SHARP PERFORMANCE BOUNDS FOR GRAPH CLUSTERING VIA CONVEX OPTIMIZATION

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ABSTRACT

The problem of finding clusters in a graph arises in several applications such as social networks, data mining and computer networks. A typical, convex optimization-approach, that is often adopted is to identify a sparse plus low-rank decomposition of the adjacency matrix of the graph, with the (dense) low-rank component representing the clusters. In this paper, we sharply characterize the conditions for successfully identifying clusters using this approach. In particular, we introduce the "effective density" of a cluster that measures its significance and we find explicit upper and lower bounds on the minimum effective density that demarcates regions of success or failure of this technique. Our conditions are in terms of (a) the size of the clusters, (b) the denseness of the graph, and (c) regularization parameter of the convex program. We also present extensive simulations that corroborate our theoretical findings.

Index Terms— Graph clustering, low rank plus sparse, convex optimization, thresholds.

1. INTRODUCTION

Given an unweighted graph, finding nodes that are well-connected with each other is a very useful problem with applications in social networks [1–3], data mining [4, 5], bioinformatics [6, 7], computer networks, sensor networks and so on. Different versions of this problem have been studied as graph clustering [8–11], correlation clustering [12–15], graph partitioning on planted partition model [16–19] etc. Developments in convex optimization techniques to recover low-rank matrices [20–24] via nuclear norm minimization has recently led to the development of several convex algorithms to recover clusters in a graph [25–32].

Let us assume that a given graph has dense clusters; we can look at its adjacency matrix as a low-rank matrix with sparse noise. That is, the graph can be viewed as a union of cliques with some edges missing inside the cliques and extra edges between the cliques. Our aim is to recover the low-rank matrix since it is equivalent to finding clusters. In this paper, we will look at the following well known convex program which decomposes the adjacency matrix ($\mathbf{A}$) as the sum of a low-rank ($\mathbf{L}$) and a sparse ($\mathbf{S}$) component.

$$\underset{\mathbf{L},\mathbf{S}}{\text{minimize}} \ \|\mathbf{L}\|_\star + \lambda\|\mathbf{S}\|_1 \tag{1.1}$$

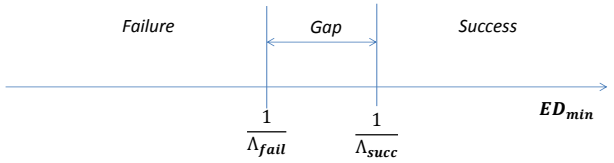
subject to

$$1 \geq \mathbf{L}_{i,j} \geq 0 \ \text{ for all } i,j \in \{1,2,\dots n\} \tag{1.2}$$

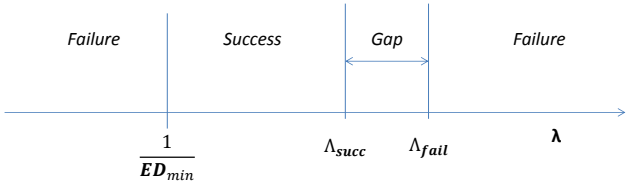
$$\mathbf{L} + \mathbf{S} = \mathbf{A} \tag{1.3}$$

where $\lambda > 0$ is a regularization parameter. $\|\mathbf{X}\|_\star$ and $\|\mathbf{X}\|_1$ denote the nuclear norm (maximum singular value) and the l_1-norm (sum of absolute values of all entries) respectively of the matrix $\mathbf{X}$. This program is very intuitive and requires the knowledge of only the adjacency matrix. Program 1.1 has been proposed in many works [28–30].

We consider the popular *stochastic block model* (also called the planted partition model) for the graph. Under



(a) Feasibility of Program 1.1 in terms of the minimum effective density (ED_{min}).



(b) Feasibility of Program 1.1 in terms of the regularization parameter (λ).

Fig. 1: Characterization of the feasibility of Program (1.1) in terms of the minimum effective density and the value of the regularization parameter. The feasibility is determined by the values of these parameters in comparison with two constants Λ_{succ} and Λ_{fail}, derived in Theorem 1 and Theorem 2. The thresholds guaranteeing the success or failure of Program 1.1 derived in this paper are fairly close to each other.

This work was supported in part by the National Science Foundation under grants CCF-0729203, CNS-0932428 and CIF-1018927, by the Office of Naval Research under the MURI grant N00014-08-1-0747, and by a grant from Qualcomm Inc. The first author is also supported by the Schlumberger Foundation Faculty for the Future Program Grant.

this model of generating random graphs, the existence of an edge between any pair of vertices is independent of the other edges. The probability of the existence of an edge is identical within any individual cluster, but may vary across clusters. One may think of this as a heterogeneous form of the Erdos-Renyi model. We characterize the conditions under which Program (1.1) can successfully recover the correct clustering, and when it cannot. Our analysis reveals the dependence of its success on a metric that we term the *minimum effective density* of the graph. While defined more formally later in the paper, in a nutshell, the minimum effective density of a random graph tries to capture the density of edges in the sparsest cluster. We derive explicit upper and lower bounds on the value of this metric that determine the success or failure of Program 1.1 (as illustrated in Fig. 1a).

A second contribution of this paper is to explicitly characterize the efficacy of Program 1.1 with respect to the regularization parameter λ. We obtain bounds on the values of λ that permit the recovery of the clusters, or those that necessitate Program 1.1 to fail (as illustrated in Fig. 1b). Our results thus lead to a more principled approach towards the choice of the regularization parameter for the problem at hand.

Most of the convex algorithms proposed for graph clustering, for example, the recent works by Xu et al. [25], Ames and Vavasis [26, 27], Jalali et al. [28], Oymak and Hassibi [29], Chen et al. [30], Ames [31], Ailon et al. [32] are variants of Program 1.1. These results show that planted clusters can be identified via tractable convex programs as long as the cluster size is proportional to the square-root of the size of the adjacency matrix. However, the exact requirements on the cluster size are not known. In this work, we find sharp bounds for identifiability as a function of cluster sizes, inter cluster density and intra cluster density. To the best of our knowledge, this is the first explicit characterization of the feasibility of the convex optimization based approach 1.1 towards this problem.

The rest of the paper is organized as follows. Section 2 formally introduces the model considered in this paper. Section 3 presents the main results of the paper: an analytical characterization of the feasibility of the low rank plus sparse based approximation for identifying clusters. Section 4 presents simulations that corroborate our theoretical results. Finally, the Appendix contains an outline of the proofs of the theoretical results presented in the paper.

2. MODEL

For any positive integer m, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. Let $\mathcal{G}$ be an unweighted graph on n nodes, $[n]$, with K disjoint (dense) clusters. Let $\mathcal{C}_i$ denote the set of nodes in the i^{th} cluster. Let n_i denote the size of the i^{th} cluster, i.e., the number of nodes in $\mathcal{C}_i$. We shall term the set of nodes that do not fall in any of these K clusters as *outliers* and denote them as $\mathcal{C}_{K+1} := [n] - \bigcup_{i=1}^{K} \mathcal{C}_i$. The number of outliers is thus $n_{K+1} := n - \sum_{i=1}^{K} n_i$. Since the clusters are assumed to be disjoint, we have $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for all $i, j \in [n]$.

Let $\mathcal{R}$ be the region corresponding to the union of regions induced by the clusters, i.e., $\mathcal{R} = \bigcup_{i=1}^{K} \mathcal{C}_i \times \mathcal{C}_i \subseteq [n] \times [n]$. So, $\mathcal{R}^c = [n] \times [n] - \mathcal{R}$ is the region corresponding to out of cluster regions. Note that $|\mathcal{R}| = \sum_{i=1}^{K} n_i^2$ and $|\mathcal{R}^c| = n^2 - \sum_{i=1}^{K} n_i^2$. Let $n_{min} := \min_{1 \le i \le K} n_i$.

Let $\mathbf{A} = \mathbf{A}^T$ denote the adjacency matrix of the graph $\mathcal{G}$. The diagonal entries of $\mathbf{A}$ are 1. We use the following probabilistic model. We consider a more general version of the popular stochastic block model [16, 33].

Definition 2.1 (Stochastic Block Model)**.** *Let $\{p_i\}_{i=1}^{K}, q$ be constants between 0 and 1. Then, a random graph $\mathcal{G}$, generated according to stochastic block model, has the following adjacency matrix. Entries of $\mathbf{A}$ on the lower triangular part are independent random variables and for any $i > j$:*

$$\mathbf{A}_{i,j} = \begin{cases} \textit{Bernoulli}(p_l) & \textit{if both } \{i,j\} \in \mathcal{C}_l \textit{ for some } l \le K \\ \textit{Bernoulli}(q) & \textit{otherwise.} \end{cases} \tag{2.1}$$

So, an edge inside i^{th} cluster exists with probability p_i and an edge outside clusters exists with probability q. Let $p_{min} := \min_{1 \le i \le K} p_i$. We assume that the clusters are dense and the density of edges inside clusters is greater than outside, i.e., $p_{\min} > \frac{1}{2} > q > 0$. We note that the program 1.1 does not require the knowledge of $\{p_i\}_{i=1}^{K}, q$ or K, and uses only the adjacency matrix $\mathbf{A}$ for its operation.

3. MAIN RESULTS

The desired solution to Program 1.1 is $(\mathbf{L}^0, \mathbf{S}^0)$ where $\mathbf{L}^0$ corresponds to the full cliques, when missing edges inside $\mathcal{R}$ are completed, and $\mathbf{S}^0$ corresponds to the missing edges and the extra edges between the clusters. In particular we want:

$$\mathbf{L}^0_{i,j} = \begin{cases} 1 & \text{if both } \{i,j\} \in \mathcal{C}_l \text{ for some } l \le K, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

$$\mathbf{S}^0_{i,j} = \begin{cases} -1 & \text{if both } \{i,j\} \in \mathcal{C}_l \text{ for some } l \le K, \text{ and } \mathbf{A}_{i,j} = 0, \\ 1 & \text{if } \{i,j\} \text{ are not in the same cluster and } \mathbf{A}_{i,j} = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

It is easy to see that the $(\mathbf{L}^0, \mathbf{S}^0)$ pair is feasible. We say that Program 1.1 *succeeds* when $(\mathbf{L}^0, \mathbf{S}^0)$ is the optimal solution to Program 1.1. In this section we present two theorems which give the conditions under which Program 1.1 succeeds or fails.

