16-748 Problem Set
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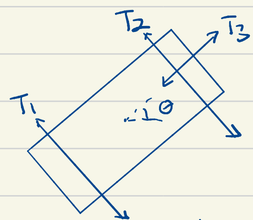
## Problem #1

a) Assume the system has no damping (single rigid body)

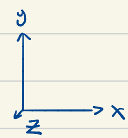
the dynamics of the system can be represented by $[\ddot{x} \quad \ddot{y} \quad \ddot{\theta}]^T$

$2^{nd}$ order diff equation:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = A \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$




the force can be summarized:




$$F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} T_1 \cdot \sin\theta + T_2 \sin\theta + T_3 \cdot \cos\theta \\ T_1 \cdot \cos\theta + T_2 \cos\theta + T_3 \cdot \sin\theta \\ 0 \end{bmatrix}$$

the torque acting on center

$$\tau = L \cdot F = L \cdot T_1 + L \cdot T_2$$
$$= \underset{\uparrow \, 1m}{L} \, (T_1 + T_2)$$

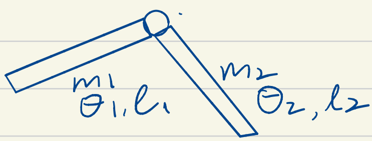
$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \sin\theta & \sin\theta & \cos\theta \\ \cos\theta & \cos\theta & \sin\theta \\ 1 & 1 & 0 \end{bmatrix}}_{A} \cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

Since there exist $\theta$

where $\text{rank}(A) = 2 < 3$

thus the $\boxed{\text{system is underactuated}}$

b)




$m_1 = m_2$

$\theta_1 = \theta$

$l_1 = l_2$

kinetic energy $T = \frac{1}{2} m_1 l_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2 \theta_2^2$

potential energy $U \to 0 \to$ planar system

$$L = T - V = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

Dynamics

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right) - \frac{\partial L}{\partial \theta} = F \leftarrow \text{force by rotational joint}$$

because two identical links are actuated by a single link

$\therefore \dot{\theta}_1 = \dot{\theta}_2$

$\therefore m l^2 \ddot{\theta}_1 = F$

$$\ddot{\theta} = \underbrace{\frac{1}{m l^2}}_{\neq 0 \ \text{rank}(1)} F$$

$\boxed{\text{Full rank Fully Actuated}}$

Problem 2

a) $$\begin{bmatrix} \dot\theta \\ \ddot\theta \end{bmatrix} = \begin{bmatrix} \dot\theta \\ (u - m \cdot g \cdot l \cdot \sin\theta - b\dot\theta) \cdot \frac{1}{ml^2} \end{bmatrix}$$

to find stable points

$$\begin{bmatrix} \dot\theta \\ \ddot\theta \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix} \qquad \begin{matrix} \dot\theta = 0 \\ (u - m \cdot g \cdot l \cdot \sin\theta - b\dot\theta) \cdot \frac{1}{ml^2} = 0 \end{matrix}$$

(i) $\{ b = u = 0 \}$

$$\cancel{u}^{0} - m \cdot g \cdot l \cdot \sin\theta - \cancel{b\dot\theta}^{0} = 0$$

$$\therefore \; m \cdot g \cdot l \cdot \sin\theta$$

fixed points are $\begin{bmatrix} \theta \\ \dot\theta \end{bmatrix} = \begin{bmatrix} \alpha \cdot \pi \\ 0 \end{bmatrix}$ ← where

$$\alpha = \underbrace{-\infty, \dots, -1}_{\text{unstable equilibrium}}, 0, 1, \dots, +\infty$$

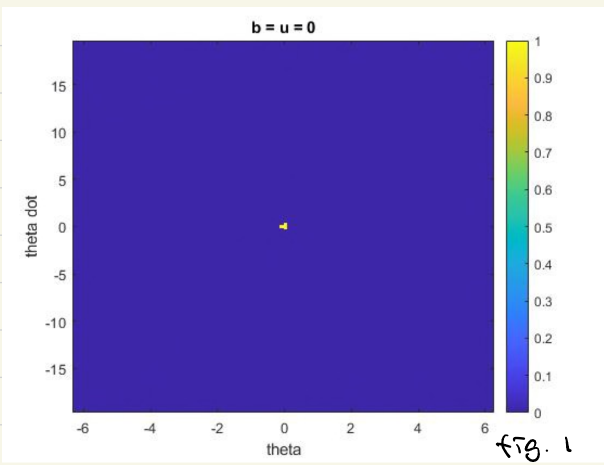



fig. 1

the basin attraction of the fixed point is a point which is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

