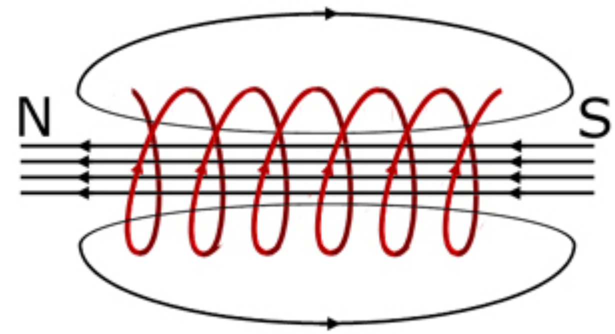
Special Topic: Solenoids

Applications of Solenoidal Fields



- Solenoids can provide an arbitrarily uniform magnetic field through a very large and/or extended volume
- Solenoids have long been used to create momentum tracking volumes in central high energy physics detectors.
- Solenoids can also be used to contain and transport low momentum particles (p up to a few 10's of MeV) by "trapping" in helical trajectories along the field lines
  - Concept originally applied to plasma containment
  - Currently drawing a great deal of interest as a way to transport very large emittance beams of low momentum particles
    - Particularly useful pions and muons for neutrino physics or muon application.

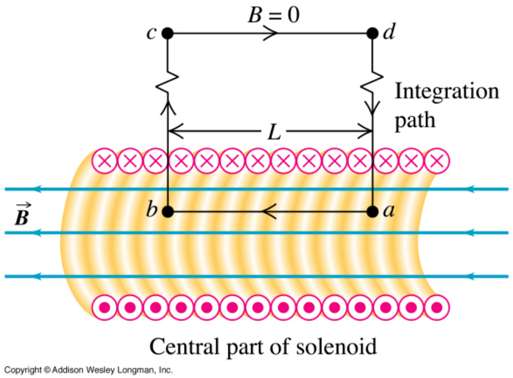
The fields in a solenoid

- Within a long solenoid, the magnetic field is more or less uniform, calculated with

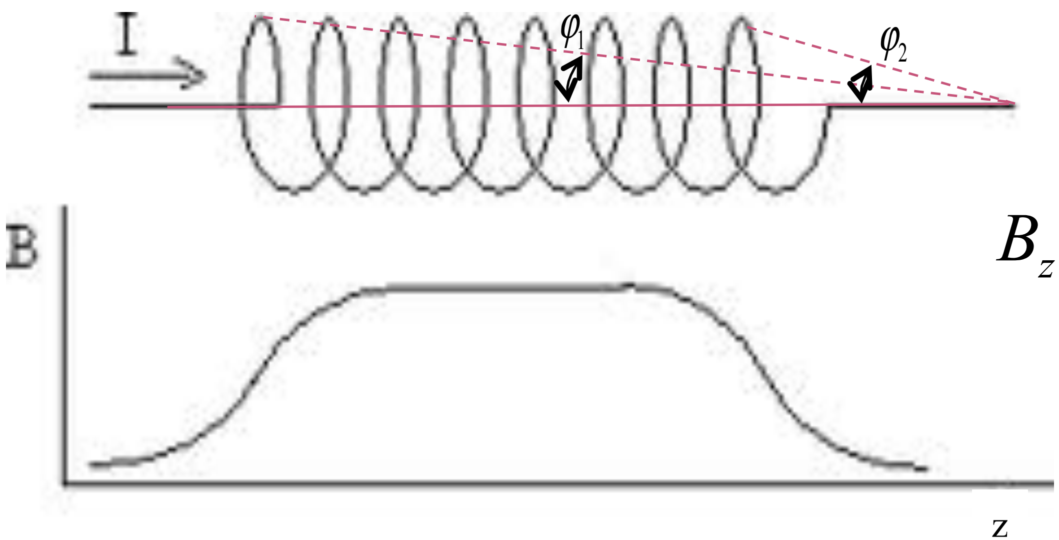
$$\oint \vec{B} \bullet d\vec{l} = BL = \mu_0 I_{\text{enc}} = \mu_0 InL$$

