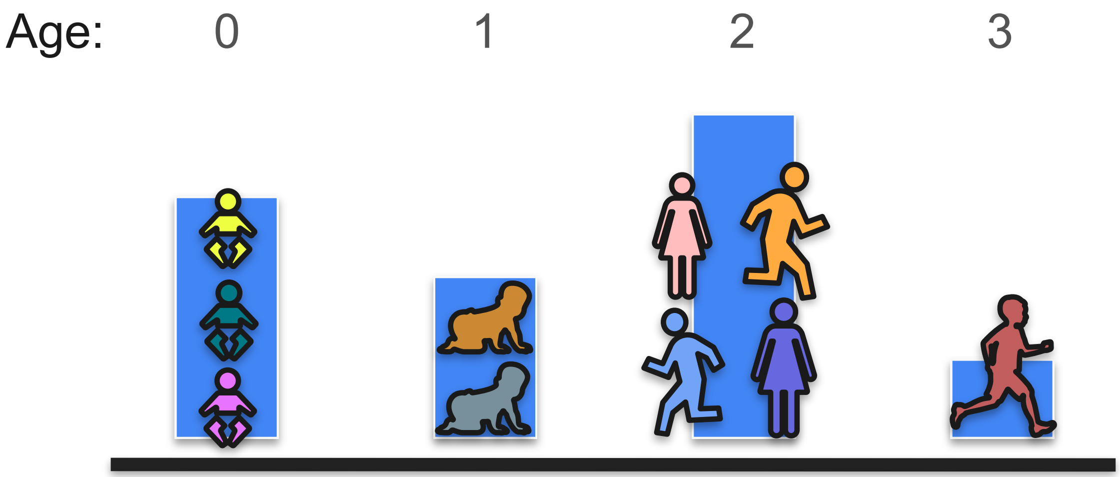
W2 Lesson 1

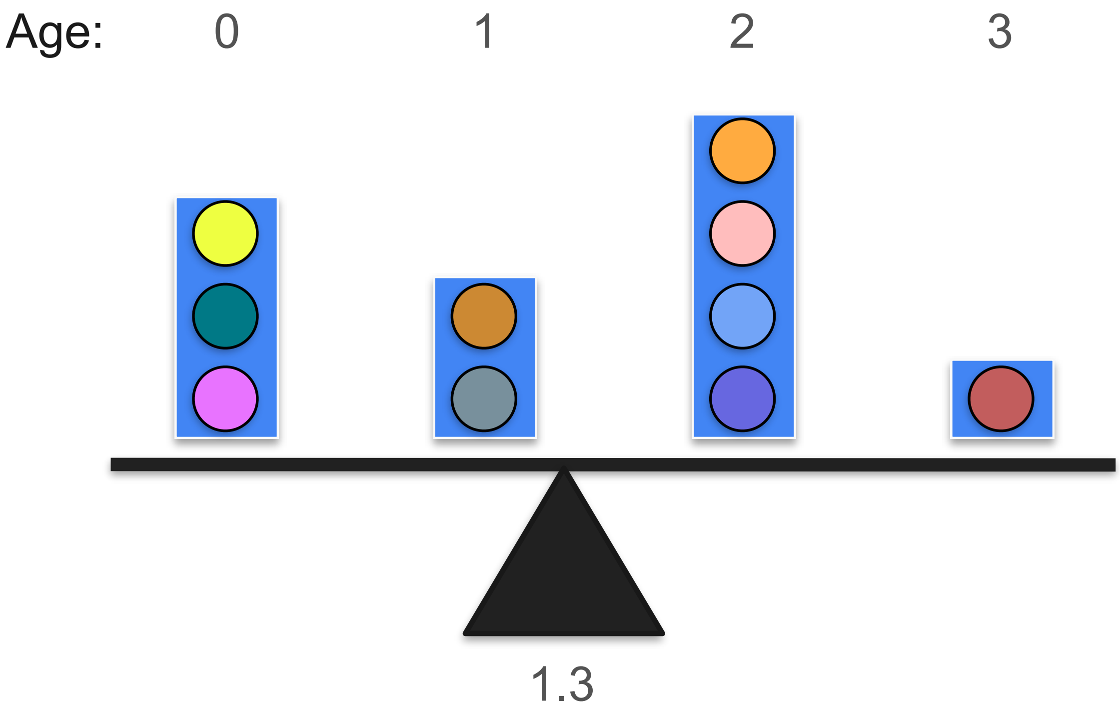
Describing Distributions

Expected value

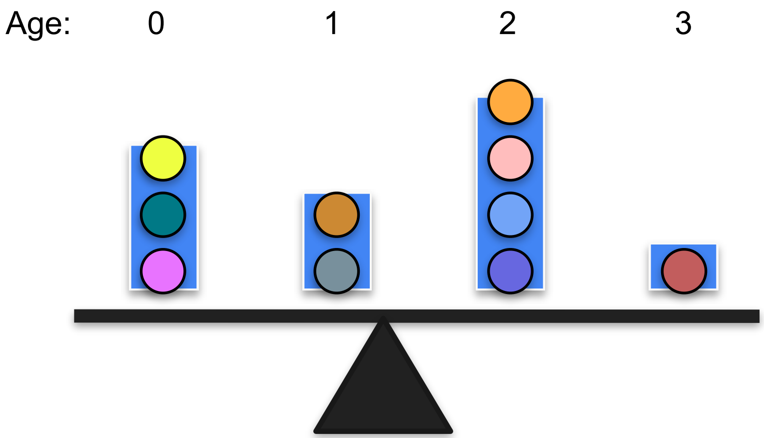
Mean: Example



Mean: Example



Mean: Example

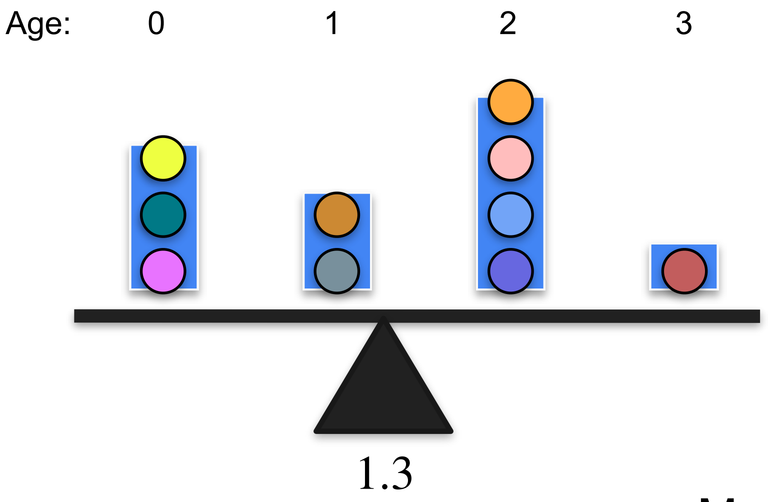


$$\frac{0+0+0+1+1+2+2+2+2+3}{10}$$

$$=\frac{13}{10}$$

$$=1.3$$

Children in a Room



Mean

Expected Value

$\mathbb{E}[X]$

$$\frac{0+0+0+1+1+2+2+2+2+3}{10}$$

$$= \frac{13}{10} \quad = 1.3$$

$$= \frac{3 \cdot 0 + 2 \cdot 1 + 4 \cdot 2 + 1 \cdot 3}{10}$$

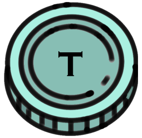
Weighted average

$$= \frac{3}{10} \cdot 0 + \frac{2}{10} \cdot 1 + \frac{4}{10} \cdot 2 + \frac{1}{10} \cdot 3 = 1.3$$

Expected Value: Motivation Example 1

You play a game with a friend

You win 10 dollars



You win nothing

Game cost:

$6

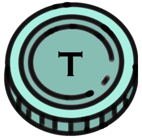
Do you play the game?

What is the maximum amount of money you would pay to play this game?

Expected Value: Motivation Example 1

You play a game with a friend

You win 10 dollars



You win nothing

Game cost:

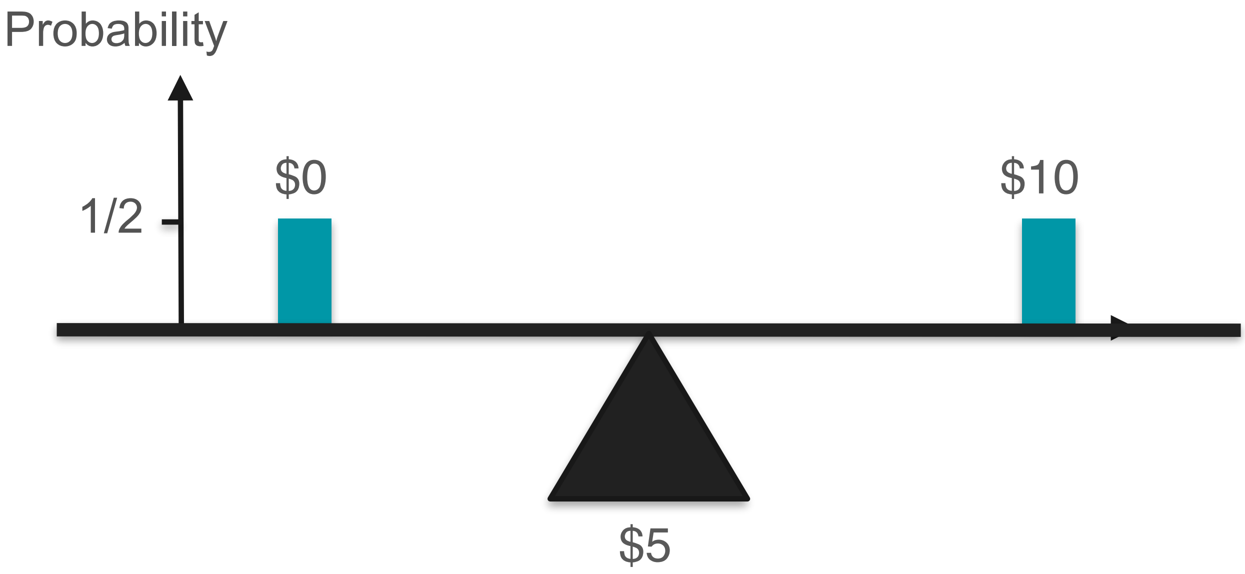
\$5

Long term: $0.5 \cdot \$10 \; + 0.5 \cdot \$0 = \$5$

You expect to win $\$5$ on average

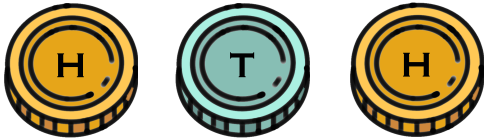
$\mathbb{E}[X] = 5$

Expected Value: Motivation Example 1



Expected Value: Motivation Example 2

Play another game



Flip three coins. For each heads you win $1

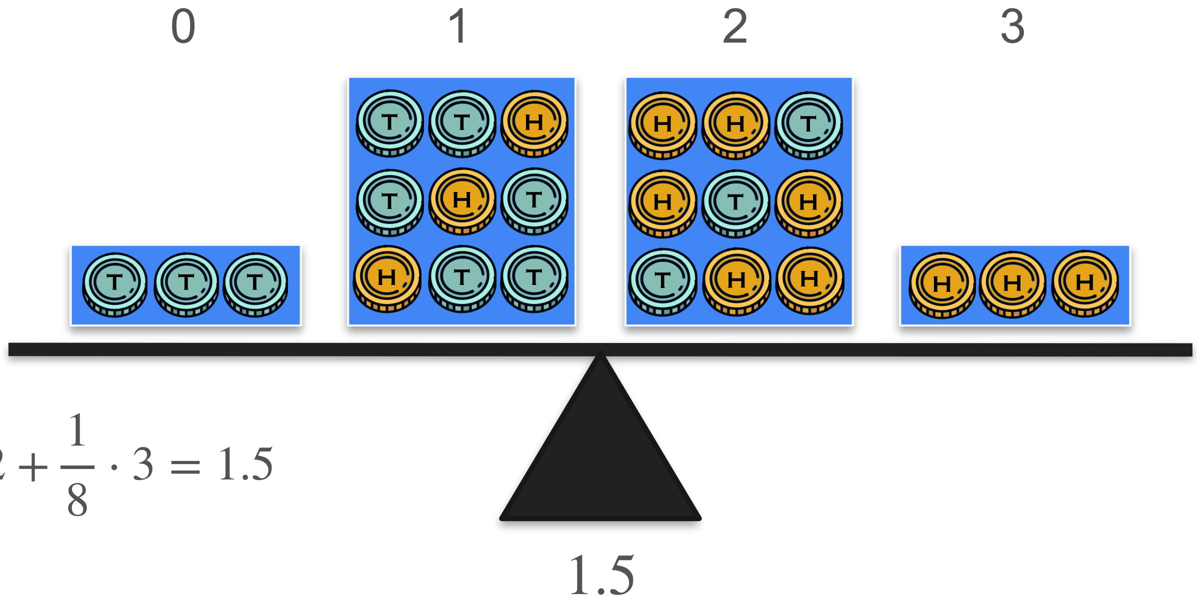
What is the maximum amount of money you would pay to play this game?

Expected Value: Motivation Example 2

X: Number of heads

$$\mathbb{E}[X] = 1.5$$

$$\mathbb{E}[X] = \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = 1.5$$



Expected Value: Discrete Case

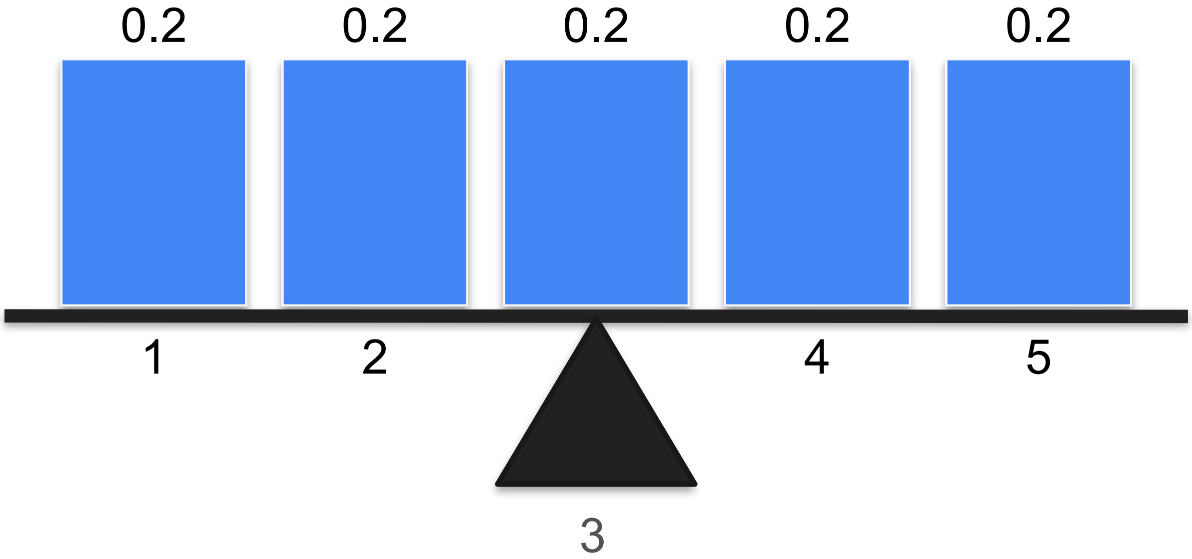
X a discrete random variable

PMF of X

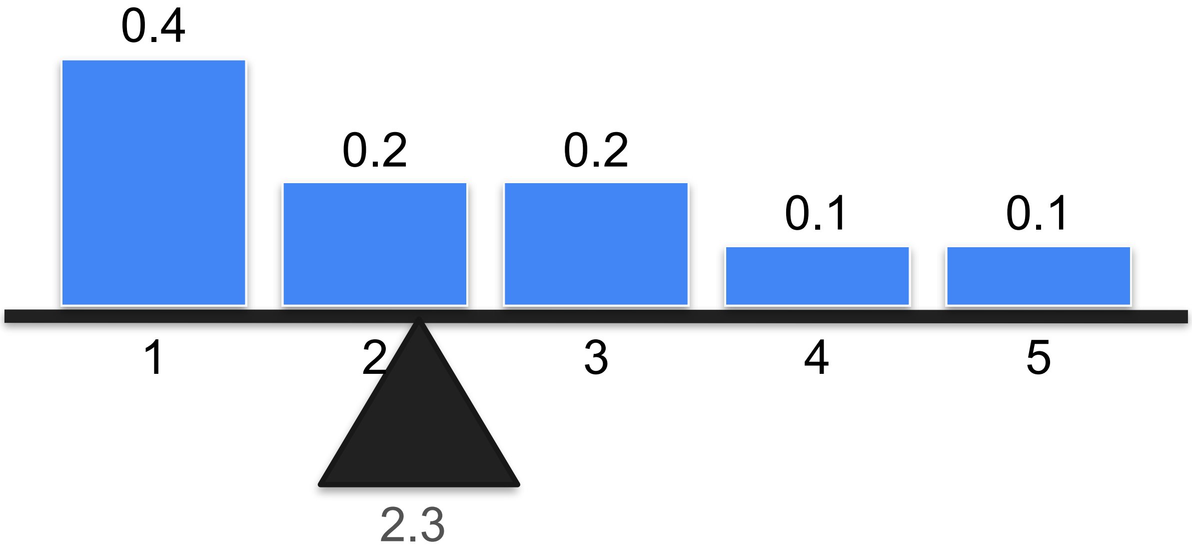
$$p_X(x) = \mathbf{P}(X = x)$$

$$\mathbb{E}[X] = \sum_x x\, p_X(x)$$

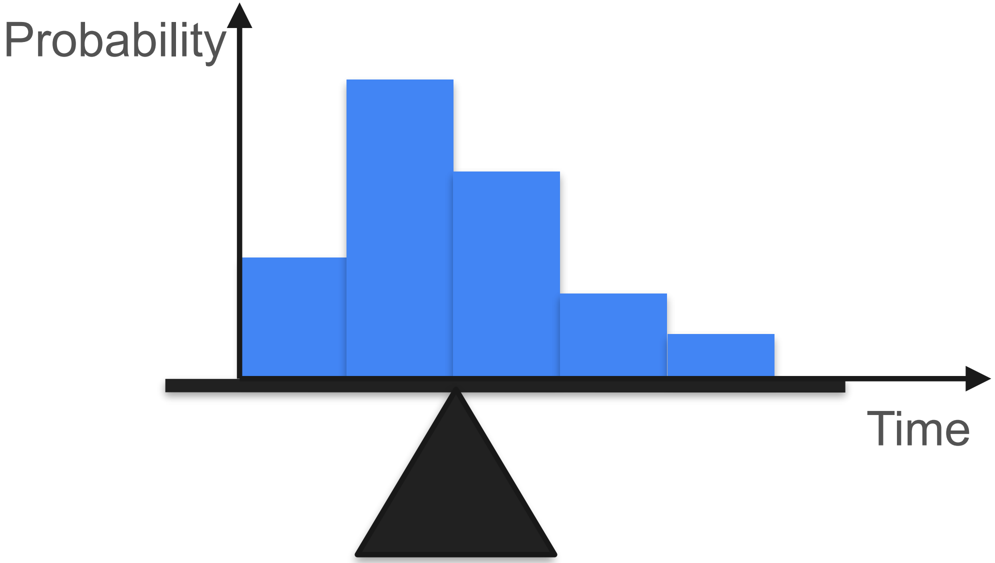
Expected Value



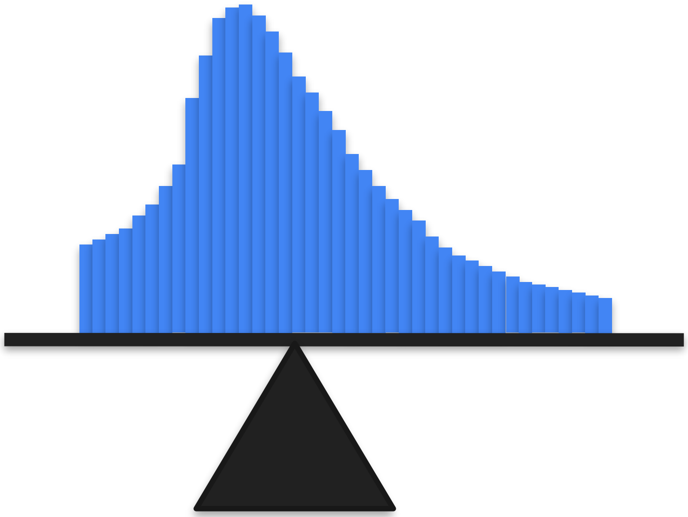
Expected Value



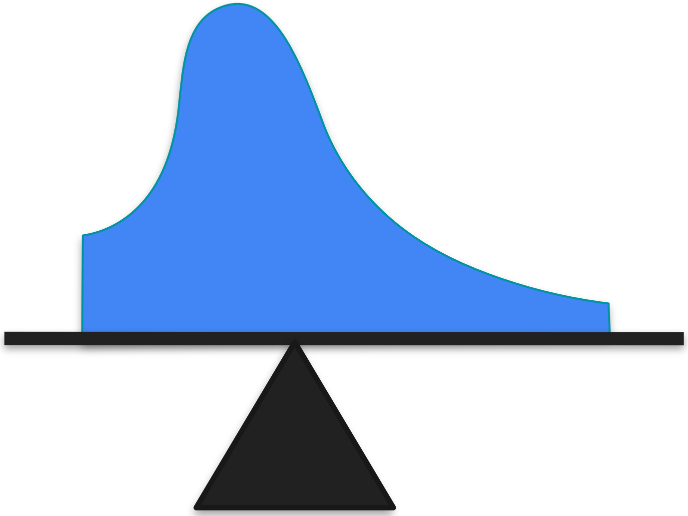
Expected Value - Continuous



Expected Value - Continuous



Expected Value - Continuous



Expected Value - Continuous

Discrete random variables

$$\mathbb{E}[X] = \sum_x x \, p_X(x)$$

Weighted using PMF

Continuous random variables

$$\int_{-\infty}^{\infty} x \cdot f_X(x) \, dx$$

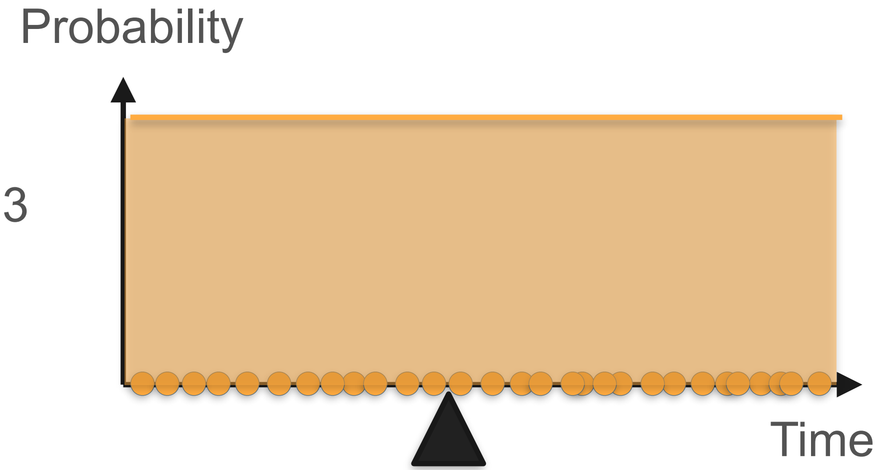
Integrals

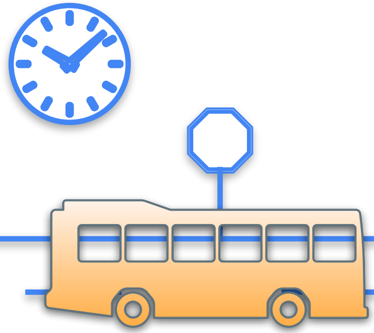
Weighted using PDF

Expected Value

Waiting Time
15 min
32 min
58 min
1 min
47 min
14 min
37 min
8 min
29 min
55 min
…

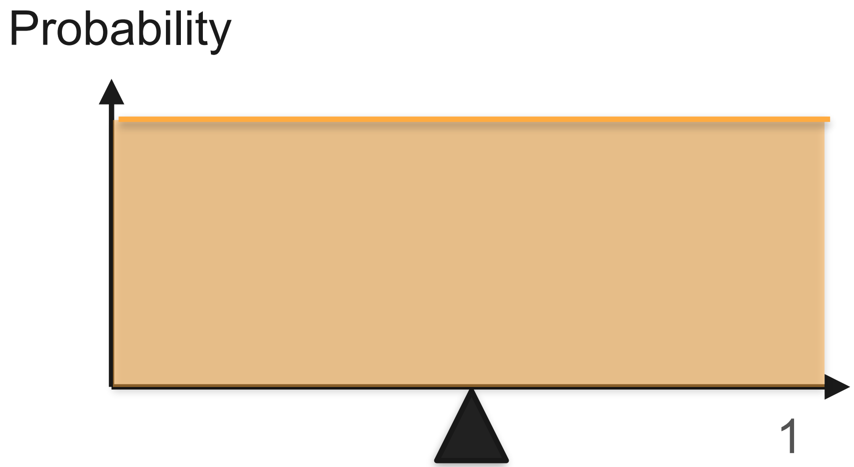
Average = 30.833





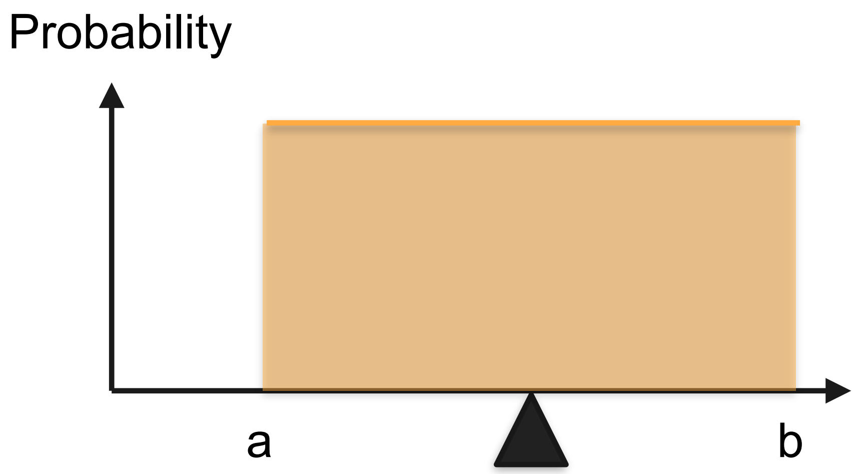
Expected Value: Uniform Distribution

Average = 0.5

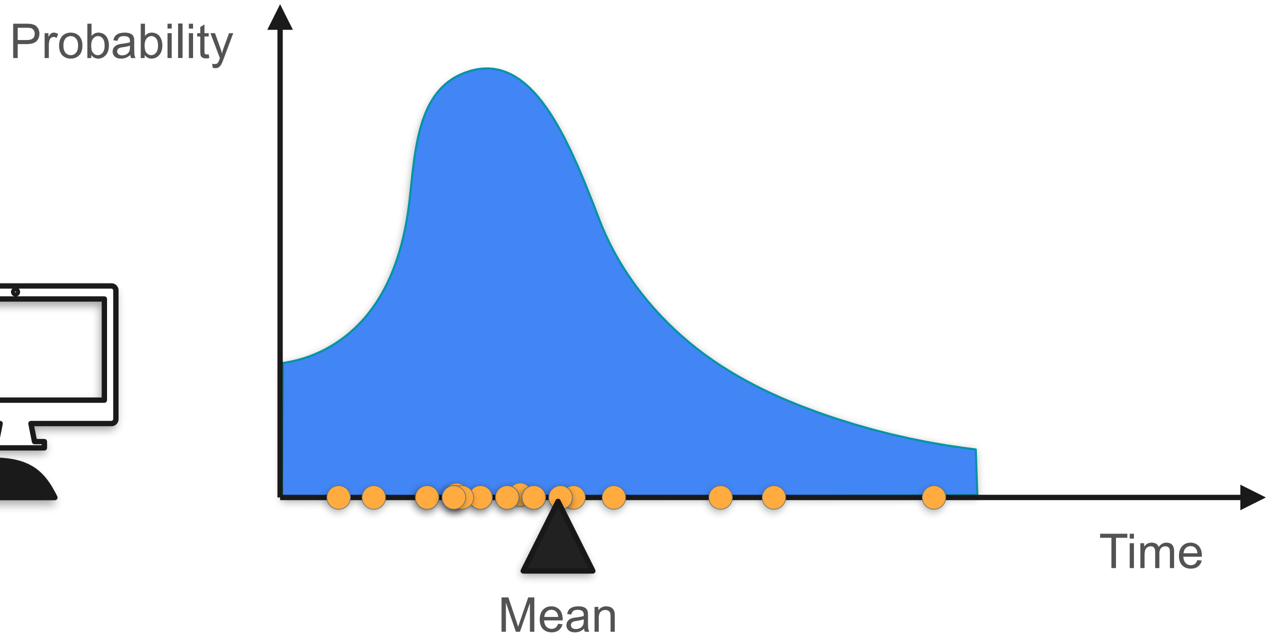


Expected Value: Uniform Distribution

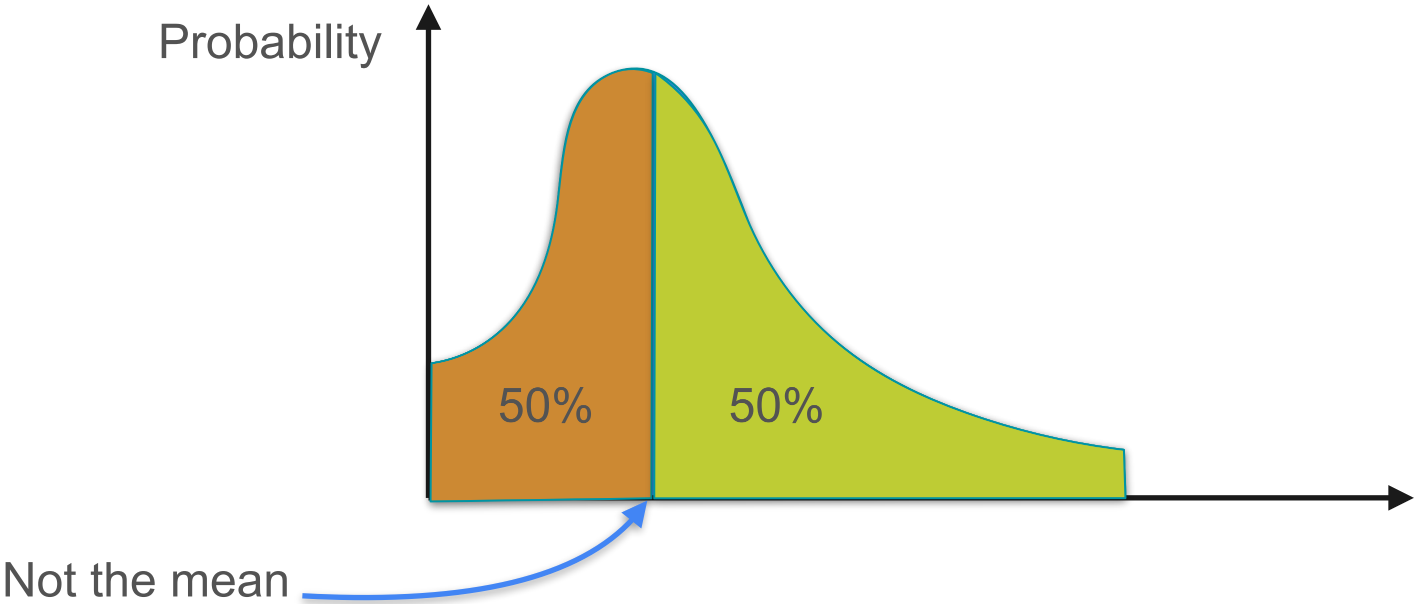
Average = $\frac{a+b}{2}$



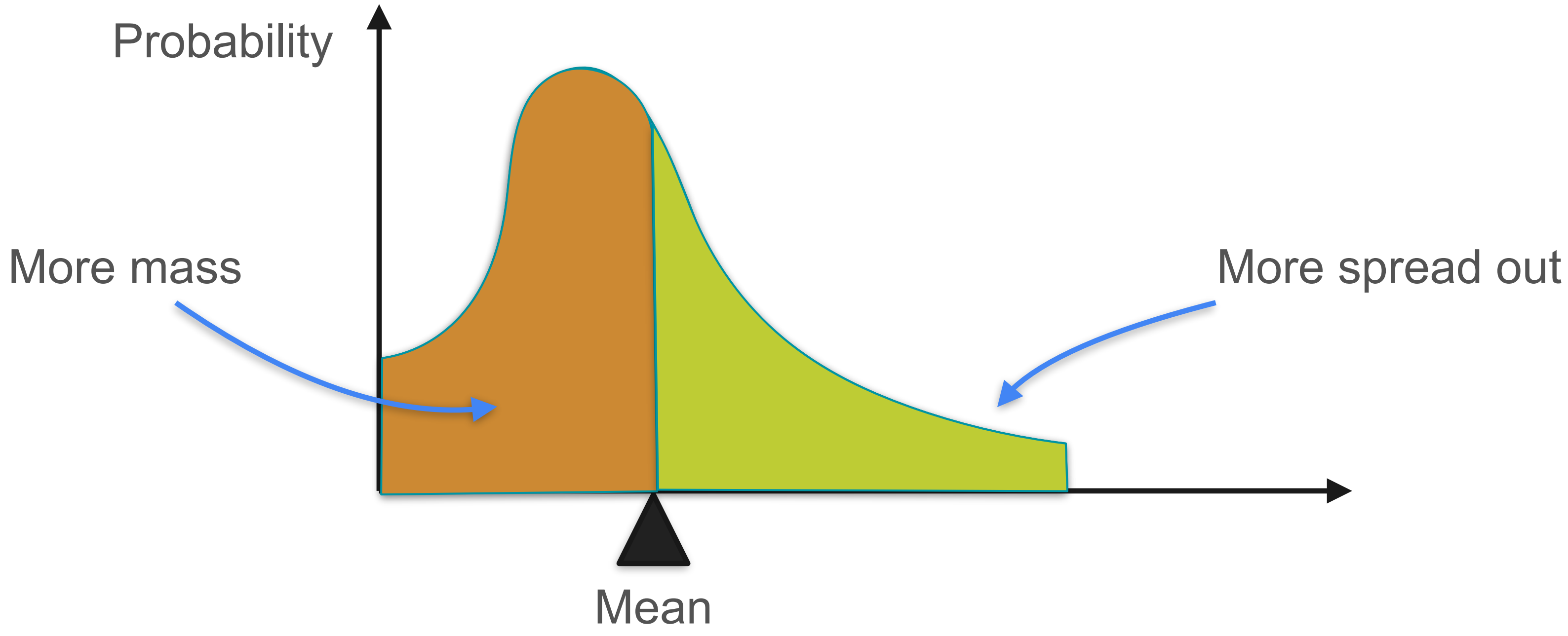
Expected Value



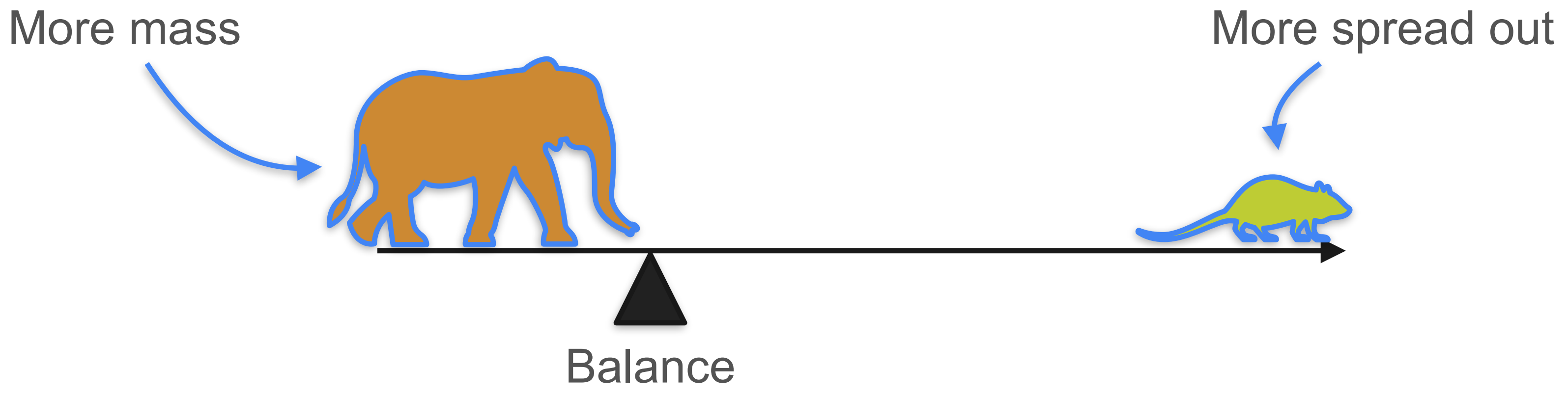
Expected Value: Common Misconception



Expected Value: Common Misconception



Expected Value: Common Misconception



Expected Value

- $\mathbb{E}[X]$
- Mean / Balancing point
- Defined for discrete and continuous random variables
- Weighted average of the PMF / PDF

