Supplementary Material

1 Convex Program for Similarity Clustering

Recall the convex program:

$$\underset{\mathbf{X}}{\text{minimize}} \ \frac{1}{2}\|\mathbf{A}-\mathbf{X}\|_F^2 + \lambda \ \text{trace}(\mathbf{X}) \tag{1.1}$$

subject to

$$\mathbf{X} \succcurlyeq 0 \tag{1.2}$$

$$\sum_j \mathbf{X}_{i,j} \leq 1, \quad \text{for all } i \tag{1.3}$$

$$\mathbf{X}_{i,j} \geq 0 \ \ \text{for all } i,j \in \{1,2,\dots n\} \tag{1.4}$$

where $||.||_F$ is the Frobenius norm (square root of the sum of the squares of the entries of the matrix), and $\lambda > 0$ is a regularization parameter. Also, by $\mathbf{X} \succcurlyeq 0$, we mean that $\mathbf{X}$ is symmetric and has non-negative eigenvalues.

2 Proof Sketches

First we note that the objective function in Program 1.1 is strongly convex in $\mathbf{X}$, and hence has an unique optimal solution. So, it is enough to produce an optimal solution.

Define dual variables for the constraints,

1. $\mathbf{Y} \in \mathbb{R}^{n\times n}, \ \mathbf{Y} \succcurlyeq 0$ for constraint (1.2).
2. $\nu \in \mathbb{R}^n, \ 2\nu \geq 0$ for constraints (1.3).
3. $\mathbf{Z} \in \mathbb{R}^{n\times n}, \ \mathbf{Z} \geq 0$ for constraints (1.4), where $\geq$ is entry-wise.

Lagrange can be written as,

$$\mathcal{L}(\mathbf{X};\mathbf{Y},\mathbf{Z},\nu) = \max_{\mathbf{Z}\geq 0,\mathbf{Y}\succcurlyeq 0,\nu\geq 0} - 2\nu^T \ \mathbb{1}\min_{\mathbf{X}} \ \ \frac{1}{2}||\mathbf{A}-\mathbf{X}||_F^2 + \text{trace}(\lambda\mathbf{I} - \mathbf{Z} - \mathbf{Y} + \mathbb{1}\nu^T + \nu\mathbb{1}^T + \mathbf{X}). \tag{2.1}$$

where $\mathbb{1} \in \mathbb{R}^n$ is a vector of all 1's.

If a feasible $\hat{\mathbf{X}}$ is an optimal solution to Program (1.1), then the following conditions have to hold (from KKT conditions and complementary slackness):

$$\mathbf{Z} + \mathbf{Y} = \lambda\mathbf{I} + \hat{\mathbf{X}} + \mathbb{1}\nu^T - \mathbf{A} + \nu\mathbb{1}^T \tag{2.2}$$

$$\text{trace}(\hat{\mathbf{X}}\mathbf{Y}) = 0, \tag{2.3}$$

$$\text{trace}(\hat{\mathbf{X}}\mathbf{Z}) = 0, \tag{2.4}$$

$$\nu^T \left(\hat{\mathbf{X}}\mathbb{1} - \mathbb{1}\right) = 0. \tag{2.5}$$

Since $\hat{\mathbf{X}}, \mathbf{Y} \succcurlyeq 0$, from(2.3), we get

$$\hat{\mathbf{X}}\mathbf{Y} = \mathbf{Y}\hat{\mathbf{X}} = 0. \tag{2.6}$$

We first construct dual variables that satisfy the conditions (2.2) (2.3) (2.4) and (2.5). The dual variables $\mathbf{Z}, \mathbf{Y}, \nu$ thus obtained are functions of the problem parameters $\{\mu_i\}_{i\in[K]}, \mu_{out}, \sigma, \{n_i\}_{i\in[K]}, n_{K+1}$. The condition $\mathbf{Y} \succcurlyeq 0$ will give the lower bound on λ of the form Λ or $\Lambda + \mu_{out}n_{out}$ depending on the case. The conditions $\nu \geq 0$ gives the lower bounds on the cluster densities ρ. The conditions $\mathbf{Z} \geq 0$ gives the lower bounds on cross-cluster densities γ and the effective cluster densities η.

Notation: Let $[m] := \{1, 2, \cdots, m\}$. Let n be the number of data points denoted by $[n]$ with K disjoint clusters. Let $\mathcal{C}_i \subset [n]$ be the set of data points in cluster i and the size (number of data points in the cluster) be n_i. μ_i is the mean similarity between nodes inside the same cluster i. μ_{out} is the mean similarity between nodes that are not in the same cluster, and σ^2 is the noise variance.

For a vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v}_{[i]}$ denotes a vector zero-entries everywhere except for the indices corresponding to $\mathcal{C}_i$ where it is equal to the entries of $\mathbf{v}$ corresponding to $\mathcal{C}_i$. Similarly for a matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_{[i][j]}$ denotes a matrix with zero entries everywhere except the block corresponding to $\mathcal{C}_i \times \mathcal{C}_j$ where it the entries equal to the block corresponding to $\mathcal{C}_i \times \mathcal{C}_j$ in $\mathbf{B}$.

We define the following quantities:

- *Cluster Density:* For each cluster $i \in \{1, \cdots, K\}$, define the *cluster density* as $\rho_i := n_i\mu_i$
- *Cross Cluster Density:* For each pair of clusters $i \neq j \in \{1, \cdots, K\}$, define the *cross cluster density* as $\gamma_{ij} := 2\left(\frac{\mu_i+\mu_j}{2} - \mu_{out}\right)\left(\frac{1}{n_i} + \frac{1}{n_j}\right)^{-1}$.
- *Minimum Cluster Density* is defined as $\rho_{min} := \min_{i\leq K} \rho_i$ and *Minimum Cross Cluster Density:* is defined as $\gamma_{min} := \min_{i\neq j\leq K} \gamma_{ij}$.
- *Effective cluster density* for each cluster i as $\eta_i := (\mu_i - 2\mu_{out})\, n_i$.
- *Minimum effective density* be $\eta_{min} := \min_{i\leq K}\eta_i$.
- *Outlier Density:* $\rho_{K+1} := \mu_{out}n_{K+1}$.
- *Cross Cluster-Outlier Density:* For $i \in [K]$, $\gamma_{i,K+1} := (\mu_i - \mu_{out})\left(\frac{1}{n_i} + \frac{1}{n_{K+1}}\right)^{-1}$.
- *Minimum cluster density* in the presence of *outliers* as $\rho^{out}_{min} := \min_{i\leq K+1} \rho_i$.
- *Minimum cross cluster density* in the presence of *outliers* $\gamma^{out}_{min} := \min_{i\neq j\leq K+1}\gamma_{ij}$.
- Noise threshold $\Lambda := 2\,\sigma\sqrt{n}$ which depends only on the noise variance and number of data points.

Let us first look at the case without outliers. The solution to the case with outliers builds on the case without outliers.

3 No Outliers

For the case with K disjoint clusters and no outliers, the desired solution is

$$\mathbf{X}^*_{l,m} = \begin{cases} \frac{1}{n_i}\ , & \text{if both nodes } l,m \text{ are in the same cluster } i \\ 0\ , & \text{if nodes } l,m \text{ are not in the same cluster} \end{cases} \tag{3.1}$$

which has non-zero entries only in the region corresponding to clusters.

In terms of the regularization parameter λ, the conditions for the success and failure of Program 1.1 is given by the following theorem:

Theorem 1 *[No Outliers] If the regularizer λ is within the following range,*

$$\Lambda < \lambda < \min\{\rho_{min}, \gamma_{min}\} - 1 \tag{3.2}$$

then, $\mathbf{X}^*$ is the unique optimal solution to Program 1.1 with high probability.

