Macroparticle Models

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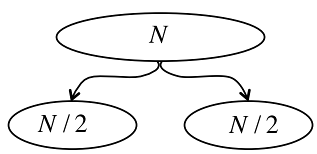
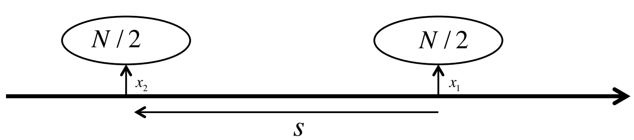
A common approach to understanding simple instabilities is to break a bunch into two "macrobunches"



As an example, we will apply this to an electron linac. At high γ, $\nu_s \rightarrow 0$ (ie, no synchrotron motion) , so the longitudinal positions of the particles remain fixed



From the last lecture, we have

$$F_r = eQ_m m r^{m-1}\cos(m\theta)W_m(s)$$

Consider the lowest order (transverse) mode due to the leading macroparticle

$$\begin{aligned} Q_1 &= \int \rho(r,\theta,z) r\cos\theta r\,dr\,d\theta\,dz \\ &= \int \frac{Ne}{2}\delta(x-x')\delta(y)\delta(z-ct)x\,dx\,dy\,dz \\ &= \frac{Ne}{2}x_1 \end{aligned}$$

The force on the second macroparticle will then be

$$\begin{aligned} F_x &= F_r(\cos\theta = 1) \\ &= eQ_1W_1(s) \\ &= \frac{Ne^2}{2}W_1(s)x_1 \end{aligned}$$

If x_1 is executing β oscillations, then

$$x_1 = A_1\cos\omega_\beta t$$

so the second particle sees

driving term

$$\begin{aligned} \ddot{x}_2 + \omega_\beta^2 &= \frac{F_z}{m\gamma} \\ &= \frac{Ne^2}{2m\gamma}W_1(s)x_1 \\ &= \frac{Ne^2}{2m\gamma}W_1(s)A_1\cos\omega_b t \end{aligned}$$

If the two have the same betatron frequency, then the solution is

particular solution

$$x_2(t) = A_2\cos\omega_\beta t + p(t)$$

homogeneous solution

Try $$p(t) = kt\sin\omega_\beta t$$
$$\dot{p}(t) = k\sin\omega_\beta t + kt\omega_\beta\cos\omega_\beta t$$
$$\ddot{p}(t) = 2k\omega_\beta\cos\omega_\beta t - kt\omega_\beta^2\sin\omega_\beta t$$

Plug this in and we find

$$\begin{aligned}\ddot{x} + \omega_b^2 x &= 2k\omega_\beta\cos\omega_\beta t - \cancel{kt\omega_\beta^2\sin\omega_\beta t} \\ &\quad + \cancel{kt\omega_\beta^2\sin\omega_\beta t} \\ &= A_1\frac{Ne^2}{2m\gamma}W_1\cos\omega_\beta t\end{aligned}$$

$$\longrightarrow k = A_1\frac{Ne^2}{4\omega_b m\gamma}W_1$$

grows with time!

$$\longrightarrow x_2(t) = A_2\cos\omega_\beta t + A_1\frac{Ne^2}{4\omega_\beta m\gamma}W_1 t\sin\omega_\beta t$$

This is a problem in linacs, which can cause beam to break up in a length

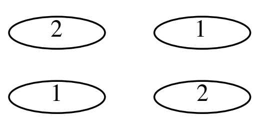
$$\frac{Ne^2}{4\omega_\beta m\gamma}W_1 t \sim 1 \rightarrow L_{max} = ct \sim \frac{4c\omega_\beta m\gamma}{Ne^2 W_1(l_b/2)}$$

wake function ~half a bunch length behind

Must keep wake functions as low as possible in design!

Strong Head-Tail Instability

In a machine undergoing synchrotron oscillations, this problem is alleviated somewhat, in that the leading and trailing macroparticles change places every ~half synchrotron period



$0 < t < T_s/2$

$T_s/2 < t < T_s$

$$0 < t < \frac{T_s}{2}: \quad \ddot{x}_1 + \omega_\beta^2 x_1 = 0$$
$$\ddot{x}_2 + \omega_\beta^2 x_2 = \frac{Ne^2}{2m\gamma}W_1 x_1$$
$$\frac{T_s}{2} < t < T_s: \quad \ddot{x}_1 + \omega_\beta^2 x_1 = \frac{Ne^2}{2m\gamma}W_1 x_2$$
$$\ddot{x}_2 + \omega_\beta^2 x_2 = 0$$

In and unperturbed system

complex form

$$\begin{aligned}x(t) &= x_0\cos\omega_\beta t + \frac{\dot{x}_0}{\omega_\beta}\sin\omega_\beta t \\ \dot{x}(t) &= \dot{x}_0\cos\omega_\beta t - x_0\omega_\beta\,\mathrm{in}\,\omega_\beta t\end{aligned} \Longrightarrow \tilde{x}(t) \equiv x + \frac{i}{\omega_\beta}\dot{x} = \tilde{x}_0 e^{-i\omega_\beta t}$$

