УРМФ

Vlad Leonov

5 декабря 2025 г.

Содержание

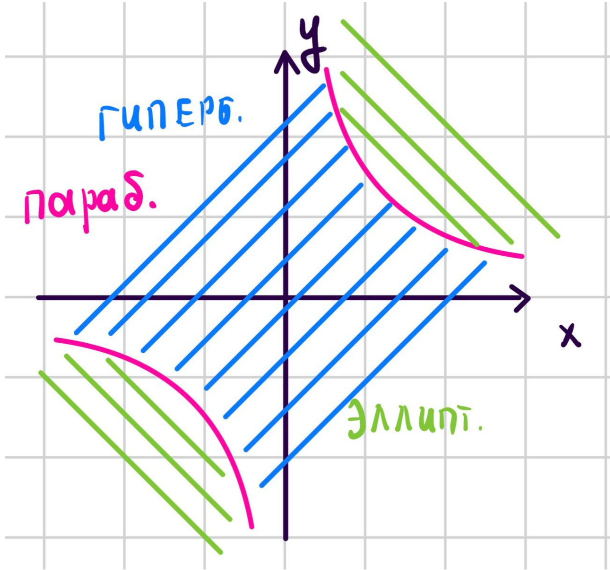
Часть I

Задание 1

Классификация уравнений 2-го порядка, характеристики

1

a) $yu_{xx} + 2u_{xy} + xu_{yy} - u_y = 5x$;



$$\begin{pmatrix} y & 1 \\ 1 & x \end{pmatrix}$$

$\Delta_1 = y, \Delta_2 = xy - 1$

Эллиптический, если

$$\begin{cases} y > 0 \\ xy - 1 > 0 \end{cases} \quad \text{либо} \begin{cases} y < 0 \\ xy - 1 > 0 \end{cases}$$

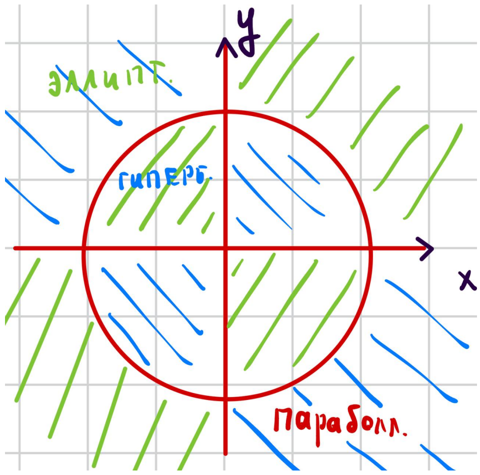
Гиперболический, если

$$\begin{cases} y < 0 \\ xy - 1 < 0 \end{cases} \quad \text{либо} \begin{cases} y > 0 \\ xy - 1 < 0 \end{cases}$$

Параболлический, если

$xy - 1 = 0$

б) $(x^2+y^2-1)u_{xx}+xyu_{yy}-u_x=0.$



$$\begin{pmatrix} x^2+y^2-1 & 0 \\ 0 & xy \end{pmatrix}$$

Эллиптический, если

$xy(x^2+y^2-1)>0$

Гиперболический, если

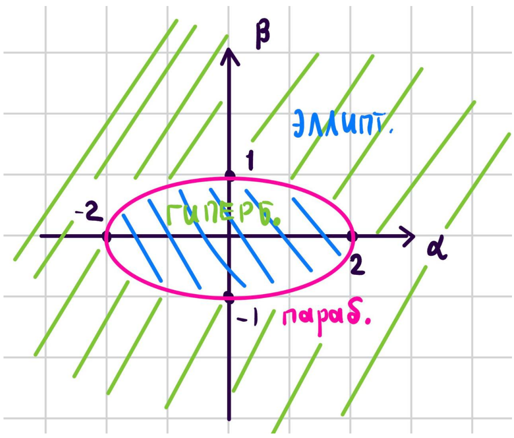
$xy(x^2+y^2-1)<0$

Параболлический, если

$xy(x^2+y^2-1)=0$

2

$u_{xx}+2\alpha u_{xz}+u_{yy}+4\beta u_{yz}+4u_{zz}=0.$



$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 2\beta \\ \alpha & 2\beta & 4 \end{pmatrix}$$

$\Delta_3=4-4\beta^2-\alpha^2$

Эллиптический, если

$4-4\beta^2-\alpha^2>0$

Гиперболический, если

$4-4\beta^2-\alpha^2<0$

Параболлический, если

$4-4\beta^2-\alpha^2=0$

2.1(2)

$4u_{xx} - 4u_{xy} - 2u_{yz} + u_y + u_z = 0$

$$\left(4\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} - 2\frac{\partial^2}{\partial y \partial z}\right) u + u_y + u_z = 0$$

$$\left(\left(\underbrace{2\frac{\partial}{\partial x} - \frac{\partial}{\partial y}}_{\frac{\partial}{\partial \xi}}\right)^2 - \left(\underbrace{\frac{\partial}{\partial y} + \frac{\partial}{\partial z}}_{\frac{\partial}{\partial \mu}}\right)^2 + \left(\underbrace{\frac{\partial}{\partial z}}_{\frac{\partial}{\partial \nu}}\right)^2\right) u + u_y + u_z = 0$$

$u_{\xi\xi} - u_{\mu\mu} + u_{\nu\nu} + u_\mu = 0$ гиперболический

$u = u(x(\xi, \mu\nu), y(\xi, \mu, \nu), z(\xi, \mu, \nu))$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \underbrace{\frac{\partial x}{\partial \xi}}_{2} + \frac{\partial u}{\partial y} \underbrace{\frac{\partial y}{\partial \xi}}_{-1} + \frac{\partial u}{\partial z} \underbrace{\frac{\partial z}{\partial \xi}}_{0}$$

$$\frac{\partial u}{\partial \mu} = \frac{\partial u}{\partial x} \underbrace{\frac{\partial x}{\partial \mu}}_{0} + \frac{\partial u}{\partial y} \underbrace{\frac{\partial y}{\partial \mu}}_{1} + \frac{\partial u}{\partial z} \underbrace{\frac{\partial z}{\partial \mu}}_{1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x} \underbrace{\frac{\partial x}{\partial \nu}}_{0} + \frac{\partial u}{\partial y} \underbrace{\frac{\partial y}{\partial \nu}}_{0} + \frac{\partial u}{\partial z} \underbrace{\frac{\partial z}{\partial \nu}}_{1}$$

Замена

$$\begin{cases} x = 2\xi \\ y = -\xi + \mu \\ z = \mu + \nu \end{cases}$$

Приведение к каноническому виду уравнений 1-го порядка в случае двух независимых переменных в области

2.11(6)

$u_{xy} + 2xyu_y - 2xu = 0$

$$u_y = v$$

$$v_x + 2xyv - 2xu = 0 \quad \text{Продиффиринцируем по } y$$

$$\frac{\partial}{\partial y}(v_x + 2xyv - 2xu) = 0 \;\Rightarrow\; v_{xy} + 2xyv_y = 0$$

$$v_y = \tau$$

$$\tau_x + 2xy\tau = 0 \quad \frac{d\tau}{dx} = -2xy\tau \Rightarrow \frac{d\tau}{\tau} = -2xydx$$

$$\tau = \exp(-x^2y)C(y) \qquad v = \int_0^y C(t)e^{-x^2t}\,dt + f(x)$$

$$u = \frac{1}{2x}(v_x + 2xyv)$$

$$\boxed{u = \frac{1}{2x}\left(\int_0^y C(t)(-2xt)e^{-x^2t}\,dt + f'(x) + 2xy\int_0^y C(t)e^{-x^2t}\,dt + f(x)\right)}$$

1

a) $y^3u_{xy} - yu_{yy} - 3y^5u_x + (2+3y^3)u_y = 0,\ y>0,\ x\in\mathbb{R}^1;$
$u|_{y=1} = 1+3x, \quad u_y|_{y=1} = 3(4+3x),\ x\in\mathbb{R}^1.$

Ур-е характеристик

$$-y^3dxdy - y(dx)^2 = 0 \quad dx(y^3dy + ydx) = 0$$

$$\begin{cases} dx = 0 \\ y^3dy + ydx = 0 \end{cases} \Leftrightarrow \begin{cases} x = C_1 \\ y(y'y^2+1) = 0 \end{cases} \Leftrightarrow \begin{cases} C_1 = x \\ C_2 = x + \frac{y^3}{3} \end{cases}$$

Делаем замену

$$\begin{cases} \xi = x \\ \mu = x + \frac{y^3}{3} \end{cases}$$

$$u_x = u_\xi + u_\mu$$

$$u_y = y^2u_\mu$$

$$u_{yy} = y^4u_{\mu\mu} + 2yu_\mu \quad u_{xy} = u_{\xi\mu}y^2 + u_{\mu\mu}y^2$$