We will prove the following lemmas which together prove Theorem 1:

Lemma 3.1 *If $\lambda > \sigma 2\sqrt{n} := \Lambda$, then $\mathbf{Y} \succcurlyeq 0$ with at least $1 - \exp(-\Omega(n))$.*

Lemma 3.2 *If $\frac{-(\lambda+1)+n_i\mu_i}{2n_i} > 0,\ \forall i \in [K]$, then $\nu \geq 0$ with probability at least $1 - n\exp^{\frac{-\delta^2 n_{\min}}{2\sigma^2}}$, for $\delta = \min_{i\in[K]} \frac{-(\lambda+1)+n_i\mu_i}{2n_i}$.*

The condition $\frac{-(\lambda+1)+n_i\mu_i}{2n_i} > 0$ implies $\rho_i > \lambda+1,\ i \in [K]$ from the definition of ρ_i and hence the condition $\rho_{\min} > \lambda + 1$.

Lemma 3.3 *If $\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right) > 0$, for all $i \neq j \in [K]$, then $\mathbf{Z} \geq 0$ with probability at least $1 - n^2\exp^{-(\delta')^2\Omega(n_{\min})/\sigma^2}$, for $\delta' = \min_{i,j}\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right)$.*

The condition, for all $i \neq j \in [K]$, $\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right) > 0$ implies $\gamma_{ij} > \lambda + 1$ for all $i \neq j \in [K]$ by the definition of γ_{ij}, and hence the condition $\gamma_{\min} > \lambda + 1$.

Our solution does not depend on the order in which the data points are arranged, but the description of the proof will be easy to visualize when they are arranged in the order. Hence to aid visual understanding, assume that all the data points are arranged in the order of which cluster they belong to. In this case, $\mathbf{X}^*$ has block diagonal structure with $K \times K$ blocks. Diagonal blocks, $\mathbf{X}^*_{[i][i]} = \frac{1}{n_i}\mathbb{1}_{[i]}\mathbb{1}^T_{[i]}$ and off diagonal blocks, $\mathbf{X}^*_{[i][j]} = 0$ for $i \neq j,\ i,j \in [K]$.

Let $\mathbf{A} = \mathbf{M} + \sigma\,\mathbf{\Phi}$, where $\mathbf{\Phi}$ has i.i.d entries with zero mean and variance σ^2 and $\mathbf{M}$ is the matrix of mean similarities. Note that $\mathbf{M}_{[i][i]} = \mu_i\mathbb{1}_{[i]}\mathbb{1}^T_{[i]}$ and $\mathbf{M}_{[i][j]} = \mu_{out}\mathbb{1}_{[i]}\mathbb{1}^T_{[j]}$ for $i \neq j \in [K]$.

The condition (2.6) implies, for $i,j \in [K]$,

$$\mathbf{Y}_{[i][i]}\mathbb{1}_{[i]} = 0, \tag{3.3}$$

and

$$\mathbf{Y}_{[i][j]}\mathbb{1}_{[j]} = 0. \tag{3.4}$$

Also, since $\text{trace}(\mathbf{X}^*\mathbf{Z}) = 0$ and $\mathbf{X}^*_{[i][i]} > 0$, we have

$$\mathbf{Z}_{[i][i]} = 0. \tag{3.5}$$

3.1 Expression for ν

From (2.2), for $i \in [K]$,

$$\mathbf{Z}_{[i][i]} + \mathbf{Y}_{[i][i]} = \lambda \mathbf{I}_{[i][i]} + \mathbf{X}^*_{[i][i]} + \mathbb{1}_{[i]}\nu_{[i]}^T + \nu_{[i]}\mathbb{1}_{[i]} - \mathbf{M}_{[i][i]} - \sigma\ \mathbf{\Phi}_{[i][i]}. \tag{3.6}$$

From (3.5), we have

$$\mathbf{Y}_{[i][i]} = \lambda \mathbf{I}_{[i][i]} + \frac{1}{n_i}\mathbb{1}_{[i]}\mathbb{1}_{[i]}^T - \mu_i \mathbb{1}_{[i]}\mathbb{1}_{[i]}^T - \sigma\ \mathbf{\Phi}_{[i][i]} + \mathbb{1}_{[i]}\nu_{[i]}^T + \nu_{[i]}\mathbb{1}_{[i]}^T.$$

From (3.3) and from the equation above,

$$0 = \lambda \mathbb{1}_{[i]} + \frac{1}{n_i}\mathbb{1}_{[i]}n_i - \mu_i \mathbb{1}_{[i]}n_i - \sigma\ \mathbf{\Phi}_{[i][i]}\mathbb{1}_{[i]} + \mathbb{1}_{[i]}\left(\nu_{[i]}^T \mathbb{1}_{[1]}\right) + \nu_{[i]}n_i. \tag{3.7}$$

Observing that, $\nu_{[i]}^T \mathbb{1}_{[1]} = \mathbb{1}_{[1]}^T \nu_{[i]}$ (3.7) can be written as,

$$\nu_{[i]}n_i + \mathbb{1}_{[i]}\left(\mathbb{1}_{[1]}^T\nu_{[i]}\right) = -\lambda\mathbb{1}_{[i]} - \mathbb{1}_{[i]} + \mu_i n_i \mathbb{1}_{[i]} + \sigma\ \mathbf{\Phi}_{[i][i]}\mathbb{1}_{[i]}.$$

So we have,

$$\left(n_i \mathbf{I}_{[i][i]} + \mathbb{1}_{[i]}\mathbb{1}_{[i]}^T\right)\nu_{[i]} = \left(\left(-(\lambda+1) + \mu_i n_i\right)\mathbf{I}_{[i][i]} + \sigma\ \mathbf{\Phi}_{[i][i]}\right)\mathbb{1}_{[i]}$$

Using matrix inversion lemma,

$$\left(n_i \mathbf{I}_{[i][i]} + \mathbb{1}_{[i]}\mathbb{1}_{[i]}^T\right)^{-1} = \frac{1}{n_i}\mathbf{I}_{[i][i]} - \frac{1}{2n_i^2}\mathbb{1}_{[i]}\mathbb{1}_{[i]}^T.$$

Thus we can solve for $\nu_{[i]}$,

$$\nu_{[i]} = \left(\frac{1}{n_i}\mathbf{I}_{[i][i]} - \frac{1}{2n_i^2}\mathbb{1}_{[i]}\mathbb{1}_{[i]}^T\right)\left(\left(-(\lambda+1) + \mu_i n_i\right)\mathbf{I}_{[i][i]} + \sigma\ \mathbf{\Phi}_{[i][i]}\right)\mathbb{1}_{[i]}$$

which can be simplified to,

$$\nu_{[i]} = \left(\frac{\mu_i n_i - (\lambda+1)}{2n_i}\right)\mathbb{1}_{[i]} + \left(-\frac{\sigma}{2n_i^2}\left(\mathbb{1}_{[i]}^T\mathbf{\Phi}_{[i][i]}\mathbb{1}_{[i]}\right)\mathbf{I}_{[i][i]} + \frac{\sigma}{n_i}\mathbf{\Phi}_{[i][i]}\right)\mathbb{1}_{[i]}, \tag{3.8}$$

3.2 Expression for Z

Now consider the off-diagonal blocks of (2.2), for $i \neq j \in [K]$,

$$\mathbf{Z}_{[i][j]} + \mathbf{Y}_{[i][j]} = \underbrace{\lambda \mathbf{I}_{[i][j]}}_{0} + \underbrace{\mathbf{X}^*_{[i][j]}}_{0} - \mathbf{M}_{[i][j]} - \sigma\ \mathbf{\Phi}_{[i][j]} + \mathbb{1}_{[i]}\nu_{[j]}^T + \nu_{[i]}\mathbb{1}_{[j]}^T.$$