The following definitions are critical to our results.

- Define $\mathbf{D}_i := n_i\,(2p_i - 1)$ as the effective density of cluster $\mathcal{C}_i$ and $\mathbf{D}_{\min} = \min_{1 \le i \le K} \mathbf{D}_i$.

- Let $\gamma_{\text{succ}} := \max_{1\le i\le K} 4\sqrt{(q(1-q)+p_i(1-p_i))n_i}$, $\gamma_{\text{fail}} := \sum_{i=1}^{K} \frac{n_i^2}{n}$
- $\Lambda_{\text{fail}} := \frac{1}{\sqrt{q(n-\gamma_{\text{fail}})}}$ and $\Lambda_{\text{succ}} := \frac{1}{4\sqrt{q(1-q)n}+\gamma_{\text{succ}}}$.

Theorem 1. *Let $\mathcal{G}$ be a random graph generated according to the stochastic block model 2.1 with K clusters of sizes $\{n_i\}_{i=1}^K$ and probabilities $\{p_i\}_{i=1}^K$ and q, such that $p_{min} > \frac{1}{2} > q > 0$. Given $\epsilon > 0$, there exists positive constants δ, c_1, c_2 such that,*

1. *Whenever $\mathbf{D}_{\min} \ge (1+\epsilon)\Lambda_{succ}^{-1}$, for $\lambda = (1-\delta)\Lambda_{succ}$, Program (1.1) succeeds with probability $1-c_1 n^2 \exp(-c_2 n_{\min})$.*
2. *For any given $\lambda \ge 0$, if $\mathbf{D}_{\min} \le (1-\epsilon)\Lambda_{fail}^{-1}$ then Program (1.1) fails with probability $1 - c_1 \exp(-c_2|\mathcal{R}^c|)$.*

Theorem 2. *Let $\mathcal{G}$ be a random graph generated according to the stochastic block model 2.1 with K clusters of sizes $\{n_i\}_{i=1}^K$ and probabilities $\{p_i\}_{i=1}^K$ and q, such that $p_{min} > \frac{1}{2} > q > 0$. Given $\epsilon > 0$, there exists positive constants c_1', c_2' such that,*

1. *If $\lambda \ge (1+\epsilon)\Lambda_{fail}$, then Program (1.1) fails with probability $1 - c_1' \exp(-c_2'|\mathcal{R}^c|)$.*
2. *If $\lambda \le (1-\epsilon)\Lambda_{succ}$ then,*
   - *If $\mathbf{D}_{\min} \le (1-\epsilon)\frac{1}{\lambda}$, then Program (1.1) fails with probability $1 - c_1' \exp(-c_2' n_{\min})$.*
   - *If $\mathbf{D}_{\min} \ge (1+\epsilon)\frac{1}{\lambda}$, then Program (1.1) succeeds with probability $1 - c_1' n^2 \exp(-c_2' n_{\min})$.*

We see that the minimum effective density $\mathbf{D}_{\min}, \Lambda_{\text{succ}}$ and Λ_{fail} play a fundamental role in determining the success of Program 1.1. Theorem 1 gives criteria for the inherent success of Program 1.1 whereas theorem 2 characterizes the conditions for the success of Program 1.1 as a function of the regularization parameter λ. We illustrate these results in Figures 1a and 1b.

Sharp performance bounds: From the forward and converse, we see that there is a gap between Λ_{fail} and Λ_{succ}. The gap is $\frac{\Lambda_{\text{fail}}}{\Lambda_{\text{succ}}} = \frac{4\sqrt{q(1-q)n}+\gamma_{\text{succ}}}{\sqrt{q(n-\gamma_{\text{fail}})}}$ times. In the small cluster regime where $\max_{1\le i\le K} n_i = o(n)$ and $\sum_{i=1}^K n_i^2 = o(n^2)$, the ratio $\frac{\Lambda_{\text{fail}}}{\Lambda_{\text{succ}}}$ takes an extremely simple form as we have $\gamma_{\text{fail}} \ll n$ and $\gamma_{\text{succ}} \ll \sqrt{n}$. In particular, $\frac{\Lambda_{\text{fail}}}{\Lambda_{\text{succ}}} = 4\sqrt{1-q}+o(1)$, which is at most 4 times in the worst case.

4. SIMULATIONS

We implement Program 1.1 using the inexact augmented Lagrangian multiplier method algorithm by Li et al. [34]. We note that this algorithm solves the program approximately.

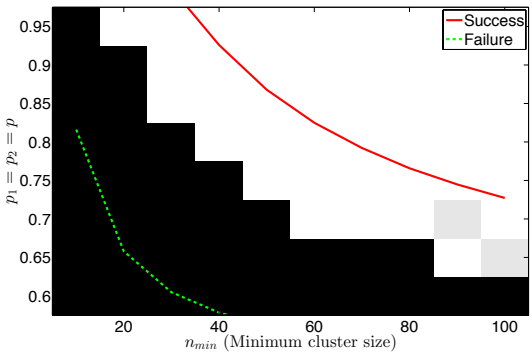


Fig. 2: Simulation results showing the region of success (white region) and failure (black region) of Program 1.1 with $\lambda = 0.99\Lambda_{\text{succ}}$. Also depicted are the thresholds for success (solid red curve on the top-right) and failure (dashed green curve on the bottom-left) predicted by Theorem 1.

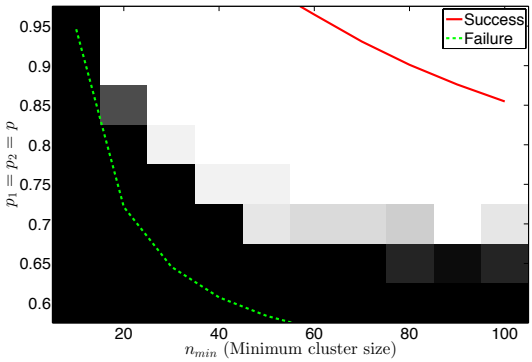


Fig. 3: Simulation results showing the region of success (white region) and failure (black region) of Program 1.1 with $\lambda = 2\mathbf{D}_{\min}^{-1}$. Also depicted are the thresholds for success (solid red curve on the top-right) and failure (dashed green curve on the bottom-left) predicted by Theorem 2.

Moreover, numerical imprecision prevents the output of the algorithm from being strictly 1 or 0. Hence we round each entry to 1 or 0 by comparing it with the mean of all entries of the output. In other words, if an entry is greater than the mean, we round it to 1 and to 0 otherwise. We declare success if the number of entries that are wrong in the rounded output compared to $\mathbf{L}^0$ (recall from 3.1) is less than 0.1%.

We consider the set up with $n = 200$ nodes and two clusters of equal sizes, $n_1 = n_2$. We vary the cluster sizes from 10 to 100 in steps of 10. We fix $q = 0.1$ and vary the probability of edge inside clusters $p_1 = p_2 = p$ from 0.6 to 0.95 in steps of 0.5. We run the experiments 20 times and average the over the outcomes. In the first set of experiments, we run the program with $\lambda = 0.99\Lambda_{succ}$ which ensures that $\lambda < \Lambda_{\text{succ}}$. Figure 2 shows the region of success (white region) and failure (black region) for this experiment. From Theorem 1, we expect the program to succeed when $\mathbf{D}_{\min} > \Lambda_{\text{succ}}^{-1}$ which is

the region above the solid red curve in Figure 2 and fail when $\mathbf{D}_{\min} < \Lambda_{\text{fail}}^{-1}$ which is the region below the dashed green curve in Figure 2.

In the second set of experiments, we run the program with $\lambda = \frac{2}{\mathbf{D}_{\min}}$. This ensures that $\mathbf{D}_{\min} > \frac{1}{\lambda}$. Figure 3 shows the region of success (white region) and failure (black region) for this experiment. From Theorem 2, we expect the program to succeed when $\lambda < \Lambda_{\text{succ}}$ which is the region above the solid red curve in Figure 3 and fail when $\lambda > \Lambda_{\text{fail}}$ which is the region below the dashed green curve in Figure 3.

We see that the transition indeed happens between the solid red curve and the dashed green curve in both Figure 2 and Figure 3 as predicted by Theorem 1 and Theorem 2 respectively.

5. DISCUSSION AND CONCLUSION

We provided sharp analysis of Program 1.1 which is commonly used to identify clusters in a graph and more generally, to decompose a matrix into low-rank and sparse components. We believe, our technique can be extended to tightly analyze variants of this approach. As a future work, we are looking at the extensions of Problem 1.1 where the adjacency matrix $\mathbf{A}$ is partially observed and also modifying Program 1.1 for clustering weighted graphs, where the adjacency matrix $\mathbf{A}$ with $\{0, 1\}$-entries is replaced by a similarity matrix with real entries.

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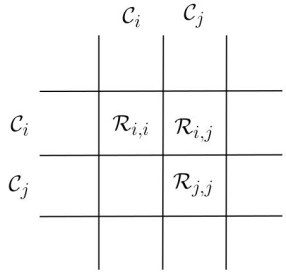

Fig. 4: Illustration of $\{\mathcal{R}_{i,j}\}$ dividing $[n] \times [n]$ into disjoint regions similar to a grid

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7. APPENDIX: PROOF

The theorems in the Section 3, provide the conditions under which Program 1.1 succeeds and when it fails. In this section, we provide the proof of these results. We first provide the proof for the conditions for the success in the section 7.1, and then in the section 7.2, we go on to prove the conditions for the failure. Later, in the section 7.3 we combine these results to conclude the results presented in the Theorem 1 and Theorem 2.

