

Thermal Modeling of Wire-coil Insert

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1 Introduction

Many high-heat load components in Advanced Photon Source(APS) utilize forced convection heat transfer in order to reduce the temperature on beam interacting surfaces. The wire-coil inserts are routinely used in APS front end and beamline and high-heat-load components to significantly enhance convection heat transfer, up to 400% compared to open passages [1].

Wire-coil inserts are usually made of thermally conducting materials, such as copper, and have a spring-like geometry as shown in Fig. 1. The geometry of the inserts can be described by three parameters: the wire diameter (e), the cooling passage diameter (D), and the wire-coil pitch (p).

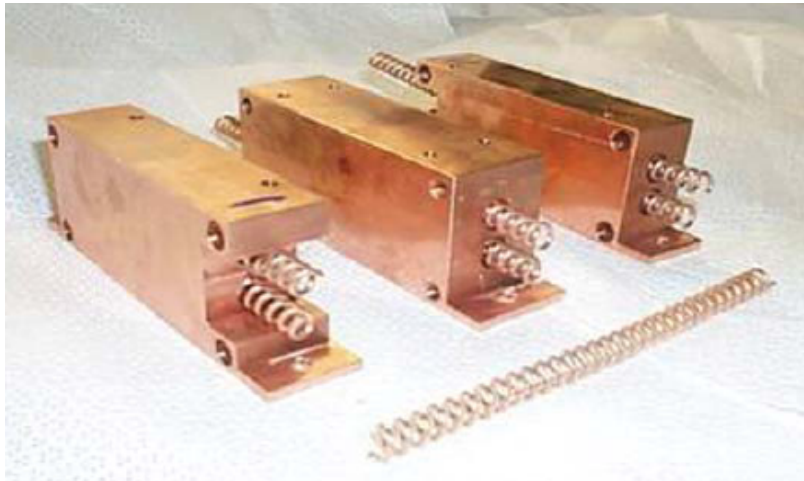


Figure 1: Example of wire coil inert[1]

Previous experiments [2] tested the heat transfer performance and pressure loss of wire coil inserts in cooling pipes over a range of geometric parameters. The results showed that wire-coil inserts can enhance the heat transfer performance at reasonably low flow rates. The wire-coil inserts showed reduced pressure losses compared to previously utilized wire-mesh inserts while reducing concerns for water clogging and water quality issues. The results also showed that pitch length can be optimized in order to lower the flow rate required for a desired heat transfer coefficient(as

shown in Fig. 2). However, the physical mechanisms behind these results are still unclear, making it difficult to generalize the results to pipes with arbitrary size and thermal properties.

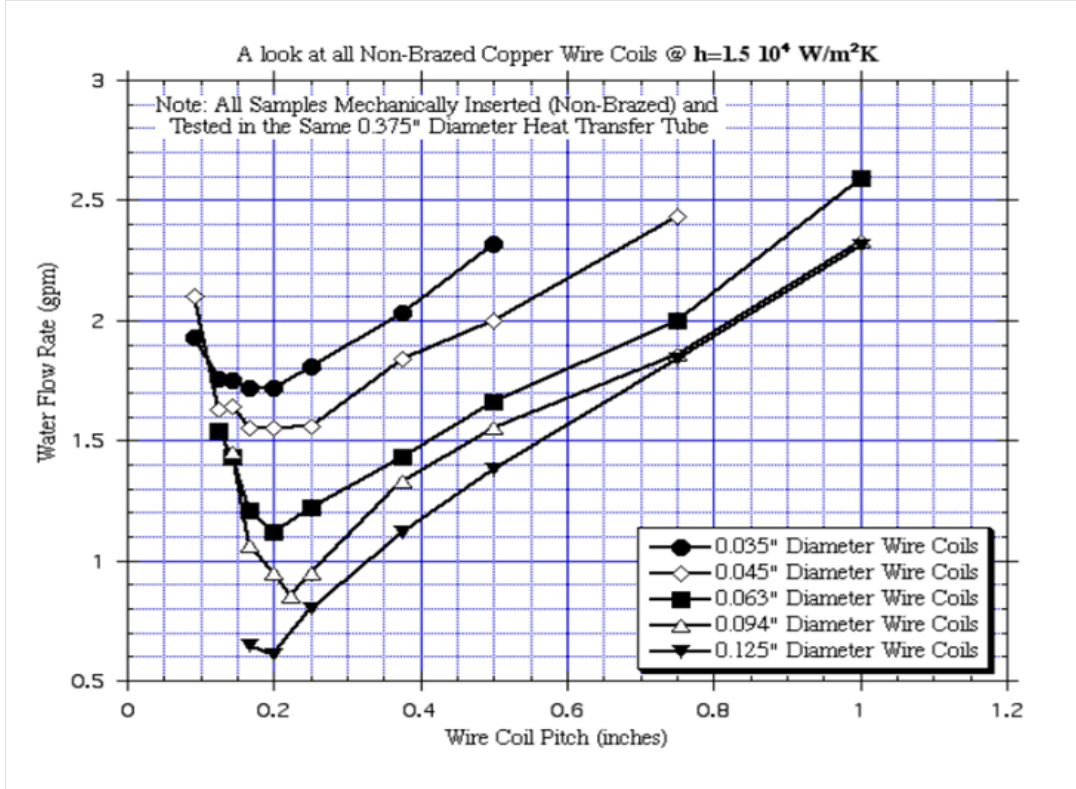


Figure 2: Experimental result for flow rate needed to reach a fixed heat transfer performance in relation to pitch length [2]

In this project, our goal is to develop a computational simulation for the heat transfer performance of wire-coil inserts and validate the model with the existing experimental data. This project will give indications for the design of coil-insert cooling systems in APS as well as the generalization of this heat transfer enhancement in other applications.

