A CONSTRUCTION OF Q-VARIANCE IN CONTINUOUS TIME

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CONTENTS

1 Q-VARIANCE

Suppose an asset price dynamics is written as

$$dS_t = \sigma_t dW_t, \quad S_0 = 0, \tag{1}$$

the quadratic variation $QV_T = \int_0^T \sigma_t^2 dt$ and the terminal price $S_T = \int_0^T \sigma_t dW_t$ is related by the Itô isometry in a global way. The *q-variance* concerns the fine structure of a model such that the expectation of the quadratic variation conditioned on the price takes the following form for all $T > 0$:

$$\mathbb{E}\left[QV_T | S_T = x\right] = \sigma_0^2 T + \frac{1}{2}x^2, \tag{2}$$

or in annualized terms:

$$\mathbb{E}\left[\frac{1}{T}QV_T \middle| S_T = x\right] = \sigma_0^2 + \frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2. \tag{3}$$

2 CONSTRUCTION IN SCENARIOS

A construction of q-variance in continuous time goes as follows:

- At $t = 0$, a Bachelier volatility σ is drawn from a distribution of density $w(\sigma)$:

$$\int_0^\infty w(\sigma)d\sigma = 1; \quad w(\sigma) \geqslant 0, \quad \forall \sigma \in [0, \infty). \tag{4}$$

- For $t > 0$, the price evolves according to a Bachelier model of the drawn σ:

$$dS_t = \sigma\, dW_t, \quad S_0 = 0. \tag{5}$$

Conditioned on a drawn Bachelier volatility σ, the density of $S_T = x$ at time T is given by a Gaussian:

$$p(x|\sigma) = \frac{1}{\sqrt{2\pi T}\sigma} \exp\left(-\frac{x^2}{2\sigma^2 T}\right). \tag{6}$$

Conditioned on $S_T = x$, the posterior density of the drawn Bachelier volatility σ is given by Bayes:

$$p(\sigma|S_T = x) = \frac{w(\sigma)\frac{1}{\sqrt{2\pi T}\sigma} \exp\left(-\frac{x^2}{2\sigma^2 T}\right)}{\int_0^\infty w(\sigma')\frac{1}{\sqrt{2\pi T}\sigma'} \exp\left(-\frac{x^2}{2\sigma'^2 T}\right) d\sigma'}. \tag{7}$$

The conditional quadratic variation is then

$$\mathbb{E}\left[QV_T|S_T=x\right]=\int_0^\infty p(\sigma|S_T=x)(\sigma^2T)d\sigma=\frac{\int_0^\infty w(\sigma)\frac{1}{\sqrt{2\pi T}\sigma}\exp\left(-\frac{x^2}{2\sigma^2T}\right)(\sigma^2T)\,d\sigma}{\int_0^\infty w(\sigma)\frac{1}{\sqrt{2\pi T}\sigma}\exp\left(-\frac{x^2}{2\sigma^2T}\right)d\sigma}.$$

In annualized terms, the q-variance Eq.(14) requires that

$$\mathbb{E}\left[\frac{1}{T}QV_T\middle|S_T=x\right]=\frac{\int_0^\infty w(\sigma)\exp\left(-\frac{x^2}{2\sigma^2T}\right)\sigma d\sigma}{\int_0^\infty w(\sigma)\exp\left(-\frac{x^2}{2\sigma^2T}\right)\frac{d\sigma}{\sigma}}=\sigma_0^2+\frac{x^2}{2T}. \tag{8}$$

The solution to Eq.(8) is

$$\boxed{w(\sigma)=\frac{4}{\sqrt{\pi}}\frac{\sigma_0^3}{\sigma^4}e^{-\left(\frac{\sigma_0}{\sigma}\right)^2}.} \tag{9}$$

More details are given in Appendix 5. One can verify that the density given by Eq.(9) is normalized, has finite mean and variance:

$$\int_0^\infty w(\sigma)d\sigma=1,\quad \int_0^\infty \sigma w(\sigma)d\sigma=\frac{2}{\sqrt{\pi}}\sigma_0,\quad \int_0^\infty \sigma^2 w(\sigma)d\sigma=2\sigma_0^2. \tag{10}$$

Expressed in units of σ_0^2, the density of the variance $v \triangleq \sigma^2$:

$$w(v)=\frac{2}{\sqrt{\pi}}\frac{1}{v^{\frac{5}{2}}}e^{-\frac{1}{v}}. \tag{11}$$

is known as that of the *inverse gamma* with shape parameter $\frac{3}{2}$, the second moment of which does not exist. Note, the density of volatility $w(\sigma)$ of Eq.(9) does not depend on T, the model of Eq.(4)-(5) produces continuous paths and the exact q-variance relation Eq.(8) for all $T>0$. Due to the factor $e^{-\left(\frac{\sigma_0}{\sigma}\right)^2}$, the density $w(\sigma)$ is heavily suppressed for small σ and has essentially a floor near $\frac{1}{3}\sigma_0$ or $\frac{1}{4}\sigma_0$, as shown in Figure (1).