Before we proceed, we need some additional notations. Let $\mathcal{R}_{i,j} = \mathcal{C}_i \times \mathcal{C}_j$ for $1 \leq i, j \leq K+1$. One can see that $\{\mathcal{R}_{i,j}\}$ divides $[n] \times [n]$ into $(K+1)^2$ disjoint regions similar to a grid which is illustrated in the figure 4. Thus, $\mathcal{R}_{i,i}$ is the region induced by i'th cluster for any $i \leq K$.

Let $\mathcal{A} \subseteq [n] \times [n]$ be the set of nonzero coordinates of $\mathbf{A}$. Then the sets,

1. $\mathcal{A} \cap \mathcal{R}$ corresponds to the edges inside the clusters.
2. $\mathcal{A}^c \cap \mathcal{R}$ corresponds to the missing edges inside the clusters.
3. $\mathcal{A} \cap \mathcal{R}^c$ corresponds to the set of edges outside the clusters, which should be ideally not present.

Let c and d be positive integers. Consider a matrix, $\mathbf{X} \in \mathbb{R}^{c \times d}$. Let β be a subset of $[c] \times [d]$. Then, let $\mathbf{X}_\beta$ denote the matrix induced by the entries of $\mathbf{X}$ on β i.e.,

$$(\mathbf{X}_\beta)_{i,j} = \begin{cases} \mathbf{X}_{i,j} & \text{if } (i,j) \in \beta \\ 0 & \text{otherwise .} \end{cases}$$

In other words, $\mathbf{X}_\beta$ is a matrix whose entries match those of $\mathbf{X}$ in the positions $(i,j) \in \beta$ and zero otherwise. For example, $\mathbb{1}_{\mathcal{A}}^{n \times n} = \mathbf{A}$.

7.1. Conditions for Success

In order to show that $(\mathbf{L}^0, \mathbf{S}^0)$ is the unique optimal solution to the program 1.1, we need to prove the following,

$$(\|\mathbf{L}^0 + \mathbf{E}^L\|_\star + \lambda \|\mathbf{S}^0 + \mathbf{E}^S\|_1) - (\|\mathbf{L}^0\|_\star + \lambda \|\mathbf{S}^0\|_1) > 0, \tag{7.1}$$

for all feasible perturbations $(\mathbf{E}^L, \mathbf{E}^S)$.

We can lower bound the LHS of the equation 7.1 using the subgradients as follows,

$$\begin{aligned} &\left(\|\mathbf{L}^0 + \mathbf{E}^L\|_\star + \lambda \|\mathbf{S}^0 + \mathbf{E}^S\|_1\right) - \left(\|\mathbf{L}^0\|_\star + \lambda \|\mathbf{S}^0\|_1\right) \\ &\geq \langle \partial\|\mathbf{L}^0\|_\star, \mathbf{E}^L\rangle + \lambda\langle \partial\|\mathbf{S}^0\|_1, \mathbf{E}^S\rangle, \end{aligned} \tag{7.2}$$

where $\partial\|\mathbf{L}^0\|_\star$ and $\partial\|\mathbf{S}^0\|_1$ are subgradients of nuclear norm and $l1$-norm respectively at the point $\left(\mathbf{L}^0, \mathbf{S}^0\right)$.

Let $\mathbf{L}^0 = \mathbf{U}\Lambda\mathbf{U}^T$, where $\Lambda = \text{diag}\{n_1, n_2, \ldots, n_K\}$ and $\mathbf{U} = [\mathbf{u}_1 \ \ldots \ \mathbf{u}_K] \in \mathbb{R}^{n \times K}$, with

$$\mathbf{u}_{l,i} = \begin{cases} \frac{1}{\sqrt{n_l}} & \text{if } i \in \mathcal{C}_l \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

Then subgradient $\partial\|\mathbf{L}^0\|_\star$ is of the form $\mathbf{U}\mathbf{U}^T + \mathbf{W}$ such that $\mathbf{W} \in \mathcal{M}_U := \{\mathbf{X} : \mathbf{X}\mathbf{U} = \mathbf{U}^T\mathbf{X} = 0, \|\mathbf{X}\| \leq 1\}$. The subgradient $\partial\|\mathbf{S}^0\|_1$ is of the form $\text{sign}(\mathbf{S}^0) + \mathbf{Q}$ where $\mathbf{Q}_{i,j} = 0$ if $\mathbf{S}^0_{i,j} \neq 0$ and $\|\mathbf{Q}\|_\infty \leq 1$. We note that since $\mathbf{L} + \mathbf{S} = \mathbf{A}$, $\mathbf{E}^L = -\mathbf{E}^S$. Note that $\text{sign}(\mathbf{S}^0) = \mathbb{1}_{\mathcal{A} \cap \mathcal{R}^c}^{n \times n} - \mathbb{1}_{\mathcal{A}^c \cap \mathcal{R}}^{n \times n}$. Choosing $\mathbf{Q} = \mathbb{1}_{\mathcal{A} \cap \mathcal{R}}^{n \times n} - \mathbb{1}_{\mathcal{A}^c \cap \mathcal{R}^c}^{n \times n}$, we get,

$$\begin{aligned} &\|\mathbf{L}^0 + \mathbf{E}^L\|_\star + \lambda \|\mathbf{S}^0 + \mathbf{E}^S\|_1 - (\|\mathbf{L}^0\|_\star + \lambda \|\mathbf{S}^0\|_1) \\ &\geq \langle \partial\|\mathbf{L}^0\|_\star, \mathbf{E}^L\rangle + \lambda\langle \partial\|\mathbf{S}^0\|_1, \mathbf{E}^S\rangle \\ &= \langle \mathbf{U}\mathbf{U}^T + \mathbf{W}, \mathbf{E}^L\rangle + \lambda\langle \text{sign}(\mathbf{S}^0) + \mathbf{Q}, \mathbf{E}^S\rangle \\ &= \underbrace{\sum_{i=1}^{K} \frac{1}{n_i} \text{sum}(\mathbf{E}_{R_{i,i}}) + \lambda \left(\text{sum}(\mathbf{E}^L_{\mathcal{A}^c}) - \text{sum}(\mathbf{E}^L_{\mathcal{A}})\right)}_{:= g(\mathbf{E}^L)} \\ &+ \left\langle \mathbf{W}, \mathbf{E}^L \right\rangle. \end{aligned} \tag{7.4}$$

Define,

$$g(\mathbf{E}^L) := \sum_{i=1}^{K} \frac{1}{n_i} \text{sum}(\mathbf{E}^L_{\mathcal{R}_{i,i}}) + \lambda \text{sum}(\mathbf{E}^L_{\mathcal{A}^c}). \tag{7.5}$$

Also, define $f\left(\mathbf{E}^L, \mathbf{W}\right) := g\left(\mathbf{E}^L\right) + \left\langle \mathbf{W}, \mathbf{E}^L \right\rangle$. So, our aim is to show that for all feasible perturbations $\mathbf{E}^L$, there exists $\mathbf{W}$ such that,

$$f\left(\mathbf{E}^L, \mathbf{W}\right) = g\left(\mathbf{E}^L\right) + \left\langle \mathbf{W}, \mathbf{E}^L \right\rangle > 0. \tag{7.6}$$

Note that $g\left(\mathbf{E}^L\right)$ does not depend on $\mathbf{W}$.

Lemma 7.1. *Given $\mathbf{E}^L$, assume there exists $\mathbf{W} \in \mathcal{M}_{\mathbf{U}}$ with $\|\mathbf{W}\| < 1$ such that $f(\mathbf{E}^L, \mathbf{W}) \geq 0$. Then at least one of the followings holds:*

- *There exists $\mathbf{W}^* \in \mathcal{M}_{\mathbf{U}}$ with $\|\mathbf{W}^*\| \leq 1$ and $f(\mathbf{E}^L, \mathbf{W}^*) > 0$.*
- *For all $\mathbf{W} \in \mathcal{M}_{\mathbf{U}}$, $\left\langle \mathbf{E}^L, \mathbf{W} \right\rangle = 0$.*

Proof. Let $c = 1 - \|\mathbf{W}\|$. Assume $\left\langle \mathbf{E}^L, \mathbf{W}' \right\rangle \neq 0$ for some $\mathbf{W}' \in \mathcal{M}_{\mathbf{U}}$. Since $\left\langle \mathbf{E}^L, \mathbf{W}' \right\rangle$ is linear in $\mathbf{W}'$, WLOG, let $\left\langle \mathbf{E}^L, \mathbf{W}' \right\rangle > 0$, $\|\mathbf{W}'\| =$. Then choose $\mathbf{W}^* = \mathbf{W} + c\mathbf{W}'$. Clearly, $\|\mathbf{W}^*\| \leq 1$, $\mathbf{W}^* \in \mathcal{M}_{\mathbf{U}}$ and

$$\begin{aligned} f(\mathbf{E}^L, \mathbf{W}^*) &= f(\mathbf{E}^L, \mathbf{W}) + \left\langle \mathbf{E}^L, c\mathbf{W}' \right\rangle \\ &> f(\mathbf{E}^L, \mathbf{W}) \geq 0 \end{aligned} \tag{7.7}$$

■

Notice that, for all $\mathbf{W} \in \mathcal{M}_{\mathbf{U}}$, $\left\langle \mathbf{E}^L, \mathbf{W} \right\rangle = 0$ is equivalent to $\mathbf{E}^L \in \mathcal{M}_{\mathbf{U}}^{\perp}$ which is the orthogonal complement of $\mathcal{M}_{\mathbf{U}}$ in $\mathbb{R}^{n \times n}$. $\mathcal{M}_{\mathbf{U}}^{\perp}$ has the following simple characterization:

$$\begin{aligned} \mathcal{M}_{\mathbf{U}}^{\perp} = \{\mathbf{X} \in \mathbb{R}^{n \times n} : \mathbf{X} = \mathbf{U}\mathbf{M}^T + \mathbf{N}\mathbf{U}^T \\ \text{for some } \mathbf{M}, \mathbf{N} \in \mathbb{R}^{n \times K}\}. \end{aligned} \tag{7.8}$$

Now we have broken down our aim into two steps.

