Synchrotron Radiation

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For a relativistic particle, the total radiated power (S&E 8.1) is

$$P = \frac{1}{6\pi\epsilon_0}\frac{e^2a^2}{c^3}\gamma^4$$

$$a = \text{acceleration} = \frac{v^2}{\rho} \approx \frac{c^2}{\rho}$$

$$\approx \boxed{\frac{1}{6\pi\epsilon_0}\frac{e^2c}{\rho^2}\gamma^4} = \frac{1}{6\pi\epsilon_0}\frac{e^2c}{\rho^2}\left(\frac{E}{m_0c^2}\right)^4$$

For a fixed energy and geometry, power goes as the inverse *fourth power* of the mass!

In a magnetic field $\rho = \dfrac{m\gamma c}{eB}$ $\rightarrow P$

$$\boxed{\begin{aligned} &= \frac{e^4}{6\pi\epsilon_0}\frac{B^2}{m_0^2c}\gamma^2 \\ &= \frac{e^4c}{6\pi\epsilon_0 m^4c^5}B^2E^2 \end{aligned}}$$

Effects of Synchrotron Radiation

- Two competing effects
  - Damping

$$\tau_{\Delta E} \approx \tau \frac{E_s}{U_s}$$

damping time

period

energy

energy lost per turn

  - Quantum effects related to the statistics of the photons

$$N_p = \dot{N}\tau \quad \rightarrow \quad \sigma_{\Delta E} = \sqrt{\dot{N}\tau_{\Delta E}\left\langle u^2 \right\rangle}$$

Number of photons per period

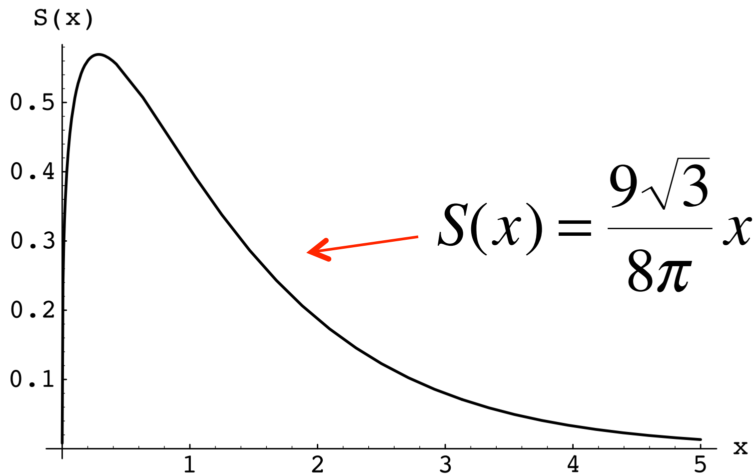
Rate of photon emission

Average photon energy

The power spectrum of radiation is

$$\frac{dP}{d\omega} = \frac{P}{\omega_c} S\left(\frac{\omega}{\omega_c}\right); \qquad \omega_c = \frac{3\gamma^3}{2}\frac{c}{\rho} \qquad \dot{n} = \frac{d\dot{N}}{du}$$

"critical energy"



$$S(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^\infty K_{5/3}(u)\,du$$

$$\frac{dP}{d\omega} d\omega = \dot{n}\hbar\omega du; \qquad d\omega = \frac{du}{\hbar}$$

$$\rightarrow \frac{dP}{d\omega}\frac{1}{\hbar} = \dot{n}\hbar\omega$$

Calculate the photon rate per unit energy

$$\rightarrow \dot{n}(u) = \frac{1}{\hbar\omega}\frac{dP}{d\omega} = \frac{P}{(\hbar\omega)(\hbar\omega)} S\left(\frac{u}{u_c}\right)$$

$$= \frac{P}{u u_c} S\left(\frac{u}{u_c}\right)$$

$$u_c \equiv \hbar\omega_c$$

The total rate is:

$$\dot{N} = \int_0^\infty \dot{n}(u)du = \frac{15\sqrt{3}}{8}\frac{P}{u_c}$$

The mean photon energy is then

$$\langle u \rangle = \frac{P}{\dot{N}} = \frac{8}{15\sqrt{3}}u_c$$

The mean square of the photon energy is

$$\langle u^2 \rangle = \frac{1}{\dot{N}}\int_0^\infty u^2 \dot{n}(u)\, du = \frac{P}{\dot{N}}\int_0^\infty \frac{u}{u_c} S\left(\frac{u}{u_c}\right) du$$

