REDUCED DISCRETE ADVECTED EQUATIONS

DDP-II

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Lagrangian Mechanics

- Lagrangian $L : TQ \rightarrow \mathbb{R}$, is a real valued function on the Tangent bundle TQ of manifold Q.
- For a given Lagrangian, action A maps a smooth curve on Q to $\mathbb{R}$.

$$A(q) = \int_0^T L\left(q(t), \dot{q}(t)\right) dt$$

(Smooth curve is called a path)

- For a given path $q : [0, T] \rightarrow Q$, a variation $\delta q : [0, T] \times (-\epsilon, \epsilon) \rightarrow Q$ is a smooth function, satisfying

$$\delta q(t, 0) = q(t) \quad \forall t \in [0, T]$$

$$\delta q(0, s) = q(0) \ , \ \delta q(T, s) = q(T) \ \ \forall s \in (-\epsilon, \epsilon)$$

Variation

- Given end points q_0, q_1 on the manifold, we intend to find the path $q : [0, T] \to Q$ which extremizes action locally and satisfies

$$q(0) = q_0$$
$$q(T) = q_1$$

- Above statement can be restated as

$$\left.\frac{d}{ds}\right|_{s=0} A\left(\delta q(\cdot, s)\right) = 0$$

- Any path extremising action necessarily satisfy Euler-Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Variational Integrator

- For the purpose of integrators, a discrete Lagrangian $L_d^h : Q \times Q \to \mathbb{R}$ is introduced. L_d^h approximates action A

$$L_d^h(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t))\, dt$$

  where $q : [0, h] \to Q$ is the actual path satisfying Euler-Lagrange equations.

- Variational integrators are constructed by extremizing the discrete action

$$A_d(q_d) = \sum_{i=1}^{N-1} L_d^h(q_i, q_{i+1})$$

  where $q_d : \left\{ i^* h \middle| \ i \in \{1, 2, \dots N-1\} \right\} \to Q$ is the discrete path.

Variational Integrator

- Analogous to continuous case, we introduce a variation $\delta q_d : \{0, h, 2h, \cdots Nh\} \times (-\epsilon, \epsilon) \to Q$ satisfying

  (We'll denote $q_d(kh)$ as q_k whenever there is no confusion. q_d always denotes discrete path, q_i, $i \neq d$ denotes a point on Q)

$$\delta q_d(kh, 0) = q_k$$

$$\delta q_d(0, s) = q_0 \ \ , \ \ \delta q(T, s) = q_N \ \ \forall s \in (-\epsilon, \epsilon)$$

- discrete path q_d extremising A_d satisfy

$$\left.\frac{d}{ds}\right|_{s=0} A_d\left(\delta q_d(\cdot, s)\right) = 0$$

Variational Integrator

- discrete path q_d of a variational integrator satisfy

$$-D_1 L_d\left(q_i, q_{i+1}\right) = D_2 L_d\left(q_{i-1}, q_i\right)$$

  ($D_j L_d$ is partial derivative of L_d w.r.t j^{th} input parameter)

  Above terms belong to the co-tangent bundle T^*Q

- The discrete Legendre Transforms $\mathbb{F}l_d^{\pm} : Q \times Q \to T^*Q$ are defined as

$$\mathbb{F}L_d^{-}\left(q_1, q_2\right) = -D_1 L_d\left(q_1, q_2\right)$$
$$\mathbb{F}L_d^{+}\left(q_1, q_2\right) = D_2 L_d\left(q_1, q_2\right)$$

Variational Integrator

$$
\begin{tikzcd}
& Q \times Q \arrow[rr, "F_{L_d}"] \arrow[dl, "\mathbb{F}L_d^-"'] \arrow[dr, "\mathbb{F}L_d^+"] & & Q \times Q \arrow[dl, "\mathbb{F}L_d^-"'] \arrow[dr, "\mathbb{F}L_d^+"] & \\
T^*Q \arrow[rr, "F_{H_d}"] & & T^*Q \arrow[rr, "F_{H_d}"] & & T^*Q
\end{tikzcd}
$$

- The discrete Lagrangian flow $F_{L_d} : Q \times Q \to Q \times Q$ satisfies

$$F_{L_d}(q_0, q_1) = (q_1, q_2)$$

- The discrete Hamiltonian flow $F_{H_d} : T^*Q \to T^*Q$ is defined such that the diagram is closed

$$F_{h_d} = \mathbb{F}L_d^+ \circ \left(\mathbb{F}L_d^-\right)^{-1}$$

Lie Group

Definition (Group)

A group G, is a set along with binary operation $(\cdot)$

$$(\cdot) : G \times G \to G$$

1. Identity: $\exists e \in G$, s.t $g \cdot e = e \cdot g = g$; e is called the identity.
2. Inverse: $\forall g \in G$, $\exists g^{-1} \in G$, s.t. $g^{-1} \cdot g = g \cdot g^{-1} = e$; g^{-1} is called the inverse of g
3. Association: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

Definition (Lie Group)

A manifold with group structure is said to be a Lie Group when the binary operation is diffeomoerphic.