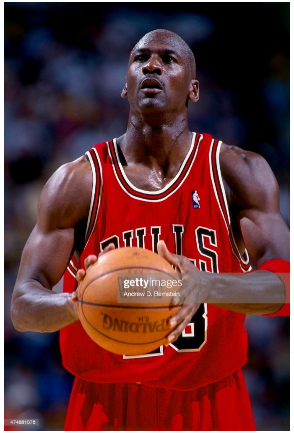
DeepLearning.AI

Describing Distributions

Other measures of central tendency

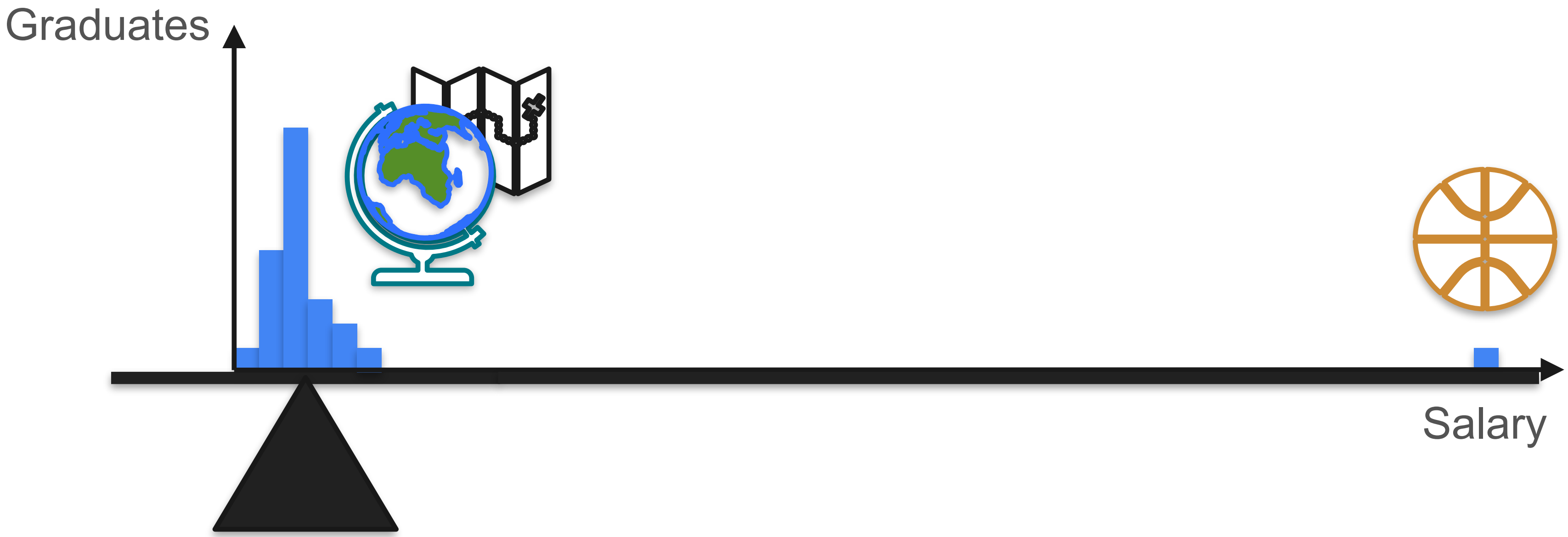
Median: Motivation

- In the 1980s, the starting salary for a geography graduate at the University of North Carolina was $250,000.
- For the rest of the country, the starting salary for a geography graduate was $22,000.
- Why?
  1. The program was really good
  2. The university had great connections
  3. One student made lots of money



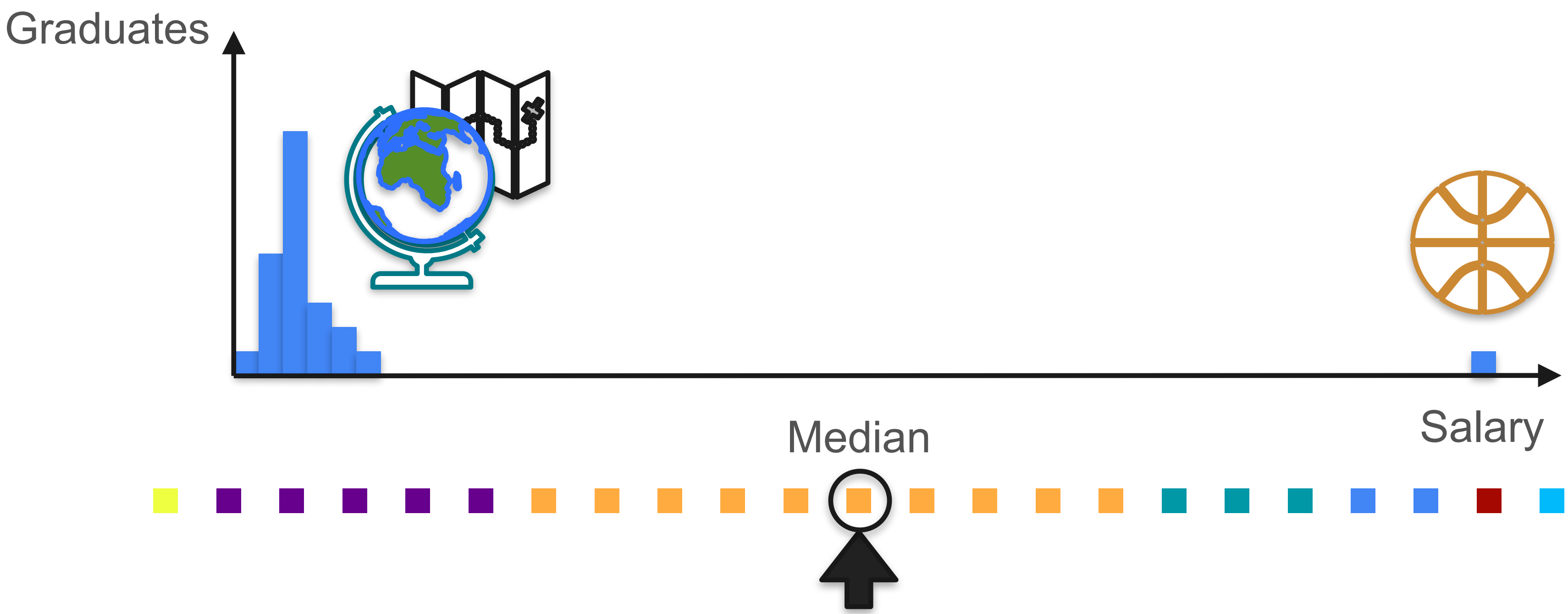
Michael Jordan

Outliers

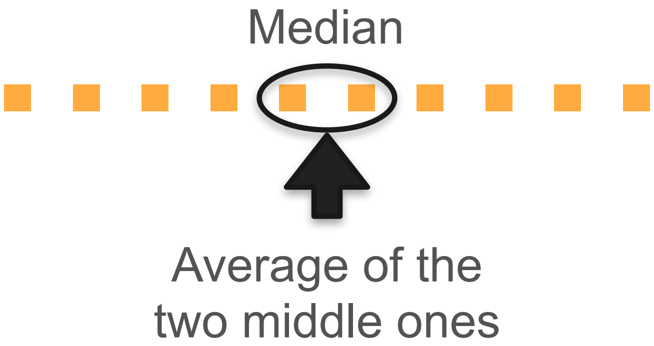


Outliers

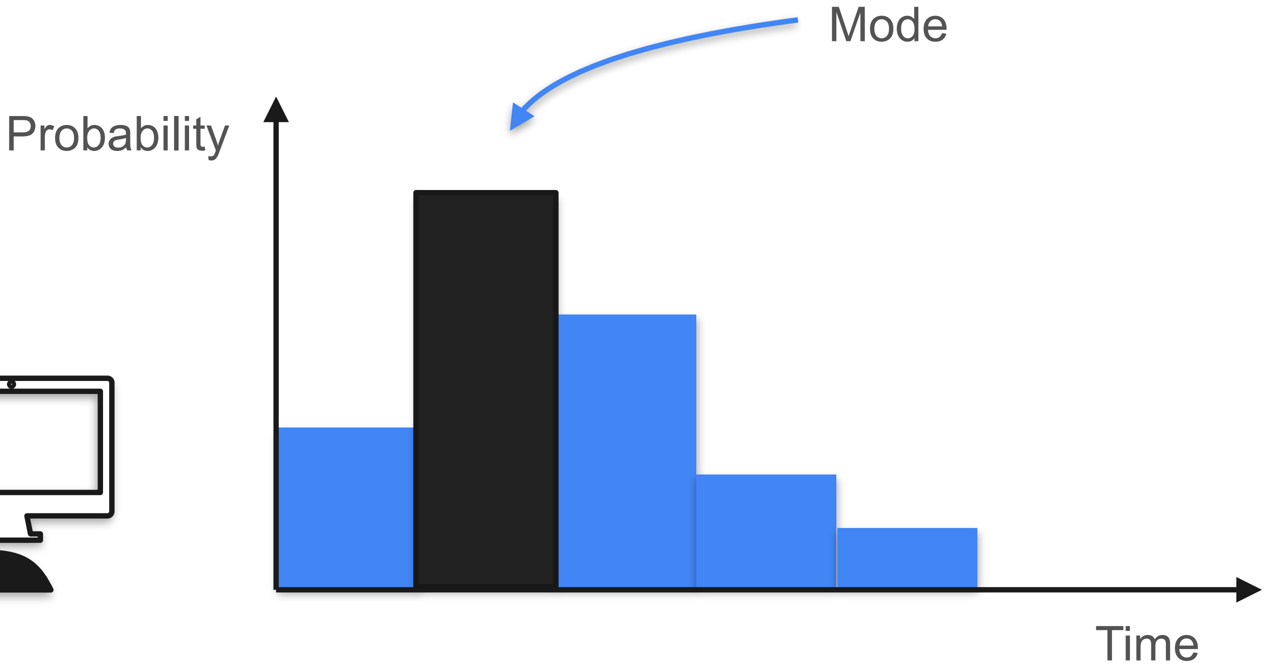
Median



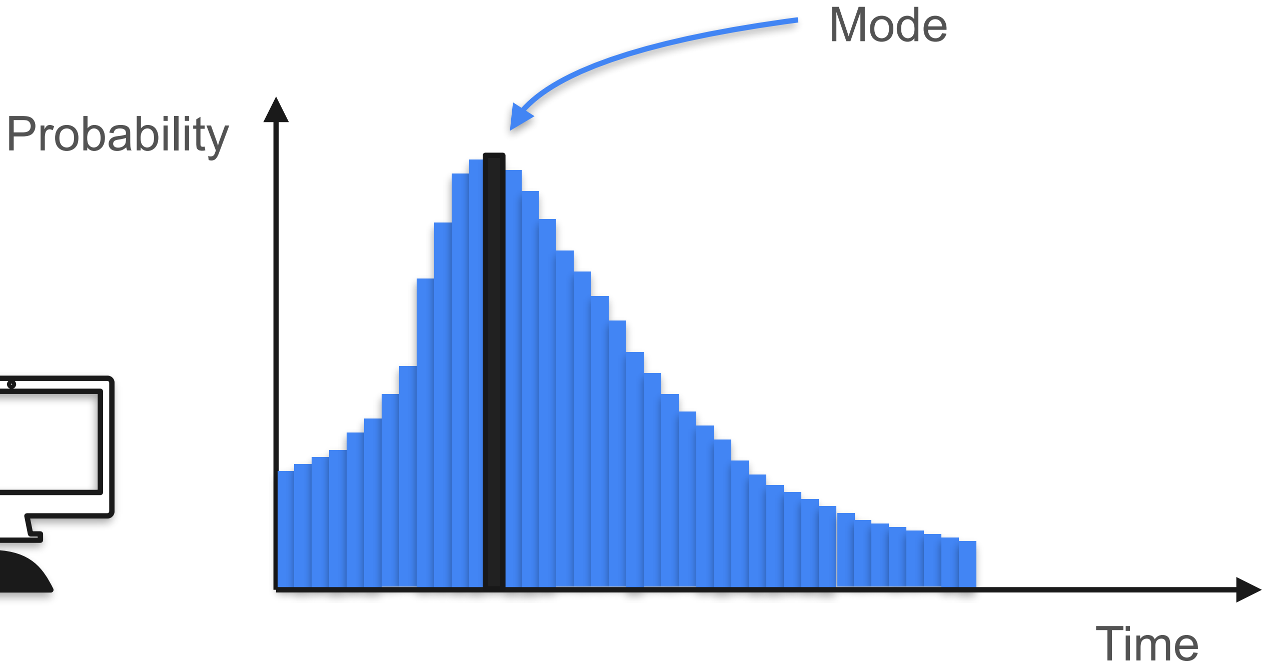
Median



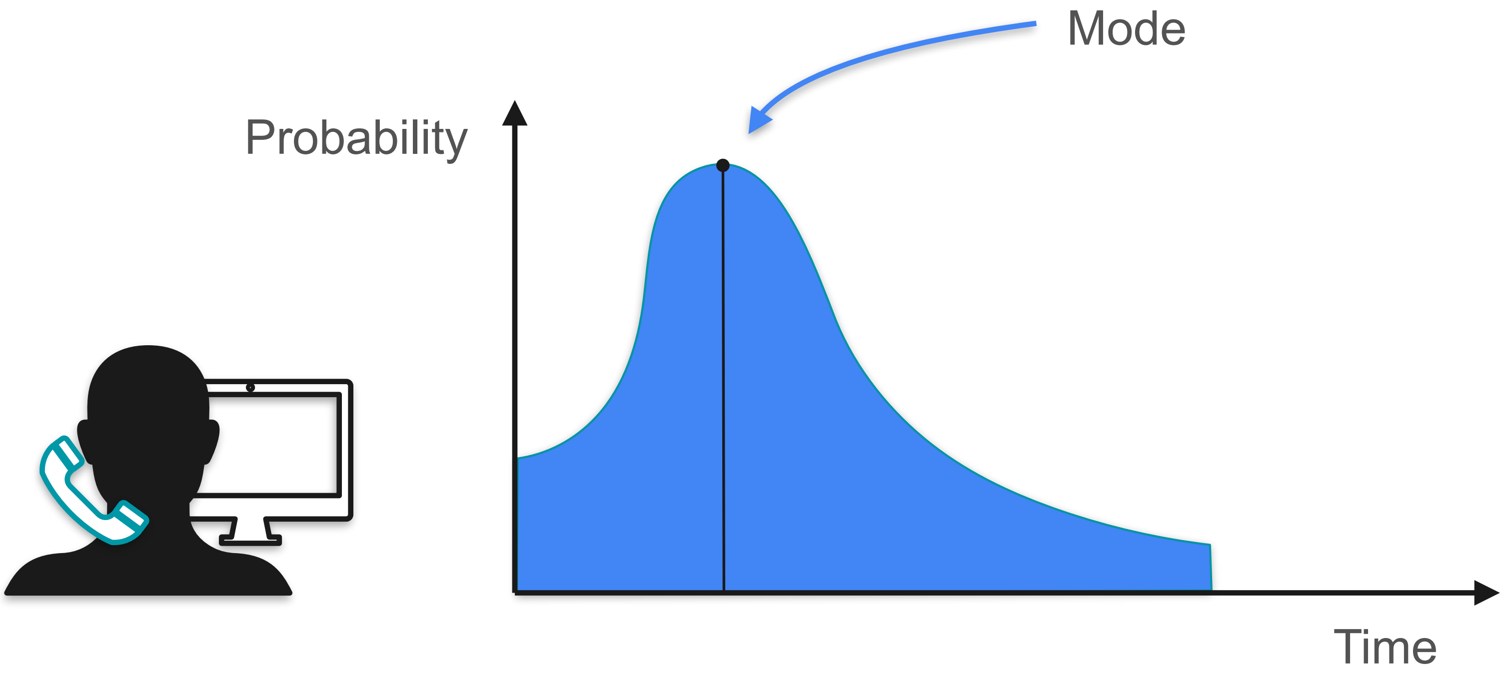
Mode



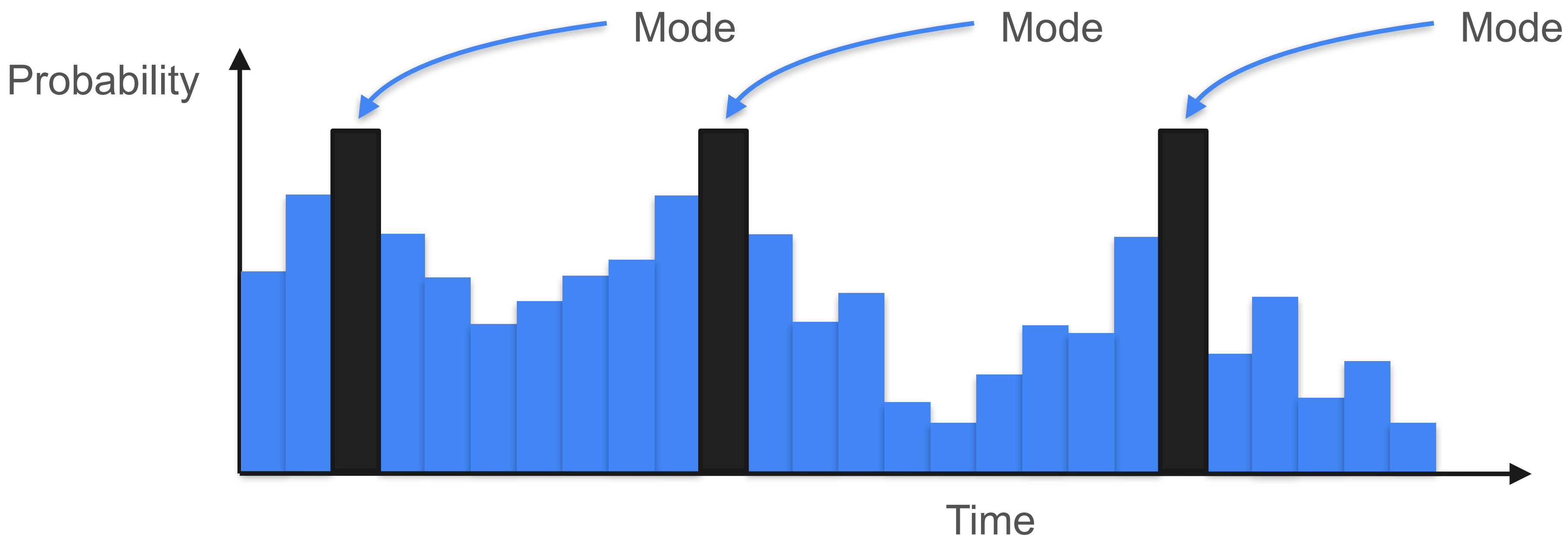
Mode



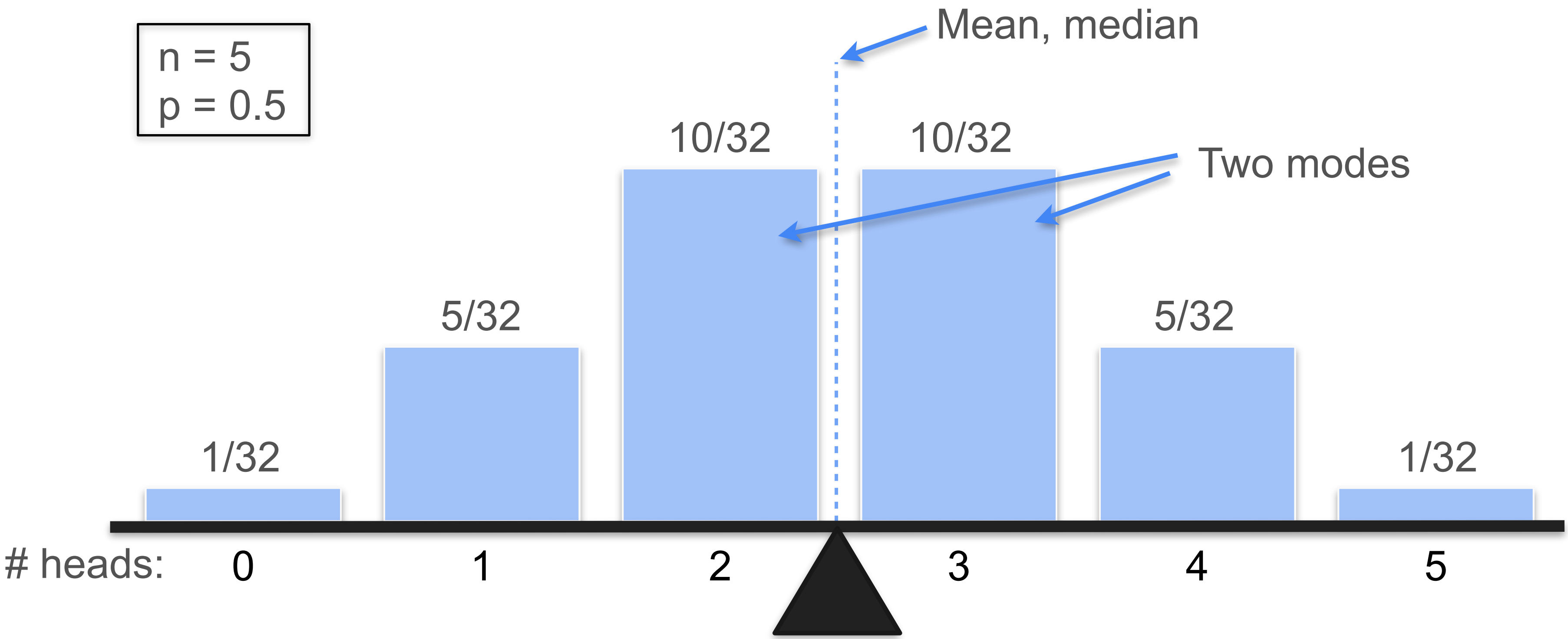
Mode



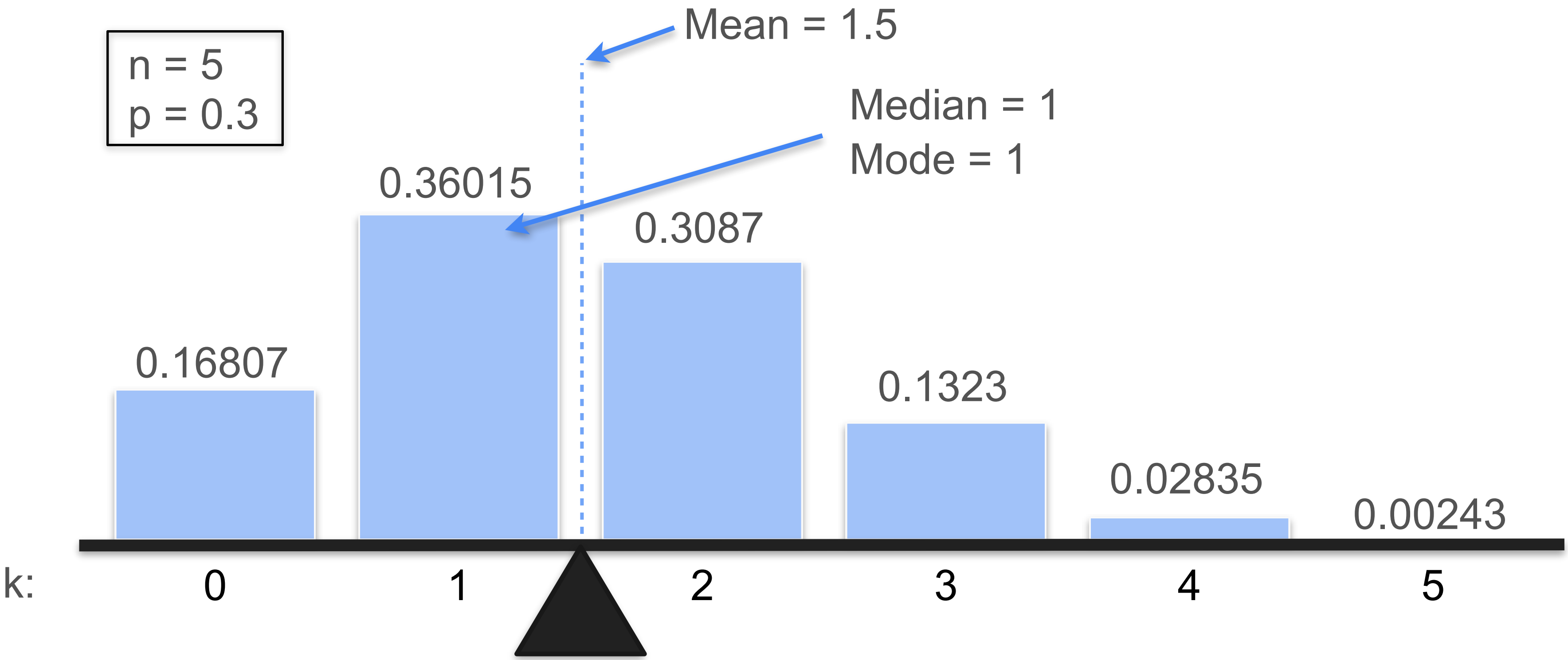
Mode: Multimodal Distribution



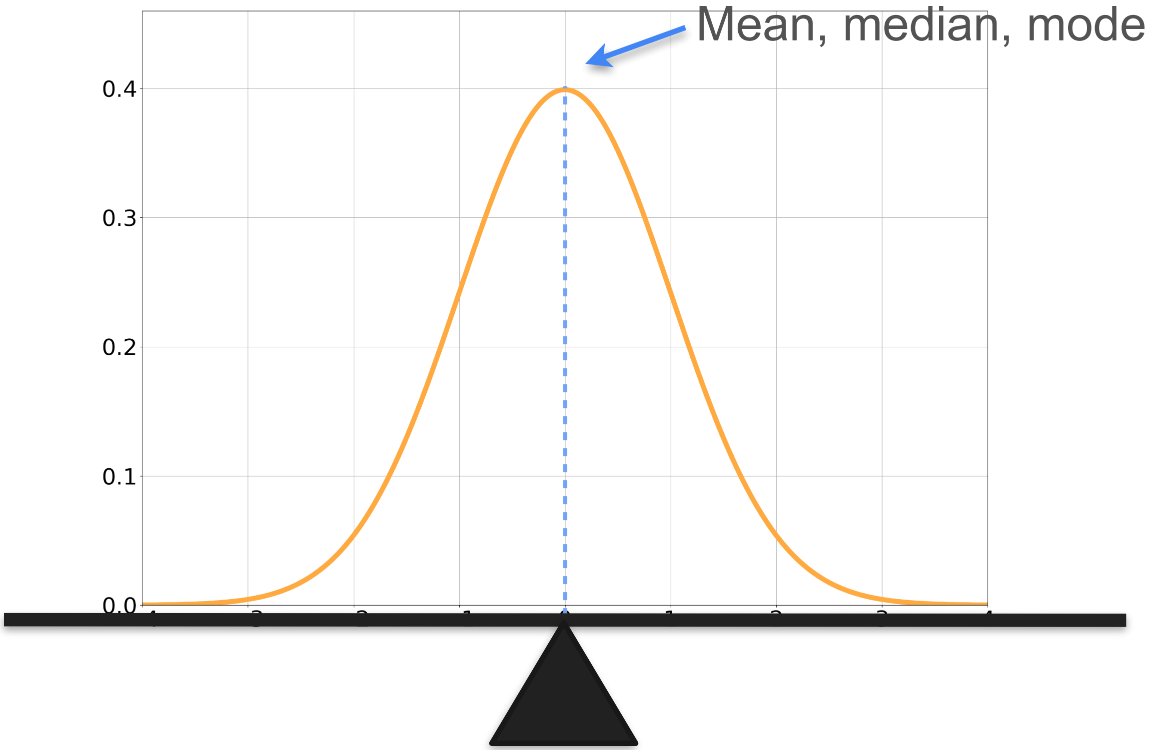
Mean, Median and Mode in Binomial Distribution



Mean, Median and Mode in Binomial Distribution



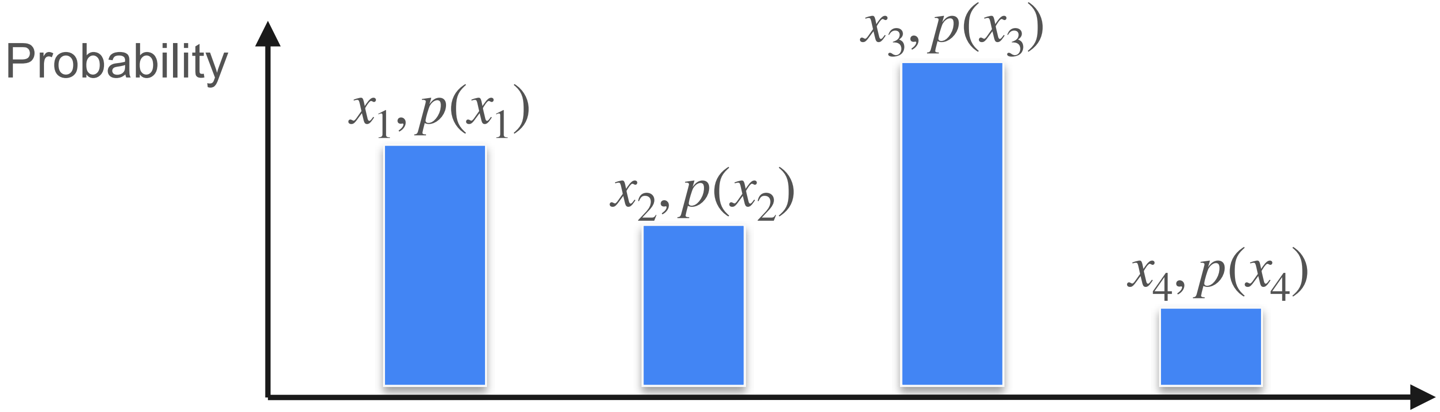
Mean, Median and Mode in Normal Distribution



Describing Distributions

Expected value of a function

Expected Value of a Function

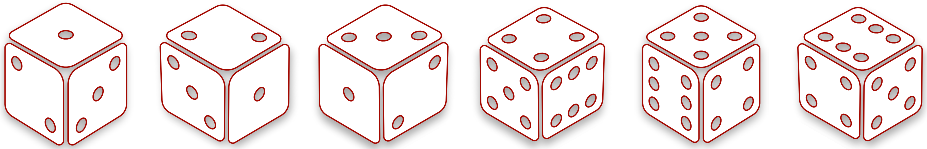


$$\mathbb{E}[X] = x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3) + x_4 p(x_4)$$

$$E[g(X)] = g(x_1)p(x_1) + g(x_2)p(x_2) + g(x_3)p(x_3) + g(x_4)p(x_4)$$

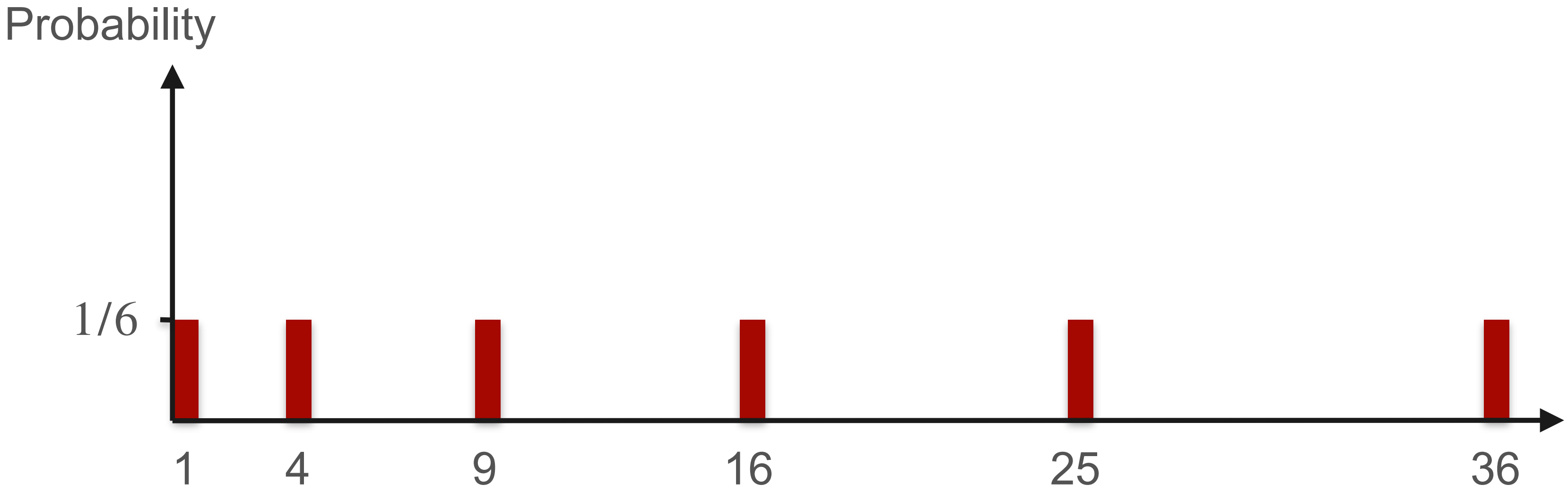
Expected Value of a Function

Probability:	1/6	1/6	1/6	1/6	1/6	1/6
Roll:	1	2	3	4	5	6



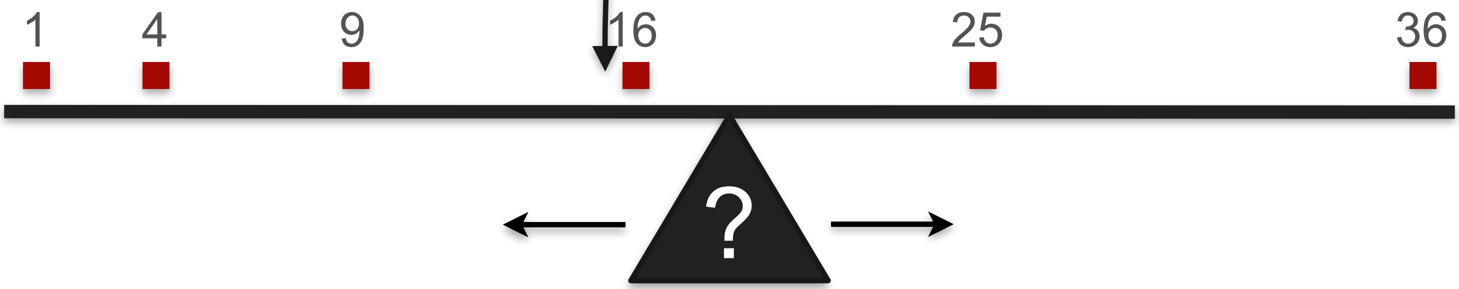
Square:	1	4	9	16	25	36

Expected Value of a Function



Expected Value of a Function

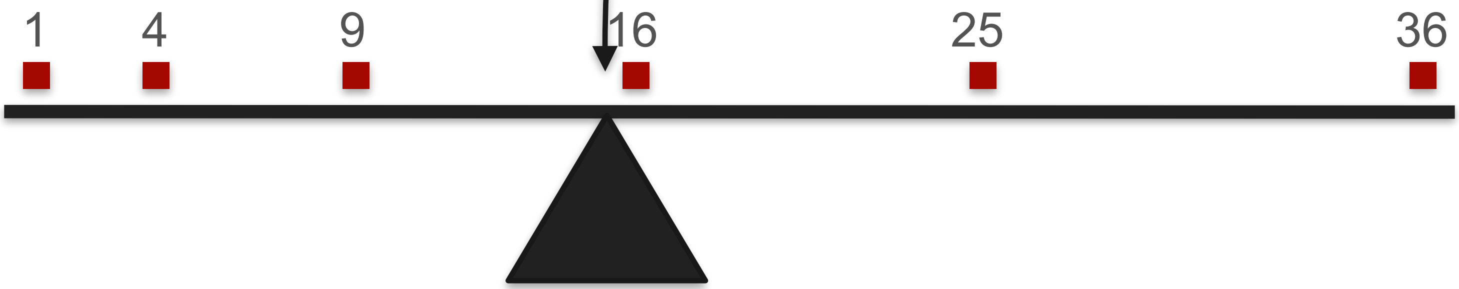
$$\frac{1 + 4 + 9 + 16 + 25 + 36}{6} = \frac{91}{6}$$