$$= \frac{11}{27}u_c^2$$

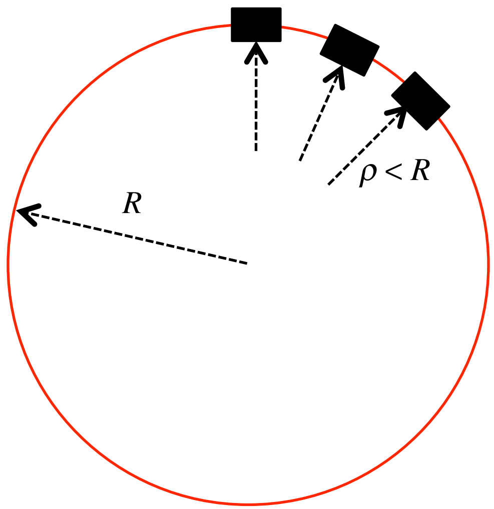
The energy lost per turn is

$$U_s = \oint P\, dt = \frac{e^2 c\gamma^4}{6\pi\epsilon_0}\oint \frac{1}{\rho^2}\left(\frac{dt}{ds}\right) ds$$

$$= \frac{e^2\gamma^4}{6\pi\epsilon_0}\oint \frac{1}{\rho^2} ds$$

$$\frac{1}{c}$$

It's important to remember that ρ is *not* the curvature of the accelerator as a whole, but rather the curvature of individual magnets.



$$\Delta\theta = \frac{\Delta s}{\rho} \rightarrow \oint \frac{ds}{\rho} = 2\pi$$

So if an accelerator is built using magnets of a fixed radius ρ_0, then the energy lost per turn is

$$U_s = \frac{e^2\gamma^4}{6\pi\epsilon_0}\oint\frac{1}{\rho^2}ds = \frac{e^2\gamma^4}{3\epsilon_0\rho_0}$$

For electrons

$$U_s[\text{MeV}] = .0885\frac{E^4[\text{GeV}]}{\rho_0[\text{m}]}$$

$$u_c = \hbar\omega_c = \frac{3\gamma^3\hbar}{2}\frac{c}{\rho_0}$$

$$u_c[\text{keV}] = 2.218\frac{E^3[\text{GeV}]}{\rho_0[\text{m}]}$$

$$N_s = \dot{N}\tau = \frac{15\sqrt{3}}{8}\frac{P}{u_c}\tau = \frac{15\sqrt{3}}{8}\frac{U_s}{u_c}$$

$$= .1296E[\text{GeV}]$$

photons/turn

For CESR

$$E = 5.29 \text{ GeV}$$

$$\rho_0 = 98 \text{ m}$$

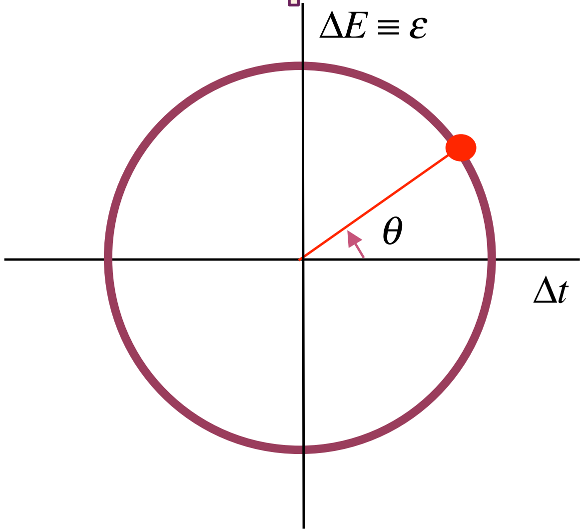
$$U_s = .71 \text{ MeV}$$

$$\langle u\rangle = \frac{8}{15\sqrt{3}}u_c = .98 \text{ keV}$$

$$\sqrt{\langle u^2\rangle} = \sqrt{\frac{11}{27}}u_c = 2.0 \text{ keV}$$

$$\text{N}_s = 721$$

Small Amplitude Longitudinal Motion



$$\varepsilon = \varepsilon_0 \sin(2\pi\nu_s n + \delta)$$

$$\Delta t = \beta_L \varepsilon_0 \cos(2\pi\nu_s n + \delta)$$

$$\frac{1}{\beta_L}(\Delta t)^2 + \beta_L \varepsilon^2 = \epsilon_L = \text{ constant w/o radiation}$$

$$\frac{1}{\beta_L^2}(\Delta t)^2 + \varepsilon^2 = \varepsilon_0^2 \leftarrow$$ amplitude of energy oscillation