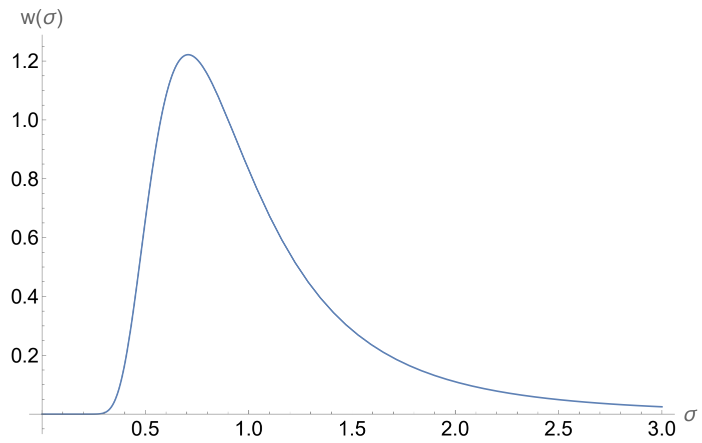

Figure 1: The density $w(\sigma)$ of σ, measured in units of σ_0, given by Eq.(9).

The marginal distribution of $Z_T \triangleq \frac{S_T}{\sigma_0\sqrt{T}}$ is given by the following time-invariant density:

$$p(z)=\int_0^\infty w(\sigma)p(z|\sigma)d\sigma=\frac{\sqrt{2}}{\pi}\frac{1}{\left(1+\frac{1}{2}z^2\right)^2}. \tag{12}$$

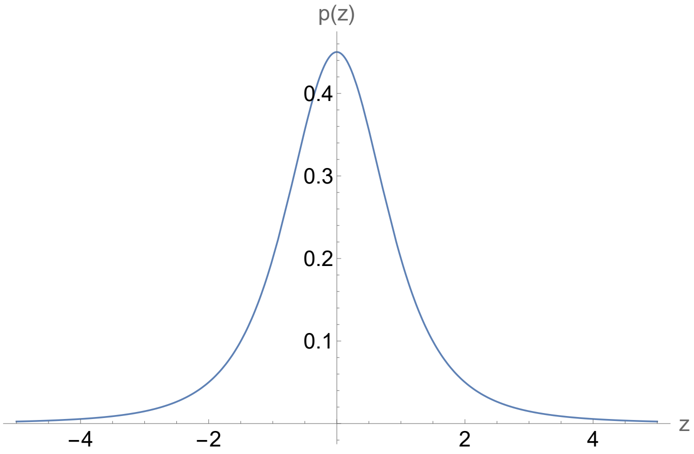

Figure 2: The density $Z_T \triangleq \frac{S_T}{\sigma_0 \sqrt{T}}$.

<u>*Simulation 1*</u>. To verify the construction, we draw $M = 1,000,000$ samples of σ from Eq.(9) with $\sigma_0 = 1$ and simulate correspondingly $1,000,000$ Bachelier paths on a grid of $N = 100$ time points in an interval $t \in [0, 1]$, equal spaced at $\delta = \frac{1}{N}$. We collect the statistics of the prices and the quadratic variations:

- $\frac{1}{\sqrt{T}} S_T = \frac{1}{\sqrt{T}} \sum_{j=1}^{T/\Delta t} \sigma_i \delta W_{i,j}$;
- $\frac{1}{T} QV_T = \frac{1}{T} \sum_{j=1}^{T/\Delta t} \sigma_i^2 \delta W_{i,j}^2$,

for each path $i = 1, 2, \cdots, M$, where $\delta W_{i,j} \triangleq W_{i,j} - W_{i,j-1}$ is the Brownian motion increment of path $\#i$ at j^{th} time step. In Figure (3), we make a scatter plot for each $T \in \{0.25, 0.5, 0.75, 1.0\}$. Each sample point in the scatter plot corresponds to one path color-coded by the Bachelier σ of the path — the blue (yellow) points correspond to low (high) values of σ, respectively. On top of the scatter plots, the gray lines are kernel regressions of the sample points and the yellow lines are $y = 1 + \frac{1}{2} z^2$ for all $T \in \{0.25, 0.5, 0.75, 1.0\}$. Notice that, the quadratic relationship does not depend on T.