In the mathematical model, we consider a 3D problem where a copper wire-coil insert with uniform thermal properties is inserted into a cylindrical pipe of the same material. The cooling fluid is water, as is used in APS cooling systems, and is modeled as an incompressible, Newtonian fluid. The flow is assumed turbulent as this is a requirement in APS cooling systems. We assume perfect thermal contact at all contact points in the system. This includes contact between the pipe wall and fluid, the pipe wall and wire-coil insert, and the coil insert and water. Finally, we assume a uniform prescribed heat flux at outer pipe wall.

Two different internal contact cases are considered: one being the wire-coil having a finite contact with the pipe wall, the other being the wire-coil insert isolated from the pipe wall by a thin gap. The solution of wire-coil having a point contact with the pipe should be bounded by the two solutions. Since in APS applications, the wire-coils are not soldered onto the pipes, but only mechanically inserted. Thus we think the second case is closer to reality.

2 Nondimensionalization

To set up the coordinate system for this problem, we let the end circular cross section be on the x - y plane, and axial direction be z -direction.

To model system of fluid, pipe, and wire-coil insert, we consider incompressible Navier-Stokes equations, which solves for the velocity field; and convection-diffusion equation, which solves for temperature field

The convection-diffusion equation is

$$(\rho c_p)_j \left(\frac{d\tilde{T}}{dt} + \tilde{\mathbf{u}} \cdot \nabla \tilde{T} \right) = k_j \nabla^2 \tilde{T} + \tilde{Q}_j \quad (1)$$

where $j = f$ (fluid) or s (solid). \tilde{T} is temperature, $\tilde{\mathbf{u}}$ is the fluid velocity, ρ is density, c_p is the specific heat, k is the thermal conductivity, and \tilde{Q} is prescribed heat flux on system.

We nondimensionalize fluid part of (1) by firstly scaling length over the diameter of pipe (D), where the nondimensional lengths are:

$$x = \frac{\tilde{x}}{D}, \quad y = \frac{\tilde{y}}{D}, \quad \text{and} \quad z = \frac{\tilde{z}}{D}.$$

Then we define nondimensional time: $t = \tilde{t} \cdot U/D$; nondimensional heat flux: $Q = \tilde{Q}D/(U\rho_f \cdot c_{pf})$; and nondimensional velocity: $u = \tilde{u}/U$, where U is the average velocity of fluid on axial direction.

Using these nondimensional variables, equation for fluid part of (1) can be expressed in the nondimensional form:

$$\frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T + \frac{1}{\text{Pe}} \nabla^2 T + Q_f, \quad (2)$$

where $\text{Pe} = \alpha_f/(U \cdot D)$ is the Peclet number, $\alpha_f = k_f/(\rho_f \cdot c_{pf})$ is thermal diffusivity of fluid. With the same scaling, the nondimensional convection-diffusion equation for the solid part is

$$\frac{\alpha_f}{\alpha_s} \frac{dT}{dt} = \frac{1}{\text{Pe}} \nabla^2 T + Q_s. \quad (3)$$

3 Numerical Method

In order to solve the incompressible Navier-Stokes equations coupled with nondimensional convection-diffusion equation (2), (3), we use an open source 3D fluid dynamics solver, Nek5000, which considers both of the equations.

We generate the turbulent initial condition of fluid profile by applying periodic boundary conditions, that is, using the profile at the exit of the pipe in the previous iteration as the initial condition profile at the entrance of the pipe for the current iteration. Also, the effect of the wire coil insert on the fluid velocity field is incorporated into the model as perturbation.

Before we solve the problem of wire-coil insert, we solve a few related problems, in most of which computational solutions can be validated with analytical solutions, so that we have confidence in our final computational model that is heavily based on these problems. In order to reach an effective comparison, we use the Nusselt number as the comparison quantity, which represents the heat transfer efficiency, and is computed using

$$\text{Nu} = \frac{q''}{T_{\text{wall}} - T_{\text{bulk}}}, \quad (4)$$

where q'' is the heat flux per unit area on the inner wall, and T_{wall} is average wall temperature and T_{bulk} is average fluid temperature.

3.1 2D channel flow problem

We first tested the computational result of a 2D problem with its analytical solution. We first consider a 2D channel flow problem with laminar flow profile and prescribed heat flux on inside pipe wall. We begin with finding the Nusselt number analytically.

We start with (2), the non-dimensional convection diffusion equation in the fluid:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{Pe} \nabla^2 T,$$

with boundary conditions

$$\begin{aligned} T(x+L, y, t) &= T(x, y, t) + \gamma L \text{ (periodicity in } x), \\ \frac{1}{Pe} \frac{\partial T}{\partial y} \Big|_{y=0} &= 0 \text{ (symmetry at } y=0), \\ \frac{1}{Pe} \frac{\partial T}{\partial y} \Big|_{y=1} &= q'' \text{ (prescribed heat flux at } y=1). \end{aligned}$$

For our chosen non-dimensionalization, width of the channel, $H = 1$, and prescribed heat flux per unit area, $q'' = 1$.