If we radiate a photon of energy *u*, then

$$\varepsilon_{0,new}^2 = \frac{1}{\beta_L^2}(\Delta t)^2 + (\varepsilon - u)^2$$

$$= \frac{1}{\beta_L^2}(\Delta t)^2 + \varepsilon^2 - 2\varepsilon u + u^2$$

$$= \varepsilon_0^2 - 2\varepsilon u + u^2$$

$$\rightarrow \quad \Delta\varepsilon_0^2 = -2\varepsilon u + u^2$$

$$\rightarrow \quad \frac{d\varepsilon_0^2}{dt} = -2\varepsilon \dot{N}\langle u \rangle + \dot{N}\langle u^2 \rangle$$

$$\left\langle \frac{d\varepsilon_0^2}{dt} \right\rangle = \frac{1}{\tau_s}\oint \frac{d\varepsilon_0^2}{dt} dt$$

$$= -\frac{2}{\tau_s}\oint \langle \varepsilon P \rangle dt + \frac{1}{\tau_s}\oint \dot{N}\langle u^2 \rangle dt$$

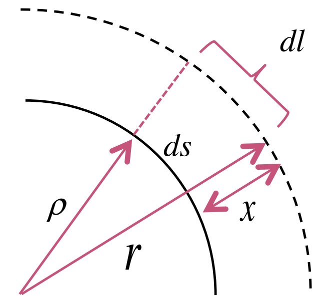
damping term

Heating term

Evaluate integral in damping term

$$\oint \langle \varepsilon P \rangle dt = \frac{1}{c}\oint \left(1+\frac{x}{\rho}\right)\langle \varepsilon P \rangle ds$$

$$\approx \frac{1}{c}\oint \left(1+D\frac{\varepsilon}{\rho E_s}\right)\langle \varepsilon P \rangle ds$$



$$dl = (\rho + x)ds = cdt$$

$$\to dt = \frac{1}{c}(\rho + x)ds$$

use $x = D\frac{\Delta p}{p} \approx D\frac{\varepsilon}{E_s}$

Recall

$$P = \frac{e^4 c}{6\pi\epsilon_0 m^4 c^5} B^2 E^2$$

$$\to P(\varepsilon) = P_s\left(1 + 2\frac{1}{B_0}\frac{dB}{dE} + 2\frac{1}{E_s}\varepsilon\right)$$

Dependence of field

$$\frac{dB}{dx} = B'$$
$$= \kappa(B\rho)$$

$$\to \frac{dB}{dE} = \frac{dB}{dx}\frac{dx}{dE}$$
$$= \frac{\kappa(B\rho)D}{E_s}$$

$$\to P(\varepsilon) = P_s\left(1 + \frac{2\varepsilon}{E_s}(\kappa\rho D + 1)\right)$$

Putting it all together…

$$\oint\langle \varepsilon P\rangle dt = \frac{1}{c}\oint\left\langle \varepsilon P_s\left(1+\frac{\varepsilon}{E_s}\frac{D}{\rho}\right)\left(1+\frac{2\varepsilon}{E_s}(\kappa\rho D+1)\right)\right\rangle ds$$

$$= \frac{1}{c}\oint\left\langle P_s\left(\cancelto{0}{\varepsilon}+\frac{\varepsilon^2}{E_s}\left(2+2\kappa\rho D+\frac{D}{\rho}\right)+\cancelto{0}{\varepsilon^3}\frac{2D(\kappa\rho D+1)}{E_s\rho}\right)\right\rangle ds$$

$$= \frac{1}{c}\frac{\varepsilon_0^2}{2E_s}\oint P_s\left(2+2\kappa\rho D+\frac{D}{\rho}\right)ds$$

$$= \frac{\varepsilon_0^2 U_s}{E_s}+\frac{\varepsilon_0^2}{2E_s}\frac{1}{c}\oint P_s\left(2\kappa\rho D+\frac{D}{\rho}\right)ds$$

$$= \frac{\varepsilon_0^2 U_s}{E_s}+\frac{\varepsilon_0^2 U_s}{2E_s}\mathcal{D}$$