② $\{ b = 0.25 \quad u = 0 \}$

solve the equation: $\dot{\theta} = 0$

$$\cancel{u}^{\,0} - m \cdot g \cdot l \cdot \sin\theta - \cancel{b}^{\,0}\dot{\theta} = 0$$

∴ fixed point is equal to $\theta$

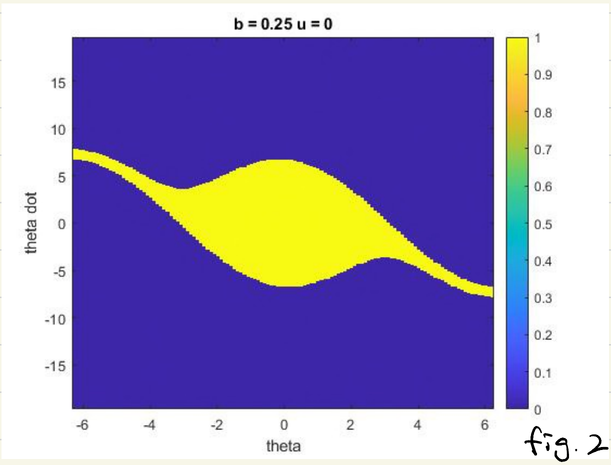



fig. 2

the basin attraction is show in figure 2, yellow region represents converge, purple region re presents diverge

③ $\left\{ b = 0.25 \qquad u = \frac{g}{2l} \right\}$

solve equations $\quad \dot{\Theta} = 0$

$$\frac{g}{2l} - m \cdot g \cdot l \cdot \sin\Theta - b\dot{\Theta} = 0$$

$$\frac{9.8}{2} - 9.8 \cdot \sin\Theta - 0 = 0$$

$$\sin\Theta = \frac{1}{2}$$

$$\Theta = \frac{\pi}{6}, \frac{5}{6}\pi, \frac{7}{6}\pi, \dots$$

$\therefore$ fixed point $\begin{bmatrix} \frac{\pi}{6} \\ 0 \end{bmatrix}$

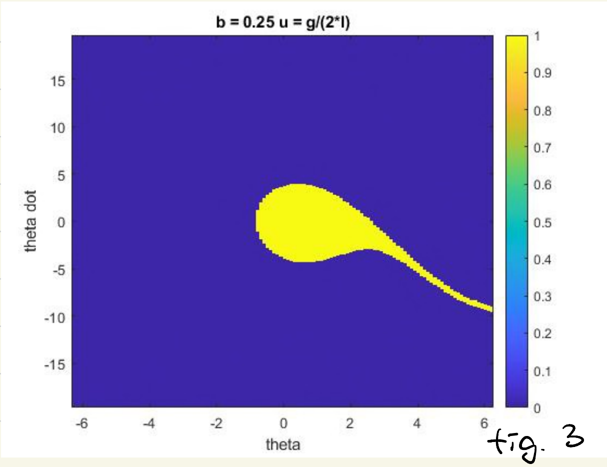
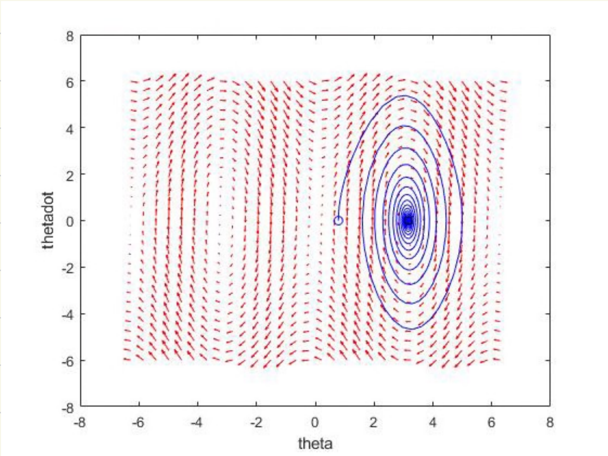


fig. 3

b) $m \ell^2 \ddot{\theta} + m \cdot g \cdot \ell \cdot \sin\theta + b\dot{\theta} = u$

when $\boxed{u = 2 \cdot m \cdot g \cdot \ell \cdot \sin\theta}$

$$m \ell^2 \ddot{\theta} = -b\dot{\theta} + m \cdot g \cdot \ell \cdot \sin\theta$$