Expected Value of a Function

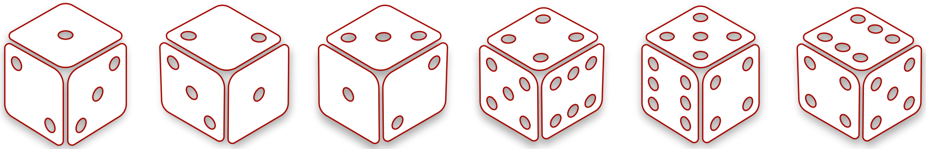
$$\frac{1+4+9+16+25+36}{6} = \frac{91}{6} = \frac{1^2}{6} + \frac{2^2}{6} + \frac{3^2}{6} + \frac{4^2}{6} + \frac{5^2}{6} + \frac{6^2}{6}$$

$$= \mathbb{E}\left[X^2\right]$$



Expectation of Linear Function

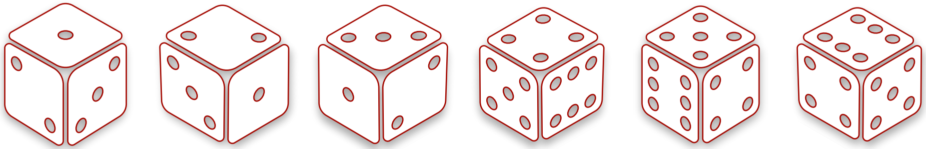
Probability:	1/6	1/6	1/6	1/6	1/6	1/6
Roll:	1	2	3	4	5	6



Double:	2	4	6	8	10	12
Wins	2 - 5	4 - 5	6 - 5	8 - 5	10 - 5	12 - 5

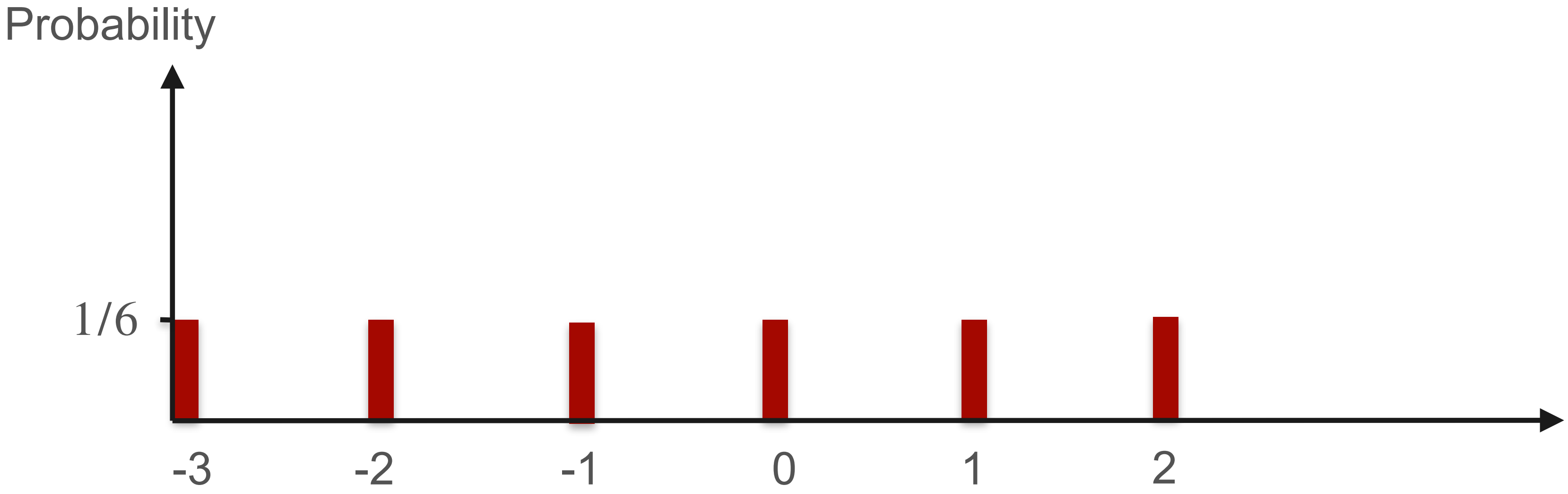
Expectation of Linear Function

Probability:	1/6	1/6	1/6	1/6	1/6	1/6
Roll:	1	2	3	4	5	6



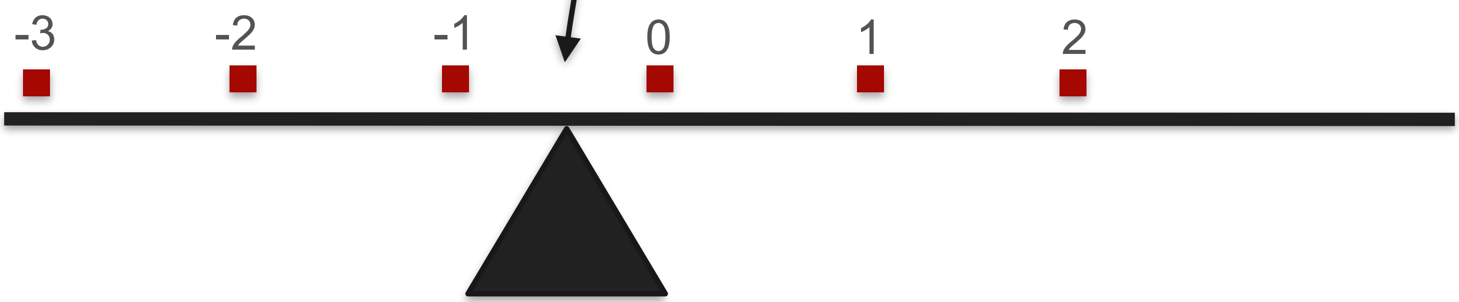
Double:	2	4	6	8	10	12
Wins	-3	-2	-1	0	1	2

Expected Value of a Function



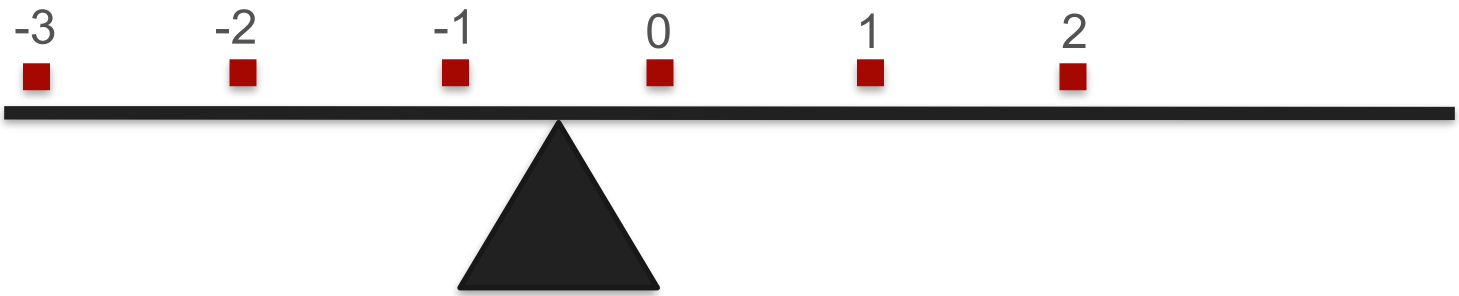
Expected Value of a Function

$$\frac{-3+-2+-1+0+1+2}{6} = \frac{-3}{6} = -0.5$$



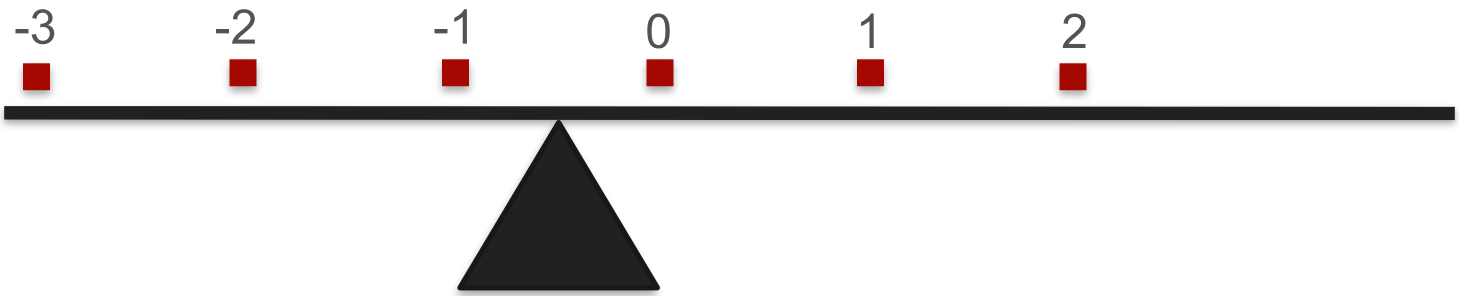
Expected Value of a Function

$$\frac{(2\cdot 1-5)+(2\cdot 2-5)+(2\cdot 3-5)+(2\cdot 4-5)+(2\cdot 5-5)+(2\cdot 6-5)}{6} = \frac{-3}{6} = -0.5$$



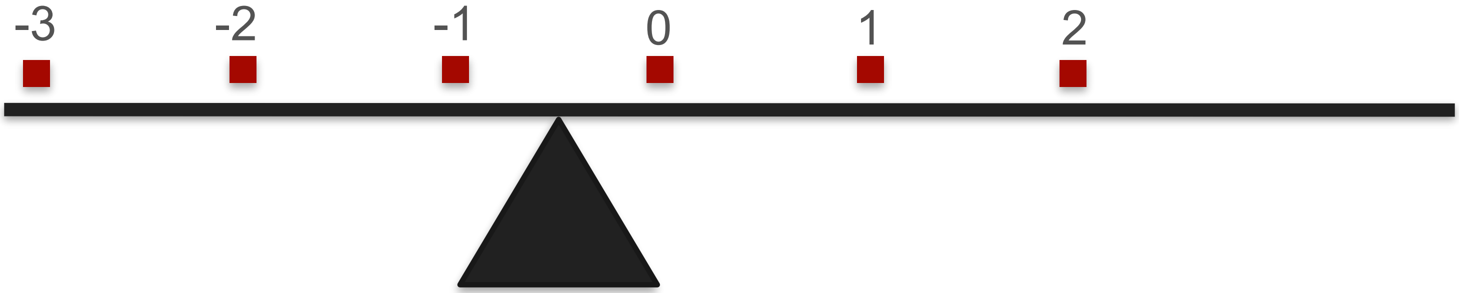
Expected Value of a Function

$$\frac{(2\cdot 1)+(2\cdot 2)+(2\cdot 3)+(2\cdot 4)+(2\cdot 5)+(2\cdot 6)+6\cdot(-5)}{6} = \frac{-3}{6} = -0.5$$



Expected Value of a Function

$$\frac{(2\cdot 1)+(2\cdot 2)+(2\cdot 3)+(2\cdot 4)+(2\cdot 5)+(2\cdot 6)}{6}+(-5) \quad = \frac{-3}{6} = -0.5$$



Expected Value of a Function

$$\mathbb{E}[2 \cdot X + (-5)] = \frac{2 \cdot \boxed{(1+2+3+4+5+6)}}{\boxed{6}} + (-5) = \frac{-3}{6} = -0.5$$

$\mathbb{E}[X]$

$$= 2 \cdot \mathbb{E}[X] + (-5)$$

In general:
$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

$\mathbb{E}[aX] = a\mathbb{E}[X]$

$\mathbb{E}[b] = b$

DeepLearning.AI

Describing Distributions

Sum of expectations

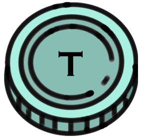
Sum of Expectations

You play a game:

Flip a coin. If heads you win $ 1, else you win nothing.

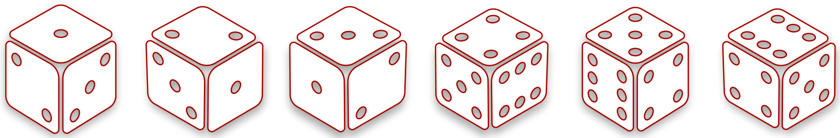
Then roll a die. You win the amount you roll.

Win $1



Win nothing

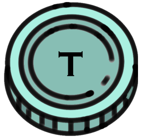
Win $1 $2 $3 $4 $5 $6



What are your expected winnings for the game?

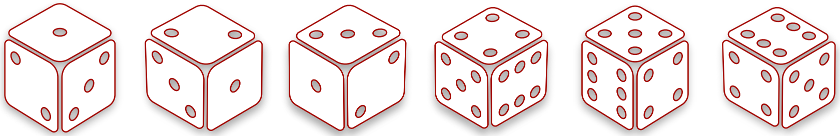
Sum of Expectations

Win $1



Win nothing

Win $1 $2 $3 $4 $5 $6



$$\mathbb{E}\left[X_{coin}\right] = \$0.5 \qquad \mathbb{E}\left[X_{dice}\right] = \$3.5$$

$$\mathbb{E}[X] = \mathbb{E}\left[X_{coin}\right] + \mathbb{E}\left[X_{dice}\right] = \$0.5 + \$3.5 = \$4$$

In general: $\mathbb{E}\left[X_1 + X_2\right] = \mathbb{E}\left[X_1\right] + \mathbb{E}\left[X_2\right]$

Sum of Expectations

Expected number of correct assignments?

8 billion people

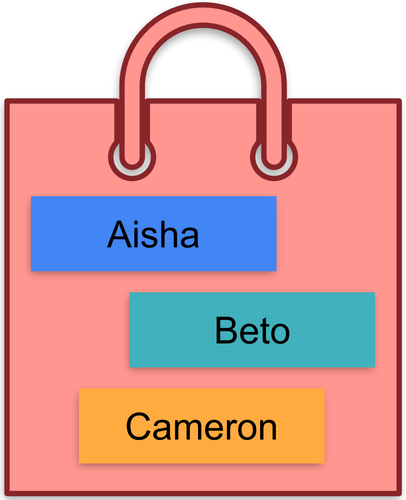
Sum of Expectations

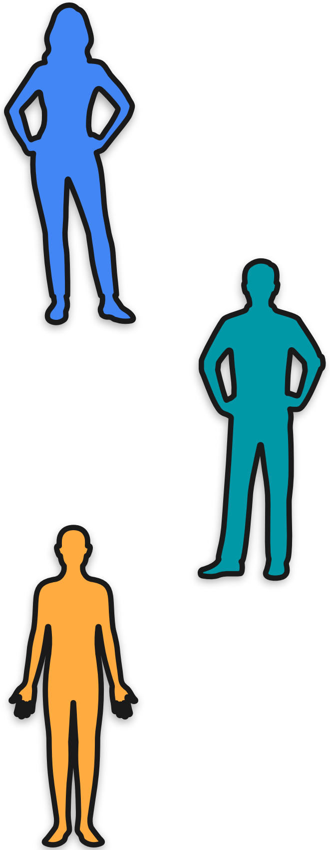
1

Expected number of correct assignments?

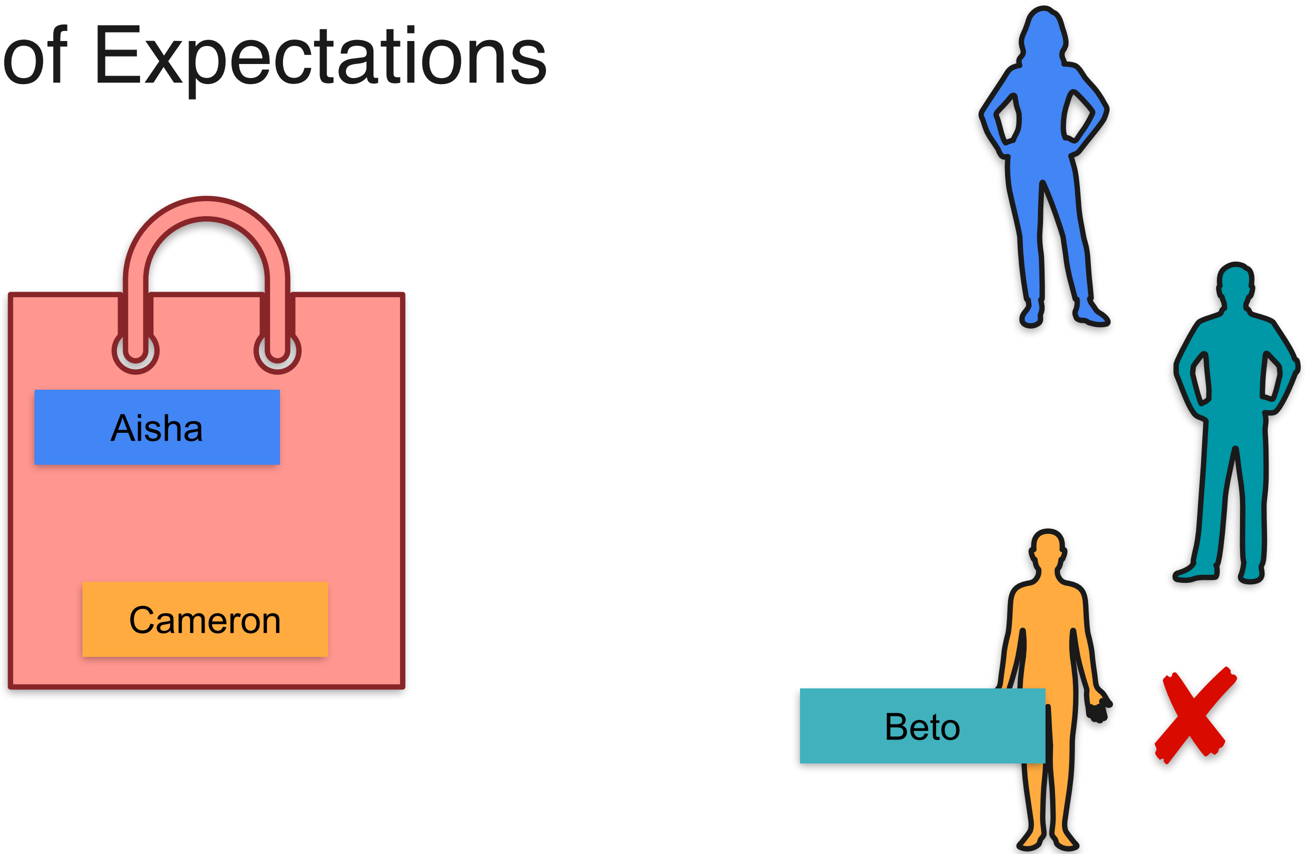
8 billion people

Sum of Expectations

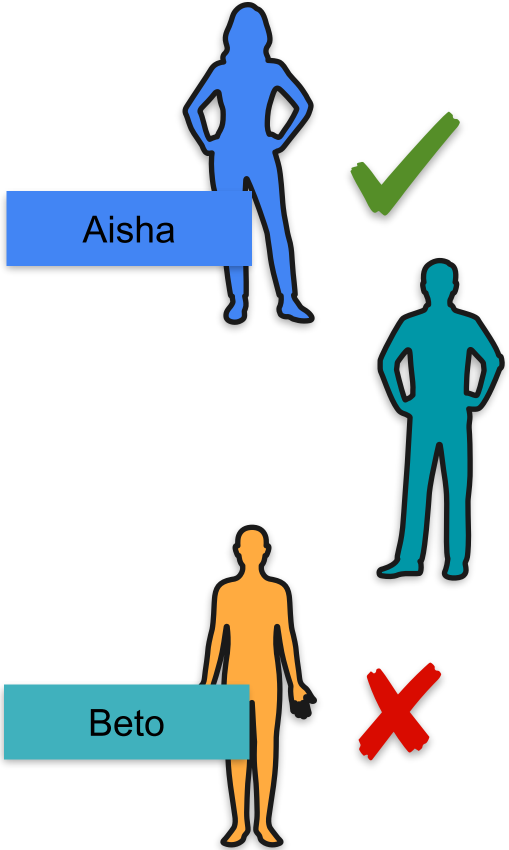




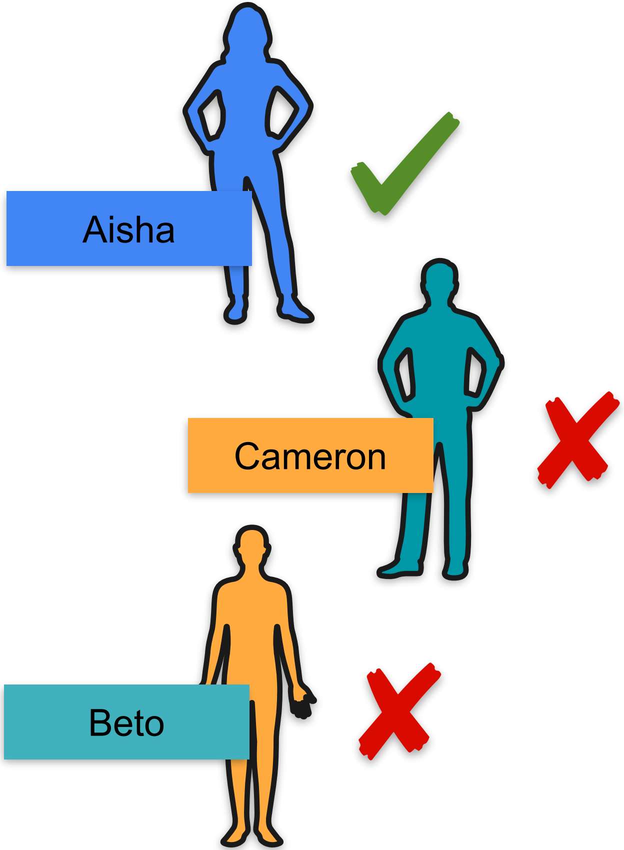
Sum of Expectations



Sum of Expectations

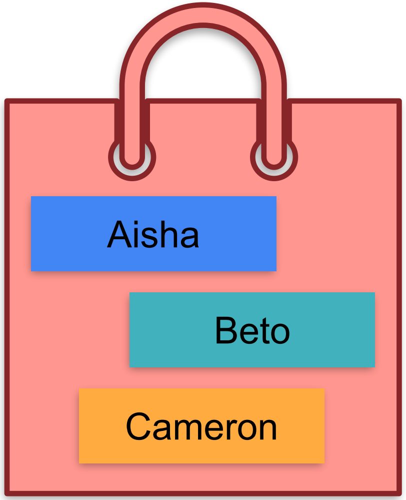


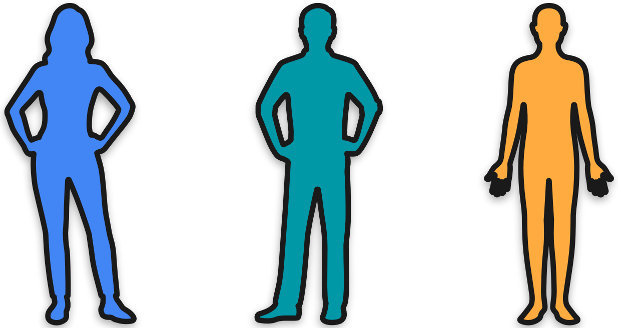
Sum of Expectations



Sum of Expectations

Average
1





Correct

6

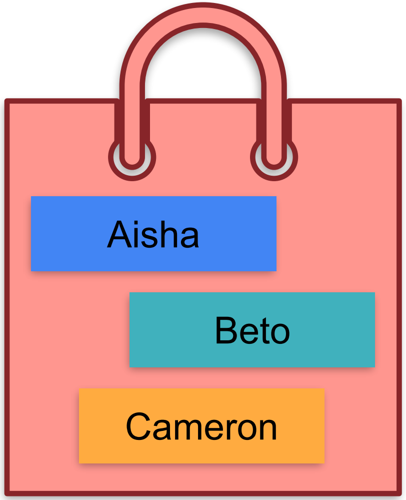
Correct			
3	Aisha	Beto	Cameron
1	Aisha	Cameron	Beto
1	Beto	Aisha	Cameron
0	Beto	Cameron	Aisha
0	Cameron	Aisha	Beto
1	Cameron	Beto	Aisha

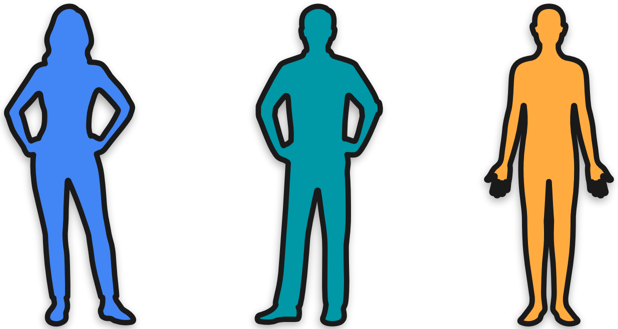
Sum of Expectations

1/3

1/3

1/3



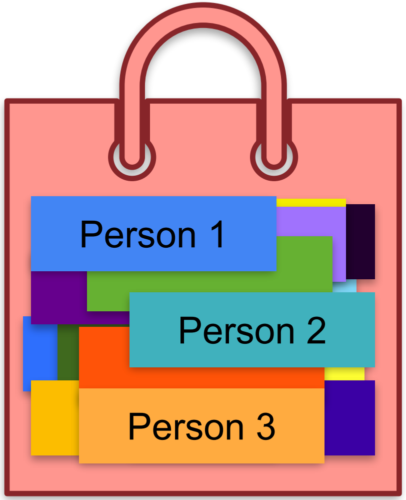


$$\mathbb{E}[\text{Matches}] = \mathbb{E}[A] + \mathbb{E}[B] + \mathbb{E}[C]$$
$$= 1/3 + 1/3 + 1/3$$
$$= 1$$

Average
1

Sum of Expectations

Expected number = ?



8 billion people

Sum of Expectations

$$\mathbb{E}\,[\text{Matches}] = \overbrace{\mathbb{E}\left[X_{\text{person}_1}\right] + \mathbb{E}\left[X_{\text{person}_2}\right] + \ldots + \mathbb{E}\left[X_{\text{person}_n}\right]}^{\text{n people (n = 8 billion)}} = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$$

$$= n \cdot \frac{1}{n} = 1$$

In general:

$$\mathbb{E}\left[X_1 + X_2 + \ldots + X_n\right] = \mathbb{E}\left[X_1\right] + \mathbb{E}\left[X_2\right] + \ldots + \mathbb{E}\left[X_n\right]$$

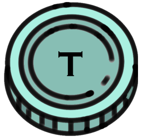
Describing Distributions

Variance

Variance Motivation: Fair Price To Play the Game

You play a game with a friend

You win 1 dollar



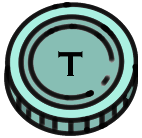
You lose 1 dollar

What is the fair amount of money to pay to play this game?

Variance Motivation: Fair Price To Play the Game

You play a game with a friend

You win 1 dollar



You lose 1 dollar

Game cost:

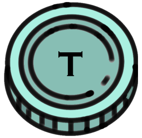
$0

What is the fair amount of money to pay to play this game?

Variance Motivation: Fair Price To Play the Game

You play a game with a friend

You win 100 dollars



You lose 100 dollars

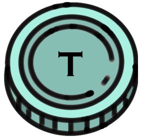
What is the fair amount of money to pay to play this game?

Variance Motivation: Fair Price To Play the Game

You play a game with a friend

Ga cost:

You win 100 dollars



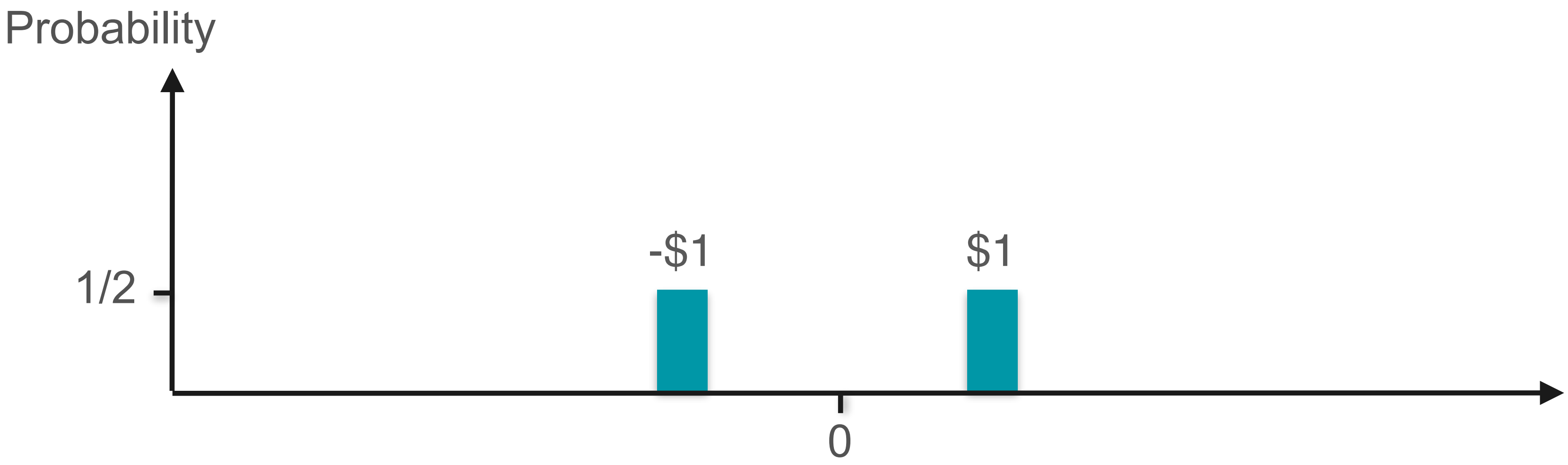
You lose 10

0

Variance!

What is the fair amount of money to pay to play this game?