$$\boxed{= \frac{\varepsilon_0^2 U_s}{2E_s}\left(2+\mathcal{D}\right)}$$

use

$$\varepsilon = \varepsilon_0\sin\left(2\pi\nu_s n+\delta\right)$$

$$\rightarrow \langle\varepsilon\rangle = \langle\varepsilon^3\rangle = 0$$

$$\langle\varepsilon^2\rangle = \frac{\varepsilon_0^2}{2}$$

note $\displaystyle \frac{1}{c}\oint P_s\, ds = \frac{1}{c}(\text{const})\oint\frac{1}{\rho^2}ds$

$$= U_s$$

$$\frac{1}{c}\oint P_s\left(2\kappa\rho D+\frac{D}{\rho}\right)ds = \frac{1}{c}(\text{const})\oint\frac{1}{\rho^2}\left(2\kappa\rho D+\frac{D}{\rho}\right)ds$$

$$= U_s\mathcal{D}$$

where $\displaystyle \mathcal{D} \equiv \frac{\oint\frac{1}{\rho^2}\left(2\kappa\rho D+\frac{D}{\rho}\right)ds}{\oint\frac{1}{\rho^2}ds}$

Going *way* back to our original equation (p. 7)

$$\left\langle \frac{d\varepsilon_0^2}{dt} \right\rangle = -\frac{2}{\tau_s}\oint \langle \varepsilon P \rangle dt + \frac{1}{\tau_s}\oint \dot{N}\left\langle u^2 \right\rangle dt$$

$$= \underbrace{\frac{\varepsilon_0^2 U_s}{\tau_s E_s}(2+\mathcal{D})}_{\text{damping}} + \underbrace{\frac{1}{\tau_s}\oint \dot{N}\left\langle u^2 \right\rangle dt}_{\text{heating}}$$

$$\varepsilon_0^2(t) = \varepsilon_0^2(0) e^{-t/\tau_{\varepsilon^2}} + \varepsilon_0^2(\infty) e^{-t/\tau_\varepsilon^2}\left(1 - e^{-t/\tau_{\varepsilon^2}}\right)$$

where $\dfrac{1}{\tau_{\varepsilon^2}} = \dfrac{U_s}{\tau_s E_s}(2+\mathcal{D})$

$$\varepsilon_0^2(\infty) = \frac{\tau_{\varepsilon^2}}{\tau_s}\oint \dot{N}\left\langle u^2 \right\rangle dt$$

The energy then decays in a time

$$\tau_\varepsilon = 2\tau_{\varepsilon^2}$$

$$\frac{1}{\tau_\varepsilon} = \frac{U_s}{2\tau_s E_s}(2+\mathcal{D})$$

In a separated function lattice, there is no bend in the quads, so $K \neq 0 \rightarrow \rho \rightarrow \infty$
Further assume uniform dipole field ($\rho=\rho_0$)

$$\mathcal{D} \equiv \frac{\oint \frac{1}{\rho^2}\left(2\kappa\rho D + \frac{D}{\rho}\right) ds}{\oint \frac{1}{\rho^2} ds} = \frac{\frac{1}{\rho_0^2}\oint \frac{D}{\rho_0} ds}{\frac{1}{\rho_0}\oint \frac{1}{\rho_0} ds} = \frac{\frac{1}{\rho_0^2}(C\alpha)}{\frac{1}{\rho_0}(2\pi)}$$

$$= \frac{C\alpha}{2\pi\rho_0} \ll 1$$

$$\boxed{\frac{1}{\tau_\varepsilon} \approx \frac{U_s}{\tau_s E_s}}$$

probably the answer you would have guessed without doing any calculations.

Equilibrium energy spread will be

$$\sigma_\varepsilon^2 = \left\langle \varepsilon_0^2(\infty) \right\rangle = \frac{1}{2}\varepsilon_0^2(\infty)$$

$$= \frac{\tau_\varepsilon}{4\tau_s} \oint \left\langle \dot{N} u^2 \right\rangle dt$$

$$\text{Use } \dot{N} = \frac{15\sqrt{3}}{8}\frac{P}{u_c}, \quad \left\langle u^2 \right\rangle = \frac{11}{27} u_c^2, \quad u_c = \frac{3}{2}\frac{\hbar\gamma^3}{\rho} c$$

$$\longrightarrow \oint \left\langle \dot{N} u^2 \right\rangle = \frac{55}{16\sqrt{3}} \frac{e^2 \hbar c \gamma^7}{6\pi\epsilon_0} \oint \frac{1}{\rho^3} ds$$

