Imperfections (corrected)

Dipole Error (or Correction)

- Recall our generic transfer matrix

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}\left(\cos\Delta\psi + \alpha_0 \sin\Delta\psi\right) & \sqrt{\beta_0\beta_1}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta_1}}\left((\alpha_0-\alpha_1)\cos\Delta\psi - (1+\alpha_0\alpha_1)\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta_1}}\left(\cos\Delta\psi - \alpha_1\sin\Delta\psi\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

- If we use a dipole to introduce a small bend Θ at one point, it will in general propagate as

$$\begin{pmatrix} x(\Delta\psi) \\ x'(\Delta\psi) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\Delta\psi + \alpha_0 \sin\Delta\psi\right) & \sqrt{\beta_0\beta(s)}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta(s)}}\left((\alpha_0-\alpha(s))\cos\Delta\psi - (1+\alpha_0\alpha(s))\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\psi - \alpha(s)\sin\Delta\psi\right) \end{pmatrix} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

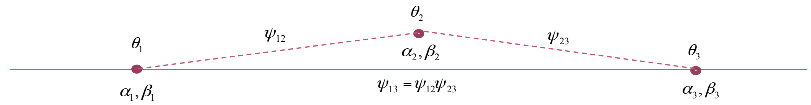
$$\boxed{x(\Delta\psi) = \theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi}$$

Remember this one forever

$$x'(\Delta\psi) = \theta\sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\psi - \alpha(s)\sin\Delta\psi\right)$$

Example: Local Correction ("Three Bump")

- Consider a particle going down a beam line. By using a combination of three magnets, we can localize the beam motion to one area of the line



- We require

$$x_3 = \theta_1\sqrt{\beta_1\beta_3}\sin\psi_{13} + \theta_2\sqrt{\beta_2\beta_3}\sin\psi_{23} = 0$$

$$\boxed{\Rightarrow \theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}}$$

$$\theta_3 = -\left(\theta_1\sqrt{\frac{\beta_1}{\beta_3}}(\cos\psi_{13} - \alpha_3\sin\psi_{13}) + \theta_2\sqrt{\frac{\beta_2}{\beta_3}}(\cos\psi_{23} - \alpha_3\sin\psi_{23})\right)$$

$$= -\theta_1\left(\sqrt{\frac{\beta_1}{\beta_3}}(\cos\psi_{13} - \alpha_3\sin\psi_{13}) - \sqrt{\frac{\beta_1}{\beta_2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}\sqrt{\frac{\beta_2}{\beta_3}}(\cos\psi_{23} - \alpha_3\sin\psi_{23})\right)$$

$$= -\theta_1\sqrt{\frac{\beta_1}{\beta_3}}\left(\cos\psi_{13} - \frac{\sin\psi_{13}}{\sin\psi_{23}}\cos\psi_{23}\right) = -\theta\sqrt{\frac{\beta_1}{\beta_3}}\left(\frac{\sin\psi_{23}\cos\psi_{13} - \cos\psi_{23}\sin\psi_{13}}{\sin\psi_{23}}\right) = -\theta_1\sqrt{\frac{\beta_1}{\beta_3}}\left(\frac{\sin(\psi_{23} - \psi_{13})}{\sin\psi_{23}}\right)$$

$$\boxed{\Rightarrow \theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\left(\frac{\sin\psi_{12}}{\sin\psi_{23}}\right)}$$

Local Bumps are an *extremely* powerful tool in beam tuning!!