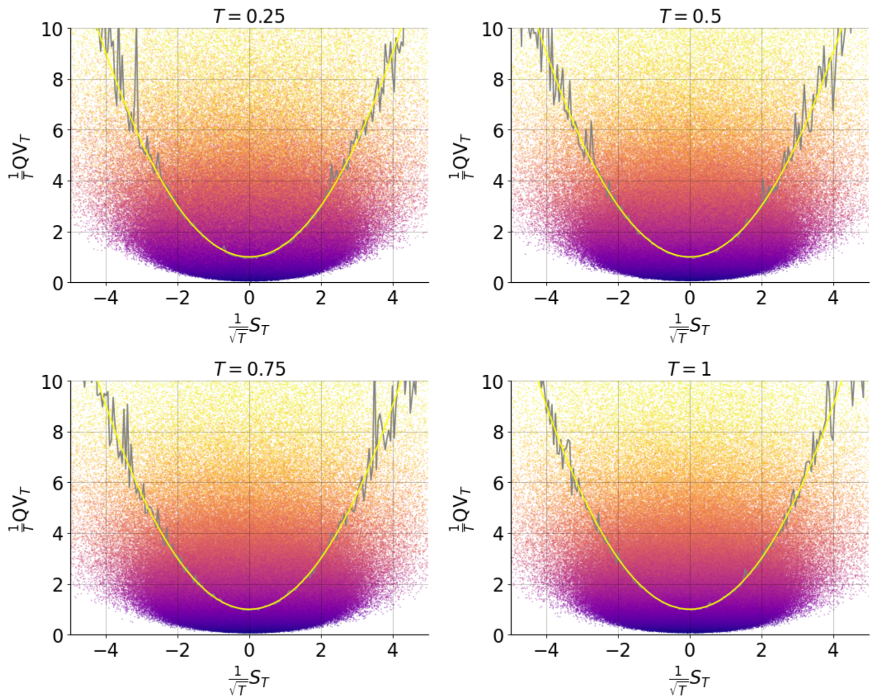

Figure 3: Scatter plot of $\frac{1}{\sqrt{T}} S_T$ versus $\frac{1}{T} QV_T$ for $T \in \{0.25, 0.5, 0.75, 1.0\}$.

3 CONSTRUCTION IN TIME SERIES

To collect the statistics of $\frac{1}{\sqrt{T}}S_T$ and $\frac{1}{T}QV_T$ on the (non-overlapping) segments of various lengths of one path, one needs to construct a time series model. Before attempting to construct a time series model that satisfies the q-variance relationship Eq.(14), one may notice some difficulties, detailed in Appendix 5. Instead of generating multiple Bachelier paths each having a volatility drawn from the density $w(\sigma)$ of Eq.(9) at time zero, as in Section 2, one can generate one long path with **slowly** varying volatility regimes. To be more specific,

- the volatility regime change is triggered by a Poisson process with arrival rate λ such that $\lambda T_{max} \ll 1$, where T_{max} is the maximum time horizon over which the q-variance is to be tested, for example, $T_{max} = 160$ days, $\lambda = \frac{1}{500}\text{days}^{-1}$.
- at time zero and every time a volatility regime change is triggered, one draws a new Bachelier σ from the density $w(\sigma)$ of Eq.(9) and simulate a Bachelier path with volatility σ until the next volatility regime change.

Denote by $\tau_1 < \tau_2 < \cdots < \tau_{N_t}$ be the stochastic arrival times of the Poisson counting process N_t, the price process can be written as the following SDE:

$$dS_t = \begin{cases} \sqrt{\nu_0}dW_t, & 0 \leqslant t < \tau_1 \\ \sqrt{\nu_1}dW_t, & \tau_1 \leqslant t < \tau_2 \\ \sqrt{\nu_2}dW_t, & \tau_2 \leqslant t < \tau_3 \\ \vdots & \\ \sqrt{\nu_{N_t}}dW_t, & t \geqslant \tau_{N_t}, \end{cases} \tag{13}$$

where the instantaneous variances ν_i, $i = 0, 1, 2, \cdots, N_t$ are drawn from the inverse gamma distribution Eq.(11), up to a scaling factor. A Poisson process is chosen to trigger the volatility regime changes to ensure that the constructed time series model is time-homogeneous. For any segment of length T between time t and $t+T$, the probability of no volatility regime change during the time interval $[t, t+T]$ is $e^{-\lambda T} \approx 1$ because $\lambda T_{max} \ll 1$. There will be some residual probabilities that one or more volatility regime changes may occur during $[t, t+T]$, especially for relatively large T, so the q-variance relationship will not be exact but the approximation is controlled by the Poisson λ, an additional hyper-parameter.

Simulation 2. To verify the construction, we set $\sigma_0 = 1$, $\lambda = \frac{1}{500}\text{days}^{-1}$ and simulate a long daily time series of $N = 500{,}000{,}000$ time steps, with time step width $\Delta t = 1$ day. The expected number of volatility regime changes is about $\lambda N = 1{,}000{,}000$. Such a long path is generated so that roughly $N/T_{max} \approx 3{,}000{,}000$ non-overlapping segments are obtained for $T_{max} = 160$ days. The statistics of $\frac{1}{\sqrt{T}}S_T$ and $\frac{1}{T}QV_T$ are collected from the non-overlapping segments of lengths $T \in \{5, 10, 20, 40, 80, 160\}$ days:

- $\frac{1}{\sqrt{T}}S_T = \frac{1}{\sqrt{T}}\sum_{j=i}^{i+T}\sigma_j\delta W_j$;
- $\frac{1}{T}QV_T = \frac{1}{T}\sum_{j=i}^{i+T}\sigma_j^2\delta W_j^2$,

for each time point $i = 0, T, 2T, \cdots$, on the path. Here, the maximum time horizon over which the q-variance is tested is $T_{max} = 160$ days, so $\lambda T_{max} = \frac{160}{500} = 0.32$. During a period of 160 days, the probability of having no volatility regime change about $e^{-0.32} \approx 0.73$; the probability of having at most one volatility regime change is about 0.96. The results are shown in Figure (4).