Variance Motivation: Measuring Spread



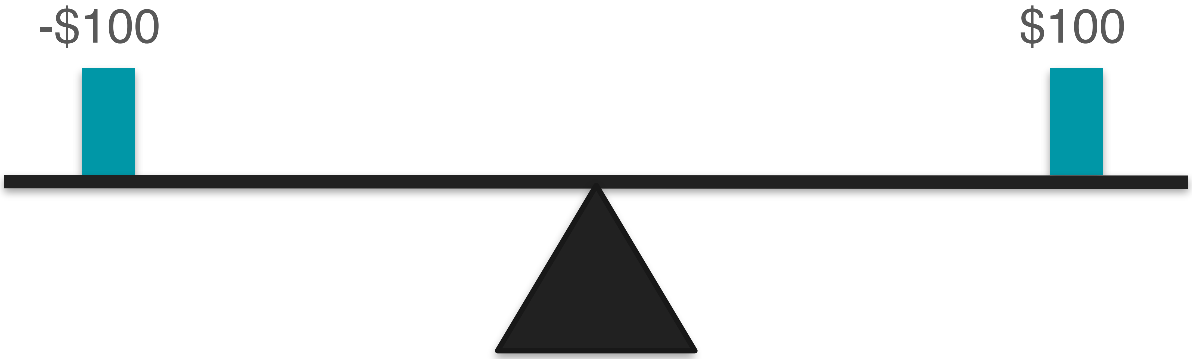
Variance Motivation: Measuring Spread

X_1 = amount of money gained in game 1

$$\mathbb{E}[X_1] = 0$$

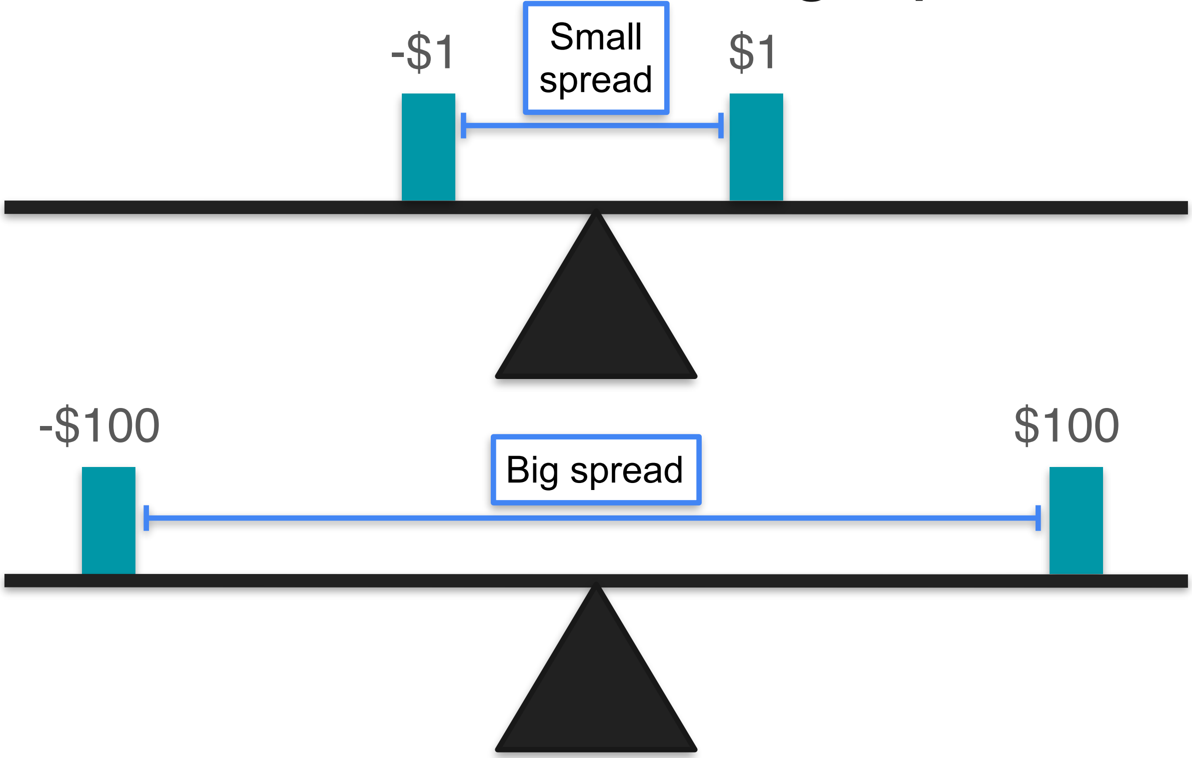
Variance Motivation: Measuring Spread

X_2 = amount of money gained in game 2



$$\mathbb{E}[X_2] = 0$$

Variance Motivation: Measuring Spread



$\mathbb{E}[X_1] = 0$

$\mathbb{E}[X_2] = 0$

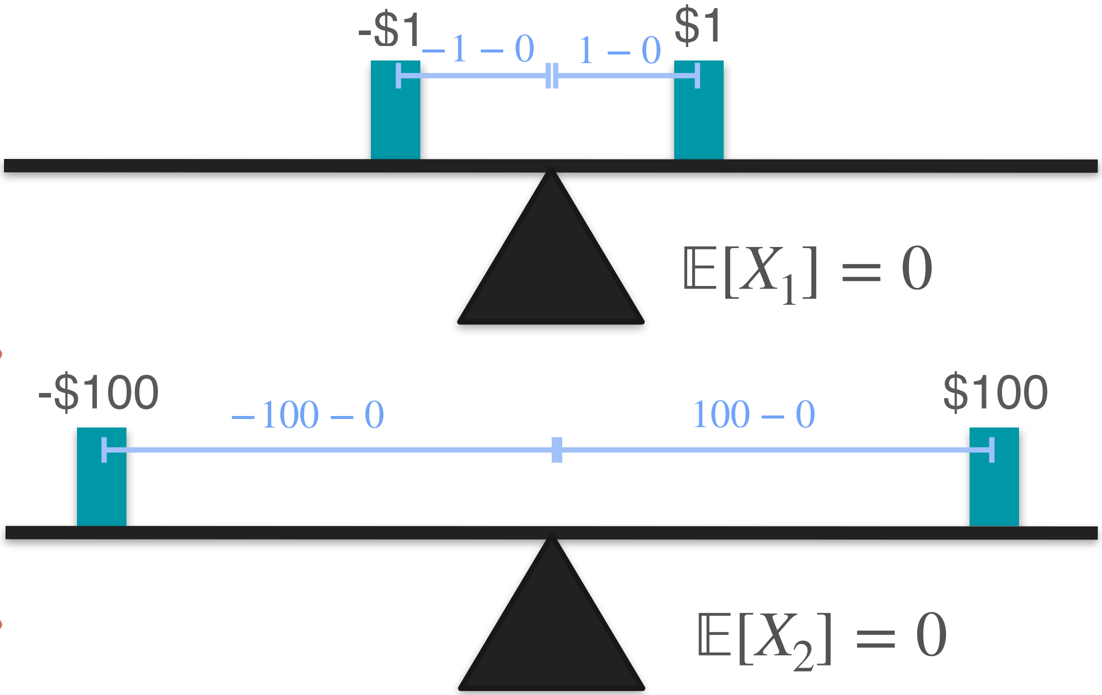
Variance Motivation: Measuring Spread

Deviation
$x - \mathbb{E}[X]$

Expected Deviation
$\cancel{\mathbb{E}\left[X - \mathbb{E}[X]\right]}$

$\cancel{\dfrac{-1+1}{2} = 0}$

Absolute deviation
$\cancel{|x - \mathbb{E}[X]|}$

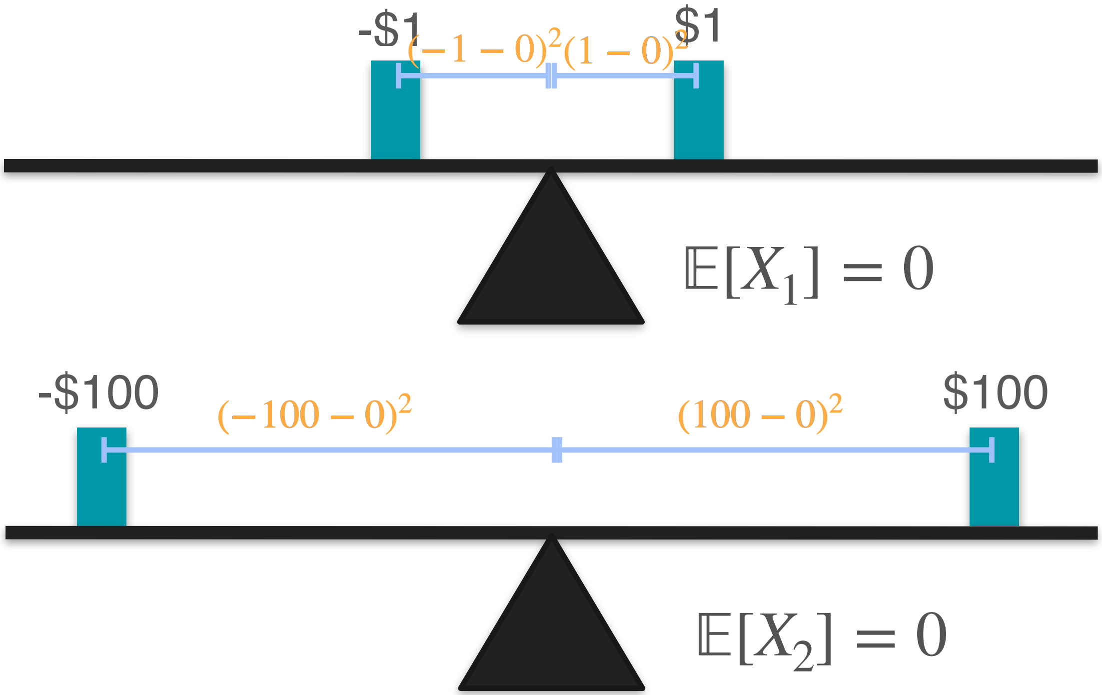


Variance Motivation: Measuring Spread

Deviation
$x - \mathbb{E}[X]$

Squared deviation
$(x - \mathbb{E}[X])^2$

Expected squared deviation
$\mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$



Variance Motivation: Measuring Spread

Deviation

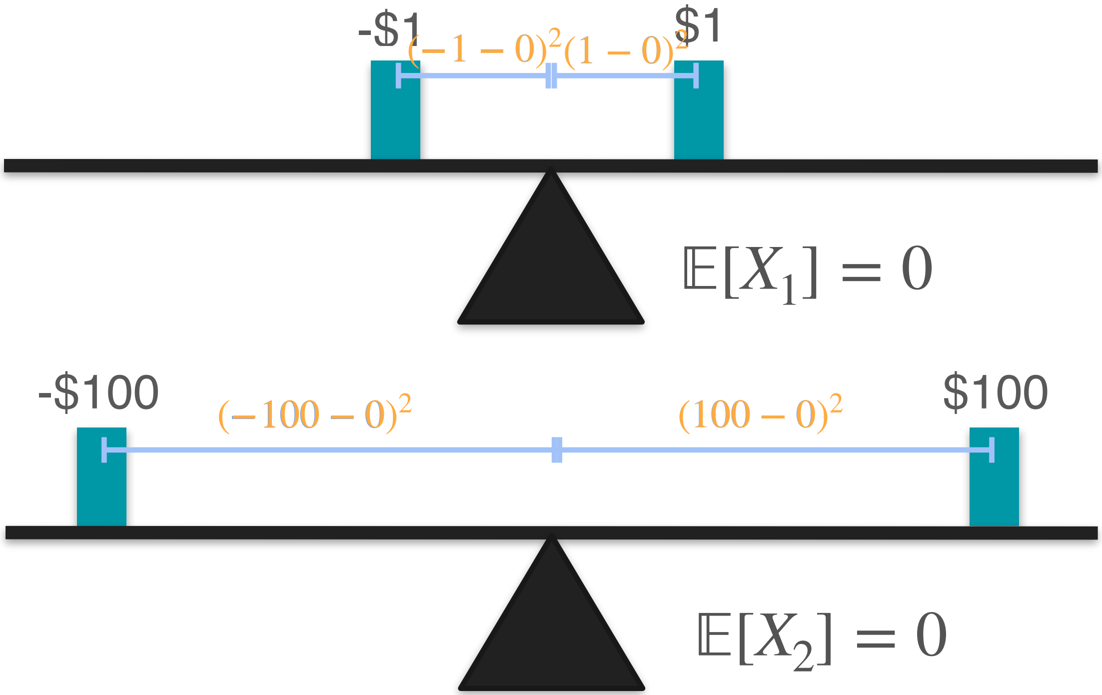
$x - \mathbb{E}[X]$

Squared deviation

$(x - \mathbb{E}[X])^2$

Expected squared deviation

$\mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$



$$\frac{(-1-0)^2 + (1-0)^2}{2}$$

Variance: 1

$$\frac{(-100-0)^2 + (100-0)^2}{2}$$

Variance: 10,000

Variance Formula

$$\text{Variance} \quad Var(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

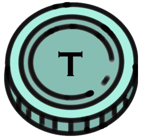
1. Find X's mean
2. Find the deviation from that mean for every value of X
3. Square those deviations
4. Average those squared deviations

"Average squared deviation"

Variance Motivation: Centering With Mean

Game 1

You win 2 dollars



You lose 2 dollars

Game 2

You win 3 dollars

You lose 1 dollar

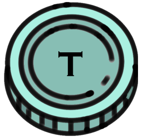
Which of these games has greater variance?

Hint: Think of the spread

Variance Motivation: Centering With Mean

Game 1

You win 2 dollars



You lose 2 dollars

Game 2

You win 3 dollars

You lose 1 dollar

They have the same variance

Variance Motivation: Centering With Mean

Game 1

You win 2 dollars

You lose 2 dollars

$$\mathbb{E}[X_1] = \frac{1}{2} \cdot (-2) + \frac{1}{2} \cdot 2 \ = 0$$

Game 2

You win 3 dollars

You lose 1 dollar

$$\mathbb{E}[X_2] = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 3 \ = 1$$

Variance Motivation: Centering With Mean

Game 1

You win 2 dollars

You lose 2 dollars

$$\mathbb{E}[X_1] = \mathbf{0} \qquad \frac{1}{2}(-2-\quad)^2+\frac{1}{2}(2-\quad)^2= 4$$

Different price, but same spread

Game 2

You win 3 dollars

You lose 1 dollar

$$\mathbb{E}[X_2] = \mathbf{1} \qquad \frac{1}{2}(-1-\quad)^2+\frac{1}{2}(3-\quad)^2= 4$$

Variance Formula

$$
\begin{aligned}
Var(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2
\end{aligned}
$$

Variance Formula

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2\mathbb{E}[X]X + \mathbb{E}[X]^2]$$

$\mathbb{E}[\text{constant} \cdot X] = \text{constant} \cdot \mathbb{E}[X]$

$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$$= \mathbb{E}[X^2] - \mathbb{E}[2\mathbb{E}[X]X] + \mathbb{E}[\mathbb{E}[X]^2]$$

$\mathbb{E}[X]$ is a constant

$\mathbb{E}[\text{constant}] = \text{constant}$

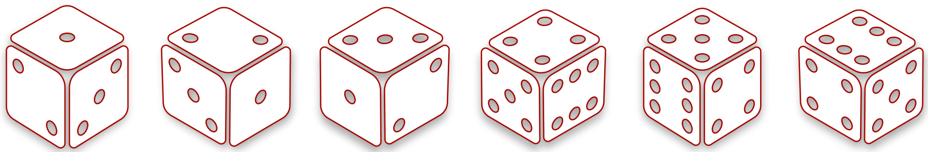
$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\,\mathbb{E}[X] + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Properties of the Variance

Probability:	1/6	1/6	1/6	1/6	1/6	1/6
Roll:	1	2	3	4	5	6



Win Double:	$2	$4	$6	$8	$10	$12
Net Amount:	-$3	-$1	$1	$3	$5	$7

Properties of the Variance

changes the spread

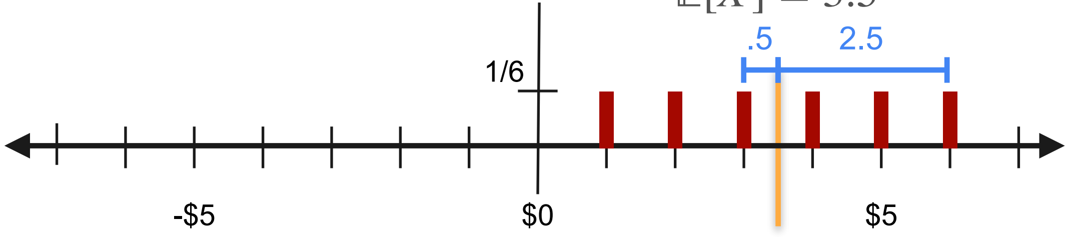
$$Var(\boxed{aX} + \boxed{b}) = a^2Var(X)$$

doesn't change the spread

Variance: "Average squared deviation"

Dice roll is random variable: X

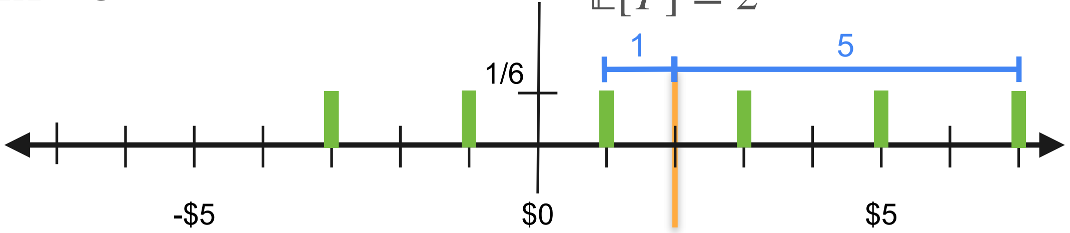
$\mathbb{E}[X] = 3.5$



Net winnings is random variable: $Y = 2X - 5$

$Var(Y) = Var(2X - 5) = 4Var(X)$

$\mathbb{E}[Y] = 2$



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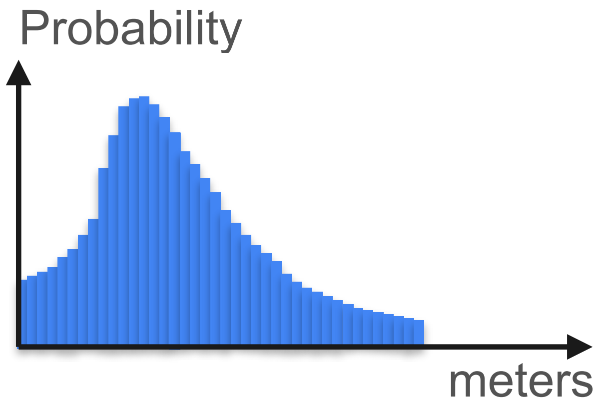
Describing Distributions

Standard deviation

Standard Deviation

$$\overset{m^2}{\boxed{Var(X)}} = \mathbb{E}[(X-\mu)^2] = \overset{m^2}{\boxed{\mathbb{E}[X^2]}} - \overset{m^2}{\boxed{\mathbb{E}[X]^2}}$$



Say X is measured in meters.

Then $\mathbb{E}[X]$ is measured in meters.

Then $Var(X)$ is measured in meters^2.

Then $\sqrt{Var(X)}$ is measured in meters.

Let's call $std(X) = \sqrt{Var(X)}$, the *standard deviation* of X

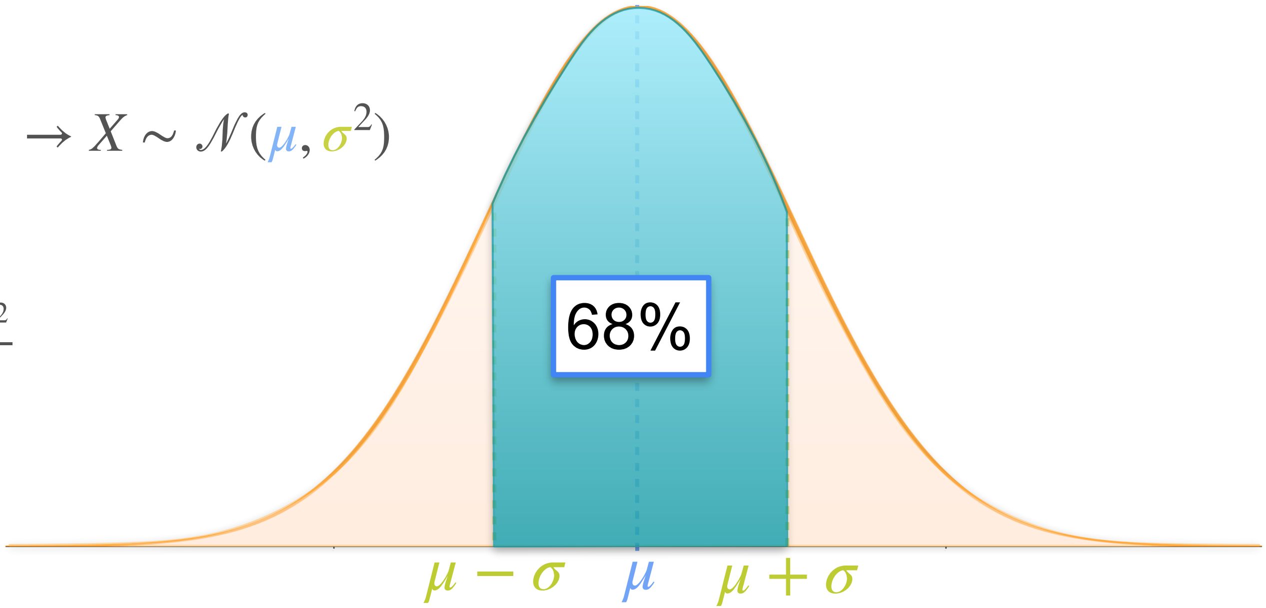
Normal Distribution: 68-95-99.7 Rule

Parameters:

- μ: center of the bell
- σ: spread of the bell

$\rightarrow X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$



Normal Distribution: 68-95-99.7 Rule

Parameters:

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- σ: spread of the bell

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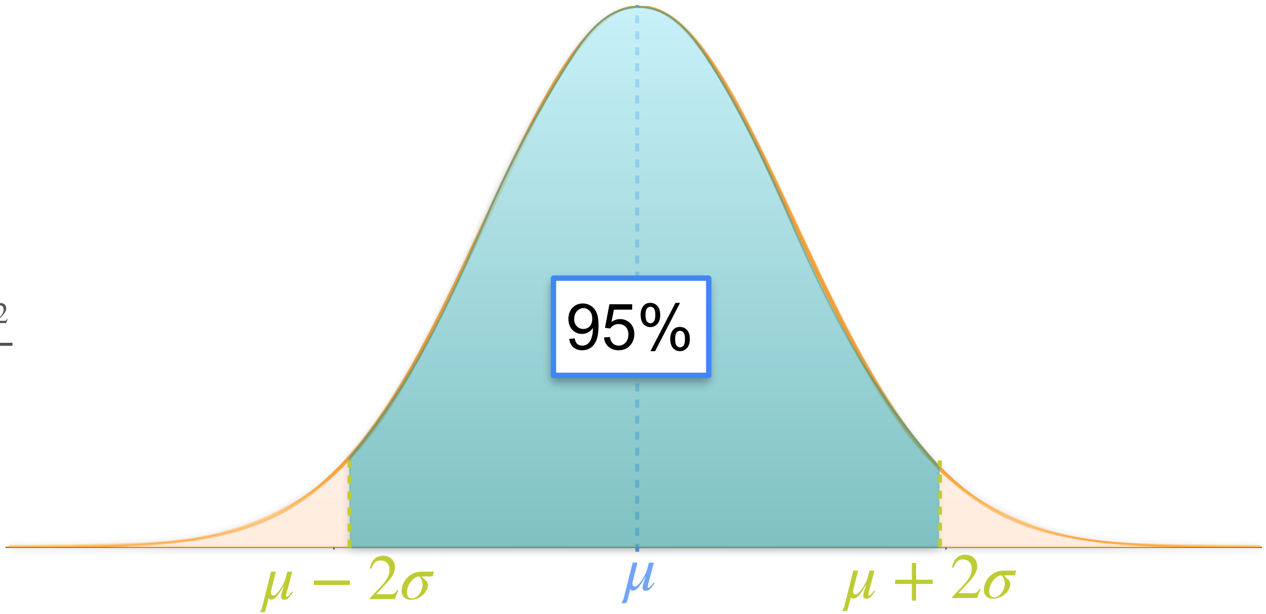
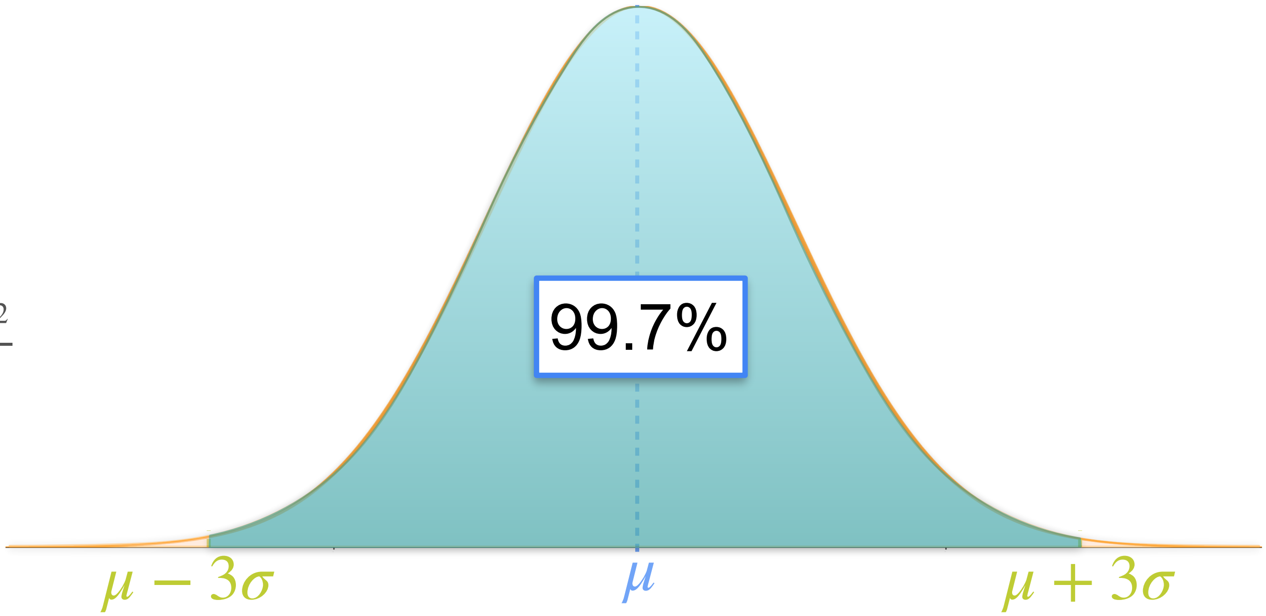


Normal Distribution: 68-95-99.7 Rule

Parameters:

- μ: center of the bell
- σ: spread of the bell

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$



Normal Distribution: 68-95-99.7 Rule

Parameters:

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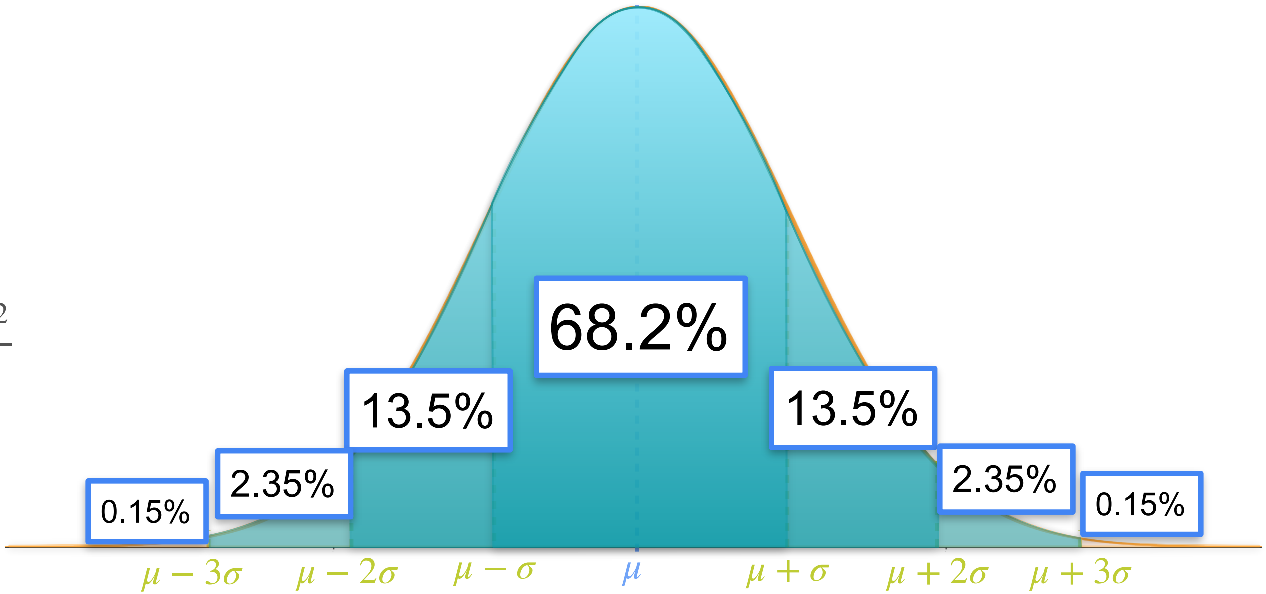

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Describing Distributions

Sum of Gaussians

Sum of Gaussians: an Example

R : Total response time of a computer system

Independent

T : Processing time

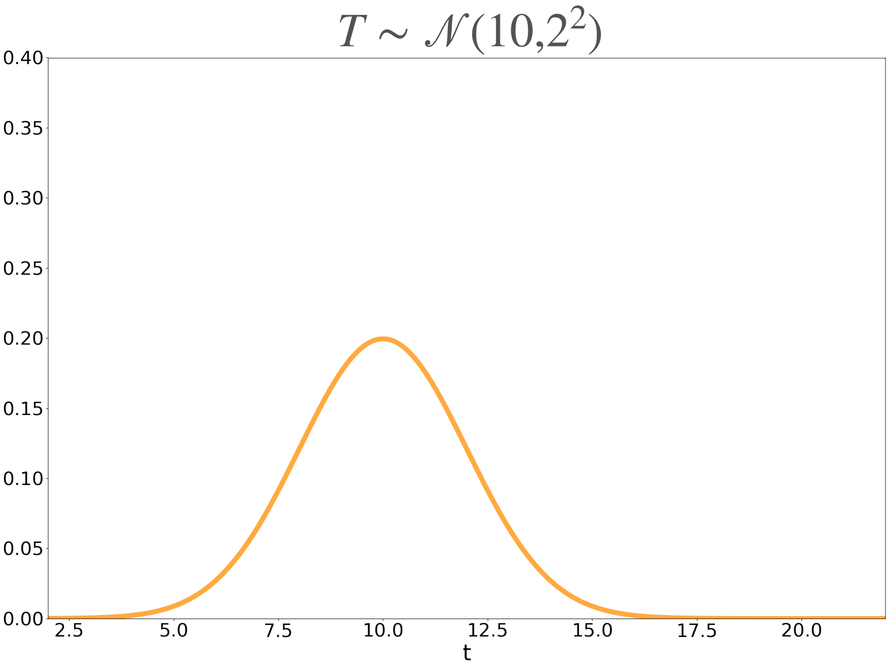
L : Network Latency

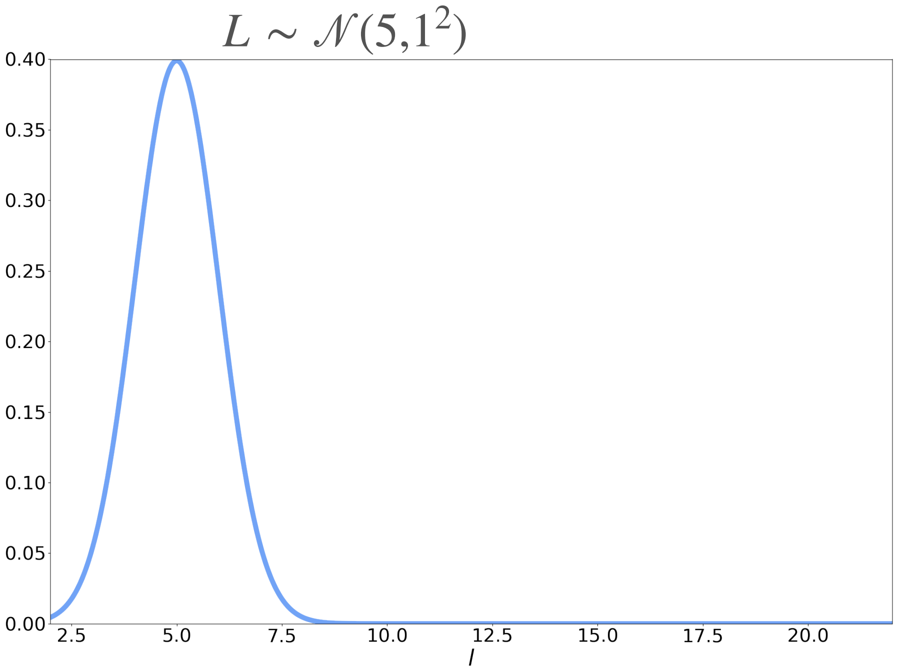
$T \sim \mathcal{N}(10, 2^2)$

$L \sim \mathcal{N}(5, 1^2)$

$$R = T + L$$

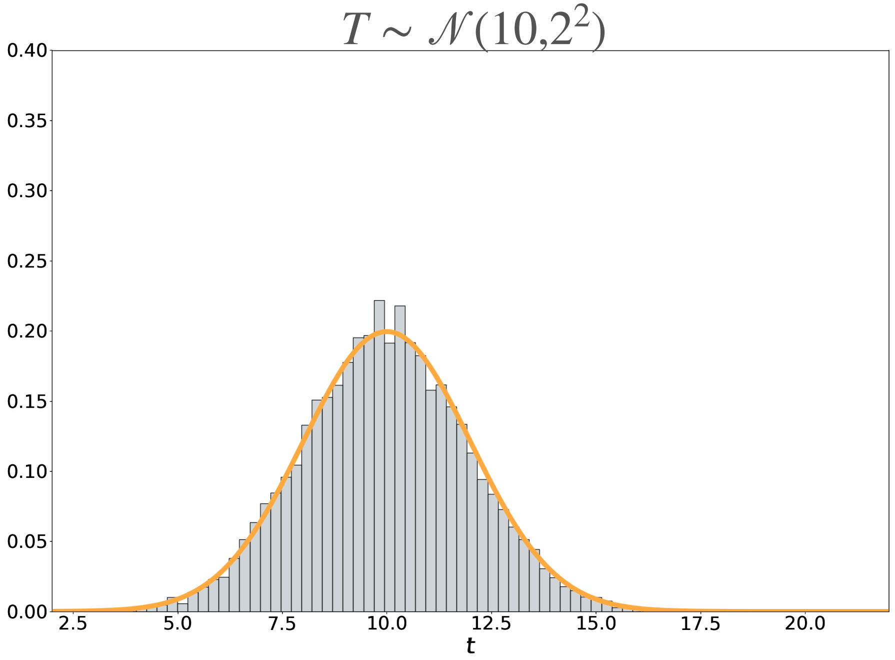
Sum of Gaussians

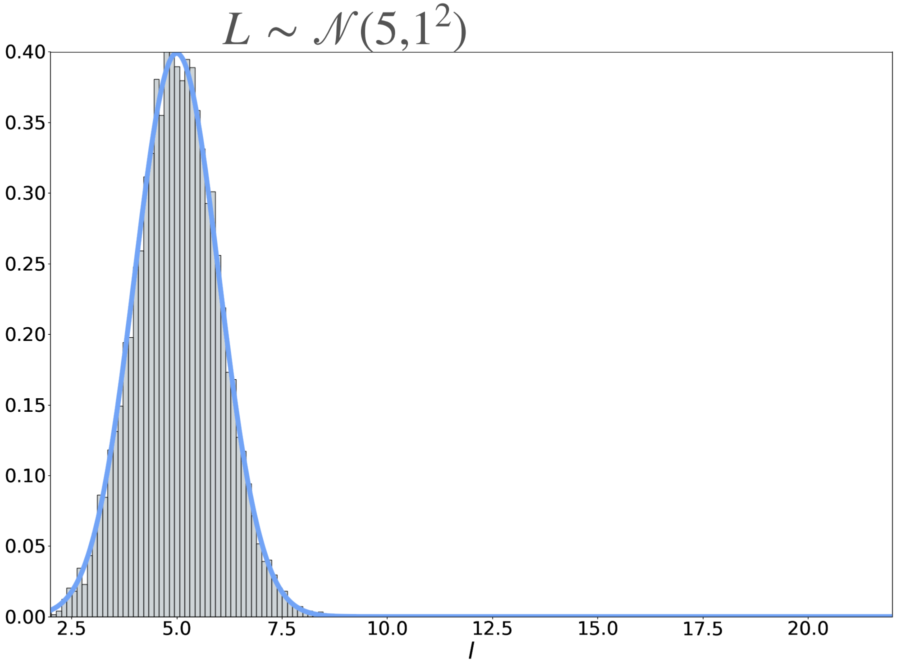




Sum of Gaussians

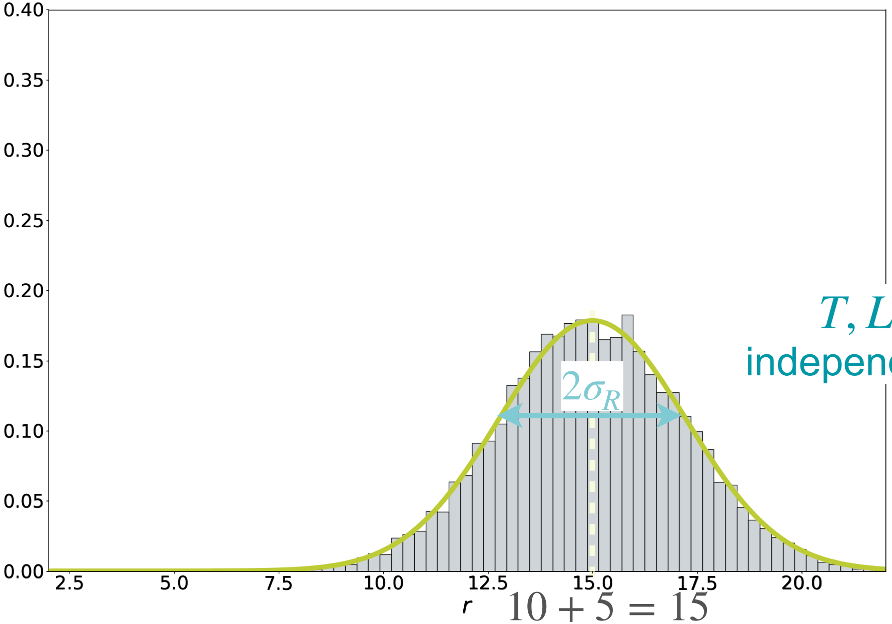
Sample each variable 10000 times





Sum of Gaussians

$$R = T + L$$



$10 + 5 = 15$

R is still Gaussian!