Подставим

$$y^3(u_{\xi\mu}y^2 + u_{\mu\mu}y^2) - y(y^4u_{\mu\mu} + 2yu_\mu) - 3y^5(u_\xi + u_\mu) + (2+3y^3)y^2u_\mu = 0$$

$$y^5u_{\xi\mu} - 3y^5u_\xi = 0 \quad u_{\xi\mu} - 3u_\xi = 0$$

$$u_\xi = v$$

$$v_\mu - 3v = 0 \quad \frac{dv}{d\mu} = 3v \quad v = e^{3\mu}C(\xi)$$

$$u_\xi = e^{3\mu}C(\xi) \quad du = e^{3\mu}C(\xi)d\xi$$

$$u = \int C(\xi)e^{3\mu}\,d\xi + f(\mu) = e^{3\mu}g(\xi) + f(\mu)$$

$$u = e^{3x+y^3} g(x) + f(x + \frac{y^3}{3})$$

$$u|_{y=1} = e^{3x+1} g(x) + f(x + \frac{1}{3}) = 1 + 3x$$

$$u_y = 3y^2 e^{3x+y^3} g(x) + y^2 f'(x + \frac{y^3}{3})$$

$$u_y|_{y=1} = 3e^{3x+1} g(x) + f'(x + \frac{1}{3}) = 3(4 + 3x)$$

$$3 + 9x - 3f(x + \frac{1}{3}) = 3(4 + 3x) - f'(x + \frac{1}{3}),\ p = x + \frac{1}{3}$$

$$f'(p) = 3f(p) + 9 \qquad f(p) = e^{3p} C - 3 \qquad g(x) = \frac{4 + 3x - Ce^{3p}}{e^{3x+1}}$$

$$g(x) = (4 + 3x) e^{-3x-1} - C$$

$$u = e^{3x+y^3}((4 + 3x)e^{-3x-1} - C) + Ce^{3x+y^3} - 3 = 3x + 1$$

$$\boxed{u = e^{y^3-1}(4 + 3x) - 3}$$

б) $x^2 u_{xx} - 9y^2 u_{yy} + 3xu_x - 3yu_y = 0, \quad x > 1,\ y > 1$
$u|_{x=y} = y^{2/3}, u_x|_{x=y} = y^{-3} + y^{-1/3}, y > 1$

Ур-е характеристик

$$x^2(dy)^2 - 9y^2(dx)^2 = 0$$

$$(\frac{dy}{dx})^2 = (\frac{3y}{x})^2$$

$$\begin{cases} \frac{dy}{y} = \frac{3dx}{x} \\ \frac{dy}{y} = -\frac{3dx}{x} \end{cases} \Leftrightarrow \begin{cases} C_1 = \frac{y}{x^3} \\ C_2 = yx^3 \end{cases} \Leftrightarrow \begin{cases} \xi = \frac{y}{x^3} \\ \mu = yx^3 \end{cases}$$

$$u_x = -3u_\xi \frac{y}{x^4} + 3u_\mu yx^2,\ u_y = \frac{u_\xi}{x^3} + u_\mu x^3$$

$$u_{xx} = 9u_{\xi\xi} y^2 x^{-8} + 9u_{\mu\mu} y^2 x^4 - 18u_{\xi\mu} x^{-2} y^2 + 12u_\xi yx^{-5} + 6u_\mu yx$$

$$u_{yy} = u_{\xi\xi} x^{-6} + u_{\mu\mu} x^6 + 2u_{\xi\mu}$$

$$-36u_{\xi\mu}y^2 + u_\xi(12yx^{-3} - 9x^{-3}y - 3x^{-3}y) + u_\mu(6yx^3 + 9yx^3 - 3x^3y) = 0$$

$$-36u_{\xi\mu}y^2 + 12u_\mu yx^3 = 0$$

$$u_\mu = v$$

$$3v_\xi y = vx^3 \quad 3\frac{dv}{d\xi} = \xi^{-1}v \quad \frac{dv}{v} = \frac{d\xi}{3\xi}$$

$$v = \xi^{\frac{1}{3}}C(\mu)$$

$$u = \xi^{\frac{1}{3}}f(\mu) + g(\xi) \quad u = (x^{-3}y)^{\frac{1}{3}}f(yx^3) + g(yx^{-3})$$

$$u_x = -y^{1/3}x^{-2}f(yx^3) + 3xy^{4/3}f' - 3x^{-4}yg'$$

$$u|_{x=y} = y^{-\frac{2}{3}}f(y^4) + g(y^{-2}) = y^{2/3}$$

$$u_x|_{x=y} = -y^{-5/3}f(y^4) + 3y^{7/3}f'(y^4) - 3y^{-3}g'(y^{-2}) = y^{-3} + y^{-1/3}$$

$$\frac{2}{3}y^{-1/3} = -\frac{2}{3}y^{-5/3}f(y^4) + 4y^{7/3}f'(y^4) - 2y^{-3}g'(y^{-2})$$

$$-6y^{7/3}f'(y^4) + 3y^{7/3}f'(y^4) = y^{-3} \quad -3y^{7/3}f'(y^4) = y^{-3}$$

$$3\frac{df(\mu)}{dy} = -y^{-16/3} \; 3f'(\mu) = -\mu^{-4/3}$$

$$f(y^4) = (y^4)^{-1/3} + C$$

$$y^{-2/3}(y^{-4/3} + C) + g(y^{-2}) = y^{2/3} \quad g(y^{-2}) = y^{2/3} - Cy^{-2/3} - y^{-2}$$

$$f(\mu) = \mu^{-1/3} + C \quad g(\xi) = \xi^{-1/3} - C\xi^{1/3} - \xi$$

$$u = \xi^{1/3}(\mu^{-1/3} + C) + \xi^{-1/3} - C\xi^{1/3} - \xi$$

$$u = \xi^{1/3}\mu^{-1/3} + \xi^{-1/3} - \xi$$

$$\boxed{u = x^{-2} + y^{-1/3}x - yx^{-3}}$$

в) $x^2u_{xx} - xyu_{xy} - 2y^2u_{yy} + xu_x - 2yu_y = 9xy^2,\ x > 0,\ y > 0$
$u|_{x=1} = 3e^y, \quad u_x|_{x=1} = -y^2$

Ур-е хар-тик

$$x^2(dy)^2 + xydxdy - 2y^2(dx)^2 = 0$$

$$(\frac{dy}{dx})^2 + \frac{y}{x}\frac{dy}{dx} - 2(\frac{y}{x})^2 = 0$$

$$\begin{cases} \frac{dy}{dx} = -\frac{2y}{x} \\ \frac{dy}{dx} = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} C_1 = yx^2 \\ C_2 = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} \xi = yx^2 \\ \mu = \frac{y}{x} \end{cases}$$

$$u_x = 2u_\xi yx - u_\mu yx^{-2} \qquad u_y = u_\xi x^2 + u_\mu x^{-1}$$

$$u_{xx} = 4u_{\xi\xi}y^2x^2 + u_{\mu\mu}yx^{-4} - 4u_{\xi\mu}y^2x^{-1} + 2u_\xi y + 2u_\mu yx^{-3}$$

$$u_{yy} = u_{\xi\xi}x^4 + u_{\mu\mu}x^{-2} + 2u_{\xi\mu}x$$

$$u_{xy} = 2u_{\xi\xi}yx^3 - u_{\mu\mu}yx^{-3} + u_{\xi\mu}y + 2u_\xi x - u_\mu x^{-2}$$

После подстановки

$$u_{\xi\mu} = -1 \qquad u_\xi = v \qquad v_\mu = -1$$

$$v = -\mu + C(\xi) \qquad u = -\mu\xi + F(\xi) + G(\mu)$$

$$u = -y^2x + F(yx^2) + G(yx^{-1}) \qquad u_x = -y^2 + 2yxF'(yx^2) - yx^{-2}G'(yx^{-1})$$

$$u|_{x=1} = -y^2 + F(y) + G(y) = 3e^y \qquad u_x|_{x=1} = -y^2 + 2yF'(y) - yG'(y) = -y^2$$

$$\begin{cases} -y^2 + F(y) + G(y) = 3e^y \\ 2F'(y) = G'(y) \end{cases} \Rightarrow \begin{cases} -2y + F'(y) + G'(y) = 3e^y \\ 2F'(y) = G'(y) \end{cases} \Rightarrow$$

$$-2y + 3F'(y) = 3e^y \ \Rightarrow \ F(y) = e^y + \frac{y^2}{3} + C$$

$$G(y) = 3e^y + y^2 - e^y - \frac{y^2}{3} - C = 2e^y + \frac{2y^2}{3} - C$$

$$\boxed{u = -xy^2 + e^{yx^2} + \frac{y^2x^4}{3} + 2e^{yx^{-1}} + \frac{2y^2}{3x^2}}$$