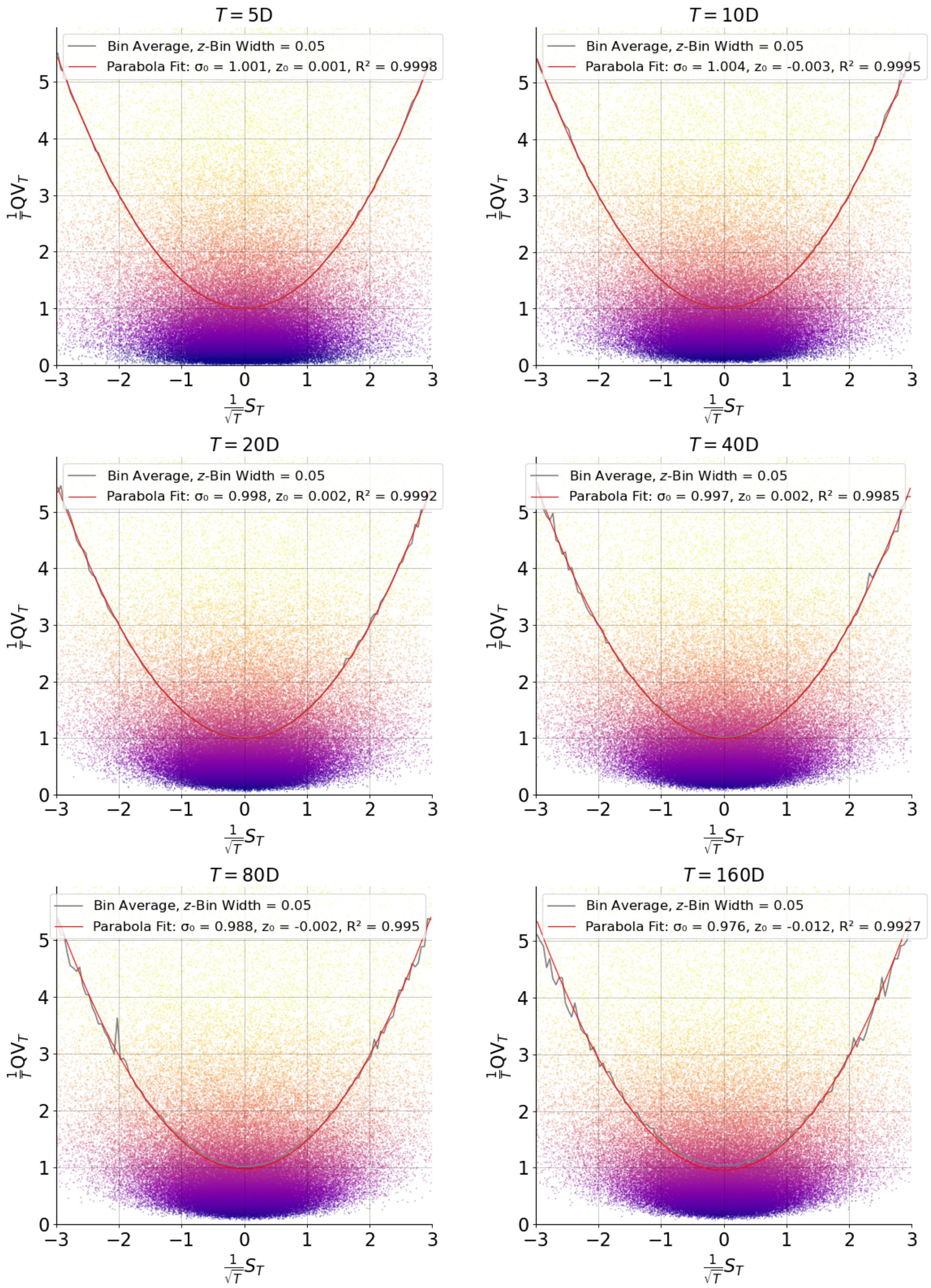

Figure 4: Scatter plot of $\frac{1}{\sqrt{T}}S_T$ versus $\frac{1}{T}QV_T$ for $T \in \{5, 10, 20, 40, 80, 160\}$ days.