Lie Algebra

Definition (Lie Algebra)

Lie algebra $\mathfrak{g}$ is the Tangent space at e i.e.

$$\mathfrak{g} = T_e G$$

On a Lie-Group, it is possible to pull back any Tangent space $T_g G$ to $\mathfrak{g}$ via the Right or Left Action

- Right Action

$$\begin{aligned} R : G \times G &\to G \\ R(g, h) &\mapsto hg \end{aligned}$$

- Left Action

$$\begin{aligned} L : G \times G &\to G \\ L(g, h) &\mapsto gh \end{aligned}$$

Lie Group

- Pull back w.r.t $Lg := L(g, \cdot)$ is denoted as $(Lg)^*$

$$(Lg)^* : T_g G \to \mathfrak{g}$$

- Push Forward w.r.t Lg is denoted as $(Lg)_*$

$$(Lg)_* : T_{g^{-1}} G \to \mathfrak{g}$$

- $(Lg)^* = \left(Lg^{-1}\right)_*$

Example

Special orthogonal matrix Group $SO(3) = \left\{R \in \mathbb{R}_{3\times 3} | RR^T = R^T R = I\right\}$ is a Lie-Group

- binary operation $(\cdot)$ is the matrix multiplication
- $I_{3\times 3}$ is the identity element, R^T is the inverse of R
- The Lie-Algebra $\mathfrak{so}(3)$ is the set of all skew symmetric matrices
- If $\dot{R} \in T_R SO(3)$, then $(LR)^* \dot{R} = R^T \dot{R}$

Reduction

- The Tangent Lift TLg of Lg is a diffeomorphism on TG

$$TLg : TG \to TG$$
$$TLg\left(h, \dot{h}\right) = \left(gh, (Lg)_* h\right)$$

When the Lagrangian $L : TG \to \mathbb{R}$ is *G-invariant* under the tangent lifted left multiplication, then a reduced lagrangian $l : \mathfrak{g} \to \mathbb{R}$ can be introduced as follows

$$l(\xi) = L(e, \xi) \quad \xi \in \mathfrak{g}$$

Also, Lagrangian mechanics can be studied directly from the reduced lagrangian l.

Euler-Poincaré Reduction

Theorem

If $L : TG \to \mathbb{R}$ is left G-invariant under the tangent lifted action of Lie group G and l is the reduced larangian, then the following statements are equivalent

- *The variational principle*

$$\delta \int_a^b L\left(g, \dot{g}\right) dt = 0$$

  holds on TG for variations with fixed end points.

- *The variational principle*

$$\delta \int_a^b l\left(\xi\right) dt = 0$$

  where $\xi = g^{-1}\dot{g}$, holds on $\mathfrak{g}$, for variations of the form

$$\delta\xi = ad_{\xi}\eta + \dot{\eta} \quad \eta \text{ arbitrary}$$

Euler-Poincaré Reduction

Theorem

- *The Euler-Lagrange equations are given by*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{g}}\right) = \frac{\partial L}{\partial g}$$

- *The reduced Euler-Lagrange equations are*

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) = ad^*_{\xi}\frac{\partial l}{\partial \xi}$$

The path on TG can be reconstructed using the relation

$$\dot{g}(t) = g(t) \cdot \xi(t)$$

Note: On $SO(3)$, $ad^*_{\xi}\frac{\partial l}{\partial \xi} = \frac{\partial l}{\partial \xi} \times \xi$

Example (Free Rigid Body)

Consider a free Rigid body fixed at the origin. The phase space of the system is $TSO(3)$. The Lagrangian for this system is given as

$$L\left(R,\dot{R}\right)=\frac{1}{2}tr\left(\dot{R}\mathbb{J}\dot{R}^{T}\right)$$

where, $\mathbb{J}$ is the inertia of body calculated w.r.t pivot

Above Lagrangian is G-invariant and can be studied via Euler-Poincaré reduction.

Note that for $\Omega\in\mathbb{R}^3$, $\hat{\Omega}\in\mathfrak{so}(3)$ is defined such that

$$\hat{\Omega}\,v=\Omega\times v \qquad \forall\, v\in\mathbb{R}^3$$

Example

Example (Free Rigid Body)

We will use the following identities on $SO(3)$.

1. $ad_{\hat{\Omega}}\hat{\eta} = \widehat{\Omega \times \eta}$
2. $ad^*_{\hat{\Omega}}\hat{\Pi} = \widehat{\Pi \times \Omega}$

where $\hat{\eta}, \hat{\Omega} \in \mathfrak{so}(3),\ \hat{\Pi} \in \mathfrak{so}^*(3)$

Note: Π & Ω are treated as column vectors
Note: α is used in place of $\hat{\alpha}$ in the following derivations

Euler-Poincaré reduction:
The reduced Lagrangian $l : \mathfrak{so}(3) \rightarrow \mathbb{R}$ is given as

$$l(\Omega) = \frac{1}{2}\Omega^T \mathbb{I}\Omega$$

Example

Example (Free Rigid Body)

Euler-Poincaré equation is

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \Omega}\right) = ad^*_{\Omega}\frac{\delta l}{\delta \Omega}$$

$$l(\Omega) = \frac{1}{2}\Omega^T \mathbb{I}\Omega$$

$$\frac{\delta l}{\delta \Omega} = \mathbb{I}\Omega$$

$$ad^*_{\hat{\Omega}}\hat{\Pi} = \widehat{\Pi \times \Omega} \qquad \hat{\Pi} \in \mathfrak{so}^*(3)$$

$$\Longrightarrow \mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega$$

Discrete Reduction

$$L_d\left(g_0, g_1\right) \approx \int_0^h L\left(g(s), \dot{g}(s)\right) ds$$

where $g(t)$ is the actual path satisfying Euler-Lagrange equations, with (g_0, g_1) as the end points.

