}{2y} \\ \frac{dx}{x} = -\frac{dy}{2y} \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{y}C_1 \\ x = \frac{1}{\sqrt{y}}C_2 \end{cases}$$

Замена

$$\begin{cases} \mu = \sqrt{y}x \\ \xi = \frac{x}{\sqrt{y}} \end{cases}$$

$$u_x = u_\xi \frac{1}{\sqrt{y}} + u_\mu \sqrt{y}, \quad u_y = -\frac{x}{2\sqrt{y^3}}u_\xi + u_\mu \frac{x}{2\sqrt{y}}$$

$$u_{xx} = \frac{u_{\xi\xi}}{y} + u_{\mu\mu}y + 2u_{\xi\mu}$$

$$u_{yy} = u_{\xi\xi} \cdot \frac{x^2}{4y^3} + u_{\mu\mu} \cdot \frac{x^2}{4y} - u_{\xi\mu} \cdot \frac{x^2}{2y^2} - u_\mu \frac{x}{4\sqrt{y^3}} + u_\xi \frac{3x}{4\sqrt{y^5}}$$

Подставим

$$x^2(\frac{u_{\xi\xi}}{y} + u_{\mu\mu}y + 2u_{\xi\mu}) - 4y^2(u_{\xi\xi} \cdot \frac{1}{4y^3} + u_{\mu\mu} \cdot \frac{x^2}{4y} - u_{\xi\mu} \cdot \frac{x}{2y^2} - u_\mu \frac{x}{4\sqrt{y^3}} + u_\xi \frac{3}{4\sqrt{y^5}})+$$

$$+x(u_\xi \frac{1}{\sqrt{y}} + u_\mu \sqrt{y}) - 4y(-\frac{1}{2\sqrt{y^3}}u_\xi + u_\mu \frac{x}{2\sqrt{y}}) = 0$$

$$4x^2u_{\xi\mu} = 0 \ \Rightarrow \ u_{\xi\mu} = 0$$

$$u = f(\xi) + g(\mu) \qquad u = f(\frac{x}{\sqrt{y}}) + g(\sqrt{y}x)$$

$$u|_{y=x^{-2}} = f(x^2) + g(1) = 1 + 2x^4 \qquad u|_{y=x^2} = f(1) + g(x^2) = 2 + x^4$$

$$\begin{cases} f(x^2) + g(1) = 1 + 2x^4 \\ f(1) + g(x^2) = 2 + x^4 \end{cases} \Rightarrow \begin{cases} f(x^2) = 1 + 2x^4 - g(1) \\ g(x^2) = x^4 - 1 + g(1) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(a) = 2a^2 + C \\ g(b) = b^2 - C \end{cases}$$

$$\boxed{u = 2\frac{x^2}{y} + yx^2}$$

3

$u_{yy} - u_{xx} = 0$
$u|_{y=0} = u_0(x),\ u_y|_{y=0} = u_1(x),\ 0 < x < 1,\ u_0(x) \in C^2(0;1), u_1(x) \in C_1(0;1)$

максимальная область, где $\exists!$ реш

$$\begin{cases} 0 < x + y < 1 \\ 0 < x - y < 1 \end{cases}$$

Волновое уравнение

1

$$\begin{cases} 4u_{tt} = u_{xx} + 4t^2\cos(2x) \\ u|_{t=0} = e^x,\ u_t|_{t=0} = x^2 \end{cases}$$

$$u_{\text{частн}} = f(t)\cos(2x)$$

$$4f''\cos(2x) = -4f\cos(2x) + 4t^2\cos(2x)$$

$$f'' = -f + t^2$$

$$f = \alpha t^2 + \beta t + \gamma \quad 2\alpha = -\alpha t^2 - \beta t - \gamma + t^2$$

$$\alpha = 1,\ \beta = 0,\ \gamma = -2$$

$$u_{\text{частн}} = (t^2 - 2)\cos(2x)$$

$$u = (t^2 - 2)\cos(2x) + v(x,t)$$

$$\begin{cases} 4v_{tt} = v_{xx} \\ v|_{t=0} = e^x + 2\cos(2x) \\ v_t|_{t=0} = x^2 \end{cases}$$

$$\boxed{v(x,t) = \frac{1}{2}(e^{x+\frac{t}{2}} + 2\cos(2x+t) + e^{x-\frac{t}{2}} + 2\cos(2x-t)) + \int_{x-\frac{t}{2}}^{x+\frac{t}{2}} \xi^2\, d\xi}$$