From (3.4), we have

$$\mathbf{Z}_{[i][j]}\mathbb{1}_{[j]} + \underbrace{\mathbf{Y}_{[i][j]}\mathbb{1}_{[j]}}_{0} = -\mu_{out}\mathbb{1}_{[i]}\mathbb{1}_{[j]}^T\mathbb{1}_{[j]} - \sigma\ \mathbf{\Phi}_{[i][j]}\mathbb{1}_{[j]} + \mathbb{1}_{[i]}\nu_{[j]}^T\mathbb{1}_{[j]} + \nu_{[i]}\mathbb{1}_{[j]}^T\mathbb{1}_{[j]}$$

which simplifies to

$$\mathbf{Z}_{[i][j]}\mathbb{1}_{[j]} = -\mu_{out} n_j \mathbb{1}_{[i]} - \sigma\ \mathbf{\Phi}_{[i][j]}\mathbb{1}_{[j]} + \mathbb{1}_{[i]}\nu_{[j]}^T\mathbb{1}_{[j]} + n_j \nu_{[i]}. \tag{3.9}$$

Define,

$$\mathbf{N}_{[i][j]} = -\mathbf{M}_{[i][j]} - \sigma\,\mathbf{\Phi}_{[i][j]} + \mathbb{1}_{[i]}\nu_{[j]}^T + \nu_{[i]}\mathbb{1}_{[j]}^T. \tag{3.10}$$

The expected value of $\mathbf{N}_{[i][j]}$ is,

$$\hat{\mathbf{N}}_{[i][j]} = \left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right)\mathbb{1}_{[i]}\mathbb{1}_{[j]}^T. \tag{3.11}$$

We now proceed to construct $\mathbf{Z}_{[i][j]}$ based on $\hat{\mathbf{N}}$ as follows,

$$\mathbf{Z}_{[i][j]} = \hat{\mathbf{N}}_{[i][j]} + \mathbf{w}_{[i]}^j\mathbb{1}_{[j]}^T + \mathbb{1}_{[i]}\left(\mathbf{u}_{[j]}^i\right)^T \tag{3.12}$$

where $\mathbf{w}_{[i]}^j$ is variable vector with non-zero entries only for indices in $\mathcal{C}_i$ (the superscript j is to identify its association to the block matrix $\mathbf{Z}_{[i][j]}$) and $\mathbf{u}_{[j]}^i$ is a variable vector with non-zero entries only for indices in $\mathcal{C}_j$ (the superscript i is to identify its association to the block matrix $\mathbf{Z}_{[i][j]}$). The variables in $\mathbf{w}_{[i]}^j$ and $\mathbf{u}_{[j]}^i$ can be found as a solution to the following system of equations obtained from $\mathbf{Z}_{[i][j]}\mathbb{1}_{[j]} = \mathbf{N}_{[i][j]}\mathbb{1}_{[j]}$ and $\mathbb{1}_{[i]}^T\mathbf{Z}_{[i][j]} = \mathbb{1}_{[i]}^T\mathbf{N}_{[i][j]}$:

$$\mathbf{w}_{[i]}^j n_j + \mathbb{1}_{[i]}\left(\mathbf{u}_{[j]}^i\right)^T\mathbb{1}_{[j]} = \underbrace{\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\mathbb{1}_{[j]}}_{:=\mathbf{t}_1} \tag{3.13}$$

$$\mathbb{1}_{[j]}\left(\mathbf{w}_{[i]}^j\right)^T\mathbb{1}_{[i]} + n_i\mathbf{u}_{[j]}^i = \underbrace{\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)^T\mathbb{1}_{[i]}}_{:=\mathbf{t}_2} \tag{3.14}$$

which can be written as,

$$\begin{bmatrix} n_j\mathbf{I}_{[i][i]} & \mathbb{1}_{[i]}\mathbb{1}_{[j]}^T \\ \mathbb{1}_{[j]}\mathbb{1}_{[i]}^T & n_i\mathbf{I}_{[j][j]} \end{bmatrix}\begin{bmatrix}\mathbf{w}_{[i]}^j \\ \mathbf{u}_{[j]}^i\end{bmatrix} = \begin{bmatrix}\mathbf{t}_1 \\ \mathbf{t}_2\end{bmatrix} \tag{3.15}$$

Since the system of equations (3.15) is singular with null space spanned by $(\mathbb{1}_{[i]}; -\mathbb{1}_{[j]})$, we proceed by solving the following perturbed system for $\theta > 0$,

$$\begin{bmatrix} n_j\mathbf{I}_{[i][i]} + \theta\mathbb{1}_{[i]}\mathbb{1}_{[i]}^T & (1-\theta)\mathbb{1}_{[i]}\mathbb{1}_{[j]}^T \\ (1-\theta)\mathbb{1}_{[j]}\mathbb{1}_{[i]}^T & n_i\mathbf{I}_{[j][j]} + \theta\mathbb{1}_{[j]}\mathbb{1}_{[j]}^T \end{bmatrix}\begin{bmatrix}\mathbf{w}_{[i]}^j \\ \mathbf{u}_{[j]}^i\end{bmatrix} = \begin{bmatrix}\mathbf{t}_1 \\ \mathbf{t}_2\end{bmatrix} \tag{3.16}$$

Left multiplying the system (3.16) by $(\mathbb{1}_{[i]}; -\mathbb{1}_{[j]})^T$ and observing that $(\mathbb{1}_{[i]}; -\mathbb{1}_{[j]})^T(\mathbf{t}_1; \mathbf{t}_2) = 0$, we obtain,

$$\underbrace{\theta(n_i + n_j)}_{>0}\left(\mathbb{1}_{[i]}^T\mathbf{w}_{[i]}^j - \mathbb{1}_{[j]}^T\mathbf{u}_{[j]}^T\right) = 0$$

and hence,

$$\mathbb{1}_{[i]}^T\mathbf{w}_{[i]}^j = \mathbb{1}_{[j]}^T\mathbf{u}_{[j]}. \tag{3.17}$$

Using (3.17) in the equations (3.13) and (3.14), we get,

$$\left(n_j\mathbf{I}_{[i][i]} + \mathbb{1}_{[i]}\mathbb{1}_{[i]}^T\right)\mathbf{w}_{[i]}^j = \left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\mathbb{1}_{[j]}$$

and

$$\left(\mathbb{1}_{[j]}\mathbb{1}_{[j]}^T + n_i\mathbf{I}_{[j][j]}\right)\mathbf{u}_{[j]}^i = \left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)^T\mathbb{1}_{[i]}$$

which can be solved to get,

$$\mathbf{w}^j_{[i]} = \frac{1}{n_j}\left(\mathbf{I}_{[i][i]} - \frac{\mathbb{1}_{[i]}\mathbb{1}^T_{[i]}}{n_i+n_j}\right)\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\mathbb{1}_{[j]} \tag{3.18}$$

and

$$\mathbf{u}^i_{[j]} = \frac{1}{n_i}\left(\mathbf{I}_{[j][j]} - \frac{\mathbb{1}_{[j]}\mathbb{1}^T_{[j]}}{n_i+n_j}\right)\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)^T\mathbb{1}_{[i]}. \tag{3.19}$$

Substituting in (3.12), we get the following expression for $\mathbf{Z}_{[i][j]}$,

$$\begin{aligned}\mathbf{Z}_{[i][j]} = &\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right)\mathbb{1}_{[i]}\mathbb{1}^T_{[j]} \\ &+ \frac{1}{n_j}\left(\mathbf{I}_{[i][i]} - \frac{\mathbb{1}_{[i]}\mathbb{1}^T_{[i]}}{n_i+n_j}\right)\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\mathbb{1}_{[j]}\mathbb{1}^T_{[j]} \\ &+ \frac{1}{n_i}\mathbb{1}_{[i]}\mathbb{1}^T_{[i]}\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\left(\mathbf{I}_{[j][j]} - \frac{\mathbb{1}_{[j]}\mathbb{1}^T_{[j]}}{n_i+n_j}\right).\end{aligned} \tag{3.20}$$

3.3 Expression for $\mathbf{Y}$

From (2.2), and the expressions for ν and $\mathbf{Z}$ (equations (3.8) and (3.20)), we have a construction for $\mathbf{Y}$ since,

$$\mathbf{Y} = -\mathbf{Z} + \lambda\mathbf{I} + \mathbf{X}^* - \mathbf{M} - \sigma\mathbf{\Phi} + \mathbb{1}\nu^T + \nu\mathbb{1}^T.$$

Now that we have expressions for candidate dual variables $\mathbf{Z}, \mathbf{Y}, \nu$, we proceed to show that under reasonable conditions on the problem parameters $\{\{\mu_i\}_{i\in[K]}, \mu_{out}, \sigma, \{n_i\}_{i\in[K]}\}$ and the regularization parameter λ, the conditions, $\mathbf{Y} \succcurlyeq 0,\ \nu \geq 0, \mathbf{Z} \geq 0$ hold with high probability.