The constant γ arises from the balance between the energy entering via the surface flux, $E_{in} = Lq''$, and net energy removed by convection

$$E_{out} = \int_{x=L} uT dy - \int_{x=0} uT dy = \gamma HL.$$

Equating these fluxes we find

$$\gamma = \frac{q''L}{UHL} = 1,$$

where $U = 1$ is the average velocity in the channel.

To simplify numerical solution, we write $T(\mathbf{x}, t) = \theta(\mathbf{x}, t) + \gamma x$, where θ satisfies

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \frac{1}{Pe} \nabla^2 \theta - \gamma u \quad (5)$$

subject to periodic boundary conditions in x , $\theta(x, y, t) = \theta(x+L, y, t)$.

The heat transfer coefficient, h , is defined by the relationship

$$q'' = h(T_w - T_b) = h(\theta_w - \theta_b),$$

where θ_w is the temperature at the wall ($y = 1$) and θ_b is the bulk (mixing-cup) temperature

$$\theta_b := \frac{\int_{\Omega} u \theta dV}{\int_{\Omega} u dV},$$

with u the x -component of the convecting field \mathbf{u} . In the general case, one must consider temporal and/or spatial averages of θ_w and θ_b , but the above definitions suffice for the particular case considered here. For a fluid with conductivity k , the Nusselt number is

$$Nu := \frac{Hh}{k} = \frac{Pe}{\theta_w - \theta_b}.$$

As an initial test, we take prescribed steady parallel flows of the form $\mathbf{u} = (u, v)$ with $v = 0$ and

$$u = \frac{m+1}{m} (1 - y^m)$$

for integer m . The case $m = 2$ is standard plane Poiseuille flow, while $m = \infty$ corresponds to uniform flow, $u \equiv 1$. The temperature $\theta(y)$ satisfies

$$\frac{1}{Pe} \frac{d^2 \theta}{dy^2} = \frac{m+1}{m} (1 - y^m), \quad \theta'(0) = 0, \quad \theta'(1) = Pe.$$

Integrating twice and applying the boundary conditions yields

$$\theta = Pe \cdot y^2 \left(\frac{m+1}{2m} - \frac{y^m}{m(m+2)} \right).$$

To compute Nu , we need θ_w and θ_b .

$$\theta_w = Pe \cdot \theta(1) = \left(\frac{m+1}{2m} - \frac{1}{m(m+2)} \right).$$

and

$$\begin{aligned} \theta_b &= Pe \frac{\int \theta u dy}{\int u dy} = Pe \int_0^1 \theta u dy \\ &= Pe \frac{m+1}{m} \int_0^1 (1 - y^m) y^2 \left(\frac{m+1}{2m} - \frac{y^m}{m(m+2)} \right) dy \\ &= Pe \frac{m+1}{m} \left[\frac{m+1}{6m} - \frac{1}{m+3} \left(\frac{1}{m(m+2)} + \frac{m+1}{2m} \right) + \frac{1}{m(m+2)(2m+3)} \right]. \end{aligned}$$

Finally, we have

$$Nu = \frac{1}{\theta_w - \theta_b}. \tag{6}$$

For $m=1, 2, 3$, and ∞ , we find respective Nusselt numbers, $Nu=1.8750, 2.0588, 2.1892$, and 3 .

The Nusselt number for parabolic flow profile calculated by Nek5000 code is 2.0694698.

3.2 Simulating 3D conjugate heat transfer with laminar flow, and deriving Nusselt number

Since the wire-coil insert problem involves 3D conjugate heat transfer in a cylindrical pipe, we firstly considered a 3D conjugate heat transfer problem in a plain tube with prescribed heat flux on outer wall.

Since the Nusselt number in (4) is defined in terms of temperature and heat flux of inner wall, and the inner wall of the wire-coil insert has a complicated geometry and is difficult to define, we need to express Nusselt number in terms of temperature and heat flux on the outer wall of the tube.

We start by expressing the total energy input into the system per unit time:

$$Q = \pi q_o'' D_o L = \frac{2\pi k_s L (T_o - T_i)}{\ln(D_o/D_i)} [4], \quad (7)$$

where subscript "o" denotes parameters for the outside pipe wall, and subscript "i" denotes those for the inner pipe wall. Q is heat input into the system by heat flux on outer wall, L is length of the pipe.

So we can express inner wall temperature of pa

$$T_i = T_o - \frac{Q_{in} \ln(D_o/D_i)}{2\pi k_s L} = T_o - q_o'' \frac{D_o \ln(D_o/D_i)}{2k_p}. \quad (8)$$

By definition of Nusselt number,

$$\text{Nu} = \frac{D_i h}{k_f},$$

in which h is heat transfer coefficient, defined as:

$$h = \frac{q_i''}{T_i - T_b} = \frac{D_o}{D_i} \frac{q_o''}{T_i - T_b},$$

where T_b is bulk (mixing cup) temperature of fluid.

Then Nusselt number is

$$\text{Nu} = \frac{D_o \cdot q_o''}{k_f (T_i - T_b)}. \quad (9)$$

When we use (8) in (9) to solve for Nusselt number, since the denominator of the result will become complicated, we work with the inverse Nusselt number:

$$\text{Nu}^{-1} = \frac{k_f}{D_o q_o''} (T_o - T_b) - \frac{k_f}{2k_p} \ln\left(\frac{D_o}{D_i}\right). \quad (10)$$

In this expression of inverse Nusselt number, we can see that the first term represents the inverse Nusselt number as if it is defined regarding the outer wall temperature and heat flux; and the second term serves as a correction.