- Assume Lagrangian L is G-invariant, when the end points (g_0, g_1) are translated to (hg_0, hg_1) above solution path $g(t)$ also translates to $hg(t)$ where
$$hg(s) = h \cdot g(s) \quad s \in [0, h]$$
- So, R.H.S in above approximation is also invariant. Hence, it is imperative to consider a G-invariant discrete Lagrangian L_d i.e.

$$L_d\left(hg_0, hg_1\right) = L_d\left(g_0, g_1\right)$$

Variational Integrator Properties

1. If the discrete Lagrangian is invariant under the left action of a Lie-group G, then the discrete Lagrange momentum map $J_{L_d} : Q \times Q \to \mathfrak{g}^*$ can be defined as

$$\begin{aligned} \left\langle J_{L_d}(q_1, q_2), \xi \right\rangle &= \left\langle -D_1 L_d(q_1, q_2), \xi \cdot q_1 \right\rangle \\ &= \left\langle D_2 L_d(q_1, q_2), \xi \cdot q_2 \right\rangle \end{aligned}$$

   J_{L_d} is conserved along F_{L_d} whenever L_d is G-invariant.

2. A discrete Lagrangian is said to be exact, when it approximates Action exactly. For an exact discrete Lagrangian, the discrete path satisfies

$$q_d(k) = q(k) \qquad \text{where } q \text{ is the continuous solution}$$

3. Variational integrators are symplectic on T^*Q w.r.t the canonical symplectic form.

Discrete Euler-Poincaré Equations

Theorem

Let $L_d : G \times G \to \mathbb{R}$, L_d is G-invariant under the left multiplicative action, then a reduced discrete Lagrangian $l_d : G \to \mathbb{R}$ can be introduced as

$$l_d(f) = L_d\left(g_1, g_2\right)$$

where $f = g_1^{-1} g_2$

The discrete Euler-Poincaré equations are given as

$$\left(Rf_{i+1}\right)^* \frac{\delta l_d}{\delta f}\left(f_{i+1}\right) = \left(Lf_i\right)^* \frac{\delta l_d}{\delta f}\left(f_i\right)$$

The discrete path $g_d \in \underbrace{G \times G \cdots \times G}_{N \text{ times}}$ *can be reconstructed using*

$$g_{i+1} = g_i f_i$$

g_d *extremizes the discrete action* A_d

Discrete Reduction

discrete Euler-Poincaré flow $F_{l_d} : G \to G$ is defined such that

$$F_{l_d}(f) = g$$
$$\Longleftrightarrow (Rg)^* \frac{\delta l_d}{\delta g} = (Lf)^* \frac{\delta l_d}{\delta f}$$

Reduced discrete Legendre Transforms: $\mathbb{F}l_d^{\pm} : G \to \mathfrak{g}^*$ are defined as

$$\mathbb{F}l_d^{-}(f) = (Rf)^* \frac{\delta l_d}{\delta f}$$
$$\mathbb{F}l_d^{+}(f) = (Lf)^* \frac{\delta l_d}{\delta f}$$

Discrete Reduction

Reduced discrete Momentum: $\mu : G \to \mathfrak{g}^*$ is defined as

$$\mu(f) = \mathbb{F}L_d^{-1}(f) = (Rf)^* \frac{\delta l_d}{\delta f}$$

With above definitions in place, it is easy to verify that the Discrete Lie-Poisson Equation

$$\mu_{i+1} = Ad^*_{f_i} \mu_i$$

is equivalent to Discrete Euler-Poincaré Equations

Discrete Reduction

<u>discrete Lie-Poisson flow</u> $F_{h_d} : \mathfrak{g}^* \to \mathfrak{g}^*$ is reminiscent of F_{H_d}

$$F_{h_d} = \mathbb{F}l_d^+ \circ \left(\mathbb{F}l_d^-\right)^{-1}$$

$$\begin{tikzcd}
 & G \arrow[rr, "F_{l_d}"] \arrow[dl, "\mathbb{F}l_d^-"'] \arrow[dr, "\mathbb{F}l_d^+"] & & G \arrow[dl, "\mathbb{F}l_d^-"'] \arrow[dr, "\mathbb{F}l_d^+"] & \\
\mathfrak{g}^* \arrow[rr, "F_{h_d}"] & & \mathfrak{g}^* \arrow[rr, "F_{h_d}"] & & \mathfrak{g}^*
\end{tikzcd}$$