For the first half period, we plug in the term from the linac case

$$x_1(t) = A_1 \cos\omega_\beta t$$
$$x_2(t) = A_2 \cos\omega_\beta t + \frac{Ne^2}{4\omega_\beta m\gamma} W_1 A_1 t \sin\omega_\beta t \quad \Longrightarrow \quad \begin{aligned}\tilde{x}_1(t) &= \tilde{x}_1(0) e^{-i\omega_\beta t} \\ \tilde{x}_2(t) &= \tilde{x}_2(0) e^{-i\omega_\beta t} + i\frac{Ne^2}{4\omega_\beta m\gamma} W_1 t \tilde{x}_1(0) e^{-i\omega_\beta t}\end{aligned}$$

pull out sin() term

We can express this as a matrix. For the first half period, we have

$$\begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i\frac{Ne^2}{4\omega_\beta m\gamma} W_1 t & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} e^{-i\omega_\beta t}$$

After half a synchrotron period, we have

$$\begin{pmatrix} \tilde{x}_1(T_s/2) \\ \tilde{x}_2(T_s/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i\kappa & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} e^{-i\omega_\beta (T_s/2)}; \quad \kappa \equiv \frac{Ne^2 W_1 T_s}{8\omega_\beta m\gamma}$$

For the second half of the synchrotron period, we get.

$$\begin{pmatrix} \tilde{x}_1(T_s) \\ \tilde{x}_2(T_s) \end{pmatrix} = \begin{pmatrix} 1 & i\kappa \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(T_s/2) \\ \tilde{x}_2(T_s/2) \end{pmatrix} e^{-i\omega_\beta (T_s/2)}$$

For the second half of the synchrotron period, we get.

$$\begin{pmatrix} \tilde{x}_1(T_s) \\ \tilde{x}_2(T_s) \end{pmatrix} = \begin{pmatrix} 1 & i\kappa \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ i\kappa & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} e^{-i\omega_\beta T_s}$$
$$= \begin{pmatrix} 1-\kappa^2 & i\kappa \\ i\kappa & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} e^{-i\omega_\beta T_s}$$

We proved a long time ago that after many cycles, motion will only be stable if

"strong head-tail instability" threshold

$$\left|\mathrm{Tr}(M)\right| = \left|2-\kappa^2\right| \le 2 \rightarrow \boxed{\frac{Ne^2 W_1 T_s}{16\omega_\beta m\gamma} \le 1}$$

We now consider the tune differences cause by chromaticity

$$\omega_\beta = 2\pi\nu f$$

revolution frequency

tune

If the momentum changes by $\delta = \frac{\Delta p}{p}$

$$\omega_\beta(\delta) = 2\pi\nu(\delta) f(\delta)$$

$$= 2\pi\nu(\nu_0 + \xi\delta) f_0(1-\eta\delta)$$

slip factor

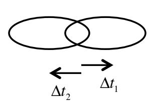
chromaticity

$$= 2\pi\nu_0 f_0 + 2\pi f_0\xi\delta - 2\pi f_0\nu_0\eta\delta + \mathcal{O}(\delta^2)$$

$$\approx \omega_\beta + \omega_0\xi\delta$$

revolution angular frequency

Now we write the positions of our macroparticles as

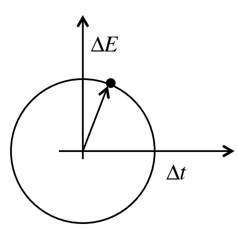


$$\Delta t_1 = \Delta t_0 \sin\omega_s t$$
$$\Delta t_2 = -\Delta t_0 \sin\omega_s t$$

Make *s* the independent variable

$$\Delta t_1 = \Delta t_0 \sin\omega_s \frac{s}{c}$$

$$\Delta t_2 = -\Delta t_0 \sin\omega_s \frac{s}{c}$$



$$\frac{\Delta E}{E} \approx \frac{\Delta p}{p} = \delta$$

$$\eta\delta_{max} = \frac{\Delta t_{max}}{T_s}$$

We calculate the accumulated phase angle

$$\phi = \int^t \omega_\beta(\delta)\,dt' = \frac{1}{c}\int^s \omega_\beta(\delta)\,ds'$$

$$= \frac{1}{c}\int^s \omega_\beta\,ds' + \omega_0\xi\frac{1}{c}\int^s \delta\,ds'$$

$$= \frac{\omega_\beta s}{c} - \frac{\omega_0\xi}{\eta}\Delta t_0 \sin\left(\frac{\omega_\beta s}{c}\right)$$

So we can write

$$x_1(s) = \tilde{x}_1(0) e^{-i\left(\frac{\omega_\beta s}{c} - \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

$$x_2(s) = \tilde{x}_2(0) e^{-i\left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