Effects of synchrotron radiation

- Damping in both planes
- Heating in bend plane

Behavior of beams

We're going to derive two important results

1. Robinson's Theorem

$$\frac{1}{\tau_\varepsilon}+\frac{1}{\tau_x}+\frac{1}{\tau_y}=\frac{2U_s}{E_s\tau_s}$$

transverse damping times

For a separated function lattice

$$\tau_x=\tau_y$$

$$\tau_\varepsilon=\frac{E_s\tau_s}{U_S}\rightarrow\tau_x=\tau_y=2\tau_\varepsilon=\frac{2E_s\tau_s}{U_s}$$

2. The equilibrium horizontal emittance

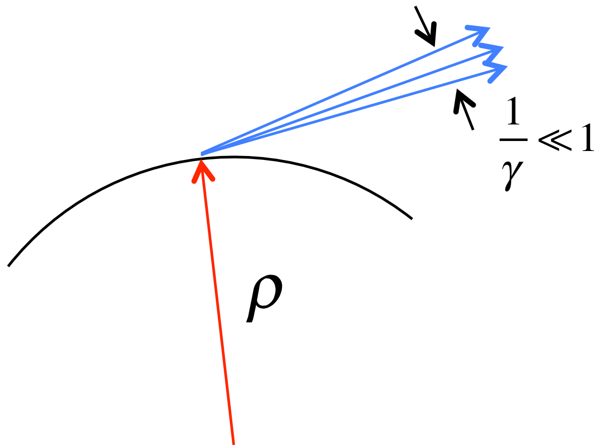
$$\sigma_x=\sqrt{\dot{N}\tau_x}\,\frac{\sqrt{\langle u^2\rangle}}{E_s}\langle D\rangle$$

photons emitted in a damping period

Mean dispersion

Here we go...

Synchrotron radiation



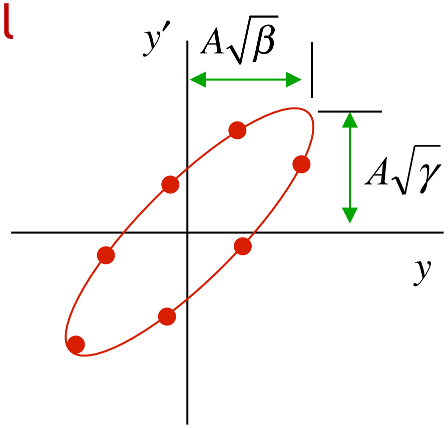
Energy lost along trajectory, so radiated power will reduce momentum along flight path

$$\frac{d\vec{p}}{dt} \approx -\frac{P}{c}\hat{\theta}$$

If we assume that the RF system restores the energy lost each turn, then

Energy lost along the path $\rightarrow \Delta y = \Delta y' = 0$
Energy restored along nominal path $\hat{s}$ ➔"adiabatic damping"

Recall



$$y = a\sqrt{\beta}\cos\left(\psi(s)+\delta\right)$$

$$y' = -\frac{a}{\sqrt{\beta}}\left(\alpha\cos\left(\psi(s)+\delta\right)+\sin\left(\psi(s)+\delta\right)\right)$$

$$y' = \frac{p_y}{p} \rightarrow y' + \Delta y' = \frac{p_y}{p+\Delta p} \approx \left(1-\frac{\Delta p}{p}\right)$$

$$\frac{\Delta y'}{dn} = -y'\frac{\Delta p}{p} = -y'\frac{\Delta E}{E} = \boxed{-y'\frac{U_s}{E_s}}$$

$$a^2 = \gamma y^2 + 2\alpha y y' + \beta y'^2$$

$$\frac{da^2}{dy'} = 2\alpha y + 2\beta y'$$

$$\frac{d(a^2)}{dn} = -2\left(\alpha y + \beta y'\right)y'\left(\frac{U_s}{E_s}\right)$$

$$= -\frac{2U_s}{E_s}\left(\alpha y y' + \beta y'^2\right)$$

$$= -\frac{2U_s}{E_s}a^2\left(-\left(\alpha^2C^2+\alpha SC\right)+\left(\alpha^2C^2+2\alpha SC+S^2\right)\right)$$

$$= -\frac{2U_s}{E_s}a^2\left(\alpha SC + S^2\right)$$

$$= -\frac{2U_s}{E_s}a^2\left(\frac{\alpha}{s}\sin\left(2\left(\psi(s)+\delta\right)\right)+\sin^2\left(\psi(s)+\delta\right)\right)$$