3.4 Positive semidefiniteness of $\mathbf{Y}$

The expression for $\mathbf{Y}$ is as follows,

$$\mathbf{Y} = \lambda\mathbf{I} - \sigma\mathbf{\Phi}\underbrace{-\mathbf{Z} + \mathbf{X}^* - \mathbf{M} + \mathbb{1}\nu^T + \nu\mathbb{1}^T}_{\text{all have}\,\mathbb{1}_{[i]}\,\text{or}\,\mathbb{1}^T_{[i]}\,\text{by construction}}$$

For any vector $\mathbf{x} \in \mathbb{R}^n$, consider the decomposition,

$$\mathbf{x} = \sum_{i=1}^{K} x_i\mathbb{1}_{[i]} + \mathbf{x}_\perp \tag{3.21}$$

where $\mathbf{x}_\perp$ is sum of components perpendicular to $\mathbb{1}_{[i]},\ i \in [K]$. From KKT, $\mathbf{Y}\mathbb{1}_{[i]} = 0$, $\mathbb{1}^T_{[i]}\mathbf{Y} = 0$. Also, from construction of $\mathbf{Z}$, ν and forms of $\mathbf{X}^*$ and $\mathbf{M}$,

$$\mathbf{x}^T_\perp\left(-\mathbf{Z} + \mathbf{X}^* - \mathbf{M} + \mathbb{1}\nu^T + \nu\mathbb{1}^T\right) = \mathbf{0}^T$$

$$\left(-\mathbf{Z} + \mathbf{X}^* - \mathbf{M} + \mathbb{1}\nu^T + \nu\mathbb{1}^T\right)\mathbf{x}_\perp = \mathbf{0}$$

So, $\mathbf{x}^T\mathbf{Y}\mathbf{x} = \mathbf{x}^T_\perp(\lambda\mathbf{I} - \sigma\mathbf{\Phi})\mathbf{x}_\perp \geq (\lambda - \sigma||\mathbf{\Phi}||)||\mathbf{x}_\perp||_2^2$. Since $\mathbf{\Phi}$ is a random matrix with bounded i.i.d. entries with zero mean and unit variance, using the standard results in random matrix theory [1], with high probability $(1-\exp(-\Omega(n)))$, $||\mathbf{\Phi}|| = 2\sqrt{n}$. Hence, if $\lambda > \sigma 2\sqrt{n}$, then $\mathbf{Y}$ is positive semidefinite with high probability (Lemma 3.1).

3.5 Non-negativity of ν

Recall the expression for ν from 3.8, for $i \in [K]$,

$$\nu_{[i]} = \underbrace{\left(\frac{\mu_i n_i - (\lambda+1)}{2n_i}\right)\mathbb{1}_{[i]}}_{\text{expected value of}\nu_{[i]}} + \underbrace{\left(-\frac{\sigma}{2n_i^2}\left(\mathbb{1}_{[i]}^T \mathbf{\Phi}_{[i][i]}\mathbb{1}_{[i]}\right)\mathbf{I}_{[i][i]} + \frac{\sigma}{n_i}\mathbf{\Phi}_{[i][i]}\right)\mathbb{1}_{[i]}}_{\text{perturbation term}}.$$

Without loss of generality, consider the first entry of $\nu_{[i]}$.

$$\nu_{[i]1} = \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \underbrace{\left(-\frac{\sigma}{2n_i^2}\sum_{a,b=1}^{n_i}\mathbf{\Phi}_{[i][i]a,b} + \frac{\sigma}{n_i}\sum_{d=1}^{n_i}\mathbf{\Phi}_{[i][i]1,d}\right)}_{:=\nu_{[i]1,p}} \tag{3.22}$$

$$\begin{aligned}
\nu_{[i]1,p} &= -\frac{\sigma}{2n_i^2}\sum_{a,b=2}^{n_i}\mathbf{\Phi}_{[i][i]a,b} - \frac{\sigma}{2n_i^2}\sum_{a=2}^{n_i}\mathbf{\Phi}_{[i][i]a,1} - \frac{\sigma}{2n_i^2}\sum_{b=1}^{n_i}\mathbf{\Phi}_{[i][i]1,b} + \frac{\sigma}{n_i}\sum_{d=1}^{n_i}\mathbf{\Phi}_{[i][i]1,d} \\
&= -\frac{\sigma}{2n_i^2}\sum_{a,b=2}^{n_i}\mathbf{\Phi}_{[i][i]a,b} - \frac{\sigma}{2n_i^2}\sum_{a=2}^{n_i}\mathbf{\Phi}_{[i][i]a,1} + \left(\frac{\sigma}{n_i} - \frac{\sigma}{2n_i^2}\right)\sum_{b=1}^{n_i}\mathbf{\Phi}_{[i][i]1,b} \\
&= -\frac{\sigma}{2n_i^2}\underbrace{\sum_{a=2}^{n_i}\mathbf{\Phi}_{[i][i]a,a}}_{n_i-1\text{ terms}} - \frac{2\sigma}{2n_i^2}\underbrace{\sum_{a,b=2,a<b}^{n_i}\mathbf{\Phi}_{[i][i]a,b}}_{\frac{(n_i-1)(n_i-2)}{2}\text{ terms}} + \left(\frac{\sigma}{n_i} - \frac{\sigma}{2n_i^2}\right)\mathbf{\Phi}_{[i][i]1,1} \\
&\quad + \left(\frac{\sigma}{n_i} - \frac{2\sigma}{2n_i^2}\right)\underbrace{\sum_{b=1}^{n_i}\mathbf{\Phi}_{[i][i]1,b}}_{n_i-1\text{ terms}}.
\end{aligned} \tag{3.23}$$

Theorem 2 (Hoeffding) *Let $W_1, \cdots, W_m$ be independent random variables with zero mean and $|W_i| \le R_i$ almost surely for all i. Define*

$$S := \sum_{i=1}^{m} W_i \quad \tau^2 := \sum_{i=1}^{m} R_i.$$

Then, Var$(S) \le \tau^2$ and $\mathbb{P}\{|S| > \delta\} \le 2\exp^{-\delta^2/\tau^2}$.

From (3.23), we see that the perturbation term is a sum of $\binom{n_i}{2} + n_i$ bounded independent zero mean random variables. A conservative bound on $\mathbf{\Phi}$ is $|\mathbf{\Phi}_{lm}| \le 1$ since $\mathbf{A}_{lm} \in [0,1]$. So, we can compute τ as follows

$$\begin{aligned}
\tau^2 &= (n_i-1)\frac{\sigma^2}{4n_i^4} + \binom{n_i-1}{2}\frac{\sigma^2}{n_i^4} + \frac{\sigma^2}{n_i^2}\left(1-\frac{1}{2n_i}\right)^2 + (n_i-1)\frac{\sigma^2}{n_i^2}\left(1-\frac{1}{n_i}\right)^2 \\
&= \frac{\sigma^2}{n_i}\left(1-\frac{3}{2n_i}+\frac{3}{4n_i^2}\right) < \frac{2\sigma^2}{n_i}.
\end{aligned} \tag{3.24}$$

So, by Hoeffding's inequality 2,for $\delta > 0$, $\mathbb{P}\{\nu_{[i]1,p} < -\delta\} \le \exp^{\frac{-\delta^2 n_i}{2\sigma^2}}$. Using union bound we can conclude Lemma 3.2

3.6 Non-negativity of Z

Recall,

$$\mathbf{Z}_{[i][j]} = \underbrace{\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right)\mathbb{1}_{[i]}\mathbb{1}_{[j]}^T}_{\text{expected value of } \mathbf{Z}_{[i][j]}}$$
$$+ \underbrace{\frac{1}{n_j}\left(\mathbf{I}_{[i][i]} - \frac{\mathbb{1}_{[i]}\mathbb{1}_{[i]}^T}{n_i+n_j}\right)\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\mathbb{1}_{[j]}}_{=\mathbf{w}_{[i]}^j}\mathbb{1}_{[j]}^T$$
$$+ \mathbb{1}_{[i]}\underbrace{\frac{1}{n_i}\mathbb{1}_{[i]}^T\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\left(\mathbf{I}_{[j][j]} - \frac{\mathbb{1}_{[j]}\mathbb{1}_{[j]}^T}{n_i+n_j}\right)}_{=\left(\mathbf{u}_{[j]}^i\right)^T}.$$

Each column of $\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)\mathbb{1}_{[j]}\mathbb{1}_{[j]}^T$ is row sums of $\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)$. Each row of $\mathbb{1}_{[i]}\mathbb{1}_{[i]}^T\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)$ is column sum of $\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)$.