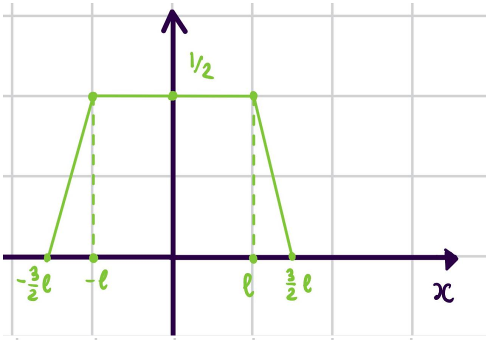
2

$u_{tt} = a^2 u_{xx}, \quad x \in \mathbb{R}^1, \quad t > 0$

а) $u|_{t=0} = \varphi(x),\ u_t|_{t=0} = 0;$

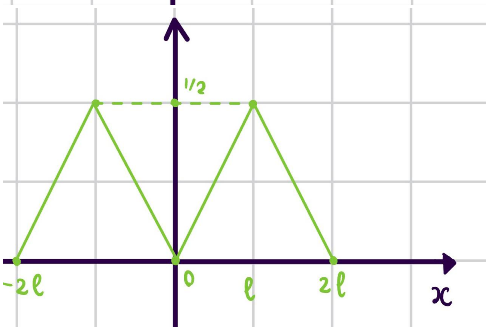
Ф-ла Даламбера

$$u(x,t) = \frac{\varphi(x+at) - \varphi(x-at)}{2}$$



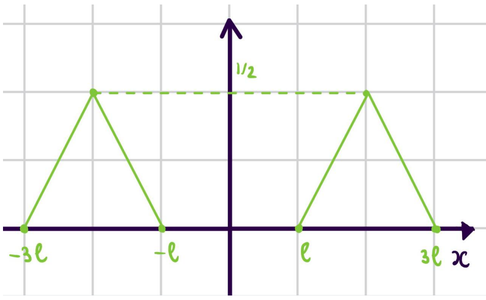
$$t = \frac{l}{2a}$$

$$u(x,t) = \frac{\varphi(x+\frac{l}{2}) - \varphi(x-\frac{l}{2})}{2}$$



$$t = \frac{l}{a}$$

$$u(x,t) = \frac{\varphi(x+l) - \varphi(x-l)}{2}$$



$$t = \frac{2l}{a}$$

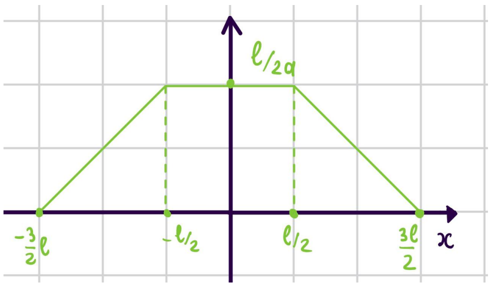
$$u(x,t) = \frac{\varphi(x+2l) - \varphi(x-l)}{2}$$

б) $u|_{t=0} = 0,\ u_t|_{t=0} = \psi(x) = \theta(l - |x|)$

$$l = \text{const} > 0 \quad \theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Общее решение

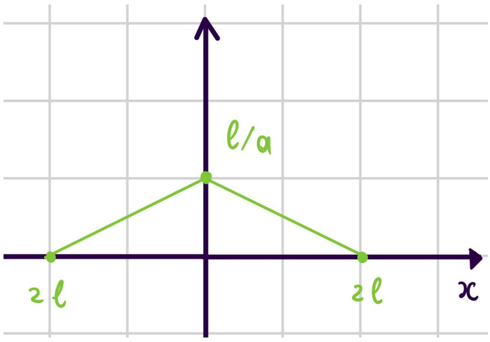
$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,d\xi$$



$$t = \frac{l}{2a}$$

$$u(x,t) = \frac{1}{2a}\int_{x-\frac{l}{2}}^{x+\frac{l}{2}} \theta(l - |\xi|)\, d\xi$$