From the graph pendulum eventually stables at $[\theta, \dot{\theta}]^T = [\pi, 0]^T$

total energy $E = \int_{t_1}^{t_2} \tau \dot{\theta} \, dt$

$$= \int_{t_1}^{t_2} 2 \cdot m \cdot g \cdot \ell \cdot \sin\theta \cdot \dot{\theta} \, dt$$

$$= \boxed{10.8851}$$

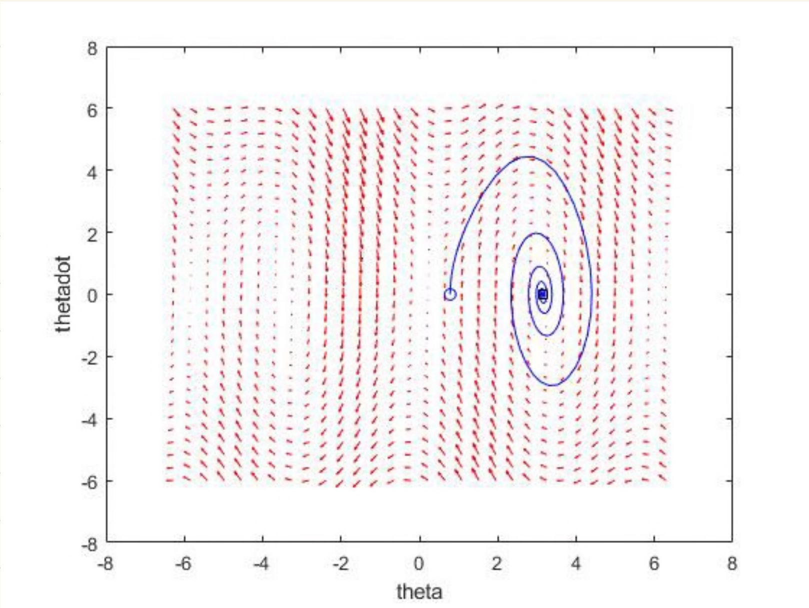
from matlab

c) $m l^2 \ddot{\Theta} + m \cdot g \cdot l \cdot \sin(\Theta) + b\dot{\Theta} = u$

PD control feedback controller

$u = K_d (\Theta - \pi) + K_p (\dot{\Theta})$

where $K_d = -0.5$ $\quad K_d = -0.5$



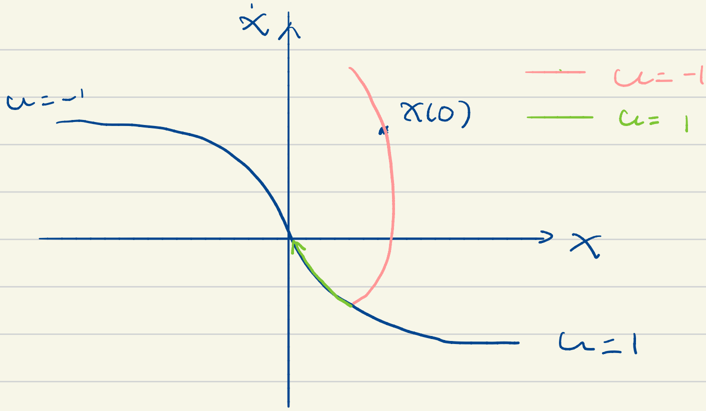

total energy $E = \int_{t_1}^{t_2} \tau \, \dot{\Theta} \, dt$

$= (-0.5 \cdot (\Theta - \pi) - 0.5(\dot{\Theta}))\dot{\Theta}$

$= \boxed{-2.4675}$

3

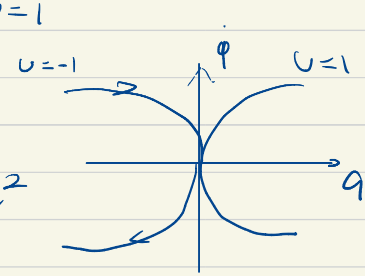
a)




the minimum-time policy is presented above
( check matlab function )

