Evolution of the Distribution Function

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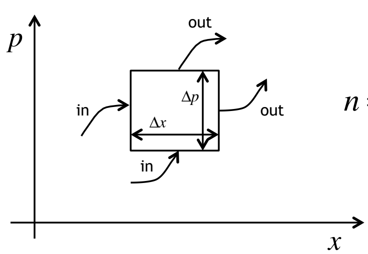
Our simple model can only go so far. Now we have to develop the tools to help us deal with real particle distributions in phase space.

Phase density:



$$n = \psi(x,p,t)\Delta x \Delta p$$

$$n(t+\Delta t) = n(t) + \text{(flow in) - (flow out)}$$
$$\text{(flow in)} = \psi(x,p,t)(\Delta p \dot{x} \Delta t + \Delta x \dot{p} \Delta t)$$
$$\text{(flow out)} = \psi(x+\Delta x, p+\Delta p, t)(\Delta p \dot{x} \Delta t + \Delta x \dot{p} \Delta t)$$

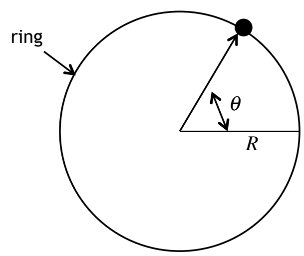
$$
\begin{aligned}
\frac{n(t+\Delta t)-n(t)}{\Delta t} &= \frac{\left(\psi(x,p,t+\Delta t)-\psi(x,p,t)\right)\Delta x\Delta p}{\Delta t} \\
&= \psi(x,p,t)\Delta p\dot{x}+\psi(x,p,t)\Delta x\dot{p} \\
&\quad -\psi(x+\Delta x,p,t)\Delta p\dot{x}-\psi(x,p+\Delta p,t)\Delta x\dot{p} \\
&= -\left(\dot{x}\Delta p\frac{\partial\psi}{\partial x}\Delta x+\dot{p}\Delta x\frac{\partial\psi}{\partial p}\Delta p\right) \\
&= -\left(\dot{x}\frac{\partial\psi}{\partial x}+\dot{p}\frac{\partial\psi}{\partial p}\right)\Delta x\Delta p
\end{aligned}
$$

$$\longrightarrow \boxed{\frac{\partial\psi}{\partial t}+\dot{x}\frac{\partial\psi}{\partial x}+\dot{p}\frac{\partial\psi}{\partial p}=0}$$

Vlasov Equation

Dispersion Relation

Apply a special case of the Vlasov Equation



$$\frac{\partial\psi}{\partial t}+\dot{x}\frac{\partial\psi}{\partial x}+\dot{p}\frac{\partial\psi}{\partial p}=0$$

$$
\begin{array}{ll}
x=R\theta & p=p_0(1+\delta) \\
\dot{x}=R\dot{\theta} & \dot{p}=p_0\dot{\delta} \\
\partial x=R\,\partial\theta & \partial p=p_0\,\partial\delta
\end{array}
$$

$$
\begin{aligned}
\theta &= \frac{S}{R} \\
\dot{\theta} &= \omega \\
\delta &= \frac{\delta p}{p} \\
\dot{\delta} &= \frac{\dot{p}}{p_0}=\frac{1}{\beta^2}\frac{\Delta\dot{E}}{E}
\end{aligned}
$$

$$\longrightarrow \frac{\partial\psi}{\partial t}+\dot{\theta}\frac{\partial\psi}{\partial\theta}+\dot{\delta}\frac{\partial\psi}{\partial\delta}=0$$

Recall that in our discussion of the Distribution Function, we found that if the current is described by

$$I = I_0 + I_1 e^{i(\Omega t - n\theta)}$$

frequency of oscillation

mode

then

$$\frac{dE}{dt} = \text{(energy lost per turn)(turns/sec)}$$

$$= -eI_1 Z_{\parallel} e^{i(\Omega t - n\theta)} \left(\frac{\omega_0}{2\pi}\right)$$

$$\longrightarrow \dot{\delta} = \frac{1}{\beta^2}\frac{\Delta \dot{E}}{E}$$

$$= -\frac{e\omega_0 I_1 Z_{\parallel}}{2\pi\beta^2 E} e^{i(\Omega t - n\theta)}$$

Just as we did with the current, we will define the density to have a constant part and an oscillatory part.

$$\psi(\delta,\theta,t) = \psi_0(\delta) + \psi_1(\delta) e^{i(\Omega t - n\theta)}$$