As we average this over many turns, we must average over all phase angles

$$\left\langle \sin\left(2\left(\psi(s)+\delta\right)\right)\right\rangle = 0$$

$$\left\langle \sin^2\left(\psi(s)+\delta\right)\right\rangle = \frac{1}{2}$$

$$\left\langle \frac{d(a^2)}{dn}\right\rangle = -\frac{2U_s}{E_s}a^2\left(\frac{1}{2}\right) = -a^2\frac{U_s}{E_s}$$

$$\frac{d(a^2)}{dt} = \left\langle \frac{d(a^2)}{dn}\right\rangle\left(\frac{dn}{dt}\right)$$

$$\boxed{= -a^2\frac{U_s}{\tau_s E_s}}$$

Calculating beam size

$$y = a\sqrt{\beta}\cos(\psi+\delta) \rightarrow \langle y^2 \rangle = \beta\frac{a^2}{2}$$

$$\frac{d\sigma_y^2}{dt} = \frac{\beta}{2}\frac{da^2}{dt} = -\sigma_y^2\frac{U_s}{\tau_s E_s}$$

$$\frac{d\sigma_y^2}{dt} = 2\sigma_y\frac{d\sigma}{dt}$$

$$\frac{d\sigma}{dt} = -\sigma_y\frac{U_s}{2\tau_s E_s} = -\frac{\sigma_y}{\tau_y}$$

$$\boxed{\tau_y = 2\tau_s\frac{E_S}{U_S} = 2\tau_\varepsilon}$$

Note, in the absence of any heating terms or emittance exchange, this will damp to a very small value. This is why electron machines typically have flat beams. Allowing it to get too small can cause problems (discussed shortly)

Horizontal Plane

Things in the horizontal plane are a bit more complicated because position depends on the energy

betatron motion

ΔE

$$x = x_\beta + D\frac{\epsilon}{E_s}$$

$$x' = x'_\beta + D'\frac{\epsilon}{E_s}$$

where

$$x_\beta = a\sqrt{\beta}\cos\left(\psi(s)+\delta\right) \equiv a\sqrt{b}C$$

$$x'_\beta = -\frac{a}{\sqrt{\beta}}\left(\alpha\cos\left(\psi(s)+\delta\right)+\sin\left(\psi(s)+\delta\right)\right) \equiv -\frac{a}{\sqrt{\beta}}\left(\alpha C + S\right)$$

Now since the radiated photon changes the energy, but not the position or the angle, the betatron orbit must be modified; that is

$$\Delta x = \left[\left(x_\beta + \Delta x_\beta\right) + D\frac{(\varepsilon - u)}{E_s}\right] - \left[x_\beta + D\frac{\varepsilon}{E_s}\right]$$

$$= \Delta x_\beta - D\frac{u}{E_s} = 0$$

$$\longrightarrow \quad \Delta x_\beta = D\frac{u}{E_s}$$

$$\Delta x' = \Delta x'_\beta - D'\frac{u}{E_s} = 0$$

$$\longrightarrow \quad \Delta x'_\beta = D'\frac{u}{E_s}$$

Going back to the motion in phase space

$$a^2 = \gamma x_\beta^2 + 2\alpha x_\beta x'_\beta + \beta x'^2_\beta$$

$$\Delta a^2 = \gamma\left(x_\beta + \frac{u}{E_s}D\right)^2 + 2\alpha\left(x_\beta + \frac{u}{E_s}D\right)\left(x'_\beta + \frac{u}{E_s}D'\right) + \beta\left(x'_\beta + \frac{u}{E_s}D'\right)^2$$

$$-\left(\gamma x_\beta^2 + 2\alpha x_\beta x'_\beta + \beta x'^2_\beta\right)$$

$$= 2\frac{u}{E_s}\left[\alpha\left(D'x_\beta + Dx'_\beta\right) + \gamma x_\beta D + \beta x'_\beta D'\right]$$

$$+\left(\frac{u}{E_s}\right)^2 \left[\gamma D^2 + 2\alpha DD' + \beta D^2\right] \equiv \mathcal{H}$$