г) $yu_{xx} + (x-y)u_{xy} - xu_{yy} - u_x + u_y = 0$
$u|_{y=0} = 2x^2, \quad u_y|_{y=0} = 2x, \quad 1 < x < 4$

Ур-е хар-тик

$$y(dy)^2 - (x-y)dxdy - x(dx)^2 = 0$$

$$\begin{cases} \frac{dy}{dx} = \frac{x}{y} \\ \frac{dy}{dx} = -1 \end{cases} \Leftrightarrow \begin{cases} y^2 = x^2 + C_1 \\ y = -x + C_2 \end{cases} \Leftrightarrow \begin{cases} \xi = y^2 - x^2 \\ \mu = y + x \end{cases}$$

$$u_x = -2xu_\xi + u_\mu \qquad u_y = 2yu_\xi + u_\mu$$

$$u_{xx} = 4x^2u_{\xi\xi} + u_{\mu\mu} - 4xu_{\xi\mu} - 2u_\xi$$

$$u_{yy} = 4y^2u_{\xi\xi} + u_{\mu\mu} + 4yu_{\xi\mu} + 2u_\xi$$

$$u_{xy} = -4yxu_{\xi\xi} + u_{\mu\mu} + 2(y-x)u_{\xi\mu}$$

После подстановки

$$2u_{\xi\mu}\mu^2 = 0 \quad u_\xi = v \quad v_\mu = 0$$

$$v = C(\xi) \qquad u = f(\xi) + g(\mu) \qquad u = f(y^2 - x^2) + g(y+x)$$

$$u_y = 2yf'(y^2 - x^2) + g'(y+x)$$

$$u|_{y=0} = f(-x^2) + g(x) = 2x^2 \qquad u_y|_{y=0} = g'(x) = 2x$$

$$g(x) = x^2 + C \qquad f(-x^2) + x^2 + C = 2x^2 \;\Rightarrow\; f(-x^2) = x^2 - C$$

$$u = (y+x)^2 + x^2 - y^2$$

$$\boxed{u = 2x^2 + 2xy}$$

д) $y^4u_{yy} + y^2u_{xy} - 2u_{xx} + 2y^3u_y = 0$
$u|_{y=1} = x^2 + 5, \quad u_y|_{y=1} = 2x - 6, \ 1 < x < 2$

Ур-е хар-тик

$$y^4(dx)^2 - y^2dxdy - 2(dy)^2 = 0$$

$$\begin{cases} y' = -y^2 \\ y' = \frac{1}{2}y^2 \end{cases} \Leftrightarrow \begin{cases} C_1 = x - \frac{1}{y} \\ C_2 = x + \frac{2}{y} \end{cases} \Rightarrow \begin{cases} \xi = x - \frac{1}{y} \\ \mu = x + \frac{2}{y} \end{cases}$$

$$u_x = u_\xi + u_\mu \qquad u_y = \frac{u_\xi}{y^2} - \frac{2u_\mu}{y^2}$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\mu} + u_{\mu\mu} \qquad u_{yy} = \frac{u_{\xi\xi}}{y^4} + \frac{4u_{\mu\mu}}{y^4} - \frac{4u_{\xi\mu}}{y^4} - 2\frac{u_\xi}{y^3} + 4\frac{u_\mu}{y^3}$$

$$u_{xy} = \frac{u_{\xi\xi}}{y^2} - 2\frac{u_{\mu\mu}}{y^2} - \frac{u_{\xi\mu}}{y^2}$$

$$u_{\xi\mu} = 0$$

$$u = f(\xi) + g(\mu) \quad u = f(x - \frac{1}{y}) + g(x + \frac{2}{y})$$

$$u_y = y^{-2}f'(x - \frac{1}{y}) - 2y^{-2}g'(x + \frac{2}{y})$$

$$u|_{y=1} = f(x-1) + g(x+2) = x^2 + 5$$

$$u_y|_{y=1} = f'(x-1) - 2g'(x+2) = 2x - 6$$

$$\begin{cases} f(x-1) + g(x+2) = x^2 + 5 \\ f'(x-1) - 2g'(x+2) = 2x - 6 \end{cases} \Rightarrow \begin{cases} f(x-1) = x^2 + 5 - g(x+2) \\ 2x - g'(x+2) - 2g'(x) = 2x - 6 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(x-1) = x^2 + 5 - g(x+2) \\ g'(x+2) = 2 \end{cases} \Rightarrow \begin{cases} f(x-1) = x^2 - 2x + 1 - C \\ g(x+2) = 2x + 4 + C \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(x-1) = (x-1)^2 - C \\ g(x+2) = 2(x+2) + C \end{cases} \Rightarrow \begin{cases} f(a) = a^2 - C \\ g(b) = 2b + C \end{cases}$$

$$\boxed{u = (x - \frac{1}{y})^2 + 2(x + \frac{2}{y})}$$

2

$x^2u_{xx} - 4y^2u_{yy} + xu_x - 4yu_y = 0, \quad \frac{1}{x^2} < y < x^2,\ x > 0$
$u|_{y=\frac{1}{x^2}} = 1 + 2x^4,\ u|_{y=x^2} = 2 + x^4$

Ур-е характеристик

$$x^2(dy)^2 - 4y^2(dx)^2$$

$$\begin{cases} \frac{dx}{x} = \frac{dy}{2y} \\ \frac{dx}{x} = -\frac{dy}{2y} \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{y}C_1 \\ x = \frac{1}{\sqrt{y}}C_2 \end{cases}$$

Замена

$$\begin{cases} \mu = \sqrt{y}x \\ \xi = \frac{x}{\sqrt{y}} \end{cases}$$

$$u_x = u_\xi \frac{1}{\sqrt{y}} + u_\mu \sqrt{y}, \quad u_y = -\frac{x}{2\sqrt{y^3}} u_\xi + u_\mu \frac{x}{2\sqrt{y}}$$

$$u_{xx} = \frac{u_{\xi\xi}}{y} + u_{\mu\mu}y + 2u_{\xi\mu}$$

$$u_{yy} = u_{\xi\xi} \cdot \frac{x^2}{4y^3} + u_{\mu\mu} \cdot \frac{x^2}{4y} - u_{\xi\mu} \cdot \frac{x^2}{2y^2} - u_\mu \frac{x}{4\sqrt{y^3}} + u_\xi \frac{3x}{4\sqrt{y^5}}$$

Подставим

$$x^2(\frac{u_{\xi\xi}}{y} + u_{\mu\mu}y + 2u_{\xi\mu}) - 4y^2(u_{\xi\xi} \cdot \frac{1}{4y^3} + u_{\mu\mu} \cdot \frac{x^2}{4y} - u_{\xi\mu} \cdot \frac{x}{2y^2} - u_\mu \frac{x}{4\sqrt{y^3}} + u_\xi \frac{3}{4\sqrt{y^5}}) +$$

$$+ x(u_\xi \frac{1}{\sqrt{y}} + u_\mu \sqrt{y}) - 4y(-\frac{1}{2\sqrt{y^3}} u_\xi + u_\mu \frac{x}{2\sqrt{y}}) = 0$$

$$4x^2u_{\xi\mu} = 0 \ \Rightarrow \ u_{\xi\mu} = 0$$

$$u = f(\xi) + g(\mu) \qquad u = f(\frac{x}{\sqrt{y}}) + g(\sqrt{y}x)$$

$$u|_{y=x^{-2}} = f(x^2) + g(1) = 1 + 2x^4 \qquad u|_{y=x^2} = f(1) + g(x^2) = 2 + x^4$$

$$\begin{cases} f(x^2) + g(1) = 1 + 2x^4 \\ f(1) + g(x^2) = 2 + x^4 \end{cases} \Rightarrow \begin{cases} f(x^2) = 1 + 2x^4 - g(1) \\ g(x^2) = x^4 - 1 + g(1) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(a) = 2a^2 + C \\ g(b) = b^2 - C \end{cases}$$

$$\boxed{u = 2\frac{x^2}{y} + yx^2}$$

3

$u_{yy} - u_{xx} = 0$
$u|_{y=0} = u_0(x),\ u_y|_{y=0} = u_1(x),\ 0 < x < 1,\ u_0(x) \in C^2(0;1), u_1(x) \in C_1(0;1)$

