REDUCED DISCRETE ADVECTED EQUATIONS

A Project Report
Submitted in partial fulfillment of
the requirements for the degree of
Master of Technology
by

Prasanth YSS
(160010047)

Supervisor:
Prof. Amuthan A. Ramabathiran

Department of Aerospace Engineering
Indian Institute of Technology Bombay
Mumbai 400076 (India)

1 July 2021

Thesis Approval Sheet

Department of Aerospace Engineering
Indian Institute of Technology, Bombay

The thesis entitled “Reduced Discrete Advected Equations” submitted by Yeduru Satya Sai Prasanth (160010047) is approved for the degree of Master of Technology in Aerospace Engineering from Indian Institute of Technology, Bombay.

Date: 1 July 2021

Prof. Amuthan A. Ramabathiran
(Supervisor)

Rohit 06/07/2021

Prof. Rohit Gupta
(Internal examiner)

R. N. Banavar 6/7/21

Prof. Ravi N Banavar
(External examiner)

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Prasanth YSS
(160010047)

Date: 6 July 2021

Abstract

Governing dynamics of an invariant Lagrangian system, evolving on Lie-group, can be reduced to Lie-algebra. A similar reduction to the Lie-group is possible for the discrete cases. Starting from an invariant discrete Lagrangian, we derive equations that can be used as an integrator. Previous theory can be extended to reduce such systems when the Lagrangian is not fully invariant due to an advected term. Reducing discrete equations to Lie-group and reconstructing the path from there, makes the integrator inherently geometric. An integrator developed from these extended equations is equivalent to a variational integrator. “Reduced discrete advected equation” have been applied to the case of heavy top and precession behavior is modeled accurately.

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Chapter 1

Reduction

Let G be a Lie Group and TG, $\mathfrak{g}$ be it's Tangent bundle and Lie algebra respectively. If G acts on TG by the tangent lifted left multiplication, then the following are diffeomorphic[1]

$$TG/G \simeq (G \times \mathfrak{g})\,/G \simeq \mathfrak{g}$$
$$[g, \dot{g}] \mapsto [g, g^{-1}\dot{g}] \mapsto g^{-1}\dot{g}$$

Similarly,

$$T^*G/G \simeq (G \times \mathfrak{g}^*)\,/G \simeq \mathfrak{g}^*$$
$$[g, \alpha] \mapsto [g, g^{-1}\alpha] \mapsto g^{-1}\alpha$$

where TG^* is the cotangent bundle and $\mathfrak{g}^*$ is the dual of Lie algebra

When the Lagrangian $L : TG \to \mathbb{R}$ is *G-invariant* under the tangent lifted left multiplication, then a reduced lagrangian $l : \mathfrak{g} \to \mathbb{R}$ can be introduced as follows

$$l(\xi) = L(e, \xi) \tag{1.1}$$

Also, Lagrangian mechanics can be studied directly from the reduced lagrangian l. This is studied under *Euler Poincaré reduction*. Similar reduction is also possible for Hamiltonian dynamics on a cotangent bundle, *Lie Poisson reduction* studies this case. Both these reduction theories can be related using the *reduced Legendre transform*.

[1]refer section 6.3 in Darryl D. Holm (2009)

1.1 Euler-Poincaré Reduction

Theorem 1.1.1. *If $L : TG \to \mathbb{R}$ is left G-invariant under the tangent lifted action of Lie group G and l is the reduced larangian from* (1.1)*, then the following statements are equivalent*

1. *The variational principle*
$$\delta \int_a^b L(g, \dot{g})\, dt = 0$$
holds on TG for variations with fixed end points.

2. *The Euler-Lagrange equations are given by*
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{g}}\right) = \frac{\partial L}{\partial g}$$

3. *The variational principle*
$$\delta \int_a^b l(\xi)\, dt = 0 \tag{1.2}$$
where $\xi = g^{-1}\dot{g}$, holds on $\mathfrak{g}$, for variations of the form
$$\delta\xi = ad_{\xi}\eta + \dot{\eta} \tag{1.3}$$
where η is arbitrary vanishing at the end points.

4. *The reduced Euler-Lagrange equations are*
$$\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) = ad^*_{\xi}\frac{\partial l}{\partial \xi} \tag{1.4}$$
The path on TG can be regenerated using the relation
$$\dot{g}(t) = g(t) \cdot \xi(t) \tag{1.5}$$

Proof. 1 & 2 are known to be equivalent from previous work. From the definition of reduced lagrangian (1.1), it follows that

$$\delta \int_a^b L(g, \dot{g})\, dt = \delta \int_a^b l(\xi)\, dt$$

Let

$$g(\epsilon, t) = g(t) exp\left(\epsilon\eta(t)\right) \tag{1.6}$$

be the variation on G with η vanishing at end points, then[2]

$$\begin{aligned}
\xi &= g^{-1}\dot{g} \\
\implies \delta\xi &= \delta\left(g^{-1}\dot{g}\right) \\
&= \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(g^{-1}(\epsilon)\cdot\dot{g}(\epsilon)\right) \\
&= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g^{-1}(\epsilon)\cdot\dot{g} + g^{-1}\cdot\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\dot{g}(\epsilon)
\end{aligned}$$

Now[3],

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} g^{-1}(\epsilon) = -g^{-1}\delta g\cdot g^{-1}$$

Also, from (1.6)

$$\begin{aligned}
\dot{g}(\epsilon) &= \frac{d}{dt}\left(g(t)exp\left(\epsilon\eta(t)\right)\right) \\
&= g(t)\cdot\xi\cdot exp\left(\epsilon\eta(t)\right) + \epsilon g(t)exp\left(\epsilon\eta(t)\right)\cdot\dot{\eta}(t) \\
\implies \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\dot{g}(\epsilon) &= g\cdot\xi\eta + g\cdot\dot{\eta}
\end{aligned}$$

Substituting back in $\delta\xi$, we have

$$\begin{aligned}
\delta\xi &= \left(-g^{-1}\delta g\cdot g^{-1}\right)\cdot(g\xi) + g^{-1}\cdot(g\cdot\xi\eta + g\cdot\dot{\eta}) \\
&= \xi\eta - \eta\xi + \dot{\eta} \\
&= ad_{\xi}\eta + \dot{\eta}
\end{aligned}$$

Since η is aritrary, this proves equivalence of 1 & 3[4].

Now for 3 $\cong$ 4,

$$\begin{aligned}
\delta\int_a^b l(\xi)dt &= \int_a^b\left\langle\frac{\partial l}{\partial\xi},\delta\xi\right\rangle dt \\
&= \int_a^b\left\langle\frac{\partial l}{\partial\xi}, ad_{\xi}\eta + \dot{\eta}\right\rangle dt \\
&= \int_a^b\left\langle ad^*_{\xi}\frac{\partial l}{\partial\xi} - \frac{d}{dt}\frac{\partial l}{\partial\xi},\eta\right\rangle dt
\end{aligned}$$

Since η is arbitrary, it follows from (1.2) that

$$\frac{d}{dt}\left(\frac{\partial l}{\partial\xi}\right) = ad^*_{\xi}\frac{\partial l}{\partial\xi}$$

□

[2] t is suppressed in the following equations for readability

[3] use $\delta\left(g\cdot g^{-1}\right) = 0$

[4] Here, we have only shown (1) $\implies$ (3), it is direct to show converse follows by tracing above steps backward

1.2 Lie-Poisson Reduction

When T^*Q is the cotangent bundle of a manifold Q, the canonical Poisson bracket $\{\cdot,\cdot\}$ is given as

$$\{F,G\}(q,p) \coloneqq \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i} \tag{1.7}$$

where $F, G \in C^\infty(T^*Q)$

The cotangent lift of any diffeomorphism of Q preserves the canonical Poisson structure. Assume a lie group G and its cotangent bundle T^*G, equipped with the canonical Poisson bracket. The Hamiltonian vector field Z_H of a Hamiltonian $H : T^*G \to \mathbb{R}$ is defined by

$$\{\cdot, H\} = Z_H(\cdot) \tag{1.8}$$

which are given in coordinates by

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial p} \end{aligned} \tag{1.9}$$

We will induce $\mathfrak{g}^*$ with a poisson structure and derive Hamiltonian dynamics directly on $\mathfrak{g}^*$. We have seen that

$$T^*G/G \simeq \mathfrak{g}^*$$

For $F : T^*G \to \mathbb{R}$, which is G-invariant under the cotangent lift of left multiplication, let $f : \mathfrak{g}^* \to \mathbb{R}$ and $r : T^*G \to \mathfrak{g}^*$ be defined such that the following diagram commutes

$$\begin{tikzcd} T^*G \arrow[r, "F"] \arrow[d] \arrow[dr, "r"] & \mathbb{R} \\ T^*G/G \arrow[r, "\simeq"] & \mathfrak{g}^* \arrow[u, "f"'] \end{tikzcd}$$

It is clear that above definition is well-defined for a G-invariant F, conversely a G-invariant F can be defined for a given f.

The Lie-Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is defined as

$$\{f,g\}_{\mathfrak{g}^*} \circ r = \{F,G\} \tag{1.10}$$

where R.H.S is the canonical Poisson bracket on T^*G

Lemma 1.2.1. *The Lie-Poisson backet on $\mathfrak{g}^*$ is given as*

$$\{f,g\}_{\mathfrak{g}^*}(\mu) = -\left\langle ad_{\frac{\delta f}{\delta \mu}} \frac{\delta g}{\delta \mu}, \mu \right\rangle \tag{1.11}$$

where $F, G : \mathfrak{g}^* \to \mathbb{R}$

Proof. (Darryl D. Holm, 2009) Poisson bracket depends only on the first derivatives of the function[5]. So, define linear functions $F_L, G_L : T^*G \to \mathbb{R}$ s.t their first derivatives match with F, G respectively. i.e.

$$F_L(q,p) := \left\langle q \cdot \frac{\delta f}{\delta \mu}, p \right\rangle$$
$$G_L(q,p) := \left\langle q \cdot \frac{\delta g}{\delta \mu}, p \right\rangle$$

then,

$$\begin{aligned}
\{F, G\} &= \{F_L, G_L\} \\
\implies \{f, g\}_{\mathfrak{g}^*}(\mu) &= \{F_L, G_L\}(e, \mu) \\
&= \left(\frac{\partial F_L}{\partial q^i} \frac{\partial G_L}{\partial p_i} - \frac{\partial G_L}{\partial q^i} \frac{\partial F_L}{\partial p_i} \right)(e, \mu) \\
&= p_j \left(\frac{\partial X^j}{\partial q^i} Y^i - \frac{\partial Y^j}{\partial q^i} X^i \right)(e, \mu) \\
&= -\left\langle [X, Y], p \right\rangle (e, \mu) \\
&= -\left\langle \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right], \mu \right\rangle
\end{aligned}$$

In above equations X, Y are vector fields generated by $\frac{\delta f}{\delta \mu}$ & $\frac{\delta q}{\delta \mu}$ respectively. Second last step follows from the Lie derivative identity[6].

