R_1 Derivation

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1 Ratios

$$R_1 = \frac{P_B \cdot k_f}{P_B \cdot k_i} \tag{1}$$

$$P_B = \left(\frac{M_B^2 + z_N^2 q_T^2}{\sqrt{2} z_N Q}, \frac{z_N Q}{\sqrt{2}}, -z_N q_T \right) \tag{2}$$

$$k_i = \left(\frac{Q}{\hat{x_N}\sqrt{2}}, \frac{\hat{x_N}(k_i^2 + k_{iT}^2)}{\sqrt{2} Q}, k_{iT} \right) \tag{3}$$

$$k_f = \left(\frac{k_{fT}^2 + k_f^2}{\sqrt{2}\hat{z_N} Q}, \frac{\hat{z_N} Q}{\sqrt{2}}, k_{fT} \right) \tag{4}$$

$$P_B \cdot k_f = P_B^+ k_f^- + P_B^- k_f^+ - P_{BT} k_{fT} \tag{5}$$

$$P_B \cdot k_f = \left(\frac{M_B^2 + z_N^2 q_T^2}{\sqrt{2} z_N Q} \right) \left(\frac{\hat{z_N} Q}{\sqrt{2}} \right) + \left(\frac{z_N Q}{\sqrt{2}} \right) \left(\frac{k_{fT}^2 + k_f^2}{\sqrt{2}\hat{z_N} Q} \right) - (P_{BT}) \cdot k_{fT} \tag{6}$$

$$P_B \cdot k_f = (M_B^2 + z_N^2 q_T^2) \frac{\hat{z_N}}{2 z_N} + \frac{z_N}{2\hat{z_N}} (k_{fT}^2 + k_f^2) - P_{BT} \cdot k_{fT} \tag{7}$$

$$k_{fT} = \big((\delta_{kT} cos\phi - \hat{z_N} q_T, \delta_{kT} sin\phi\big) \tag{8}$$

$$P_{BT} = (-z_N qT, 0) \tag{9}$$

$$P_{BT} \cdot k_{fT} = -z_N q_T(\delta_{kT} \cos\phi - \hat{z_N} q_T) \tag{10}$$

$$k_{fT}^2 = k_{fT} \cdot k_{fT} = (\delta_{kT} \cos\phi - \hat{z_N} q_T)^2 + \delta_{kT}^2 \sin^2\phi \tag{11}$$

$$k_{fT}^2 = \delta_{kT}^2 \cos^2\phi - 2\delta_{kT} \cos\phi \hat{z_N} q_T + \hat{z_N}^2 q_T^2 + \delta_{kT}^2 \sin^2\phi \tag{12}$$

$$k_{fT}^2 = \delta_{kT}^2 - 2\delta_{kT} \cos\phi \hat{z_N} q_T + \hat{z_N}^2 q_T^2 \tag{13}$$

Plugging in for k_{fT}

$$P_B \cdot k_f = (M_B^2 + z_N^2 q_T^2)\frac{\hat{z_N}}{2z_N} + \frac{z_N}{2\hat{z_N}}(\hat{z_N}^2 q_T^2 + \delta_{kT}^2 - 2\hat{z_N} q_T \delta_{kT} \cos\phi + k_f^2) + z_N q_T(\delta_{kT} \cos\phi - \hat{z_N} q_T) \tag{14}$$

$$P_B \cdot k_f = (M_B^2 + z_N^2 q_T^2)\frac{\hat{z_N}}{2z_N} + \frac{z_N \hat{z_N} q_T^2}{2} + \frac{z_N}{2\hat{z_N}}(\delta_{kT}^2 + k_f^2) - z_N q_T \delta_{kT} \cos\phi - z_N q_T \hat{z_N} q_T + z_N q_T \delta_{kT} \cos\phi \tag{15}$$

Note: $\hat{z_N} = \frac{z_N}{\zeta}$

$$P_B \cdot k_f = (M_B^2 + z_N^2 q_T^2)\frac{1}{2\zeta} + \frac{z_N^2 q_T^2}{2\zeta} + \frac{\zeta}{2}(\delta_{kT}^2 + k_f^2) - \frac{z_N^2 q_T^2}{\zeta} \tag{16}$$

$$P_B \cdot k_f = (M_B^2 + z_N^2 q_T^2 + z_N^2 q_T^2 - 2z_N^2 q_T^2)\frac{1}{2\zeta} + + \frac{\zeta}{2}(\delta_{kT}^2 + k_f^2) \tag{17}$$

$$P_B \cdot k_f = \frac{M_B^2}{2\zeta} + + \frac{\zeta}{2}(\delta_{kT}^2 + k_f^2) \tag{18}$$

$$\boxed{P_B \cdot k_f = \frac{1}{2\zeta}\left(M_B^2 + \zeta^2(\delta_{kT}^2 + k_f^2)\right)} \tag{19}$$

$$P_B \cdot k_i = P_B^+ k_i^- + P_B^- k_i^+ - P_{BT} k_{iT} \tag{20}$$

$$P_{BT} \cdot k_{iT} = (-z_N q_T, 0) \cdot (k_{iT} \cos\phi_i, k_{iT} \sin\phi_i) = -z_N q_T k_{iT} \cos\phi_i \tag{21}$$

$$P_B \cdot k_i = \left(\frac{M_B^2 + z_N^2 q_T^2}{\sqrt{2} z_N Q}\right)\left(\frac{\hat{x_N}(k_i^2 + k_{iT}^2)}{\sqrt{2}Q}\right) + \left(\frac{z_N Q}{\sqrt{2}}\right)\left(\frac{Q}{\hat{x_N}\sqrt{2}}\right) - (-z_N q_T) k_{iT} \cos\phi_i \tag{22}$$

Note: $\hat{x_N} = \frac{x_N}{\xi}$

$$P_B \cdot k_i = (M_B^2 + z_N^2 q_T^2)(k_i^2 + k_{iT}^2)\frac{x_N}{2 z_N Q^2 \xi} + \frac{z_N Q^2 \xi}{2 x_N} + z_N q_T k_{iT} \cos\phi_i \tag{23}$$

$$\boxed{P_B \cdot k_i = \frac{1}{2}\left((M_B^2 + z_N^2 q_T^2)(k_i^2 + k_{iT}^2)\frac{x_N}{z_N Q^2 \xi} + \frac{z_N Q^2 \xi}{x_N} + 2 z_N q_T k_{iT} \cos\phi_i\right)} \tag{24}$$

$$R_1 = \frac{P_B \cdot k_f}{P_B \cdot k_i} = \frac{1}{\zeta}\frac{M_B^2 + \zeta^2(\delta_{kT}^2 + k_f^2)}{(M_B^2 + z_N^2 q_T^2)(k_i^2 + k_{iT}^2)\left(\frac{x_N}{z_N Q^2 \xi}\right) + \frac{z_N Q^2 \xi}{x_N} + 2 z_N q_T k_{iT} \cos\phi_i} \tag{25}$$