максимальная область, где $\exists!$ реш

$$\begin{cases} 0 < x + y < 1 \\ 0 < x - y < 1 \end{cases}$$

Волновое уравнение

1

$$\begin{cases} 4u_{tt} = u_{xx} + 4t^2\cos(2x) \\ u|_{t=0} = e^x,\ u_t|_{t=0} = x^2 \end{cases}$$

$$u_{\text{частн}} = f(t)\cos(2x)$$

$$4f''\cos(2x) = -4f\cos(2x) + 4t^2\cos(2x)$$

$$f'' = -f + t^2$$

$$f = \alpha t^2 + \beta t + \gamma \quad 2\alpha = -\alpha t^2 - \beta t - \gamma + t^2$$

$$\alpha = 1,\ \beta = 0,\ \gamma = -2$$

$$u_{\text{частн}} = (t^2 - 2)\cos(2x)$$

$$u = (t^2 - 2)\cos(2x) + v(x,t)$$

$$\begin{cases} 4v_{tt} = v_{xx} \\ v|_{t=0} = e^x + 2\cos(2x) \\ v_t|_{t=0} = x^2 \end{cases}$$

$$\boxed{v(x,t) = \frac{1}{2}(e^{x+\frac{t}{2}} + 2\cos(2x+t) + e^{x-\frac{t}{2}} + 2\cos(2x-t)) + \int_{x-\frac{t}{2}}^{x+\frac{t}{2}} \xi^2\, d\xi}$$

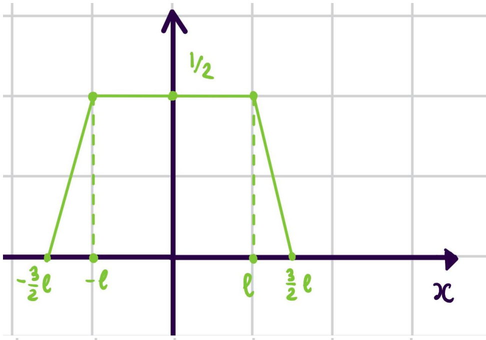
2

$u_{tt} = a^2 u_{xx}, \quad x \in \mathbb{R}^1, \quad t > 0$

a) $u|_{t=0} = \varphi(x),\ u_t|_{t=0} = 0;$

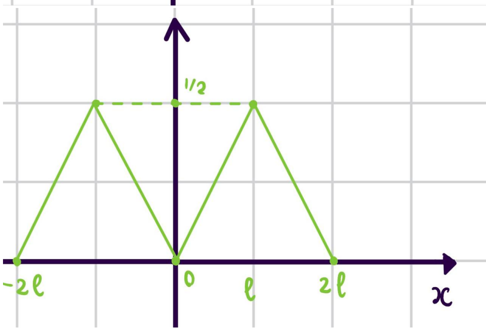
Ф-ла Даламбера

$$u(x,t) = \frac{\varphi(x+at) - \varphi(x-at)}{2}$$



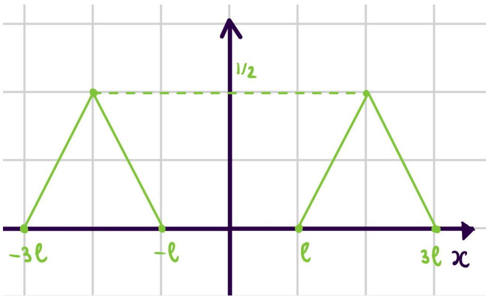
$$t = \frac{l}{2a}$$

$$u(x,t) = \frac{\varphi(x+\frac{l}{2}) - \varphi(x-\frac{l}{2})}{2}$$



$$t = \frac{l}{a}$$

$$u(x,t) = \frac{\varphi(x+l) - \varphi(x-l)}{2}$$



$$t = \frac{2l}{a}$$

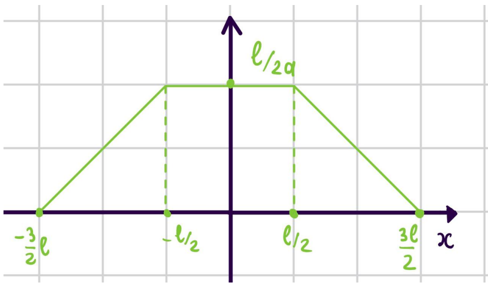
$$u(x,t) = \frac{\varphi(x+2l) - \varphi(x-l)}{2}$$

б) $u|_{t=0} = 0,\ u_t|_{t=0} = \psi(x) = \theta(l - |x|)$

$$l = \text{const} > 0 \quad \theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Общее решение

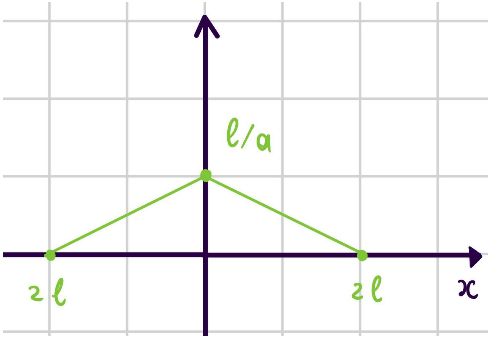
$$u(x,t) = \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi)\, d\xi$$



$$t = \frac{l}{2a}$$

$$u(x,t) = \frac{1}{2a}\int_{x-\frac{l}{2}}^{x+\frac{l}{2}} \theta(l-|\xi|)\,d\xi$$

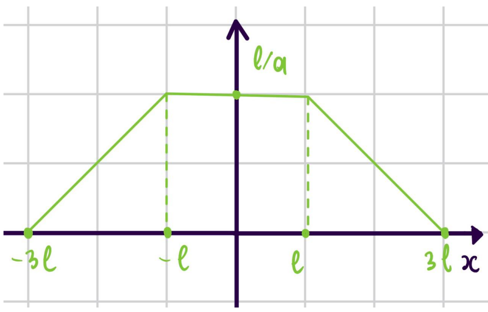
$$= \frac{1}{2a}\text{length}\left([x-\frac{l}{2}, x+\frac{l}{2}] \cap [-l,l]\right)$$



$$t = \frac{l}{a}$$

$$u(x,t) = \frac{1}{2a}\int_{x-l}^{x+l} \theta(l-|\xi|)\,d\xi$$

$$= \frac{1}{2a}\text{length}\left([x-l, x+l] \cap [-l,l]\right)$$



$$t = \frac{2l}{a}$$

$$u(x,t) = \frac{1}{2a}\int_{x-2l}^{x+2l} \theta(l-|\xi|)\,d\xi$$

$$= \frac{1}{2a}\text{length}\left([x-2l, x+2l] \cap [-l,l]\right)$$

3

a) 21.19
$u_{tt} = 3u_{xx} + 2(1-6t^2)e^{-2x},\ t>0,\ x>0$
$u|_{t=0} = 1,\ u_t|_{t=0} = x,\ (u_x - 2u)|_{x=0} = -2 + t - 4t^2$

Ищем частное решение

$$u_{\text{частн}} = e^{-2x}(At^2 + Bt + C)$$

$$2Ae^{-2x} = 12e^{-2x}(At^2 + Bt + C) + 2(1-6t^2)e^{-2x}$$

$$2A = 12At^2 + 12Bt + 12C + 2 - 12t^2$$

$$\begin{cases} A = 1, \\ B = 0, \\ C = 0; \end{cases}$$

$$u_{\text{частн}} = t^2 e^{-2x}$$

Сводим ур-е к однородному

$$u = t^2 e^{-2x} + v(x,t)$$

$$v_{tt} = 3v_{xx}$$

$$v|_{t=0} = 1, \quad v_t|_{t=0} = x, \quad (v_x - 2v)|_{x=0} = -2 + t$$

$$v = f(x + \sqrt{3}t) + g(x - \sqrt{3}t)$$

Ищем решение в области $x \geq \sqrt{3}t$

$$v|_{t=0} = f(x) + g(x) = 1 \qquad v_t|_{t=0} = \sqrt{3}f'(x) - \sqrt{3}g'(x) = x$$

$$f'(x) + g'(x) = 0$$

$$2\sqrt{3}f'(x) = x$$

$$\boxed{f(x) = \frac{x^2}{4\sqrt{3}} + C, \quad g(x) = 1 - \frac{x^2}{4\sqrt{3}} - C}$$

$$v(x,t) = 1 - \frac{(x + \sqrt{3}t)^2 - (x - \sqrt{3}t)^2}{4\sqrt{3}} = 1 - xt$$

Ищем решение в области $x < \sqrt{3}t$

$$(v_x - 2v)|_{x=0} = f'(\sqrt{3}t) + g'(-\sqrt{3}t) - 2f(\sqrt{3}t) - 2g(-\sqrt{3}t) - 2C = -2 + t$$