Reduced Variational Integrator

$$\begin{tikzcd}
 & G \times G \arrow[rr, "F_{L_d}"] \arrow[dl, "\mathbb{F}L_d^-"'] \arrow[dr, "\mathbb{F}L_d^+"] & & G \times G \arrow[dl, "\mathbb{F}L_d^-"'] \arrow[dr, "\mathbb{F}L_d^+"] & \\
T^*G \arrow[rr, "F_{H_d}"] & & T^*G \arrow[rr, "F_{H_d}"] & & T^*G
\end{tikzcd}$$

Variational Integrator

Example

Example (Free Rigid Body)

For a fixed free rigid body, the Lagrangian is defined on $TSO(3)$.

$$L\left(R, R\hat{\Omega}\right) = \frac{1}{2} tr\left(\hat{\Omega} J_d \hat{\Omega}^T\right)$$

Consider a reduce discrete Lagrangian $l_d : SO(3) \to \mathbb{R}$ as

$$l_d\left(F\right) = \frac{1}{2h} tr\left((F - I) J_d (F - I)^T\right)$$

The reduced discrete momentum $\hat{\Pi}$ is

$$\begin{aligned} \hat{\Pi} &= (RF)^* \frac{\delta l_d}{\delta F} \\ &= \frac{1}{2h}\left(F J_d - J_d F^T\right) \end{aligned}$$

Example

Example (Free Rigid Body)

Since L_d is G-invariant, we expect the discrete Lagrange Momentum $J_{L_d} = \widehat{R\Pi}$ to be conserved. Conservation of J_{L_d} can be confirmed directly from discrete Lie-Poisson equations.

$$\begin{aligned}
\hat{\Pi}_{i+1} &= Ad^*_{F_i}\hat{\Pi}_i \\
&= F_i^T \hat{\Pi}_i F_i \\
\text{where, } F_i &= R_i^T R_{i+1} \\
\text{i.e. } R_{i+1}\hat{\Pi}_{i+1}R_{i+1}^T &= R_i\hat{\Pi}_i R_i^T \\
\implies R_{i+1}\Pi_{i+1} &= R_i\Pi_i
\end{aligned}$$

Above derivation is independent of the choice of discrete Lagrangian! i.e. $\widehat{R\Pi}$ is conserved irrespective of L_d. However, the momentum $\hat{\Pi}$ itself depends on the choice of L_d. $\widehat{R_i\Pi_i}$ can be treated as discrete version of the spatial angular momentum from continuous case.

Discrete Advected Reduction

So far we developed tools to study dynamics on Lie-Groups. These are applicable only when the discrete Lagrangian is G-invariant. But Lagrangian of many important physical systems is not invariant.

For example Lagrangian of a Heavy Top is not invariant

$$L(R, R\hat{\Omega}) = \frac{1}{2} tr \left(\hat{\Omega} J_d \hat{\Omega}^T \right) - mg \langle e_3, R\rho \rangle$$

When the Lagrangian has broken symmetry, it is possible to study the reduced dynamics by restricting the action of Lie-group G, in above case to $SO(2)$. A more elegant reduction is possible by embedding the manifold in a higher dimension space, in above case by treating e_3 as a variable. We will study one such case when L has an advected term.

Diamond Operator

Let V be a vector space with the Lie-group G acting linearly on it i.e.

$$g \cdot (v + w) = g \cdot v + g \cdot w \quad \forall\, g \in G \;\&\; v, w \in V$$

The action of $\xi \in \mathfrak{g}$ on V is defined as

$$\xi \cdot v = \frac{d}{ds}\bigg|_{s=0} exp(s\xi) \cdot v$$

Note: On $SO(3)$, $exp(s\xi) = I + s\xi + \frac{(s\xi)^2}{2!} + \cdots$

We can then define a map $\rho_v : \mathfrak{g} \rightarrow V$ as

$$\rho_v(\xi) = \xi \cdot v$$

This defines the pull-back $\rho_v^* : V^* \rightarrow \mathfrak{g}^*$ as

$$\left\langle \rho_v^*(w), \xi \right\rangle = \langle w, \xi \cdot v \rangle$$

Diamond Operator: The diamond operator $\diamond : V \times V^* \rightarrow \mathfrak{g}^*$ is defined as

$$\langle v \diamond w, \xi \rangle = \langle w, \xi \cdot v \rangle$$

Discrete Advected Reduction

Theorem

Let $\tilde{L}_d : G \times G \to \mathbb{R}$ be a discrete Lagrangian. Assume G acts linearly on a vector space V, and there exists a function $L_d : G \times G \times V^* \to \mathbb{R}$ such that

$$L_d\left(g_1, g_2, w_0\right) = \tilde{L}\left(g_1, g_2\right)$$

If L_d is G-invariant, then we can introduce a reduced Lagrangian $l_d : G \times V^* \to \mathbb{R}$ satisfying

$$l_d(f, w) = L(g_1, g_2, g_1 \cdot w)$$

where $f = g_1^{-1} g_2$

Discrete Advected Reduction

Theorem

Then the following equations are equivalent to the evolution of $\tilde{L}_d$ on $G \times G$

$$w_{i+1} = f_i^{-1} w_i$$

$$(Rf_{i+1})^* \frac{\delta l_d}{\delta f}(f_{i+1}) - \left(\frac{\delta l_d}{\delta w} \diamond w \right)(f_{i+1}, w_{i+1}) = (Lf_i)^* \frac{\delta l_d}{\delta f}(f_i, w_i)$$