winding pitch

$$\Rightarrow B = \mu_0 nI$$



- The exact formula is

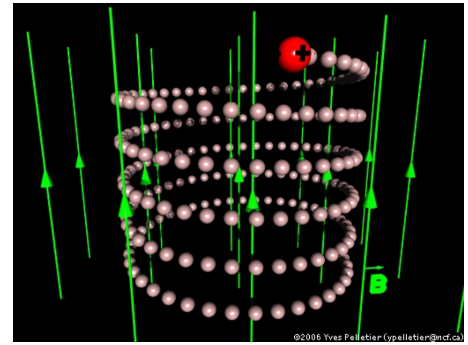


$$B_z = \frac{1}{2}\mu_0 nI\left(\cos\varphi_1 - \cos\varphi_2\right)$$

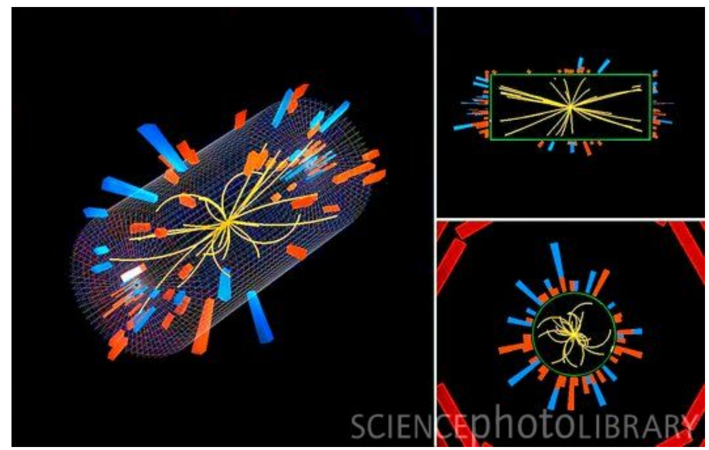
Particle motion in a solenoidal field

- Generally, particles move in a helical trajectory

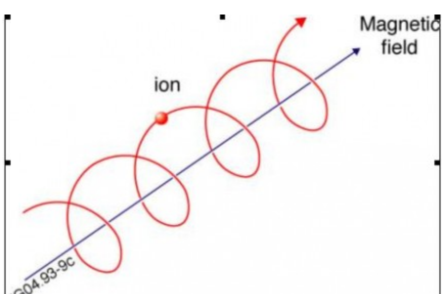
$$\rho = \frac{p}{qB}; \rho[m] = \frac{p[MeV/c]/299}{B[T]}$$



- For high momentum particles, the curvature is used to measure the momentum



- Low momentum particles are effectively "trapped" along the field lines
  - 10 MeV/c particle will have a radius of 3 cm in a 1 T field



Solenoids are a powerful tool to transport low momentum particles and can accommodate beams with very large emittances.

Constants of the motion

- Both total momentum and angular momentum are conserved

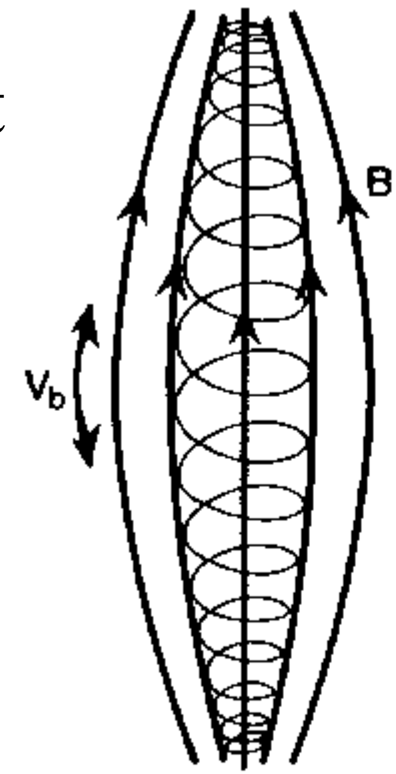
$$p_0^2 = p_\perp^2 + p_{||}^2 = \mathrm{constant}$$

$$L = p_\perp \rho = \frac{p_\perp^2}{qB_{||}} = \mathrm{constant}$$

$$\Rightarrow \quad p_\perp^2 = qLB_{||}$$

$$p_{||}^2 = p_0^2 - qLB_{||}$$

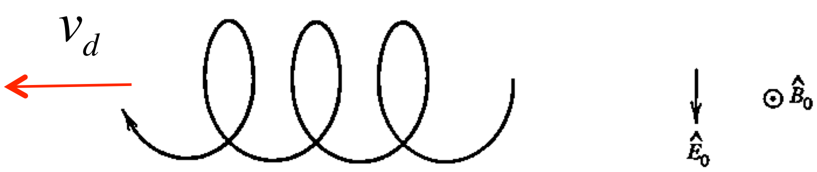
- If $qLB_{||} > p_0^2$, then particle will be reflected
  - Basis of “pinch confinement”