Заметим, что уравнение полученное в области $x \geq \sqrt{3}t$ также подходит

Следовательно, решения на полученных областях совпадают

$$\boxed{u(x,t) = 1 - xt + t^2 e^{-2x}}$$

21.2
$u_{tt} - a^2 u_{xx} = 0$
$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x)$
$u_x|_{x=0} = 0$

$$u(x,t) = f(x+at) + g(x-at)$$

$$x \geq at$$

$$f(x) + g(x) = u_0(x), \quad u_t|_{t=0} = af'(x) - ag'(x) = u_1(x)$$

$$f(x) - g(x) = \frac{1}{a}\int_0^x u_1(\xi)\,d\xi + A$$

$$f(x) = \frac{u_0(x)}{2} + \frac{1}{2a}\int_0^x u_1(\xi)\,d\xi + \frac{A}{2} \qquad g(x) = \frac{u_0}{2} - \frac{1}{2a}\int_0^x u_1(\xi)\,d\xi - \frac{A}{2}$$

$$\boxed{u(x,t) = \frac{u_0(x+at) + u_0(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} u_1(\xi)\,d\xi}$$

$$x < at$$

$$u_x|_{x=0} = f'(at) - g'(-at) = 0 \Rightarrow f(at) + g(-at) = C$$

$$g(-at) = C - \frac{u_0(at)}{2} = \frac{1}{2a}\int_0^{at} u_1(\xi)\,d\xi - \frac{A}{2}$$

$$g(b) = C - \frac{u_0(-b)}{2} = \frac{1}{2a}\int_0^{-b} u_1(\xi)\,d\xi - \frac{A}{2}$$

Сшивка

$$g(+0) = g(-0)$$

$$C - \frac{A}{2} = u_0 - \frac{A}{2} \Rightarrow C = u_0(0) = 0$$

Показали, что

$$\boxed{u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} u_1(\xi)\,d\xi + \begin{cases} \frac{u_0(x+at)+u_0(x-at)}{2}, & x \geq at \\ \frac{u_0(x+at)-u_0(x-at)}{2}, & x < at \end{cases}}$$

б) $u_{tt} = u_{xx} + xe^t, \quad x > 0, t > 0$
$u|_{t=0} = 1 + x, \quad u_t|_{t=0} = 4 - 5x, x \geq 0$
$(2u + u_x)|_{x=0} = (1+t)e^t + 2 + t - 3t^2, \quad t \geq 0$

Ищем частные решения

$$u_{\text{частн}} = xe^t$$

Сводим ур-е к однородному

$$u = xe^t + v(x,t)$$

$$v_{tt} = v_{xx} \qquad v|_{t=0} = (u - xe^t)|_{t=0} = 1 + x - x = 1$$

$$v_t|_{t=0} = (u_t - xe^t)|_{t=0} = 4 - 5x - x = 4 - 6x$$

$$(2v + v_x)|_{x=0} = (2u + u_x - 2xe^t - e^t)|_{x=0} = te^t + 2 + t - 3t^2,\ t \geq 0$$

$$v = f(x+t) + g(x-t)$$

Ищем решение в области $x \geq t$

$$v|_{t=0} = f(x) + g(x) = 1 \qquad v_t|_{t=0} = f'(x) - g'(x) = 4 - 6x,\ x \geq 0$$

$$f'(x) + g'(x) = 0, \qquad \text{складываем}$$

$$2f'(x) = 4 - 6x \qquad 2f'(x) = 4 - 6x \qquad f'(x) = 2 - 3x$$

$$\boxed{f(x) = 2x - \frac{3x^2}{2} + A; \qquad g(x) = 1 - 2x + \frac{3x^2}{2} - A}$$

$$v = 2(x+t) - \frac{3}{2}(x+t)^2 + A + 1 - 2(x-t) + \frac{3}{2}(x-t)^2 - A,\ x \geq t$$

Ищем решения при $x < t$

$$(2v + v_x)|_{x=0} = 2f(t) + 2g(-t) + f'(t) + g'(-t) =$$

$$= te^t + 2 + t - 3t^2,\ t \geq 0$$

$$4t - 3t^2 + 2A + 2g(-t) + 2 - 3t + g'(-t) = te^t + 2 + t - 3t^2$$

$$g'(-t) + 2g(-t) = te^t - 2A \qquad -t = p < 0$$

$$g'(p) + 2g(p) = -pe^t - 2A$$

$$g_{\text{частн}_1} = (\alpha p + \beta)e^{-p}$$

$$\alpha e^{-p} - (\alpha p + \beta)e^{-p} + 2(\alpha p + \beta)e^{-p} = -pe^{-p}$$

$$\alpha + 2\alpha = -1 \qquad \alpha = -1 \qquad \alpha - \beta + 2\beta = 0$$

$$\beta = -\alpha = 1$$

$$g_{\text{частн}_1} = (1-p)e^{-p} \qquad g_{\text{частн}_2} = -A$$

$$g(p) = ce^{-2p} + (1-p)e^{-p} - A, \quad p < 0$$

Сшивка(склейка)

$$g(+0) = g(-0)$$

$$1 - A = C + 1 - A \Rightarrow C = 0$$

Ответ

$$\boxed{u(x,t) = xe^t + 2(x+t) - \frac{3}{2}(x+t)^2 + \begin{cases} 1 - 2(x-t) + \frac{3}{2}(x-t)^2, & x \geq t \\ (1-(x-t))e^{-(x-t)}, & x < t \end{cases}}$$

в) $4u_{tt} = u_{xx} - 3\sin(x+t),\ x > 0,\ t > 0$
$u|_{t=0} = \sin(x) + \sin(2x), \quad u_t|_{t=0} = 2 + \cos(x),\ x \geq 0$
$(u_x - 2u)|_{x=0} = 2 - 4t - 2\sin(t) + \cos(t),\ t \geq 0$

Ищем частное решение

$$u_{\text{частн}} = \sin(x+t)$$

Сводим уравнение к однородному

$$u = \sin(x+t) + v(x,t)$$

$$v_{tt} = \left(\frac{1}{2}\right)^2 v_{xx}, \quad v|_{t=0} = \sin(2x), \quad v_t|_{t=0} = 2$$

$$(v_x - 2v)|_{x=0} = 2 - 4t,\ t \geq 0$$

$$v = f(x + \frac{t}{2}) + g(x - \frac{t}{2})$$

$$\text{Ищем решение в области } x \geq \frac{t}{2}$$

$$v|_{t=0} = f(x) + g(x) = \sin(2x)$$

$$v_t|_{t=0} = \frac{1}{2}f'(x) - \frac{1}{2}g'(x) = 2$$

$$f'(x) + g'(x) = 2\cos(2x) \qquad f'(x) = \cos(2x) + 2$$

$$\boxed{f(x) = \frac{1}{2}\sin(2x) + 2x + C, \quad g(x) = \frac{1}{2}\sin(2x) - 2x - C}$$

$$v(x,t) = \frac{1}{2}\sin(2x+t) + \frac{1}{2}\sin(2x-t),\ x \geq \frac{t}{2}$$

$$\text{Область } x < \frac{t}{2}$$

$$(v_x - 2v)\,|_{x=0} = f'\left(\frac{t}{2}\right) + g'\left(-\frac{t}{2}\right) - 2f\left(\frac{t}{2}\right) - 2g\left(-\frac{t}{2}\right) = 2 - 4t$$

$$\cos(t) + 2 - 2\left(\frac{1}{2}\sin(t) + t + C\right) + g'\left(-\frac{t}{2}\right) + g\left(-\frac{t}{2}\right) = 2 - 4t$$

$$g'(b) + g(b) + \sin(2b) + \cos(2b) - 4b - 2C = 0, \quad b = -\frac{t}{2}$$

Решая дифф. ур-е

$$g(b) = Ae^{2b} - 2b - 1 + \frac{1}{2}\cos(2b) - C$$

Сшивка

$$g(+0) = g(-0)$$

$$A - 1 + \frac{1}{2} - C = -C \Rightarrow A = \frac{1}{2}$$

$$\boxed{u = \sin(x+t) + \frac{1}{2}\sin(2x+t) + \begin{cases} \frac{1}{2}\sin(2x-t),\ x \geq \frac{t}{2} \\ \frac{1}{2}\cos(2x-t) + 2t - 1 + \frac{1}{2}e^{2x-1},\ x < \frac{t}{2} \end{cases}}$$

