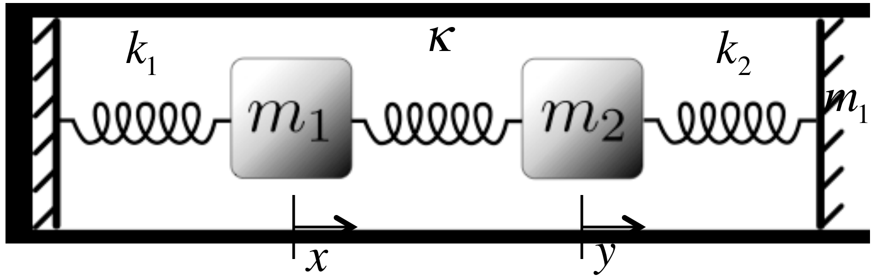
Coupled Oscillations

Coupled Harmonic Oscillators



$m_1 = m_2 = m$

Equations of motion

$$m\ddot{x} + k_1 x + \kappa(x - y) = 0$$

$$m\ddot{x} + (k_1 + \kappa)x - \kappa y = 0$$

$$m\ddot{y} + (k_2 + \kappa)y - \kappa x = 0$$

$$\ddot{x} + \omega_1^2 x - q^2 y = 0$$

$$\ddot{y} + \omega_2^2 y - q^2 x = 0$$

$$a\left(\omega_1^2 - \omega^2\right)e^{i\omega t} = bq^2 e^{i\omega t}$$

$$b\left(\omega_2^2 - \omega^2\right)e^{i\omega t} = aq^2 e^{i\omega t}$$

Define uncoupled frequencies:

$$\omega_1^2 \equiv \frac{(k_1 + \kappa)}{m};\omega_2^2 \equiv \frac{(k_2 + \kappa)}{m};q^2 \equiv \frac{\kappa}{m}$$

Try a solution of he form:

$$x = ae^{i\omega t} \Rightarrow \ddot{x} = -\omega^2 ae^{i\omega t}$$

$$y = be^{i\omega t} \Rightarrow \ddot{y} = -\omega^2 be^{i\omega t}$$

Multiply the top by the bottom:

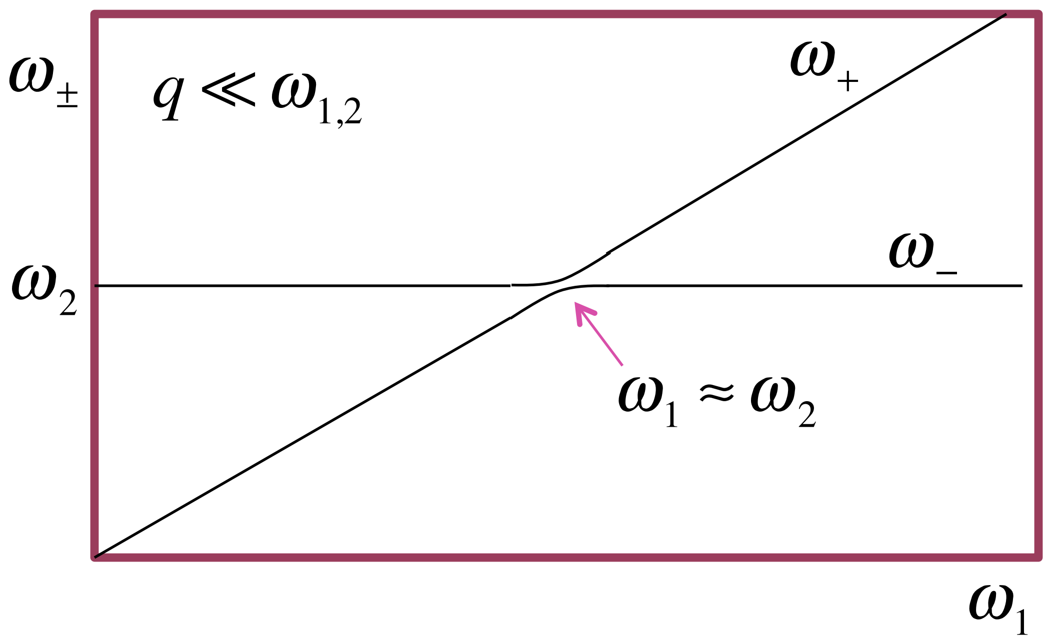
$$\left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right) = q^4$$