The discrete path g_d on G can be reconstructed via

$$g_{i+1} = g_i f_i$$

Note: "evolution" here is defined by the discrete path of corresponding Variational Integrator

Discrete Advected Reduction

Proof.

$$A_d = \sum_{i=1}^{n-1} \tilde{L}_d\left(g_i, g_{i+1}\right)$$
$$= \sum_{i=1}^{n-1} L_d\left(g_i, g_{i+1}, w_0\right) = \sum_{i=1}^{n-1} l_d\left(g_i^{-1} g_{i+1}, g_i^{-1} w_0\right)$$

let $f_j = g_j^{-1} g_{j+1}$ & $w_k = g_k^{-1} w_0$, then

$$dA_d \cdot \delta g_d = \sum_{i=1}^{n-1} \left\langle \frac{\delta l_d}{\delta f_i}, \delta f_i \right\rangle + \left\langle \frac{\delta l_d}{\delta w_i}, \delta w_i \right\rangle$$

$$\delta f_i = \delta g_i^{-1} \cdot g_{i+1} + g_i^{-1} \cdot \delta g_{i+1}$$
$$\delta w_i = \delta g_i^{-1} w_0$$

$$\delta g_i^{-1} = -g_i^{-1} \delta g_i \cdot g_i^{-1}$$

Discrete Advected Reduction

Proof.

$$\text{let } \delta g_j = g_j \cdot \eta_j$$
$$\implies \delta f_i = f_i \cdot \eta_{i+1} - \eta_i \cdot f_i$$
$$\delta w_i = -\eta_i \cdot w_i$$

Substituting back,

$$\begin{aligned} dA_d \cdot \delta g_d &= \sum_{i=1}^{n-1} \left\langle \frac{\delta l_d}{\delta f_i}, f_i \cdot \eta_{i+1} - \eta_i \cdot f_i \right\rangle + \left\langle \frac{\delta l_d}{\delta w_i}, -\eta_i \cdot w_i \right\rangle \\ &= \sum_{i=1}^{n-1} \left\langle (Lf_i)^* \frac{\delta l_d}{\delta f_i} - (Rf_{i+1})^* \frac{\delta l_d}{\delta f_{i+1}} + \frac{\delta l_d}{\delta w_{i+1}} \diamond w_{i+1}, \eta_{i+1} \right\rangle \end{aligned}$$

Since η_j are arbitrary, we have

$$(Rf_{i+1})^* \frac{\delta l_d}{\delta f}(f_{i+1}) - \left(\frac{\delta l_d}{\delta w} \diamond w \right)(f_{i+1}, w_{i+1}) = (Lf_i)^* \frac{\delta l_d}{\delta f}(f_i, w_i)$$

Discrete Advected Reduction

discrete Advected Euler-Poincaré flow $\tilde{F}_{l_d} : G \times V^* \to G \times V^*$ is defined as

$$\tilde{F}_{l_d}(f,u) = (g,v)$$
$$\iff (Rg)^* \frac{\delta l_d}{\delta f}(g) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)(g,v) = (Lf)^* \frac{\delta l_d}{\delta f}(f,u)$$

discrete advected Legendre Transforms $\mathbb{F}l_d^{\pm} : G \times V^* \to \mathfrak{g}^* \times V^*$ are defined as

$$\tilde{\mathbb{F}}l_d^-(f,w) = \left((Rf)^* \frac{\delta l_d}{\delta f}(f) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)(f,w), w\right)$$
$$\tilde{\mathbb{F}}l_d^+(f,w) = \left((Lf)^* \frac{\delta l_d}{\delta f}, f^{-1}w\right)$$

Discrete Advected Reduction

discrete advected Momentum: $\mu(f,w) : G \times V^* \to \mathfrak{g}^*$ is defined as

$$\mu = (Rf)^* \frac{\delta l_d}{\delta f}(f) - \left(\frac{\delta l_d}{\delta w} \diamond w\right)(f,w)$$

With above definitions in place, it is easy to verify that the Discrete advected Lie-Poisson Equation

$$(f_i, w_i) = \left(\tilde{\mathbb{F}}^{-}_{l_d}\right)^{-1}(\mu_i, w_i)$$

$$(\mu_{i+1}, w_{i+1}) = \mathbb{F}L_d^{+}(\mu_i, w_i) = \left((Lf_i)^* \frac{\delta l_d}{\delta f_i}, f_i^{-1} w_i\right)$$

The discrete path g_d is reconstructed using

$$g_{i+1} = g_i f_i$$

are equivalent to Euler-Poincaré version

Discrete Advected Reduction

<u>discrete advected Lie-Poisson flow</u> $\tilde{F}_{h_d}$ is defined such that the following diagram is closed

$$\begin{tikzcd}
 & G \times V^* \arrow[rr, "\tilde{F}_{l_d}"] \arrow[dl, "\tilde{\mathbb{F}} l_d^-"'] \arrow[dr, "\tilde{\mathbb{F}} l_d^+"] & & G \times V^* \arrow[dl, "\tilde{\mathbb{F}} l_d^-"'] \arrow[dr, "\tilde{\mathbb{F}} l_d^+"] & \\
\mathfrak{g}^* \times V^* \arrow[rr, "\tilde{F}_{h_d}"] & & \mathfrak{g}^* \times V^* \arrow[rr, "\tilde{F}_{h_d}"] & & \mathfrak{g}^* \times V^*
\end{tikzcd}$$