1. Construct $\mathbf{W} \in \mathcal{M}_{\mathbf{U}}$ with $\|\mathbf{W}\| < 1$, such that $f(\mathbf{E}^L, \mathbf{W}) \geq 0$ for all feasible perturbations $\mathbf{E}^L$.
2. For all non-zero feasible $\mathbf{E}^L \in \mathcal{M}_{\mathbf{U}}^{\perp}$, show that $g(\mathbf{E}^L) > 0$.

As a first step, in the section 7.1.1, we will argue that, under certain conditions, there exists a $\mathbf{W} \in \mathcal{M}_{\mathbf{U}}$ with $\|\mathbf{W}\| < 1$ such that with high probability, $f(\mathbf{E}^L, \mathbf{W}) \geq 0$ for all feasible $\mathbf{E}^L$. This $\mathbf{W}$ is called the dual certificate. Secondly, in section 7.1.2, we will show that, under certain conditions, for all $\mathbf{E}^L \in \mathcal{M}_{\mathbf{U}}^{\perp}$ with high probability. $g(\mathbf{E}^L) > 0$. Finally, combining these two arguments, and using Lemma 7.1 we will conclude that $(\mathbf{L}^0, \mathbf{S}^0)$ is the unique optimal with high probability.

7.1.1. Showing existence of the dual certificate

Recall that

$$\begin{aligned} f(\mathbf{E}^L, \mathbf{W}) &= \sum_{i=1}^{K} \frac{1}{n_i} \text{sum}(\mathbf{E}^L_{\mathcal{R}_{i,i}}) + \left\langle \mathbf{E}^L, \mathbf{W} \right\rangle \\ &\quad + \lambda \left(\text{sum}\left(\mathbf{E}^L_{\mathcal{A}^c}\right) - \text{sum}\left(\mathbf{E}^L_{\mathcal{A}}\right)\right) \end{aligned}$$

$\mathbf{W}$ will be constructed from the candidate $\mathbf{W}_0$ as follows.

***Candidate* $\mathbf{W}_0$:** Based on Program 1.1, we propose the following,

$$\mathbf{W}_0 = \sum_{i=1}^{K} c_i \mathbb{1}^{n \times n}_{\mathcal{R}_{i,i}} + c\mathbb{1}^{n \times n}_{\mathcal{R}^c} + \lambda \left(\mathbb{1}^{n \times n}_{\mathcal{A}} - \mathbb{1}^{n \times n}_{\mathcal{A}^c}\right), \tag{7.9}$$

where $\{c_i\}_{i=1}^K, c$ are variables.

We now have to find a bound on spectral norm of $\mathbf{W}_0$. Note that $\mathbf{W}_0$ is a random matrix where randomness is due to $\mathcal{A}$. In order to ensure a small spectral norm we set its expectation to 0.

Expectation of an entry of $\mathbf{W}_0$ on $\mathcal{R}_{i,i}$ (region corresponding to cluster i) and $\mathcal{R}^c$ (region outside the clusters) is $c_i + \lambda(2p_i - 1)$ and $c + \lambda(2q - 1)$ respectively. Hence, we set,

$$c_i = -\lambda(2p_i - 1) \quad \text{and} \quad c = -\lambda(2q - 1),$$

and $\mathbf{W}_0$ and f take the following forms:

$$\begin{aligned} \mathbf{W}_0 &= 2\lambda \left[\sum_{i=1}^{K} (1 - p_i)\ \mathbb{1}^{n \times n}_{\mathcal{R}_{i,i} \cap \mathcal{A}} - p_i\ \mathbb{1}^{n \times n}_{\mathcal{R}_{i,i} \cap \mathcal{A}^c}\right] \\ &\quad + 2\lambda \left[(1 - q)\ \mathbb{1}^{n \times n}_{\mathcal{R}^c \cap \mathcal{A}} - q\ \mathbb{1}^{n \times n}_{\mathcal{R}^c \cap \mathcal{A}^c}\right] \end{aligned} \tag{7.10}$$

$$\begin{aligned} f(\mathbf{E}^L, \mathbf{W}_0) &= \lambda \left[(1 - 2q)\ \text{sum}(\mathbf{E}^L_{\mathcal{R}^c})\right] \\ &\quad - \lambda \left[\sum_{i=1}^{K} \left(2p_i - 1 - \frac{1}{\lambda n_i}\right)\ \text{sum}(\mathbf{E}^L_{\mathcal{R}_{i,i}})\right] \end{aligned} \tag{7.11}$$

From $\mathbf{L}^0$ and (1.2) it follows that

$$\begin{aligned} &\mathbf{E}^L_{\mathcal{R}^c} \text{ is (entrywise) nonnegative.} \\ &\mathbf{E}^L_{\mathcal{R}} \text{ is (entrywise) nonpositive.} \end{aligned} \tag{7.12}$$

Thus, $\text{sum}(\mathbf{E}^L_{\mathcal{R}^c}) \leq 0$ and $\text{sum}(\mathbf{E}^L_{\mathcal{R}_{i,i}}) \geq 0$. When $\lambda(2p_i - 1) - \frac{1}{n_i} \geq 0$ and $\lambda(2q - 1) \leq 0$ we will have $f(\mathbf{E}^L, \mathbf{W}_0) \geq 0$ for all feasible $\mathbf{E}^L$. Hence, we require $\lambda(2p_i - 1) \geq \frac{1}{n_{min}}$ and $1 \geq 2q$.

We will now proceed to find a bound on the spectral norm of $\mathbf{W}^0$.

Theorem 3. *Assume* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *obeys the stochastic block model 2.1 and let* $\mathbf{M} \in \mathbb{R}^{n\times n}$. *Let entries of* $\mathbf{M}$ *be as follows.*

$$\mathbf{M}_{i,j} \sim \begin{cases} Bern(p_k, 1-p_k, 1-p_k) & if \quad (i,j) \in \mathcal{R}_{k,k} \\ Bern(q, 1-q, 1-q) & if \quad (i,j) \in \mathcal{R}^c \end{cases} \tag{7.13}$$

Then, each of the following holds with probability $1 - \exp(-\Omega(n))$.

- $\|\mathbf{M}\| \le (1+\epsilon)\sqrt{n}$.
- $\|\mathbf{M}\| \le 2\sqrt{q(1-q)}\sqrt{n} + \max_{i\le K} 2\sqrt{q(1-q)+p_i(1-p_i)}\sqrt{n_i} + \epsilon\sqrt{n}$.
- *Assume* $\max_{1\le i\le K} n_i = o(n)$. *Then, for sufficiently large* n, *with high probability,*

$$\|\mathbf{M}\| \le (2\sqrt{q(1-q)}+\epsilon)\sqrt{n}. \tag{7.14}$$

Proof. Entries of $\mathbf{M}$ are i.i.d. with maximum variance of $1/4$. Hence, the first statement follows directly from [35].

For the second statement, let,

$$\mathbf{M}_1(i,j) = \begin{cases} \mathbf{M}(i,j) & \text{if } i,j \in \mathbb{R}^c \\ \text{Bern}(q, 1-q, 0) \text{ else} \end{cases} \tag{7.15}$$

and let $\mathbf{M}_2 = \mathbf{M} - \mathbf{M}_1$. Observe that, $\mathbf{M}_1$ has i.i.d. $\text{Bern}(q, 1-q, 0)$ entries which means,

$$\|\mathbf{M}_1\| \le (2\sqrt{q(1-q)}+\epsilon)\sqrt{n} \tag{7.16}$$

with the desired probability.

For $\mathbf{M}_2$, first observe that over $\mathcal{R}_{i,i}$ $\mathbf{M}_2$ has i.i.d. entries with variance $q(1-q)+p_i(1-p_i)$. This gives,

$$\|\mathbf{M}_{2,\mathcal{R}_{i,i}}\| \le 2\sqrt{q(1-q)+p_i(1-p_i)}\sqrt{k_i} + \epsilon\sqrt{n} \tag{7.17}$$

Now, observing, $\|\mathbf{M}_2\| = \sup_{i\le K} \|\mathbf{M}_{2,\mathcal{R}_{i,i}}\|$ and using a union bound over $i \le K$ we have,

$$\|\mathbf{M}_2\| \le \max_{i\le K} 2\sqrt{q(1-q)+p_i(1-p_i)}\sqrt{k_i} + \epsilon\sqrt{n} \tag{7.18}$$

Finally, we use $\|\mathbf{M}\| \le \|\mathbf{M}_1\| + \|\mathbf{M}_2\|$ to conclude. ■

Following lemma gives a bound on $\|\mathbf{W}_0\|$.

Lemma 7.2. $\mathbf{W}_0$ *is a random matrix where randomness is on* $\mathcal{A}$. $\mathbf{W}_0$ *is given by*

$$\begin{aligned} \mathbf{W}_0 &= 2\lambda \sum_{i=1}^{K} \left[(1-p_i)\mathbb{1}^{n\times n}_{\mathcal{A}\cap\mathcal{R}_{i,i}} - p_i \mathbb{1}^{n\times n}_{\mathcal{A}^c\cap\mathcal{R}_{i,i}}\right] \\ &\quad + 2\lambda\left[(1-q)\mathbb{1}^{n\times n}_{\mathcal{A}\cap\mathcal{R}^c} - q\mathbb{1}^{n\times n}_{\mathcal{A}^c\cap\mathcal{R}^c}\right] \end{aligned} \tag{7.19}$$

Then, for any $\epsilon > 0$, *with probability at least* $1 - \exp(-\Omega(n)))$ *we have*

$$\begin{aligned} \|\mathbf{W}_0\| &\le 4\lambda\sqrt{q(1-q)}\sqrt{n} \\ &\quad + \max_{i\le K} 4\lambda\sqrt{q(1-q)+p_i(1-p_i)}\sqrt{n_i} + \epsilon\lambda\sqrt{n} \end{aligned} \tag{7.20}$$

Further, if $\max_{1\le i\le K} n_i = o(n)$. *Then, for sufficiently large* n, *with high probability,*

$$\|\mathbf{W}_0\| \le 2\lambda(2\sqrt{q(1-q)}+\epsilon)\sqrt{n}. \tag{7.21}$$

Proof. $\frac{1}{2\lambda}\mathbf{W}_0$ is a random matrix whose entries are i.i.d. and distributed as $\text{Bern}(-p_i, 1-p_i, 1-p_i)$ on $\mathcal{R}_{i,i}$ and $\text{Bern}(-q, 1-q, 1-q)$ on $\mathcal{R}^c$. Consequently, using Theorem 3 we obtain the result. ■

Lemma 7.2 verifies that asymptotically with high probability we can make $\|\mathbf{W}_0\| < 1$ as long as we choose a proper λ. However, $\mathbf{W}_0$ itself is not sufficient for construction of the desired $\mathbf{W}$, since we do not have any guarantee that $\mathbf{W}_0 \in \mathcal{M}_{\mathbf{U}}$. In order to achieve this, we will *correct* $\mathbf{W}_0$ by projecting it onto $\mathcal{M}_{\mathbf{U}}$. Following lemma suggests that we do not lose much by such a correction.