Expectation is linear

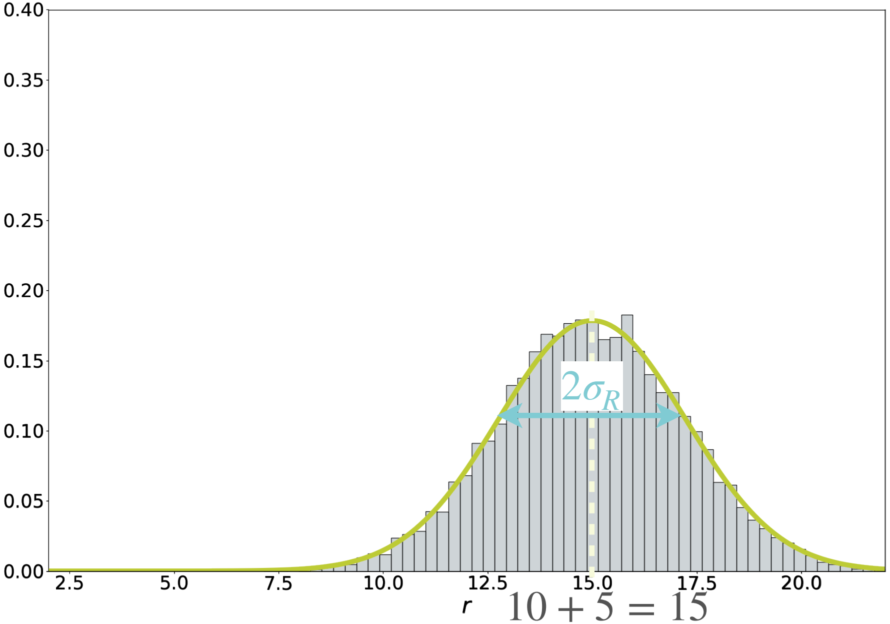
$$\mu_R = \mathbb{E}[R] = \mathbb{E}[T + L] = \mathbb{E}[T] + \mathbb{E}[L]$$
$$= \mu_T + \mu_L = 10 + 5 = 15$$

T, L independent

$$\sigma_R^2 = Var(R) \quad Var(T + L)$$
$$= Var(T) + Var(L) \quad = \sigma_T^2 + \sigma_L^2$$
$$= 4 + 1 \quad = 5$$

Sum of Gaussians

$$R = T + L$$



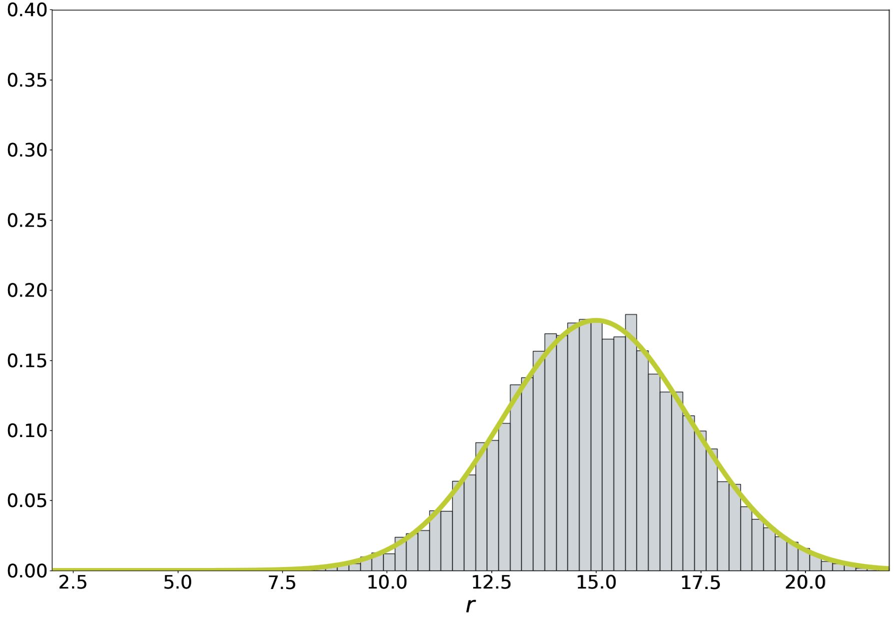
$$10 + 5 = 15$$

R is still Gaussian!

$$R = (T + L) \sim \mathcal{N}\left(10 + 5, 4 + 1\right)$$

Sum of Gaussians

$$R = T + L$$



In general: $W = aX + bY$

Independent $\begin{cases} X \sim \mathcal{N}(\mu_X, \sigma_X^2) \\ Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \end{cases}$

$$\rightarrow W \sim \mathcal{N}\left(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)$$

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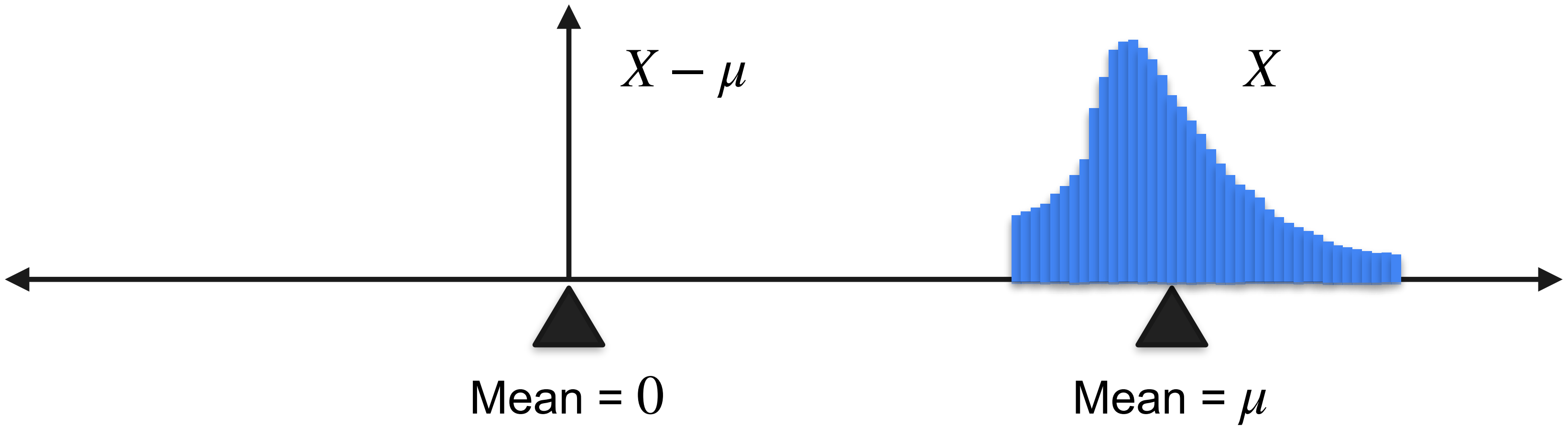
Describing Distributions

Standardizing a Distribution

Everything Is Nicer When the Mean Is 0



$$X \to X - \mu$$

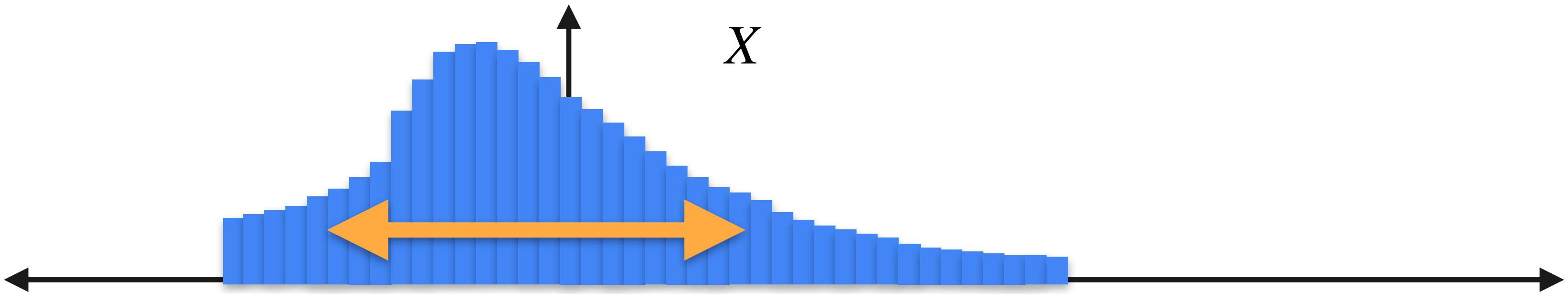
Everything Is Nicer When the Mean Is 0

Why?

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\begin{aligned}\mathbb{E}[X-\mu] &= \mathbb{E}[X] - \mathbb{E}[\mu] \\ &= \mathbb{E}[X] - \mu \\ &= 0\end{aligned}$$

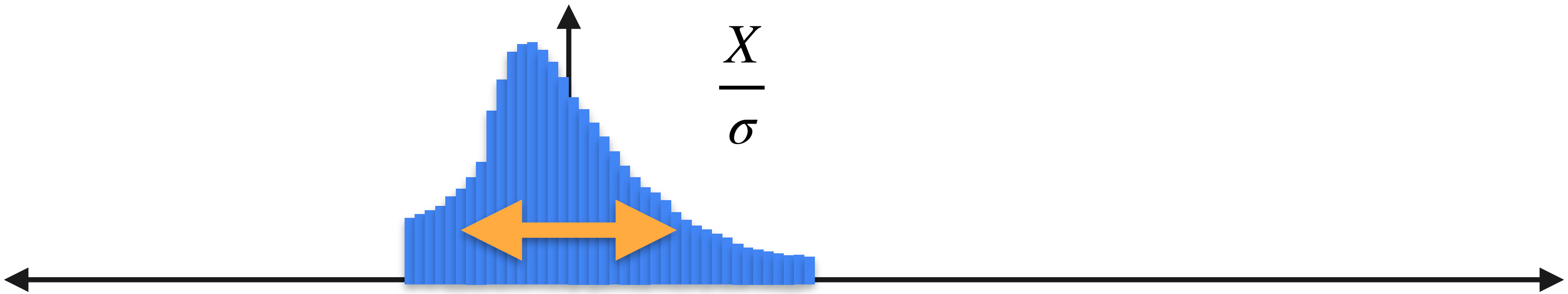
Everything Is Nicer When the Standard Deviation Is 1



Mean = 0

Standard deviation = σ

Everything Is Nicer When the Standard Deviation Is 1



Mean = 0

Standard deviation = 1

$$X \rightarrow \frac{X}{\sigma}$$

Everything Is Nicer When the Standard Deviation Is 1

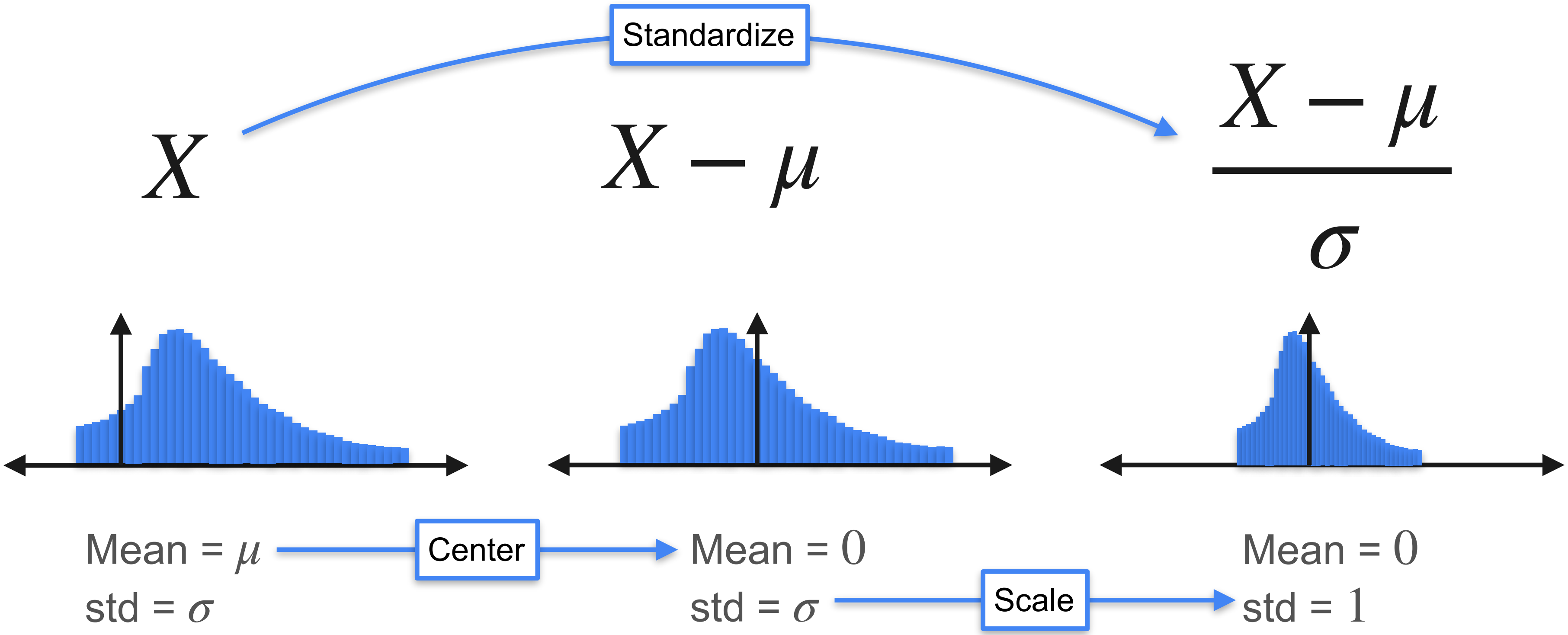
Why?

$$
\begin{aligned}
Var(cX) &= \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2 \\
&= \mathbb{E}[c^2X^2] - (c\mathbb{E}[X])^2 \\
&= c^2\mathbb{E}[X^2] - c^2\mathbb{E}[X]^2 \\
&= c^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2) \\
&= c^2 Var(X)
\end{aligned}
$$

$$
Var\left(\frac{X}{\sigma}\right) = \frac{1}{\sigma^2} Var(X)
$$

$$
\begin{aligned}
std\left(\frac{X}{\sigma}\right) &= \frac{1}{\sigma} std(X) \\
&= \frac{\sigma}{\sigma} \\
&= 1
\end{aligned}
$$

Standardize a Distribution



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Describing Distributions

Skewness and Kurtosis: Moments of a Distribution

Moments of a Distribution

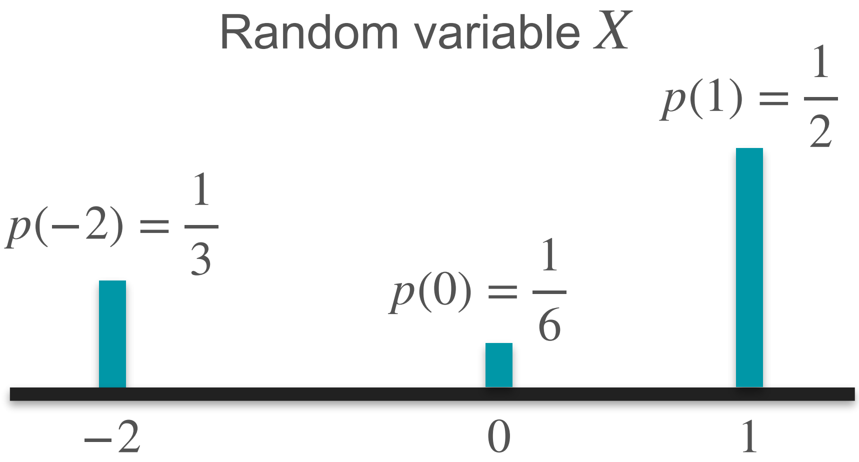
$$\mathbb{E}[X] = \frac{1}{3}(-2) + \frac{1}{6}(0) + \frac{1}{2}(1)$$

$$\mathbb{E}[X^2] = \frac{1}{3}(-2)^2 + \frac{1}{6}(0)^2 + \frac{1}{2}(1)^2$$

$$\mathbb{E}[X^3] = \frac{1}{3}(-2)^3 + \frac{1}{6}(0)^3 + \frac{1}{2}(1)^3$$

$$\dots$$

$$\mathbb{E}[X^k] = \frac{1}{3}(-2)^k + \frac{1}{6}(0)^k + \frac{1}{2}(1)^k$$



Moments of a Distribution

$$\mathbb{E}[X] = p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

$$\mathbb{E}[X^2] = p_1x_1^2 + p_2x_2^2 + \cdots + p_nx_n^2$$

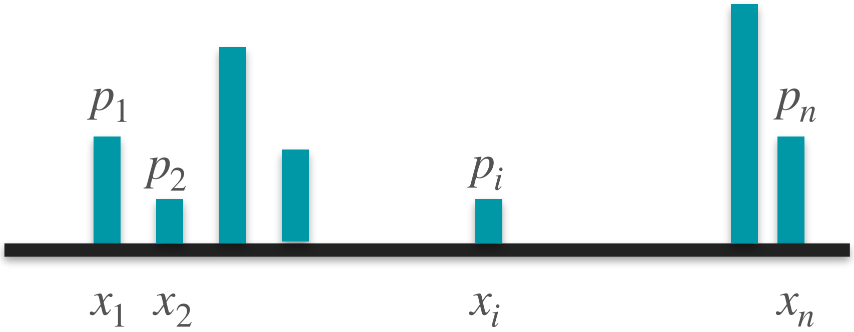
$$\mathbb{E}[X^3] = p_1x_1^3 + p_2x_2^3 + \cdots + p_nx_n^3$$

$$\mathbb{E}[X^4] = p_1x_1^4 + p_2x_2^4 + \cdots + p_nx_n^4$$

$$\cdots$$

$$\mathbb{E}[X^k] = p_1x_1^k + p_2x_2^k + \cdots + p_nx_n^k$$

Random variable X



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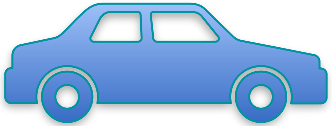
Describing Distributions

Skewness and Kurtosis: Skewness

Lottery vs Insurance

Ticket: $1
Jackpot: $100

Lottery



Cost: $1
Crash Reparation: $100

Car insurance

You **win** $99 with 1% probability

You **lose** $1 with 99% probability

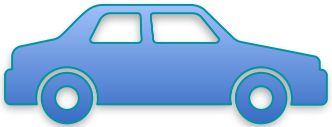
You **win** $1 with 99% probability

You **lose** $99 with 1% probability

Lottery vs Insurance

Lottery

Ticket: $1
Jackpot: $100

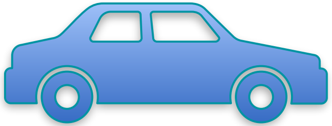


Car insurance

Cost: $1
Crash Reparation: $100

Ticket: $1
Jackpot: $100

Lottery



Cost: $1
Crash Reparation: $100

Car insurance

$$\mathbb{E}[X_1] = -1 \cdot 0.99 + 99 \cdot 0.01 = 0$$

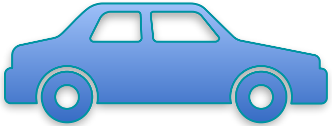
Lose 1 With probability 0.99	Win 99 With probability 0.01

$$\mathbb{E}[X_2] = -99 \cdot 0.01 + 1 \cdot 0.99 = 0$$

Lose 99 With probability 0.01	Win 1 With probability 0.99

Ticket: \$1
Jackpot: \$100

Lottery



Cost: \$1
Crash Reparation: \$100

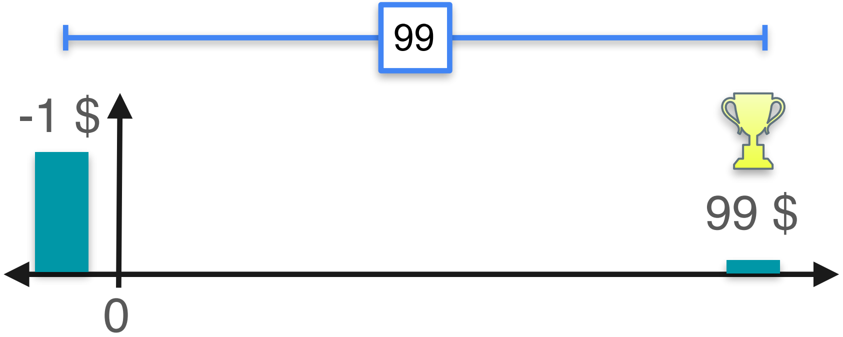
Car insurance

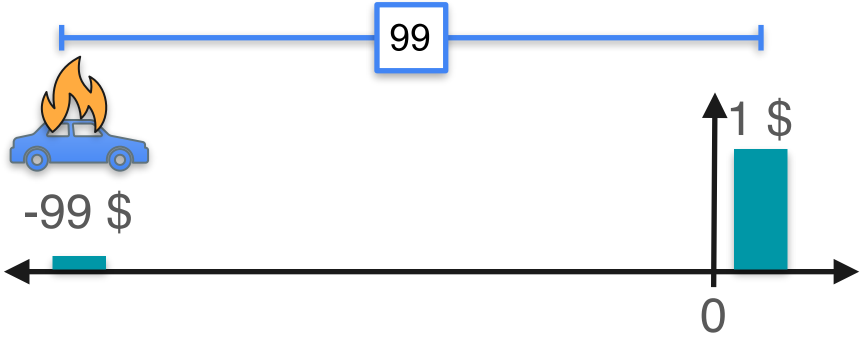
$$\mathbb{E}[X_1] = -1 \cdot 0.99 + 99 \cdot 0.01 = 0$$

$$\mathbb{E}[X_2] = -99 \cdot 0.01 + 1 \cdot 0.99 = 0$$

$$Var(X_1) = (-1)^2 \cdot 0.99 + (99)^2 \cdot 0.01 = 99$$

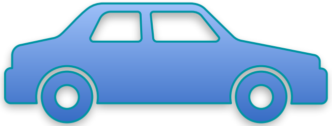
$$Var(X_2) = (-99)^2 \cdot 0.01 + (1)^2 \cdot 0.99 = 99$$





Ticket: $1
Jackpot: $100

Lottery



Cost: $1
Crash Reparation: $100

Car insurance

$$\mathbb{E}[X_1] = 0$$

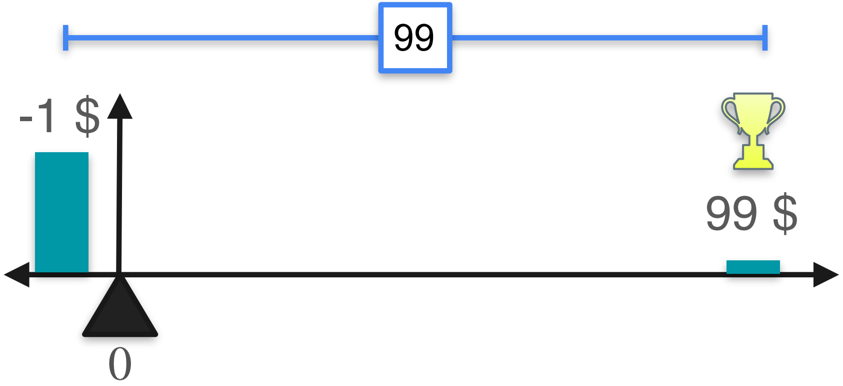
$$Var(X_1) = 99$$

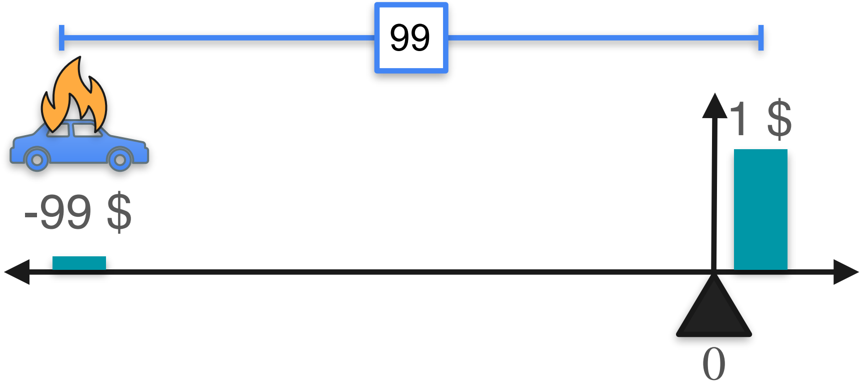
Same expectation
Same variance

How to tell them apart?

$$\mathbb{E}[X_2] = 0$$

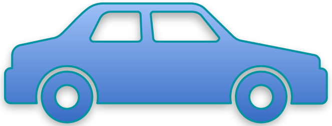
$$Var(X_2) = 99$$





Ticket: $1
Jackpot: $99

Lottery



Cost: $1
Crash Reparation: $99

Car insurance

Same expectation
Same variance

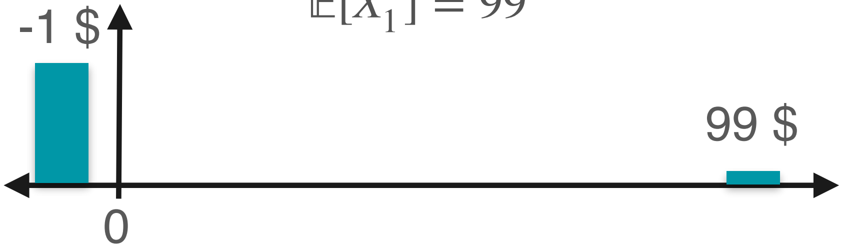
How to tell them apart?

$$\mathbb{E}[X_1] = 0$$

$$Var(X_1) = 99$$

$$\mathbb{E}[X_1^2] - \cancel{\mathbb{E}[X_1]^2} = 99$$

$$\mathbb{E}[X_1^2] = 99$$

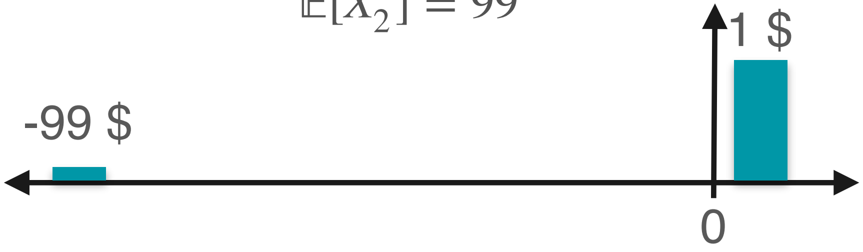


$$\mathbb{E}[X_2] = 0$$

$$Var(X_2) = 99$$

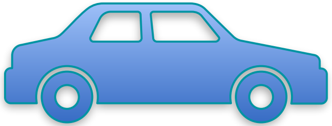
$$\mathbb{E}[X_2^2] - \cancel{\mathbb{E}[X_2]^2} = 99$$

$$\mathbb{E}[X_2^2] = 99$$



Ticket: $1
Jackpot: $99

Lottery



Cost: $1
Crash Reparation: $99

Car insurance

Same expectation
Same variance

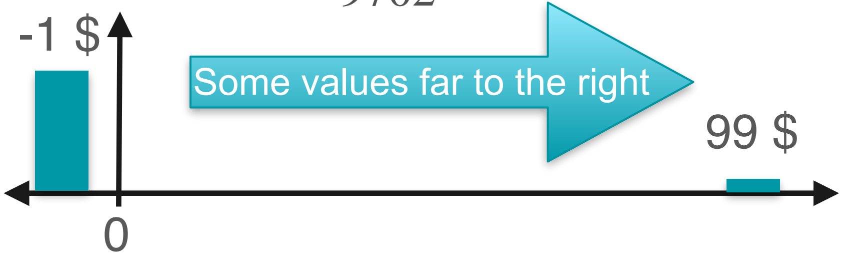
How to tell them apart?

Lottery

$$\mathbb{E}[X_1] = 0$$

$$\mathbb{E}[X_1^2] = 99$$

$$\mathbb{E}[X_1^3] = (-1)^3 \cdot 0.99 + (99)^3 \cdot 0.01$$
$$= 9702$$

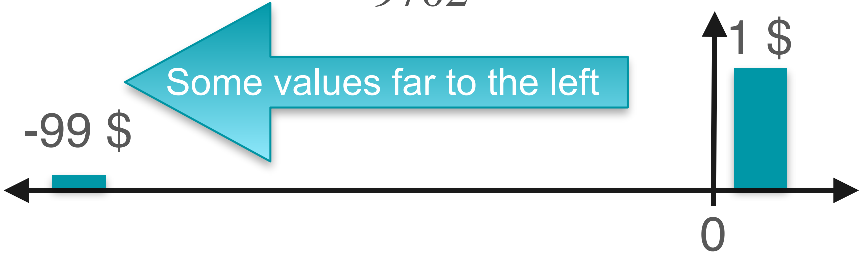


Car insurance

$$\mathbb{E}[X_2] = 0$$

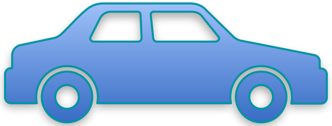
$$\mathbb{E}[X_2^2] = 99$$

$$\mathbb{E}[X_2^3] = (1)^3 \cdot 0.99 + (-99)^3 \cdot 0.01$$
$$= -9702$$



Ticket: \$1
Jackpot: \$99

Lottery



Cost: \$1
Crash Reparation: \$99

Car insurance

$\mathbb{E}[X_1] = 0$

$\mathbb{E}[X_1^2] = 99$

$\mathbb{E}[X_1^3]$ = Large positive value

Same expectation
Same variance

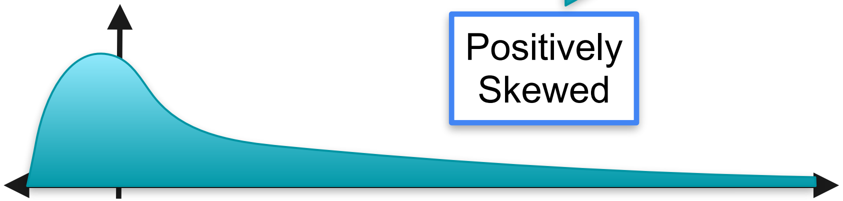
How to tell them apart?

$\mathbb{E}[X_2] = 0$

$\mathbb{E}[X_1^2] = 99$

$\mathbb{E}[X_1^3]$ = Large negative value

Positively Skewed



Negatively Skewed

Skewness

$$\mathbb{E}[X^3]$$

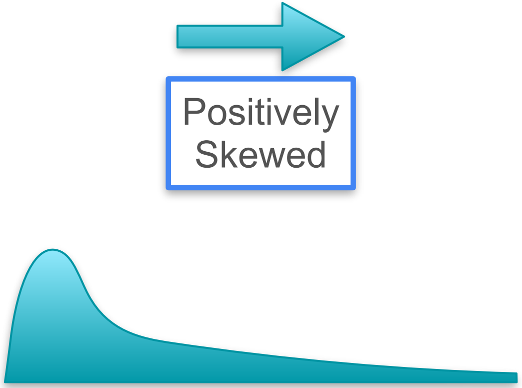
Almost…

Need to standardize…

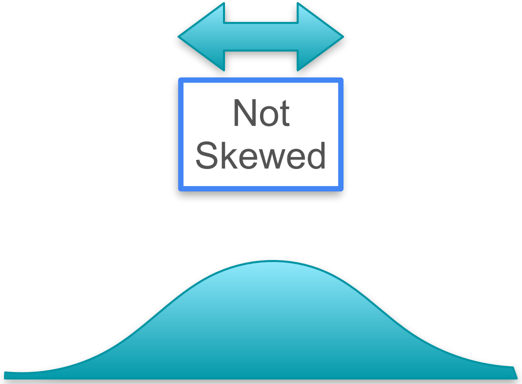
Skewness

$$\text{Skewness} = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

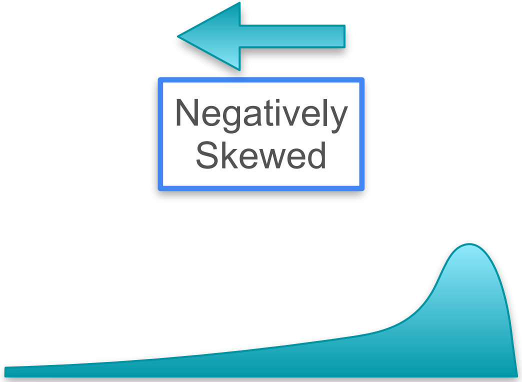
Skewness



$$E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] > 0$$



$$E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = 0$$



$$E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] < 0$$

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Describing Distributions

Skewness and Kurtosis: Kurtosis

Kurtosis: Example

Game 1

Which one is riskier?

probability $\frac{1}{2}$: You win 1 dollar

probability $\frac{1}{2}$: You lose 1 dollar

Game 2

probability $\frac{100}{202}$: You win 10 cents

probability $\frac{100}{202}$: You lose 10 cents

probability $\frac{1}{202}$: You win 10 dollars

probability $\frac{1}{202}$: You lose 10 dollars

Kurtosis: Example

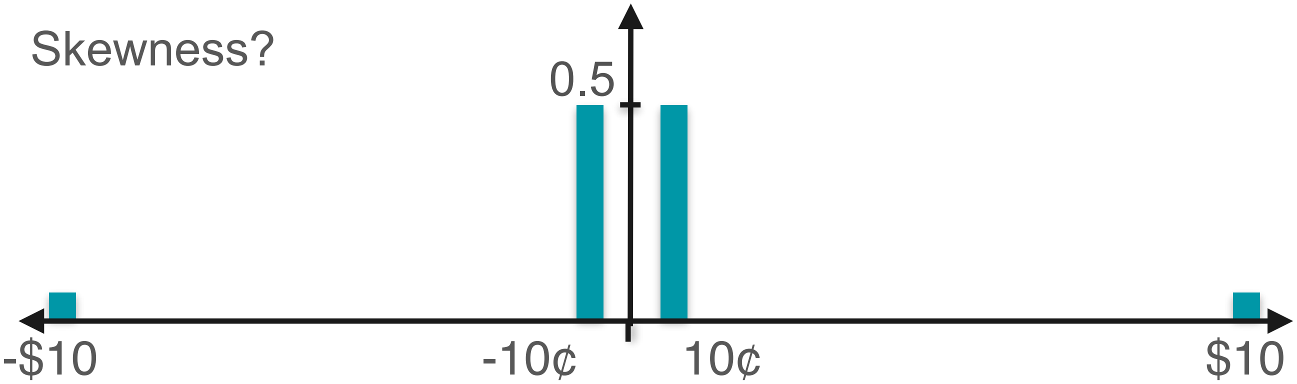
Game 1

Expected value?

Standard deviation?

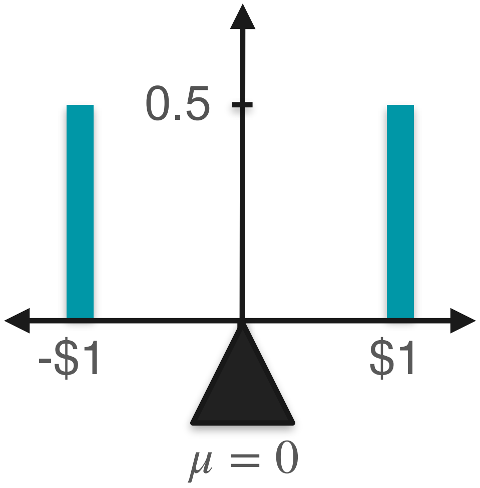
Skewness?