Lagrangian with an advected term

$$\begin{tikzcd}
 & G \arrow[rr, "F_{l_d}"] \arrow[dl, "\mathbb{F} l_d^-"'] \arrow[dr, "\mathbb{F} l_d^+"] & & G \arrow[dl, "\mathbb{F} l_d^-"'] \arrow[dr, "\mathbb{F} l_d^+"] & \\
\mathfrak{g}^* \arrow[rr, "F_{h_d}"] & & \mathfrak{g}^* \arrow[rr, "F_{h_d}"] & & \mathfrak{g}^*
\end{tikzcd}$$

Fully Invariant Lagrangian

Heavy Top

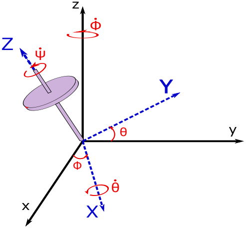


Figure: Heavy Top

$$\text{spin rate} = \dot{\psi}, \text{ precession rate} = \dot{\phi}, \text{ nutation rate} = \dot{\theta}$$

- A Heavy Top spins about it's symmetric axis in the presence of gravity
- Torque due to gravity causes the top to precess about z-axis. This behaviour is called "Precession"
- One would intutively expect an object in gravity to drop, but tops maintain their path due to precession. This behaviour is distinctive to spinning objects.

Governing Equations of a Heavy Top

Governing equations for a spinning top are as follows

$$I_0\left(\ddot{\theta}-\dot{\phi}^2 sin\theta cos\theta\right)+I\dot{\phi}sin\theta\left(\dot{\phi}cos\theta+\dot{\psi}\right)=mg\rho sin\theta$$
$$I_0\left(\ddot{\phi}sin\theta+2\dot{\phi}\dot{\theta}cos\theta\right)-I\dot{\theta}\left(\dot{\phi}cos\theta+\dot{\psi}\right)=0$$
$$I\left(\ddot{\psi}+\ddot{\phi}cos\theta-\dot{\phi}\dot{\theta}sin\theta\right)=0$$

where $I_0 = I_{xx} = I_{yy}$, $I = I_{zz}$,
m is mass and ρ is coordinates of centre of mass in body frame

Above equations, for a general case, are not analytically solvable.

Precession

Qualitatively, the path traced by the tip of a heavy top can be classified into three categories of precession.

(i) Looping Precession (ii) Cuspidial Precession (iii) Unidirectional Precession

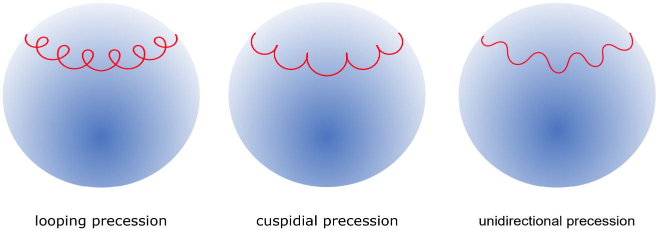


Figure: Precession

"Reduced discrete advected equations" can be applied to derive a Variational integrator and model above three cases.

Discrete Lagrangian for a Heavy Top

- Lagrangian L of the system is given as

$$L(R, R\hat{\Omega}, w) = \frac{1}{2} tr\left(\hat{\Omega} J_d \hat{\Omega}^T\right) - mg\langle w, R\rho\rangle$$

  where J_d is inertia matrix w.r.t the pivot and ρ is the coordinate of centre of mass of the body w.r.t the pivot in body frame.

- Consider the discrete Lagrangian L_d

$$L_d\left(R_i, R_{i+1}, w_i\right) \approx \frac{h}{2}\left\{L\left(R_i, R_i\hat{\Omega}_i, w_i\right) + L\left(R_{i+1}, R_{i+1}\hat{\Omega}_i, w_i\right)\right\}$$

  where

$$\hat{\Omega}_i = R_i^{-1}\dot{R}_i \approx R_i^T\left(\frac{R_{i+1} - R_i}{h}\right) = \frac{F_i - I}{h}$$

Derivation

- Above choice of discrete Lagrangian is left G-invariant. So, we can introduce reduce discrete Lagrangian l_d
- The reduced discrete Lagrangian l_d is

$$l_d\left(F_i, w_i\right) = \frac{1}{2h} tr\left(\left(F_i - I\right) J_d \left(F_i - I\right)^T\right) - \frac{h}{2} mg \langle w_i, \rho + F_i \rho \rangle$$

- Following terms are derived for above l_d
  1. $(LF)^* \frac{\delta l_d}{\delta F} = \frac{1}{2}\left[\frac{1}{h}\left(J_d F - F^T J_d\right) - \frac{h}{2} mg \left(F^T w \cdot \rho^T - \rho w^T F\right)\right]$
  2. $(RF)^* \frac{\delta l_d}{\delta F} = \frac{1}{2}\left[\frac{1}{h}\left(F J_d - J_d F^T\right) - \frac{h}{2} mg \left(w \cdot \rho^T F^T - F \rho w^T\right)\right]$
  3. $\frac{\delta l_d}{\delta w} \diamond w = -\frac{h}{2} mg \left[\frac{1}{2}\left(w\rho^T - \rho w^T + w\rho^T F^T - F\rho w^T\right)\right]$

Variational Integrator

In practice, initial momentum Π and position R of the system are readily available. For this reason, we develop integrator using the discrete Lie-Poisson approach.