Since

$$\left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] = ad_{\frac{\delta f}{\delta \mu}} \frac{\delta g}{\delta \mu}$$

we have

$$\{f, g\}_{\mathfrak{g}^*}(\mu) = -\left\langle ad_{\frac{\delta f}{\delta \mu}} \frac{\delta g}{\delta \mu}, \mu \right\rangle$$

□

For a Hamiltonian $h : \mathfrak{g}^* \to \mathbb{R}$, the Hamiltonian vector field Z_h is defined as in (1.8) i.e.

$$\{\cdot, h\}_{\mathfrak{g}^*} = Z_h(\cdot)$$

Let us now derive the coordinate expression of Z_h

Theorem 1.2.2. *For a reduced Hamiltonian $h : \mathfrak{g}^* \to \mathbb{R}$, the reduced Hamiltonian equations are given as*

$$\dot{\mu} = ad^*_{\frac{\delta h}{\delta \mu}} \mu \tag{1.12}$$

[5] refer Definition 4.26 in (Darryl D. Holm, 2009)

[6] refer theorem 3.24 in (Darryl D. Holm, 2009)

Proof.

$$\begin{aligned}\{f,h\}_{\mathfrak{g}^*} &= Z_h(f)\\ &= \left\langle \frac{\delta f}{\delta \mu}, \dot{\mu} \right\rangle\end{aligned}$$

From 1.2.1 we have

$$\begin{aligned}\{f,h\}_{\mathfrak{g}^*} &= -\left\langle \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}\right], \mu \right\rangle\\ &= \left\langle ad_{\frac{\delta h}{\delta \mu}} \frac{\delta f}{\delta \mu}, \mu \right\rangle\\ &= \left\langle \frac{\delta f}{\delta \mu}, ad^*_{\frac{\delta h}{\delta \mu}} \mu \right\rangle\\ \implies \dot{\mu} &= ad^*_{\frac{\delta h}{\delta \mu}} \mu\end{aligned}$$

□

1.3 Reduced Legendre Transform

So far, we developed reduced Lagrangian & Hamiltonian mechanics independently, in this section we will introduce the reduced Legendre transform $\mathbb{F}l : \mathfrak{g} \to \mathfrak{g}^*$. The Legendre transform $\mathbb{F}L : TQ \to T^*Q$ is defined as

$$\langle \mathbb{F}L(q,v), w \rangle = \left.\frac{d}{ds}\right|_{s=0} L(q, v + sw) \tag{1.13}$$

Assume that the Lagrangian is G-invariant, then the reduced legendre transform $\mathbb{F}l : \mathfrak{g} \to \mathfrak{g}^*$ is defined such that the following diagram commutes

$$\begin{tikzcd} TG \arrow[r, "\mathbb{F}L"] \arrow[d] & T^*G \arrow[d] \\ TG/G \arrow[d, "\simeq"] & T^*G/G \arrow[d, "\simeq"] \\ \mathfrak{g} \arrow[r, "\mathbb{F}l"] & \mathfrak{g}^* \end{tikzcd}$$

Check that the coordinate expression of $\mathbb{F}l$ is given as

$$\mathbb{F}l(\xi) = \frac{\delta l}{\delta \xi} \tag{1.14}$$

If the Hamiltonian and Lagrangian are related by

$$H(g, \alpha) = \langle \mathbb{F}L(g, \dot{g}), \dot{g} \rangle - L(g, \dot{g}) \tag{1.15}$$

where $\alpha = \mathbb{F}L(g, \dot{g})$,
Then we know that the Hamiltonian vector field Z_H is the push forward of the Lagrangian vector field X_L. If (1.15) holds, H is G-invariant whenever L is invariant. In such cases, following lemma shows that $\mathbb{F}l$ pushes forward Euler-Poincaré equations into Lie-Poisson equations.

Lemma 1.3.1. *Assume that* (1.15) *holds, and l, h are reduced versions of L, H respectively. Then*

$$h(\mu) = \mu \cdot \xi - l(\xi) \tag{1.16}$$

where $\mu = \mathbb{F}l(\xi)$
and Lie-Poisson equation (1.12) *is the push-forward of Euler-Poincaré equation* (1.4) *under* $\mathbb{F}l$

Proof.

$$\mu = \mathbb{F}l(\xi)$$
$$\implies \mu = \frac{\delta l}{\delta \xi}$$

Proof of (1.16) is straight-forward and it follows from there that

$$\frac{\delta h}{\delta \mu} = \xi$$

Substituting above equations in Euler-Poincaré equation (1.4)

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) = ad^*_{\xi}\frac{\partial l}{\partial \xi}$$

we have

$$\frac{d\mu}{dt} = ad^*_{\frac{\delta h}{\delta \mu}}\mu$$
$$\implies \dot{\mu} = ad^*_{\frac{\delta h}{\delta \mu}}\mu$$

□

In conclusion, when G is a Lie group and the Lagrangian $L : TG \to \mathbb{R}$ is G-invariant the system can be studied by it's evolution on the Lie algebra. Euler-Poincaré equations dictate evolution on Lie algebra, while Lie-Poisson equations study the corresponding hamiltonian evolution on $\mathfrak{g}^*$. Reduced Legendre transform bridges both these theories. It is easier to deal with differential equations and coordinates on a linear vector space. Since Lie algebra is a vector space unlike the Lie group G, it is advantageous to have these equations.

Example 1 (Free rigid body)**.** *Consider a free rigid body fixed at the origin. The phase space of the system is* $TSO(3)$*. The Lagrangian for this system is given as*

$$L\left(R,\dot{R}\right) = \frac{1}{2}tr\left(\dot{R}\mathbb{J}\dot{R}^T\right)$$

where, $\mathbb{J}$ *is the inertia of body calculated w.r.t pivot*
Above Lagrangian is G-invariant and can be studied via Euler-Poincaré and Lie-Poisson reductions.
Note that for $\Omega \in \mathbb{R}^3$*,* $\hat{\Omega} \in \mathfrak{so}(3)$ *is defined such that*

$$\hat{\Omega} v = \Omega \times v \qquad \forall\, v \in \mathbb{R}^3$$

Before proceeding to solve, we need to derive following identities on $SO(3)$*.*
Note: Π *&* Ω *are treated as column vectors*

1. $ad_{\hat{\Omega}}\hat{\eta} = \widehat{\Omega \times \eta}$

$$ad_{\hat{\Omega}}\hat{\eta} = \mathcal{L}_X^Y$$

where $X(g) = g \cdot \hat{\Omega}$ $Y(g) = g \cdot \hat{\eta}$

$$= \left.\frac{d}{ds}\right|_{s=0}\left[\left.\frac{d}{dt}\right|_{t=0} exp(t\hat{\Omega}) \cdot exp(s\hat{\eta}) \cdot exp(-t\hat{\Omega})\right]$$
$$= \hat{\Omega}\hat{\eta} - \hat{\eta}\hat{\Omega}$$
$$= \widehat{\Omega \times \eta}$$

2. $ad^*_{\hat{\Omega}}\hat{\Pi} = \widehat{\Pi \times \Omega}$

$$\left\langle ad^*_{\hat{\Omega}}\hat{\Pi}, \eta\right\rangle = \left\langle \hat{\Pi}, ad_{\hat{\Omega}}\hat{\eta}\right\rangle$$
$$= \langle \hat{\Pi}, \widehat{\Omega \times \eta}\rangle$$
$$= \Pi \cdot (\Omega \times \eta) = (\Pi \times \Omega) \cdot \eta$$
$$\implies ad^*_{\hat{\Omega}}\hat{\Pi} = \widehat{\Pi \times \Omega} \tag{1.17}$$

Note: α *is used in place of* $\hat{\alpha} \in \mathfrak{so}^*(3)$ *in the following derivations*

<u>Euler-Poincaré reduction:</u>

The reduced Lagrangian $l : \mathfrak{so}(3) \to \mathbb{R}$ *is given as*

$$l(\Omega) = \frac{1}{2}\Omega^T \mathbb{I}\Omega$$

Euler-Poincaré equation is

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \Omega}\right) = ad^*_{\Omega}\frac{\delta l}{\delta \Omega}$$
$$\frac{\delta l}{\delta \Omega} = \mathbb{I}\Omega$$
$$\implies \mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega \qquad (\textit{from } (1.17))$$

Lie-Poisson reduction:

The reduced Hamiltonian $h : \mathfrak{so}^*(3) \to \mathbb{R}$ *is given as*

$$\begin{aligned} h(\Pi) &= \left\langle \Pi, \mathbb{F}L^{-1}(\Pi) \right\rangle - L(\mathbb{F}L^{-1}(\Pi)) \\ \mathbb{F}L(\Omega) &= \frac{\delta l}{\delta \Omega} = \mathbb{I}\Omega \\ \implies h(\Pi) &= \frac{1}{2}\Pi^T \mathbb{I}^{-1}\Pi \\ \frac{\delta h}{\delta \Pi} &= \mathbb{I}^{-1}\Pi \end{aligned}$$

Lie-Poisson equation is

$$\begin{aligned} \dot{\Pi} &= ad^*_{\frac{\delta h}{\delta \Pi}}\Pi \\ &= \Pi \times \mathbb{I}^{-1}\Pi \\ \implies \dot{\Pi} &= \Pi \times \Omega \qquad (\textit{from } (1.17)) \end{aligned}$$