Game 2

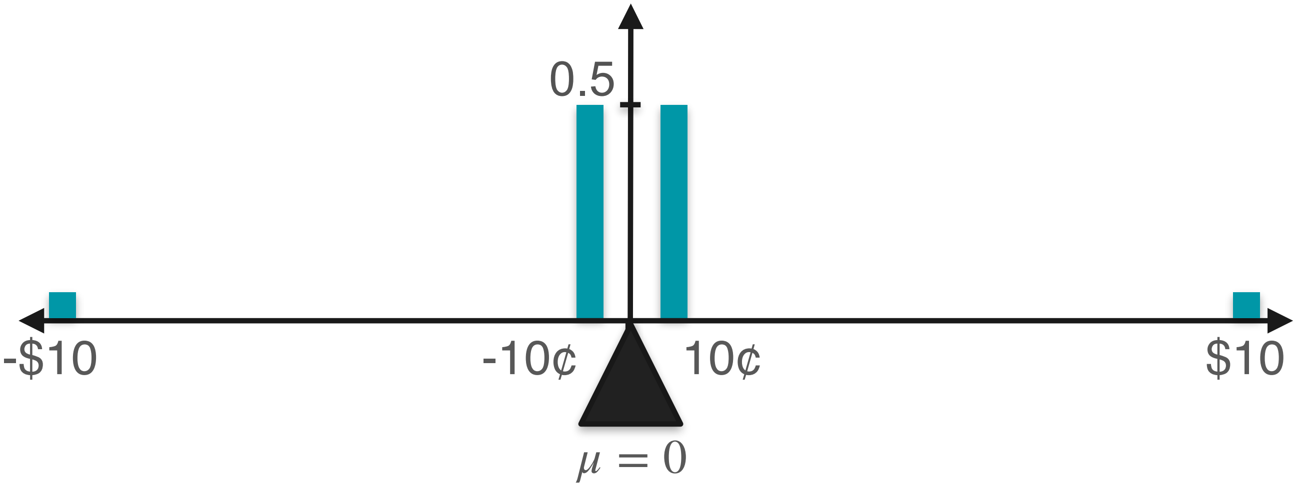


Kurtosis: Example Expected Value

Game 1 $\mathbb{E}[X_1] = 0$



Game 2 $\mathbb{E}[X_2] = 0$

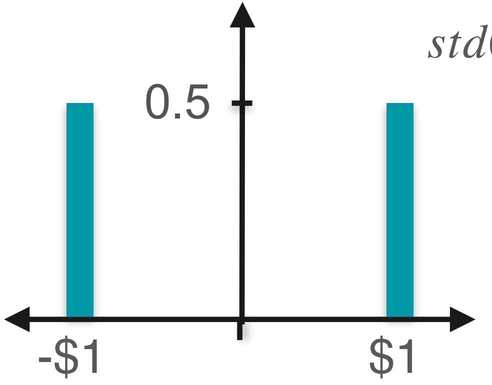


Kurtosis: Example Variance

Game 1

$Var(X_1) = 1$

$std(X_1) = 1$



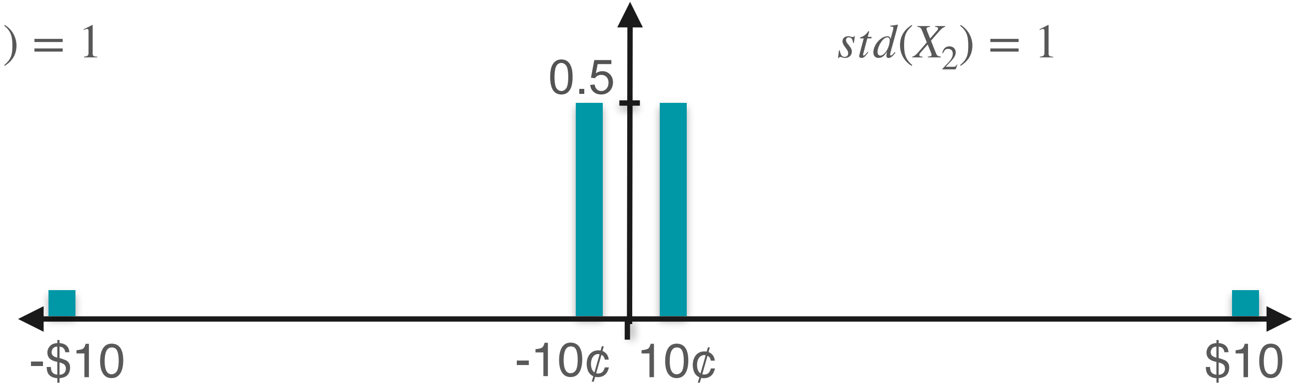
$$\mathbb{E}[X_1^2] = \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2$$

$$= 1$$

Game 2

$Var(X_2) = 1$

$std(X_2) = 1$



$$\mathbb{E}[X_2^2] = \frac{100}{202}(-0.1)^2 + \frac{100}{202}(0.1)^2 + \frac{1}{202}(-10)^2 + \frac{1}{202}(10)^2$$

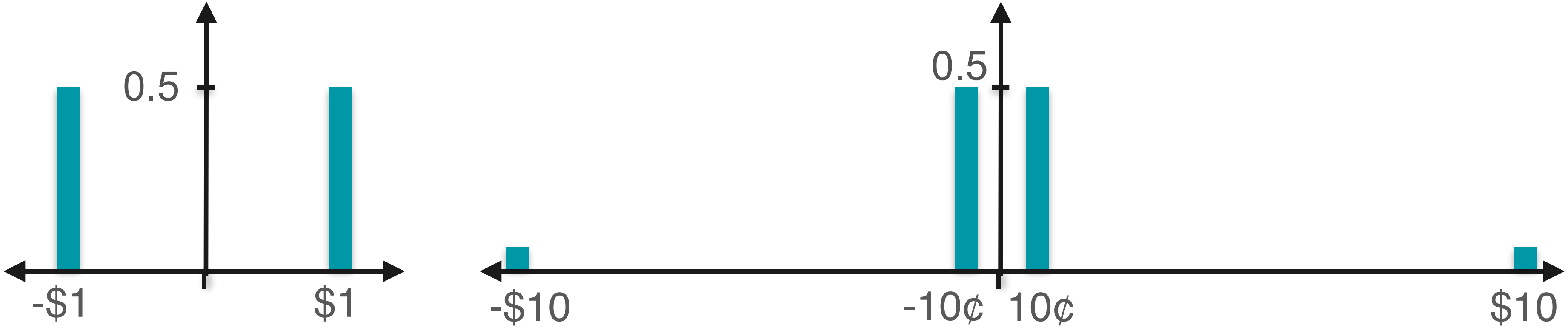
$$= \frac{100}{202}\frac{1}{100} + \frac{100}{202}\frac{1}{100} + \frac{1}{202}\cdot 100 + \frac{1}{202}\cdot 100$$

$$= \frac{1}{202} + \frac{1}{202} + \frac{100}{202} + \frac{100}{202} = 1$$

Kurtosis: Example Skewness

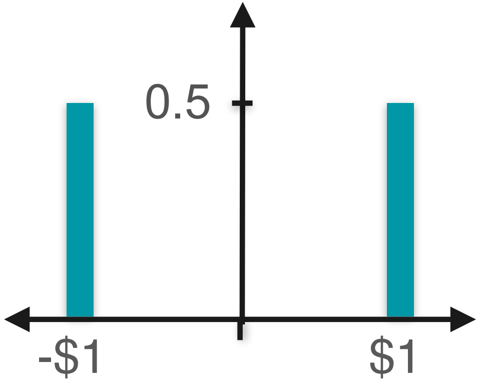
Game 1 $Skew(X_1) = 0$

Game 2 $Skew(X_2) = 0$



Kurtosis

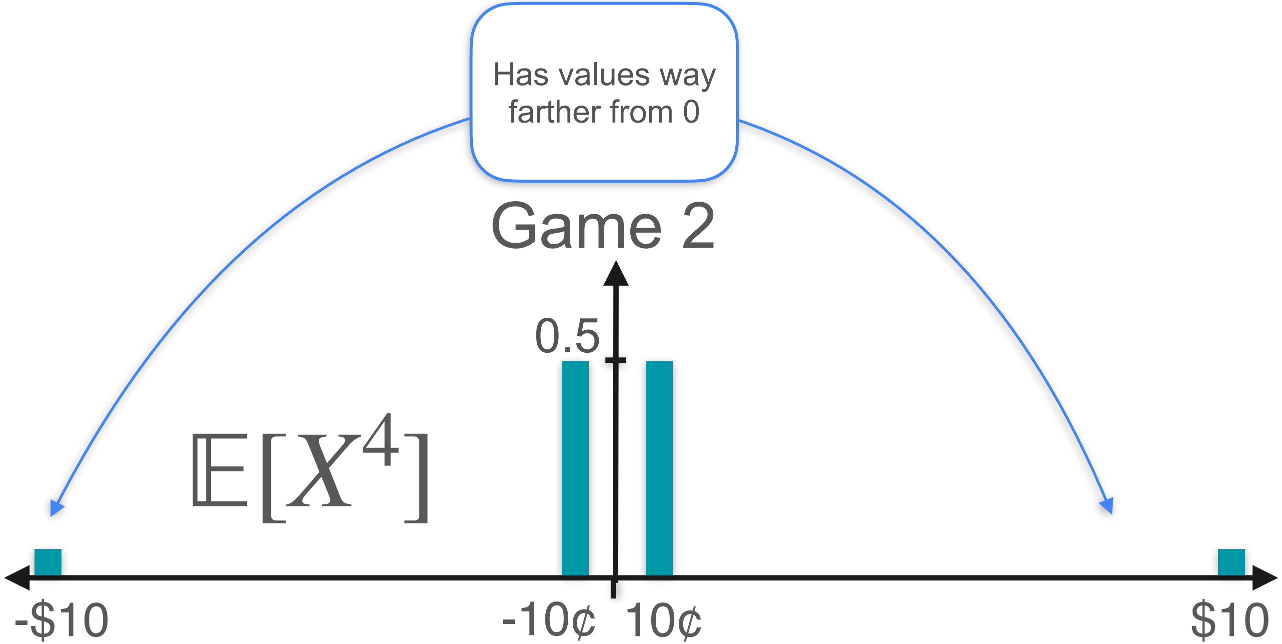
Game 1



$E[X_1] = 0$ $\quad$ $E[X_1] = 0$

$Var(X_1) = 1$ $\quad$ $E[X_1^2] = 1$

$Skew(X_1) = 0$ $\quad$ $E[X_1^3] = 0$



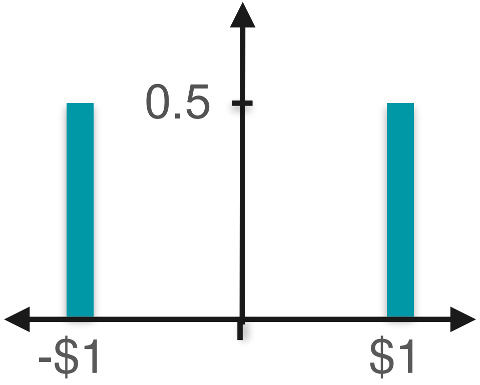
$E[X_2] = 0$ $\quad$ $E[X_2] = 0$

$Var(X_2) = 1$ $\quad$ $E[X_2^2] = 1$

$Skew(X_2) = 0$ $\quad$ $E[X_2^3] = 0$

Kurtosis

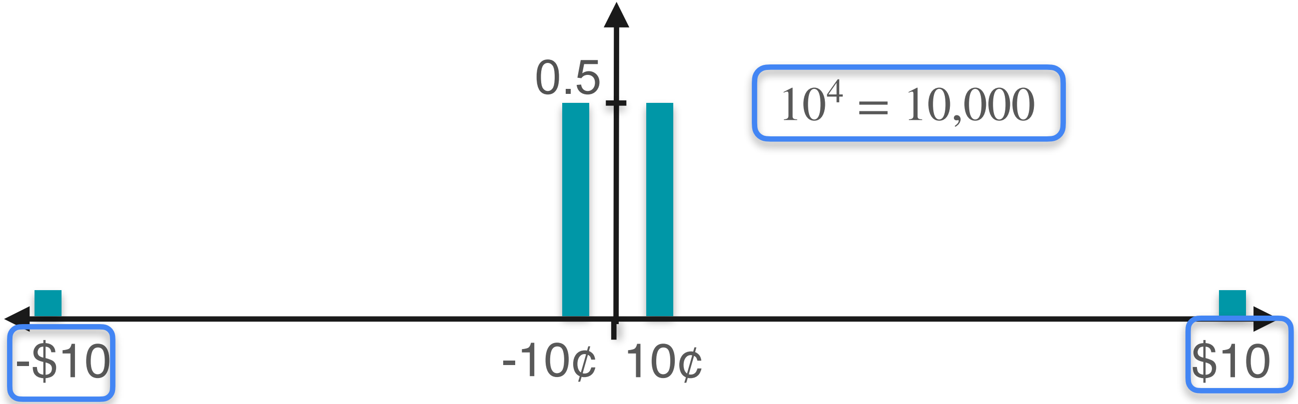
Game 1



$$\mathbb{E}[X_1^4] = \frac{1}{2}(-1)^4 + \frac{1}{2}(1)^4$$

$$= 1$$

Game 2



$$\mathbb{E}[X_2^4] = \frac{100}{202}(-0.1)^4 + \frac{100}{202}(0.1)^4 + \frac{1}{202}\boxed{(-10)^4} + \frac{1}{202}\boxed{(10)^4}$$

$$= 99.01$$

Kurtosis

$$\mathbb{E}[X^4]$$

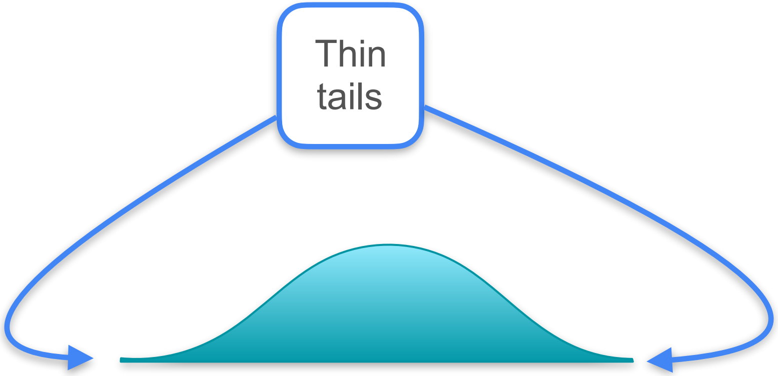
Almost…

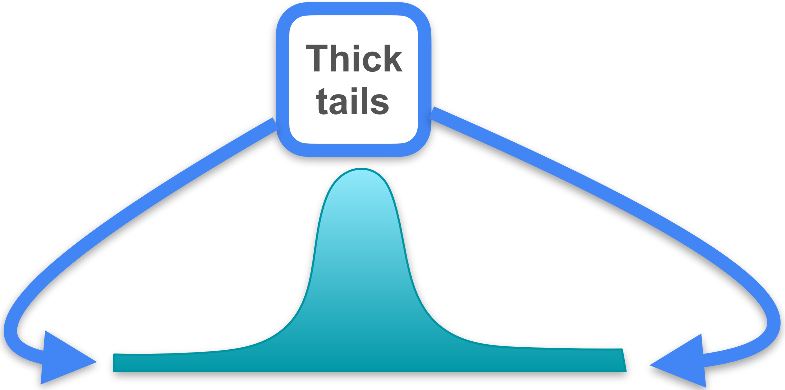
Need to standardize…

Kurtosis

$$\text{Kurtosis} = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]$$

Kurtosis: High and Low





$$E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \text{small}$$

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \text{large}$$

Even if they have the same variance!

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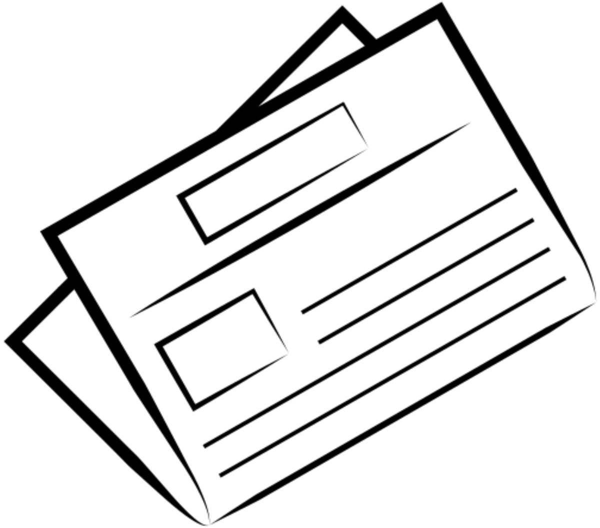
Describing Distributions

Quantiles and Box-Plots

Quantiles: Example



Newspaper advertisement

Newspaper Advertisement (X)			
18.3	18.4	23.2	51.2
35.2	29.7	75	8.7
65.9	14.2	54.7	25.9

Quantiles: Example

What is the median here?

The point that splits your data in half

Newspaper Advertisement (X)			
18.3	18.4	23.2	51.2
35.2	29.7	75	8.7
65.9	14.2	54.7	25.9

Quantiles: Example

What is the median here?

The point that splits your data in half

Median = $\dfrac{25.9 + 29.7}{2} = 27.8$

50% quantile

Second quartile

Newspaper Advertisement (X)			
8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

Quantiles: Example

What about the point that leaves 1/4 of your data to the left and 3/4 to the right?

$$q_{0.25} = Q1 = \frac{18.3 + 18.4}{2} = 18.35$$

25% quantile

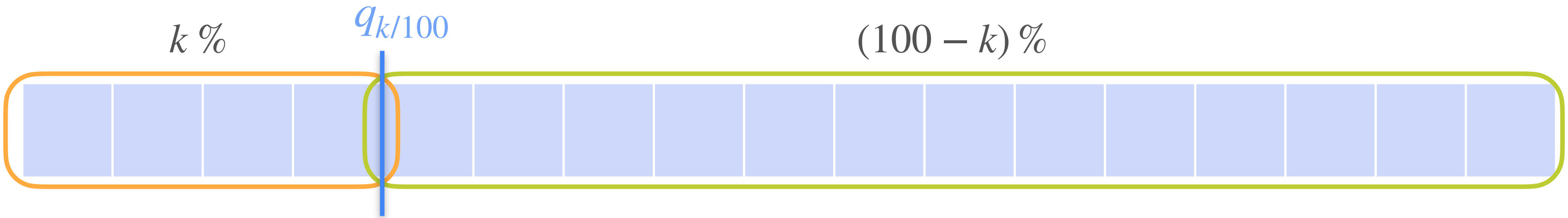
First quartile

Newspaper Advertisement (X)			
8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

Quantiles

In general:

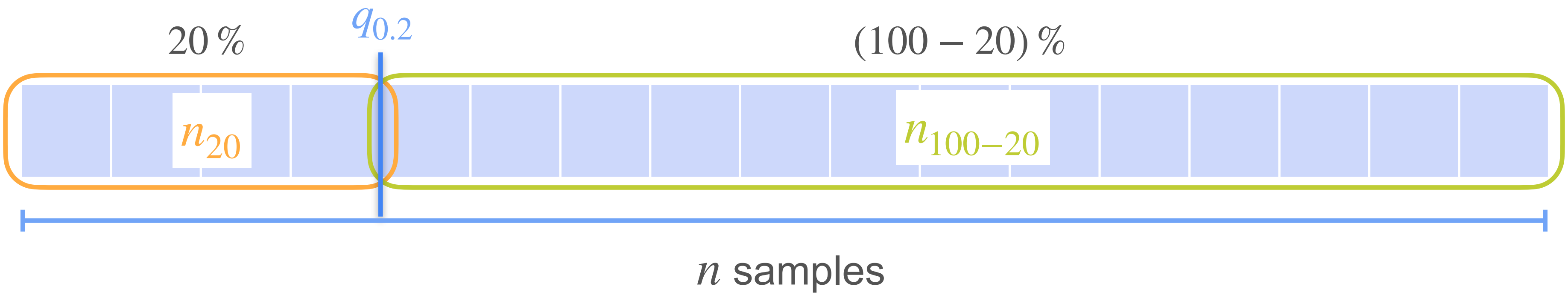
The **k% quantile ($q_{k/100}$)** is the value that leaves k% of your data to the left and (100-k)% of your data to the right



Some common quantiles:

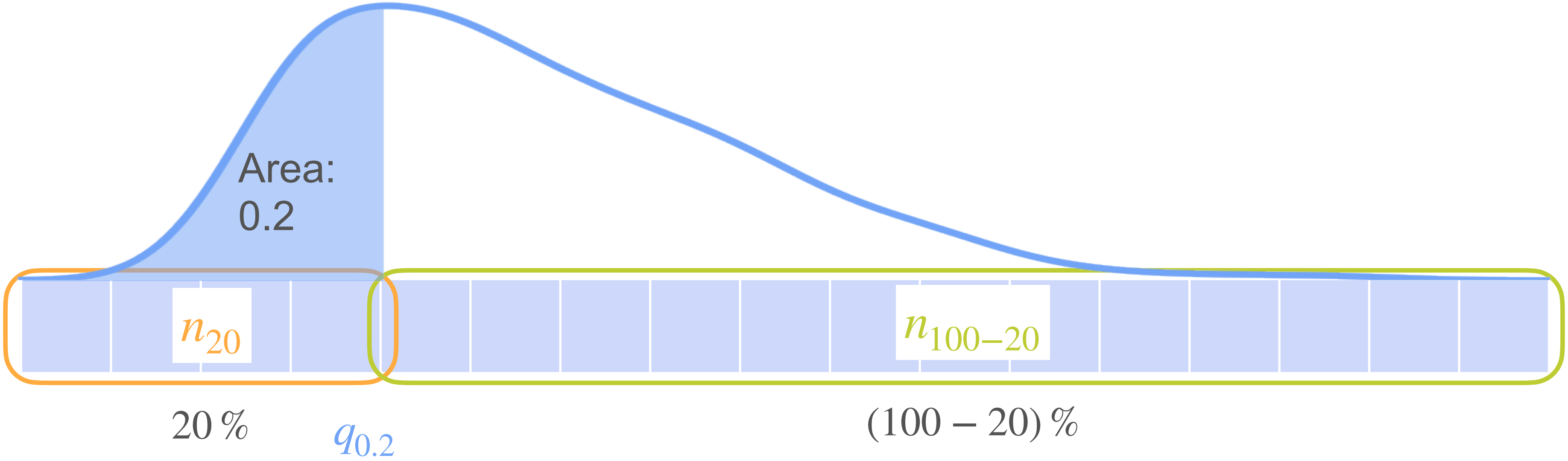
- 25% quantile (first quartile - Q1)
- 50% quantile (median - Q2)
- 75% quantile (third quartile - Q3)

Quantiles



$$\frac{20}{100} = \frac{n_{20}}{n} \approx \mathbf{P}(X \leq q_{0.2})$$

Quantiles



k% quantile ($q_{k/100}$) is the value such that $\mathbf{P}\left(X \leq q_{k/100}\right) = \frac{k}{100}$

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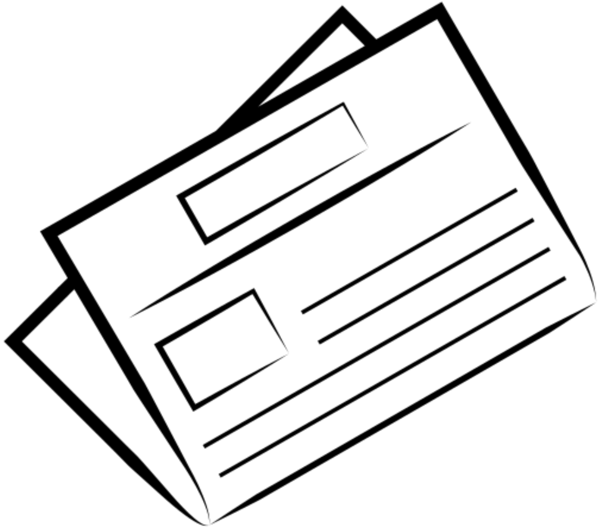
Describing Distributions

Visualizing data: Box-Plots

Box-Plots



Newspaper advertisement

Newspaper Advertisement (X)			
8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

Box-Plots

$$Q1 = q_{0.25} = \frac{18.3 + 18.4}{2} = 18.35$$

$$Q2 = q_{0.50} = \frac{25.9 + 29.7}{2} = 27.8$$

$$Q3 = q_{0.75} = \frac{51.2 + 54.7}{2} = 52.95$$

Newspaper Advertisement (X)			
8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

Box-Plots

$$Q1 = q_{0.25} = \frac{18.3 + 18.4}{2} = 18.35$$

$$Q2 = q_{0.50} = \frac{25.9 + 29.7}{2} = 27.8$$

$$Q3 = q_{0.75} = \frac{51.2 + 54.7}{2} = 52.95$$

Interquartile range (IQR)

$$IQR = Q3 - Q1 = 52.95 - 18.35 = 34.6$$

$$\text{min} = 8.7 \qquad \text{max} = 75$$

Newspaper Advertisement (X)			
8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

Box-Plots

$$Q1 = q_{0.25} = \frac{18.3 + 18.4}{2} = 18.35$$

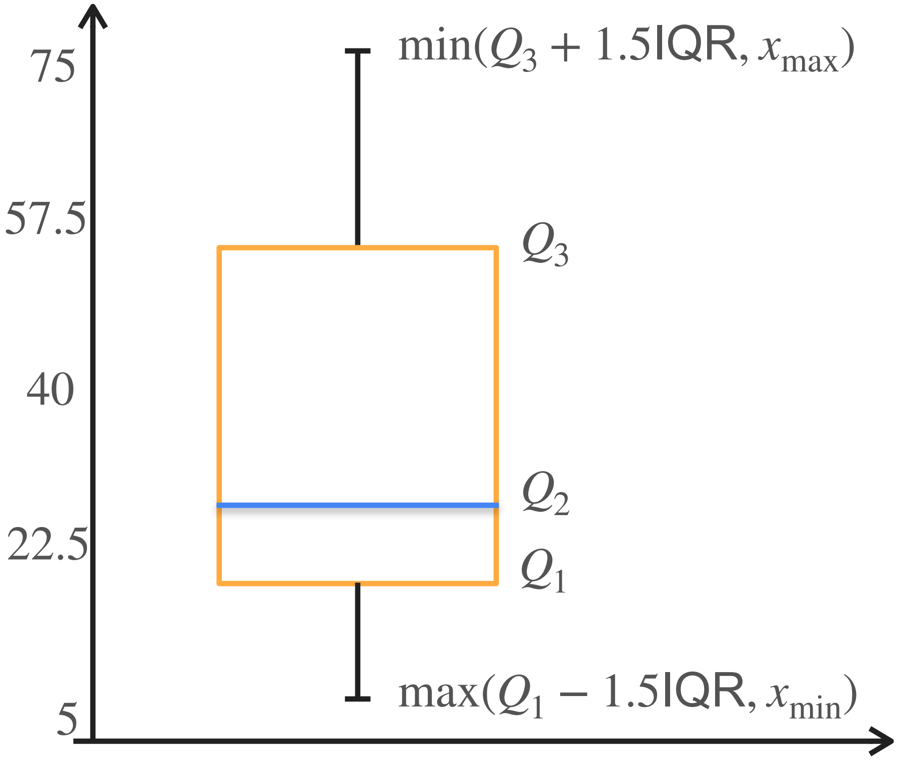
$$Q2 = q_{0.5} = \frac{25.9 + 29.7}{2} = 27.8$$

$$Q3 = q_{0.75} = \frac{51.2 + 54.7}{2} = 52.95$$

Interquartile range (IQR)

$$IQR = Q3 - Q1 = 52.95 - 18.35 = 34.6$$

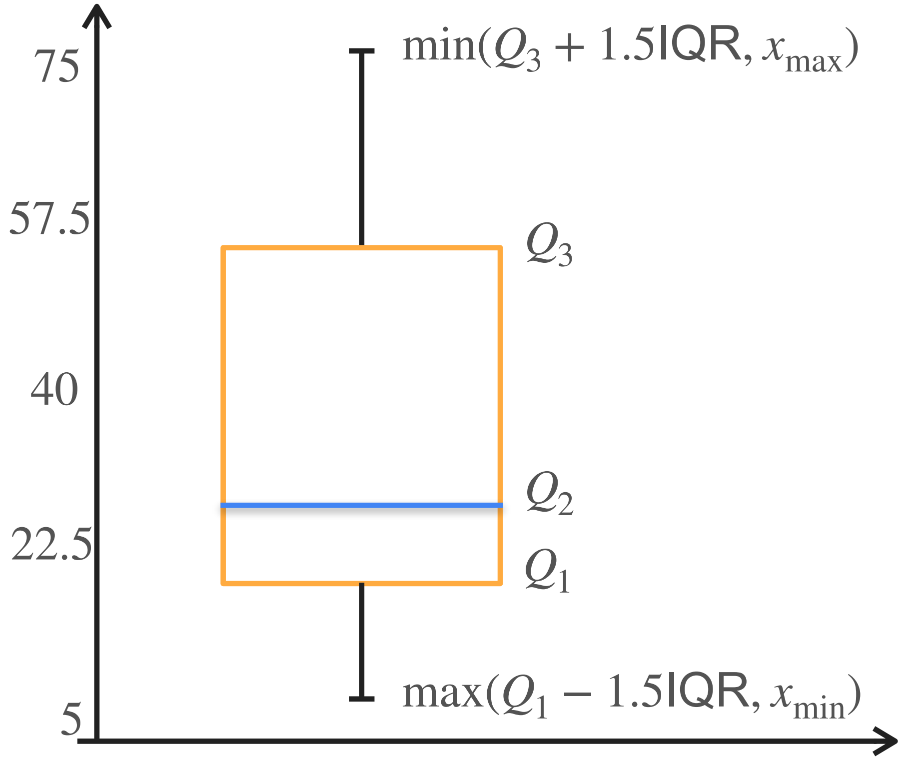
$$x_{\min} = 8.7 \qquad x_{\max} = 75$$



Box-Plots

What can you tell from this plot?

- Data is skewed
- No outliers (whiskers were cut at max and min value)
- Analyze dispersion



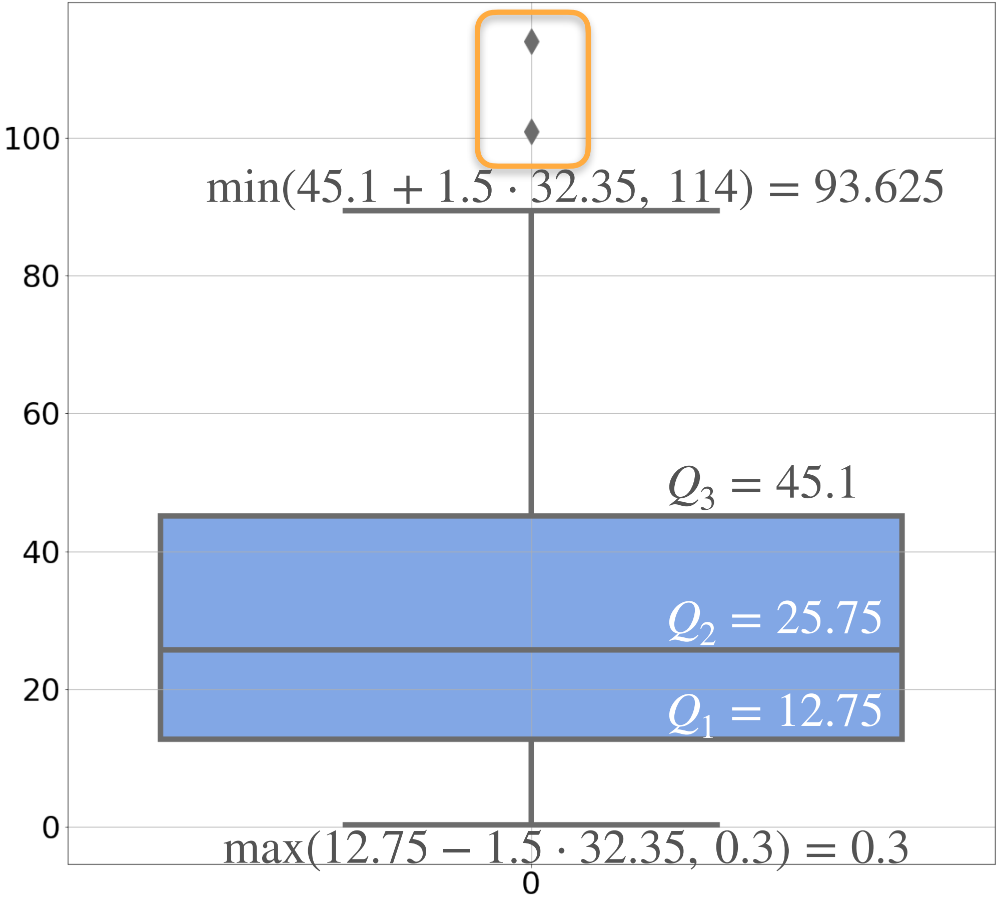
Box-Plots

Let's see how the box-plot looks for the whole dataset

$Q_1 = 12.75$ $Q_2 = 25.75$ $Q_3 = 45.1$

$\text{IQR} = 32.35$ $x_{\text{min}} = 0.3$ $x_{\text{max}} = 114$

Now you can see two outliers



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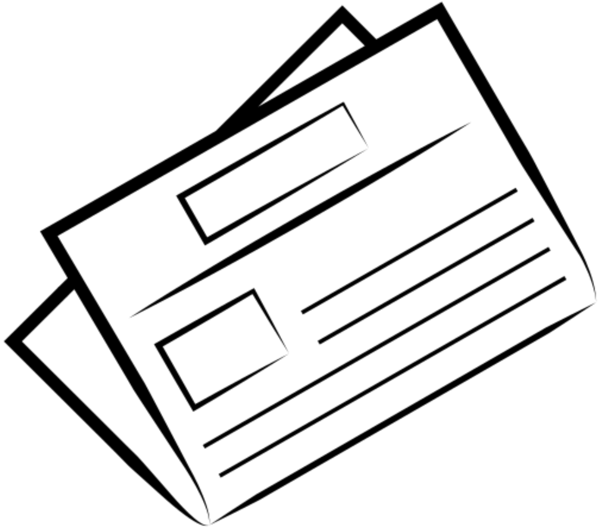
Describing Distributions

Visualizing data: Kernel density estimation

Density Estimation



Newspaper advertisement

Newspaper Advertisement (X)			
8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

