Model Assessment for Space-time Point Processes Using Super-thinning

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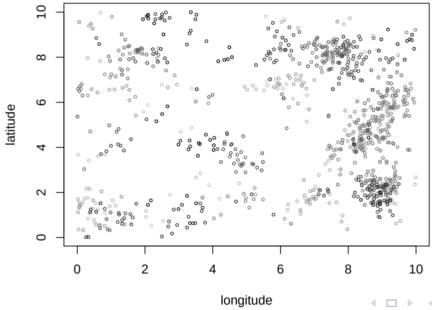
Part I

Background

Point processes

- random collections of points in some metric space
- (locally) σ-finite: finitely many points within any bounded set
- simple: points at distinct locations and times
- examples: earthquakes, wildfires, crime locations, etc.



The conditional intensity

- the conditional intensity $\lambda(x, y, t)$ is defined as the frequency with which events are expected to occur in a space S around a specific point, time and mark (for marked point processes), conditional on the prior history $\mathcal{H}_t$ (Daley and Vere-Jones, 2003)
- λ uniquely characterizes a simple point process

$$\boxed{\lim_{\Delta x, \Delta y, \Delta t \downarrow 0} \frac{E[N\{(x, x+\Delta x) \times (y, y+\Delta y) \times (t, t+\Delta t)\} | \mathcal{H}_t]}{\Delta x \Delta y \Delta t}}$$

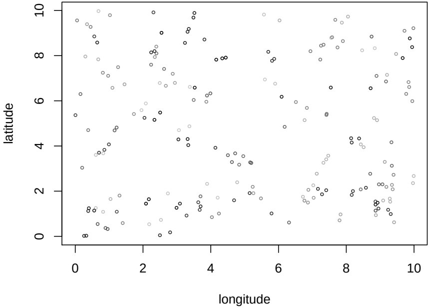
Hawkes process

- useful for modeling earthquakes (Ogata 1988, 1998)
- cluster process
  - background points (immigrants) trigger offspring points (1st generation descendants)
  - offspring points trigger their own offspring points (2nd generation descendants)
- modeled by

$$\lambda(x, y, t) = \mu(x, y, t) + \sum_{\{i:t_i<t\}} g(x - x_i, y - y_i, t - t_i)$$

Hawkes process

$$\lambda(x, y, t) = 0.02 + \sum_{\{i:t_i<t\}} K_0 \frac{\alpha\beta}{\pi} e^{-\alpha t - \beta(x^2+y^2)}$$



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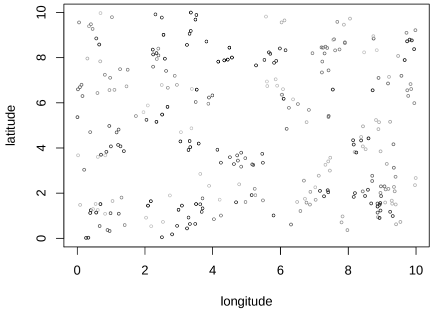

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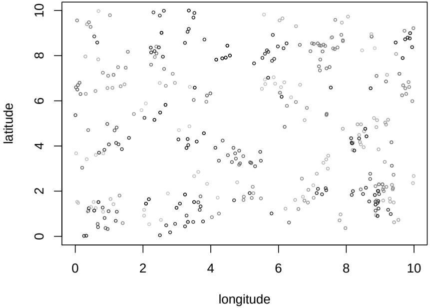

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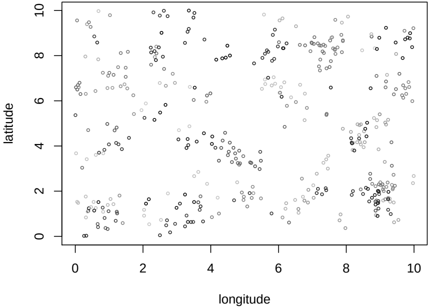

Hawkes process

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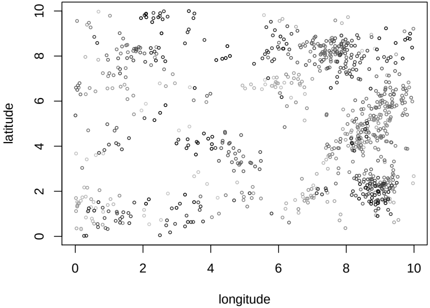


Part II

Residual Analysis

Superposed residuals

- simulate points in S with probability

$$c - \hat{\lambda}(x, y, t) \text{ where } c = \sup\{\hat{\lambda}(x, y, t)\} \text{ over } S$$

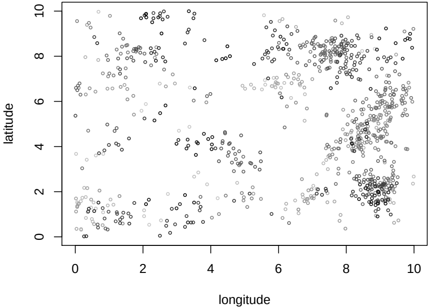
- essentially, superimpose a point process onto the existing point process, with conditional intensity

$$\hat{\lambda}_S(x, y, t) = c - \hat{\lambda}(x, y, t)$$

- the result, called *superposed* residuals, are homogeneous Poisson, with rate c, if fitted model is correct (Brémaud, 1981)