$$\omega^4 - \left(\omega_1^2 + \omega_2^2\right)\omega^2 + \left(\omega_1^2\omega_2^2 - q^4\right) = 0$$

$$\Rightarrow \quad \omega^2 = \frac{\left(\omega_1^2 + \omega_2^2\right) \pm \sqrt{\left(\omega_1^2 + \omega_2^2\right)^2 - 4\omega_1\omega_2 + 4q^4}}{2}$$

$$\boxed{= \frac{\left(\omega_1^2 + \omega_2^2\right) \pm \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + 4q^4}}{2}}$$

Weak Coupling



Degenerate Case:

$$\omega_1 = \omega_2 \equiv \omega_0$$

$$\omega^2 = \omega_0^2 \pm q^2 = \frac{k_0}{m} \pm \frac{\kappa}{m}$$

Resonance splitting

Formalism

General coupled equation

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow \quad \left(-\omega^2 \mathbf{I} + \mathbf{M}\right)\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

- i.e. ω^2 are the eigenvalues of **M** and (a,b) are the linear combinations of x and y which undergo simple harmonic motion.

Application to Accelerators

Introduce skew-quadrupole term

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y} \neq 0$$

$$x' \propto -\frac{\partial B_y}{\partial x} x - \frac{\partial B_y}{\partial y} y$$

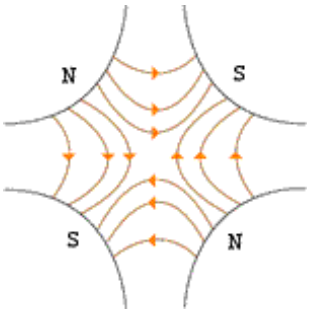
$$y' \propto \frac{\partial B_x}{\partial y} y + \frac{\partial B_x}{\partial x} x$$

Planes coupled
x and y motion *not* independent

General Transfer Matrix

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = M \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix}$$

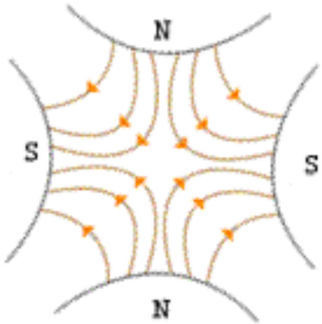
Normal Quad



$$\frac{1}{f} \equiv q = \frac{B'l}{(B\rho)}$$

$$\mathbf{M_Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & q & 1 \end{pmatrix}$$

Skew quad



$$B_x = \tilde{B}'x \to \Delta y' = \frac{\tilde{B}'l}{(B\rho)} x \equiv \tilde{q}x$$

$$B_y = -\tilde{B}'y \to \Delta x' = \frac{\tilde{B}'l}{(B\rho)} y \equiv \tilde{q}y$$

So the transfer matrix for a skew quad would be:

$$\mathbf{M}_{\tilde{\mathbf{Q}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \tilde{q} & 0 \\ 0 & 0 & 1 & 0 \\ \tilde{q} & 0 & 0 & 1 \end{pmatrix}$$

For a normal quad rotated by φ it would be (homework)

$$\mathbf{M}_{\mathbf{Q}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q\cos 2\phi & 1 & -q\sin 2\phi & 0 \\ 0 & 0 & 1 & 0 \\ -q\sin 2\phi & 0 & q\cos 2\phi & 1 \end{pmatrix}$$