Correcting the candidate $\mathbf{W}_0$**:**

Lemma 7.3. $\mathbf{W}_0$ *is as described previously in (7.19). Let* $\mathbf{W}^H$ *be projection of* $\mathbf{W}_0$ *on* $\mathcal{M}_{\mathbf{U}}$. *Then*

- $\|\mathbf{W}^H\| \le \|\mathbf{W}_0\|$
- *For any* $\epsilon > 0$, *with probability at least* $1 - 6n^2\exp(-2\epsilon^2 n_{min})$ *we have*

$$\|\mathbf{W}_0 - \mathbf{W}^H\|_\infty \le 3\lambda\epsilon \tag{7.22}$$

Proof. Choose arbitrary vectors $\{\mathbf{u}_i\}_{i=K+1}^n$ to make $\{\mathbf{u}_i\}_{i=1}^n$ an orthonormal basis in $\mathbb{R}^n$. Call $\mathbf{U}_2 = [\mathbf{u}_{K+1} \ \dots \ \mathbf{u}_n]$ and $\mathbf{P} = \mathbf{U}\mathbf{U}^T$, $\mathbf{P}_2 = \mathbf{U}_2\mathbf{U}_2^T$. Now notice that for any matrix $\mathbf{X} \in \mathbb{R}^{n\times n}$, $\mathbf{P}_2\mathbf{X}\mathbf{P}_2$ is in $\mathcal{M}_{\mathbf{U}}$ since $\mathbf{U}^T\mathbf{U}_2 = 0$. Let $\mathbf{I}$ denote the identity matrix. Then

$$\begin{aligned} \mathbf{X} - \mathbf{P}_2\mathbf{X}\mathbf{P}_2 &= \mathbf{X} - (\mathbf{I}-\mathbf{P})\mathbf{X}(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{P}\mathbf{X} + \mathbf{X}\mathbf{P} - \mathbf{P}\mathbf{X}\mathbf{P} \in \mathcal{M}_{\mathbf{U}}^{\perp} \end{aligned} \tag{7.23}$$

Hence, $\mathbf{P}_2\mathbf{X}\mathbf{P}_2$ is the orthogonal projection on $\mathcal{M}_{\mathbf{U}}$. Clearly

$$\|\mathbf{W}^H\| = \|\mathbf{P}_2\mathbf{W}_0\mathbf{P}_2\| \le \|\mathbf{P}_2\|^2\|\mathbf{W}_0\| \le \|\mathbf{W}_0\| \tag{7.24}$$

For analysis of $\|\mathbf{W}_0 - \mathbf{W}^H\|_\infty$ we can consider terms on right hand side of (7.23) separately as we have:

$$\|\mathbf{W}_0 - \mathbf{W}^H\|_\infty \le \|\mathbf{P}\mathbf{W}_0\|_\infty + \|\mathbf{W}_0\mathbf{P}\|_\infty + \|\mathbf{P}\mathbf{W}_0\mathbf{P}\|_\infty \tag{7.25}$$

Clearly $\mathbf{P} = \sum_{i=1}^K \frac{1}{n_i} \mathbb{1}_{\mathbb{R}_{i,i}}^{n\times n}$. Then, each entry of $\frac{1}{\lambda}\mathbf{P}\mathbf{W}_0$ is either a summation of n_i i.i.d. Bern$(-p_i, 1-p_i, 1-p_i)$ or Bern$(-q, 1-q, 1-q)$ random variables scaled by n_i^{-1} for some $i \le K$ or 0. Hence any $c, d \in [n]$ and $\epsilon > 0$

$$\mathbb{P}[|(\mathbf{P}\mathbf{W}_0)_{c,d}| \ge \lambda\epsilon] \le 2\exp(-2\epsilon^2 n_{min}) \tag{7.26}$$

Same (or better) bounds holds for entries of $\mathbf{W}_0\mathbf{P}$ and $\mathbf{P}\mathbf{W}_0\mathbf{P}$. Then a union bound over all entries of the three matrices will give w.p.a.l. $1 - 6n^2\exp(-2\epsilon^2 n_{min})$, we have $\|\mathbf{W}_0 - \mathbf{W}^H\|_\infty \le 3\lambda\epsilon$. ■

Recall that,
Let $\gamma_{\text{succ}} := \max_{1\le i\le K} 4\sqrt{(q(1-q) + p_i(1-p_i))n_i}$, and
$\Lambda_{\text{succ}} := \frac{1}{4\sqrt{q(1-q)n} + \gamma_{\text{succ}}}$.
We can summarize our discussion so far in the following lemma,

Lemma 7.4. *$\mathbf{W}_0$ is as described previously in (7.10). Choose $\mathbf{W}$ to be projection of $\mathbf{W}_0$ on $\mathcal{M}_{\mathbf{U}}$. Also set $\lambda = (1-\delta)\Lambda_{succ}$ and let e be same as in Theorem 7.9 and assume $\{n_i\}$ is such that $e > 0$. Then, w.p.a.l. $1 - 6n^2\exp(-\frac{2}{9}e^2 n_{min}) - 4\exp(-\frac{n}{100})$ we have*

- $\|\mathbf{W}\| < 1$
- *For all feasible $\mathbf{E}^L$, $f(\mathbf{E}^L, \mathbf{W}) \ge 0$.*

Proof. With probability at least $1 - 4\exp(-\frac{n}{100})$ we have $\|\mathbf{W}\| < 1$. Secondly, from Lemma 7.3 with probability at least $1 - 6n^2\exp(-\frac{2}{9}e^2 n_{min})$ we have $\|\mathbf{W}_0 - \mathbf{W}\|_\infty \le \lambda e$. Then based on (7.11) for all $\mathbf{E}^L$

$$\begin{aligned} f(\mathbf{E}^L, \mathbf{W}) &= f(\mathbf{E}^L, \mathbf{W}_0) - \langle \mathbf{W}_0 - \mathbf{W}, \mathbf{E}^L \rangle \\ &\ge f(\mathbf{E}^L, \mathbf{W}_0) - \lambda e \left(\text{sum}(\mathbf{E}^L_{\mathcal{R}}) - \text{sum}(\mathbf{E}^L_{\mathcal{R}^c})\right) \\ &= \lambda\left[(1-2q-e)\text{sum}(\mathbf{E}^L_{\mathcal{R}^c})\right] \\ &\quad -\lambda\sum_{i=1}^K \left[(2p_i - 1 - \frac{1}{\lambda n_i} - e)\text{sum}(\mathbf{E}^L_{\mathcal{R}_{i,i}})\right] \\ &\ge 0 \end{aligned}$$

Hence by a union bound $\mathbf{W}$ satisfies both of the desired conditions. ■

7.1.2. *Solving for $\mathbf{E}^L \in \mathcal{M}_{\mathbf{U}}^{\perp}$ case*

Recall that

$$g\left(\mathbf{E}^L\right) = \sum_{i=}^K \frac{1}{n_i}\text{sum}(\mathbf{E}_{R_{i,i}}) + \lambda\left(\text{sum}(\mathbf{E}^L_{\mathcal{A}^c}) - \text{sum}(\mathbf{E}^L_{\mathcal{A}})\right)$$

Let us define,

$$g_1(\mathbf{X}) := \sum_{l=1}^K \frac{1}{n_l}\text{sum}(\mathbf{X}_{\mathcal{R}_{l,l}}), \tag{7.27}$$

$$g_2(\mathbf{X}) := \text{sum}(\mathbf{X}_{\mathcal{A}^c}) - \text{sum}(\mathbf{X}_{\mathcal{A}}),$$

so that, $g(\mathbf{X}) = g_(\mathbf{X}) + \lambda g_2(\mathbf{X})$. Also let $\mathbf{V} = [\mathbf{v}_1 \ \dots \ \mathbf{v}_t]$ where $\mathbf{v}_i = \sqrt{k_i}\mathbf{u}_i$. Thus, $\mathbf{V}$ is basically obtained by, normalizing columns of $\mathbf{U}$ to make its nonzero entries 1. Assume $\mathbf{E}^L \in \mathcal{M}_{\mathbf{U}}^{\perp}$. Then, we can write

$$\mathbf{E}^L = \mathbf{V}\mathbf{M}^T + \mathbf{N}\mathbf{V}^T. \tag{7.28}$$

Let $\mathbf{m}_i, \mathbf{n}_i$ denote i'th columns of $\mathbf{M}, \mathbf{N}$ respectively.
From $\mathbf{L}^0$ and (1.2) it follows that

$$\begin{aligned} &\mathbf{E}^L_{\mathcal{R}^c} \text{ is (entrywise) nonnegative} \\ &\mathbf{E}^L_{\mathcal{R}} \text{ is (entrywise) nonpositive} \end{aligned}$$

Now, we list some simple observations regarding structure of $\mathbf{E}^L$. We can write

$$\mathbf{E}^L = \sum_{i=1}^K (\mathbf{v}_i\mathbf{m}_i^T + \mathbf{n}_i\mathbf{v}_i^T) = \sum_{i=1}^{K+1}\sum_{j=1}^{K+1} \mathbf{E}^L_{\mathcal{R}_{i,j}} \tag{7.29}$$

Notice that only two components : $\mathbf{v}_i\mathbf{m}_i^T$ and $\mathbf{n}_j\mathbf{v}_j^T$, contribute to the term $\mathbf{E}^L_{\mathcal{R}_{i,j}}$.