Without loss of generality, consider the $(1,1)$-th term of $\mathbf{Z}_{[i][j]}$,

$$\mathbf{Z}_{[i][j]1,1} = \left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right)$$
$$+ \underbrace{\mathbf{w}_{[i]}^j\mathbb{1}_{[j]}^T + \mathbb{1}_{[i]}\left(\mathbf{u}_{[i]}^j\right)^T}_{:=\mathbf{Z}_{[i][j]1,1,p}(\text{perturbation term})}. \tag{3.25}$$

So, $\mathbf{Z}_{[i][j]1,1,p} = \left(\mathbf{w}_{[i]}^j\mathbb{1}_{[j]}^T\right)_{1,1} + \left(\mathbb{1}_{[i]}\left(\mathbf{u}_{[i]}^j\right)^T\right)_{1,1}$.

$(1,1)$-th term of $\mathbf{w}_{[i]}^j\mathbb{1}_{[i]}^T$ is

$$\left(\mathbf{w}_{[i]}^j\mathbb{1}_{[j]}^T\right)_{1,1} = \frac{1}{n_j}\left(1 - \frac{1}{n_i+n_j}\right)\sum_{t=1}^{n_j}\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)_{1,t} - \frac{1}{n_j}\frac{1}{n_i+n_j}\sum_{s=2}^{n_i}\left(\sum_{t=1}^{n_j}\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)_{s,t}\right) \tag{3.26}$$

$(1,1)$-th term of $\mathbb{1}_{[i]}\left(\mathbf{u}_{[j]}^i\right)^T$ is

$$\left(\mathbb{1}_{[i]}\left(\mathbf{u}_{[i]}^j\right)^T\right)_{1,1} = \frac{1}{n_i}\left(1 - \frac{1}{n_i+n_j}\right)\sum_{s=1}^{n_i}\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)_{s,1} - \frac{1}{n_i}\frac{1}{n_i+n_j}\sum_{t=2}^{n_j}\left(\sum_{s=1}^{n_i}\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)_{s,t}\right) \tag{3.27}$$

where,

$$\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]} = -\sigma\mathbf{\Phi}_{[i][j]} - \mathbb{1}_{[i]}\mathbb{1}_{[j]}^T\frac{\sigma}{2n_j^2}\mathbb{1}_{[j]}^T\mathbf{\Phi}_{[j][j]}\mathbb{1}_{[j]} + \mathbb{1}_{[i]}\mathbb{1}_{[j]}^T\frac{\sigma}{n_j}\mathbf{\Phi}_{[j][j]}$$
$$- \mathbb{1}_{[i]}\mathbb{1}_{[j]}^T\frac{\sigma}{2n_i^2}\mathbb{1}_{[i]}^T\mathbf{\Phi}_{[i][i]}\mathbb{1}_{[i]} + \frac{\sigma}{n_i}\mathbf{\Phi}_{[i][i]}\mathbb{1}_{[i]}\mathbb{1}_{[j]}^T$$

Every row of $\mathbb{1}_{[i]}\mathbb{1}_{[j]}^T \frac{\sigma}{n_j}\mathbf{\Phi}_{[j][j]}$ is column sum of $\frac{\sigma}{n_j}\mathbf{\Phi}_{[j][j]}$, and every column is the column sum of the corresponding column in $\frac{\sigma}{n_j}\mathbf{\Phi}_{[j][j]}$. Similarly, every column of $\frac{\sigma}{n_i}\mathbf{\Phi}_{[i][i]}\mathbb{1}_{[i]}\mathbb{1}_{[j]}^T$ is row sum of $\frac{\sigma}{n_i}\mathbf{\Phi}_{[i][i]}$, and every row is the row sum of the corresponding row in $\frac{\sigma}{n_j}\mathbf{\Phi}_{[j][j]}$.

Row sum of a-th row of $\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}$,

$$\sum_{t=1}^{n_j}\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)_{a,t} = -\sigma \sum_{t=1}^{n_j}\mathbf{\Phi}_{[i][j]a,t} + \frac{\sigma}{2n_j}\sum_{r=1}^{n_j}\sum_{c=1}^{n_j}\mathbf{\Phi}_{[j][j]r,c} - \frac{n_j\sigma}{2n_i^2}\sum_{r=1}^{n_i}\sum_{c=1}^{n_i}\mathbf{\Phi}_{[i][i]r,c} + \frac{n_j\sigma}{n_i}\sum_{t=1}^{n_i}\mathbf{\Phi}_{[i][i]a,t}$$

Column sum of b-th column of $\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}$,

$$\sum_{s=1}^{n_i}\left(\mathbf{N}_{[i][j]} - \hat{\mathbf{N}}_{[i][j]}\right)_{s,b} = -\sigma \sum_{s=1}^{n_i}\mathbf{\Phi}_{[i][j]s,b} - \frac{n_i\sigma}{2n_j^2}\sum_{r=1}^{n_j}\sum_{c=1}^{n_j}\mathbf{\Phi}_{[j][j]r,c} + \frac{n_i\sigma}{n_j}\sum_{s=1}^{n_j}\mathbf{\Phi}_{[j][j]s,b} + \frac{\sigma}{2n_i}\sum_{r=1}^{n_i}\sum_{c=1}^{n_i}\mathbf{\Phi}_{[i][i]r,c}$$

Let $\sum_{r,c}\mathbf{B}$ denote the sum over all the entries of the matrix $\mathbf{B}$. So, (1,1)-th entry of $\mathbf{Z}_{[i][j]1,1,p}$ is

$$\begin{aligned}\mathbf{Z}_{[i][j]1,1,p} = {} & \frac{1}{n_j}\left(1 - \frac{1}{n_i+n_j}\right)\left(-\sigma\sum_{t=1}^{n_j}\mathbf{\Phi}_{[i][j]1,t} + \frac{\sigma}{2n_j}\sum_{r,c}\mathbf{\Phi}_{[j][j]} - \frac{n_j\sigma}{2n_i^2}\sum_{r,c}\mathbf{\Phi}_{[i][i]} + \frac{n_j\sigma}{n_i}\sum_{t=1}^{n_i}\mathbf{\Phi}_{[i][i]1,t}\right) \\ & - \frac{1}{n_j}\frac{1}{n_i+n_j}\sum_{s=2}^{n_i}\left(-\sigma\sum_{t=1}^{n_j}\mathbf{\Phi}_{[i][j]s,t} + \frac{\sigma}{2n_j}\sum_{r,c}\mathbf{\Phi}_{[j][j]} - \frac{n_j\sigma}{2n_i^2}\sum_{r,c}\mathbf{\Phi}_{[i][i]} + \frac{n_j\sigma}{n_i}\sum_{t=1}^{n_i}\mathbf{\Phi}_{[i][i]s,t}\right) \\ & + \frac{1}{n_i}\left(1 - \frac{1}{n_i+n_j}\right)\left(-\sigma\sum_{s=1}^{n_i}\mathbf{\Phi}_{[i][j]s,1} - \frac{n_i\sigma}{2n_j^2}\sum_{r,c}\mathbf{\Phi}_{[j][j]} + \frac{n_i\sigma}{n_j}\sum_{s=1}^{n_i}\mathbf{\Phi}_{[j][j]s,1} + \frac{\sigma}{2n_i}\sum_{r,c}\mathbf{\Phi}_{[i][i]}\right) \\ & - \frac{1}{n_i}\frac{1}{n_i+n_j}\sum_{t=2}^{n_j}\left(-\sigma\sum_{s=1}^{n_i}\mathbf{\Phi}_{[i][j]s,t} - \frac{n_i\sigma}{2n_j^2}\sum_{r,c}\mathbf{\Phi}_{[j][j]} + \frac{n_i\sigma}{n_j}\sum_{s=1}^{n_i}\mathbf{\Phi}_{[j][j]s,t} + \frac{\sigma}{2n_i}\sum_{r,c}\mathbf{\Phi}_{[i][i]}\right)\end{aligned}$$