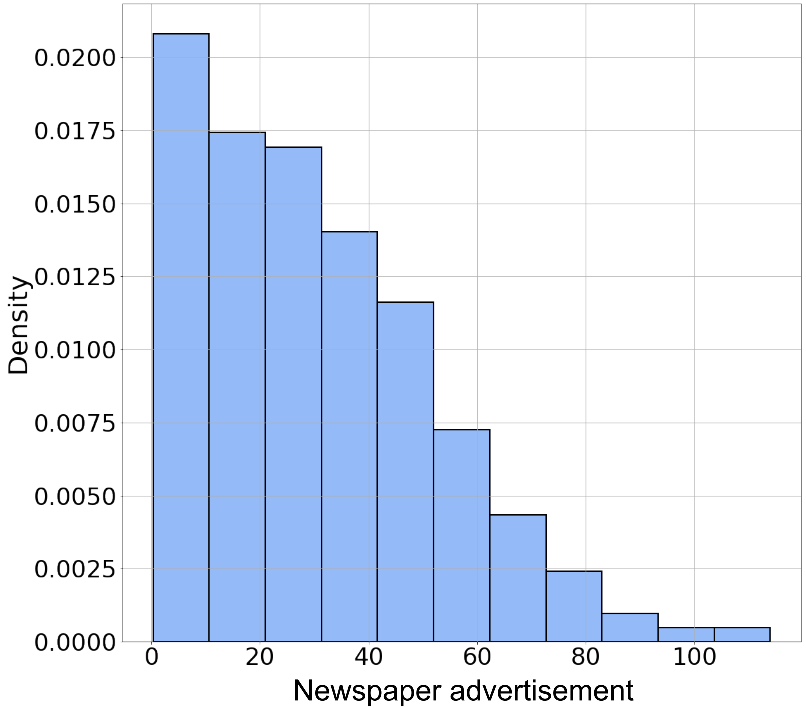
Histograms

It represents a density function

- It is positive
- Area under the curve is 1

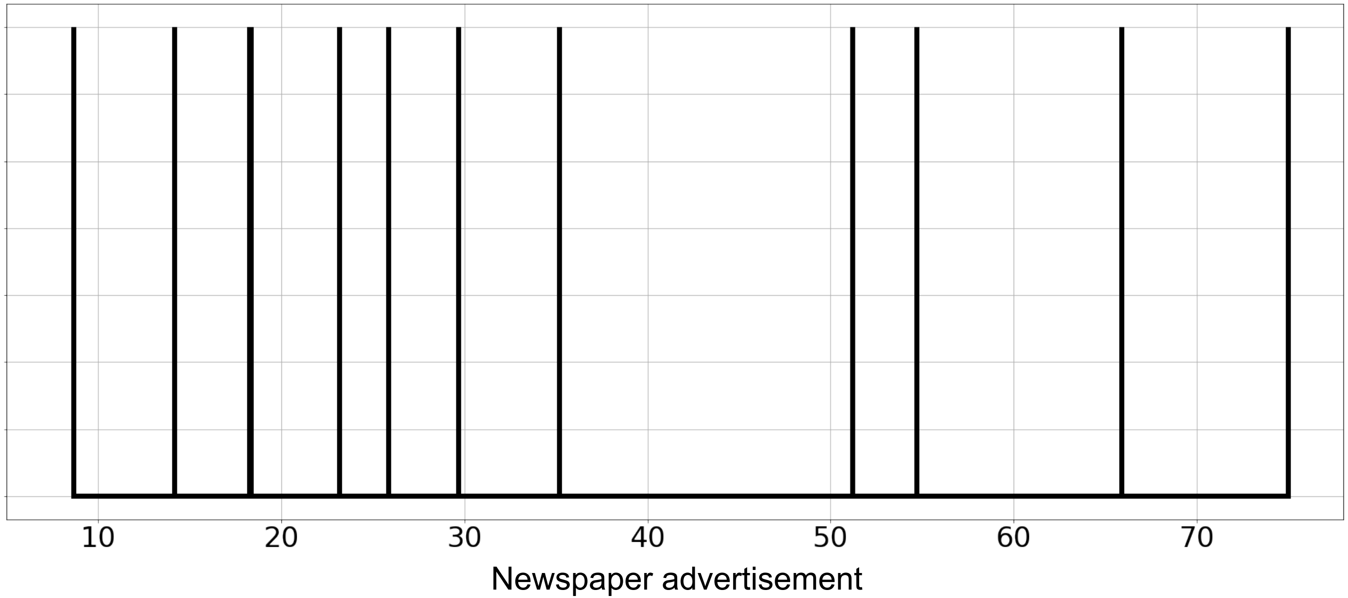
But…

- PDFs are usually smooth function
- The discontinuities come from the method and not the data



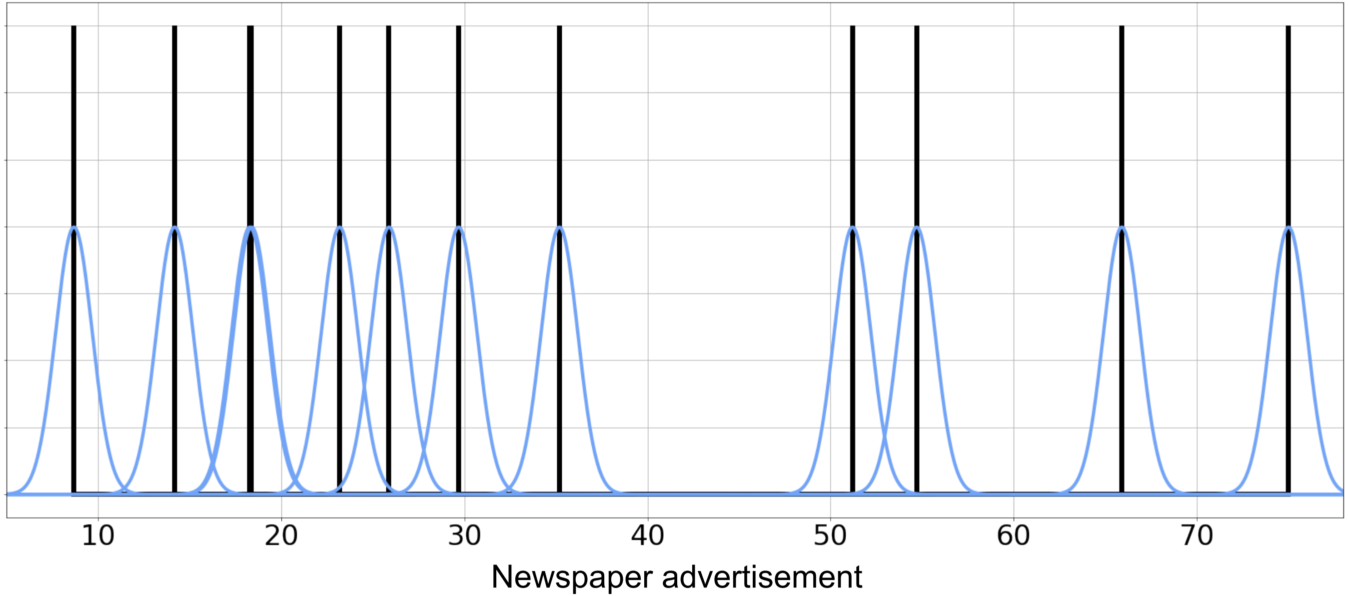
Kernel Density Estimation

First: draw your observations along the x axis



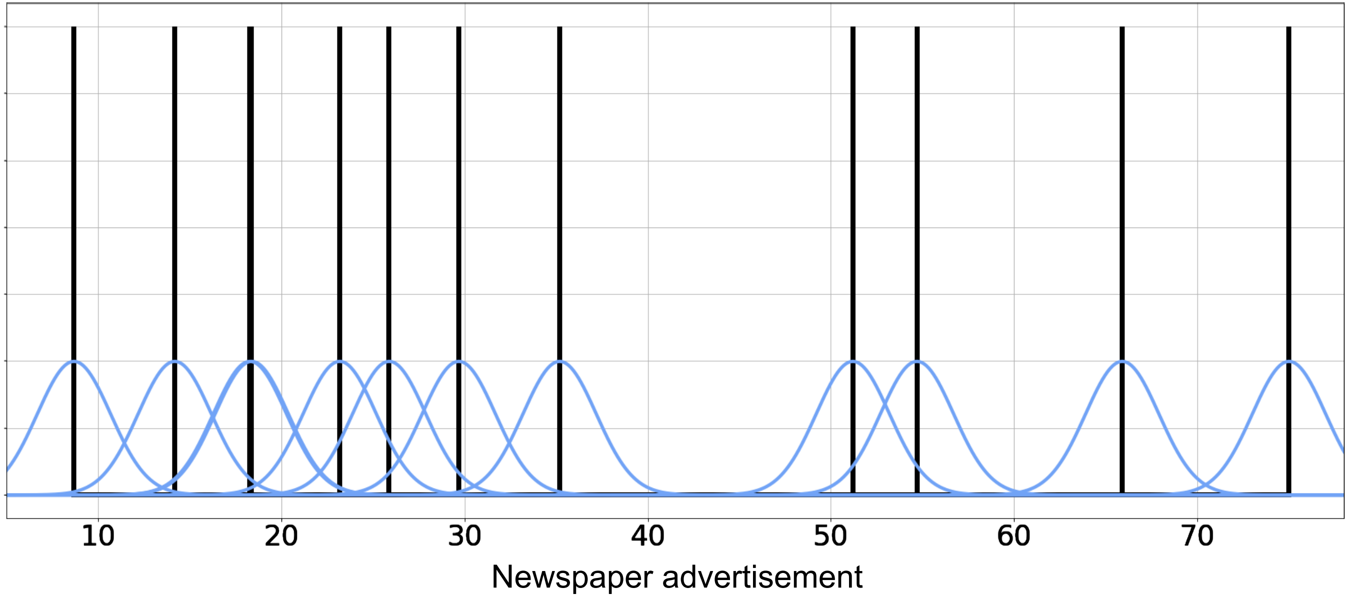
Kernel Density Estimation

Second: draw a gaussian centered at each observation



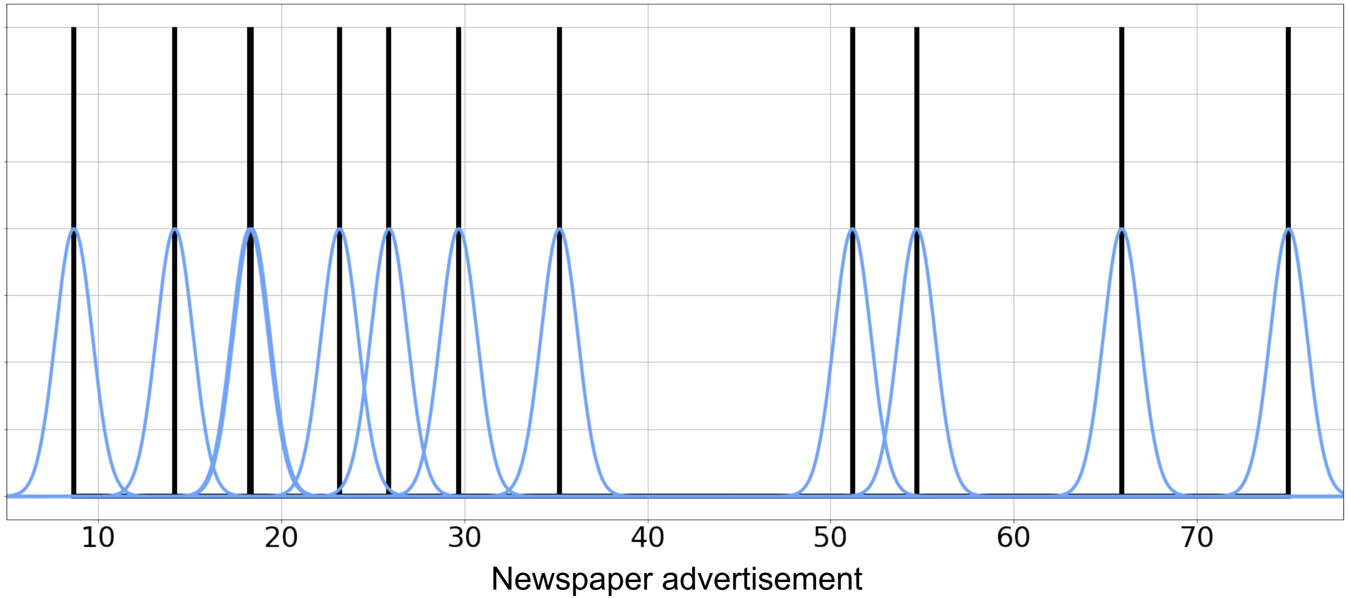
Kernel Density Estimation

Second: draw a gaussian centered at each observation



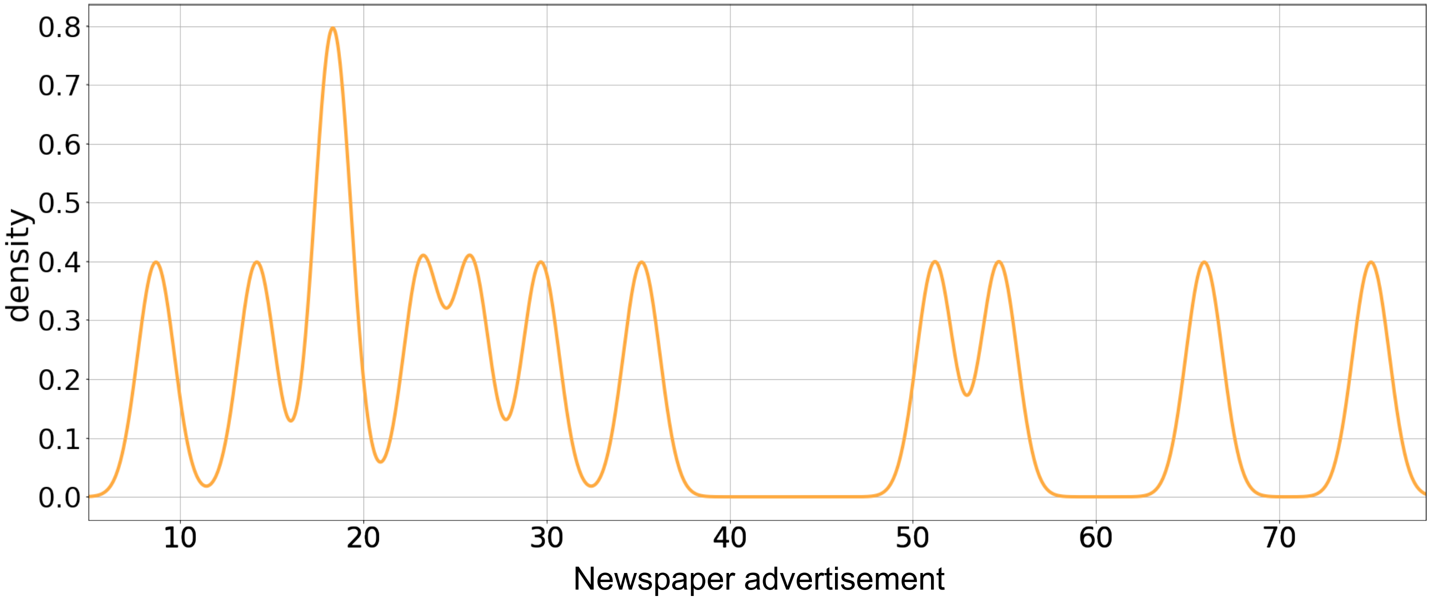
Kernel Density Estimation

Second: draw a gaussian centered at each observation



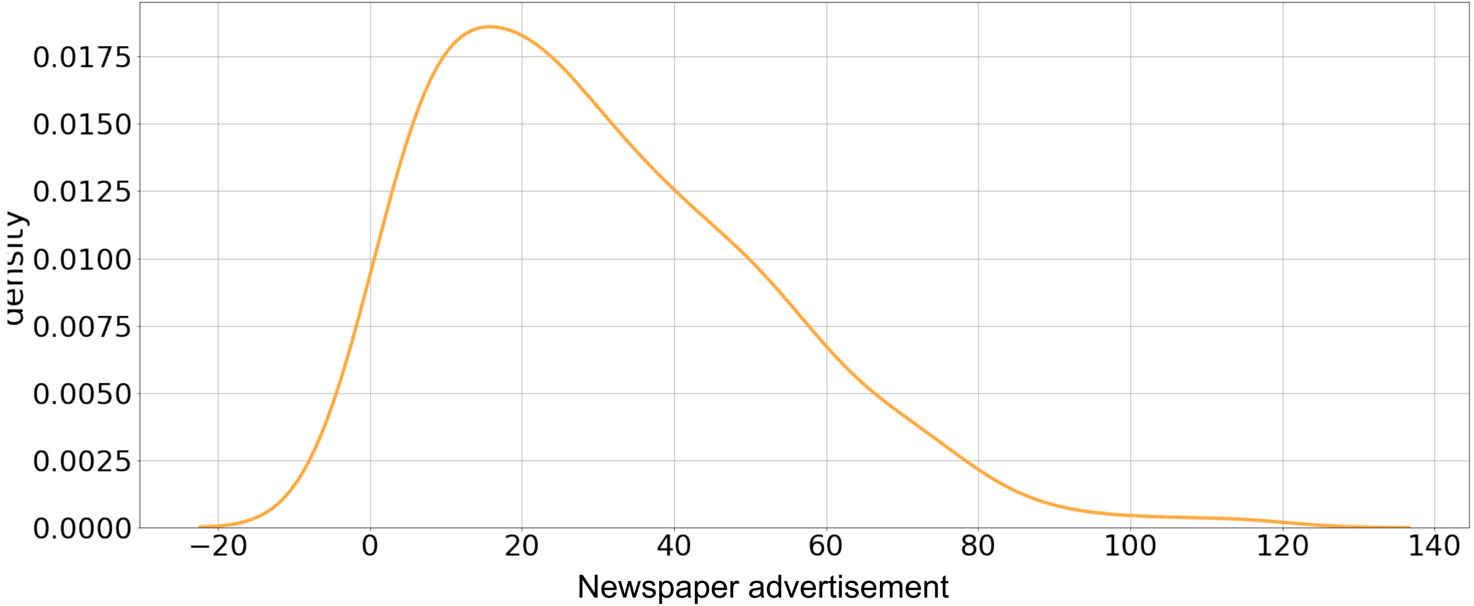
Kernel Density Estimation

Third: multiply everything by 1/n and sum the curves



Kernel Density Estimation

What if you used all the dataset?

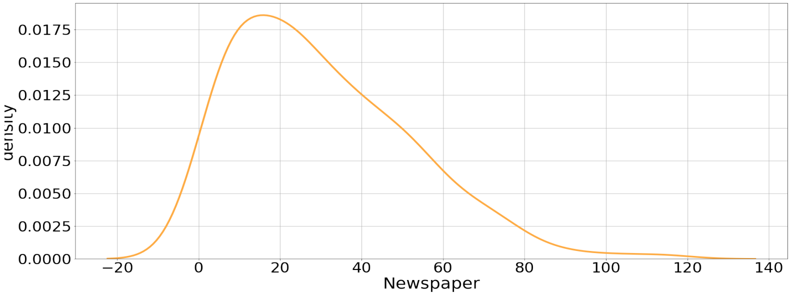


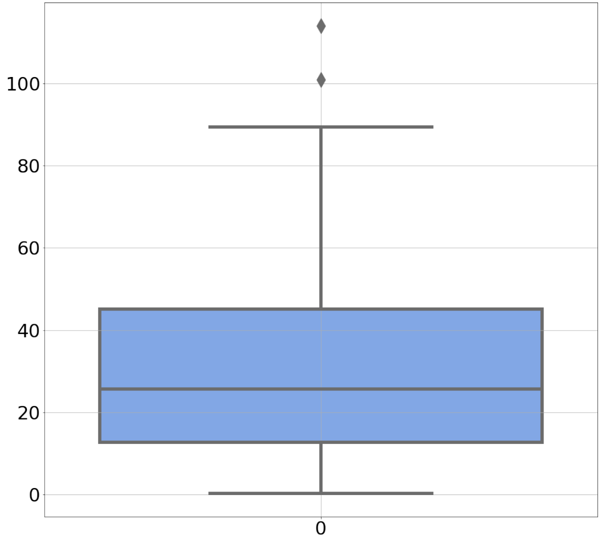
DeepLearning.AI

Describing Distributions

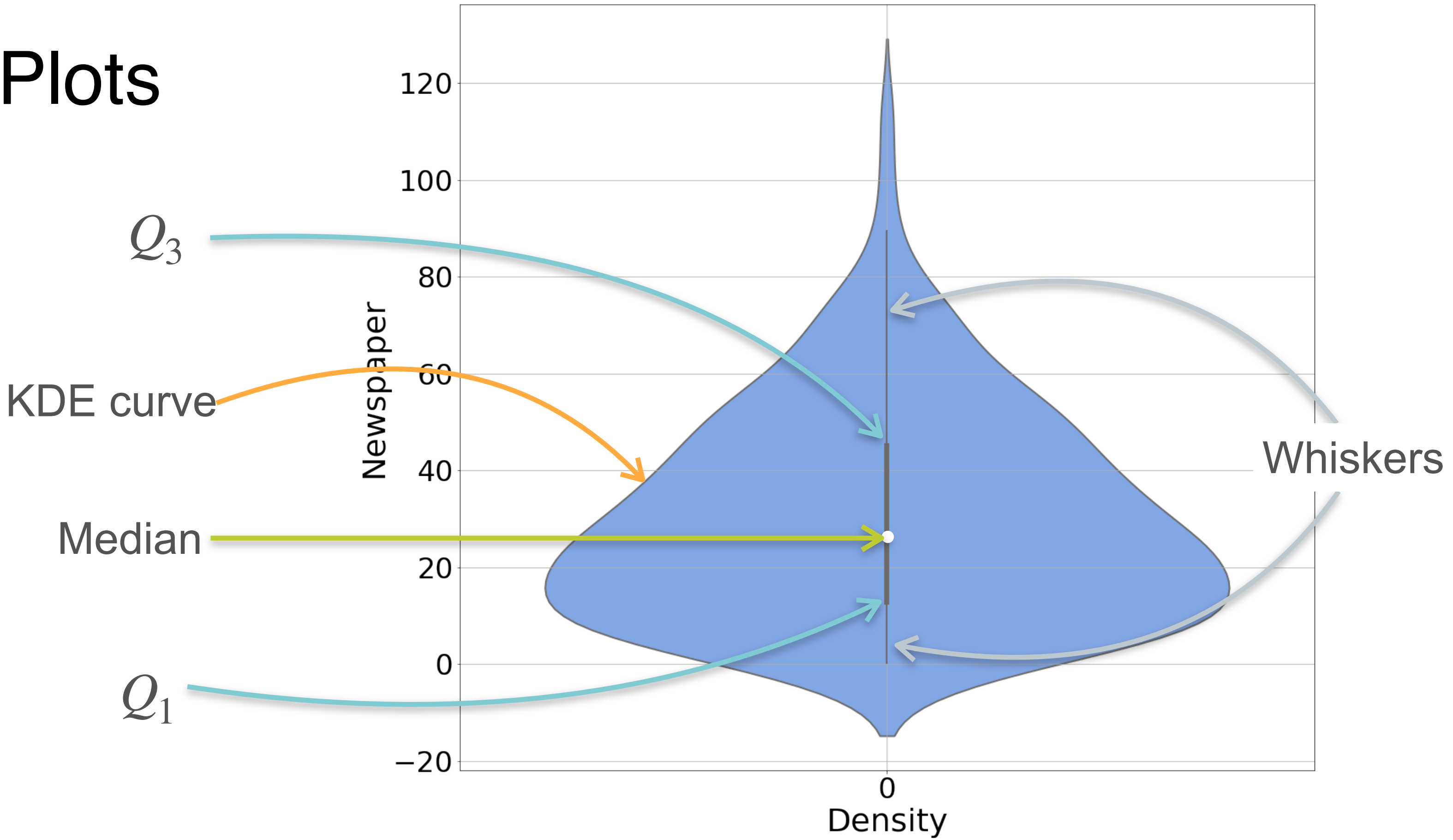
Visualizing data: Violin Plots

Violin Plots





Violin Plots



DeepLearning.AI

Describing Distributions

Visualizing data: QQ plots

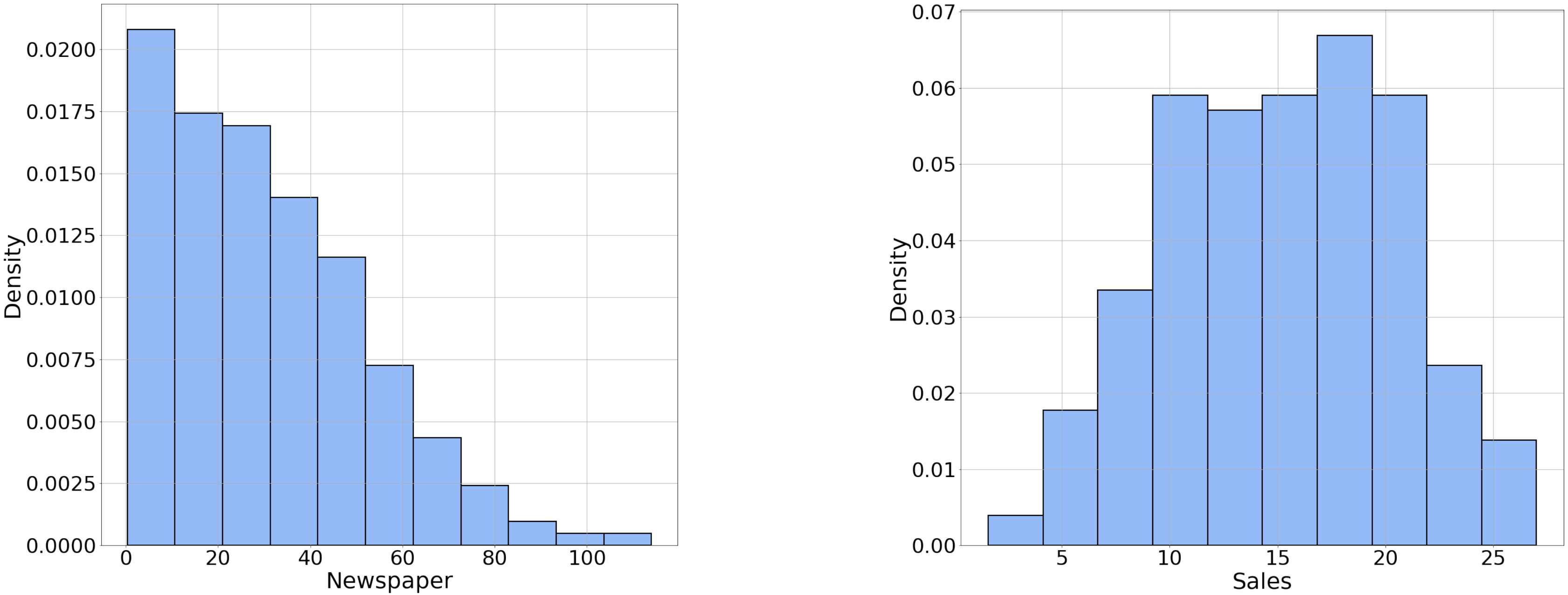
Assessing Normality of Data

Some models assume normally distributed data

- Linear regression
- Logistic regression
- Gaussian Naive Bayes
- Others

Some tests used in Data Science also assume normality.

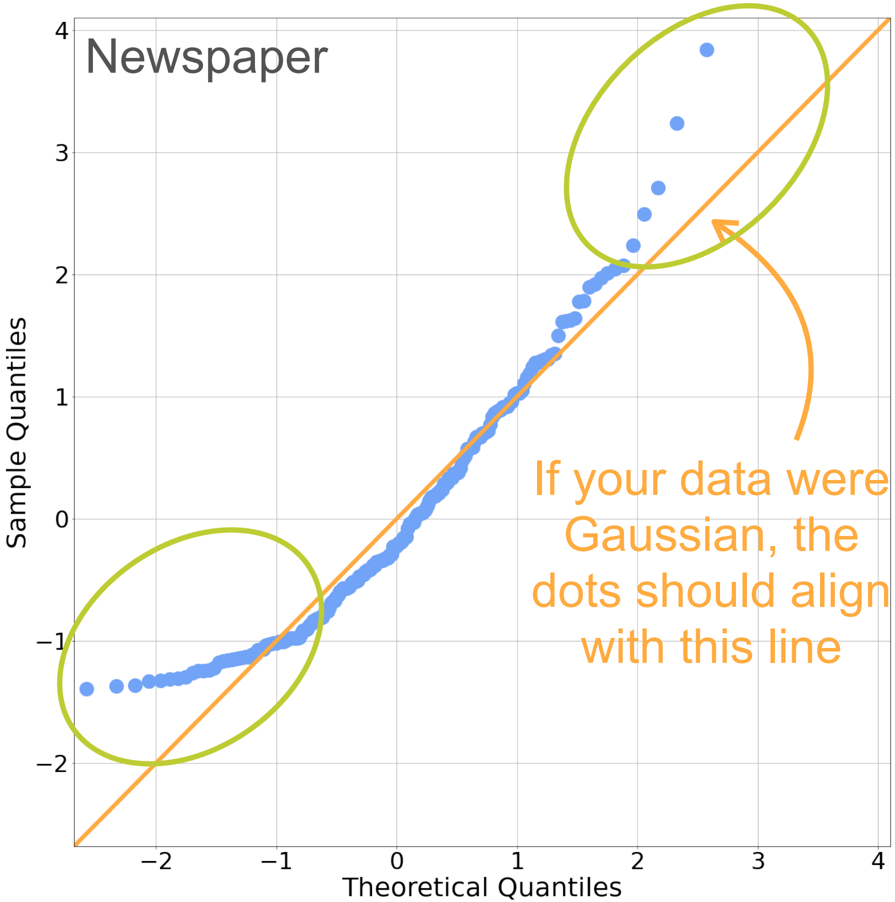
Assessing Normality of Data



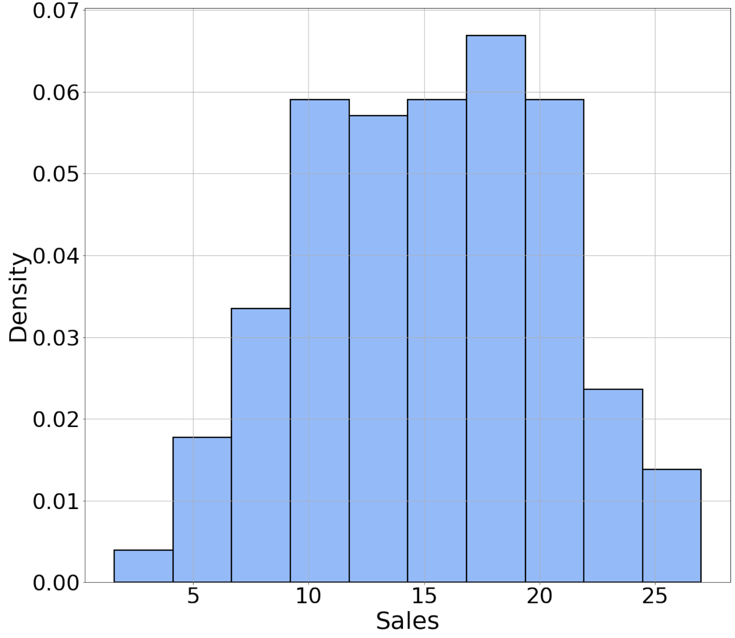
QQ Plots

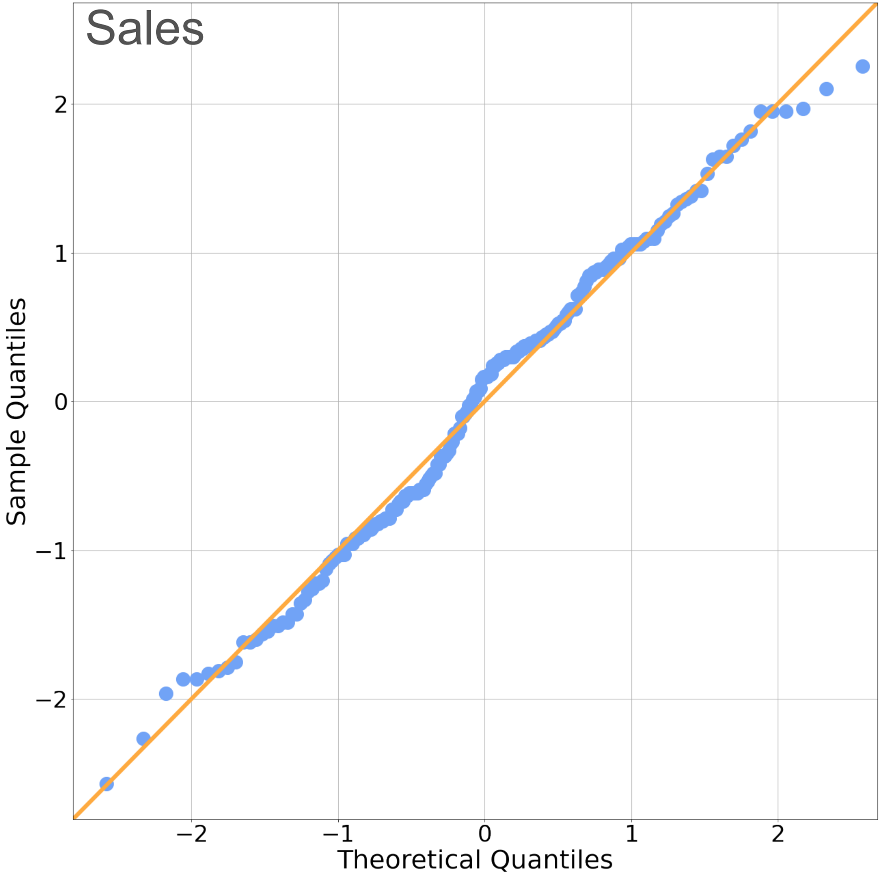
Quantile-Quantile plots (QQ Plots) compare quantiles

- Standardize your data: $\left(\frac{x-\mu}{\sigma}\right)$
- Compute quantiles
- Compare to gaussian quantiles



QQ Plots


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
Density
5
10
15
20
25
Sales


