Edge split ADMM with TV nodes, augmented Lagrangian and proximal edge fusion

Problem. Let $G = (V, E)$ be a connected undirected graph. Each node $i \in V$ holds a vector $x_i \in \mathbb{R}^n$. Each undirected edge $(i, j) \in E$ carries an auxiliary variable $z_{ij} \in \mathbb{R}^n$. The local objective at node i is

$$f_i(x_i) \;=\; \tfrac{1}{2}\|A_i x_i - b_i\|_2^2 \;+\; \lambda_i \|K x_i\|_{2,1},$$

where K is a discrete gradient operator and $\|Kx\|_{2,1}$ is the usual isotropic total variation. The consensus constraints are

$$x_i = z_{ij}, \qquad x_j = z_{ij}, \qquad \forall (i, j) \in E.$$

We also fix positive definite matrices $Q_{ij} \succ 0$ for edges and $W_i \succ 0$ for nodes.

Constrained problem.

$$\min_{\{x_i\},\{z_{ij}\}} \sum_{i \in V} f_i(x_i) \quad \text{s.t.} \quad x_i - z_{ij} = 0, \qquad x_j - z_{ij} = 0, \qquad \forall (i, j) \in E.$$

Augmented Lagrangian with unscaled duals

Introduce multipliers $\lambda_{ij,i}$ for $x_i - z_{ij} = 0$ and $\lambda_{ij,j}$ for $x_j - z_{ij} = 0$. Fix a penalty $\rho > 0$ and weight the quadratic penalties in the edge metric Q_{ij}. The augmented Lagrangian is

$$\begin{aligned}
\mathcal{L}_\rho(x, z, \lambda) = \sum_{i \in V} &\left(\tfrac{1}{2}\|A_i x_i - b_i\|_2^2 + \lambda_i \|K x_i\|_{2,1} \right) \\
&+ \sum_{(i,j) \in E} \left[\lambda_{ij,i}^\top (x_i - z_{ij}) + \lambda_{ij,j}^\top (x_j - z_{ij}) + \tfrac{\rho}{2}\|x_i - z_{ij}\|_{Q_{ij}}^2 + \tfrac{\rho}{2}\|x_j - z_{ij}\|_{Q_{ij}}^2 \right],
\end{aligned}$$

with the convention $\|u\|_M^2 = u^\top M u$ for $M \succ 0$.

Scaled duals. Define

$$y_{ij,i} = \rho^{-1} Q_{ij}^{-1} \lambda_{ij,i}, \qquad y_{ij,j} = \rho^{-1} Q_{ij}^{-1} \lambda_{ij,j}.$$

Up to constants that do not depend on (x, z) the scaled form is

$$\tilde{\mathcal{L}}_\rho(x, z, y) = \sum_{i \in V} \left(\tfrac{1}{2} \|A_i x_i - b_i\|_2^2 + \lambda_i \|K x_i\|_{2,1} \right) + \sum_{(i,j) \in E} \left[\tfrac{\rho}{2} \|x_i - z_{ij} + y_{ij,i}\|_{Q_{ij}}^2 + \tfrac{\rho}{2} \|x_j - z_{ij} + y_{ij,j}\|_{Q_{ij}}^2 \right].$$

Classical ADMM steps from the scaled form

At iteration k define the edge proposals

$$a_i^k = x_i^{k+1} + y_{ij,i}^k, \qquad a_j^k = x_j^{k+1} + y_{ij,j}^k.$$

Node step. For each $i \in V$,

$$x_i^{k+1} = \arg\min_{x_i} \tfrac{1}{2} \|A_i x_i - b_i\|_2^2 + \lambda_i \|K x_i\|_{2,1} + \tfrac{\rho}{2} \sum_{j:(i,j) \in E} \|x_i - (z_{ij}^k - y_{ij,i}^k)\|_{Q_{ij}}^2.$$

Edge step, classical form with Q_{ij}. For each $(i, j) \in E$,

$$z_{ij}^{k+1} = \arg\min_{z} \tfrac{\rho}{2} \|z - a_i^k\|_{Q_{ij}}^2 + \tfrac{\rho}{2} \|z - a_j^k\|_{Q_{ij}}^2.$$