Let $\{a_{i,j}\}_{j=1}^{n_i}$ be an (arbitrary) indexing of elements of $\mathcal{C}_i$ i.e. $\mathcal{C}_i = \{a_{i,1}, \dots, a_{i,n_i}\}$. For a vector $\mathbf{z} \in \mathbb{R}^n$, let $\mathbf{z}^i \in \mathbb{R}^{n_i}$ denote the vector induced by entries of $\mathbf{z}$ in $\mathcal{C}_i$. Basically, for any $1 \le j \le n_i$, $\mathbf{z}^i_j = \mathbf{z}_{a_{i,j}}$. Also, let $\mathbf{E}^{i,j} \in \mathbb{R}^{n_i \times n_j}$ which is $\mathbf{E}^L$ induced by entries on $\mathcal{R}_{i,j}$.

In other words,

$$\begin{aligned} \mathbf{E}^{i,j}_{c,d} = \mathbf{E}^L_{a_{i,c},a_{j,d}} \quad &\text{for any } (i,j) \in \mathcal{C}_i \times \mathcal{C}_j \\ &\text{and for any } 1 \le c \le n_i,\ 1 \le d \le n_j \end{aligned} \tag{7.30}$$

Basically, $\mathbf{E}^{i,j}$ is same as $\mathbf{E}^L_{\mathcal{R}_{i,j}}$ when we get rid of trivial zero rows and zero columns. Then

$$\mathbf{E}^{i,j} = \mathbb{1}^{n_i}\mathbf{m}_i^{jT} + \mathbf{n}_j^i\mathbb{1}^{n_j T} \tag{7.31}$$

Clearly, given $\{\mathbf{E}^{i,j}\}_{1\le i,j\le n}$, $\mathbf{E}^L$ is uniquely determined. Now, assume we fix $\text{sum}(\mathbf{E}^{i,j})$ for all i, j and we would like to find the *worst* $\mathbf{E}^L$ subject to these constraints. Variables in such an optimization are $\mathbf{m}_i, \mathbf{n}_i$. Basically we are interested in

$$\min g(\mathbf{E}^L) \tag{7.32}$$

$$\text{subject to} \tag{7.33}$$

$$\text{sum}(\mathbf{E}^{i,j}) = c_{i,j} \text{ for all } i, j \tag{7.34}$$

$$\mathbf{E}^{i,j} \begin{cases} \text{nonnegative if } i \ne j \\ \text{nonpositive if } i = j \end{cases} \tag{7.35}$$

where $\{c_{i,j}\}$ are constants. Constraint (7.34) follows from (7.12). **Remark:** For the special case of $i = j = K+1$, notice that $\mathbf{E}^{i,j} = 0$.

In (7.31), $g_1(\mathbf{E}^L)$ is fixed and equal to $\sum_{i=1}^{K} \frac{1}{n_i} c_{i,i}$. Consequently, we just need to do the optimization with the objective $g_2(\mathbf{E}^L) = \text{sum}(\mathbf{E}^L_{\mathcal{A}^c}) - \text{sum}(\mathbf{E}^L_{\mathcal{A}})$.

Let $\beta_{i,j} \subseteq [n_i] \times [n_j]$ be a set of coordinates defined as follows. For any $(c,d) \in [n_i] \times [n_j]$

$$(c,d) \in \beta_{i,j} \text{ iff } (a_{i,c}, a_{j,d}) \in \mathcal{A} \tag{7.36}$$

For $(i_1, j_1) \neq (i_2, j_2)$, $(\mathbf{m}^{j_1}_{i_1}, \mathbf{n}^{i_1}_{j_1})$ and $(\mathbf{m}^{j_2}_{i_2}, \mathbf{n}^{i_2}_{j_2})$ are independent variables. Consequently, due to (7.30), we can partition problem (7.31) into the following smaller disjoint problems.

$$\min_{\mathbf{m}^j_i, \mathbf{n}^i_j} \text{sum}(\mathbf{E}^{i,j}_{\beta^c_{i,j}}) - \text{sum}(\mathbf{E}^{i,j}_{\beta_{i,j}}) \tag{7.37}$$

$$\text{subject to} \tag{7.38}$$

$$\text{sum}(\mathbf{E}^{i,j}) = c_{i,j} \tag{7.39}$$

$$\mathbf{E}^{i,j} \begin{cases} \text{nonnegative if } i \neq j \\ \text{nonpositive if } i = j \end{cases} \tag{7.40}$$

Then, we can solve these problems locally (for each i,j) to finally obtain

$$g_2(\mathbf{E}^{L,*}) = \sum_{i,j} \text{sum}(\mathbf{E}^{i,j,*}_{\beta^c_{i,j}}) - \sum_{i,j} \text{sum}(\mathbf{E}^{i,j,*}_{\beta_{i,j}}) \tag{7.41}$$

to find the overall result of problem (7.31), where $*$ denotes the optimal solutions in problems (7.31) and (7.36). The following lemma will be useful for analysis of these local optimizations.

Lemma 7.5. *Let $\mathbf{a} \in \mathbb{R}^c$, $\mathbf{b} \in \mathbb{R}^d$ and $X = \mathbb{1}^c \mathbf{b}^T + \mathbf{a} \mathbb{1}^{d^T}$ be variables and $C_0 \geq 0$ be a constant. Also let $\beta \subseteq [c] \times [d]$. Consider the following optimization problem*

$$\min_{\mathbf{a},\mathbf{b}} \textit{sum}(\mathbf{X}_{\beta^c}) - \textit{sum}(\mathbf{X}_{\beta}) \tag{7.42}$$

$$\textit{subject to} \tag{7.43}$$

$$\mathbf{X}_{i,j} \geq 0 \textit{ for all } i,j \tag{7.44}$$

$$\textit{sum}(\mathbf{X}) = C_0 \tag{7.45}$$

For this problem there exists a (entrywise) nonnegative minimizer $(\mathbf{a}^0, \mathbf{b}^0)$.

Proof. Let x_i denotes i'th entry of vector $\mathbf{x}$. Assume $(\mathbf{a}^*, \mathbf{b}^*)$ is a minimizer. WLOG assume $b^*_1 = \min_{i,j}\{\mathbf{a}^*_i, \mathbf{b}^*_j\}$. If $b^*_1 \geq 0$ we are done. Otherwise, since $\mathbf{X}_{i,j} \geq 0$ we have $a^*_i \geq -b^*_1$ for all $i \leq c$. Then set $\mathbf{a}^0 = \mathbf{a}^* + \mathbb{1}^c b^*_1$ and $\mathbf{b}^0 = \mathbf{b}^* - \mathbb{1}^d b^*_1$. Clearly, $(\mathbf{a}^0, \mathbf{b}^0)$ is nonnegative. On the other hand, we have:

$$\begin{aligned} \mathbf{X}^* = \mathbb{1}^c \mathbf{b}^{*T} + \mathbf{a}^* \mathbb{1}^{d^T} &= \mathbb{1}^c \mathbf{b}^{0^T} + \mathbf{a}^0 \mathbb{1}^{d^T} = \mathbf{X}^0 \\ \implies \text{sum}(\mathbf{X}^*_\beta) - \text{sum}(\mathbf{X}^*_{\beta^c}) &= \text{sum}(\mathbf{X}^0_\beta) - \text{sum}(\mathbf{X}^0_{\beta^c}) \\ &= \text{minimum value} \end{aligned}$$

■

Lemma 7.6. *A direct consequence of Lemma 7.5 is the fact that in the local optimizations (7.36), WLOG we can assume $(\mathbf{m}^j_i, \mathbf{n}^i_j)$ entrywise nonnegative whenever $i \neq j$ and entrywise nonpositive when $i = j$. This follows from the structure of $\mathbf{E}^{i,j}$ given in (7.30) and (7.12).*

Following lemma will help us characterize the relationship between $\text{sum}(\mathbf{E}^{i,j})$ and $\text{sum}(\mathbf{E}^{i,j}_{\beta^c_{i,j}})$.

Lemma 7.7. *Let $\beta \in \mathbb{R}^{c \times d}$ be a random support with parameter $0 \leq r \leq 1$. Then for any $\epsilon > 0$ w.p.a.l. $1 - d\exp(-2\epsilon^2 c)$ for all nonzero and entrywise nonnegative $\mathbf{a} \in \mathbb{R}^d$ we'll have:*

$$\textit{sum}(\mathbf{X}_\beta) > (r - \epsilon)\textit{sum}(\mathbf{X}) \tag{7.46}$$

where $\mathbf{X} = \mathbb{1}^c \mathbf{a}^T$. Similarly, with same probability, for all such $\mathbf{a}$, we'll have $\textit{sum}(\mathbf{X}_\beta) < (r + \epsilon)\textit{sum}(\mathbf{X})$

Proof. We'll only prove the first statement (7.45) as proofs are identical. For each $i \leq d$, a_i occurs exactly c times in $\mathbf{X}$ as i'th column of X is $\mathbb{1}^c a_i$. By using a Chernoff bound, we can estimate the number of coordinates of i'th column which are element of β (call this number C_i) as we can view this number as a sum of c i.i.d. $\text{Bern}(1, 0, r)$ random variables. Then

$$\mathbb{P}(C_i \leq c(r - \epsilon)) \leq \exp(-2\epsilon^2 c) \tag{7.47}$$

Now, we can use a union bound over all columns to make sure for all i, $C_i > c(r - \epsilon)$

$$\mathbb{P}(C_i > c(r - \epsilon) \text{ for all } i \leq d) \geq 1 - d\exp(-2\epsilon^2 c) \tag{7.48}$$

On the other hand if each $C_i > c(r - \epsilon)$ then for any nonnegative $\mathbf{a} \neq 0$

$$\begin{aligned} \text{sum}(\mathbf{X}_\beta) &= \sum_{(i,j) \in \beta} \mathbf{X}_{i,j} = \sum_{i=}^{d} C_i a_i \\ &> c(r - \epsilon) \sum_{i=1}^{d} a_i \\ &= (r - \epsilon)\text{sum}(\mathbf{X}) \end{aligned} \tag{7.49}$$

■

Using Lemma 7.7, we can calculate a lower bound for $g(\mathbf{E}^L)$ with high probability as long as cluster sizes are sufficiently large. Due to (7.29) and the linearity of $g(\mathbf{E}^L)$, we can focus on contributions due to specific clusters i.e. $\mathbf{v}_i\mathbf{m}_i^T + \mathbf{n}_i\mathbf{v}_i^T$ for the i'th cluster. We additionally know the simple structure of $\mathbf{m}_i, \mathbf{n}_i$ from Lemma 7.6. In particular, subvectors $\mathbf{m}_i^i$ and $\mathbf{n}_i^i$ of $\mathbf{m}_i, \mathbf{n}_i$ can be assumed to be nonpositive and rest of the entries are nonnegative.