derivation:

the policy is to accelerate then brake ("bang-bang")
in this case: $u=-1$ then $u=1$




$$\ddot{q} = u \qquad u=-1$$

$$\dot{q}(t) = \dot{q}(0) - t$$

$$q(t) = q(0) + t\dot{q}(0) - \frac{1}{2}t^2$$

$u=1$

$$\dot{q}(t) = \dot{q}(0) + t$$

$$q(t) = q(0) + t\dot{q}(0) + \frac{1}{2}t^2$$

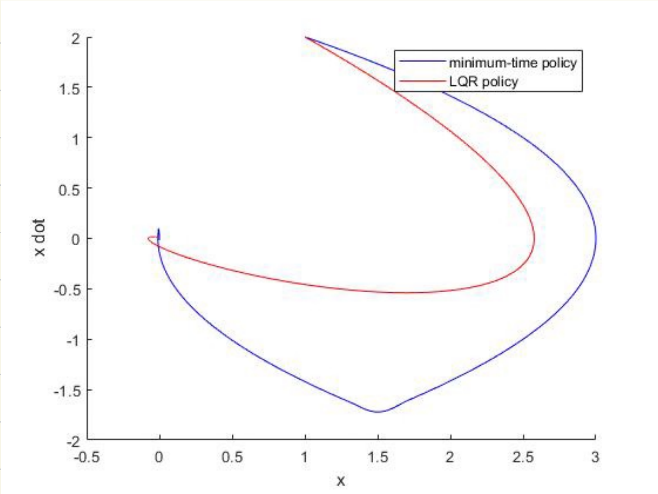
when $x \geq 0$ :
$$\begin{cases} u = -1 & \dot{x} > -|\sqrt{2x}| \\ u = 1 & \dot{x} \leq -|\sqrt{2x}| \end{cases}$$

when $x < 0$ :
$$\begin{cases} u = -1 & \dot{x} > |-\sqrt{2x}| \\ u = 1 & \dot{x} \leq |-\sqrt{2x}| \end{cases}$$

b) Plot two policies in the same graph

$Q = 0.25I$

$R = 5$



c)

using ode 45, time to get to 10.051 within the goal is calculated.

$t = 5.42\ s$ using minimum-time

$t = 14.38\ s$ using LQR

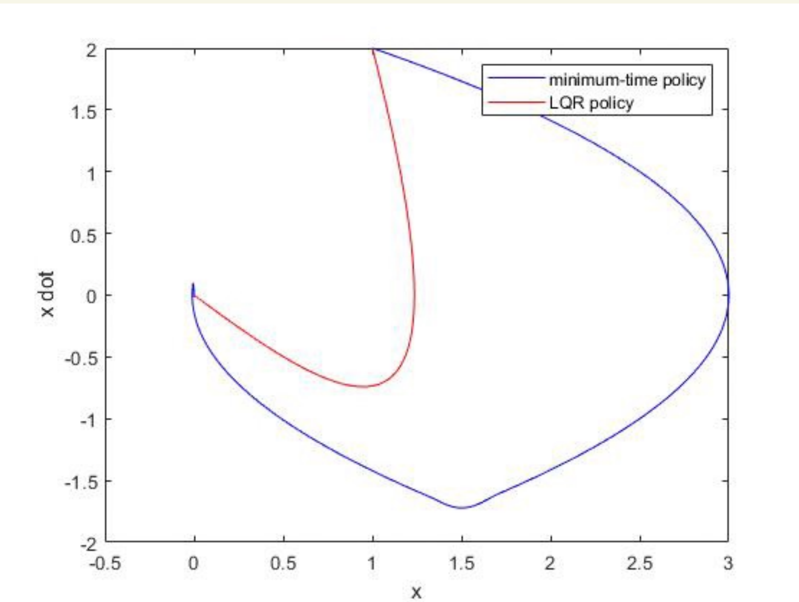
time decreases as Q is increased because the system punishes more when the $x$ is not at the desired location.

time increases as R is increased because the system focuses more on the control input then the states. Thus less input can result in more converge time.

when $Q = 100I$ $R = 10I$

$t = 4.08s$ using LQR

the result make sense because LQR doesn't have constrain on u while minimum-time policy must satisfy $|u| \le 1$. Thus, a higher input can result in less time to converge.




$Q = 100I$

$R = 10$