Collecting all the terms corresponding to $\frac{\sigma}{2n_j}\sum_{r,c}\mathbf{\Phi}_{[j][j]}$,

$$\frac{1}{n_j}\left(1 - \frac{1}{n_i+n_j}\right) - \frac{1}{n_j}\frac{1}{n_i+n_j}(n_i - 1) + \left(-\frac{n_i}{n_j}\right)\left(\frac{1}{n_i}\left(1 - \frac{1}{n_i+n_j}\right) - \frac{1}{n_i}\frac{1}{n_i+n_j}(n_j - 1)\right)$$
$$= \frac{1}{n_i+n_j}\left(1 - \frac{n_i}{n_j}\right).$$

Similarly, collecting all the terms corresponding to $\frac{\sigma}{2n_i}\sum_{r,c}\mathbf{\Phi}_{[i][i]}$,

$$\left(-\frac{n_j}{n_i}\right)\left(\frac{1}{n_j}\left(1 - \frac{1}{n_i+n_j}\right) - \frac{1}{n_j}\frac{1}{n_i+n_j}(n_i - 1)\right) + \frac{1}{n_i}\left(1 - \frac{1}{n_i+n_j}\right) - \frac{1}{n_i}\frac{1}{n_i+n_j}(n_j - 1)$$
$$= \frac{1}{n_i+n_j}\left(1 - \frac{n_j}{n_i}\right).$$

So we have,

$$
\begin{aligned}
\mathbf{Z}_{[i][j]1,1,p} =& \frac{1}{n_j}\left(1-\frac{1}{n_i+n_j}\right)\left(-\sigma\sum_{t=1}^{n_j}\mathbf{\Phi}_{[i][j]1,t}+\frac{n_j\sigma}{n_i}\sum_{t=1}^{n_i}\mathbf{\Phi}_{[i][i]1,t}\right)\\
&-\frac{1}{n_j}\frac{1}{n_i+n_j}\sum_{s=2}^{n_i}\left(-\sigma\sum_{t=1}^{n_j}\mathbf{\Phi}_{[i][j]s,t}+\frac{n_j\sigma}{n_i}\sum_{t=1}^{n_i}\mathbf{\Phi}_{[i][i]s,t}\right)\\
&+\frac{1}{n_i}\left(1-\frac{1}{n_i+n_j}\right)\left(-\sigma\sum_{s=1}^{n_i}\mathbf{\Phi}_{[i][j]s,1}+\frac{n_i\sigma}{n_j}\sum_{s=1}^{n_j}\mathbf{\Phi}_{[j][j]s,1}\right)\\
&-\frac{1}{n_i}\frac{1}{n_i+n_j}\sum_{t=2}^{n_j}\left(-\sigma\sum_{s=1}^{n_i}\mathbf{\Phi}_{[i][j]s,t}+\frac{n_i\sigma}{n_j}\sum_{s=1}^{n_j}\mathbf{\Phi}_{[j][j]s,t}\right)\\
&+\frac{1}{n_i+n_j}\left(1-\frac{n_j}{n_i}\right)\frac{\sigma}{2n_i}\sum_{r,c}\mathbf{\Phi}_{[i][i]}+\frac{1}{n_i+n_j}\left(1-\frac{n_i}{n_j}\right)\frac{\sigma}{2n_j}\sum_{r,c}\mathbf{\Phi}_{[j][j]}
\end{aligned}
$$

Rearranging the terms,

$$
\begin{aligned}
\mathbf{Z}_{[i][j]1,1,p} =& \frac{\sigma}{n_i}\left(1-\frac{1}{2n_i}\right)\sum_{t=1}^{n_i}\mathbf{\Phi}_{[i][i]1,t}-\frac{\sigma}{2n_i^2}\sum_{s=2}^{n_i}\sum_{t=1}^{n_i}\mathbf{\Phi}_{[i][i]s,t}\\
&+\frac{\sigma}{n_j}\left(1-\frac{1}{2n_j}\right)\sum_{s=1}^{n_j}\mathbf{\Phi}_{[j][j]s,1}-\frac{\sigma}{2n_j^2}\sum_{s=1}^{n_j}\sum_{t=2}^{n_j}\mathbf{\Phi}_{[j][j]s,t}\\
&-\frac{\sigma}{n_j}\left(1-\frac{1}{n_i+n_j}\right)\sum_{t=1}^{n_j}\mathbf{\Phi}_{[i][j]1,t}-\frac{\sigma}{n_i}\left(1-\frac{1}{n_i+n_j}\right)\sum_{s=1}^{n_i}\mathbf{\Phi}_{[i][j]s,1}\\
&+\frac{\sigma}{n_j\left(n_i+n_j\right)}\sum_{s=2}^{n_i}\sum_{t=1}^{n_j}\mathbf{\Phi}_{[i][j]s,t}+\frac{\sigma}{n_i\left(n_i+n_j\right)}\sum_{s=1}^{n_i}\sum_{t=2}^{n_j}\mathbf{\Phi}_{[i][j]s,t}
\end{aligned}
$$

Putting together the independent terms,

$$
\begin{aligned}
&\mathbf{Z}_{[i][j]1,1,p}\\
&=\frac{\sigma}{n_i}\left(1-\frac{1}{2n_i}\right)\mathbf{\Phi}_{[i][i]1,1}+\frac{\sigma}{n_i}\left(1-\frac{1}{n_i}\right)\underbrace{\sum_{t=2}^{n_i}\mathbf{\Phi}_{[i][i]1,t}}_{n_i-1\text{ terms}}-\frac{\sigma}{2n_i^2}\underbrace{\sum_{s=2}^{n_i}\mathbf{\Phi}_{[i][i]s,s}}_{n_i-1\text{ terms}}-\frac{\sigma}{n_i^2}\underbrace{\sum_{s,t=2,s<t}^{n_i}\mathbf{\Phi}_{[i][i]s,t}}_{(n_i-1)(n_i-2)/2\text{ terms}}\\
&+\frac{\sigma}{n_j}\left(1-\frac{1}{2n_j}\right)\mathbf{\Phi}_{[j][j]1,1}+\frac{\sigma}{n_j}\left(1-\frac{1}{n_j}\right)\underbrace{\sum_{t=2}^{n_j}\mathbf{\Phi}_{[j][j]1,t}}_{n_j-1\text{ terms}}-\frac{\sigma}{2n_j^2}\underbrace{\sum_{s=2}^{n_j}\mathbf{\Phi}_{[j][j]s,s}}_{n_j-1\text{ terms}}-\frac{\sigma}{n_j^2}\underbrace{\sum_{s,t=2,s<t}^{n_j}\mathbf{\Phi}_{[j][j]s,t}}_{(n_j-1)(n_j-2)/2\text{ terms}}\\
&-\sigma\left(\frac{1}{n_i}+\frac{1}{n_j}-\frac{1}{n_in_j}\right)\mathbf{\Phi}_{[i][j]1,1}-\frac{\sigma}{n_j}\left(1-\frac{1}{n_i}\right)\underbrace{\sum_{t=2}^{n_j}\mathbf{\Phi}_{[i][j]1,t}}_{n_j-1\text{ terms}}-\frac{\sigma}{n_i}\left(1-\frac{1}{n_j}\right)\underbrace{\sum_{s=2}^{n_i}\mathbf{\Phi}_{[i][j]s,1}}_{n_i-1\text{ terms}}\\
&+\frac{\sigma}{n_in_j}\underbrace{\sum_{s=2}^{n_i}\sum_{t=2}^{n_j}\mathbf{\Phi}_{[i][j]s,t}}_{(n_i-1)(n_j-1)\text{ terms}} \qquad (3.28)
\end{aligned}
$$