Now, we define an important parameter which will be useful for subsequent analysis. This parameter can be seen as a measure of distinctness of the "worst" cluster from the "background noise". Here background noise corresponds to the edges over $\mathcal{R}^c$.

$$e = \frac{1}{2}\min\left\{1-2q, \left\{2p_l - \frac{1}{\lambda n_l} - 1\right\}_{l=1}^{K}\right\} \tag{7.50}$$

Lemma 7.8. *Assume, $l \leq K$, $e > 0$. Then, w.p.a.l. $1 - n\exp(-2e^2(n_l - 1))$, we have $g(\mathbf{v}_l\mathbf{m}_l^T) \geq 0$ for all $\mathbf{m}_l$. Also, if $\mathbf{m}_l \neq 0$ then inequality is strict.*

Proof. Recall that $\mathbf{m}_l$ satisfies $\mathbf{m}_l^i$ is nonpositive/nonnegative when $i = l/i \neq l$ for all i. Call $\mathbf{X}^i = \mathbb{1}^{n_l}\mathbf{m}_l^{iT}$. We can write

$$g(\mathbf{v}_l\mathbf{m}_l^T) = \frac{1}{n_l}\text{sum}(\mathbf{X}^l) + \sum_{i=1}^{K}\lambda h(\mathbf{X}^i, \beta_{l,i}^c) \tag{7.51}$$

where $h(\mathbf{X}^i, \beta_{l,i}^c) = \text{sum}(\mathbf{X}^i_{\beta_{l,i}^c}) - \text{sum}(\mathbf{X}^i_{\beta_{l,i}})$. Now assume $i \neq l$. Using Lemma 7.7 and the fact that $\beta_{l.i}$ is a random support with q w.p.a.l. $1 - n_i\exp(-2\epsilon^2 n_l)$, for all $\mathbf{X}^i$, we have

$$\begin{aligned} h(\mathbf{X}^i, \beta_{l,i}^c) &\geq (1-q-\epsilon)\text{sum}(\mathbf{X}^i) - (q+\epsilon)\text{sum}(\mathbf{X}^i) \\ &= (1-2q-2\epsilon)\text{sum}(\mathbf{X}^i) \end{aligned} \tag{7.52}$$

where inequality is strict if $X^i \neq 0$. Similarly when $i = l$ with probability at least $1 - n_l\exp(-2\epsilon^2(n_l-1))$, we have,

$$\begin{aligned} &\frac{1}{\lambda n_l}\text{sum}(\mathbf{X}^l) + h(\mathbf{X}^l, \beta_{l,l}^c) \\ &\geq \left(1 - p_l + \epsilon + \frac{1}{\lambda n_l}\right)\text{sum}(\mathbf{X}^l) \\ &\quad - (p_l - \epsilon)\,\text{sum}(\mathbf{X}^l) \\ &= -\left(2p_l - 1 - \frac{1}{\lambda n_l} - 2\epsilon\right)\text{sum}(\mathbf{X}^l) \end{aligned} \tag{7.53}$$

(7.54)

Choosing $\epsilon = e$ and using the facts $1 - 2q - 2e \geq 0$, $2p_l - 1 - \frac{1}{\lambda n_l} - 2e \geq 0$ and a union bound w.p.a.l. $1 - n\exp(-2e^2(n_l-1))$ we have $g(\mathbf{v}_l\mathbf{m}_l^T) \geq 0$ and inequality is strict when $\mathbf{m}_l \neq 0$ as at least one of the $\mathbf{X}^i$'s will be nonzero. ■

Following lemma immediately follows from Lemma 7.8 and summarizes the main result of the section.

Lemma 7.9. *Let e be as defined in (7.49) and assume $e > 0$. Then with probability at least $1 - 2nK\exp(-2e^2(n_{min}-1))$ we have $g(\mathbf{E}^L) > 0$ for all nonzero feasible $\mathbf{E}^L \in \mathcal{M}_U^{\perp}$.*

7.1.3. Final Step

Lemma 7.10. *Let $p_{min} > \frac{1}{2} > q$ and $\mathcal{G}$ be a random graph generated according to the stochastic block model 2.1 with cluster sizes $\{n_i\}_{i=1}^K$. If $\lambda < \Lambda_{succ}$ and $\mathbf{D}_{\min} = \min_{1\leq i\leq n}(2p_i - 1)\,n_i > \frac{1}{\lambda}$, then $\left(\mathbf{L}^0, \mathbf{S}^0\right)$ is the unique optimal solution to Program 1.1 with high probability.*

Proof. Notice that $\lambda < \Lambda_{\text{succ}}$ and $(2p_i - 1)\,n_i \geq \frac{1}{\lambda}$ implies

$$\begin{aligned} 2e &= \min\left\{1-2q, \left\{2p_i - 1 - \frac{1}{\lambda n_i}\right\}_{i=1}^{K}\right\} \\ &\geq \min\left\{1-2q, \left\{\frac{2p_i - 1}{2}\right\}_{i=1}^{K}\right\} \\ &= \min\{1-2q, p_{min} - 1/2\} \end{aligned} \tag{7.55}$$

Then based on Lemma 7.9 and Lemma 7.4 with probability at least $1 - cn^2\exp(-C\left(\min\{1-2q, 2p_{min}-1\}\right)^2 n_{min})$,

- There exists $\mathbf{W} \in \mathcal{M}_{\mathbf{U}}$ with $\|\mathbf{W}\| < 1$ such that for all feasible $\mathbf{E}^L$, $f(\mathbf{E}^L, \mathbf{W}) \geq 0$.
- For all nonzero $\mathbf{E}^L \in \mathcal{M}_{\mathbf{U}}^{\perp}$ we have $g(\mathbf{E}^L) > 0$.

Consequently based on Lemma 7.1, $(\mathbf{L}^0, \mathbf{S}^0)$ is the unique optimal of problem 1.1. ■

7.2. Conditions for Failure

Lemma 7.11. *Let $p_{min} > \frac{1}{2} > q$ and $\mathcal{G}$ be a random graph generated according to the stochastic block model 2.1 with cluster sizes $\{n_i\}_{i=1}^K$.*

1. *If $\min_i\{n_i(2p_i - 1)\} < \frac{1}{\lambda}$, then $\left(\mathbf{L}^0, \mathbf{S}^0\right)$ is not an optimal solution to the program 1.1 with probability at least $1 - K\exp\left(-\Omega(k_{\min}^2)\right)$.*
2. *If $\lambda > \sqrt{\frac{n}{q\left(n^2 - \sum_{i=1}^K n_i^2\right)}}$, then $\left(\mathbf{L}^0, \mathbf{S}^0\right)$ is not an optimal solution to the program 1.1 with high probability.*

Proof. Lagrange for the problem 1.1 can be written as follows

$$\begin{aligned} &\mathscr{L}(\mathbf{L}, \mathbf{S}; \mathbf{M}, \mathbf{N}) \\ &\quad = \|\mathbf{L}\|_\star + \lambda\|\mathbf{S}\|_1 + \text{trace}(\mathbf{M}(\mathbf{L} - \mathbb{1}\mathbb{1}^T)) \\ &\quad\quad -\text{trace}(\mathbf{NL}). \end{aligned} \tag{7.56}$$

where $\mathbf{M}$ and $\mathbf{N}$ are dual variables corresponding to the inequality constraints 1.2.

For $\mathbf{L}^0$ to be an optimal solution to 1.1, it has to satisfy the KKT conditions. Therefore, the subgradient of 7.55 at $\mathbf{L}^0$ has to be 0, i.e.,

$$\partial\|\mathbf{L}^0\|_\star + \lambda\,\partial\|\mathbf{A}-\mathbf{L}^0\|_1 + \mathbf{M}^0 - \mathbf{N}^0 = 0. \tag{7.57}$$

where $\mathbf{M}^0$ and $\mathbf{N}^0$ are optimal dual variables.

Also, by complementary slackness,

$$\text{trace}(\mathbf{M}^0(\mathbf{L}^0 - \mathbb{1}\mathbb{1}^T)) = 0, \tag{7.58}$$

and

$$\text{trace}(\mathbf{N}^0\mathbf{L}^0) = 0. \tag{7.59}$$

From 3.1 and 7.57, 7.58, we have $(\mathbf{M}^0)_{\mathcal{R}} \geq 0$, $(\mathbf{M}^0)_{\mathcal{R}^c} = 0$, $(\mathbf{N}^0)_{\mathcal{R}} = 0$ and $(\mathbf{N}^0)_{\mathcal{R}^c} \geq 0$. Hence $(\mathbf{M}^0 - \mathbf{N}^0)_{\mathcal{R}} \geq 0$ and $(\mathbf{M}^0 - \mathbf{N}^0)_{\mathcal{R}^c} = 0$.