1.4 Discrete Reduction

Consider a system with Lagrangian $L : TQ \to \mathbb{R}$, action $A : C(Q) \to \mathbb{R}$ is a functional on the set of smooth curves on Q.

$$A = \int_0^T L\left(q(s), \dot{q}(s)\right) ds$$

For the purpose of integrators, a discrete Lagrangian $L_d^h : Q \times Q \to \mathbb{R}$ is introduced. L_d^h approximates action A

$$L_d^h\left(q_0, q_1\right) \approx \int_0^h L\left(q(t), \dot{q}(t)\right) dt \tag{1.18}$$

Variational integrators are constructed by extremizing the discrete action

$$A_d\left(q_d\right) = \sum_{i=1}^{N-1} L_d^h\left(q_i, q_{i+1}\right)$$

where $q_d : \left\{ ih \middle| \; i \in \{1, 2, \dots N-1\} \right\} \to Q$ is the discrete path. The discrete Lagrangian flow $F_{L_d} : Q \times Q \to Q \times Q$ satisfies

$$F_{L_d}\left(q_i, q_{i+1}\right) = \left(q_{i+1}, q_{i+2}\right)$$

The discrete Hamiltonian version is constructed by introducing the discrete Legendre transforms $\mathbb{F}L_d^{\pm} : Q \times Q \to T^*Q$

$$\begin{aligned} \mathbb{F}L_d^-\left(q_1, q_2\right) &= -D_1 L_d\left(q_1, q_2\right) \\ \mathbb{F}L_d^+\left(q_1, q_2\right) &= D_2 L_d\left(q_1, q_2\right) \end{aligned} \tag{1.19}$$

The discrete Hamiltonian flow $F_{H_d} : T^*Q \to T^*Q$ is defined such that the following diagram commutes

$$\begin{tikzcd}
 & Q\times Q \arrow[rr, "F_{L_d}"] \arrow[dl, "\mathbb{F}L_d^-"'] \arrow[dr, "\mathbb{F}L_d^+"] & & Q\times Q \arrow[dl, "\mathbb{F}L_d^-"'] \arrow[dr, "\mathbb{F}L_d^+"] & \\
T^*Q \arrow[rr, "F_{H_d}"] & & T^*Q \arrow[rr, "F_{H_d}"] & & T^*Q
\end{tikzcd}$$

i.e.

$$F_{h_d} = \mathbb{F}L_d^+ \circ \left(\mathbb{F}L_d^-\right)^{-1} \tag{1.20}$$

We have proved following properties in our previous study from (Marsden and West, 2001)

1. Variational integrators are symplectic on T^*Q w.r.t the canonical symplectic form.

2. If the discrete Lagrangian is invariant under the left action of a Lie group G, then the discrete Lagrange momentum map $J_{L_d} : Q \times Q \to \mathfrak{g}^*$ can be defined as

$$\begin{aligned} \left\langle J_{L_d}(q_1, q_2), \xi \right\rangle &= \left\langle -D_1 L_d(q_1, q_2), \xi \cdot q_1 \right\rangle \\ &= \left\langle \; D_2 L_d(q_1, q_2), \; \xi \cdot q_2 \right\rangle \end{aligned} \tag{1.21}$$

   J_{L_d} is conserved along F_{L_d} whenever L_d is G-invariant.

3. A discrete Lagrangian is said to be exact, when approximation in (1.18) is satisfied. For an exact discrete Lagrangian, the discrete path satisfies

$$q_d(k) = q(k) \qquad \text{where } q \text{ is the continuous solution}$$

   .

In this section, we will see that when G is a Lie group G and $L_d : G \times G \to \mathbb{R}$ is G-invariant, we can study reduced dynamics on G. Analogous to continuous case, discrete versions of Euler-Poincaré and Lie-Poisson reductions are introduced. We will prove that the reduced discrete version is equivalent to the unreduced discrete case. Hence above properties of variational integrators and momentum conservation are satisfied automatically.

1.4.1 Discrete Euler-Poincaré Reduction

Theorem 1.4.1. *Let $L_d : G \times G \to \mathbb{R}$, L_d is G-invariant under the left multiplicative action, then a reduced discrete Lagrangian $l_d : G \to \mathbb{R}$ can be introduced as*

$$l_d(f) = L_d(g_1, g_2) \tag{1.22}$$

where $f = g_1^{-1} g_2$
The discrete Euler-Poincaré equations are given as

$$\left(R_{f_{i+1}}\right)^* \frac{\delta l_d}{\delta f}(f_{i+1}) = \left(L_{f_i}\right)^* \frac{\delta l_d}{\delta f}(f_i) \tag{1.23}$$

where, $R_f : G \to G$, $R_f(g) = gf$ is the right multiplicative action. Similary $L_f(g) = fg$ is the left multiplicative action.
The discrete path $g_d \in \underbrace{G \times G \cdots \times G}_{N\ times}$ can be reconstructed using the following equation

$$g_{i+1} = g_i f_i \tag{1.24}$$

g_d extremizes the discrete action A_d

Proof. We will show that the path f_d satisfies (1.23) whenever g_d defined by (1.24) extremizes A_d

$$\begin{aligned} A_d &= \sum_{i=1}^{n-1} L_d(g_i, g_{i+1}) \\ &= \sum_{i=1}^{n-1} l_d\left(g_i^{-1} g_{i+1}\right) \end{aligned}$$

let $f_i = g_i^{-1} g_{i+1}$, then

$$\begin{aligned} \delta f_i &= \delta\left(g_i^{-1} g_{i+1}\right) \\ &= \delta g_i^{-1} g_{i+1} + g_i^{-1} \delta g_{i+1} \\ &= -g_i^{-1} \delta g_i g_i^{-1} g_{i+1} + g_i^{-1} \delta g_{i+1} \end{aligned}$$

let $\delta g_j = g_j \cdot \eta_j$

$$\implies \delta f_i = f_i \cdot \eta_{i+1} - \eta_i \cdot f_i$$

$$dA_d \cdot \delta g_d = \sum_{i=1}^{n-1} \left\langle \frac{\delta l_d}{\delta f_i}, \delta f_i \right\rangle$$

Substituting δf_i,

$$\begin{aligned} dA_d \cdot &= \sum_{i=1}^{n-1} \left\langle \frac{\delta l_d}{\delta f_i}, f_i \cdot \eta_{i+1} - \eta_i \cdot f_i \right\rangle \\ &= \sum_{i=1}^{n-1} \left\langle \left(L_{f_i}\right)^* \frac{\delta l_d}{\delta f_i} - \left(R_{f_{i+1}}\right)^* \frac{\delta l_d}{\delta f_{i+1}}, \eta_{i+1} \right\rangle \end{aligned}$$

Since η_j are arbitrary, we have

$$\left(R_{f_{i+1}}\right)^* \frac{\delta l_d}{\delta f_{i+1}} = \left(L_{f_i}\right)^* \frac{\delta l_d}{\delta f_i}$$

□

Discrete Euler-Poincaré flow

The discrete Euler-Poincaré flow $F_{l_d} : G \to G$ is defined such that

$$\begin{aligned} & F_{l_d}(f) = g \\ \iff & \left(R_g\right)^* \frac{\delta l_d}{\delta g} = \left(L_f\right)^* \frac{\delta l_d}{\delta f} \end{aligned} \tag{1.25}$$

1.4.2 Discrete Lie-Poisson Reduction

Discrete Lie-Poisson reduction evolves on $\mathfrak{g}^*$ and can be viewed as the reduction of discrete Hamiltonian dynamics. To be in parallel with the study of discrete dynamics done earlier, we introduce the corresponding Legendre transforms.

Reduced discrete Legendre transforms:

The reduced discrete Legendre transforms $\mathbb{F}l_d^{\pm} : G \to \mathfrak{g}^*$ are defined as

$$\begin{aligned} \mathbb{F}l_d^-(f) &= \left(R_f\right)^* \frac{\delta l_d}{\delta f} \\ \mathbb{F}l_d^+(f) &= \left(L_f\right)^* \frac{\delta l_d}{\delta f} \end{aligned} \tag{1.26}$$

These equations are equal to left multiplicative pull-back of (1.19), where $f = g_1^{-1} g_2$

Reduced discrete momentum: The reduced discrete momentum is defined as $\mu : G \to \mathfrak{g}^*$

$$\mu(f) = \mathbb{F}L_d^{-1}(f) = \left(R_f\right)^* \frac{\delta l_d}{\delta f} \tag{1.27}$$

Discrete Lie-Poisson flow The discrete Lie-Poisson flow $F_{h_d} : \mathfrak{g}^* \to \mathfrak{g}^*$ is reminiscent of F_{H_d} in (1.20)

$$F_{h_d} = \mathbb{F}l_d^+ \circ \left(\mathbb{F}l_d^-\right)^{-1} \tag{1.28}$$

$$\begin{tikzcd}
 & G \arrow[rr, "F_{l_d}"] \arrow[dl, "\mathbb{F}l_d^-"'] \arrow[dr, "\mathbb{F}l_d^+"] & & G \arrow[dl, "\mathbb{F}l_d^-"'] \arrow[dr, "\mathbb{F}l_d^+"] & \\
\mathfrak{g}^* \arrow[rr, "F_{h_d}"] & & \mathfrak{g}^* \arrow[rr, "F_{h_d}"] & & \mathfrak{g}^*
\end{tikzcd}$$

Discrete Lie-Poisson equation: With above definitions in place, it is easy to verify that the discrete Lie-Poisson equation

$$\mu_{i+1} = Ad_{f_i}^* \mu_i \tag{1.29}$$

is equivalent to (1.23)

The discrete path g_d can be constructed recursively from (μ_1, g_1)

$$\begin{aligned} f_i &= \left(\mathbb{F}L_d^-\right)^{-1}(\mu_i) \\ g_{i+1} &= g_i f_i \end{aligned} \tag{1.30}$$