4 OFFSET z_0 VIA ASYMMETRIC JUMPS

A simple mechanism of introducing an offset z_0 to the q-variance Eq.(14):

$$\mathbb{E}\left[\frac{1}{T}QV_T \middle| \frac{S_T}{\sqrt{T}} = z\right] = \sigma_0^2 + \frac{1}{2}(z - z_0)^2. \tag{14}$$

is via jumps of a small constant size δ, triggered by the same Poisson process that triggers the volatility resampling. To compensate for the asymmetric distribution of the jump amplitude (in this case a point mass to one side), a compensator needs to be added to the price process. To be more precise,

$$dS_t = \underbrace{\delta(dN_t - \lambda dt)}_{\text{compensated jumps}} + \begin{cases} \sqrt{v_0}dW_t, & 0 \leqslant t < \tau_1 \\ \sqrt{v_1}dW_t, & \tau_1 \leqslant t < \tau_2 \\ \sqrt{v_2}dW_t, & \tau_2 \leqslant t < \tau_3 \ . \\ \vdots & \\ \sqrt{v_{N_t}}dW_t, & t \geqslant \tau_{N_t}, \end{cases} \tag{15}$$

where $\tau_1 < \tau_2 < \cdots < \tau_{N_t}$ are the successive arrival times of the Poisson process N_t, as in Eq.(13). A small positive offset $z_0 > 0$ in the q-variance will be generated by the jumps to the negative direction $\delta < 0$, without perturbing the q-variance significantly.

Simulation 3. To verify the construction, we simulate a long daily time series of $N = 5,000,000$ time steps, with time step width $\Delta t = 1$ day. The diffusion variance parameter v_0 and the jump amplitude parameter δ are calibrated to yield a q-variance relationship with $\sigma_0 = 0.259$ and $z_0 = 0.021$, while the Poisson arrival rate λ is set to $1/500\text{days}^{-1}$ as usual. The statistics of $\frac{1}{\sqrt{T}}S_T$ and $\frac{1}{T}QV_T$ are collected from the non-overlapping segments of lengths $T \in \{1, 2, \cdots, 26\}$ weeks, pooled together.

- The calibrated parameters: $v_0 \approx 0.5$, $\delta \approx -0.11$.
- The Poisson parameter: $\lambda = \frac{1}{500}\text{days}^{-1}$; the inverse gamma shape parameter: $\frac{3}{2}$.

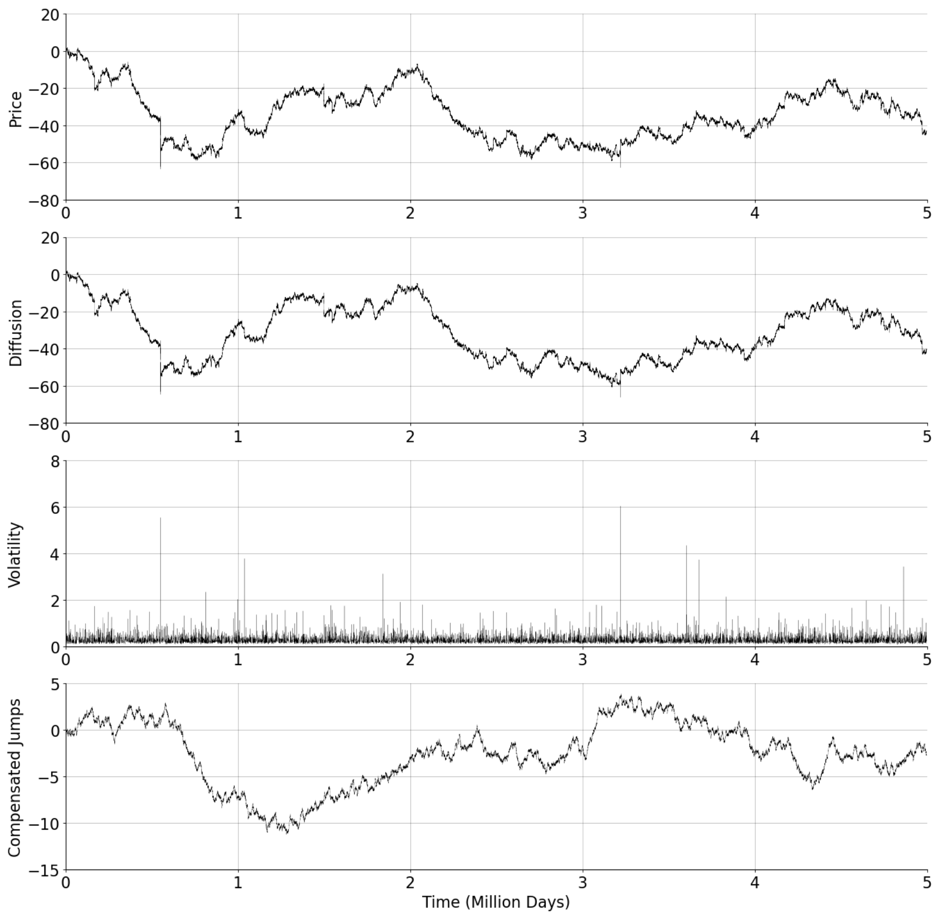

Figure 5: Bachelier Price time series over $N = 5,000,000$ days based the calibrated parameters $v_0 \approx 0.5$, $\delta \approx -0.11$ and $\lambda = \frac{1}{500}\text{days}^{-1}$. In this simulation, there are 10,040 jumps. The big moves are caused by the volatility spikes rather than the jumps.