Plug this into the Vlasov equation, and we get

$$\frac{\partial \psi}{\partial t} = i\Omega \psi_1 e^{i(\Omega t - n\theta)}$$

$$\dot{\theta}\frac{\partial \psi}{\partial \theta} = -i\omega n \psi_1 e^{i(\Omega t - n\theta)}$$

$$\dot{\delta}\frac{\partial \psi}{\partial \delta} \approx -\frac{e\omega_0 I_1 Z_{\parallel}}{2\pi\beta^2 E} e^{i(\Omega t - n\theta)} \frac{\partial \psi_0}{\partial \delta}$$

$$\longrightarrow \quad i(\Omega - n\omega)\psi_1 - \frac{\partial \psi_0}{\partial \delta}\frac{e\omega_0 I_1 Z_{\parallel}}{2\pi\beta^2 E} = 0$$

Convert $\partial\delta$ to $\partial\omega$

$$\omega T = 2\pi \rightarrow \frac{d\omega}{\omega} = -\frac{dT}{T} = -\delta\eta$$

$$\longrightarrow \frac{\partial}{\partial \delta} \approx -\omega_0 \eta \frac{\partial}{\partial \omega}$$

Combining, we get

$$i(\Omega - n\omega)\psi_1 + \frac{\partial\psi_0}{\partial\omega}\frac{e\omega_0^2 I_1 Z_\parallel \eta}{2\pi\beta^2 E} = 0$$

$$\psi_1 = i\frac{e\omega_0^2 I_1 Z_\parallel \eta}{2\pi\beta^2 E}\frac{1}{(\Omega - n\omega)}\frac{\partial\psi_0}{\partial\omega}$$

Integrating the LHS, we have

$$\int_{-\infty}^{\infty}\psi_1(\omega)d\omega = -\omega_0\eta\int_{-\infty}^{\infty}\psi_1(\delta)d\delta = -\eta\int_{-\infty}^{\infty}\psi_1(\delta)\omega_0\, d\delta$$
$$= -\frac{\eta I_1}{e}$$

Integrating the RHS and equating, we get

dispersion relation

$$\boxed{1 = -i\frac{e^2\omega_0^2 Z_\parallel}{2\pi\beta^2 E}\int_{-\infty}^{\infty}\frac{\left(\frac{\partial\psi_0}{\partial\omega}\right)}{(\Omega - n\omega)}d\omega}$$

Evolution of the Distribution Function

Application to the negative mass instability
Unbunched beam with δ=0

yeah, I know. Deal with it

$$\rightarrow \psi_0(\delta,\theta,t) = \frac{N}{2\pi}\delta(\delta)$$

Write this in terms of ω, we have

$$\psi_0(\omega,\theta) = -\frac{N\eta\omega_0}{2\pi}\delta(\omega - \omega_0)$$

$$\longrightarrow \int\frac{(\partial\psi_0/\partial\omega)}{(\Omega - n\omega)}d\omega = -\frac{N\eta\omega_0}{2\pi}\int\frac{\delta'(\omega-\omega_0)}{(\Omega - n\omega)}d\omega$$
$$= \frac{N\eta\omega_0}{2\pi}\frac{n}{(\Omega - n\omega)^2}$$

recall $\int f(x)\delta'(x-x_0)dx = f'(x_0)$

Dispersion relation gives

$$1 = -i\frac{e^2\omega_0^2 Z_\parallel}{2\pi\beta^2 E}\int_{-\infty}^{\infty}\frac{\left(\frac{\partial\psi_0}{\partial\omega}\right)}{(\Omega - n\omega)}d\omega = -i\frac{e\omega_0^2 Z_\parallel}{2\pi\beta^2 E}\frac{\eta}{2\pi}\frac{n}{(\Omega - n\omega)^2}\left(\frac{eN\omega_0}{2\pi}\right) = -i\frac{e\omega_0^2 Z_\parallel \eta n}{2\pi\beta^2 E}\frac{I_0}{(\Omega - n\omega}$$

$$\boxed{(\Omega - n\omega)^2 = -i\frac{e\eta I_0 Z_\parallel \omega_0^2 n}{2\pi\beta^2 E}}$$ as before!