From (3.28), we see that $\mathbf{Z}_{[i][j]1,1,p}$ is a sum of $\binom{n_i}{2}+n_i+\binom{n_j}{2}+n_j+n_in_j$ bounded independent zero-mean random variables.

$$
\begin{aligned}
\tau^2 &= \frac{\sigma^2}{n_i^2}\left(1-\frac{1}{2n_i}\right)^2 + (n_i-1)\frac{\sigma^2}{n_i^2}\left(1-\frac{1}{n_i}\right)^2 + (n_i-1)\frac{\sigma^2}{4n_i^4} + \frac{(n_i-1)(n_i-2)}{2}\frac{\sigma^2}{n_i^4} \\
&+ \frac{\sigma^2}{n_j^2}\left(1-\frac{1}{2n_j}\right)^2 + (n_j-1)\frac{\sigma^2}{n_j^2}\left(1-\frac{1}{n_j}\right)^2 + (n_j-1)\frac{\sigma^2}{4n_j^4} + \frac{(n_j-1)(n_j-2)}{2}\frac{\sigma^2}{n_j^4} \\
&+ \sigma^2\left(\frac{1}{n_i}+\frac{1}{n_j}-\frac{1}{n_in_j}\right)^2 + (n_j-1)\frac{\sigma^2}{n_j^2}\left(1-\frac{1}{n_i}\right)^2 \\
&+(n_i-1)\frac{\sigma^2}{n_i^2}\left(1-\frac{1}{n_j}\right)^2 + (n_i-1)(n_j-1)\frac{\sigma^2}{n_i^2n_j^2} \\
&= \frac{\sigma^2}{n_i}\left(1-\frac{3}{2n_i}+\frac{3}{4n_i^2}\right) + \frac{\sigma^2}{n_j}\left(1-\frac{3}{2n_j}+\frac{3}{4n_j^2}\right) + \frac{\sigma^2}{n_in_j}(n_i+n_j-1) \\
&< \frac{2\sigma^2}{n_i} + \frac{2\sigma^2}{n_j} + \frac{\sigma^2}{n_in_j}(n_i+n_j) = 3\sigma^2\left(\frac{1}{n_i}+\frac{1}{n_j}\right)
\end{aligned}
\tag{3.29}
$$

Using a conservative bound on $\mathbf{\Phi}$ as $|\mathbf{\Phi}| \le 1$, from Hoeffding's inequality $\mathbb{P}\{\mathbf{Z}_{[i][j]} < -\delta'\} \le \exp^{-(\delta')^2\Omega(n_{\min})/\sigma^2}$. Using union bound, we can conclude Lemma 3.3.

4 Large Number of Outliers

For the case with K disjoint clusters with large number of outliers, the desired solution is

$$
\tilde{\mathbf{X}}_{l,m} = \begin{cases} \frac{1}{n_i}\ , & \text{if both nodes } l,m \text{ are in the same cluster } i. \\ 0\ , & \text{if nodes } l,m \text{ are in different clusters.} \\ \frac{1}{n_{K+1}} & \text{if both nodes } l,m \text{ are outliers.} \end{cases}
\tag{4.1}
$$

The outliers together get considered as a new cluster $K+1$. We have,

$$
\hat{\mathbf{X}}_{[K+1][K+1]} = n_{K+1}\mathbb{1}_{[K+1]}\mathbb{1}^T_{[K+1]},\ \mathbf{M}_{[K+1][K+1]} = \mu_{out}\mathbb{1}_{[K+1]}\mathbb{1}^T_{[K+1]},\ \mu_{K+1} = \mu_{out}.
$$

So, $\mathbf{Z}_{[K+1][K+1]} = 0$ and expressions for $\nu_{[K+1]}$, $\mathbf{Z}_{[i][K+1]}$ obtained just the same way as for other clusters as in equations (3.8) and (3.20). The conditions $\rho_{K+1} > \lambda+1$ and $\gamma_{i,K} > \lambda+1$ for all $i \in [K]$ are required for $\nu_{K+1} > 0$ and $\mathbf{Z}_{[i][K+1]} > 0$ respectively.

Lemma 4.1 *If $\lambda > \sigma 2\sqrt{n} := \Lambda$, then $\mathbf{Y} \succcurlyeq 0$ with at least $1-\exp(-\Omega(n))$.*

Lemma 4.2 *If $\frac{-(\lambda+1)+n_i\mu_i}{2n_i} > 0,\ \forall i \in [K+1]$, then $\nu \ge 0$ with probability at least $1 - n\exp^{\frac{-\delta^2 n_{\min}}{2\sigma^2}}$, for $\delta = \min_{i\in[K+1]} \frac{-(\lambda+1)+n_i\mu_i}{2n_i}$.*

The condition $\frac{-(\lambda+1)+n_i\mu_i}{2n_i} > 0$ implies $\rho_i > \lambda+1,\ i \in [K+1]$ from the definition of ρ_i and hence the condition $\rho_{\min} > \lambda+1$.

Lemma 4.3 *If $\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right) > 0$, for all $i \neq j \in [K+1]$, then $\mathbf{Z} \ge 0$ with probability at least $1-n^2\exp^{-(\delta')^2\Omega(n_{\min})/\sigma^2}$, for $\delta' = \min_{i,j}\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right)$.*

The condition, for all $i \neq j \in [K+1]$, $\left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} + \frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right) > 0$ implies $\gamma_{ij} > \lambda + 1$ for all $i \neq j \in [K+1]$ by the definition of γ_{ij}, and hence the condition $\gamma_{\min} > \lambda + 1$.

From Lemma 4.1, 4.2 and 4.3 we can conclude Theorem 3 as follows:

Theorem 3 *[Large Number of Outliers] If the regularizer λ is within the following range,*

$$\Lambda < \lambda < \min\left\{\rho_{min}^{out}, \gamma_{\min}^{out}\right\} - 1, \tag{4.2}$$

then $\tilde{\mathbf{X}}$ is the unique optimal solution to Program 1.1 with high probability.

5 Small Number of Outliers

In the case where the number of outliers is small, the desired solution is (3.1). So, $\mathbf{X}^*$ has non-zero entries only in the diagonal blocks for $i \in [K]$ and $\mathbf{X}^*_{[K+1][K+1]} = \mathbf{0}$. So, from (2.5),

$$\nu^T\left(\mathbf{X}^*\mathbb{1} - \mathbb{1}\right) = \mathbf{0}.$$

and $\nu \geq 0$, we get, $\nu_{[K+1]} = 0$. The expressions for $\nu_{[i]}, i \in [K]$ remain the same as in the case of no outliers (Equation (3.8)). The expressions for $\mathbf{Z}_{[i][j]}$ for $i \neq j \in [K]$ remain the same as in (3.20). From steps similar to the case of no outliers, we can find the expression for $\mathbf{Z}_{[i][K+1]},\ i \in [K]$ as,

$$\mathbf{Z}_{[i][K+1]} = \hat{\mathbf{N}}_{[i][K+1]} + \mathbf{w}_{[i]}^{K+1}\mathbb{1}_{[K+1]}^T + \mathbb{1}_{[i]}\left(\mathbf{u}_{[K+1]}^i\right)^T \tag{5.1}$$

where,

$$\hat{\mathbf{N}}_{[i][K+1]} = \left(-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i}\right)\mathbb{1}_{[i]}\mathbb{1}_{[K+1]}^T. \tag{5.2}$$

since $\nu_{[K+1]} = 0$ and $\mathbf{w}$ and $\mathbf{u}$ are defined as in (3.18) and (3.19). For $\mathbf{Z}_{[i][K+1]} > 0$ to hold with high probability we will require, $-\mu_{out} + \frac{-(\lambda+1)+n_i\mu_i}{2n_i} > 0$, which gives the condition $\eta_i > \lambda + 1$ from the definition of η_i.