Recall, $\mathbf{L}^0 = \mathbf{U}\Lambda\mathbf{U}^T$, where $\mathbf{U} = [\mathbf{u}_1\ \ldots\ \mathbf{u}_t] \in \mathbb{R}^{n\times t}$,

$$\mathbf{u}_{l,i} = \begin{cases} \frac{1}{\sqrt{k_l}} & \text{if } i \in \mathcal{C}_l \\ 0 & \text{else.} \end{cases} \tag{7.60}$$

Also recall that the subgradient $\partial\|\mathbf{L}^0\|_\star$ is of the form $\mathbf{U}\mathbf{U}^T + \mathbf{W}$ such that $\mathbf{W} \in \{\mathbf{X} : \mathbf{X}\mathbf{U} = \mathbf{U}^T\mathbf{X} = 0, \|\mathbf{X}\| \leq 1\}$. The subgradient $\partial\|\mathbf{S}^0\|_1$ is of the form $\text{sign}(\mathbf{S}^0) + \mathbf{Q}$ where $\mathbf{Q}_{i,j} = 0$ if $\mathbf{S}_{i,j} \neq 0$ and $\|\mathbf{Q}\|_\infty \leq 1$.

From 7.56, we have

$$\mathbf{U}\mathbf{U}^T + \mathbf{W} - \lambda\left(\text{sign}(\mathbf{S}^0) + \mathbf{Q}\right) + (\mathbf{M}^0 - \mathbf{N}^0) = 0. \tag{7.61}$$

Consider the sum of the entires corresponding $\mathcal{R}_{i,i}$, i.e.,

$$\underbrace{\text{sum}\left(\mathbf{L}^0\right)_{\mathcal{R}_{i,i}}}_{n_i} - \text{sum}\left(\lambda\left(\text{sign}(\mathbf{S}^0) + \mathbf{Q}\right)_{\mathcal{R}_{i,i}}\right) + \underbrace{\text{sum}\left(\mathbf{M}^0 - \mathbf{N}^0\right)_{\mathcal{R}_{i,i}}}_{\geq 0} = 0 \tag{7.62}$$

Then by Bernstein's inequality and using $\|\mathbf{Q}\|_\infty \leq 1$, with probability $1 - \exp\left(-\Omega(n_i^2)\right)$ we have

$$\text{sum}\left(\text{sign}(\mathbf{S}^0)\right) = -n_i^2(1-p_i)$$

and $\text{sum}\left(\mathbf{Q}\right) \leq n_i^2 p_i$.

Thus, $-\text{sum}\left(\lambda\left(\text{sign}(\mathbf{S}^0) + \mathbf{Q}\right)_{\mathcal{R}_{i,i}}\right) \geq \lambda n_i^2(1-2p_i)$ and hence,

$$\underbrace{\text{sum}\left(\mathbf{L}^0\right)_{\mathcal{R}_{i,i}}}_{n_i} - \text{sum}\left(\lambda\left(\text{sign}(\mathbf{S}^0) + \mathbf{Q}\right)_{\mathcal{R}_{i,i}}\right) + \underbrace{\text{sum}\left(\mathbf{M}^0 - \mathbf{N}^0\right)_{\mathcal{R}_{i,i}}}_{\geq 0} \geq n_i + \lambda n_i^2(1-2p_i) \tag{7.63}$$

If $n_i + \lambda n_i^2(1-2p_i) > 0$, then the equation 7.56 does not hold and hence $\mathbf{L}^0$ cannot be an optimal solution to the program 1.1. $n_i + \lambda n_i^2(1-2p_i) > 0$ would require $n_i\,(2p_i - 1) < \frac{1}{\lambda}$. (Note that, $p_i > \frac{1}{2}$ and hence $2p_i - 1 > 0$.) The same argument holds for each cluster $i = 1 \ldots K$, and by union bound we get the result 1 in the Lemma 7.11.

Notice that $\left(\mathbf{U}\mathbf{U}^T\right)_{\mathcal{R}^c} = 0$ and entries of $\text{sign}(\mathbf{S}^0) + \mathbf{Q}$ and $\mathbf{M}^0 - \mathbf{N}^0$ over $\mathcal{R}^c \cap \mathcal{A}$ are negative. Hence from the equation 7.60,

$$\begin{aligned} \|\mathbf{W}\|_F^2 &\geq \|\left(\mathbf{U}\mathbf{U}^T + \mathbf{W}\right)_{(\mathcal{R}^c\cap\mathcal{A})}\|_F^2 \\ &\geq \|\lambda\left(\text{sign}(\mathbf{S}^0) + \mathbf{Q}\right)_{(\mathcal{R}^c\cap\mathcal{A})}\|_F^2. \end{aligned} \tag{7.64}$$

Recall that $\mathbf{S}^0_{(\mathcal{R}_c\cap\mathcal{A})} \neq 0$ and hence $\mathbf{Q}_{(\mathcal{R}_c\cap\mathcal{A})} = 0$. Further, recall that by the stochastic block model 2.1, each entry of $\mathbf{A}$ over $\mathcal{R}_c$ is non-zero with probability q. Hence with probability at least $1 - \exp\left(-\Omega(|\mathcal{R}_c|)\right)$, $|\mathcal{R}_c \cap \mathcal{A}| = q(n^2 - \sum_{i=1}^K n_i^2)$. Thus from equation 7.63 we have,

$$\|\mathbf{W}\|_F^2 \geq \lambda^2 q(n^2 - \sum_{i=1}^K n_i^2), \tag{7.65}$$

Recall that $\|\mathbf{W}\| \leq 1$ should hold true for $\left(\mathbf{L}^0, \mathbf{S}^0\right)$ to be an optimal solution to the program 1.1.

$$\|\mathbf{W}\| = |\sigma_{\max}(\mathbf{W})| \geq \frac{\|\mathbf{W}\|_F}{\sqrt{n}},$$

which on combining with equation 7.64 gives us,

$$\|\mathbf{W}\| \geq \lambda\sqrt{\frac{q\left(n^2 - \sum_{i=1}^K n_i^2\right)}{n}}.$$

So, if $\lambda\sqrt{q\left(n^2 - \sum_{i=1}^K n_i^2\right)/n} > 1$ then, $\left(\mathbf{L}^0, \mathbf{S}^0\right)$ cannot be an optimal solution to Program 1.1. This gives us result 2 in the Lemma 7.11.

■

7.3. Proof of Main Results

Recall the following definitions :

- Define $\mathbf{D}_i := n_i\,(2p_i - 1)$ as the effective density of cluster $\mathcal{C}_i$ and $\mathbf{D}_{\min} = \min_{1\leq i\leq K} \mathbf{D}_i$.
- Let $\gamma_{\text{succ}} := \max_{1\leq i\leq K} 4\sqrt{(q(1-q) + p_i(1-p_i))n_i}$, $\gamma_{\text{fail}} := \sum_{i=1}^K \frac{n_i^2}{n}$
- $\Lambda_{\text{fail}} := \frac{1}{\sqrt{q(n-\gamma_{\text{fail}})}}$ and $\Lambda_{\text{succ}} := \frac{1}{4\sqrt{q(1-q)n} + \gamma_{\text{succ}}}$.

Theorem 4. *Let $\mathcal{G}$ be a random graph generated according to the stochastic block model 2.1 with K clusters of sizes $\{n_i\}_{i=1}^K$ and probabilities $\{p_i\}_{i=1}^K$ and q, such that $p_{min} > \frac{1}{2} > q > 0$. Given $\epsilon > 0$, there exists positive constants δ, c_1, c_2 such that,*

1. *Whenever $\mathbf{D}_{\min} \geq (1+\epsilon)\Lambda_{succ}^{-1}$, for $\lambda = (1-\delta)\Lambda_{succ}$, Program (1.1) succeeds with probability $1-c_1 n^2 \exp(-c_2 n_{\min})$.*
2. *For any given $\lambda \geq 0$, if $\mathbf{D}_{\min} \leq (1-\epsilon)\Lambda_{fail}^{-1}$ then Program (1.1) fails with probability $1 - c_1 \exp(-c_2|\mathcal{R}^c|)$.*

Proof. 1. $\mathbf{D}_{\min} \geq (1+\epsilon)\Lambda_{\text{succ}}^{-1}$ and $\lambda = (1-\delta)\Lambda_{\text{succ}}$. Hence the first part of the theorem follows from the Lemma 7.10.

2. If $\mathbf{D}_{\min} \leq (1-\epsilon)\Lambda_{\text{fail}}^{-1}$, then from Lemma 7.11, Program 1.1 fails with high probability.

■

Theorem 5. *Let $\mathcal{G}$ be a random graph generated according to the stochastic block model 2.1 with K clusters of sizes $\{n_i\}_{i=1}^K$ and probabilities $\{p_i\}_{i=1}^K$ and q, such that $p_{min} > \frac{1}{2} > q > 0$. Given $\epsilon > 0$, there exists positive constants c_1', c_2' such that,*

1. *If $\lambda \geq (1+\epsilon)\Lambda_{fail}$, then (1.1) fails with probability $1 - c_1' \exp(-c_2'|\mathcal{R}^c|)$.*
2. *If $\lambda \leq (1-\epsilon)\Lambda_{succ}$ then,*
   - *If $\mathbf{D}_{\min} \leq (1-\epsilon)\frac{1}{\lambda}$, then Program (1.1) fails with probability $1 - c_1' \exp(-c_2' n_{\min})$.*
   - *If $\mathbf{D}_{\min} \geq (1+\epsilon)\frac{1}{\lambda}$, then Program (1.1) succeeds with probability $1 - c_1' n^2 \exp(-c_2' n_{\min})$.*

Proof. 1. The first part of the theorem follows from the Lemma 7.11(2).

2. If $\mathbf{D}_{\min} \leq (1-\epsilon)\frac{1}{\lambda}$, then from Lemma 7.11(1), Program (1.1) fails with probability $1 - c_1' \exp(-c_2' n_{\min})$.

■