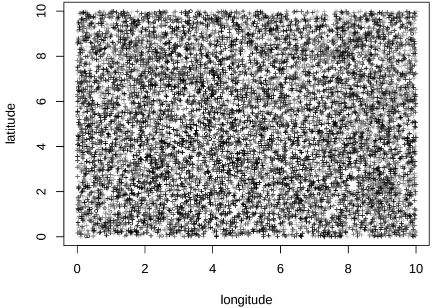
Superposed residuals

$$\hat{\lambda}(x,y,t) = .02 + \sum_{\{i:t_i<t\}} K_0 \frac{\alpha}{2\sigma^2\pi} exp\left\{-\alpha t - \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}}{\sigma}\right)^2\right\}$$



Superposed residuals

$$\hat{\lambda}(x, y, t) = .02 + \sum_{\{i : t_i < t\}} K_0 \frac{\alpha}{2\sigma^2\pi} exp\left\{ -\alpha t - \frac{1}{2}\left(\frac{\sqrt{x^2 + y^2}}{\sigma}\right)^2 \right\}$$



Thinned residuals

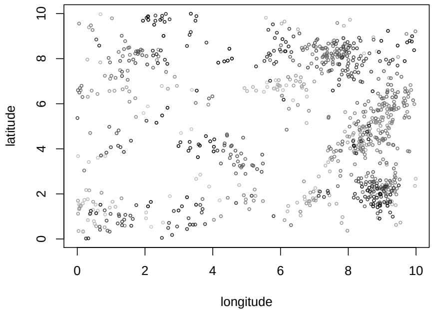
- keep each observation in S with probability

$$\frac{b}{\hat{\lambda}(x_i, y_i, t_i)} \text{ where } b = \inf\{\hat{\lambda}(x, y, t)\} \text{ over } S$$

- remaining points, called *thinned* residuals, are homogeneous Poisson, with rate b, if fitted model is correct (Schoenberg, 2003)

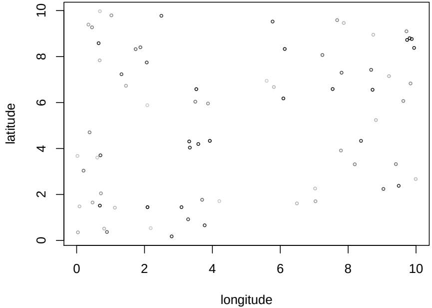
Thinned residuals

$$\hat{\lambda}(x,y,t) = .02 + \sum_{\{i:t_i<t\}} K_0 \frac{\alpha}{2\sigma^2\pi} exp\left\{-\alpha t - \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}}{\sigma}\right)^2\right\}$$



Thinned residuals

$$\hat{\lambda}(x,y,t) = .02 + \sum_{\{i:t_i<t\}} K_0 \frac{\alpha}{2\sigma^2\pi} exp\left\{ -\alpha t - \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}}{\sigma}\right)^2 \right\}$$

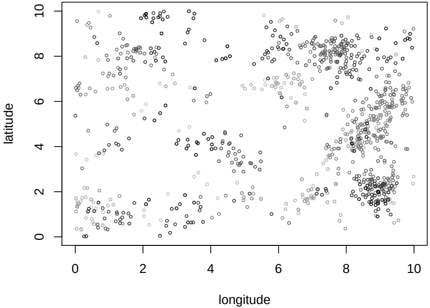


Super-thinned residuals

- thin points if $\hat{\lambda}(x_i, y_i, t_i) \geq k$
  - keep each point with probability $\frac{k}{\hat{\lambda}(x_i, y_i, t_i)}$
- simulate points if $\hat{\lambda}(x, y, t) < k$
  - simulate points with probability $k - \hat{\lambda}(x, y, t)$
- the result, called *super-thinned* residuals, is homogeneous Poisson, with rate k, if fitted model is correct

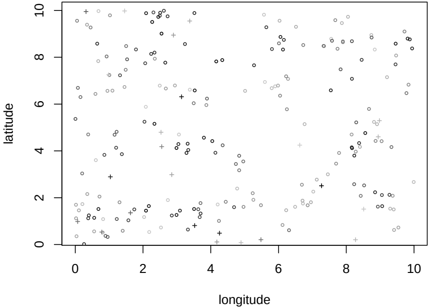
Super-thinned residuals

$$\hat{\lambda}(x,y,t) = .02 + \sum_{\{i:t_i<t\}} K_0 \frac{\alpha}{2\sigma^2\pi} exp\left\{-\alpha t - \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}}{\sigma}\right)^2\right\}$$



Super-thinned residuals

$$k = \frac{250}{|S|} = 0.0833$$

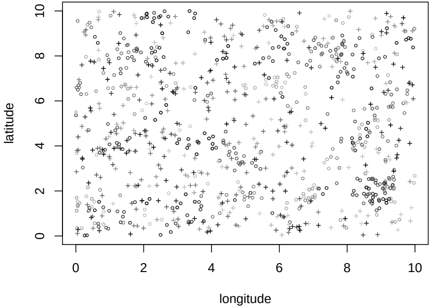


Super-thinned residuals

- some choices of k seem to be more powerful than others
- how should we choose k?
- common sense choices:
  - choose k such that the same number of points are thinned and superposed (the mean of λ)
  - choose k such that the fewest number of points are thinned and superposed (the median of λ)

Super-thinned residuals

$$k = \frac{889}{|S|} = 0.2963$$



Summary

- superposition may result in too many points, which is difficult to interpret
- thinning may result in too few points
- super-thinning is a more powerful alternative due to the tuning parameter, k
- open question: which value of k is most powerful?

Thank you for attending!

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