Coupling in Floquet Coordinates

$$\Delta x' = \tilde{q} y$$

$$\Delta x = 0 = \sqrt{\beta_x}\Delta\xi_x$$

In Floquet Coordinate (lecture 10)

$$x' = \frac{1}{\sqrt{\beta_x \nu_x}}\left(\dot{\xi}_x - \alpha_x \nu_x \xi_x\right)$$

$$\Delta x' = \frac{\Delta\dot{\xi}_x}{\sqrt{\beta_x \nu_x}} = \tilde{q} y = \tilde{q}\sqrt{\beta_y}\xi_y$$

$$\Longrightarrow$$

$$\Delta\dot{\xi}_x = \nu_x \kappa \xi_y$$

$$\Delta\dot{\xi}_y = \nu_y \kappa \xi_x$$

$$\text{where } \kappa \equiv \tilde{q}\sqrt{\beta_x \beta_y}$$

Motion given by

$$\xi_x = r_x \cos\theta_x; \xi_y = r_y \cos\theta_y;$$

$$\dot{\xi}_x = -\nu_x r_x \sin\theta_x; \dot{\xi}_y = -r_y \nu_y \cos\theta_y;$$

$$\Longrightarrow$$

$$\Delta\dot{\xi}_x = \nu_x \kappa r_y \cos\theta_y$$

$$\Delta\dot{\xi}_y = \nu_y \kappa r_x \cos\theta_x$$

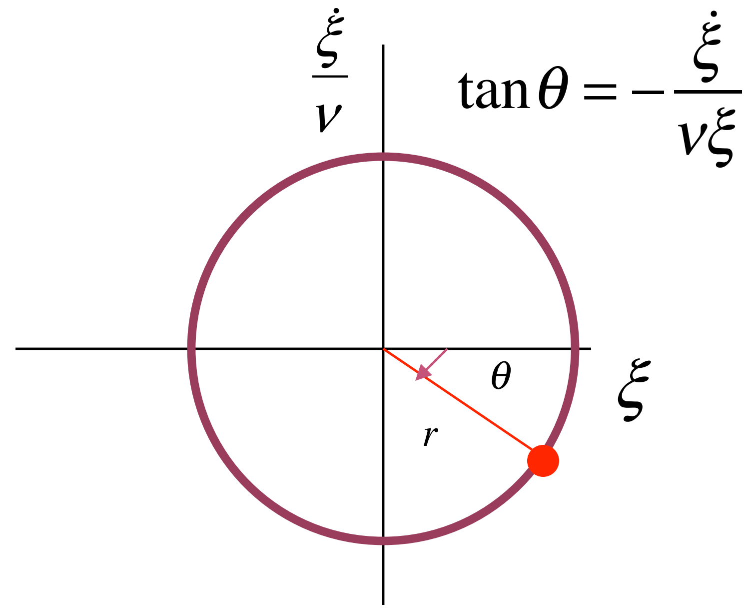
Motion in Floquet Plane

$$r^2 = \xi^2 + \left(\frac{\dot{\xi}}{\nu}\right)^2$$

$$\Delta r^2 = 2\frac{\dot{\xi}}{\nu^2}\Delta\dot{\xi} = -2\frac{r\sin\theta}{\nu}\Delta\dot{\xi}$$

$$\Rightarrow \quad \Delta r_x^2 = -2r_x r_y \kappa \sin\theta_x \cos\theta_y$$

$$\Delta r_y^2 = -2r_x r_y \kappa \sin\theta_y \cos\theta_x$$



Phase Changes

$$\theta = \tan^{-1}\left(-\frac{\dot{\xi}}{\nu\xi}\right)$$

$$\Delta\theta = \frac{1}{1+\left(\frac{\dot{\xi}}{\nu\xi}\right)^2}\left(-\frac{\Delta\dot{\xi}}{\nu\xi}\right) = -\frac{\xi}{r^2}\left(\frac{\Delta\dot{\xi}}{\nu}\right)$$

$$= -\cos\theta\left(\frac{\Delta\dot{\xi}}{\nu r}\right)$$

$$\Rightarrow \quad \Delta\theta_x = -\frac{r_y}{r_x}\kappa\cos\theta_x\cos\theta_y$$

$$\Delta\theta_y = -\frac{r_x}{r_y}\kappa\cos\theta_x\cos\theta_y$$

Evolution of Perturbation

Assume one perturbation per turn.

$$\frac{dr_x^2}{dn} = 2r_x\left(\frac{dr_x}{dn}\right) = \Delta r_x^2 = -2r_x r_y \kappa \sin\theta_x \cos\theta_y$$

$$\rightarrow \frac{dr_x}{dn} = \frac{\Delta r_x^2}{2r_x}$$

$$\boxed{\begin{aligned} \frac{dr_x}{dn} &= -r_y \kappa \sin\theta_x \cos\theta_y \\ \frac{dr_y}{dn} &= -r_x \kappa \sin\theta_y \cos\theta_x \end{aligned}}$$