In Figure (6), the q-variance for all horizons $T \in \{1, 2, \cdots, 26\}$ weeks is shown in the upper panel. The q-variance (middle panel) and the density of $z = \frac{1}{\sqrt{T}} S_T$ (lower panel) for each horizon $T \in \{1, 2, \cdots, 26\}$ weeks are shown in gray lines, compared with the global parabola fit (red line) and the invariant density given by Eq.(12).

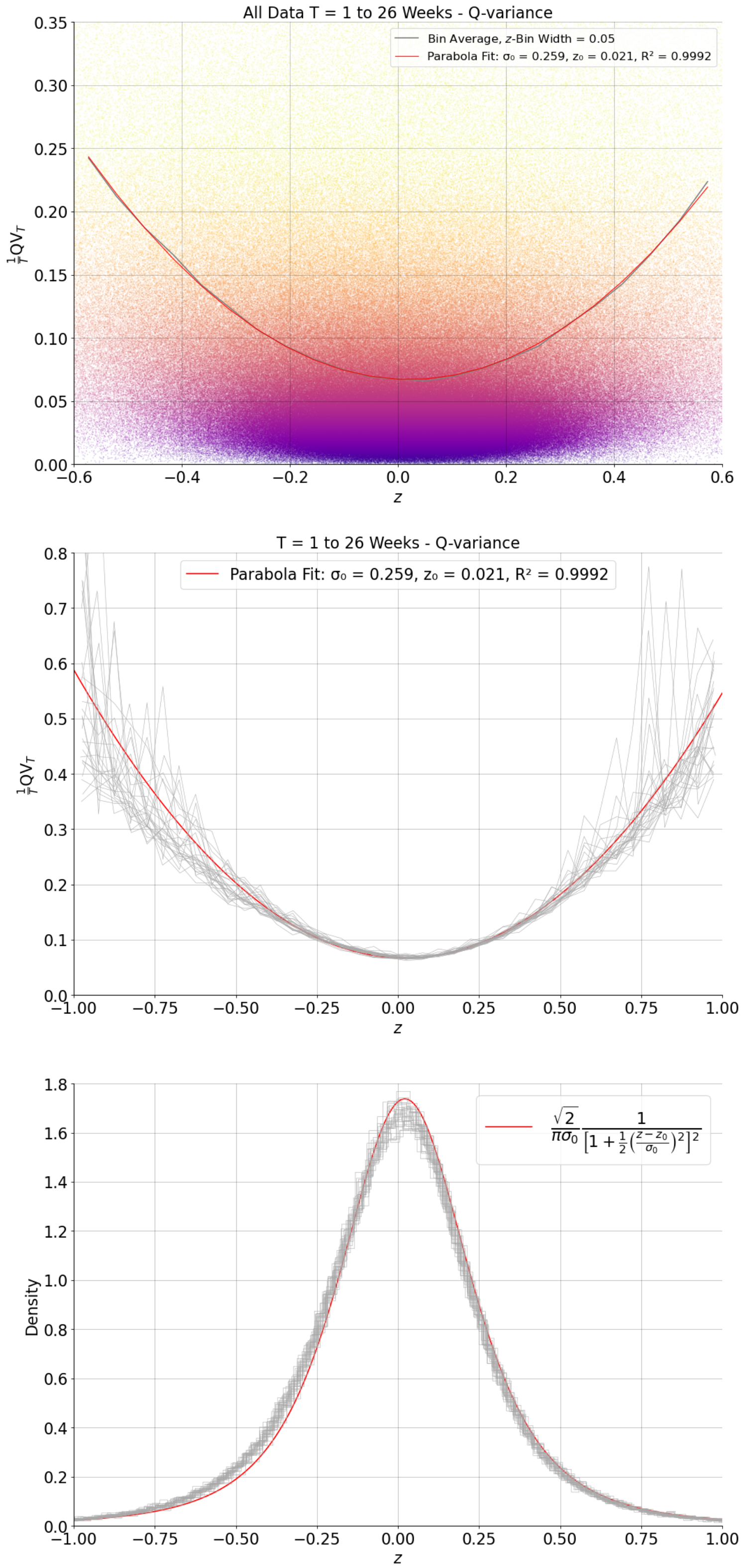

Figure 6: The q-variance for all horizons $T \in \{1, 2, \cdots, 26\}$ weeks is shown in the upper panel. The q-variance (upper panel) and the density of $z = \frac{1}{\sqrt{T}} S_T$ (lower panel) for each horizon $T \in \{1, 2, \cdots, 26\}$ weeks shown in gray lines.