Example 2 (Free rigid body)**.** *We will have a quick glance at the discrete equations for a free rigid body. A more detailed study is done for the same system under the action of gravity after we develop 'discrete advected equations' in next chapter.*

For a fixed free rigid body, the Lagrangian is defined on $TSO(3)$*.*

$$L\left(R, R\hat{\Omega}\right) = \frac{1}{2} tr\left(\hat{\Omega} J_d \hat{\Omega}^T\right)$$

Consider a reduce discrete Lagrangian $l_d : SO(3) \to \mathbb{R}$ *as*[7]

$$l_d\,(F) = \frac{1}{2h} tr\left((F - I) J_d (F - I)^T\right)$$

The reduced discrete momentum[8] $\hat{\Pi}$ *is*

$$\begin{aligned}
\hat{\Pi} &= (R_F)^* \frac{\delta l_d}{\delta F} \\
&= \frac{1}{2h}\left(F J_d - J_d F^T\right)
\end{aligned} \tag{1.31}$$

It can be checked that the discrete Lagrange momentum map J_{L_d} *in* (1.21)

$$J_{L_d}\,(R_1, R_2) = R_1 \Pi_1 \tag{1.32}$$

Since L_d *is G-invariant, we expect the discrete Lagrange momentum* J_{L_d} *to be conserved. Conservation of* J_{L_d} *can be confirmed directly from discrete Lie-Poisson equations. In this case,* (1.29) *reads*

$$\begin{aligned}
\hat{\Pi}_{i+1} &= Ad^*_{F_i} \hat{\Pi}_i \\
&= F_i^T \hat{\Pi}_i F_i \\
\textit{where, } F_i &= R_i^T R_{i+1} \\
\textit{i.e. } R_{i+1} \hat{\Pi}_{i+1} R_{i+1}^T &= R_i \hat{\Pi}_i R_i^T \\
\implies R_{i+1} \Pi_{i+1} &= R_i \Pi_i
\end{aligned}$$

Above derivation is independent of the choice of discrete Lagrngian! i.e. $R\hat{\Pi}$ *is conserved irrespective of* L_d*. However, the momentum* $\hat{\Pi}$ *itself depends on the choice of* L_d*. This momentum is discrete version of the spatial angular momentum from continuous case. Discrete Lie-Poisson equations give us an integrator which conserves this momentum along with respecting symplecticity and geometric structure.*

[7] refer (3.13)
[8] refer (3.17)

Chapter 2

Advected Parameters

So far we developed tools to study dynamics on reduced Lie groups. These are applicable only when the discrete Lagrangian is G-invariant. When the Lagrangian has broken symmetry, it is possible to study the reduced dynamics by restricting the action of Lie group G. However, a more elegant reduction is possible by embedding the manifold in a higher dimension space. We will study one such case when L has an advected term.

Let V be a vector space with the Lie group G acting linearly on it i.e.

$$g \cdot (v + w) = g \cdot v + g \cdot w \quad \forall\, g \in G \;\&\; v, w \in V$$

The action of $\xi \in \mathfrak{g}$ on V is defined as

$$\xi \cdot v = \left.\frac{d}{ds}\right|_{s=0} exp(s\xi) \cdot v$$

We can then define a map $\rho_v : \mathfrak{g} \to V$ as

$$\rho_v(\xi) = \xi \cdot v$$

This defines the dual-map $\rho_v^* : V^* \to \mathfrak{g}^*$ as

$$\left\langle \rho_v^*(w), \xi \right\rangle = \langle w, \xi \cdot v \rangle$$

Diamond operator: The diamond operator $\diamond : V \times V^* \to \mathfrak{g}^*$ is defined as

$$\langle v \diamond w, \xi \rangle = \langle w, \xi \cdot v \rangle \tag{2.1}$$

2.1 Advected Parameters

Theorem 2.1.1. *Let $\tilde{L} : TG \to \mathbb{R}$ be a Lagrangian on TG. Assume G acts linearly on a vector space V and there exists a function $L : TG \times V^* \to \mathbb{R}$ such that*

$$L(g, \dot{g}, a_0) = \tilde{L}(g, \dot{g})$$

If L is G-invariant, then we can introduce a reduced Lagrangian $l : \mathfrak{g} \times V^* \to \mathbb{R}$ *satisfying*

$$l(\xi, a) = L(e, \xi, a)$$

Then the following equations are equivalent to the evolution of $\tilde{L}$ *on* TG

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) &= ad^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a \\ \dot{a}(t) &= -\xi(t) \cdot a(t) \\ \dot{g}(t) &= g(t) \cdot \xi(t) \end{aligned} \tag{2.2}$$

Proof. [1] The action $A : C(TG) \to \mathbb{R}$ is[2]

$$\begin{aligned} A(g) &= \int_0^T \tilde{L}(g, \dot{g}) dt \\ &= \int_0^T L(g, \dot{g}, a_0) dt = \int_0^T l(g^{-1}\dot{g}, g^{-1}a_0) dt \end{aligned}$$

let $\xi = g^{-1}\dot{g}$, $a = g^{-1}a_0$

$$dA \cdot \delta g = \int_0^T \left\langle \frac{\delta l}{\delta \xi}, \delta \xi \right\rangle + \left\langle \frac{\delta l}{\delta a}, \delta a \right\rangle dt$$

When $\delta g = g \cdot \eta$, we derived in (1.3) that

$$\delta \xi = ad_\xi \eta + \dot{\eta}$$

$$\begin{aligned} a &= g^{-1} a_0 \\ \implies \delta a &= -\eta \cdot a \end{aligned}$$

Substituting variations we have

$$\begin{aligned} dA \cdot \delta g &= \int_0^T \left\langle \frac{\delta l}{\delta \xi}, ad_\xi \eta + \dot{\eta} \right\rangle + \left\langle \frac{\delta l}{\delta a}, -\eta \cdot a \right\rangle dt \\ &= \int_0^T \left\langle ad^*_\xi \frac{\delta l}{\delta \xi} - \frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right), \eta \right\rangle + \left\langle \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle dt \\ &= \int_0^T \left\langle -\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) + ad^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle dt \end{aligned}$$

Since η is arbitrary, we have

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) = ad^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a$$

□

[1] t-dependence is suppressed in notation to be succinct

[2] $C(TG)$ is the set of all continuous curves on TG

Example 3 (Heavy top)**.** *Consider a fixed 3-D Rigid body under the action of gravity. The Lagrangian $\tilde{L} : TSO(3) \to \mathbb{R}$ is given as*

$$\tilde{L}\left(R, \dot{R}\right) = \frac{1}{2} tr\left(\dot{R}\mathbb{J}\dot{R}^T\right) - mg\left\langle e_3, R\rho\right\rangle \tag{2.3}$$

where ρ is coordinate of centre of mass of the body w.r.t pivot in the body frame.

Introducing the variable $a \in \mathbb{R}^{3^*}$ and $a_0 = e_3$, we can see that the Lagrangian $L : TSO(3) \times \mathbb{R}^{3^*} \to \mathbb{R}$

$$L\left(R, \dot{R}, a\right) = \frac{1}{2} tr\left(\dot{R}\mathbb{J}\dot{R}^T\right) - mg\left\langle a, R\rho\right\rangle$$

is G-invariant and satisfies the conditions in theorem 2.1.1, so we can introduce the reduced Lagrangian $l : \mathfrak{so}(3) \times \mathbb{R}^{3^*} \to \mathbb{R}$ as

$$l(\Omega, a) = \frac{1}{2}\Omega^T \mathbb{I}\Omega - mg\left\langle a, \rho\right\rangle$$

After deriving the following results, we can substitute in (2.2) *to arrive at the governing equations*

1. $\underline{\dfrac{\delta l}{\delta \Omega}}$

$$\frac{\delta l}{\delta \Omega} = \mathbb{I}\Omega$$

2. $\underline{ad^*_\Omega \dfrac{\delta l}{\delta \Omega}}$

   It follows from (1.17) *that*

$$\begin{aligned} ad^*_{\Omega}\frac{\delta l}{\delta \Omega} &= \frac{\delta l}{\delta \Omega} \times \Omega \\ &= \mathbb{I}\Omega \times \Omega \end{aligned}$$

3. $\underline{\dfrac{\delta l}{\delta a} \diamond a}$

$$\begin{aligned} \left\langle \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle &= \left\langle a, \eta \cdot \frac{\delta l}{\delta a} \right\rangle \\ \text{substituting } \frac{\delta l}{\delta a} &= -mg\rho \\ \implies \left\langle \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle &= -mg \frac{d}{ds}\bigg|_{s=0} \left\langle a, exp\left(s\hat{\eta}\right)\rho \right\rangle \\ &= -mg\left\langle a, \eta \times \rho\right\rangle \\ &= mg\left\langle a \times \rho, \eta\right\rangle \end{aligned}$$

   Adding the constraint $R \cdot a = e_3$, we have

$$\begin{aligned} \dot{R}a + R\dot{a} &= 0 \\ \implies \dot{a} &= -R^T\dot{R}a = a \times \Omega \end{aligned}$$

Substituting above equations in (2.2)*, we have*

$$\begin{aligned}\mathbb{I}\dot{\Omega} &= \mathbb{I}\Omega \times \Omega + mg(a \times \rho)\\ \dot{a} &= a \times \Omega\\ \dot{R} &= R\hat{\Omega}\end{aligned}$$

2.2 Reduced Discrete Advected Equations

We will derive discrete version of theorem 2.1.1. This can be used to derive variational integrators for systems with advected parameters. As a demonstration, equations developed here are shown to be equivalent to integrator in (Lee *et al.*, 2005), for a particular choice of discrete Lagrangian. Definitions and flow of concept in this section are redolent of section 1.4