Bounce Motion

"E cross B Drift"

An electric field transverse to the magnetic will cause a lateral drift, but the average acceleration will be zero.



$$\left\langle \dot{\vec{p}} \right\rangle = q\vec{E}_0 + q\left(\vec{v}_d \times \vec{B}_0\right) = 0$$

averaged over a cycle

assume

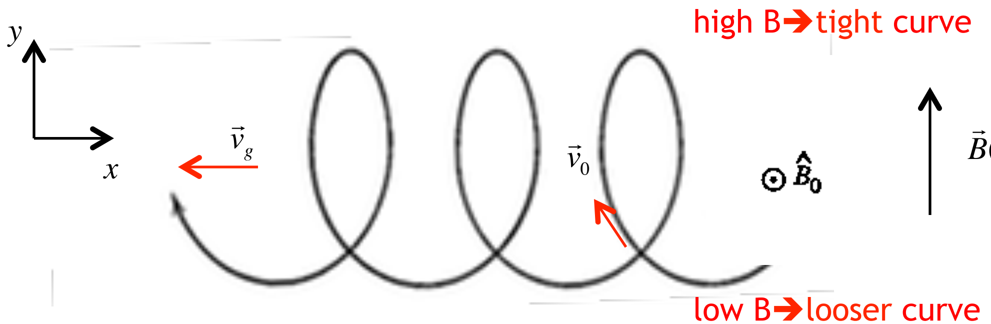
Cross the magnetic field into this

$$q\left[\vec{B}_0 \times \vec{E}_0 + \vec{B}_0 \times \left(\vec{v}_d \times \vec{B}_0\right)\right] = 0$$

$$\vec{B}_0 \times \vec{E}_0 + B_0^2 \vec{v}_d - \left(\vec{B}_0 \bullet \vec{v}_d\right)\vec{B}_0 = 0$$

$$\boxed{\vec{v}_d = \frac{\vec{E}_0 \times \vec{B}_0}{B_0^2}}$$

"Grad-B Drift"



$$\vec{B}(r) \approx \vec{B}_0 + \left(\vec{r} \bullet \vec{\nabla}\right)\vec{B}$$

We'll divide the motion in the cyclical part (v_0) and the drift (v_g)

$$\frac{\vec{F}}{q} = \vec{v}(t)\times\vec{B}(r) = \vec{v}_0(t)\times\vec{B}(r) + \vec{v}_g\times\vec{B}(r)$$

$$\approx \vec{v}_0(t)\times\vec{B}_0 + \vec{v}_0(t)\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right] + \vec{v}_g\times\vec{B}(r) + \vec{v}_g\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right]$$

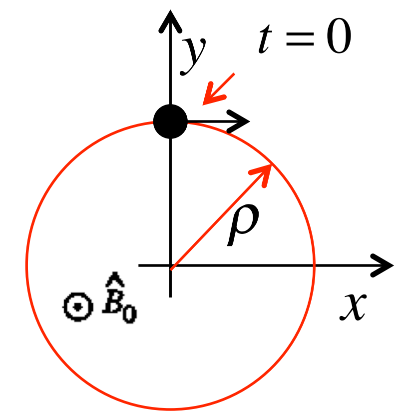
$$\longrightarrow \left\langle\frac{\vec{F}}{q}\right\rangle = \cancel{\left\langle\vec{v}_0(t)\times\vec{B}_0\right\rangle} + \left\langle\vec{v}_0(t)\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right]\right\rangle + \left\langle\vec{v}_g\times\vec{B}_0\right\rangle + \cancel{\left\langle\vec{v}_g\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right]\right\rangle}$$

$$= \left\langle\vec{v}_0(t)\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right]\right\rangle + \vec{v}_g\times\vec{B}_0 = 0$$