5 SUMMARY

Re-writing the jump-diffusion time series model Eq.(15):

$$dS_t = \underbrace{\delta(dN_t - \lambda dt)}_{\text{compensated jumps}} + \underbrace{\begin{cases} \sqrt{\nu_0}dW_t, & 0 \leqslant t < \tau_1 \\ \sqrt{\nu_1}dW_t, & \tau_1 \leqslant t < \tau_2 \\ \sqrt{\nu_2}dW_t, & \tau_2 \leqslant t < \tau_3 \\ \vdots & \\ \sqrt{\nu_{N_t}}dW_t, & t \geqslant \tau_{N_t}, \end{cases}}_{\text{diffusion with volatility resampling}} . \tag{16}$$

where $\tau_1 < \tau_2 < \cdots < \tau_{N_t}$ are the successive arrival times of the Poisson process N_t, there are three (3) sources of randomness:

- the volatility resampling from an inverse gamma distribution;
- the Brownian motion driving the diffusion component of the price dynamics; and
- the Poisson process that triggers the volatility resampling and the jumps,

as well as four (4) free parameters:

- inverse gamma distribution shape parameter $3/2$;
- inverse gamma distribution scale parameter ν_0;
- the jump amplitude δ; and
- the Poisson process arrival rate λ.

The shape parameter of the inverse gamma distribution is set to $\frac{3}{2}$ to yield the $\frac{1}{2}$ quadratic coefficient of the q-variance; the scale parameter ν_0 of the inverse gamma distribution is directly related to the minimum volatility σ_0 of the q-variance: $\nu_0 \approx \sigma_0^2$; the jump amplitude δ is a mechanism to generate skewness and a handle to the q-variance offset z_0; finally, the Poisson λ regulates a slow volatility regime change so that the volatility persists over a time horizon over 26 weeks.

In summary, the dynamics given by Eq.(16) can be described as a jump-diffusion model with volatility resampling. In the setting of randomized volatilities, the construction agrees with the q-variance in principle. As long as the Poisson arrival is slow enough, one has the q-variance to a good approximation for the time horizons shorter than the typical inter-arrival times. The corresponding drawback is that one needs to simulate a long time series for the q-variance to emerge due to the slowly varying volatility regimes. For a shorter simulation where one does not have enough non-overlapping segments to collect the statistics, the q-variance relationship could be unstable. In Table (1), the R^2's of the parabola fit are reported for three different simulations with lengths of the simulation $N = 10K$, $100K$, $1,000K$.

Random Number Generator Seed	$N = 10K$	$N = 100K$	$N = 1,000K$
4	0.561	0.959	0.987
40	0.876	0.982	0.998
400	0.959	0.980	0.992

Table 1: R^2 of parabola fit to $\sigma_0^2 + \frac{1}{2}(z - z_0)^2$, where $\sigma_0 = 0.259$ and $z_0 = 0.021$.

APPENDIX — DERIVATION OF EQ.(9)

To derive Eq.(9), it is more convenient to make a substitution of variable:

$$z \triangleq \frac{x^2}{2T}, \quad \mu \triangleq \frac{1}{\sigma^2}, \tag{17}$$

so $\sigma = \frac{1}{\sqrt{\mu}}$, $d\sigma = -\frac{d\mu}{2\mu^{3/2}}$. Eq.(8) can be re-written as

$$\mathbb{E}\left[\frac{1}{T}QV_T \middle| S_T = x\right] = \frac{\int_0^\infty \frac{w\left(\frac{1}{\sqrt{\mu}}\right)}{\mu^2} \exp(-z\mu)\, d\mu}{\int_0^\infty \frac{w\left(\frac{1}{\sqrt{\mu}}\right)}{\mu} \exp(-z\mu)\, d\mu}, \tag{18}$$

where the numerator and the denominator are the Laplace transforms of $\frac{w\left(\frac{1}{\sqrt{\mu}}\right)}{\mu^2}$ and $\frac{w\left(\frac{1}{\sqrt{\mu}}\right)}{\mu}$, respectively. Choose a parameter $\sigma_0 > 0$ and let

$$w\left(\frac{1}{\sqrt{\mu}}\right) \propto \mu^2 e^{-\sigma_0^2 \mu}, \tag{19}$$

the normalization condition of the density

$$\int_0^\infty w(\sigma) d\sigma = \frac{1}{2}\int_0^\infty w\left(\frac{1}{\sqrt{\mu}}\right) \frac{d\mu}{\mu^{3/2}} = 1. \tag{20}$$

fixes the normalization constant:

$$w\left(\frac{1}{\sqrt{\mu}}\right) = \frac{4\sigma_0^3}{\sqrt{\pi}} \mu^2 e^{-\sigma_0^2 \mu}, \tag{21}$$

or equivalently,

$$w(\sigma) = \frac{4}{\sqrt{\pi}} \frac{\sigma_0^3}{\sigma^4} e^{-\left(\frac{\sigma_0}{\sigma}\right)^2}. \tag{22}$$