Theorem 2.2.1. *Let $\tilde{L}_d^{w_0} : G \times G \to \mathbb{R}$ be a discrete Lagrangian. Assume G acts linearly on a vector space V and there exists a function $L_d : G \times G \times V^* \to \mathbb{R}$ such that*

$$L_d\left(g_1, g_2, w_0\right) = \tilde{L}^{w_0}\left(g_1, g_2\right)$$

Assume L_d is G-invariant, then we can introduce a reduced Lagrangian $l_d : G \times V^* \to \mathbb{R}$ satisfying

$$l_d(f, w) = L(g_1, g_2, g_1 \cdot w)$$

where $f = g_1^{-1}g_2$
Then the following equations are equivalent to the evolution of $\tilde{L}_d$ on $G \times G$

$$\begin{aligned} w_{i+1} &= f_i^{-1}w_i \\ \left(R_{f_{i+1}}\right)^* \frac{\delta l_d}{\delta f}\left(f_{i+1}\right) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)\left(f_{i+1}, w_{i+1}\right) &= \left(L_{f_i}\right)^* \frac{\delta l_d}{\delta f}\left(f_i, w_i\right)\end{aligned} \tag{2.4}$$

The discrete path g_d on G can be reconstructed via

$$g_{i+1} = g_i f_i$$

Proof.

$$
\begin{aligned}
A_d &= \sum_{i=1}^{n-1} \tilde{L}_d^{w_0}\left(g_i, g_{i+1}\right) \\
&= \sum_{i=1}^{n-1} L_d\left(g_i, g_{i+1}, w_0\right) = \sum_{i=1}^{n-1} l_d\left(g_i^{-1} g_{i+1}, g_i^{-1} w_0\right)
\end{aligned}
$$

let $f_j = g_j^{-1} g_{j+1}$ & $w_k = g_k^{-1} w_0$, then

$$
dA_d \cdot \delta g_d = \sum_{i=1}^{n-1} \left\langle \frac{\delta l_d}{\delta f_i}, \delta f_i \right\rangle + \left\langle \frac{\delta l_d}{\delta w_i}, \delta w_i \right\rangle
$$

$$
\begin{aligned}
&\text{let } \delta g_j = g_j \cdot \eta_j \\
\implies \; &\delta f_i = f_i \cdot \eta_{i+1} - \eta_i \cdot f_i \\
&\delta w_i = -\eta_i \cdot w_i
\end{aligned}
$$

Substituting back,

$$
\begin{aligned}
dA_d \cdot \delta g_d &= \sum_{i=1}^{n-1} \left\langle \frac{\delta l_d}{\delta f_i}, f_i \cdot \eta_{i+1} - \eta_i \cdot f_i \right\rangle + \left\langle \frac{\delta l_d}{\delta w_i}, -\eta_i \cdot w_i \right\rangle \\
&= \sum_{i=1}^{n-1} \left\langle \left(L_{f_i}\right)^* \frac{\delta l_d}{\delta f_i} - \left(R_{f_{i+1}}\right)^* \frac{\delta l_d}{\delta f_{i+1}} + \frac{\delta l_d}{\delta w_{i+1}} \diamond w_{i+1}, \eta_{i+1} \right\rangle
\end{aligned}
$$

Since η_j are arbitrary, we have

$$
\left(R_{f_{i+1}}\right)^* \frac{\delta l_d}{\delta f}\left(f_{i+1}\right) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)\left(f_{i+1}, w_{i+1}\right) = \left(L_{f_i}\right)^* \frac{\delta l_d}{\delta f}\left(f_i, w_i\right)
$$

□

Discrete advected Euler-Poincaré flow: The discrete advected Euler-Poincaré flow $\tilde{F}_{l_d}$: $G \times V^* \to G \times V^*$ is defined as

$$
\begin{aligned}
&\tilde{F}_{l_d}(f, u) = (g, v) \\
\iff &\left(R_g\right)^* \frac{\delta l_d}{\delta f}(g) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)(g, v) = \left(L_f\right)^* \frac{\delta l_d}{\delta f}(f, u)
\end{aligned}
\tag{2.5}
$$

Discrete advected Legendre transform: The Legendre tranforms for the advected case $\mathbb{F}l_d^{\pm} : G \times V^* \to \mathfrak{g}^* \times V^*$ are defined as

$$
\begin{aligned}
\tilde{\mathbb{F}}l_d^{-}(f, w) &= \left(\left(R_f\right)^* \frac{\delta l_d}{\delta f}(f) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)(f, w), w\right) \\
\tilde{\mathbb{F}}l_d^{+}(f, w) &= \left(\left(L_f\right)^* \frac{\delta l_d}{\delta f}, f^{-1} w\right)
\end{aligned}
\tag{2.6}
$$

Discrete advected momentum: Discrete advected momentum $\mu(f, w) : G \times V^* \to \mathfrak{g}^*$ is defined to study the Hamiltonian version

$$
\mu = \left(R_f\right)^* \frac{\delta l_d}{\delta f}(f) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)(f, w) \tag{2.7}
$$

<u>**Discrete advected Lie-Poisson flow:**</u> $\tilde{F}_{h_d}$ is defined so that the following diagram commutes

$$\begin{tikzcd}
 & G \times V^* \arrow[rr, "\tilde{F}_{l_d}"] \arrow[dl, "\tilde{\mathbb{F}}l_d^-"'] \arrow[dr, "\tilde{\mathbb{F}}l_d^+"] & & G \times V^* \arrow[dl, "\tilde{\mathbb{F}}l_d^-"'] \arrow[dr, "\tilde{\mathbb{F}}l_d^+"] & \\
\mathfrak{g}^* \times V^* \arrow[rr, "\tilde{F}_{h_d}"] & & \mathfrak{g}^* \times V^* \arrow[rr, "\tilde{F}_{h_d}"] & & \mathfrak{g}^* \times V^*
\end{tikzcd}$$

i.e.

$$\tilde{F}_{h_d} = \tilde{\mathbb{F}}^+_{l_d} \circ \left(\tilde{\mathbb{F}}^-_{l_d}\right)^{-1} \tag{2.8}$$

Discrete advected Lie-Poisson equations: With notations from above definitions, it is easy to verify that the following discrete advected Lie-Poisson equations are equivalent to (2.4).

$$\begin{aligned}
(f_i, w_i) &= \left(\tilde{\mathbb{F}}^-_{l_d}\right)^{-1} (\mu_i, w_i) \\
\mu_{i+1} &= (L_{f_i})^* \frac{\delta l_d}{\delta f_i} \\
w_{i+1} &= f_i^{-1} w_i
\end{aligned} \tag{2.9}$$

The discrete path g_d is reconstructed using

$$g_{i+1} = g_i f_i \tag{2.10}$$

Chapter 3

Reduced Discrete Advected Equations applied to Heavy Top

3.1 Heavy Top Dynamics

A heavy top system spins about it's symmetry axis in the presence of gravity. Tops have been interest of study since the time of Newton for their "strange" behavior. This "strange" behavior has practical applications in gyroscopes. Gyroscopes are similar to heavy tops and used in aircrafts and satellites to measure the change in pitch, roll and yaw. In this section we will develop equations of motion for an heavy top.

Here, we study heavy top dynamics from a 'Torque-Inertia'[1] approach using local coordinates. Though the Lagrangian version is more elegant and coordinate free, approach followed here provides us with a better intuition about the system.

Note on notation: The inertial coordinate axis and vectors in that frame are represented in small case letters, while the same in body frame are represented in capital letters.

Let (x, y, z) be the inertial reference frame with gravity acting in the negative z-direction. Rotations (ϕ, θ, ψ) are applied respectively to (z, x', z'') axis. These act as Euler angles in this case.

[1]Derivation followed here is adopted from MIT lecture notes on rigid body dynamics by J. Peraire & S. Widnall

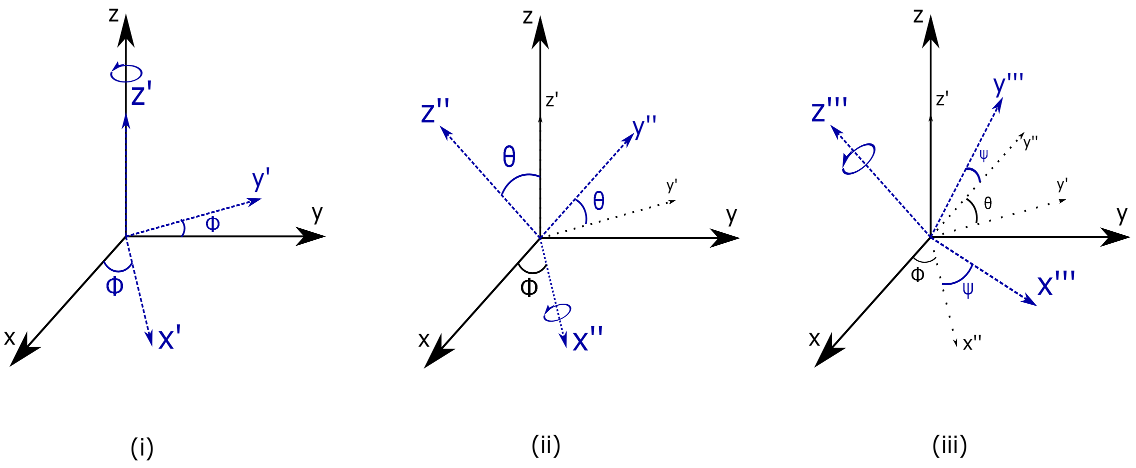


Figure 3.1: Euler Angles

Assume that the top spins about $z'' - axis$. Exploiting the symmetry of the top, we choose to develop our equations in the frame following ϕ, θ i.e. (x'', y'', z'') from above figure. Let us represent this frame by (X, Y, Z). Top spins at a rate $\dot{\psi}$ about it's Z-axis. The nutation rate $\dot{\theta}$ is measured about the $X - axis$, while precession rate $\dot{\phi}$ is measured about $z - axis$.