Again, cross B into this and we get

$$\vec{B}_0\times\left\langle\vec{v}_0(t)\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right]\right\rangle + B_0^2\vec{v}_g - \left(\vec{B}_0\bullet\vec{v}_g\right)\vec{B}_0 = 0 \rightarrow \vec{v}_g = \frac{1}{B_0^2}\left\langle\vec{v}_0(t)\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right]\right\rangle\times\vec{B}_0$$

For our example



$$\rho_0 = \frac{p}{qB_0}$$

$$y = \rho_0 \cos\left(\Omega_c t\right)$$

$$v_{0,x} = v_0 \cos(\Omega_c t)$$

$$v_{0,y} = -v_0 \sin(\Omega_c t)$$

$$\vec{B}_0 = B_0 \hat{z}$$

$$\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B} = y\frac{\partial B}{\partial y}\hat{z}$$

$$\vec{v}_0(t)\times\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_{0,x}(t) & v_{0,y}(t) & 0 \\ 0 & 0 & y\dfrac{\partial B}{\partial y} \end{vmatrix} = v_{0,y}(t)y\frac{\partial B}{\partial y}\hat{x} - v_{0,x}(t)y\frac{\partial B}{\partial y}\hat{y}$$

$$= -\rho_0 v_0 \cos\left(\Omega_s t\right)\sin\left(\Omega_s t\right)\frac{\partial B}{\partial y}\hat{x} - \rho_0 v_0 \cos^2\left(\Omega_s t\right)\frac{\partial B}{\partial y}\hat{y}$$

$$\left\langle \vec{v}_0(t)\times\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right\rangle = -\frac{1}{2}\rho_0 v_0 \frac{\partial B}{\partial y}\hat{y}$$

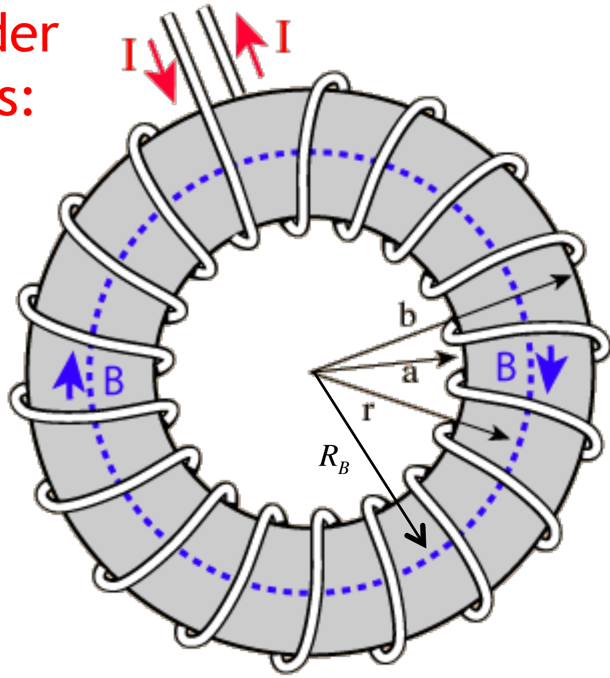
$$\vec{v}_g = \frac{1}{B_0^2}\left\langle \vec{v}_0(t)\times\left[\left(\vec{r}\bullet\vec{\nabla}\right)\vec{B}\right]\right\rangle\times\vec{B}_0$$

$$= -\frac{1}{2B_0}\rho_0 v_0 \frac{\partial B}{\partial y}\hat{x}$$

$$\boxed{= \frac{1}{2B_0^2}\rho_0 v_0 \left(\vec{B}_0\times\vec{\nabla}B\right)}$$

Fields and drift in a Curved Solenoid

Consider a torus:



$$\int \vec{B} \bullet d\vec{l} = 2\pi r B = \mu_0 N I$$

$$B = \frac{\mu_0 N I}{2\pi r}$$

$$B_0 \equiv \frac{\mu_0 N I}{2\pi R_B} = \mu_0 n I$$ field in center solenoid

Nominal radius of curvature

$$B = B_0 \frac{R_B}{r}$$

$$\approx B_0 \left(1 - \frac{x}{R_B}\right)$$

x measured outward from center of solenoid

Clearly these formulas will also hold in an area of local curvature R_0.

As the particle moves along the field lines, it will experience a (fictitious) centrifugal force outward.