Now, evaluate Eq.(18) and we obtain Eq.(8):

$$\mathbb{E}\left[\frac{1}{T}QV_T \middle| S_T = x\right] = \frac{\int_0^\infty e^{-(\sigma_0^2+z)\mu}\, d\mu}{\int_0^\infty \mu e^{-(\sigma_0^2+z)} d\mu} = \frac{\frac{1}{\sigma_0^2+z}}{\frac{1}{(\sigma_0^2+z)^2}} = \sigma_0^2 + z = \sigma_0^2 + \frac{x^2}{2T}. \tag{23}$$

In fact, Eq.(9) is the unique solution to

$$\mathbb{E}\left[\frac{1}{T}QV_T \middle| S_T = x\right] = \frac{\int_0^\infty w(\sigma) \exp\left(-\frac{z}{\sigma^2}\right) \sigma d\sigma}{\int_0^\infty w(\sigma) \exp\left(-\frac{z}{\sigma^2}\right) \frac{d\sigma}{\sigma}} = \sigma_0^2 + z. \tag{24}$$

Denote

$$\mathcal{N}(z) \triangleq \int_0^\infty w(\sigma) \exp\left(-\frac{z}{\sigma^2}\right) \sigma d\sigma, \tag{25}$$

$$\mathcal{D}(z) \triangleq \int_0^\infty w(\sigma) \exp\left(-\frac{z}{\sigma^2}\right) \frac{d\sigma}{\sigma}. \tag{26}$$

Note that, $\mathcal{N}'(z) = -\mathcal{D}(z)$, so Eq.(24) gives

$$\mathcal{N}(z) = (\sigma_0^2 + z)\mathcal{D}(z) = -(\sigma_0^2 + z)\mathcal{N}'(z) \Rightarrow \mathcal{N}(z) \propto \frac{1}{\sigma_0^2 + z}. \tag{27}$$

The density $w(\sigma)$ can then be obtained by the inverse Laplace transform of Eq.(25) and a subsequent normalization step.

APPENDIX — DIFFICULTY OF IMPOSING EXACT Q-VARIANCE IN TIME SERIES MODEL

In a restricted setting where the instantaneous volatilities, either drawn from a distribution or generated by a stochastic process, are independent of the Brownian motion W_t driving the price process Eq.(1), the density of $S_T = x$ at time T conditioned on the quadratic variation $V_T = \int_0^T \sigma_t^2 dt = V$ is given by a Gaussian:

$$p_{S_T}(x|V_T = V) = \frac{1}{\sqrt{2\pi V}} \exp\left(-\frac{x^2}{2V}\right). \tag{28}$$

Applying Bayes in the same way as in Eq.(8) and impose the exact q-variance:

$$\mathbb{E}\left[V_T|S_T = x\right] = \frac{\int_0^\infty p_{V_T}(V) \exp\left(-\frac{x^2}{2V}\right)\sqrt{V}dV}{\int_0^\infty p_{V_T}(V) \exp\left(-\frac{x^2}{2V}\right)\frac{dV}{\sqrt{V}}} = \sigma_0^2 T + \frac{1}{2}x^2, \tag{29}$$

where $p_{V_T}(\cdot)$ is the unconditional density of V_T, one is again led to the inverse gamma distribution of shape parameter $\frac{3}{2}$:

$$p_{V_T}(V) = \frac{2}{\sqrt{\pi}} \frac{(\sigma_0^2 T)^{\frac{3}{2}}}{V^{\frac{5}{2}}} e^{-\frac{\sigma_0^2 T}{V}}. \tag{30}$$

The density of the annualized quadratic variation $U_T \triangleq \frac{1}{T} V_T$ is then given by

$$p_{U_T}(U) = \frac{2}{\sqrt{\pi}} \frac{\sigma_0^3}{U^{\frac{5}{2}}} e^{-\frac{\sigma_0^2}{U}}, \tag{31}$$

and is independent of T. Consider a discretization in time, $t_n = n\Delta t$, $n = 0, 1, 2, \cdots$,

$$U_n \approx \frac{1}{n}\sum_{i=1}^{n} \nu_i \tag{32}$$

where ν_i's are the instantaneous volatilities. If the instantaneous volatilities are i.i.d. random variables, the n-independence of the distribution of $U_n \sim U_1 \sim \nu_i$ requires that ν_i's obey an α-stable distribution with $\alpha = 1$. However, the inverse gamma distribution with shape parameter $\frac{3}{2}$ is not such a distribution. A side note, the inverse gamma distribution with shape parameter $\frac{1}{2}$ is an α-stable distribution with $\alpha = \frac{1}{2}$, but this is not relevant in the present context. The conclusion is unlikely to change if the instantaneous volatilities ν_i's are generated by a Markov chain instead of being i.i.d random variables.