$\dot{N}u$

$$\frac{da^2}{dt} = 2\frac{P}{E_s}\left[\alpha\left(D'x_\beta + Dx'_\beta\right) + \gamma x_\beta D + \beta x'_\beta D'\right]$$

$$+\dot{N}\left(\frac{u}{E_s}\right)^2 \mathcal{H}$$

Averaging over one turn

$$\left\langle \frac{da^2}{dt} \right\rangle = \frac{1}{\tau_s} \oint \frac{da^2}{dt} dt$$

$$\Delta a^2 = \frac{2}{\tau_s E_s} \oint P\left[\alpha\left(D' x_\beta + D x'_\beta\right) + \gamma x_\beta D + \beta x'_\beta D'\right] dt$$

$$+ \frac{1}{\tau_s E_s} \oint \dot{N} u^2 \mathcal{H}\, dt$$

Average over all particle and phases

$$\frac{da^2}{dt} = \frac{2}{\tau_s E_s} \oint \left\langle P\left[\alpha\left(D' x_\beta + D x'_\beta\right) + \gamma x_\beta D + \beta x'_\beta D'\right]\right\rangle dt = A$$

$$+ \frac{1}{\tau_s E_s} \oint \left\langle \dot{N} u^2 \right\rangle \mathcal{H}\, dt$$

$$\text{use } dl = \left(1 + \frac{x}{\rho}\right) ds; \quad \text{c} = \frac{dl}{dt}$$

$$\longrightarrow A = \frac{1}{c} \oint \left\langle \left(1 + \frac{x}{\rho}\right) P\left[\alpha\left(D' x_\beta + D x'_\beta\right) + \gamma x_\beta D + \beta x'_\beta D'\right] \right\rangle ds$$

As before

$$P = \frac{e^4}{6\pi\epsilon_0}\frac{B^2}{m_0^2 c}\gamma^2$$

$$\frac{dP}{dB} = 2\frac{P}{B_0}$$

$$P\left(x_\beta\right) = P_s\left(1 + \frac{2}{B_0}\frac{dB}{dx}x_\beta\right)$$

$$= P_s\left(1 + \frac{2}{B_0}\kappa\rho x_\beta\right)$$

$$A = \frac{1}{c}\oint\left\langle\left(1+\frac{x}{\rho}\right)P_s\left(1+2\kappa\rho x_\beta\right)\left[\alpha\left(D'x_\beta + Dx'_\beta\right) + \gamma x_\beta D + \beta x'_\beta D'\right]\right\rangle ds$$

Same procedure as p. 9

$$= \frac{a^2}{2c}\oint P_s\frac{D}{\rho}\left(1+2\kappa\rho\right)ds = \boxed{\frac{a^2 U_s}{2}\mathcal{D}}; \quad \text{where } \mathcal{D} \equiv \frac{\oint \frac{1}{\rho^2}\left(2\kappa\rho D + \frac{D}{\rho}\right)ds}{\oint \frac{1}{\rho^2}ds}$$

Going back...

$$\frac{da^2}{dt} = \frac{2}{\tau_s E_s} A + \frac{1}{\tau_s E_s} \oint \left\langle \dot{N} u^2 \right\rangle \mathcal{H}\, dt$$

$$= \frac{a^2 U_s}{\tau_s E_s} \mathcal{D} + \frac{1}{\tau_s E_s} \oint \left\langle \dot{N} u^2 \right\rangle \mathcal{H}\, dt$$

But remember, we still have the adiabatic damping term from re-acceleration, so

$$\frac{da^2}{dt} = (\ldots) - a^2 \frac{U_s}{\tau_s E_s}$$

$$= -\frac{a^2 U_s}{\tau_s E_s} (1 - \mathcal{D}) + \frac{1}{\tau_s E_s} \oint \left\langle \dot{N} u^2 \right\rangle \mathcal{H}\, dt$$

$$a^2(t) = a^2(0) e^{-t/\tau} + a^2(\infty)\left(1 - e^{-t/\tau}\right)$$

$$\frac{1}{\tau} = -\frac{a^2 U_s}{\tau_s E_s} (1 - \mathcal{D})$$

As before...

$$\sigma_x \propto \sqrt{a^2}$$

$$\tau_x = 2\tau$$

$$\frac{1}{\tau_x} = \frac{U_s}{2\tau_s E_s} (1 - \mathcal{D})$$

$\Rightarrow$

Robinson's Theorem

$$\boxed{\begin{aligned} \frac{1}{\tau_\varepsilon} + \frac{1}{\tau_x} + \frac{1}{\tau_y} &= \frac{U_s}{2\tau_s E_s}(2 + \mathcal{D}) \\ &+ \frac{U_s}{2\tau_s E_s}(1 - \mathcal{D}) \\ &+ \frac{U_s}{2\tau_s E_s} \\ &= \frac{2U_s}{\tau_s E_s} \end{aligned}}$$

for $\mathcal{D} \ll 1$

$$\tau_x \approx \tau_y \approx 2\tau_\varepsilon$$

$$\sigma_x^2(\infty) = \langle x_\infty^2 \rangle = \frac{1}{2}\beta a^2$$

$$= \frac{\tau_x \beta}{4E_s^2 \tau_s} \oint \langle \dot{N} u^2 \rangle \mathcal{H}\, dt$$

$$= \frac{\beta}{2U_s E_s (1-\mathcal{D})} \oint \langle \dot{N} u^2 \rangle \mathcal{H}\, dt$$