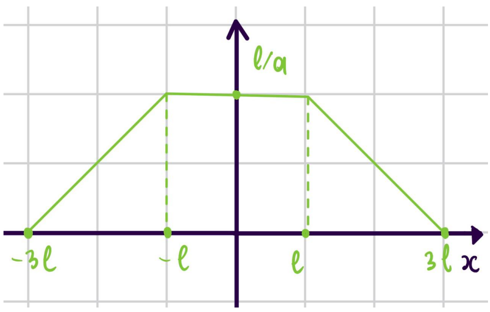
$$= \frac{1}{2a}\text{length}\left([x - \frac{l}{2}, x + \frac{l}{2}] \cap [-l, l]\right)$$



$$t = \frac{l}{a}$$

$$u(x,t) = \frac{1}{2a}\int_{x-l}^{x+l} \theta(l - |\xi|)\, d\xi$$

$$= \frac{1}{2a}\text{length}\left([x - l, x + l] \cap [-l, l]\right)$$



$$t = \frac{2l}{a}$$

$$u(x,t) = \frac{1}{2a}\int_{x-2l}^{x+2l} \theta(l - |\xi|)\, d\xi$$

$$= \frac{1}{2a}\text{length}\left([x - 2l, x + 2l] \cap [-l, l]\right)$$

3

a) 21.19
$u_{tt} = 3u_{xx} + 2(1 - 6t^2)e^{-2x},\ t > 0,\ x > 0$
$u|_{t=0} = 1,\ u_t|_{t=0} = x,\ (u_x - 2u)|_{x=0} = -2 + t - 4t^2$

Ищем частное решение

$$u_{\text{частн}} = e^{-2x}(At^2 + Bt + C)$$

$$2Ae^{-2x} = 12e^{-2x}(At^2 + Bt + C) + 2(1 - 6t^2)e^{-2x}$$

$$2A = 12At^2 + 12Bt + 12C + 2 - 12t^2$$

$$\begin{cases} A = 1, \\ B = 0, \\ C = 0; \end{cases}$$

$$u_{\text{частн}} = t^2 e^{-2x}$$

Сводим ур-е к однородному

$$u = t^2 e^{-2x} + v(x,t)$$

$$v_{tt} = 3v_{xx}$$

$$v|_{t=0} = 1, \quad v_t|_{t=0} = x, \quad (v_x - 2v)|_{x=0} = -2 + t$$

$$v = f(x + \sqrt{3}t) + g(x - \sqrt{3}t)$$

Ищем решение в области $x \geq \sqrt{3}t$

$$v|_{t=0} = f(x) + g(x) = 1 \qquad v_t|_{t=0} = \sqrt{3}f'(x) - \sqrt{3}g'(x) = x$$

$$f'(x) + g'(x) = 0$$

$$2\sqrt{3}f'(x) = x$$

$$\boxed{f(x) = \frac{x^2}{4\sqrt{3}} + C, \quad g(x) = 1 - \frac{x^2}{4\sqrt{3}} - C}$$

$$v(x,t) = 1 - \frac{(x+\sqrt{3}t)^2 - (x-\sqrt{3}t)^2}{4\sqrt{3}} = 1 - xt$$

Ищем решение в области $x < \sqrt{3}t$

$$(v_x - 2v)|_{x=0} = f'(\sqrt{3}t) + g'(-\sqrt{3}t) - 2f(\sqrt{3}t) - 2g(-\sqrt{3}t) - 2C = -2 + t$$

Заметим, что уравнение полученное в области $x \geq \sqrt{3}t$ также подходит

Следовательно, решения на полученных областях совпадают

$$\boxed{u(x,t) = 1 - xt + t^2 e^{-2x}}$$

21.2

$u_{tt} - a^2 u_{xx} = 0$

$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x)$

$u_x|_{x=0} = 0$

$$u(x,t) = f(x+at) + g(x-at)$$

$$x \geq at$$

$$f(x) + g(x) = u_0(x), \quad u_t|_{t=0} = af'(x) - ag'(x) = u_1(x)$$

$$f(x) - g(x) = \frac{1}{a}\int_0^x u_1(\xi)\,d\xi + A$$

$$f(x) = \frac{u_0(x)}{2} + \frac{1}{2a}\int_0^x u_1(\xi)\,d\xi + \frac{A}{2} \qquad g(x) = \frac{u_0}{2} - \frac{1}{2a}\int_0^x u_1(\xi)\,d\xi - \frac{A}{2}$$

$$\boxed{u(x,t) = \frac{u_0(x+at) + u_0(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} u_1(\xi)\,d\xi}$$

$$x < at$$

$$u_x|_{x=0} = f'(at) - g'(-at) = 0 \Rightarrow f(at) + g(-at) = C$$

$$g(-at) = C - \frac{u_0(at)}{2} = \frac{1}{2a}\int_0^{at} u_1(\xi)\,d\xi - \frac{A}{2}$$

$$g(b) = C - \frac{u_0(-b)}{2} = \frac{1}{2a}\int_0^{-b} u_1(\xi)\,d\xi - \frac{A}{2}$$