Now we can also solve for the parameter of linear axial temperature dependence at steady state, γ . By conservation of energy, the energy put into the system by the heat on pipe wall equals to the heat brought out of the system by the fluid, which is,

$$\pi q_o'' L D_o = \gamma L \rho c_p \int_V u_z dV, \quad (11)$$

where u_z is axial component of fluid velocity.

Then we can solve for γ from (11):

$$\gamma = \frac{q_o'' A_o}{\rho c_p \int_V u_z dV}, \quad (12)$$

where A_o is surface area of outside pipe wall.

The analytical solution [3] for Nusselt number is

$$\text{Nu} = 4.36, \text{ and } \gamma = 4.$$

And the simulation result produced by Nek5000 yields $\text{Nu} = 4.36$, and $\gamma = 4$. Thus our simulation result agrees with analysis.

And for analytic derivation of temperature profile at steady state, see Appendix A

4 Building mesh for wire-coil insert geometry

Examining the geometry of the wire coil insert, we find that all cross sections along the horizontal direction are identical shapes rotated at different angles. Thus, we generate the 3D geometry by firstly constructing the horizontal cross section of wire coil insert.

We denote the outer radius of the coil to be R , and diameter of wire to be e , and distance from coil center to center of wire to be $R_w = R - e/2$.

The wire coil insert can be constructed by curving a tilted straight wire around a cylinder.

The cross section of a straight wire is the intersection of a cylinder with the horizontal plane, which is an ellipse with minor axis as the diameter of wire, e , and major axis a , as shown in Fig. 3.

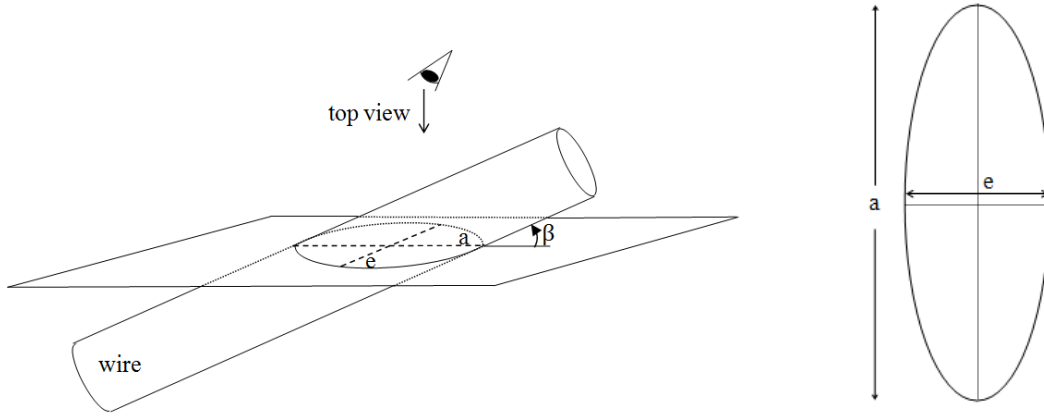


Figure 3: Left: a tilted straight wire intersecting a horizontal plane. Right: top view of elliptical cross section.

We can express length of the major axis, a , in terms of tilted angle, β and wire diameter, e :

$$a = e / \sin \beta . \quad (13)$$

To find β , we focus on one pitch of wire coil insert, and if straighten the wire, we can find the geometrical relation as shown in Fig. 4:

$$\tan \beta = \frac{p}{2\pi R_w} . \quad (14)$$

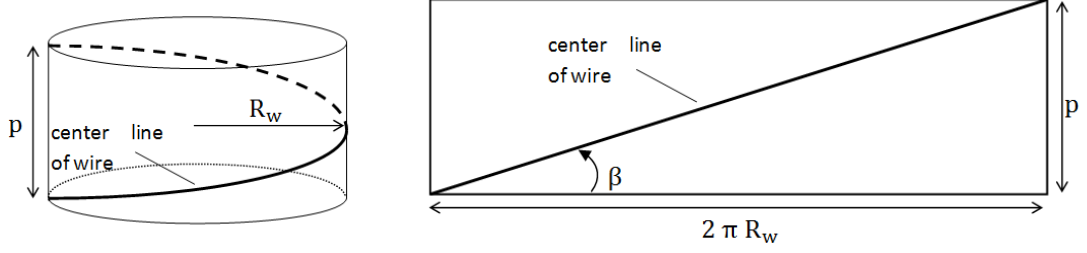


Figure 4: Left: one pitch of wire coil insert. Right: lateral surface of the wire coil insert.

where $R_w = R - e/2$, is radius of wire coil at wire center, and p is pitch of coil.

With (13) and (14), we solve for the major axis of wire cross section:

$$a = \frac{e}{p} \sqrt{p^2 + 4\pi^2 R_w^2} .$$

After obtaining the dimensions of the wire cross section, we need to find a linear mapping to map the elliptical cross section into a curved ellipse, to model the curved wire that is constrained inside a cylinder with radius R .

The mapping should satisfy that points $(R, 0)$ and $(R - e, 0)$ are mapped to themselves. For a point with coordinate (x, y) , we define the following quantity:

$$\Omega = \frac{y}{R} , \text{ and } r = x .$$

Then coordinate after mapping, (x', y') is:

$$x' = r \cos \Omega , y' = r \sin \Omega .$$

Fig. 5 left shows the transformation and the ellipse after mapping represents the cross section of the wire coil insert.