Equilibrium emittance

$$\epsilon_x = \frac{\sigma_x^2(\infty)}{\beta} = \frac{1}{2U_s E_s (1-\mathcal{D})} \oint \langle \dot{N} u^2 \rangle \mathcal{H}\, dt$$

$$\oint \langle \dot{N} u^2 \rangle \mathcal{H}\, dt = \frac{55\gamma^3 \hbar c}{16\sqrt{3}} \left(\frac{e^2 c \gamma^4}{6\pi\epsilon_0} \right) \oint \frac{\mathcal{H}}{\rho^3} dt$$

$$= \frac{55\gamma^3 \hbar c}{16\sqrt{3}} \left[\oint \left(\frac{e^2 c \gamma^4}{6\pi\epsilon_0} \right) \frac{1}{\rho^2} dt \right] \frac{\oint \frac{\mathcal{H}}{\rho^3} ds}{\oint \frac{1}{\rho^2} ds}$$

$$= \frac{55\gamma^3 \hbar c}{16\sqrt{3}} U_s \frac{\oint \frac{\mathcal{H}}{\rho^3} ds}{\oint \frac{1}{\rho^2} ds}$$

use

$$\dot{N} = \frac{15\sqrt{3}}{8} \frac{P}{u_c}; \langle u^2 \rangle = \frac{11}{27} u_c^2; u_c = \frac{3}{2} \frac{\hbar \gamma^3 c}{\rho}$$

$$P = \frac{e^2 c \gamma^4}{6\pi\epsilon_0} \frac{1}{\rho^2}$$

$$\epsilon_x(\infty) = \frac{1}{2U_s E_s (1-\mathcal{D})} \oint \langle \dot{N} u^2 \rangle dt$$

$$= \frac{1}{2U_s (\gamma mc^2)(1-\mathcal{D})} \frac{55\gamma^3 \hbar c}{16\sqrt{3}} U_s \frac{\oint \frac{\mathcal{H}}{\rho^3} ds}{\oint \frac{1}{\rho^2} ds}$$

$$\boxed{= C_q \frac{\gamma^2}{(1-\mathcal{D})} \frac{\oint \frac{\mathcal{H}}{\rho^3} ds}{\oint \frac{1}{\rho^2} ds} \quad \text{where } C_q \equiv \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.8 \times 10^{-13} \ \text{(for electrons)}}$$

For a separated function, isomagnetic machine $\rho = \rho_0$ or $\rho = \infty \rightarrow \oint \frac{1}{\rho^2} ds = \frac{1}{\rho_0} \oint \frac{ds}{\rho} = \frac{2\pi}{\rho_0}$

$$\boxed{\epsilon_x(\infty) = C_q \frac{\gamma^2}{2\pi\rho_0 (1-\mathcal{D})} \oint \frac{\mathcal{H}}{\rho} ds}$$

Approximate

$$\oint \frac{\mathcal{H}}{\rho} ds \sim 2\pi \langle \mathcal{H} \rangle; \quad \text{recall } \mathcal{H} \equiv \gamma D^2 + 2\alpha D D' + \beta D'^2$$

$$\sim 2\pi \langle \gamma D^2 \rangle \sim \left\langle \frac{D^2}{\beta} \right\rangle$$

$$\langle D \rangle \approx \alpha_c R; \quad \beta \approx \frac{R}{\nu_x}; \quad \alpha_c = \frac{1}{\gamma_t} \approx \frac{1}{\nu_x}$$

$$\epsilon_x(\infty) \approx \gamma^2 \frac{C_q}{\rho_0} \left\langle \frac{D^2}{\beta} \right\rangle$$

$$\approx C_q \gamma^2 \frac{R}{\rho_0} \alpha_c^2 \nu_x$$

$$\boxed{\approx C_q \gamma^2 \frac{R}{\rho_0} \frac{1}{\nu_x^3}}$$