Component of velocity along *B* field

$$\vec{F}_c = m \frac{v_\parallel^2}{R_B} \hat{r}$$

This is analogous to the effect of the electric field, so

$$\vec{E}_0 \to \frac{\vec{F}_c}{q} \Rightarrow \vec{v}_d = \frac{\vec{E}_0 \times \vec{B}_0}{B_0^2} \to \vec{v}_d = \frac{mv_\parallel^2}{qR_B B_0^2}\hat{r} \times \vec{B}_0$$

But we also have a gradient

$$\vec{\nabla} B = -\frac{B_0}{R_B}\hat{r}$$

velocity perpendicular to B

$$v_g = \frac{1}{2B_0^2}\rho_0 v_0\left(\vec{B}_0 \times \vec{\nabla} B\right) = \frac{1}{2B_0 R_B}\left(\frac{\gamma m v_\perp}{qB_0}\right)v_\perp\left(\hat{r} \times \vec{B}_0\right)$$

$$= \frac{\gamma m v_\perp^2}{2qB_0^2 R_B}\left(\hat{r} \times \vec{B}_0\right)$$

So the total drift velocity is

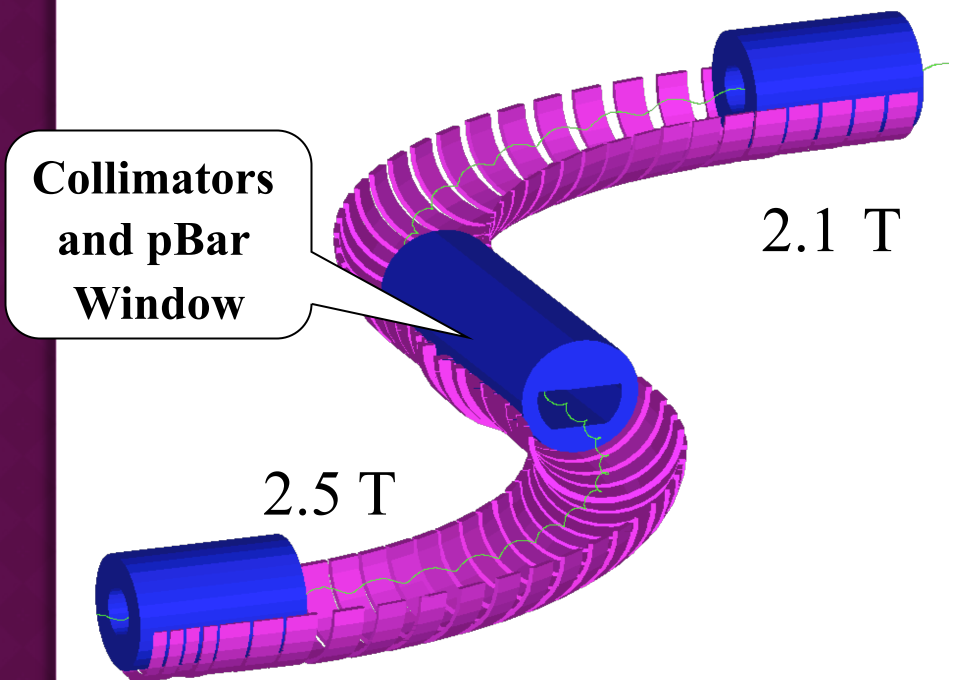
$$\boxed{\begin{aligned}\vec{v}_{tot} &= \vec{v}_d + \vec{v}_g \\ &= \frac{m}{qR_B B_0^2}\left(\hat{r} \times \vec{B}\right)\left(v_\parallel^2 + \frac{1}{2}\gamma v_\perp^2\right)\end{aligned}}$$