Then by duplicating the the curved ellipse and rotating by a certain angle for each layer, we generate a 3D mesh for the wire coil insert shown in Fig. 5 right.

5 Results

We simulated the problem with plain tube and compared with both Dittus-Boelter correlation and experimental data (Fig. 6). The simulation result is in satisfactory agreement with both experimental data and Dittus-Boelter correlation.

Also, we completed two simulations for wire coil insert with wire diameter to pitch ratio, $e/p = 0.438375$ (is shown in Fig. 6). The e/p ratio is calculated based on an empirical relation [1] to be the optimal wire diameter to pitch length ratio :

$$e/p_{\text{optimum}} = 0.207 - 0.181(e/D) + 4.426(e/D)^2 .$$

Two snap shots of temperature solutions are shown in Fig. 7. The left picture shows temperature solution early in time, and the right picture shows temperature solution later in time. The colors from blue to red represent nondimensionalized temperature from low to high. From Fig. 7, we can

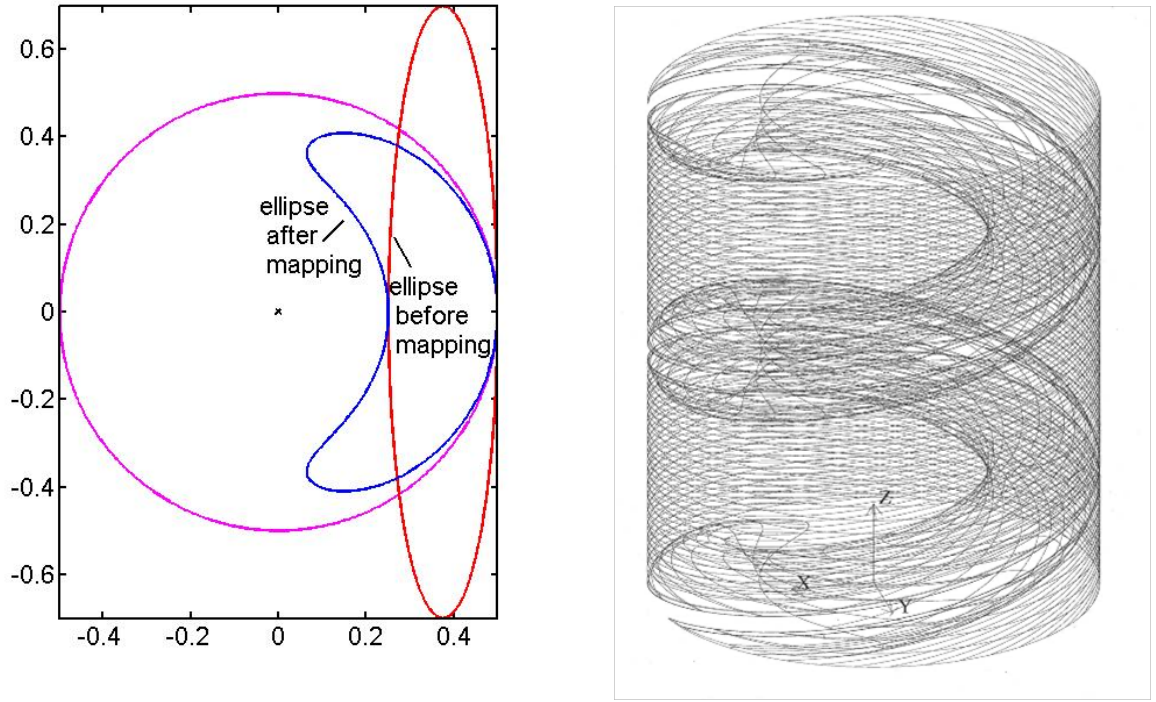


Figure 5: Left: Elliptical cross section before and after mapping. With $R = 0.5$, $e = 0.2$, and $p = 0.57$. Right: 3D mesh for wire wire coil insert