5.1 Positive semidefiniteness of Y

The expression for $\mathbf{Y}$ is as follows,

$$\mathbf{Y} = \lambda\mathbf{I} - \sigma\mathbf{\Phi}\underbrace{-\mathbf{Z} + \mathbf{X}^* - \mathbf{M} + \mathbb{1}\nu^T + \nu\mathbb{1}^T}_{\text{all have } \mathbb{1}_{[i]} \text{or} \mathbb{1}_{[i]}^T \text{by construction}}$$

Further, $\mathbf{X}^*_{[K+1][K+1]}$, $\nu_{[K+1]}$ and $\mathbf{Z}_{[K+1][K+1]}$ are all zeros.

For any vector $\mathbf{x} \in \mathbb{R}^n$, consider the decomposition,

$$\mathbf{x} = \sum_{i=1}^{K} x_i\mathbb{1}_{[i]} + \mathbf{x}_\perp + \mathbf{x}_{out} \tag{5.3}$$

where $\mathbf{x}_\perp$ is sum of components perpendicular to $\mathbb{1}_{[i]},\ i \in [K]$ and has all zero entries in the entries corresponding to $\mathcal{C}_{K+1}$, $\mathbf{x}_{out}$ has non-zero entries only in the entries corresponding to C_{K+1}.

From KKT, $\mathbf{Y}\mathbb{1}_{[i]} = 0$, $\mathbb{1}_{[i]}^T\mathbf{Y} = 0$ for $i \in [K]$. Also, from construction of $\mathbf{Z}$, ν and forms of $\mathbf{X}^*$ and $\mathbf{M}$,

$$\mathbf{x}_\perp^T\left(-\mathbf{Z} + \mathbf{X}^* - \mathbf{M} + \mathbb{1}\nu^T + \nu\mathbb{1}^T\right) = \mathbf{0}^T$$

$$\left(-\mathbf{Z}+\mathbf{X}^*-\mathbf{M}+\mathbb{1}\nu^T+\nu\mathbb{1}^T\right)\mathbf{x}_\perp=\mathbf{0}$$

$$\begin{aligned}\mathbf{x}^T\mathbf{Y}\mathbf{x} &= \mathbf{x}_\perp^T\mathbf{Y}\mathbf{x}_\perp+\mathbf{x}_{out}^T\mathbf{Y}\mathbf{x}_{out}+2\mathbf{x}_{out}^T\mathbf{Y}\mathbf{x}_\perp\\ &= \mathbf{x}_\perp^T\left(\lambda\mathbf{I}-\sigma\mathbf{\Phi}\right)\mathbf{x}_\perp+\mathbf{x}_{out}^T\left(\lambda\mathbf{I}-\sigma\mathbf{\Phi}-\mu_{out}\mathbb{1}_{[K+1]}\mathbb{1}_{[K+1]}^T\right)\mathbf{x}_{out}\\ &\quad+2\mathbf{x}_{out}^T\left(-\sigma\mathbf{\Phi}-\underbrace{\mu_{out}\mathbb{1}_{[K+1]}\mathbb{1}_{[1:K]}^T+\mathbb{1}_{[K+1]}^T\nu_{[1:K]}-Z_{[i][K+1]}}_{\text{all have}\mathbb{1}_{[i]}or\mathbb{1}_{[i]}^T}\right)\mathbf{x}_\perp\\ &= \mathbf{x}_\perp^T\left(\lambda\mathbf{I}-\sigma\mathbf{\Phi}\right)\mathbf{x}_\perp+\mathbf{x}_{out}^T\left(\lambda\mathbf{I}-\sigma\mathbf{\Phi}-\mu_{out}\mathbb{1}_{[K+1]}\mathbb{1}_{[K+1]}^T\right)\mathbf{x}_{out}+2\mathbf{x}_{out}\left(-\sigma\mathbf{\Phi}_{[K+1][1:K]}\right)\mathbf{x}_\perp\end{aligned} \tag{5.4}$$

where $\mathbf{v}_{[1:K]}$ denotes a vector with non-zero values in entries corresponding to $\cup_i\mathcal{C}_i,\ i\in[K]$ and zeros in the entries corresponding to $\mathcal{C}_{K+1}$ and similarly for a matrix, $\mathbf{B}_{[i][1:K]}$ has non-zero block corresponding to $\mathcal{C}_i\times\cup_j\mathcal{C}_j,\ j\in[K]$ and zero everywhere else.

Defining $\mathbf{q}:=(\mathbf{x}_\perp;\mathbf{x}_{out})$,

$$\begin{aligned}\mathbf{x}^T\mathbf{Y}\mathbf{x} &= \mathbf{q}^T\lambda\mathbf{I}\mathbf{q}-\mathbf{q}^T\sigma\mathbf{\Phi}\mathbf{q}-\mathbf{q}^T\mathbf{M}_{[K+1][K+1]}\mathbf{q}\\ &\geq\left(\lambda-\sigma||\mathbf{\Phi}||-\mu_{out}||\mathbb{1}_{[K+1]}\mathbb{1}_{[K+1]}^T||\right)||\mathbf{q}||_2^2.\end{aligned} \tag{5.5}$$

since $||\sigma\mathbf{\Phi}+\mu_{out}\mathbb{1}_{[K+1][K+1]}||\leq\sigma||\mathbf{\Phi}||+\mu_{out}||\mathbb{1}_{[K+1]}\mathbb{1}_{[K+1]}^T||$. $||\mathbb{1}_{[K+1]}\mathbb{1}_{[K+1]}^T||=n_{K+1}$. Since $\mathbf{\Phi}$ is a random matrix with bounded i.i.d. entries with zero mean and unit variance, using the standard results in random matrix theory [1], with high probability, $||\mathbf{\Phi}||=2\sqrt{n}$. Hence, if $\lambda>\sigma2\sqrt{n}+\mu_{out}n_{K+1}:=\Lambda+\mu_{out}n_{out}$, then $\mathbf{Y}$ is positive semidefinite with high probability.

Lemma 5.1 *(Small Outliers) If $\lambda>\sigma2\sqrt{n}+\mu_{out}n_{out}:=\Lambda+\mu_{out}n_{out}$, then $\mathbf{Y}\succcurlyeq 0$ with at least* $1-\exp\left(-\Omega(n)\right)$.

Lemma 5.2 *If $\left(-\mu_{out}+\frac{-(\lambda+1)+n_i\mu_i}{2n_i}+\frac{-(\lambda+1)+n_j\mu_j}{2n_j}\right)>0$ and $\left(-\mu_{out}+\frac{-(\lambda+1)+n_i\mu_i}{2n_i}\right)>0$ for all $i\neq j\in[K]$, then $\mathbf{Z}\geq 0$ with probability at least $1-n^2\exp^{-(\delta")^2\Omega(n_{\min})/\sigma^2}$, for $\delta''=$* $\min\{\eta_{\min},\gamma_{\min}\}$.

From Lemma 5.1, 3.2 and 5.2 we can conclude Theorem 4:

Theorem 4 *[Small Number of Outliers] If the regularizer λ is within the following range,*

$$\Lambda+\mu_{out}n_{K+1}<\lambda<\min\left\{\eta_{min},\gamma_{min}\right\}-1 \tag{5.6}$$

then $\mathbf{X}^*$ is the unique optimal solution to Program 1.1 with high probability.

References

[1] Van H. Vu. Spectral norm of random matrices. In Harold N. Gabow and Ronald Fagin, editors, *STOC*, pages 423–430. ACM, 2005.