г) $u_{tt} + u_{xt} - 2u_{xx} = 0, \quad x > 0,\ t > 0$
$u|_{t=0} = \operatorname{sh} x + \operatorname{arctg} x,\ u_t|_{t=0} = \operatorname{ch} x - \frac{2}{1+x^2},\ x \geq 0$
$u_x|_{x=0} = \operatorname{ch} x + 1,\ t \geq 0$

$$(dx)^2 - dxdt - 2(dt)^2 = 0 \quad (dx - 2dt)(dx + dt) = 0$$

$$\begin{cases} x = 2t + C_1 \\ x = -t + C_2 \end{cases} \Leftrightarrow \begin{cases} \xi = x - 2t \\ \mu = x + t \end{cases}$$

$$u_x = u_\xi + u_\mu \quad u_t = -2u_\xi + u_\mu$$

$$u_{xx} = u_{\xi\xi} + u_{\mu\mu} + 2u_{\xi\mu}$$

$$u_{tt} = 4u_{\xi\xi} + u_{\mu\mu} - 4u_{\xi\mu}$$

$$u_{xt} = -2u_{\xi\xi} + u_{\mu\mu} - u_{\xi\mu}$$

Подставляем

$$u_{\xi\mu} = 0$$

$$u = f(\xi) + g(\mu) = f(x - 2t) + g(x + t)$$

$$u|_{t=0} = f(x) + g(x) = \operatorname{sh} x + \operatorname{arctg} x \Rightarrow f'(x) + g(x) + \operatorname{ch} x = \frac{1}{1 + x^2}$$

$$u_t'|_{t=0} = -2f'(x) + g'(x) = \operatorname{ch} x - \frac{2}{1 + x^2}$$

$$u_x|_{x=0} = -2tf'(t) + g'(t) = \operatorname{ch} t + 1$$

$$f'(-2t) + \operatorname{ch} t = \operatorname{ch} t + 1$$

$$-2t = z \quad f'(z) = 1 \quad f(z) = z + C,\ z < 0$$

$$f(+0) = f(-0) \Rightarrow C = A$$

$$\boxed{u(x,t) = \operatorname{sh}(x + t) + \begin{cases} \operatorname{arctg}(x - 2t), & x - 2t \geq 0 \\ x - 2t, & x - 2t < 0 \end{cases}}$$

4

a) 12.43(5)
$u_{tt} = 2\Delta u$
$u|_{t=0} = 2x^2 - y^2,\ u_t|_{t=0} = 2x^2 + y^2$

$$u = v + w$$

$$\begin{cases} v_{tt} = 2\Delta v \\ v|_{t=0} = 2x^2 \\ v_t|_{t=0} = 2x^2 \end{cases} \qquad \begin{cases} w_{tt} = 2\Delta w \\ w|_{t=0} = -y^2 \\ w_t|_{t=0} = y^2 \end{cases}$$

$$v = v|_{t=0} + tv_t|_{t=0} + \frac{t^2}{2} v_{tt}|_{t=0} + \frac{t^3}{6} v_{ttt}|_{t=0} + \dots$$

$$v_{tt}|_{t=0} = 2\Delta v|_{t=0} = 2\Delta(v|_{t=0}) = 8$$

$$v_{ttt}|_{t=0} = 2\Delta v_t|_{t=0} = 8$$

$$v = 2x^2 + 2tx^2 + 4t^2 + \frac{4}{3}t^3$$

$$w = w|_{t=0} + tw_t|_{t=0} + \frac{t^2}{2} w_{tt}|_{t=0} + \frac{t^3}{6} w_{ttt}|_{t=0} + \dots$$

$$w_{tt}|_{t=0} = 2\Delta w|_{t=0} = 2\Delta(w|_{t=0}) = -4$$

$$w_{ttt}|_{t=0} = 2\Delta w_t|_{t=0} = 4$$

$$w = -y^2 + ty^2 - 2t^2 + \frac{2}{3}t^3$$

$$\boxed{u = 2x^2 - y^2 + (2x^2 + y^2)t + 2t^2 + 2t^3}$$

12.44(7)

$u_{tt} = a^2\Delta u + \cos x \sin y e^z$

$u|_{t=0} = x^2 e^{y+z},\ u_t|_{t=0} = \sin x e^{y+z}$

$$u = v + w$$

$$\begin{cases} v_{tt} = a^2\Delta v + \cos x \sin y e^z \\ v|_{t=0} = 0 \\ v_t|_{t=0} = \sin x e^{y+z} \end{cases} \qquad \begin{cases} w_{tt} = a^2\Delta w \\ w|_{t=0} = x^2 e^{y+z} \\ w_t|_{t=0} = 0 \end{cases}$$

$$v = h_1(t)\cos x \sin y e^z + h_2(t)\sin x e^{y+z}$$

$$h_1'' \cos x \sin y e^z + h_2'' \sin x e^{y+z} = a^2(-h_1 \cos x \sin y e^z + h_2 \sin x e^{y+z}) +$$

$$+ \cos x \sin y e^z$$

$$\begin{cases} h_1'' = -a^2 h_1 + 1 \\ h_2'' = a^2 h_2 \end{cases}$$

$$h_1 = A\cos(at) + B\sin(at) + \frac{1}{a^2}$$

$$h_2 = C\,\mathrm{sh}(at) + D\,\mathrm{ch}(at)$$

$$h_1|_{t=0} = h_2|_{t=0} = 0 \Rightarrow D = 0,\ A = -\frac{1}{a^2}$$

$$h_{1_t}|_{t=0} = 0 \Rightarrow B = 0 \qquad h_{2_t}|_{t=0} = 1 \Rightarrow C = \frac{1}{a}$$

$$h_1 = \frac{1}{a^2}(1 - \cos(at)) \qquad h_2 = \frac{1}{a}\,\mathrm{sh}(at)$$

$$v = \frac{1}{a^2}(1 - \cos(at))\cos x \sin y e^z + \frac{1}{a}\,\mathrm{sh}(at)\sin x e^{y+z}$$

$$w = h_3(t) x^2 e^{y+z} + h_4(t) e^{y+z}$$

$$h_3'' x^2 e^{y+z} + h_4'' e^{y+z} = a^2(h_3(2e^{y+z} + 2x^2 e^{y+z}) + 2h_4 e^{y+z})$$

$$\begin{cases} h_3'' = 2a^2 h_3 \\ h_4'' = 2a^2 h_3 + 2a^2 h_4 \end{cases}$$

$$h_3 = A\operatorname{ch}(\sqrt{2}at) + B\operatorname{sh}(\sqrt{2}at)$$

$$h_3|_{t=0} = 1 \Rightarrow A = 1 \qquad h_{3_t}|_{t=0} = 0 \Rightarrow B = 0$$

$$h_3 = \operatorname{ch}(\sqrt{2}at)$$

$$h_4'' - 2a^2 h_4 = 2a^2\operatorname{ch}(\sqrt{2}at)$$

$$h_{4_{\text{общ}}} = C\operatorname{ch}(\sqrt{2}at) + D\operatorname{sh}(\sqrt{2}at)$$

$$h_{4_{\text{частн}}} = t(E\operatorname{ch}(\sqrt{2}at) + F\operatorname{sh}(\sqrt{2}at))$$

$$2a^2t(E\operatorname{ch}(\sqrt{2}at) + F\operatorname{sh}(\sqrt{2}at)) + \sqrt{2}a(E\operatorname{ch}(\sqrt{2}at) + F\operatorname{sh}(\sqrt{2}at)) -$$
$$- 2a^2t(E\operatorname{ch}(\sqrt{2}at) + F\operatorname{sh}(\sqrt{2}at)) =$$
$$= 2a^2\operatorname{ch}(\sqrt{2}at)$$

$$E = 0,\ F = \frac{a}{\sqrt{2}}$$

$$h_4 = \frac{a}{\sqrt{2}}t\operatorname{sh}(\sqrt{2}at) + C\operatorname{ch}(\sqrt{2}at) + D\operatorname{sh}(\sqrt{2}at)$$

$$h_4|_{t=0} = 0 \Rightarrow C = 0 \qquad h_{4_t}|_{t=0} = 0 \Rightarrow D = 0$$

$$h_4 = \frac{at}{\sqrt{2}}\operatorname{sh}(\sqrt{2}at)$$

$$w = \operatorname{ch}(\sqrt{2}at)x^2e^{y+z} + \frac{at}{\sqrt{2}}\operatorname{sh}(\sqrt{2}at)e^{y+z}$$

$$\boxed{\begin{aligned} u = \frac{1}{a^2}(1-\cos(at))e^z\cos x\sin y + e^{y+z} \times \Bigg[\frac{1}{a}\operatorname{sh}(at)\sin x \\ + \frac{at}{\sqrt{2}}\operatorname{sh}(at\sqrt{2}) + x^2\operatorname{ch}(at\sqrt{2})\Bigg]\end{aligned}}$$