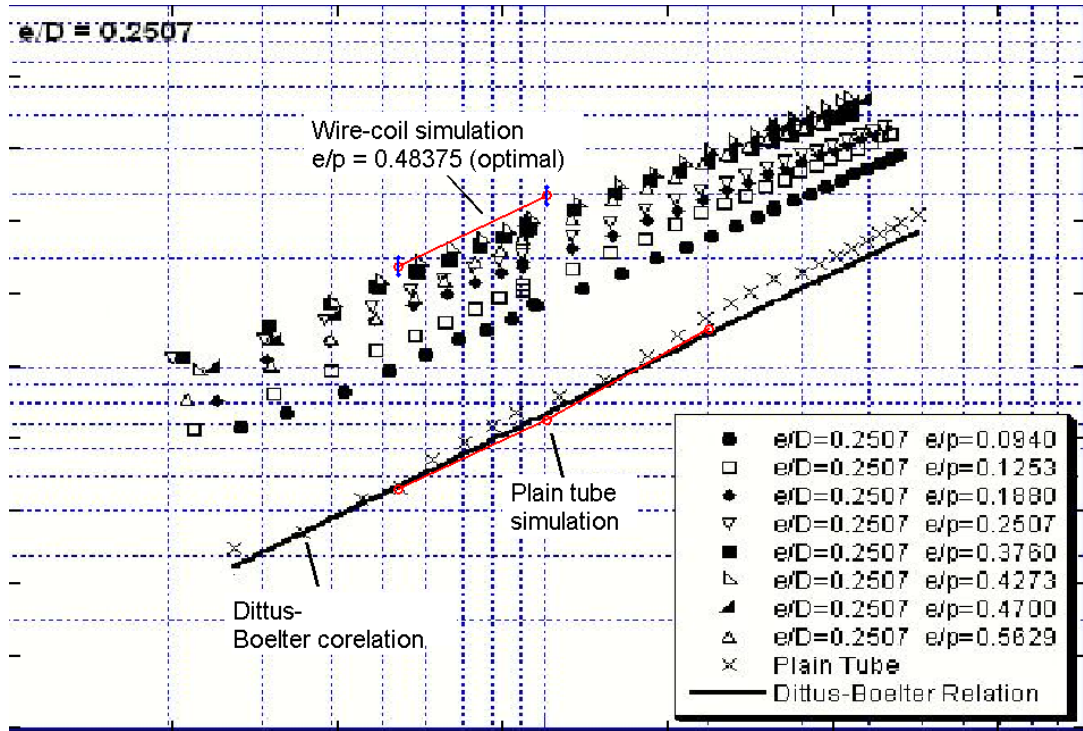


Figure 6: Simulation result (with thin gaps between wire-coil and pipe) compared with experimental result and Dittus-Boelter correlation

start understanding the reason of there existing an optimal pitch. If the pitch length is too short, there would be little vortex in between two levels of coils; if the pitch length is too long, there would

be little string effect in the tube. Thus there has to be an optimal pitch length in between these two extremes.

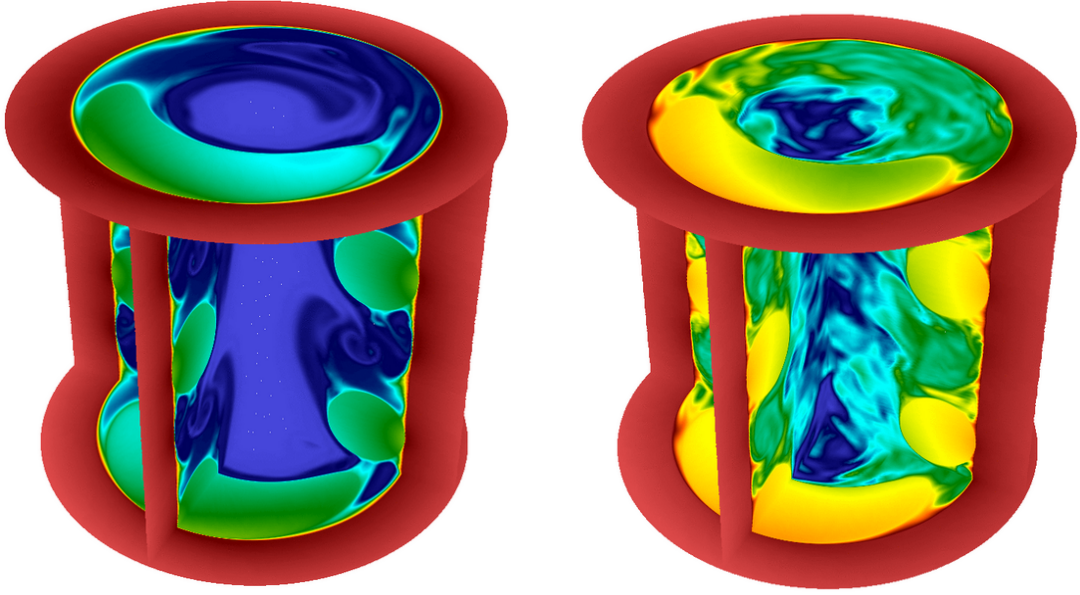


Figure 7: Temperature solution of $Re = 5300$ (thin gaps between wire-coil insert and pipe wall). Left: early in time, Right: later in time.

6 Discussion

To ensure stability, our numerical routine has requirements on time stepping size, such that the Courant number, $C < 1$. In the numerical simulation, the plain tube cases each has 4080 elements, run with polynomial order 8 and then increased to 10. We let the case run until steady state, which is about 10^5 iterations. Fig. 8 is an example of convergence for Nusselt number over time (for $Re = 20K$).

In our simulation result of wire-coil insert, the although Nusselt converges when system is close to steady state, Nusselt number still has a fluctuation of 5% after run time of more than 100 hours. An example of fluctuation for $Re = 10K$ case is shown in Fig. 9

Also, because the numerical solver we use only applies to incompressible fluid, our simulation would not apply for the cases where compressible gas is used.

Because numerical simulations cannot include a single contact point between wire-coil insert cross section and the pipe, we consider two cases: one is a finite length of contact between the wire-coil insert cross section and the pipe; the other is a thin gap between wire-coil insert and the pipe. And we think the solution for the single point contact case should be bounded by the two simulations. In real life application of wire-coil insert in APS, there is no soldering between the wire-coil insert and the inner pipe wall, so there can be thermal resistance between wire-coil insert and the pipe. Furthermore, mechanical wear can also cause gaps between wire-coil insert and pipe wall. So we think the case that considers a thin gap between wire-coil insert and pipe wall is closer to reality than considering a finite contact between the wire-coil insert and the pipe wall.