Evolution of amplitude

$$\boxed{\begin{aligned} \frac{d\theta_x}{dn} &= 2\pi\nu_x + \Delta\theta_x \\ &= 2\pi\nu_x - \frac{r_y}{r_x}\kappa \cos\theta_x \cos\theta_y \\ \frac{d\theta_y}{dn} &= 2\pi\nu_y + \Delta\theta_y \\ &= 2\pi\nu_y - \frac{r_x}{r_y}\kappa \cos\theta_y \cos\theta_x \end{aligned}}$$

Evolution of phase

Angular Gymnastics

Recall

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$$

$$\boxed{\begin{aligned}
\frac{dr_x}{dn} &= -\frac{r_y}{2}\kappa\left[\sin(\theta_x+\theta_y)+\sin(\theta_x-\theta_y)\right] \\
\frac{dr_x}{dn} &= -\frac{r_x}{2}\kappa\left[\sin(\theta_x+\theta_y)-\sin(\theta_x-\theta_y)\right] \\
\frac{d\theta_x}{dn} &= 2\pi\nu_x - \frac{r_y}{2r_x}\left[\cos(\theta_x+\theta_y)+\cos(\theta_x-\theta_y)\right] \\
\frac{d\theta_y}{dn} &= 2\pi\nu_y - \frac{r_x}{2r_y}\left[\cos(\theta_x+\theta_y)+\cos(\theta_x-\theta_y)\right]
\end{aligned}}$$

Difference Resonances

Focus on case when $\nu_x \approx \nu_y$

The sum terms will oscillate quickly, so we focus in the difference terms

$$\frac{dr_x}{dn} = -\frac{r_y}{2}\kappa \sin\left(\theta_x - \theta_y\right) \qquad \frac{d\theta_x}{dn} = 2\pi\nu_x - \frac{r_y}{2r_x}\kappa \cos\left(\theta_x - \theta_y\right)$$

$$\frac{dr_y}{dn} = +\frac{r_x}{2}\kappa \sin\left(\theta_x - \theta_y\right) \qquad \frac{d\theta_y}{dn} = 2\pi\nu_y - \frac{r_x}{2r_y}\kappa \cos\left(\theta_x - \theta_y\right)$$

Note that

$$\frac{d{r_x}^2}{dn} = r_x \frac{dr_x}{dn} = -\frac{r_x r_y}{2}\kappa \sin\left(\theta_x - \theta_y\right)$$

$$\frac{d{r_y}^2}{dn} = r_y \frac{dr_y}{dn} = +\frac{r_x r_y}{2}\kappa \sin\left(\theta_x - \theta_y\right)$$

$$\Rightarrow$$

$$\begin{aligned}\frac{d{r_x}^2}{dn} + \frac{d{r_y}^2}{dn} &= \frac{d}{dn}\left(r_x^2 + r_y^2\right) \\ &= \frac{d}{dn}\left(\epsilon_x + \epsilon_y\right) \\ &= 0\end{aligned}$$

$\boxed{\text{Sum of emittances in transverse planes stays constant!}}$

Transformation of Variable

Transform into a rotating frame

$$\tilde{\theta}_x \equiv \theta_x - \pi(\nu_x + \nu_y) \equiv \theta_x - 2\pi\bar{\nu}n$$

$$\tilde{\theta}_y \equiv \theta_y - \pi(\nu_x + \nu_y) \equiv \theta_y - 2\pi\bar{\nu}n$$