Expand both terms and differentiate. Since Q_{ij} is symmetric,

$$\begin{aligned}
\phi_Q(z) &= \tfrac{\rho}{2}(z - a_i^k)^\top Q_{ij}(z - a_i^k) + \tfrac{\rho}{2}(z - a_j^k)^\top Q_{ij}(z - a_j^k) \\
&= \rho\, z^\top Q_{ij} z - \rho\, z^\top Q_{ij}(a_i^k + a_j^k) + \tfrac{\rho}{2}\left((a_i^k)^\top Q_{ij} a_i^k + (a_j^k)^\top Q_{ij} a_j^k\right).
\end{aligned}$$

The gradient is

$$\nabla \phi_Q(z) = \rho\,(2 Q_{ij} z) - \rho\, Q_{ij}(a_i^k + a_j^k).$$

Set to zero and solve

$$2 Q_{ij} z^\star = Q_{ij}(a_i^k + a_j^k) \quad \Longrightarrow \quad z^\star = \tfrac{1}{2}(a_i^k + a_j^k).$$

Thus the factor Q_{ij} cancels and the minimizer is the simple midpoint.

Dual step. For each $(i, j) \in E$,

$$y_{ij,i}^{k+1} = y_{ij,i}^k + x_i^{k+1} - z_{ij}^{k+1}, \qquad y_{ij,j}^{k+1} = y_{ij,j}^k + x_j^{k+1} - z_{ij}^{k+1}.$$

Proximal edge step that yields a W weighted convex combination

The augmented Lagrangian above stays the same. Only the algorithmic edge subproblem is *proximalized.* Add a convex quadratic

$$\psi(z) = \tfrac{\rho}{2}\|z - a_i^k\|_{P_i}^2 + \tfrac{\rho}{2}\|z - a_j^k\|_{P_j}^2, \qquad P_i \succ 0, \ P_j \succ 0.$$

Then the modified edge objective is

$$\phi_Q(z) + \psi(z) = \tfrac{\rho}{2}\|z - a_i^k\|_{Q_{ij}+P_i}^2 + \tfrac{\rho}{2}\|z - a_j^k\|_{Q_{ij}+P_j}^2.$$

Choose $P_i = W_i - Q_{ij}$ and $P_j = W_j - Q_{ij}$ when these are positive semidefinite. This gives

$$\phi_Q(z) + \psi(z) = \tfrac{\rho}{2}\|z - a_i^k\|_{W_i}^2 + \tfrac{\rho}{2}\|z - a_j^k\|_{W_j}^2.$$

Expand and differentiate

$$\nabla\big(\phi_Q + \psi\big)(z) = \rho\,(W_i + W_j)z - \rho\,(W_i a_i^k + W_j a_j^k).$$

Set to zero and solve

$$(W_i + W_j)z^\star = W_i a_i^k + W_j a_j^k \quad \Longrightarrow \quad z^\star = (W_i + W_j)^{-1}\big(W_i a_i^k + W_j a_j^k\big).$$

If W_i and W_j are diagonal, this is a coordinate wise convex average with nonnegative weights that sum to one in each coordinate.

Remarks. The node step is unchanged and still uses the edge precisions Q_{ij} inside the quadratic pulls. The proximal modification affects only the edge subproblem and preserves the saddle point. This is standard in proximal and preconditioned ADMM theory.

Concrete Coding Algorithm

Let the undirected connected graph be $G = (V, E)$ with $|V| = N$. For each node $i \in V$, define $x_i \in \mathbb{R}^n$. For each edge $(i, j) \in E$, define $z_{ij} \in \mathbb{R}^n$. For each edge end, define scaled duals $y_{ij,i}, y_{ij,j} \in \mathbb{R}^n$.

