Regime Mixture Q-Variance Simulation Model

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We use a *regime-switching* stochastic volatility model. Time is divided into consecutive blocks called *regimes*. Within a single regime, the variance of returns is constant, so the log-price follows a standard Gaussian diffusion with fixed volatility. When the regime ends, the model draws a new variance level and continues from the current log-price.

Thus:

- inside a regime: variance is constant and increments are Gaussian,
- across regimes: variance jumps randomly, producing a mixture of Gaussian components.

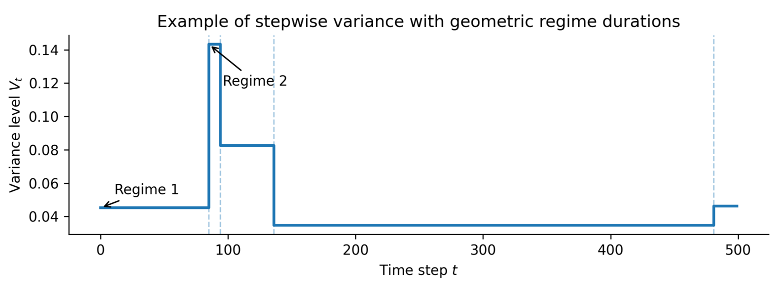

Figure 1: Illustration of a stepwise variance process with geometric regime durations. Each horizontal segment is a regime with constant variance; vertical dashed lines indicate regime switches.

This mixture of period with different volatilities is what generates both heavy tails and the characteristic Q-variance parabola observed in the challenge.

We denote by z a standardized log-return. Empirically, the Q-variance curve is not always perfectly centered at 0; we call the location of its minimum z_0 and write the curve in terms of $(z - z_0)$.

Precision and variance in a regime

Each regime j is assigned a random *precision*

$$\tau_j \sim \Gamma\big(\alpha = \tfrac{3}{2},\ \text{rate} = \sigma_0^2\big)\,,$$

where *precision* means the inverse of the variance rate:

$$V_j = \frac{1}{\tau_j}.$$

High precision means low variance; low precision means high variance.

Why this distribution?

We use this specific Gamma model because it gives exactly the behaviour needed:

- it ensures variance is always positive,
- it produces mostly calm regimes but occasionally very volatile ones,
- and most importantly, it is the simplest choice that makes the Q-variance curve a clean parabola in $(z - z_0)$.

The detailed mathematical reason for the parabola is given in the appendix.

Why the mixture creates the Q-variance parabola

Inside a regime, returns are Gaussian with variance V_j. Across time, each regime has its own variance level, so the overall return distribution becomes a mixture of Gaussians: some narrow (low V_j), some wide (high V_j).

Now fix a return size z. Small returns come mainly from calm regimes; large returns almost always come from volatile regimes:

$$\text{small } |z| \;\Rightarrow\; \text{low } V_j, \qquad \text{large } |z| \;\Rightarrow\; \text{high } V_j.$$

Thus the *typical variance level* associated with return z increases with $|z|$. For the Gamma–Gaussian construction described above (Gamma on precision, Gaussian conditional on that precision), one can show (see the appendix) that this dependence is exactly

$$\mathbb{E}[V_j \mid z] = \sigma_0^2 + \frac{(z - z_0)^2}{2},$$

a parabola with minimum at z_0.

This is why the Q-variance curve appears: small returns come from calm regimes, large returns from volatile regimes, and this particular construction makes the relationship perfectly quadratic in $(z - z_0)$.

Regime durations

We use a very simple rule for how long a regime lasts. At every internal time step there is a small, constant probability that the current regime ends and a new one begins. Because the chance of "switching" is the same at every step, the number of steps a regime survives follows a geometric distribution:

$$K_j \sim \mathrm{Geom}(p), \qquad \mathbb{E}[K_j] = \frac{1}{p}.$$

This is the discrete–time version of a constant–intensity process: there is no memory, and a regime that has lasted a long time is no more or less likely to end than one that just started.

The exact choice of duration model is not important. Any method that produces reasonably long, stable regimes will work. The geometric model is simply the easiest to implement and behaves well for simulation: long periods of nearly constant variance punctuated by occasional regime changes.

Log-price dynamics

Within regime j the log-price evolves as

$$\Delta L_t = \mu\,\Delta t + \sqrt{V_j \Delta t}\,\xi_t - \theta\,L_t\,\Delta t, \qquad \xi_t \sim N(0,1).$$

The small mean-reversion term θ is included only to prevent numerical overflow during extremely long simulations; it does not affect the mixture structure or the Q-variance curve.

Daily outputs

The simulation runs on a fine time grid with m steps per trading day. After simulation we keep one log-price per day:

$$L_n^{\text{daily}} = L_{nm}, \qquad S_n = e^{L_{nm}},$$

and compute the daily average variance over those m steps.

Appendix: Mathematical reason for the Q-variance parabola

This appendix shows why the Gamma$(3/2, \sigma_0^2)$ model together with a Gaussian conditional return implies

$$\mathbb{E}[V \mid z] = \sigma_0^2 + \frac{(z - z_0)^2}{2}.$$

The result is not needed to understand the simulation but validates the model.

We consider a single standardized return z and ignore drift and time scaling. Assume:

- a random precision (inverse variance)

$$\tau \sim \Gamma\big(\alpha = \tfrac{3}{2},\ \text{rate} = \beta = \sigma_0^2\big)\,,$$

- a Gaussian return conditional on τ with location z_0,

$$z \mid \tau \sim N\big(z_0,\ 1/\tau\big).$$

Here z_0 is the return level at which the Q-variance curve reaches its minimum; it is a centering parameter of the Gaussian, not a parameter of the Gamma prior. The variance associated with a given regime is

$$V = \frac{1}{\tau}.$$

Our goal is to compute $\mathbb{E}[V \mid z]$.

Step 1: Posterior of τ given z

The prior is

$$p(\tau) \propto \tau^{\alpha-1} e^{-\beta\tau}, \qquad \alpha = \tfrac{3}{2}.$$

The likelihood is

$$p(z \mid \tau) \propto \sqrt{\tau}\ \exp\Big(-\tfrac{1}{2}\tau(z-z_0)^2\Big).$$

Multiplying yields

$$p(\tau \mid z) \propto \tau^{(\alpha+\frac{1}{2})-1} \exp\Big(-[\beta + \tfrac{1}{2}(z-z_0)^2]\tau\Big).$$

Thus

$$\tau \mid z \sim \Gamma\big(\alpha', \beta'\big), \quad \alpha' = \alpha + \tfrac{1}{2} = 2, \quad \beta' = \beta + \tfrac{1}{2}(z-z_0)^2.$$

Step 2: Expected variance

Since $V = 1/\tau$, and for a Gamma(α', β') with $\alpha' > 1$:

$$\mathbb{E}\Big[\frac{1}{\tau}\Big] = \frac{\beta'}{\alpha' - 1},$$

we obtain

$$\mathbb{E}[V \mid z] = \frac{\beta + \frac{1}{2}(z-z_0)^2}{2-1} = \beta + \frac{(z-z_0)^2}{2}.$$

Finally, with $\beta = \sigma_0^2$:

$$\boxed{\mathbb{E}[V \mid z] = \sigma_0^2 + \frac{(z-z_0)^2}{2}}$$

This is precisely the Q-variance parabola used in the challenge: the Gamma prior controls the overall scale σ_0^2, and the Gaussian centering parameter z_0 sets where the minimum is located.