The advected momentum $\hat{\Pi}_i$ is derived as

$$\begin{aligned}
\hat{\Pi}_i &= (RF_i)^* \frac{\delta l_d}{\delta F}(F_i) - \left(\frac{\delta l_d}{\delta w} \diamond w_i\right)(F_i, w_i) \\
&= \frac{1}{2}\left[\frac{1}{h}\left(F_i J_d - J_d F_i^T\right) - \frac{h}{2} mg \left(\rho w_i^T - w_i \rho^T\right)\right]
\end{aligned} \tag{1}$$

- Given (w_i, Π_i); F_i is solved implicitly from (1). (In application, above equation is transformed into vector form using Rodrigues Formula)
- Then R_{i+1} is directly given by

$$R_{i+1} = R_i F_i$$

- w_{i+1} can be calculated from

$$w_{i+1} = R_{i+1}^T e_3$$

- Finally, $\hat{\Pi}_{i+1}$ can be calculated as

$$\begin{aligned}\hat{\Pi}_{i+1} &= (LF_i)^* \frac{\delta l_d}{\delta F_i} \\ &= \frac{1}{2}\left[\frac{1}{h}\left(J_d F_i - F_i^T J_d\right) - \frac{h}{2} mg \left(F_i^T w_i \rho^T - \rho w_i^T F_i\right)\right]\end{aligned} \tag{2}$$

This approach allows us to go from $\mathfrak{g}^* \times V^* \to \mathfrak{g}^* \times V^*$ i.e $\left(\hat{\Pi}_i, w_i\right) \mapsto \left(\hat{\Pi}_{i+1}, w_{i+1}\right)$.

From theoretical study, we expect this integrator to satisfy geometric and momentum conservation.

Geometric Behaviour

We implicitly solved for F_i on $SO(3)$ and used the reconstruction equation $R_{i+1} = R_i F_i$. This guarantees R_i to stay on $SO(3)$ i.e.

$$R_i R_i^T = I = R_i^T R_i$$

We can compare the geometric behaviour of integrators by computing $\|R^T R - I\|$.

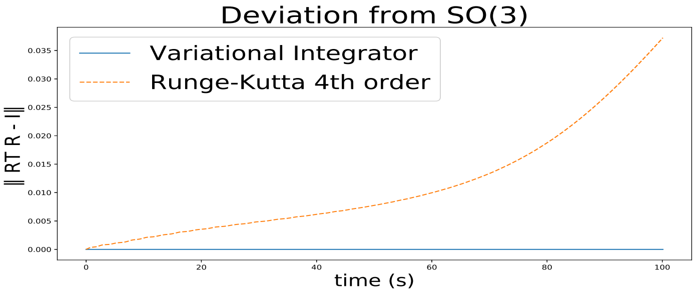


Figure: Geometric behaviour

Momentum Conservation

Since the Lagrangian $\tilde{L}$ is not G-invariant, the spatial angular momentum vector is not conserved. However the component of this momentum in direction of gravity is conserved in the physical system, which is also true in discrete case as shown below.

Acting $R_{i+1}\_\_R_{i+1}^T$ on both sides of (2) and simplifying, we have

$$R_{i+1}\Pi_{i+1} - R_i\Pi_i = \frac{h}{2}mg\left(e_3 \times (R_{i+1}\rho) + e_3 \times (R_i\rho)\right) \tag{3}$$

Taking dot product w.r.t e_3 on both sides, we have

$$e_3 \cdot (R_{i+1}\Pi_{i+1} - R_i\Pi_i) = 0 \tag{4}$$

Momentum Conservation

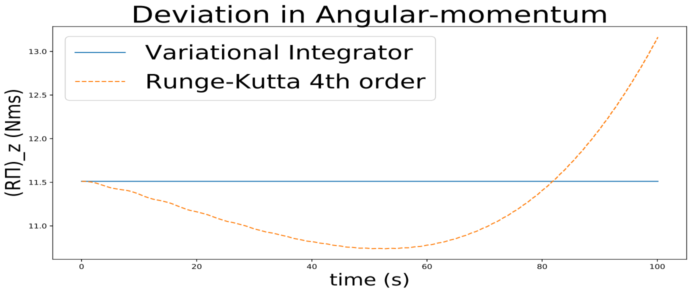


Figure: Conservation of Spatial Angular Momentum in z-direction

Energy Conservation

Though, theoretically, we did study the energy conservation property of a Variational integrator, the integrator exhibits good energy behaviour.