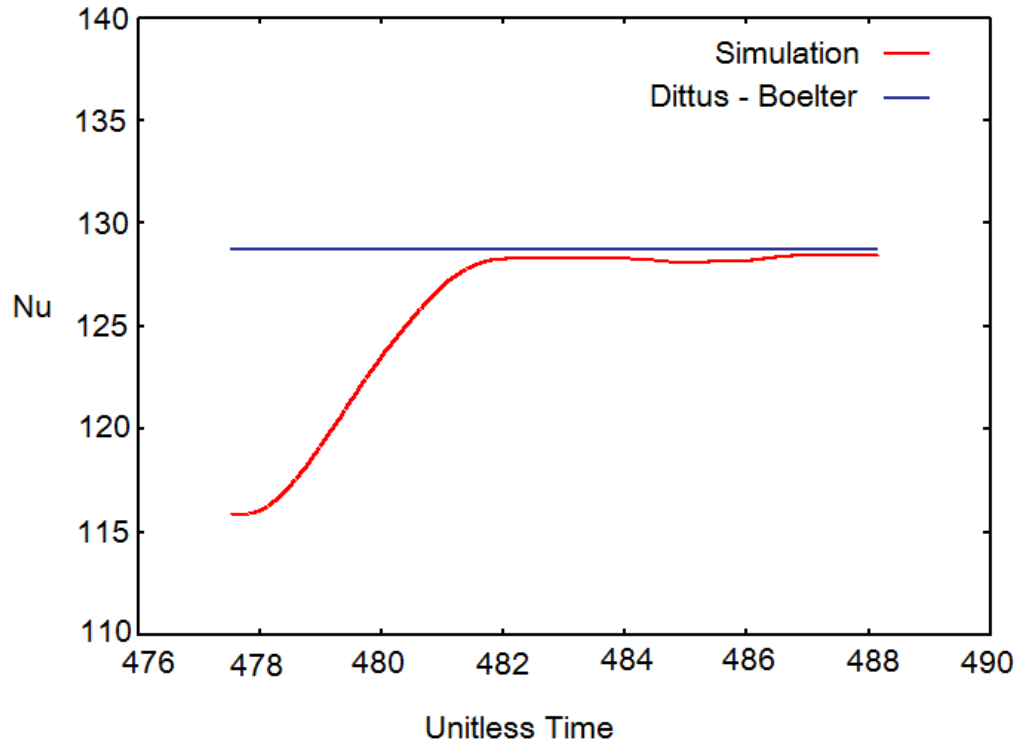


Figure 8: Convergence curve of Nusselt number of simulation, $Re = 20K$

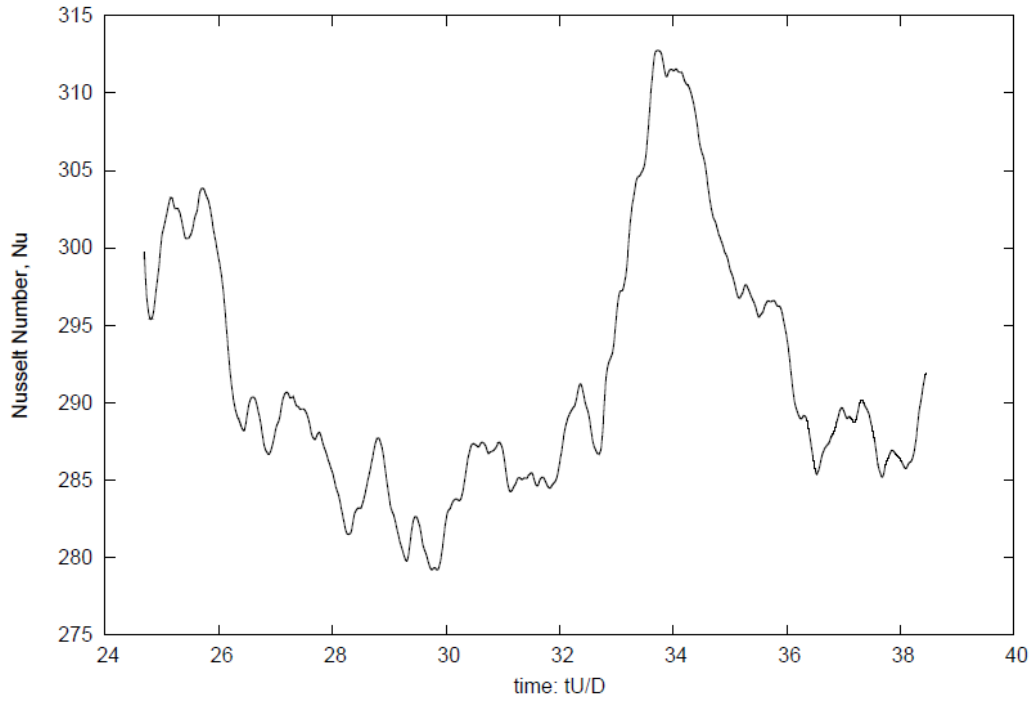


Figure 9: Nusselt number fluctuates for about 5% at $Re = 10K$ after long run time

6.1 Future work

The next step in this project will be simulating for the wire-coil inserts with the exact parameters as used in experiments, and validate with experimental data.