In most transport applications, this term will dominate

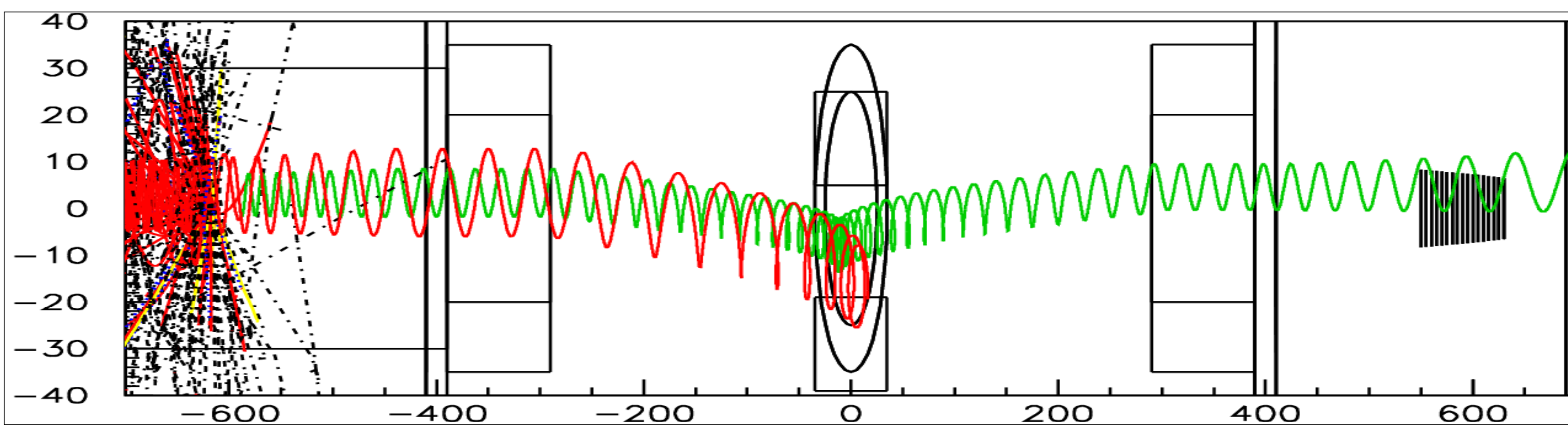
Depends on charge and direction of field, but not on direction of propagation within bend.

out of bend plane

Example: Mu2e Experiment Transport Solenoid



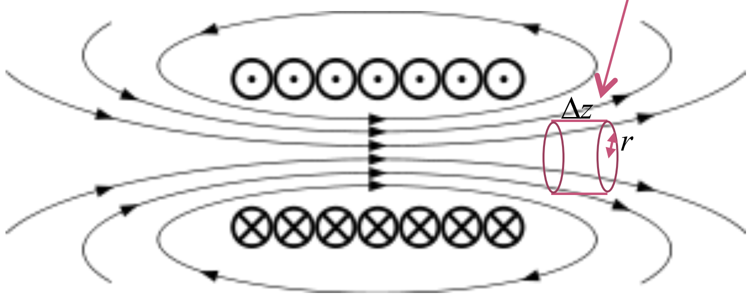
- Use curved solenoid to select negative muons with p<90 MeV/c
- Curvature drift and collimators sign and momentum select beam
- dB/ds < 0 in the straight sections to avoid trapping which would result in long transit times



Fringe field of a solenoid

- Near the ends, the field of a solenoid will have a radial component

Integration volume



$$\oint \vec{B} \bullet d\vec{A} = \pi r^2 \big(B_z(z+\Delta z) - B_z(z)\big) + 2\pi r \Delta z B_r$$

$$= \pi r^2 \Delta z B_z' + 2\pi r \Delta z B_r = 0$$

$$\Rightarrow B_r = -\frac{r}{2} B_z'$$

- For a long solenoid, this can be approximated near the end as

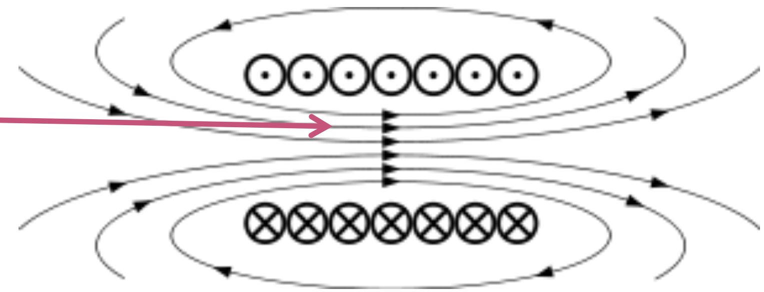
Measured from end

$$B_z \approx \frac{1}{2}\mu_0 nI\left(1-\cos\phi_2\right) = \frac{1}{2}\mu_0 nI\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

$$B_r = -\frac{r}{2}B_z' \approx \frac{r}{4}\mu_0 nI\left(\frac{1}{\sqrt{z^2+R^2}} - \frac{z}{\left(z^2+R^2\right)^{3/2}}\right)$$