б) $u_{tt} = \Delta u + (x^2+y^2)\sin t,\ t>0$
$u|_{t=0} = 0,\ u_t|_{t=0} = (x^2+y^2+z^2)^{5/2} - y^2$

$$u = v + w + p$$

$$1)\begin{cases} v_{tt} = \Delta v + (x^2+y^2)\sin t \\ v|_{t=0} = 0 \\ v_t|_{t=0} = 0 \end{cases} \qquad 2)\begin{cases} w_{tt} = \Delta w \\ w|_{t=0} = 0 \\ w_t|_{t=0} = -y^2 \end{cases}$$

$$3)\begin{cases} p_{tt} = \Delta p \\ p|_{t=0} = 0 \\ p_t|_{t=0} = (x^2+y^2+z^2)^{5/2} \end{cases}$$

$$1) v = h_1(t)(x^2+y^2) + h_2(t)$$

$$h_1''(x^2+y^2) + h_2'' = 4h_1 + (x^2+y^2)\sin t$$

$$\begin{cases} h_1'' = \sin t \\ h_2'' = 4h_1 \end{cases} \quad \begin{cases} h_1 = -\sin t + At + B \\ h_1|_{t=0} = 0 \\ h_{1_t}|_{t=0} = 0 \end{cases} \qquad \Rightarrow B = 0, \quad A = 1$$

$$\begin{cases} h_1 = t - \sin t \\ h_2'' = 4t - 4\sin t \end{cases} \quad \begin{cases} h_2 = \frac{2}{3}t^3 + 4\sin t + Ct + D \\ h_2|_{t=0} = 0 \\ h_{2_t}|_{t=0} = 0 \end{cases} \qquad \Rightarrow D = 0 \quad C = -4$$

$$h_2 = \frac{2}{3}t^3 + 4\sin t - 4t$$

$$2) w = w_t|_{t=0} + tw_t|_{t=0} + \frac{t^2}{2}w_{tt}|_{t=0} + \frac{t^3}{6}w_{ttt}|_{t=0} + \dots$$

$$w_{ttt}|_{t=0} = \Delta(w_t|_{t=0}) = -2$$

$$w = -ty^2 - \frac{t^3}{3}$$

$$3) x^2 + y^2 + z^2 = r^2$$

$$\begin{cases} p_{tt} = p_{rr} + \frac{2}{r} p \\ p|_{t=0} = 0 \\ p_t|_{t=0} = r^5 \end{cases}$$

$$p = \frac{s}{r}$$

$$\begin{cases} s_{tt} = s_{rr} \\ s|_{t=0} = 0 \\ s_t|_{t=0} = r^6 \\ s|_{r=0} = 0 \end{cases}$$

$$s = f(r+t) + g(r-t)$$

$$\begin{cases} s|_{t=0} = f(r) + g(r) = 0 \\ s_t|_{t=0} = f'(r) - g'(r) = r^6 \end{cases} \qquad \begin{cases} f(r) = \frac{r^7}{14} + C \\ g(r) = -\frac{r^7}{14} - C \end{cases}$$

$$s = \frac{(r+t)^7 - (r-t)^7}{14}$$

$$p = \frac{s}{r} = \frac{(\sqrt{x^2+y^2+z^2} + t)^7 - (\sqrt{x^2+y^2+z^2} - t)^7}{14\sqrt{x^2+y^2+z^2}}$$

$$\boxed{u = v + w + p}$$

в) $\begin{cases} u_{tt} = 8\Delta u + x \\ u|_{t=0} = e^{y-z} \\ u_t|_{t=0} = x^2 + y^2 + z^2 \end{cases}$

$$u = v + w + p$$

$$1) \begin{cases} v_{tt} = 8\Delta v + x \\ v|_{t=0} = 0 \\ v_t|_{t=0} = 0 \end{cases} \qquad v = h(t)x, \quad h''x = x$$

$$\begin{cases} h'' = 1 \\ h(0) = 0 \\ h'(0) \end{cases} \Rightarrow h(t) = \frac{t^2}{2} + At + B \quad v = \frac{t^2}{2}x$$

$$2) \begin{cases} w_{tt} = 8\Delta w \\ w|_{t=0} = e^{y-z} \\ w_t|_{t=0} = 0 \end{cases} \qquad w = h(t)e^{y-z}$$

$$\begin{cases} h''e^{y-z} = 16he^{y-z} \\ h(0) = 1 \\ h'(0) = 0 \end{cases} \qquad h'' = 16h \Rightarrow h = A\operatorname{sh}(4t) + B\operatorname{ch}(4t)$$

$$h(0) = B = 1 \quad h'(0) = 4A = 0 \Rightarrow w = \operatorname{ch}(4t)e^{y-z}$$

$$3) \begin{cases} p_{tt} = 0 \\ p|_{t=0} = 0 \\ p_t|_{t=0} = x^2 + y^2 + z^2 \end{cases} \qquad p = t \cdot \underbrace{p_t|_{t=0}}_{x^2+y^2+z^2} + \frac{t^2}{2} p_{tt}|_{t=0} + \ldots$$

$$p_{tt}|_{t=0} = 8\Delta(p|_{t=0}) = 0$$

$$p_{ttt}|_{t=0} = 8\Delta(p_t|_{t=0}) = 8\Delta(x^2 + y^2 + z^2) = 48$$

$$p = t(x^2 + y^2 + z^2) + 48\frac{t^3}{6}$$

$$\boxed{u = t(x^2 + y^2 + z^2) + 8t^3 + \operatorname{ch}(4t)e^{y-z} + \frac{xt^2}{2}}$$

г) $u_{tt} = \Delta u - 81(t+1)^2 \operatorname{sh}(2x - 2y + z)$
$u|_{t=0} = (2x + y - z)\cos y;\ u_t|_{t=0} = x^2 - y^2 + z^2$

/Fun facts...

$$\Delta(\operatorname{sh}(2x - 2z + z)) = 9\operatorname{sh}(2x - 2x + z)$$

$$\Delta((2x + y - z)\cos y) = \frac{\partial}{\partial y}(\cos y - (2x + y - z)\sin y)$$

$$= -2\sin y - (2x + y - z)\cos y$$

$$\Delta(x^2 - y^2 + z^2) = 2 = 0 \cdot (x^2 - y^2 + z^2) + 2 \cdot 1$$

/now, solution

$$u = v + w$$

$$1)\begin{cases} v_{tt} = \Delta v - 81(t+1)^2 \operatorname{sh}(2x - 2y + z) \\ v|_{t=0} = 0 \\ v_t|_{t=0} = 0 \end{cases}$$

$$v = h(t)\operatorname{sh}(2x - 2y + z)$$

$$h'' = \operatorname{sh}(2x - 2z + z) = h \cdot 9\operatorname{sh})2x - 2y + z) - 81(t+1)^2 \operatorname{sh}(2x - 2y + z)$$

$$h'' = 9h - 81(t+1)^2$$

$$\begin{cases} h'' = 9h - 81(t+1)^2 \\ h(0) = 0 \\ h'(0) = 0 \end{cases}$$

$$h = A\operatorname{sh}(3t) + B\operatorname{ch}(3t)$$

$$h_{\text{частн}} = \alpha t^2 + \beta t + \gamma$$

$$2\alpha = 9\alpha t^2 + 9\beta t + 9\gamma - 81t^2 - 162t - 81$$

$$\alpha = 9$$
$$\beta = 18$$
$$\gamma = 11$$

$$h = A\,\mathrm{sh}(3t) + B\,\mathrm{ch}(3t) + 9t^2 + 18t + 11$$

$$h(0) = B + 11 = 0 \to B = -11$$

$$h'(0) = 3A + 18 = 0 \to A = -6$$

$$\boxed{v = (-6\,\mathrm{sh}(3t) - 11\,\mathrm{ch}(3t) + 9t^2 + 18t + 11)\,\mathrm{sh}(2x - 2y + z)}$$

$$2)\ \begin{cases} w_{tt} = \Delta w \\ w|_{t=0} = (2x + y - z)\cos y \\ w_t|_{t=0} = 0 \end{cases}$$

$$w = h_1(t)(2x + y - z)\cos y + h_2(t)\sin y$$

$$h''(2x + y - z)\cos y + h_2''\sin y = h_1(-2\sin y - (2x + y - z)\cos y) + h_2(-\sin$$

$$\begin{cases} h_1'' = -h_1 \\ h_1(0) = 1 \\ h_1' = 0 \end{cases} \quad \begin{cases} h_2'' = -2h_1 - h_2 \\ h_2(0) = 0 \\ h_2'(0) = 0 \end{cases}$$

$$h_1 = A\cos t + B\sin t \quad h_1(0) = A = 1, \quad h_1'(0) = B = 0$$

$$h_1(t) = \cos t$$

$$h_2'' = -2\cos t - h_2 \quad h_{2,\text{частн}} = \alpha t\cos t + \beta t\sin t$$