Apart from further validation of our model, we also need to use our model to study the physical mechanisms behind the heat transfer enhancement. The heat transfer enhancement of wire-coil insert can be caused by two reasons: one being the increased contact area between pipe wall and fluid, the other being the stirring effect caused by the helix geometry. Future research can use our model to study which mechanism plays a larger role in heat transfer enhancement. For example, if setting the thermal property of wire-coil insert to be insulating, we can focus on solely the string effect of the helix geometry, and compare with our current result to see if it is the dominate factor.

7 Conclusion

In this project, we developed a model to simulate heat transfer performance of wire-coil insert, and the result of simulation is in good agreement with both experimental data and Dittus-Boelter correlation. This model can be used to further study the physical mechanism on the heat transfer enhancement of wire-coil insert, and can benefit designs of cooling channels in general.

References

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A 3D conjugate heat transfer problem with laminar flow and prescribed heat flux on outer pipe wall

In this problem, we consider a Poiseuille flow within a straight smooth cylindrical pipe with inner radius R_i , and outer radius R_o subject to uniform prescribed heat flux q''_o on the outer pipe wall. This section finds the steady state temperature solution analytically.

We start with non-dimensional convection-diffusion equation in the fluid

$$\frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T + \frac{1}{\text{Pe}} \nabla^2 T \quad (15)$$

with inhomogeneous Neumann condition

$$-\frac{1}{\text{Pe}} \nabla T = q''_i. \quad (16)$$

Here, T denotes temperature, t is time and $\mathbf{u} = (0, 0, u_z)$ denotes velocity of the fluid in the pipe. The Peclet number is $Pe = \bar{u}D/\alpha$, where \bar{u} is the average velocity in the pipe, D is the pipe diameter, α is the thermal diffusivity of the fluid, and $q''_i = q''_o(R_o/R_i)$, is heat flux on the inner pipe wall,

In cylindrical coordinates with radial component \hat{r} , angular component $\hat{\phi}$, and axial component \hat{z} , the gradient operators are:

$$\nabla T = \hat{r} \frac{\partial T}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial T}{\partial \phi} + \hat{z} \frac{\partial T}{\partial z}; \quad (17)$$

$$\text{and } \nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \quad (18)$$

We consider the steady state laminar case under fully-developed flow conditions, for which

$$\frac{\partial T}{\partial t} = 0, \text{ and } \frac{\partial T}{\partial \phi} = 0.$$

The fully-developed flow conditions imply $u_z = u_z(r)$, which, when coupled with the Neumann condition on temperature leads to a linear growth in temperature, which we express as

$$T_{(r,z)} = \theta_{(r)} + \gamma z, \quad (19)$$

where θ is radial dependent part of temperature, and γ is a constant.

So

$$\frac{\partial T}{\partial z} = \gamma, \quad \frac{\partial^2 T}{\partial z^2} = 0, \text{ and } \frac{\partial T}{\partial r} = \frac{d\theta}{dr}.$$

So terms in the right-hand side of (15)

$$\mathbf{u} \cdot \nabla T = \text{Pe} \cdot u_z \cdot \gamma, \quad (20)$$

$$\text{and } \nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{d\theta}{dr} \right). \quad (21)$$

Using (20) and (21) in (15), we get

$$\text{Pe} \cdot u_z \cdot \gamma = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{d\theta}{dr} \right).$$

Since in this problem, we consider a Poiseuille flow in a smooth cylindrical pipe, the fluid velocity on axial direction in relation to radial position is parabolic, and in order for the volumetric velocity to be 1, we let the velocity profile be

$$u_{z(r)} = 2(1 - 4r^2) .$$

Then we have

$$\frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) = 2\text{Pe} \cdot \gamma \cdot r(1 - 4r^2) .$$

Integrate both sides, we get

$$r \frac{d\theta}{dr} = \text{Pe} \cdot \gamma (r^2 - 2r^4) + A , \quad (22)$$

where A is integration constant. For $r = 0$, both sides of (22) should become zero, so $A = 0$.

Dividing both sides of (22) by r and integrate again , we get the general solution for θ :

$$\theta_{(r)} = \text{Pe} \cdot \gamma \left(\frac{r^2}{2} - \frac{r^4}{2} \right) + C , \quad (23)$$

where C is an integration constant.

Then we can use the boundary condition of prescribed heat flux to determine γ . By Fourier's law of conduction, we have

$$k \frac{\partial T}{\partial r} \Big|_{r=R_i} = q_i'' = q_o'' \frac{R_o}{R_i} ,$$

where k is thermal conductivity of fluid, q'' is heat flux per unit area on the inside pipe wall, and R is radius of pipe.

From Eq.(23), we have

$$\frac{\partial T}{\partial r} = \frac{\partial \theta}{\partial r} = \text{Pe} \cdot \gamma (r - 2r^3) .$$

Using in boundary condition, we find

$$\gamma = \frac{q_o'' R_o}{k R_i \cdot \text{Pe} (R_i - 2R_i^3)} .$$

From definition of Peclet number,

$$\text{Pe} = \frac{2R_i \cdot \bar{u}}{\alpha} = \frac{2R_i \cdot \rho \cdot cp}{k}$$

where \bar{u} is volumetric flow rate of water, $\alpha = k/(\rho \cdot cp)$, is diffusivity of fluid, ρ is density of fluid and cp is specific heat of fluid. In our non-dimensional problem, $q_i'' = 1$, $\rho \cdot cp = 1$, and $R_i = 0.5$, we find $\gamma = 4$. And solution for temperature is

$$T_{(r,z)} = \frac{1}{2} \gamma (r^2 - r^4) + \gamma z + C .$$