In one (unperturbed) rotation

$$\Delta\tilde{\theta}_x = 2\pi\nu_x - \pi(\nu_x + \nu_y) = \pi(\nu_x - \nu_y)$$

$$\equiv \pi\delta\nu$$

$$\Delta\tilde{\theta}_y = 2\pi\nu_y - \pi(\nu_x + \nu_y)$$

$$\equiv -\pi\delta\nu$$

Equations Become

$$\frac{dr_x}{dn} = -\frac{r_y}{2}\kappa\sin(\tilde{\theta}_x - \tilde{\theta}_y)$$

$$\frac{dr_y}{dn} = +\frac{r_x}{2}\kappa\sin(\tilde{\theta}_x - \tilde{\theta}_y)$$

$$\frac{d\tilde{\theta}_x}{dn} = \pi\delta\nu - \frac{r_y}{2r_x}\kappa\cos(\tilde{\theta}_x - \tilde{\theta}_y)$$

$$\frac{d\tilde{\theta}_y}{dn} = -\pi\delta\nu - \frac{r_x}{2r_y}\kappa\cos(\tilde{\theta}_x - \tilde{\theta}_y)$$

Trial solutions

Try $$w_x \equiv r_x e^{i\tilde{\theta}_x}; \quad w_y \equiv r_y e^{i\tilde{\theta}_y}$$

$$\frac{dw_x}{dn} = \left(-\frac{r_y}{2}\kappa \sin\left(\tilde{\theta}_x - \tilde{\theta}_y\right)\right)e^{i\tilde{\theta}_x} + ir_x\left(\pi\delta\nu - \frac{r_y}{2r_x}\kappa\cos\left(\tilde{\theta}_x - \tilde{\theta}_y\right)\right)e^{i\tilde{\theta}_x}$$

$$= i\pi\delta\nu w_x - i\frac{r_y}{2}\kappa\left(\cos\left(\tilde{\theta}_y - \tilde{\theta}_x\right) + i\sin\left(\tilde{\theta}_y - \tilde{\theta}_x\right)\right)e^{i\tilde{\theta}_x}$$

$$= i\pi\delta\nu w_x - i\frac{r_y}{2}\kappa e^{i\tilde{\theta}_y}$$

$$\boxed{\begin{aligned} &= i\pi\delta\nu w_x - i\frac{\kappa}{2}w_y \\ \frac{dw_y}{dn} &= -i\pi\delta\nu w_y - i\frac{\kappa}{2}w_x \end{aligned}}$$

Hey, this (finally!) looks like simple coupled harmonic motion

Rearrange

$$\frac{dw_x}{dn} + i\pi\delta\nu\left(-w_x + \frac{\kappa}{2\pi\delta\nu}w_y\right) = 0$$

$$\frac{dw_y}{dn} + i\pi\delta\nu\left(\frac{\kappa}{2\pi\delta\nu}w_x + w_y\right) = 0$$

$$\equiv \varepsilon$$

$$\frac{d}{dn}\begin{pmatrix} w_x \\ w_y \end{pmatrix} + i\pi\delta\nu\begin{pmatrix} -1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} w_x \\ w_y \end{pmatrix}$$

$$\equiv \left(\frac{d}{dn} + i\pi\delta\nu\mathbf{M}\right)\mathbf{w} = 0$$

Apply usual coupled oscillation formalism. Define normal modes

$$\mathbf{u} = \mathbf{S}\mathbf{w};\ \text{where}$$

$$\mathbf{\Lambda} = \mathbf{SMS}^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ is diagonal}$$

$$\Rightarrow$$

$$\mathbf{S}\left(\frac{d}{dn} + i\pi\delta\nu\mathbf{MS}^{-1}\mathbf{S}\right)\mathbf{w}$$

$$= \left(\frac{d}{dn} + i\pi\delta\nu\mathbf{SMS}^{-1}\right)\mathbf{u}$$

$$= \left(\frac{d}{dn} + i\pi\delta\nu\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right)\mathbf{u}$$

Solutions

Solve for eigenvalues of M

$$\mathbf{u} = \begin{pmatrix} u_{10} e^{-i\pi\delta\nu\lambda_1 n} \\ u_{20} e^{-i\pi\delta\nu\lambda_2 n} \end{pmatrix}$$

$$\mathbf{M}\mathbf{v}_n = \lambda_n \mathbf{v}_n \rightarrow (\mathbf{M} - \lambda_n \mathbf{I})\mathbf{v}_n = 0$$

$$\rightarrow \det(\mathbf{M} - \lambda_n \mathbf{I})$$

$$= \det \begin{pmatrix} -(1+\lambda_n) & \varepsilon \\ \varepsilon & (1-\lambda_n) \end{pmatrix}$$

$$= \lambda_n^2 - 1 - \varepsilon^2 = 0$$

$$\rightarrow \lambda_n = \pm\sqrt{1+\varepsilon^2}$$

$$\mathbf{u} = \begin{pmatrix} u_{10} e^{-i\pi\delta\nu\sqrt{1+\varepsilon^2}\,n} \\ u_{20} e^{+i\pi\delta\nu\sqrt{1+\varepsilon^2}\,n} \end{pmatrix}$$