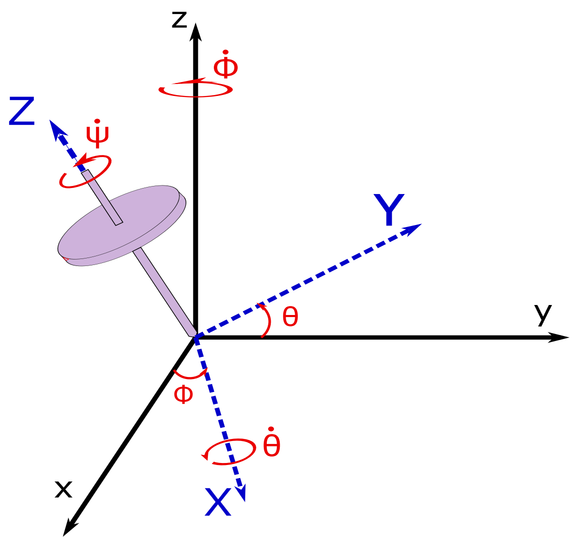


Figure 3.2: Heavy Top

The relation between vectors of both frames is given by

$$
\begin{aligned}
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} cos\phi & sin\phi & 0 \\ -cos\theta sin\phi & cos\theta cos\phi & sin\theta \\ sin\theta sin\phi & -sin\theta cos\phi & cos\theta \end{bmatrix}}_{R^T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned} \tag{3.1}
$$

$\psi \in [0, 2\pi]$, $\phi \in [0, 2\pi]$, $\theta \in [0, \pi]$

Above matrix R^T belongs to $SO(3)$

The angular velocities of top are given by

$$\omega_X = \dot{\theta} \tag{3.2}$$

$$\omega_Y = \dot{\phi} sin\theta \tag{3.3}$$

$$\omega_Z = \dot{\psi} + \dot{\phi} cos\theta \tag{3.4}$$

Inertia matrix of a top $\mathbb{I}$ in (X, Y, Z) frame, is diagonal due to symmetry. In addition, owing to the symmetry about X, Y axis we have $I_{xx} = I_{yy} = I_0$

$$\mathbb{I} = diag\left[I_0, I_0, I\right] \tag{3.5}$$

where $I_{zz} = I$

The angular momentum

$$H_X = I_0 \omega_X = I_0 \dot{\theta} \tag{3.6}$$

$$H_Y = I_0 \omega_Y = I_0 \dot{\phi} sin\theta \tag{3.7}$$

$$H_Z = I \omega_Z = I\dot{\psi} + I\dot{\phi} cos\theta \tag{3.8}$$

The moments M_X, M_Y, M_Z in body frame are

$$M_X = mg\rho sin\theta, \quad M_Y = 0, \quad M_Z = 0 \tag{3.9}$$

where m is mass of the body and ρ is coordinate of centre of mass in body frame. M_x, M_y, M_z can be calculated from (3.1), let us represent moment in inertial frame by M and moments in body frame by M', then

$$M = RM'$$

where R is as in (3.1)

The equations of motion are

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \dot{H}_x \\ \dot{H}_y \\ \dot{H}_z \end{bmatrix}$$

$$\implies M_{(X,Y,Z)} = \Omega \times \mathbb{I}\omega_{(X,Y,Z)} + \mathbb{I}\dot{\omega}_{(X,Y,Z)}$$

where Ω is angular velocity of refernce frame (X, Y, Z) w.r.t inertial frame, expressed in (X, Y, Z) coordinates

$$\Omega_X = \dot{\theta}$$
$$\Omega_Y = \dot{\phi} sin\theta$$
$$\Omega_Z = \dot{\psi} + \dot{\phi} cos\theta$$

Substituting in previous set of equations and using (3.2), (3.6) we have

$$mg\rho sin\theta = I_0 \left(\ddot{\theta} - \dot{\phi}^2 sin\theta cos\theta\right) + I\dot{\phi} sin\theta \left(\dot{\phi} cos\theta + \dot{\psi}\right) \tag{3.10}$$
$$0 = I_0 \left(\ddot{\phi} sin\theta + 2\dot{\phi}\dot{\theta} cos\theta\right) - I\dot{\theta} \left(\dot{\phi} cos\theta + \dot{\psi}\right) \tag{3.11}$$
$$0 = I \left(\ddot{\psi} + \ddot{\phi} cos\theta - \dot{\phi}\dot{\theta} sin\theta\right) \tag{3.12}$$

Above equations, for a general case, are not always analytically solvable. Qualitatively, the path traced by the tip of a heavy top can be classified into three categories of precession.

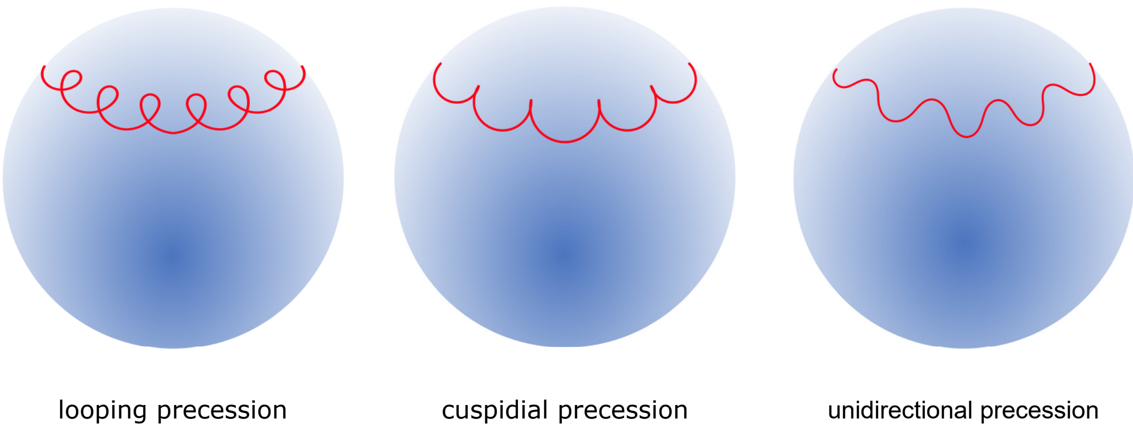


Figure 3.3: Precession

We will use the concept of “reduced discrete advected equation” from section 2.2 to develop an integrator and do a qualitative study of above three cases.

3.2 Variational Integrator

Consider a fixed rigid 3-D pendulum under the action of gravity. The Lagrangian from (2.3) $\tilde{L} : TSO(3) \to \mathbb{R}$ is not G-invariant. We can define $L : TSO(3) \times \mathbb{R}^{3^*} \to \mathbb{R}$

$$L(R, R\hat{\Omega}, w) = \frac{1}{2} tr\left(\hat{\Omega} J_d \hat{\Omega}^T\right) - mg\langle w, R\rho\rangle$$

where J_d is inertia matrix w.r.t the pivot and ρ is the coordinate of centre of mass of the body w.r.t the pivot in body frame.
Consider the discrete Lagrangian L_d

$$L_d\left(R_i, R_{i+1}, w_i\right) \approx \frac{h}{2}\left\{L\left(R_i, R_i\hat{\Omega}_i, w_i\right) + L\left(R_{i+1}, R_{i+1}\hat{\Omega}_i, w_i\right)\right\}$$

where

$$\hat{\Omega}_i = R_i^{-1}\dot{R}_i \approx R_i^T\left(\frac{R_{i+1} - R_i}{h}\right)$$

This choice of discrete Lagrangian is left G-invariant. Using $F_i = R_i^{-1}R_{i+1}$, we have

$$\hat{\Omega}_i = \frac{F_i - I}{h}$$

Substituting back in L_d, we have the reduced discrete Lagrangian l_d

$$l_d\left(F_i, w_i\right) = \frac{1}{2h} tr\left(\left(F_i - I\right) J_d \left(F_i - I\right)^T\right) - \frac{h}{2} mg\langle w_i, \rho + F_i\rho\rangle \tag{3.13}$$

3.2.1 Derivation

Variational approach to derive integrators on $SO(3)$ is implemented in (Lee *et al.*, 2005). Above reduced discrete Lagrangian l_d is equaivalent to the discrete Lagrangian L_d in this paper. Equations derived in this section, through the reduced Lagrangian approach, are expected to be equivalent to the integrator in (Lee *et al.*, 2005) which are derived via the non-reduced approach.
We will now derive each term in (2.4) for our choice of l_d in (3.13)

1. $\underline{(L_F)^* \frac{\delta l_d}{\delta F}:}$

$$\left\langle (LF)^* \frac{\delta l_d}{\delta F}, \eta \right\rangle = \left\langle \frac{\delta l_d}{\delta F}, F \cdot \eta \right\rangle$$
$$= \left.\frac{d}{ds}\right|_{s=0} l_d \left(F \cdot exp(s\eta), w\right)$$
$$= \left.\frac{d}{ds}\right|_{s=0} \left[\frac{1}{2h} tr\left(\left(F \cdot exp(s\eta) - I\right) J_d \left(F \cdot exp(s\eta) - I\right)^T\right)\right.$$
$$\left. - \frac{h}{2} mg \left\langle w, \rho + F \cdot exp(s\eta)\rho \right\rangle \right]$$
$$= \frac{1}{h} tr\left((F - I)\, J_d (F \cdot \eta)^T\right) - \frac{h}{2} mg \langle w, F\eta\rho \rangle$$
$$= \left\langle \frac{1}{h}(F - I)\, J_d - \frac{h}{2} mgw \cdot \rho^T, F\eta \right\rangle$$
$$= \left\langle \frac{1}{h}\left(I - F^T\right) J_d - \frac{h}{2} mgF^T w \cdot \rho^T, \eta \right\rangle$$

Since both $(L_F)^* \frac{\delta l_d}{\delta f}$ & η on L.H.S are are skew symmetric, we have

$$(L_F)^* \frac{\delta l_d}{\delta f} = \left\langle \frac{1}{2}\left(M - M^T\right), \eta \right\rangle$$

where M is the term in R.H.S,

$$M = \frac{1}{h}\left(I - F^T\right) J_d - \frac{h}{2} mgF^T w \cdot \rho^T$$
$$\implies (L_F)^* \frac{\delta l_d}{\delta f} = \frac{1}{2}\left[\frac{1}{h}\left(J_d F - F^T J_d\right) - \frac{h}{2} mg \left(F^T w \cdot \rho^T - \rho w^T F\right)\right]$$

2. $\underline{(R_F)^* \frac{\delta l_d}{\delta F}:}$

Proceeding as above

$$(R_F)^* \frac{\delta l_d}{\delta F} = \frac{1}{2}\left[\frac{1}{h}\left(F J_d - J_d F^T\right) - \frac{h}{2} mg \left(w \cdot \rho^T F^T - F\rho w^T\right)\right]$$