$$h'_{2,\text{частн}} = \alpha\cos t - \alpha t\sin t + \beta\sin t + \beta t\cos t$$

$$-2\alpha\sin t - \alpha t\cos t + 2\beta\cos t - \beta t\sin t = -2\cos t - \alpha t\cos t - \beta t\sin t$$

$$\alpha = 0$$
$$\beta = -1$$

$$\boxed{w = \cos t(2x + y - z)\cos y - t\sin t\sin y}$$

$$3)\ \begin{cases} p_{tt} = \Delta p \\ p|_{t=0} = 0 \\ p_t|_{2=0} = x^2 - y^2 + z^2 \end{cases}$$

$$p = h_1(t)(x^2 - y^2 + z^2) + h_2(t)$$

$$h_1''(x^2 - y^2 + z^2) + h_2'' = 2h_1$$

$$\begin{cases} h_1'' = 0, \\ h_1(0) = 0, \\ h_1'(0) = 1; \end{cases} \qquad \begin{cases} h_2'' = 2h_1, \\ h_2(0) = 0, \\ h_2'(0) = 0; \end{cases}$$

$$h_1 = At + B \;\Rightarrow\; h_1 = t$$

$$h_2 = \frac{t^3}{3} + At + B \;\Rightarrow\; h_2 = \frac{t^3}{3} + At$$

$$\boxed{p = t(x^2 - y^2 + z^2) + \frac{t^3}{3}}$$

$$\boxed{\begin{gathered} u = t(x^2 - y^2 + z^2) + \frac{t^3}{3} + \cos t(2x + y - z)\cos y - t\sin t\sin y + \\ + (-6\,\mathrm{sh}(3t) - 11\,\mathrm{ch}(3t) + 9t^2 + 18t + 11)\,\mathrm{sh}(2x - 2y + z) \end{gathered}}$$

д) $u_{tt} = \frac{1}{5}\Delta u + 2t^2\cos(x+2y)$
$u|_{t=0} = yz^3,\ u_t|_{t=0} = \frac{1}{1+(x-2z)^2}$

$$u = v + w + p$$

$$1)\begin{cases} v_{tt} = \frac{1}{5}\Delta v + 2t^2\cos(x+2y) \\ v|_{t=0} = 0 \\ v_t|_{t=0} = 0 \end{cases}$$

$$v = h(t)\cos(x+2y)$$

$$h''\cos(x+2y) = \frac{1}{5}h(-5)\cos(x+2y) + 2t^2\cos(x+2y)$$

$$\begin{cases} h'' = -h + 2t^2 \\ h(0) = 0 \\ h'(0) = 0 \end{cases}$$

$$h_{\text{частн}} = \alpha t^2 + \beta t + \gamma$$

$$2\alpha = -\alpha t^2 - \beta t - \gamma + 2t^2$$

$$\begin{aligned} \alpha &= 2 \\ \beta &= 0 \\ \gamma &= -4 \end{aligned}$$

$$h = -2t^2 - 4 + a\sin t + B\cos t$$

$$h(0) = -4 + B = 0 \to B = 4$$

$$h'(0) = A = 0$$

$$h(t) = 2t^2 - 4 + 4\cos t$$

$$v = \cos(x+2y)(2t^2 - 4 + 4\cos t)$$

$$2)\ \begin{cases} w_{tt} = \frac{1}{5}\Delta w \\ w|_{t=0} = yz^3 \\ w_t|_{t=0} = 0 \end{cases}$$

$$w = \underbrace{w|_{t=0}}_{yz^3} + t\,\underbrace{w_t|_{t=0}}_{0} + \frac{t^2}{2} w_{tt}|_{t=0} + \ldots$$

$$w_{tt} = \frac{1}{5}\Delta(w|_{t=0}) = \frac{1}{5}\Delta(yz^3) = \frac{6}{5}yz$$

$$w = yz^3 + \frac{t^2}{2}\frac{6}{5}yz$$

$$3)\ \begin{cases} p_{tt} = \frac{1}{5}\Delta p \\ p|_{t=0} = 0 \\ p_t|_{t=0} = \frac{1}{1+(x-2z)^2} \end{cases}$$

$$x - 2z = \xi$$

$$p = p(t, x-2z) = p(t, \xi)$$

$$\begin{cases} p_{tt} = \frac{1}{5}p_{\xi\xi} \cdot 5 = p_{\xi\xi} \\ p|_{t=0} = 0 \\ p_t|_{t=0} = \frac{1}{1+\xi^2} \end{cases}$$

$$p = \frac{1}{2}\int_{\xi-t}^{\xi+t} \frac{1}{1+\mu^2}\,d\mu = \frac{1}{2}(\operatorname{arctg}(\xi+t) - \operatorname{arctg}(\xi-t))$$

$$\boxed{\begin{aligned} u = \frac{1}{2}\left(\operatorname{arctg}(x-2z+t) - \operatorname{arctg}(x-2z-t)\right) + yz^3 + \frac{3t^2}{5}yz+ \\ + \cos(x+2y)(2t^2 - 4 + 4\cos t) \end{aligned}}$$

5

$u_{tt} = a^2 \Delta u,\ t > 0,\ x = (x_1, x_2, x_3) \in \mathbb{R}^3$
$u_0(x) \in C^3(\mathbb{R}^3),\ u_1(x) \in C^2(\mathbb{R}^3),\ r = \sqrt{x^2 + y^2 + z^2}$

$$\begin{cases} u_{tt} = a^2(u_{rr} + \frac{2}{r} u_r) \\ u|_{t=0} = \alpha(r) \\ u_t|_{t=0} = \beta(r) \end{cases}$$

$$u = \frac{v}{r}$$

$$u_r = \frac{v_r}{r} - \frac{v}{r^2} \qquad v_{rr} = \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3}$$

$$\frac{v_{tt}}{r} = a^2(\frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3} + \frac{2v_r}{r^2} - \frac{2v}{r^3}) = \frac{a^2 v_{rr}}{r}$$

$$\begin{cases} \frac{v_{tt}}{r} = \frac{a^2 v_{rr}}{r} \\ \frac{v|_{t=0}}{r} = \alpha(r) \\ \frac{v|_{t=0}}{r} = \beta(r) \end{cases} \qquad \begin{cases} v_{tt} = a^2 v_{rr} \\ v|_{t=0} = \alpha(r) r \\ v_t|_{t=0} = \beta(r) r \end{cases}$$

$$v = \alpha \frac{(r + at)^2 - (r - at)^2}{2} + \frac{1}{2a} \int_{r-at}^{r+at} \beta(R) R \, dR$$

6

$4u_{tt} = \Delta u,\ t > 0,\ (x, y, z) \in \mathbb{R}^3$

$u|_{t=0} = 0,\ u_t|_{t=0} = \begin{cases} \ln(1 + x^2 + y^2 + z^2), & (x, y, z) \in G, \\ 0. & (x, y, z) \in \mathbb{R}^3 \setminus G, \end{cases}$

$$u(\vec{x}, t) = \frac{1}{\pi t} \iint_S u_t|_{t=0}(\vec{y})\, dS_y + \frac{\partial}{\partial t} \frac{1}{\pi t} \iint_S u|_{t=0}(\vec{y})\, dS_y =$$

$$= \frac{1}{\pi t} \iint_{|\vec{x}-\vec{y}|=\frac{t}{2}} u_t|_{t=0}(\vec{y})\, dS_y + \frac{1}{\pi} \frac{\partial}{\partial t} \frac{1}{t} \iint_{|\vec{x}-\vec{y}|=\frac{t}{2}} u|_{t=0}(\vec{y})\, dS_y$$

$$\vec{x} = \vec{0}$$

$$u(\vec{0}, t) = \frac{1}{\pi t} \iint_{|\vec{y}|=\frac{t}{2}} u_t|_{t=0}(\vec{y})\, dS_y + \frac{1}{\pi} \frac{\partial}{\partial t} \frac{1}{t} \underbrace{\iint_{|\vec{y}|=\frac{t}{2}} u|_{t=0}(\vec{y})\, dS_y}_{0} =$$

$$= \frac{1}{\pi t} \iint_{|\vec{y}|=\frac{t}{2}} \ln(1 + (\vec{y})^2)\, dS_y = \frac{1}{\pi t} \ln(1 + \frac{t^2}{4}) \iint_{|\vec{y}|=\frac{t}{2}} dS_y =$$

$$= \frac{1}{\pi t} \ln(1 + \frac{t^2}{4}) \frac{4\pi |\vec{y}|^2}{4} = \frac{1}{\pi t} \ln(1 + \frac{t^2}{4}) \frac{\pi t^2}{4} = \frac{t}{4} \ln(1 + \frac{t^2}{4})$$

Часть II

Задание 2

Задача Коши для уравнения теплопроводности