We identify the angular term and ω and write out equation

$$\ddot{x}_2 + \omega^2 x_2 = \frac{F}{m\gamma}$$

$$\Downarrow$$

$$c^2 \frac{d^2 x_2}{ds^2} + \left[\omega_\beta + \frac{\xi\omega_0 \Delta t_0 \omega_s}{\eta}\cos\frac{\omega_s s}{c}\right]^2 x_2 = \frac{Ne^2 W_1}{2m\gamma} x_1$$

Assume that the amplitude is changing slowly over time, we look at the first term

$$x_2(s) \approx \tilde{x}_2 e^{-i\left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

$$c^2 \frac{dx_2}{ds} = c^2 \left[\frac{d\tilde{x}_2}{ds} - i\left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \frac{\omega_s}{c}\cos\frac{\omega_s s}{c}\right)\tilde{x}_2\right] e^{-i\left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

assume $\frac{d^2}{ds^2}\tilde{x}_2 \approx 0$

will cancel "spring constant" term

$$\longrightarrow c^2 \frac{d^2 x_2}{ds^2} \approx c^2 \left[-2i\left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \frac{\omega_s}{c}\cos\frac{\omega_s s}{c}\right)\frac{d\tilde{x}_2}{ds} - \left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \frac{\omega_s}{c}\cos\frac{\omega_s s}{c}\right)^2\right] e^{-i\left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

$$\longrightarrow c^2 \frac{d^2 x_2}{ds^2} + [\ldots] x_2 \approx c^2 \left[-2i\left(\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \frac{\omega_s}{c}\cos\frac{\omega_s s}{c}\right)\frac{d\tilde{x}_2}{ds}\right] e^{-i\left(\cancel{\frac{\omega_\beta s}{c}} + \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

$$= \frac{Ne^2 W_1}{2m\gamma}\tilde{x}_1 e^{-i\left(\cancel{\frac{\omega_\beta s}{c}} - \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

If we assume $\omega_s << \omega_b$

$$\frac{d\tilde{x}_2}{ds} = i\frac{Ne^2 W_1}{4m\gamma\omega_b c}\tilde{x}_1 e^{2i\left(\frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

$$\approx i\frac{Ne^2 W_1}{4m\gamma\omega_b c}\tilde{x}_1 \left(1 + 2i\left(\frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)\right)$$

Integrate

$$\tilde{x}_2(s) = \tilde{x}_2(0) + i\frac{Ne^2W_1}{4m\gamma\omega_b c}\tilde{x}_1\left(s + 2i\frac{\xi\omega_0}{\eta\omega_s}c\Delta t_0\left(1-\cos\frac{\omega_s s}{c}\right)\right)$$

Now we can obtain the evolution over half a period with

$$s = \frac{T_s}{2}c = \pi\frac{c}{\omega_s}$$

$$\begin{aligned}\tilde{x}_2(T_s/2) &= \tilde{x}_2(0) + i\frac{Ne^2W_1}{4m\gamma\omega_b c}\tilde{x}_1\left(\frac{T_s}{2}c + i\frac{4\xi\omega_0 c\Delta t_0}{\eta\omega_s}\right)\\ &= \tilde{x}_2(0) + i\frac{T_sNe^2W_1}{8m\gamma\omega_b c}\tilde{x}_1\left(1 + i\frac{4\xi\omega_0 c\Delta t_0}{\pi\eta}\right)\\ &\equiv \tilde{x}_2(0) + i\kappa\end{aligned}$$

Compare to our simple case where $\left|2-\kappa^2\right| \le 2$

$$\kappa = \frac{T_sNe^2W_1}{8m\gamma\omega_b c}$$

We have added an imaginary part due to the chromaticity

We look at our previous matrix

$$\begin{pmatrix}\tilde{x}_1(T_s)\\ \tilde{x}_2(T_s)\end{pmatrix} = \begin{pmatrix}1-\kappa^2 & i\kappa\\ i\kappa & 1\end{pmatrix}\begin{pmatrix}\tilde{x}_1(0)\\ \tilde{x}_2(0)\end{pmatrix}e^{-i\omega_\beta T_s}$$

Once more, stability requires $\left|2-\kappa^2\right| \le 2$

Define eigenvalues

$$\lambda_\pm = e^{\pm\mu}$$

$$\mathrm{Tr}(\mathbf{M}) = 2\cos\mu = 2-\kappa^2$$

For low intensity

$$\mu = \pm\kappa$$

So any the imaginary part of κ will give rise to growth

$$\tilde{x}(t) = \tilde{x}(0)e^{\pm\frac{T_sNe^2W_1\xi\omega_0\Delta t_0}{2m\gamma\omega_b\pi\eta}\frac{t}{T_s}} = \tilde{x}(0)e^{\pm\frac{Ne^2W_1\xi\omega_0\Delta t_0}{2m\gamma\omega_b\pi\eta}t}$$

In fact, we'll see that other factors, make adding chromaticity important, particularly above transition.