Controls Example

- From Fermilab "Acnet" control system
  - The B:xxxx labels indicate individual trim magnet power supplies in the Fermilab Booster
  - Defining a "MULT: *N*" will group the *N* following magnet power supplies
  - Placing the mouse over them and turning a knob on the control panel will increment the individual currents according to the ratios shown in green

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 ! INJECTION POSITION
 MULT     :6
-B:VL5T  [5]*2.45 473 f(t) values        4.933          Amps
-B:VL6T  [5]*1  6 473 f(t) values        2.117          Amps
-B:VL7T  [5]*2.47 473 f(t) values        2.058          Amps
-B:VL5T  *2.4 VL5 473 f(t) values        4.933          Amps
-B:VL6T  *1   VL6 473 f(t) values        2.117          Amps
-B:VL7T  *2.4 VL7 473 f(t) values        2.058          Amps
 MULT     :3
-B:VL5T  [1]*2.45 473 f(t) values        5.717          Amps
-B:VL6T  [1]*1  6 473 f(t) values        3.566          Amps
-B:VL7T  [1]*2.47 473 f(t) values        2.561          Amps
 MULT     :3
-B:VL5T  [2]*2.45 473 f(t) values        5.642          Amps
-B:VL6T  [2]*1  6 473 f(t) values         .427          Amps
-B:VL7T  [2]*2.47 473 f(t) values         .718          Amps
 MULT     :3
-B:VL5T  [3]*2.45 473 f(t) values        20.65          Amps
-B:VL6T  [3]*1  6 473 f(t) values        3.389          Amps
-B:VL7T  [3]*2.47 473 f(t) values        9.95           Amps
 MULT     :3
-B:VL5T  [4]*2.45 473 f(t) values        15.21          Amps
-B:VL6T  [4]*1  6 473 f(t) values        6.348          Amps
-B:VL7T  [4]*2.47 473 f(t) values        16.35          Amps
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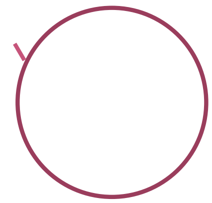
Closed Orbit Distortion ("cusp")



- We place a dipole at one point in a ring which bends the beam by an amount Θ.
- The new equilibrium orbit will be defined by a trajectory which goes once around the ring, through the dipole, and then returns to its exact initial conditions. That is

$$\mathbf{M}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \Rightarrow (\mathbf{I}-\mathbf{M})\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (\mathbf{I}-\mathbf{M})^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

- Recall that we can express the transfer matrix for a complete revolution as

$$\mathbf{M}(s+C,s) = \begin{pmatrix} \cos 2\pi\nu + \alpha(s)\sin 2\pi\nu & \beta(s)\sin 2\pi\nu \\ -\gamma(s)\sin 2\pi\nu & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu \end{pmatrix} = \mathbf{I}\cos 2\pi\nu + \mathbf{J}\sin 2\pi\nu = e^{\mathbf{J}2\pi\nu}$$

$$(\mathbf{I}-\mathbf{M}) = e^{\mathbf{J}\pi\nu}\left(e^{-\mathbf{J}\pi\nu} - e^{\mathbf{J}\pi\nu}\right) = -e^{\mathbf{J}\pi\nu}(2\sin\pi\nu\mathbf{J})$$

$$(\mathbf{I}-\mathbf{M})^{-1} = (-2\sin\pi\nu\mathbf{J})^{-1}\left(e^{\mathbf{J}\pi\nu}\right)^{-1}$$

$$= \frac{1}{2\sin\pi\nu}\mathbf{J}e^{-\mathbf{J}\pi\nu} = \frac{1}{2\sin\pi\nu}\mathbf{J}(\mathbf{I}\cos\pi\nu - \mathbf{J}\sin\pi\nu)$$

$$= \frac{1}{2\sin\pi\nu}(\mathbf{J}\cos\pi\nu + \mathbf{I}\sin\pi\nu)$$

$$= \frac{1}{2\sin\pi\nu}\begin{pmatrix} \alpha\cos\pi\nu + \sin\pi\nu & \beta\cos\pi\nu \\ -\gamma\cos\pi\nu & -\alpha\cos\pi\nu + \sin\pi\nu \end{pmatrix}$$

$$\mathbf{J} \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\mathbf{J}^2 = -\mathbf{I}$$

$$\mathbf{J}^{-1} = -\mathbf{J}$$

- Plug this back in $\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{1}{2\sin\pi\nu}\begin{pmatrix} \alpha\cos\pi\nu + \sin\pi\nu & \beta\cos\pi\nu \\ -\gamma\cos\pi\nu & -\alpha\cos\pi\nu + \sin\pi\nu \end{pmatrix}\begin{pmatrix} 0 \\ \theta \end{pmatrix}$

$$= \frac{\theta}{2\sin\pi\nu}\begin{pmatrix} \beta_0\cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