Understanding solenoidal focusing

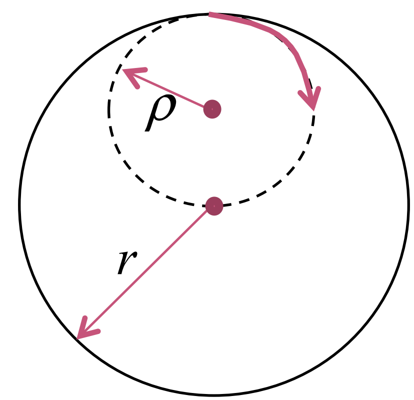
- Consider a particle coming toward a long solenoid parallel to the axis with velocity v_0
  - It will see a transverse kick (in the thin lens approximation)



$$p_\perp \approx q\int B_r dz = -q\frac{r}{2}\int B_z' dz = -q\frac{r}{2}B_0$$

  - It will begin to travel in a helix described by:

$$\rho = \frac{p_\perp}{qB_0} = \frac{r}{2}; \omega = \frac{p_\perp}{\gamma m \rho} = \frac{1}{\gamma}\Omega_s$$



  - That is, the extrema of he helix will be the radius at the point of entry and the axis of the solenoid

Focusing effect

- The radial position and velocity of the particle will be given by

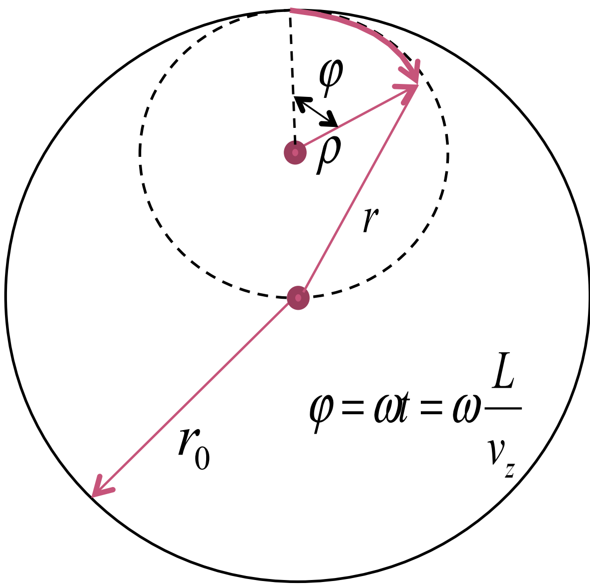
$$r^2 = 2\rho^2\left(1+\cos\phi\right) = \frac{r_0^2}{2}\left(1+\cos\phi\right) = r_0^2\cos^2\frac{\phi}{2}$$

$$\Rightarrow \quad r = r_0\cos\frac{\phi}{2}$$

$$\Rightarrow \quad v_r = \dot{r} = -\frac{r_0}{2}\omega\sin\frac{\phi}{2} = -\frac{r_0}{2}\omega\sin\frac{\omega L/v_z}{2}$$

$$\approx -\frac{r_0}{4}\omega^2\frac{L}{v_z} = \frac{r_0\Omega_s^2}{4\gamma^2 v_z}L = \frac{r_0 q^2 B_0^2}{4\gamma^2 m^2 v_z}L$$

$$\boxed{\omega\frac{L}{v_z} << \pi}$$



- This results in a focusing angle

$$\theta \approx \frac{v_r}{v_z} \equiv -r_0\frac{q^2B_0^2}{4\gamma^2 m^2 v_z^2}L \approx -r_0\frac{q^2B_0^2}{4p^2}L$$

Effective focal length and coupling

- The general form of the previous equation is

$$\theta \approx -r_0 \frac{q^2}{4p^2} \int B_0^2 dz \equiv -\frac{r_0}{f}$$

for unit charge

$$\Rightarrow \quad \frac{1}{f} = \frac{q^2}{4p^2} \int B_0^2 dz = \frac{1}{4(B\rho)^2} \int B_0^2 dz =$$

- At the exit of the solenoid, the particles will receive an opposite transverse kick, but the magnitude will be reduced by r/r_0, resulting in a coupling between the planes
- Useful in low energy beam lines
  - Eg, immediately after ion sources
- Also useful in beam lines with large emittances
  - Eg, muon beams