Local Objectives and Constraints

$$f_i(x_i) = \tfrac{1}{2}\|A_i x_i - b_i\|_2^2 + \lambda_i \|K x_i\|_{2,1}, \qquad x_i = z_{ij},\ x_j = z_{ij}, \quad \forall (i,j) \in E.$$

where the local tomographic forward matrix at node-i is $A_i \in \mathbb{R}^{m_i \times n}$ and the resulting sinogram at node-i is $b_i \in \mathbb{R}^{m_i}$. Finally $K : \mathbb{R}^n \to \mathbb{R}^{n \times 2}$, is the discrete gradient operator and the total variation penalty is $\|Kx\|_{2,1} = \sum_{j=1}^n \|(Kx)_j\|_2$.

Metrics and Parameters

- Node precisions $W_i = \mathrm{diag}\left(\eta_1^i, \ldots, \eta_n^i\right) \succ 0, \eta_j^i = \|A_i(:,j)\|_2^2, \forall j \in \{1, \ldots, n\}$

- Edge precisions $Q_{ij} = \left(W_i^{-1} + W_j^{-1}\right)^{-1} \succ 0$

- Penalty $\rho > 0$

Augmented Lagrangian (Unscaled)

$$\mathcal{L}_\rho(x, z, \lambda) = \sum_{i \in V} f_i(x_i) + \sum_{(i,j) \in E} \left[\lambda_{ij,i}^\top (x_i - z_{ij}) + \lambda_{ij,j}^\top (x_j - z_{ij}) + \tfrac{\rho}{2}\|x_i - z_{ij}\|_{Q_{ij}}^2 + \tfrac{\rho}{2}\|x_j - z_{ij}\|_{Q_{ij}}^2 \right].$$

Scaled Duals

$$y_{ij,i} = \rho^{-1} Q_{ij}^{-1} \lambda_{ij,i}, \qquad y_{ij,j} = \rho^{-1} Q_{ij}^{-1} \lambda_{ij,j}.$$

Scaled Augmented Lagrangian

$$\tilde{\mathcal{L}}_\rho(x, z, y) = \sum_{i \in V} f_i(x_i) + \sum_{(i,j) \in E} \left[\tfrac{\rho}{2}\|x_i - z_{ij} + y_{ij,i}\|_{Q_{ij}}^2 + \tfrac{\rho}{2}\|x_j - z_{ij} + y_{ij,j}\|_{Q_{ij}}^2 \right].$$

ADMM Iteration

For iteration $k = 0, 1, 2, \ldots$:

1. Node Update

$$x_i^{k+1} \in \arg\min_{x_i} \ \tfrac{1}{2}\|A_i x_i - b_i\|_2^2 + \lambda_i \|K x_i\|_{2,1} + \tfrac{\rho}{2} \sum_{j:(i,j) \in E} \|x_i - (z_{ij}^k - y_{ij,i}^k)\|_{Q_{ij}}^2. \tag{1}$$

2. Edge Update (Weighted Fusion)

$$z_{ij}^{k+1} = (W_i + W_j)^{-1}\left(W_i a_i^k + W_j a_j^k\right), \qquad a_i^k = x_i^{k+1} + y_{ij,i}^k, \ a_j^k = x_j^{k+1} + y_{ij,j}^k. \tag{2}$$

3. Dual Update

$$y_{ij,i}^{k+1} = y_{ij,i}^k + x_i^{k+1} - z_{ij}^{k+1}, \qquad y_{ij,j}^{k+1} = y_{ij,j}^k + x_j^{k+1} - z_{ij}^{k+1}. \tag{3}$$

Residuals and Stopping Criteria

Primal residual:

$$r_{ij}^{k+1} = \begin{bmatrix} x_i^{k+1} - z_{ij}^{k+1} \\ x_j^{k+1} - z_{ij}^{k+1} \end{bmatrix}. \tag{4}$$

Dual residual:

$$s_{ij}^{k+1} = \rho(z_{ij}^{k+1} - z_{ij}^k). \tag{5}$$

Stop when

$$\|r^{k+1}\|_2 < \varepsilon_{\mathrm{pri}}, \qquad \|s^{k+1}\|_2 < \varepsilon_{\mathrm{dual}}. \tag{6}$$