Recall $\tilde{\theta} \equiv \theta - 2\pi\bar{\nu}$

$$u(n) = u_0 e^{-i2\pi\left(\bar{\nu} \mp \frac{\delta\nu}{2}\sqrt{1+\varepsilon^2}\right)n}$$

$$\equiv u_0 e^{-i2\pi\nu_{\mp} n}$$

Coupled Tunes

$$\nu_\pm = \bar{\nu} \pm \frac{\delta\nu}{2}\sqrt{1+\varepsilon^2}$$

$$= \bar{\nu} \pm \frac{\delta\nu}{2}\sqrt{1+\frac{\kappa^2}{4\pi^2\delta\nu^2}}$$

$$= \bar{\nu} \pm \frac{1}{4\pi}\sqrt{4\pi^2\delta\nu^2+\kappa^2}$$

If there's no coupling, then

$$\nu_\pm = \bar{\nu} \pm \frac{\delta\nu}{2}$$

$$= \nu_{x,y}$$

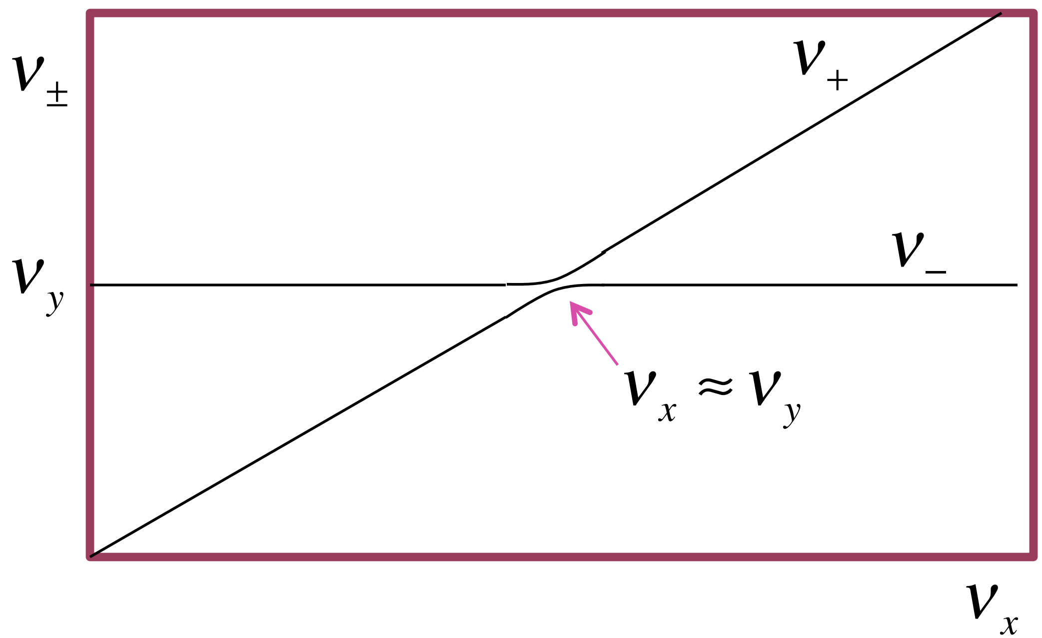
If there's coupling, then there will always be a tune split

$$\nu_x = \nu_y = \nu$$

$$\rightarrow \delta\nu = 0$$

$$\rightarrow \Delta\nu = \nu_+ - \nu_-$$

$$= \frac{\kappa}{2\pi} = \frac{\sqrt{\beta_x\beta_y}}{2\pi}\tilde{q}$$



The normal modes are given by (homework)