Сшивка

$$g(+0) = g(-0)$$

$$C - \frac{A}{2} = u_0 - \frac{A}{2} \Rightarrow C = u_0(0) = 0$$

Показали, что

$$\boxed{u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} u_1(\xi)\,d\xi + \begin{cases} \frac{u_0(x+at)+u_0(x-at)}{2}, & x \geq at \\ \frac{u_0(x+at)-u_0(x-at)}{2}, & x < at \end{cases}}$$

б) $u_{tt} = u_{xx} + xe^t, \quad x > 0, t > 0$
$u|_{t=0} = 1 + x, \quad u_t|_{t=0} = 4 - 5x, x \geq 0$
$(2u + u_x)|_{x=0} = (1+t)e^t + 2 + t - 3t^2, \quad t \geq 0$

Ищем частные решения

$$u_{\text{частн}} = xe^t$$

Сводим ур-е к однородному

$$u = xe^t + v(x,t)$$

$$v_{tt} = v_{xx} \qquad v|_{t=0} = (u - xe^t)|_{t=0} = 1 + x - x = 1$$

$$v_t|_{t=0} = (u_t - xe^t)|_{t=0} = 4 - 5x - x = 4 - 6x$$

$$(2v + v_x)|_{x=0} = (2u + u_x - 2xe^t - e^t)|_{x=0} = te^t + 2 + t - 3t^2,\ t \geq 0$$

$$v = f(x+t) + g(x-t)$$

Ищем решение в области $x \geq t$

$$v|_{t=0} = f(x) + g(x) = 1 \qquad v_t|_{t=0} = f'(x) - g'(x) = 4 - 6x,\ x \geq 0$$

$$f'(x) + g'(x) = 0, \qquad \text{складываем}$$

$$2f'(x) = 4 - 6x \qquad 2f'(x) = 4 - 6x \qquad f'(x) = 2 - 3x$$

$$\boxed{f(x) = 2x - \frac{3x^2}{2} + A; \qquad g(x) = 1 - 2x + \frac{3x^2}{2} - A}$$

$$v = 2(x+t) - \frac{3}{2}(x+t)^2 + A + 1 - 2(x-t) + \frac{3}{2}(x-t)^2 - A,\ x \geq t$$

Ищем решения при $x < t$

$$(2v + v_x)|_{x=0} = 2f(t) + 2g(-t) + f'(t) + g'(-t) =$$

$$= te^t + 2 + t - 3t^2,\ t \geq 0$$

$$4t - 3t^2 + 2A + 2g(-t) + 2 - 3t + g'(-t) = te^t + 2 + t - 3t^2$$

$$g'(-t) + 2g(-t) = te^t - 2A \qquad -t = p < 0$$

$$g'(p) + 2g(p) = -pe^t - 2A$$

$$g_{\text{частн}_1} = (\alpha p + \beta)e^{-p}$$

$$\alpha e^{-p} - (\alpha p + \beta)e^{-p} + 2(\alpha p + \beta)e^{-p} = -pe^{-p}$$

$$\alpha + 2\alpha = -1 \qquad \alpha = -1 \qquad \alpha - \beta + 2\beta = 0$$

$$\beta = -\alpha = 1$$

$$g_{\text{частн}_1} = (1-p)e^{-p} \qquad g_{\text{частн}_2} = -A$$

$$g(p) = ce^{-2p} + (1-p)e^{-p} - A, \quad p < 0$$

Сшивка(склейка)

$$g(+0) = g(-0)$$

$$1 - A = C + 1 - A \Rightarrow C = 0$$

Ответ

$$\boxed{u(x,t) = xe^t + 2(x+t) - \frac{3}{2}(x+t)^2 + \begin{cases} 1 - 2(x-t) + \frac{3}{2}(x-t)^2, & x \geq t \\ (1-(x-t))e^{-(x-t)}, & x < t \end{cases}}$$

в) $4u_{tt} = u_{xx} - 3\sin(x+t),\ x > 0,\ t > 0$
$u|_{t=0} = \sin(x) + \sin(2x), \quad u_t|_{t=0} = 2 + \cos(x),\ x \geq 0$
$(u_x - 2u)|_{x=0} = 2 - 4t - 2\sin(t) + \cos(t),\ t \geq 0$