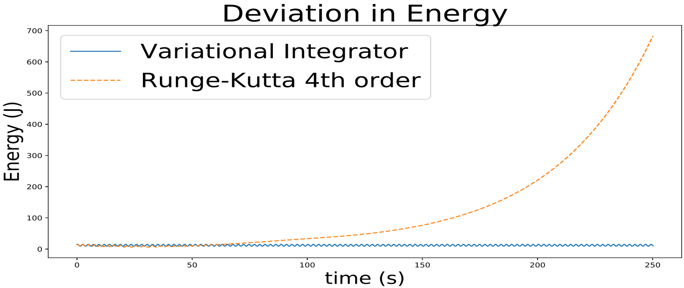
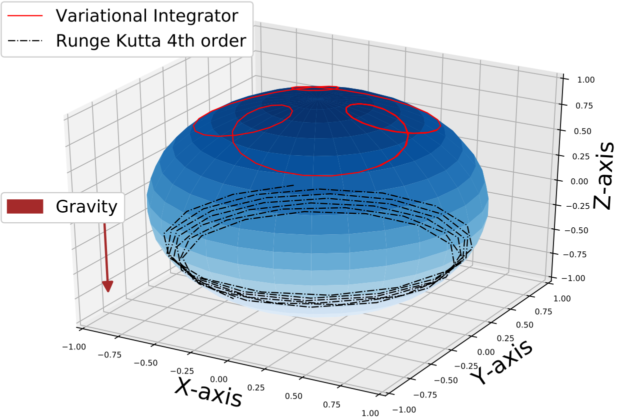


Figure: Long Term Energy Behaviour

Precession



Variational Integrator
Runge Kutta 4th order
Gravity
X-axis
Y-axis
Z-axis
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

Looping Precession

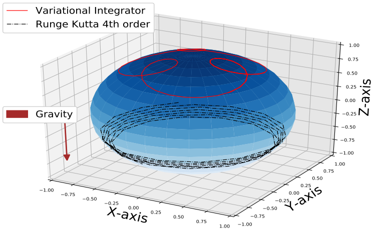


Figure: Looping Precession (100s - 200s)
Initial Conditions: $\dot{\phi}_0 = 3\,rad/s$, $\dot{\theta}_0 = 0.5\,rad/s$, $\dot{\psi}_0 = 10\,rad/s$, $\theta_0 = 0.8\,rad$.

Looping Precession occurs when initial precession and nutation rates are high

Cuspidial Precession

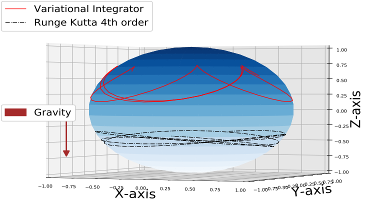


Figure: Cuspidial Precession (100s - 200s)
Initial Conditions: $\dot{\phi} = 0\, rad/s$, $\dot{\theta} = 0\, rad/s$, $\dot{\psi} = 10\, rad/s$, $\theta = 0.8\, rad$

Cuspidial Precession occurs when initial precession and nutation rates are absent

Unidirectional Precession

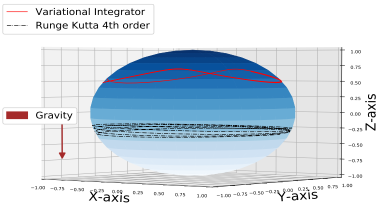


Figure: Unidirectional Precession (100s - 200s)
Initial Conditions: $\dot{\phi} = 1\, rad/s$, $\dot{\theta} = 0.01\, rad/s$, $\dot{\psi} = 10\, rad/s$, $\theta = 0.8\, rad$.

Unidirectional Precession occurs when initial precession and nutation rates are comparable

Summary

We developed discrete version of reduction for systems with invariant Lagrangian. Later, we extended the theory to systems with broken invariance and showed that the dynamics dictated by these reduced equations is equivalent to that of a variational integrator. Owing to these excellent properties of discrete reduced equations, we can develop integrators to study behaviour of important systems when explicit solution is not available.

Discrete Advected equations have been applied to a system of heavy top and chaotic behaviour of a system under unstable equilibrium is modelled. Integrator respect important characteristics of the physical system like rigidity, momentum, energy conservation and agree with our prediction for Variational Integrators. This integrator has then been used to study complex precession behaviour of a spinning heavy top. Our model was able to show good beahviour even after long intervals of time due to conservative nature of Variational Integrators.

References

- Holm, Darryl and Schmah, Tanya and Stoica, Cristina
  Geometric Mechanics and Symmetry.
  2009.
- Marsden, J. E. and West, M.
  Discrete mechanics and variational integrators
  Acta Numerica, 357–514, 10.1017/S096249290100006X, Cambridge University Press, 2001
- Taeyoung Lee and N. H. McClamroch and M. Leok
  A lie group variational integrator for the attitude dynamics of a rigid body with applications to the 3D pendulum
  Proceedings of 2005 IEEE Conference on Control Applications, 2005. CCA 2005., 962-967, 2005
- César Contreras and Tomoki Ohsawa
  Controlled Lagrangians and Stabilization of Euler–Poincaré Mechanical Systems with Broken Symmetry
  arXiv math.OC , 2003.10584, 2020

Thank You!