3. $\underline{\frac{\delta l_d}{\delta w} \diamond w}$

$$\left\langle \frac{\delta l_d}{\delta w} \diamond w, \xi \right\rangle = \left\langle w, \xi \cdot \frac{\delta l_d}{\delta w} \right\rangle$$
$$\frac{\delta l_d}{\delta w} = -\frac{h}{2} mg(\rho + F\rho)$$
$$\implies \left\langle \frac{\delta l_d}{\delta w} \diamond w, \xi \right\rangle = \left\langle w, \left.\frac{d}{ds}\right|_{s=0} \left[-\frac{h}{2} mg \left(exp(s\xi)\rho - exp(s\rho)F\rho\right)\right]\right\rangle$$
$$= -\frac{h}{2} mg \left\langle w\rho^T + w\rho^T F^T, \xi \right\rangle$$

Adding the constraint of skew-symmetricity, we have

$$\frac{\delta l_d}{\delta w} \diamond w = -\frac{h}{2} mg \left[\frac{1}{2}\left(w\rho^T - \rho w^T + w\rho^T F^T - F\rho w^T\right)\right]$$

Integrator & Numerical approach

Substituting in (2.4) and simplifying, we have

$$R_{i+1} = R_i F_i \tag{3.14}$$

$$w_{i+1} = F_i^T w_i \tag{3.15}$$

$$\frac{1}{h}\left(J_d F_{i+1}^T - F_i^T J_d - F_{i+1} J_d + J_d F_i\right) = hmg\left(F_i^T w_i \rho^T - \rho w_i^T F_i\right) \tag{3.16}$$

Above algorithm enables us the transition from $G \times V^* \rightarrow G \times V^*$ i.e. $(F_i, w_i) \mapsto (F_{i+1}, w_{i+1})$. This is the Lagrange version of discretization.

Numerical approach:

In practice, initial momentum Π and position[2] R of the system are readily available. For this reason, we develop integrator using the discrete Lie-Poisson approach. The advected momentum $\hat{\Pi}_i$ as in (2.7) is derived

$$\begin{aligned}\hat{\Pi}_i &= \left(R_{F_i}\right)^* \frac{\delta l_d}{\delta F}(F_i) - \left(\frac{\delta l_d}{\delta w} \diamond w_i\right)(F_i, w_i) \\ &= \frac{1}{2}\left[\frac{1}{h}\left(F_i J_d - J_d F_i^T\right) - \frac{h}{2} mg\left(\rho w_i^T - w_i \rho^T\right)\right]\end{aligned} \tag{3.17}$$

Given (w_i, Π_i); F_i is solved implicitly from (3.17). Then R_{i+1} is directly given by (3.14) and w_{i+1} can be calculated from (3.15).

Finally, $\hat{\Pi}_{i+1}$ can be calculated as in (2.9)

$$\begin{aligned}\hat{\Pi}_{i+1} &= (L_{F_i})^* \frac{\delta l_d}{\delta F_i} \\ &= \frac{1}{2}\left[\frac{1}{h}\left(J_d F_i - F_i^T J_d\right) - \frac{h}{2} mg\left(F_i^T w_i \rho^T - \rho w_i^T F_i\right)\right]\end{aligned} \tag{3.18}$$

This approach allows us to go from $\mathfrak{g}^* \times V^* \rightarrow \mathfrak{g}^* \times V^*$ i.e $\left(\hat{\Pi}_i, w_i\right) \mapsto \left(\hat{\Pi}_{i+1}, w_{i+1}\right)$.

Note: In application, (3.17) is solved in vector form instead of the matrix form. Refer the section *Computational approach* in (Lee *et al.*, 2005) for further details.

3.2.2 Results and Discussion

We compare the behavior of variational integrator developed here to standard Runge-Kutta 4th order forward method for the system of heavy top. The system properties are $J_d = diag[2,\ 2,\ 1], m = 1kg, \rho = [0,\ 0,\ 1]$

From theoretical study, we expect variational integrator to satisfy the properties discussed in section 1.4. Namely those of geometric and momentum conservation.

[2]In the reduced case, this is equivalent to knowing w_i

Geometric Conservation

We implicitly solved for F_i on $SO(3)$ and used the reconstruction equation $R_{i+1} = R_i F_i$. This guarantees[3] R_i to stay on $SO(3)$ i.e.

$$R_i R_i^T = I = R_i^T R_i$$

We can compare the geometric behavior of integrators by computing $||R^T R - I||$. Following graph compares variational integrator developed here, to a standard Runge-Kutta 4th order forward method for the system of heavy top.

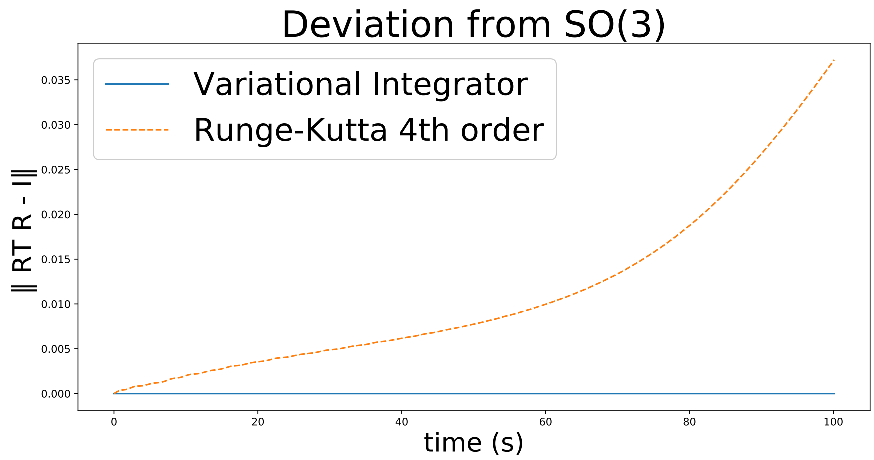


Figure 3.4: Geometric behavior

Momentum conservation

Since the Lagrangian $\tilde{L}$ is not G-invariant, the spatial angular momentum vector is not conserved. However the component of this momentum in direction of gravity is conserved in the physical system, which is also true in discrete case as shown below.

Acting $R_{i+1}\_\_R_{i+1}^T$ on both sides of (3.18) and simplifying, we have

$$\begin{aligned} R_{i+1}\widehat{\Pi_{i+1}} - \widehat{R_i\Pi_i} &= -\frac{h}{2} mgR_i\left(\widehat{\rho \times w_i} + (\widehat{F_i\rho) \times w_i}\right)R_i^T \\ &= -\frac{h}{2} mg\left((\widehat{R_{i+1}\rho) \times e_3} + (\widehat{R_i\rho) \times e_3}\right) \end{aligned}$$

Above equation tells us about the evolution of spatial angular momentum. i.e.

$$R_{i+1}\Pi_{i+1} - R_i\Pi_i = -\frac{h}{2} mg\left((R_{i+1}\rho) \times e_3 + (R_i\rho) \times e_3\right) \tag{3.19}$$

[3]note that this approach cannot eliminate the numerical error due to computation

Taking dot product w.r.t e_3 on both sides, we have

$$e_3 \cdot (R_{i+1}\Pi_{i+1} - R_i\Pi_i) = 0 \tag{3.20}$$

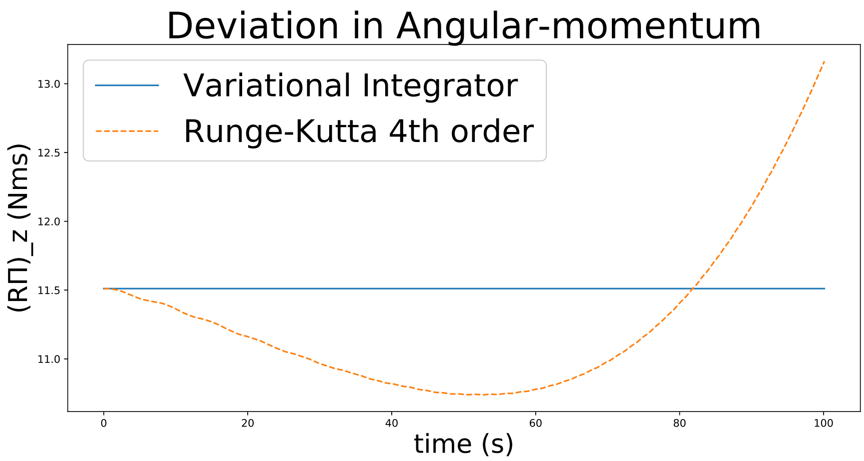
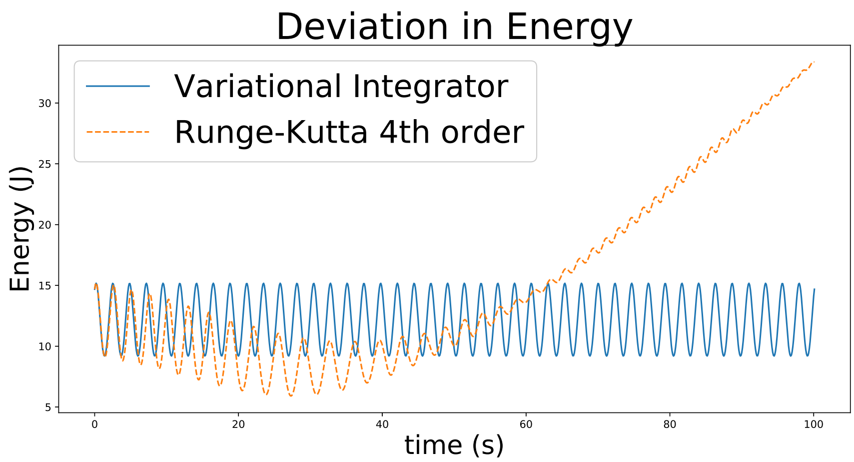