Ищем частное решение

$$u_{\text{частн}} = \sin(x+t)$$

Сводим уравнение к однородному

$$u = \sin(x+t) + v(x,t)$$

$$v_{tt} = \left(\frac{1}{2}\right)^2 v_{xx}, \quad v|_{t=0} = \sin(2x), \quad v_t|_{t=0} = 2$$

$$(v_x - 2v)|_{x=0} = 2 - 4t,\ t \geq 0$$

$$v = f(x + \frac{t}{2}) + g(x - \frac{t}{2})$$

$$\text{Ищем решение в области } x \geq \frac{t}{2}$$

$$v|_{t=0} = f(x) + g(x) = \sin(2x)$$

$$v_t|_{t=0} = \frac{1}{2}f'(x) - \frac{1}{2}g'(x) = 2$$

$$f'(x) + g'(x) = 2\cos(2x) \qquad f'(x) = \cos(2x) + 2$$

$$\boxed{f(x) = \frac{1}{2}\sin(2x) + 2x + C, \quad g(x) = \frac{1}{2}\sin(2x) - 2x - C}$$

$$v(x,t) = \frac{1}{2}\sin(2x+t) + \frac{1}{2}\sin(2x-t),\ x \geq \frac{t}{2}$$

$$\text{Область } x < \frac{t}{2}$$

$$(v_x - 2v)\,|_{x=0} = f'\left(\frac{t}{2}\right) + g'\left(-\frac{t}{2}\right) - 2f\left(\frac{t}{2}\right) - 2g\left(-\frac{t}{2}\right) = 2 - 4t$$

$$\cos(t) + 2 - 2\left(\frac{1}{2}\sin(t) + t + C\right) + g'\left(-\frac{t}{2}\right) + g\left(-\frac{t}{2}\right) = 2 - 4t$$

$$g'(b) + g(b) + \sin(2b) + \cos(2b) - 4b - 2C = 0, \quad b = -\frac{t}{2}$$

Решая дифф. ур-е

$$g(b) = Ae^{2b} - 2b - 1 + \frac{1}{2}\cos(2b) - C$$

Сшивка

$$g(+0) = g(-0)$$

$$A - 1 + \frac{1}{2} - C = -C \Rightarrow A = \frac{1}{2}$$

$$\boxed{u = \sin(x+t) + \frac{1}{2}\sin(2x+t) + \begin{cases} \frac{1}{2}\sin(2x-t),\ x \geq \frac{t}{2} \\ \frac{1}{2}\cos(2x-t) + 2t - 1 + \frac{1}{2}e^{2x-1},\ x < \frac{t}{2} \end{cases}}$$

г) $u_{tt}+u_{xt}-2u_{xx}=0, \quad x>0,\ t>0$
$u|_{t=0}=\operatorname{sh}x+\operatorname{arctg}x,\ u_t|_{t=0}=\operatorname{ch}x-\frac{2}{1+x^2},\ x\geq 0$
$u_x|_{x=0}=\operatorname{ch}x+1,\ t\geq 0$

$$(dx)^2-dxdt-2(dt)^2=0 \quad (dx-2dt)(dx+dt)=0$$

$$\begin{cases} x=2t+C_1 \\ x=-t+C_2 \end{cases} \Leftrightarrow \begin{cases} \xi=x-2t \\ \mu=x+t \end{cases}$$

$$u_x=u_\xi+u_\mu \quad u_t=-2u_\xi+u_\mu$$

$$u_{xx}=u_{\xi\xi}+u_{\mu\mu}+2u_{\xi\mu}$$

$$u_{tt}=4u_{\xi\xi}+u_{\mu\mu}-4u_{\xi\mu}$$

$$u_{xt}=-2u_{\xi\xi}+u_{\mu\mu}-u_{\xi\mu}$$

Подставляем

$$u_{\xi\mu}=0$$

$$u=f(\xi)+g(\mu)=f(x-2t)+g(x+t)$$

$$u|_{t=0}=f(x)+g(x)=\operatorname{sh}x+\operatorname{arctg}x\Rightarrow f'(x)+g(x)+\operatorname{ch}x=\frac{1}{1+x^2}$$

$$u_t'|_{t=0}=-2f'(x)+g'(x)=\operatorname{ch}x-\frac{2}{1+x^2}$$

$$u_x|_{x=0}=-2tf'(t)+g'(t)=\operatorname{ch}t+1$$

$$f'(-2t)+\operatorname{ch}t=\operatorname{ch}t+1$$

$$-2t=z \quad f'(z)=1 \quad f(z)=z+C,\ z<0$$

$$f(+0)=f(-0)\Rightarrow C=A$$

$$\boxed{u(x,t)=\operatorname{sh}(x+t)+\begin{cases} \operatorname{arctg}(x-2t), & x-2t\geq 0 \\ x-2t, & x-2t<0 \end{cases}}$$