Evolution of the Distribution Function

Now consider a more realistic beam with a momentum spread

$$\psi(\delta,\theta) = \frac{N}{2\pi}\frac{1}{\sqrt{2\pi\sigma_\delta^2}}e^{-\delta^2/2\sigma_\delta^2}$$

$$= \frac{N}{(2\pi)^{3/2}\sigma}e^{-\delta^2/2\sigma^2}$$

Drop subscript

in terms of angular frequency

$$\psi_0\omega = \frac{N}{2\pi}\frac{1}{\sqrt{2\pi\sigma_\delta^2}}e^{-\delta^2/2\sigma_\delta^2}$$

$$= \frac{N}{(2\pi)^{3/2}\sigma}e^{-(\omega-\omega)^2/2(\eta\omega_0\sigma)^2}$$

The dispersion integral becomes

$$\int \frac{\partial\psi_0}{\partial\omega}\frac{1}{\Omega-n\omega}d\omega = -\frac{N}{(2\pi)^{3/2}\sigma}\frac{1}{(\eta\omega_0\sigma)^2}\int\frac{(\omega-\omega_0)}{(\Omega-n\omega)}e^{-(\omega-\omega)^2/2(\eta\omega_0\sigma)^2}d\omega$$

$$= \frac{N}{(2\pi)^{3/2}\eta\omega_0\sigma^2 n}\int\frac{u}{u-u_0}e^{-u^2/2}du$$

where $u \equiv \frac{\omega-\omega_0}{\eta\omega_0\sigma}$, $u_0 \equiv \frac{\Omega-n\omega_0}{\eta\omega_0\sigma}$

So the dispersion relation becomes

$$1 = -i\frac{e^2\omega_0^2 Z_\parallel}{2\pi\beta^2 E}\frac{N}{(2\pi)^{3/2}\eta\omega_0\sigma^2 n}\int\frac{u}{u-u_0}e^{-u^2/2}du$$

$$\Big\Downarrow \; \frac{e\omega_0 N}{2\pi} = I_0$$

$$= -i\frac{eI_0 Z_\parallel}{2\pi\beta^2 E\eta\sigma^2 n}\overbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{u}{u-u_0}e^{-u^2/2}du\right]}^{\equiv I_D(u_0)}$$

recall

$$\psi = \psi_0 + \psi_1 e^{i(\Omega t - n\theta)}$$

$$= \psi_0 + \psi_1 e^{i((\Delta\Omega + n\omega_0)t - n\theta)}$$

$$= \psi_0 + \psi_1 e^{i(\Delta\Omega t + n(\omega_0\theta))}$$

If $\Delta\Omega$ has a negative imaginary part, then motion will be unstable.

Use trick $\frac{1}{u-u_0} = -i\int_0^\infty e^{i(u-u_0)\alpha}\,d\alpha$

$$I_D = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{u}{u-u_0}e^{-u^2/2}\,du$$

$$= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ue^{-u^2/2}\left(\int_0^\infty e^{i(u-u_0)\alpha}\,d\alpha\right)du$$

$$= \frac{-i}{\sqrt{2\pi}}\int_0^\infty e^{iu_0\alpha}\left(\int_{-\infty}^{\infty} ue^{-\left(u^2-2iu\alpha-\alpha^2\right)/2}e^{-\alpha^2/2}\,du\right)d\alpha$$

$$= \frac{-i}{\sqrt{2\pi}}\int_0^\infty e^{iu_0\alpha}e^{-\alpha^2/2}\left(\underbrace{\int_{-\infty}^{\infty} ue^{-(u-i\alpha)^2/2}\,du}_{=\sqrt{2\pi}i\alpha}\right)d\alpha$$

$$= \int_0^\infty \alpha e^{-iu_0\alpha}e^{-\alpha^2/2}$$

recall $u_0 = \frac{\Delta\Omega}{\eta\omega_0\sigma\eta}$

If $u_0 = 0$, then $I_D = 1$. If $\mathrm{Im}(u_0) < 0$ (ie unstable), then $e^{-iu_0\alpha} = e^{-i\mathrm{Re}\{u_0\}\alpha} \times e^{+\mathrm{Im}\{u_0\}\alpha}$ will decay and $I_D(u_0) < 1$

From the dispersion relation.

$$1 = -i\frac{eI_0 Z_\parallel}{2\pi\beta^2 E\eta\sigma^2 n} I_D(u_0)$$

The unstable solution *($I_D(u_0)<1$)* can only exist if

$$\left|\frac{eI_0 Z_\parallel}{2\pi\beta^2 E\eta\sigma^2 n}\right| > 0$$

So motion will be stable if

$$\sigma^2 > \frac{eI_0}{2\pi\beta^2 E\eta}\left|\frac{Z_\parallel}{n}\right|$$

More generally, motion will be stable if

$$\left|\frac{Z_\parallel}{n}\right| < \mathcal{F}\frac{2\pi\beta^2 E\eta\sigma^2}{eI_0}$$

"Keil-Schnell criterion"

Form factor which depends on details of distribution