Figure 3.5: Conservation of spatial angular momentum in z-direction

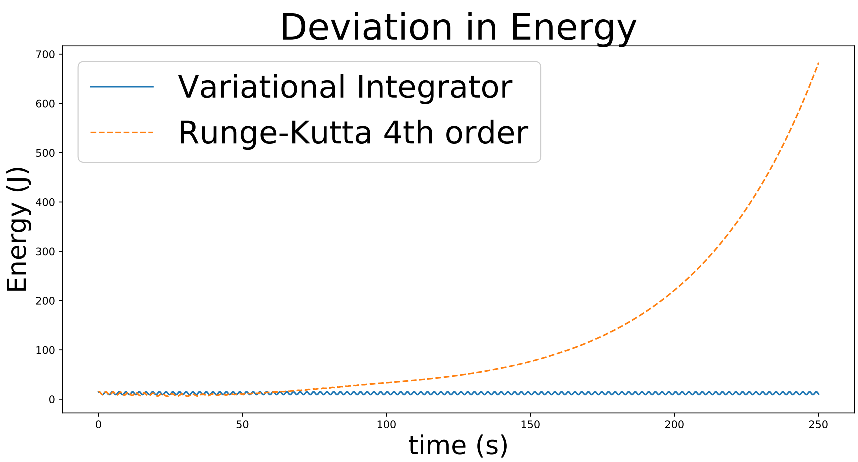
The z-component of discrete angular momentum $e_3^T R\Pi$ is conserved in figure 3.5 by the variational integrator as expected from (3.20)

Energy Conservation

Though, theoretically, we did study the energy conservation property of a variational integrator, the integrator exhibits good energy behavior.



(a) Initial behavior

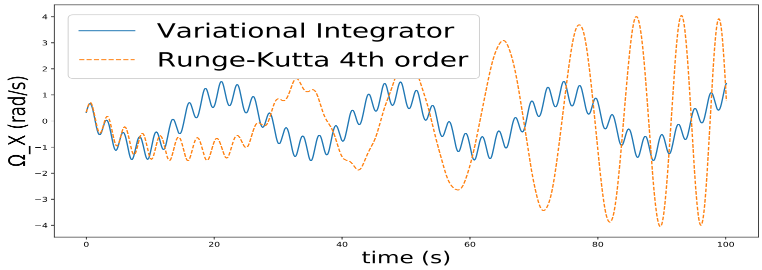


(b) Long term behavior

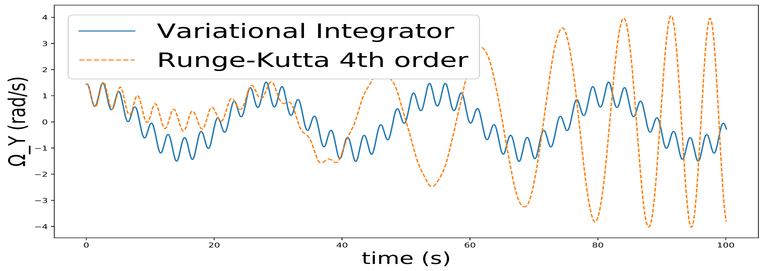
Figure 3.6: Energy behavior

Though not constant, energy does not deviate away from initial energy.

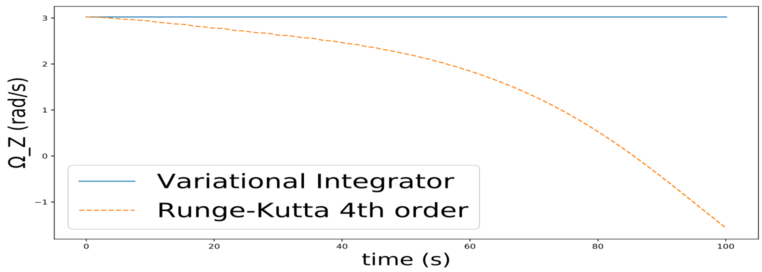
Following are the plots of angular velocity of the body under looping precession (figure 3.8). Angular velocity is expressed in body frame. Periodic oscillations of Ω_X, Ω_Y are interesting to note. Spin rate ($\dot{\psi}$) i.e. Ω_Z is unchanged over time, this keeps the top "levitating".



(a) Ω_X



(b) Ω_Y



(c) Ω_Z

Figure 3.7: Angular velocity under looping precession

Figures 3.8, 3.9, 3.10 show the long term behavior of integrators for looping, cuspidial and unidirectional precessions respectively. Standard integrator drops under gravity as time passes, while the variational integrator maintains its path.

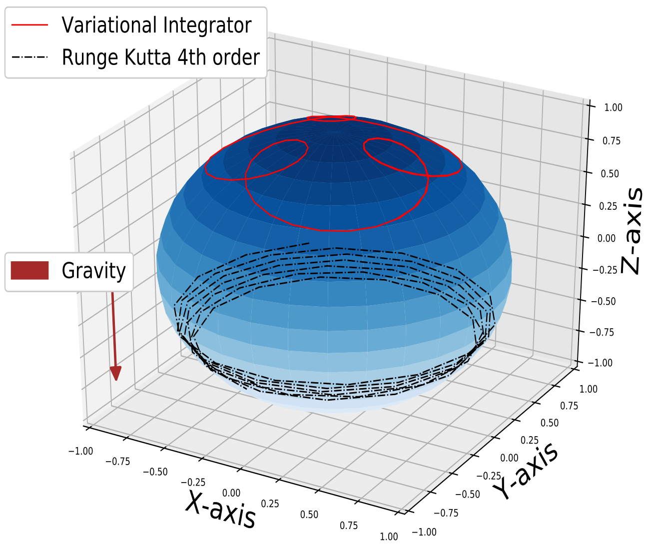


Figure 3.8: Looping precession (100s - 200s)

Initial Conditions: $\dot{\phi} = 3\,rad/s$, $\dot{\theta} = 0.5\,rad/s$, $\dot{\psi} = 10\,rad/s$, $\theta = 0.8\,rad$.

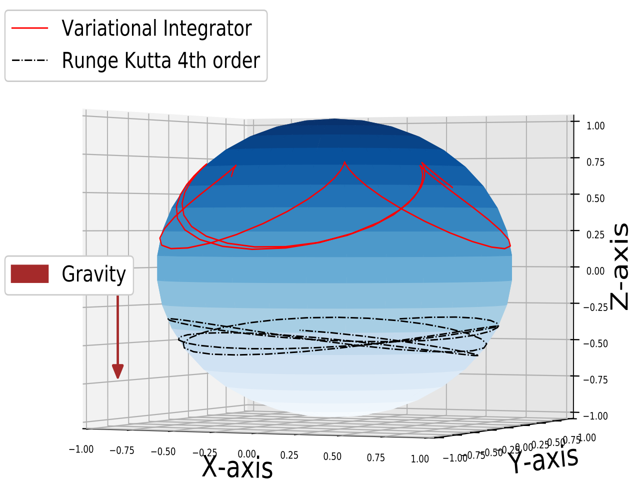


Figure 3.9: Cuspidial precession (100s - 200s)

Initial Conditions: $\dot{\phi} = 0\,rad/s$, $\dot{\theta} = 0\,rad/s$, $\dot{\psi} = 10\,rad/s$, $\theta = 0.8\,rad$

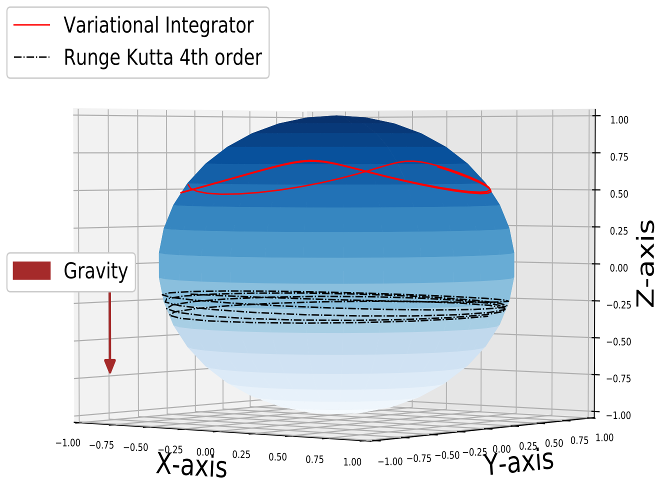


Figure 3.10: Unidirectional precession (100s - 200s)
Initial Conditions: $\dot{\phi} = 1\, rad/s$, $\dot{\theta} = 0.01\, rad/s$, $\dot{\psi} = 10\, rad/s$, $\theta = 0.8\, rad$.

3.2.3 Summary

We developed discrete version of reduction for systems with invariant Lagrangian. Later, we extended the theory to systems with broken invariance and showed that the dynamics dictated by these reduced equations is equivalent to that of a variational integrator. Owing to these excellent properties of discrete reduced equations, we can develop integrators to study behavior of important systems when explicit solution is not available.

Reduced discrete advected equations (2.5), (2.9) have been applied to a system of heavy top. This integrator (3.16) is used to model chaotic behavior of a system under unstable equilibrium. Figures 3.4, 3.5, 3.6 respect important characteristics of the physical system like rigidity, momentum, energy conservation and agree with our prediction for variational integrators. This integrator has then been used to study complex precession behavior of a spinning heavy top. Our model was able to show good beahviour even after long intervals of time due to conservative nature of variational integrators.

Acknowledgements

Geometric mechanics is mathematically rich and elegant, this made the study over past 2 years gratifying and enriching. I am grateful to Prof. Amuthan A. Ramabathiran for introducing me to the subject, and guiding me in my study.

Prasanth YSS
IIT Bombay
6 July 2021

References

Darryl D. Holm, C. S., Tanya Schmah, 2009, *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*, 1st ed. (Oxford University Press).

Lee, T., McClamroch, N., and Leok, M., 2005, "A lie group variational integrator for the attitude dynamics of a rigid body with applications to the 3d pendulum," in *Proceedings of 2005 IEEE Conference on Control Applications, 2005. CCA 2005.*, pp. 962–967.

Marsden, J. E., and West, M., 2001, "Discrete mechanics and variational integrators," *Acta Numerica* **10**, 357–514.