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \mathbf{S}\begin{pmatrix} w_x \\ w_y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} w_x \\ w_y \end{pmatrix}; \tan\alpha = \frac{1-\sqrt{1+\varepsilon^2}}{\varepsilon}$$

$$\begin{pmatrix} w_x \\ w_y \end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

$$\epsilon_x = |w_x|^2 = |\cos\alpha u_+ - \sin\alpha u_-|^2$$

$$\epsilon_y = |w_y|^2 = |\sin\alpha u_+ + \cos\alpha u_-|^2$$

$$u_\pm = u_{\pm 0} e^{\mp i\pi\delta\nu\sqrt{1+\varepsilon^2}}$$

Restrict ourselves to $u_{\pm 0}$ real

$$\epsilon_x = u_{+0}^2 \cos^2\alpha + u_{-0}^2 \sin^2\alpha - 2u_{+0}u_{-0}\cos\alpha\sin\alpha\cos\left(2\pi\delta\nu\sqrt{1+\varepsilon^2}n\right)$$

$$\epsilon_x = u_{+0}^2 \sin^2\alpha + u_{-0}^2 \cos^2\alpha + 2u_{+0}u_{-0}\cos\alpha\sin\alpha\cos\left(2\pi\delta\nu\sqrt{1+\varepsilon^2}n\right)$$

Emmittance oscillates between planes. Period = $\dfrac{1}{\delta\nu\sqrt{1+\varepsilon^2}}$ turns

Sum Resonances

Returning to the original equations, but examining the sum terms

$$\frac{dr_x}{dn} = -\frac{r_y}{2}\kappa \sin\left(\theta_x + \theta_y\right) \qquad \frac{d\theta_x}{dn} = 2\pi\nu_x - \frac{r_y}{2r_x}\kappa \cos\left(\theta_x + \theta_y\right)$$

$$\frac{dr_y}{dn} = -\frac{r_x}{2}\kappa \sin\left(\theta_x + \theta_y\right) \qquad \frac{d\theta_y}{dn} = 2\pi\nu_y - \frac{r_x}{2r_y}\kappa \cos\left(\theta_x + \theta_y\right)$$

Same sign!

Following the same math as before, we get

$$\frac{dr_x^{\ 2}}{dn} = r_x\frac{dr_x}{dn} = -\frac{r_x r_y}{2}\kappa \sin\left(\theta_x + \theta_y\right)$$

$$\frac{dr_y^{\ 2}}{dn} = r_y\frac{dr_y}{dn} = -\frac{r_x r_y}{2}\kappa \sin\left(\theta_x + \theta_y\right)$$

$$\Rightarrow \quad \frac{dr_x^{\ 2}}{dn} - \frac{dr_y^{\ 2}}{dn} = \frac{d}{dn}\left(r_x^2 - r_y^2\right) = \frac{d}{dn}\left(\epsilon_x - \epsilon_y\right) = 0$$

In other words, emittances can grow in both planes simultaneously.

We won't re-do the entire analysis for the sum resonances, but we find that the eigenmodes are

$$u(n) = u_0 e^{\pm i\pi\sqrt{(\delta\nu)^2 - \left(\frac{\kappa}{2\pi}\right)^2} n}; \quad \text{where} \quad \delta\nu \equiv m - (\nu_x + \nu_y)$$

$$\equiv u_0 e^{-i2\pi\nu_{\mp} n}$$

integer

$$\rightarrow \text{ if } \delta\nu < \frac{\kappa}{2\pi} \text{ there will be exponential growth}$$

"Stop band width". The stronger the coupling, the further away you have to keep the tune from a sum resonance.

General Case

Although we won't derive it in detail, it's clear that if motion is coupled, we can analyze the system in terms of the normal coordinates, and repeat the analysis in the last chapter. In this case, the normal tunes will be linear combinations of the tunes in the two planes, and so the general condition for resonance becomes.

$$\boxed{k_x \nu_x \pm k_y \nu_y = m \quad (k_x, k_y, m \text{ all integers})}$$

This appears as a set of crossing lines in the nx,ny "tune space". The width of individual lines depends on the details of the machine, and one tries to pick a "working point" to avoid the strongest resonances.