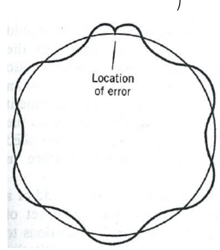
- We now propagate this around the ring

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \frac{\theta}{2\sin\pi\nu}\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta(s)}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta(s)}}\left((\alpha_0 - \alpha(s))\cos\Delta\psi - (1+\alpha_0\alpha(s))\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta(s)}}(\cos\Delta\psi - \alpha(s)\sin\Delta\psi) \end{pmatrix}\begin{pmatrix} \beta_0\cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

$$\Rightarrow x(s) = \frac{\theta}{2\sin\pi\nu}\left(\sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\psi + \alpha_0\sin\Delta\psi)\beta_0\cos\pi\nu + \sqrt{\beta_0\beta(s)}\sin\Delta\psi(\sin\pi\nu - \alpha_0\cos\pi\nu)\right)$$

$$= \frac{\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}(\cos\Delta\psi\cos\pi\nu + \sin\Delta\psi\cos\pi\nu)$$

$$= \frac{\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}\cos(\Delta\psi - \pi\nu)$$



Quadrupole Errors

- We can express the matrix for a complete revolution at a point as

$$\mathbf{M}(s) = \begin{pmatrix} \cos 2\pi\nu + \alpha(s)\sin 2\pi\nu & \beta(s)\sin 2\pi\nu \\ -\gamma(s)\sin 2\pi\nu & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu \end{pmatrix}$$

- If we add focusing quad at this point, we have

$$\mathbf{M}'(s) = \begin{pmatrix} \cos 2\pi\nu_0 + \alpha(s)\sin 2\pi\nu_0 & \beta(s)\sin 2\pi\nu_0 \\ -\gamma(s)\sin 2\pi\nu_0 & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\pi\nu_0 + \alpha(s)\sin 2\pi\nu_0 - \frac{\beta(s)}{f}\sin 2\pi\nu_0 & \beta(s)\sin 2\pi\nu_0 \\ -\gamma(s)\sin 2\pi\nu_0 - \frac{1}{f}\left(\cos 2\pi\nu_0 - \alpha(s)\sin 2\pi\nu_0\right) & \cos 2\pi\nu_0 - \alpha(s)\sin 2\pi\nu_0 \end{pmatrix}$$

- We calculate the trace to find the new tune

$$\cos 2\pi\nu = \frac{1}{2}\mathbf{M}'(s) = \cos 2\pi\nu_0 - \frac{1}{2f}\beta(s)\sin 2\pi\nu_0$$

- For small errors

$$\cos 2\pi\left(\nu_0 + \Delta\nu\right) \approx \cos 2\pi\nu_0 - 2\pi\sin 2\pi\nu_0\Delta\nu = \cos 2\pi\nu_0 - \frac{1}{2f}\beta(s)\sin 2\pi\nu_0$$

$$\Rightarrow \Delta\nu = \frac{1}{4\pi}\frac{\beta(s)}{f}$$

Total Tune Shift

- The focal length associated with a local anomalous gradient is

$$d\left(\frac{1}{f}\right) = \frac{B'}{(B\rho)}ds$$

- So the total tune shift is

$$\Delta\nu = \frac{1}{4\pi}\oint \beta(s)\frac{B'(s)}{(B\rho)}ds$$