Sales
2
1
0
−1
−2
Sample Quantiles
−2
−1
0
1
2
Theoretical Quantiles

W2 Lesson 2

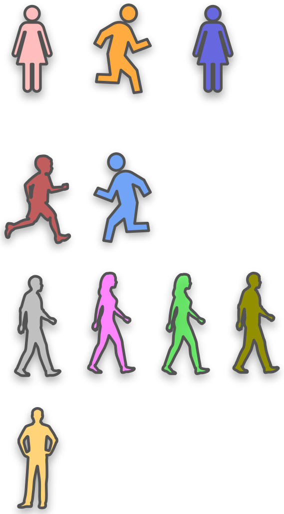
DeepLearning.AI

Probability Distributions with Multiple Variables

Joint Distribution (Discrete) Part 1

Joint Distributions (Discrete): Example 1

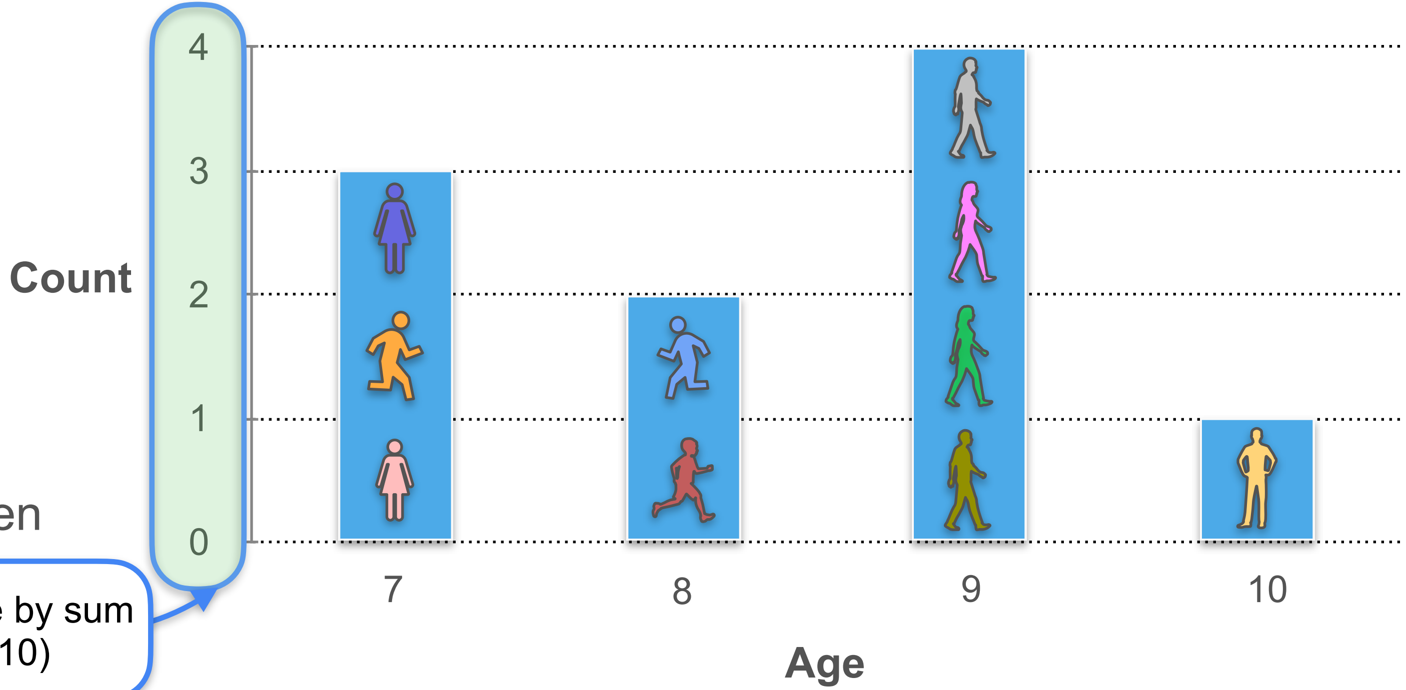
Age (Year)	Count
7	3
8	2
9	4
10	1



10

children

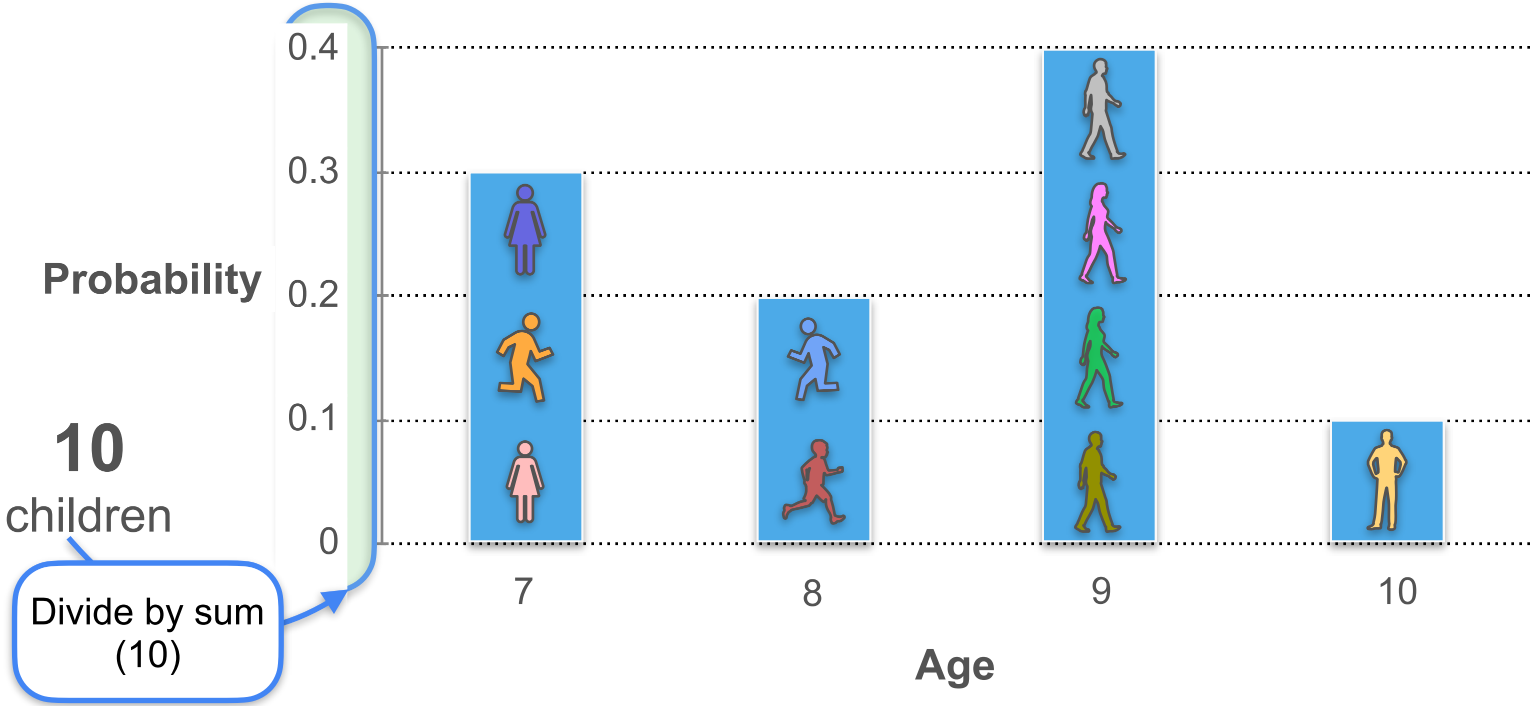
Joint Distributions (Discrete): Example 1



10
children

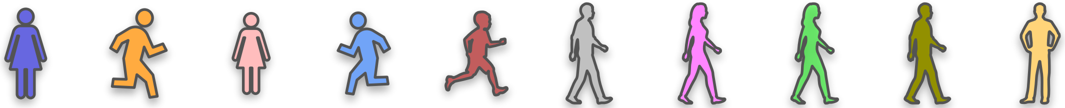
Divide by sum
(10)

Joint Distributions (Discrete): Example 1



Joint Distributions (Discrete): Example 1

Age (years):	7	7	7	8	8	9	9	9	9	10



Height (in):	45	46	46	47	47	49	49	49	50	50

Height (in)	Count
45	1
46	2
47	2
48	0
49	3
50	2

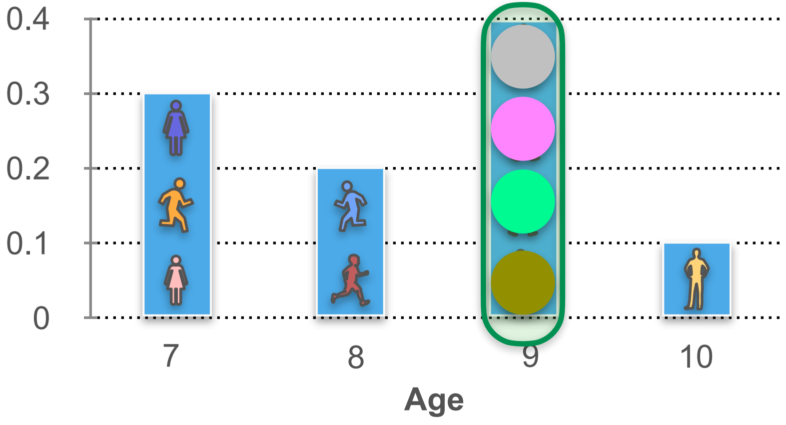
Joint Distributions (Discrete): Example 1

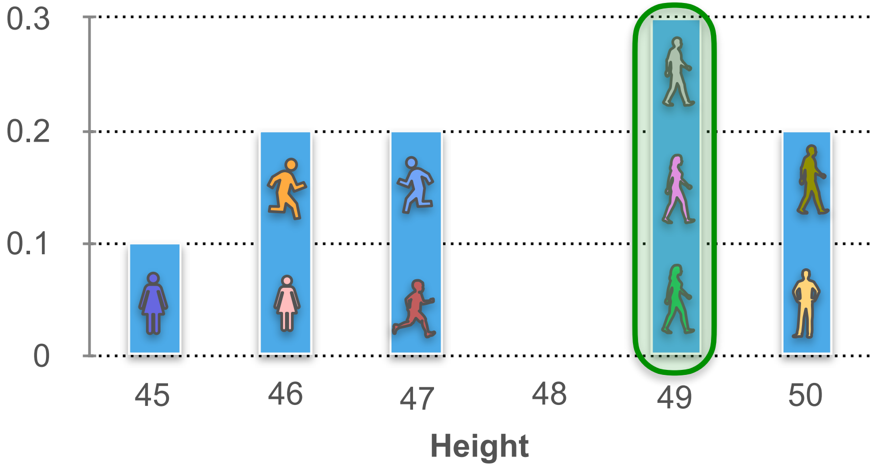
DeepLearning.AI

Joint Distributions (Discrete): Example 1

What is the probability that a child is **9 years** old and **49 inches** tall?

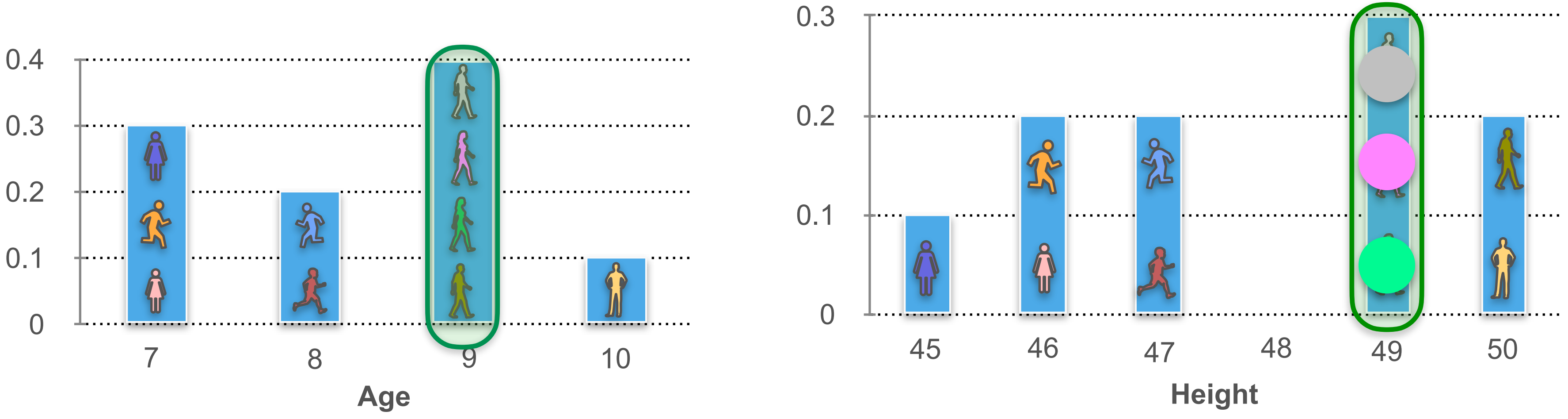
Joint Distributions (Discrete): Example 1





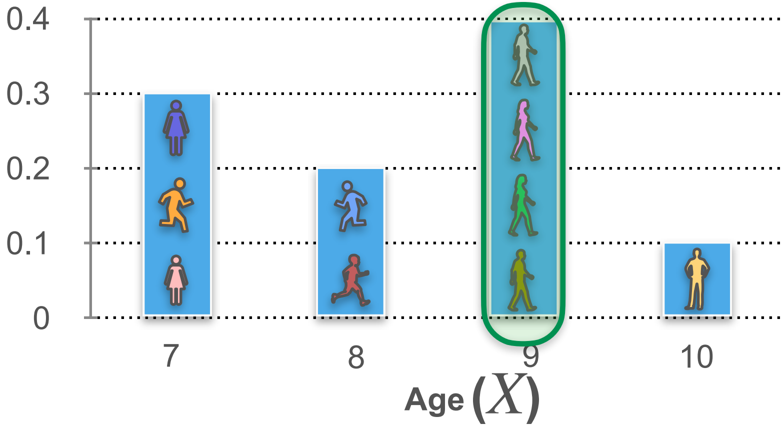
What is the probability that a child is **9 years** old and **49 inches** tall?

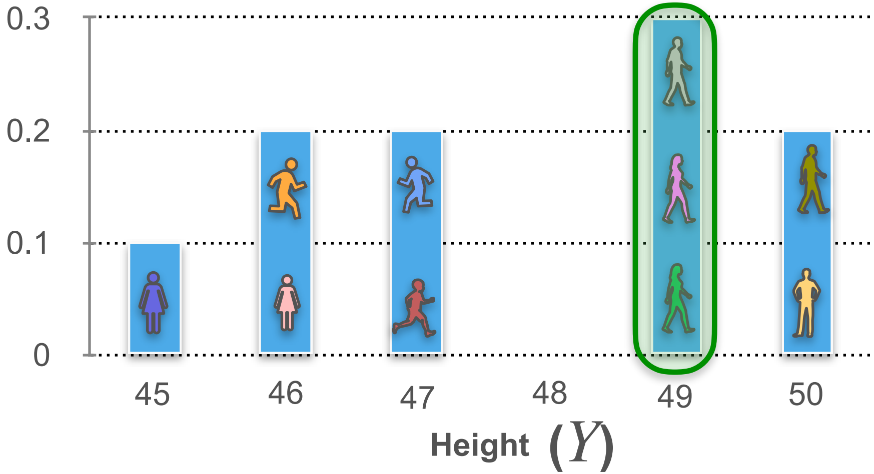
Joint Distributions (Discrete): Example 1



What is the probability that a child is **9 years** old and **49 inches** tall?

Joint Distributions (Discrete): Example 1





What is the probability that a child is **9 years** old and **49 inches** tall?

$$p_{XY}(9, 49) = \mathbf{P}(X = 9, Y = 49) = \frac{}{10} = \frac{3}{10}$$

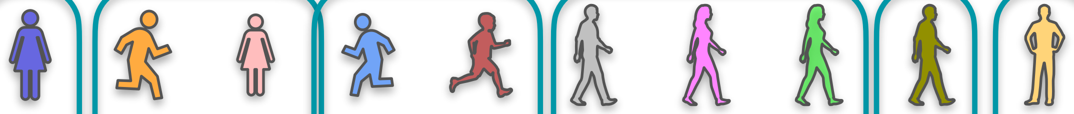
Joint Distributions (Discrete): Example 1

What is the probability that a child is **9 years** old and **49 inches** tall?

$$p_{XY}(9, 49) = \mathbf{P}(X = 9, Y = 49) = \frac{\quad}{10} = \frac{3}{10}$$

$$p_{XY}(x, y) = \mathbf{P}(X = x, Y = y)$$

Joint Distributions: Example 1



Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

Height

Age

	45	**46**	**47**	**48**	**49**	**50**
7	1	2	0	0	0	0
8	0	0	2	0	0	0
9	0	0	0	0	3	1
10	0	0	0	0	0	1

Joint Distributions: Example 1

Divide by sum (10)

Height

Age

	45	46	47	48	49	50
7	1	2	0	0	0	0
8	0	0	2	0	0	0
9	0	0	0	0	3	1
10	0	0	0	0	0	1

Counts

Height

Age

	45	46	47	48	49	50
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10

Probabilities

Joint Distributions: Example 1

Joint Distribution

All probabilities for all possible combinations of X and Y

Height (Y)

Age (X)

	45	46	47	48	49	50
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10

Probabilities

Joint Distributions: Example 1

What is the probability that a child is 8 and 48 inches tall?

$$p_{XY}(x, y) = \mathbf{P}(X = x, Y = y)$$

$$p_{XY}(8, 48) = \mathbf{P}(X = 8, Y = 48)$$

$$p_{XY}(8, 48) = 0$$

$$p_{XY}(7, 46) = \frac{2}{10}$$

Height (Y)

Age (X)

	45	46	47	48	49	50
7	1/1	2/10	0	[illegible]	0	0
8	0	0	2/1	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10

Probabilities

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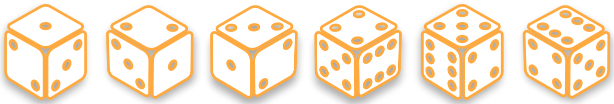
Probability Distributions with Multiple Variables

Joint Distribution (Discrete) Part 2

Joint Distributions (Discrete): Example 2

X

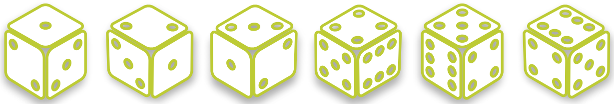
the number rolled on the 1st dice



$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

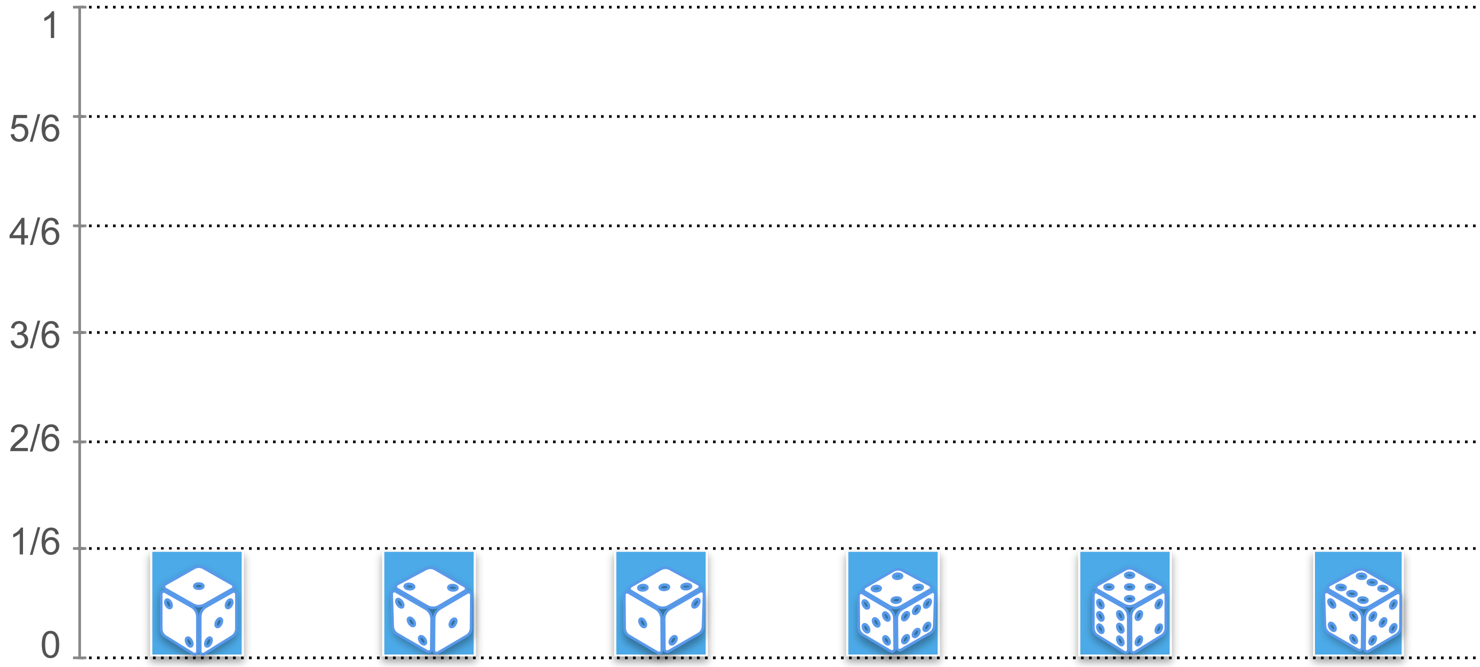
Y

the number rolled on the 2nd dice



$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

Joint Distributions: Example 2



Joint Distributions: Example 2

X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

X and Y are independent

$\mathbf{P}(x)$	1/6	1/6	1/6	1/6	1/6	1/6
$\mathbf{P}(y)$	1/6	1/6	1/6	1/6	1/6	1/6

$p_{XY}($  ,  $) = \mathbf{P}($  $) \cdot \mathbf{P}($  $) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$$p_{XY}(x, y) = \mathbf{P}(X = x, Y = y) = \frac{1}{36}$$

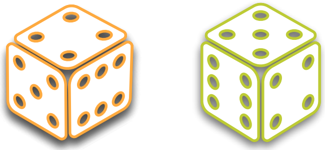
X \ Y	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	**1/36**	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Joint Distributions: Example 2

Thus for independent discrete random variables:

$$p_{XY}(x, y) = \mathbf{P}(X = x, Y = y) = \mathbf{P}(x) \cdot \mathbf{P}(y)$$

Joint Distributions (Discrete): Example 2



X

the number rolled on the 1st dice

Y

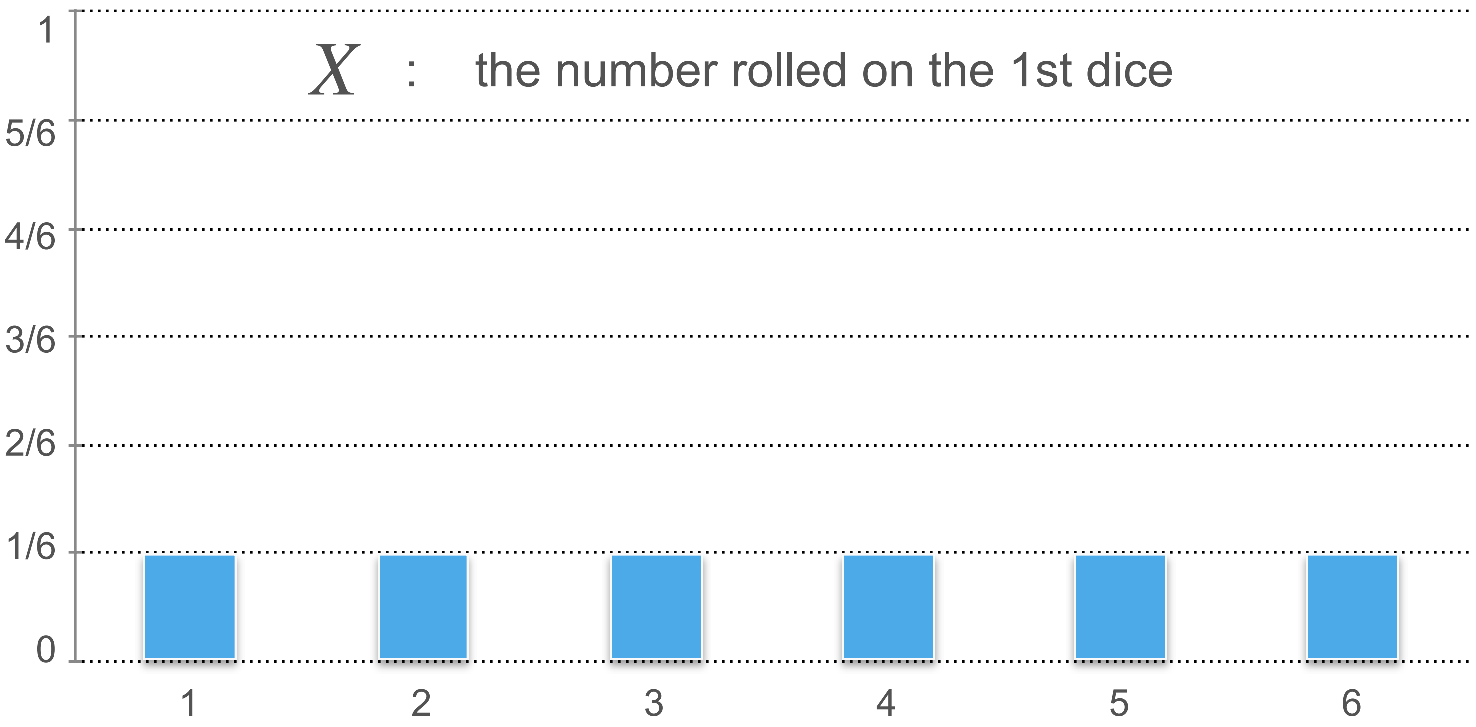
sum of the two dice

$+$

$$X = 4$$

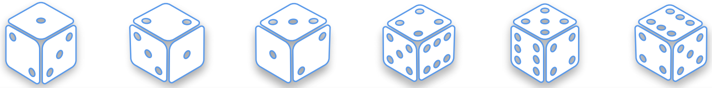
$$Y = 4 + 5$$

Joint Distributions: Example 2



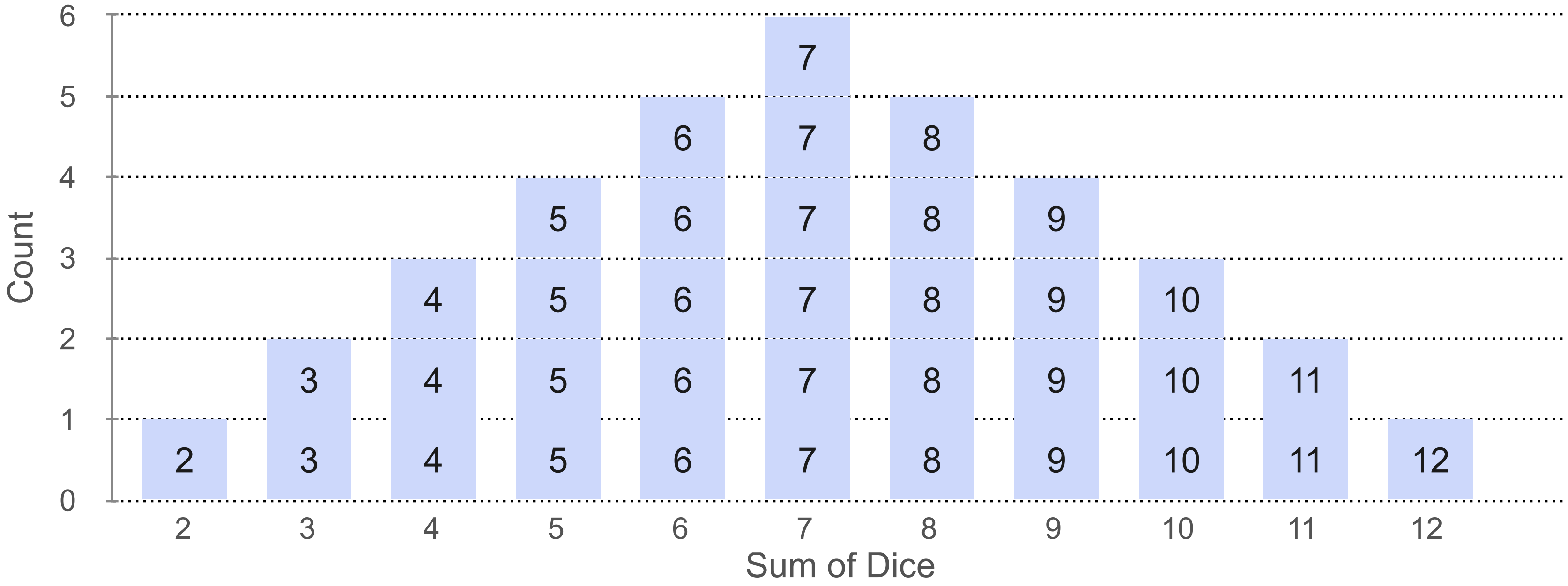
Joint Distributions - Example 3

Y: Sum of both dice

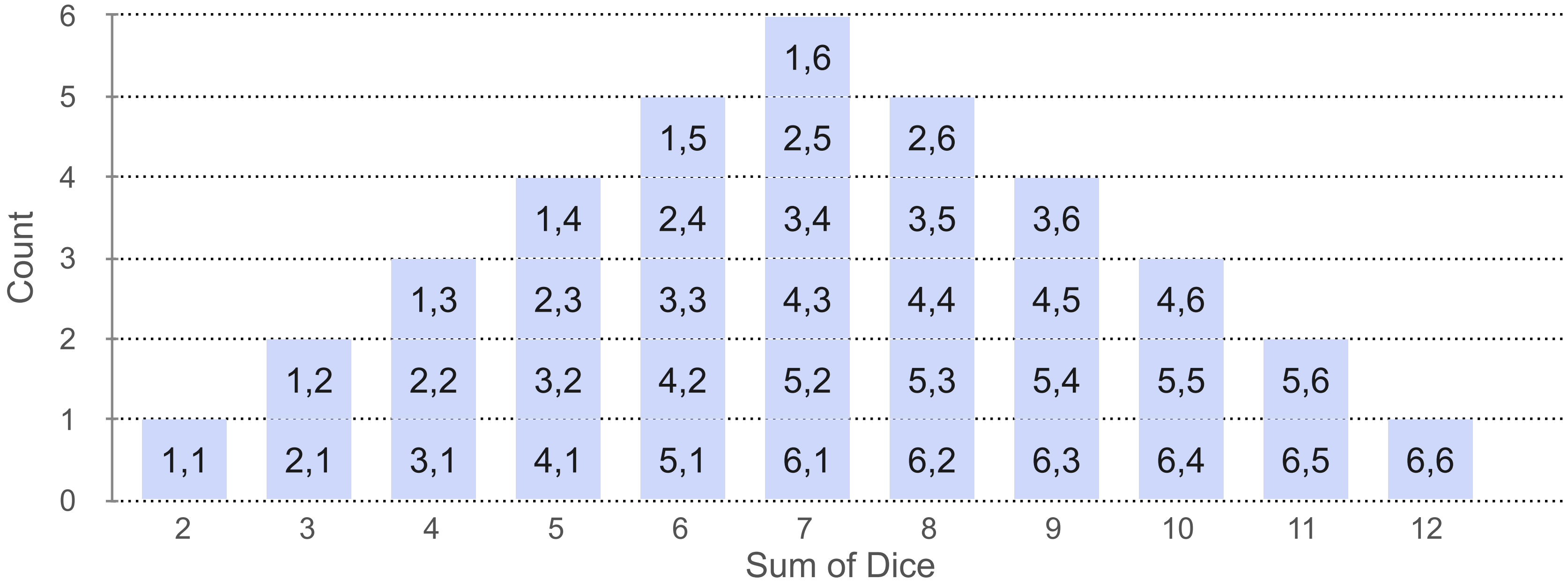


2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

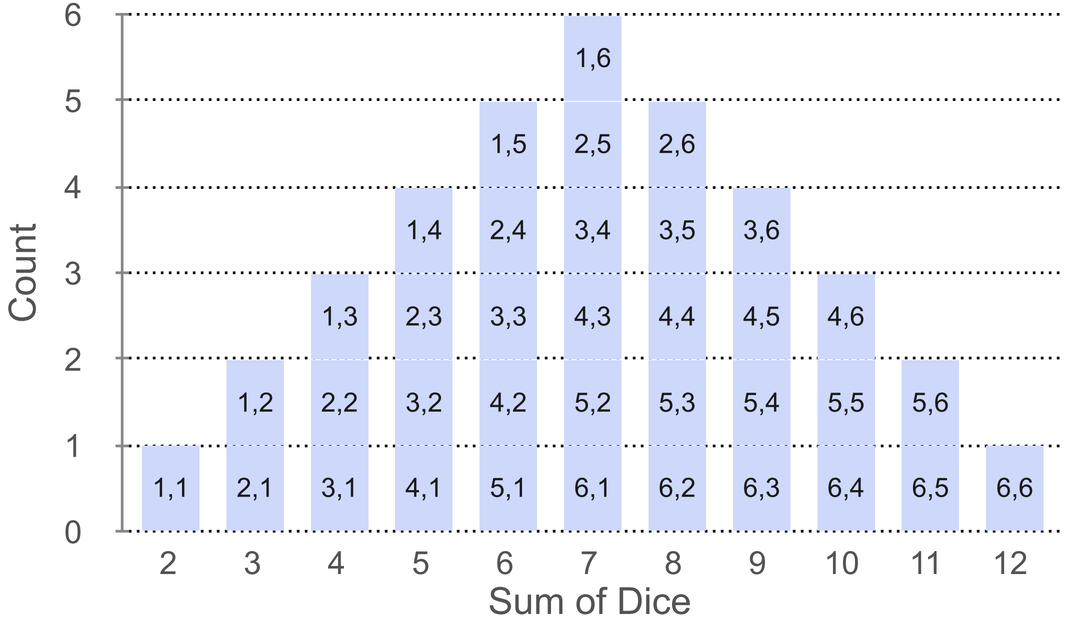
Joint Distributions: Example 3

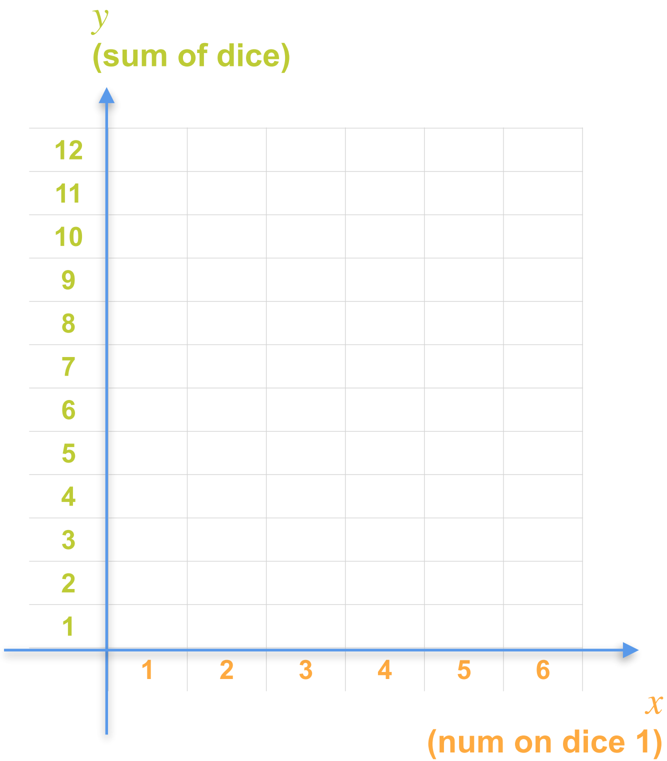


Joint Distributions: Example 3

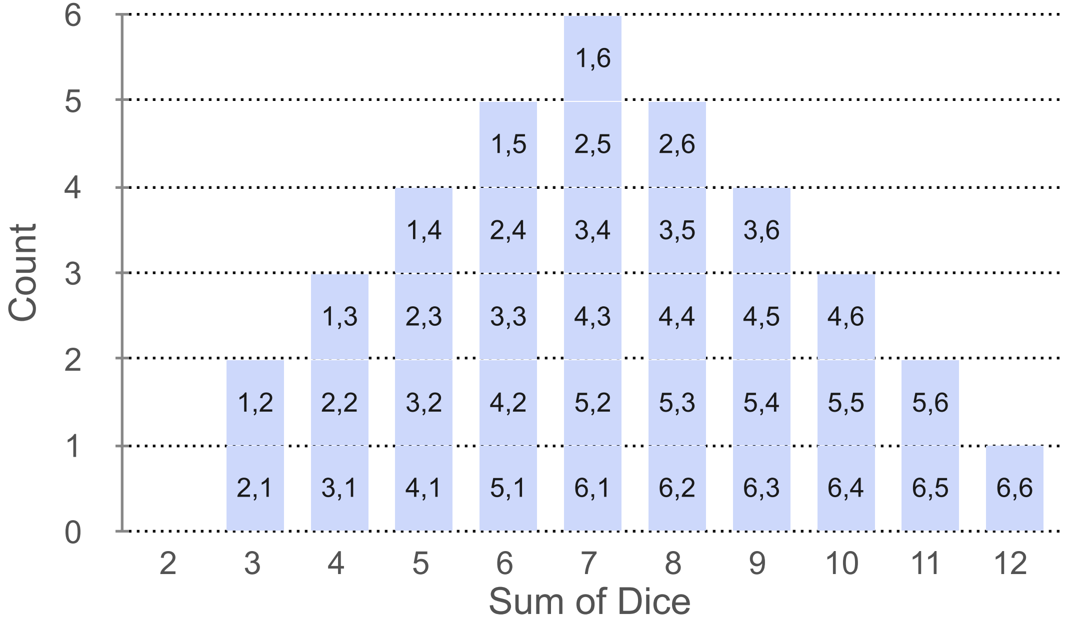


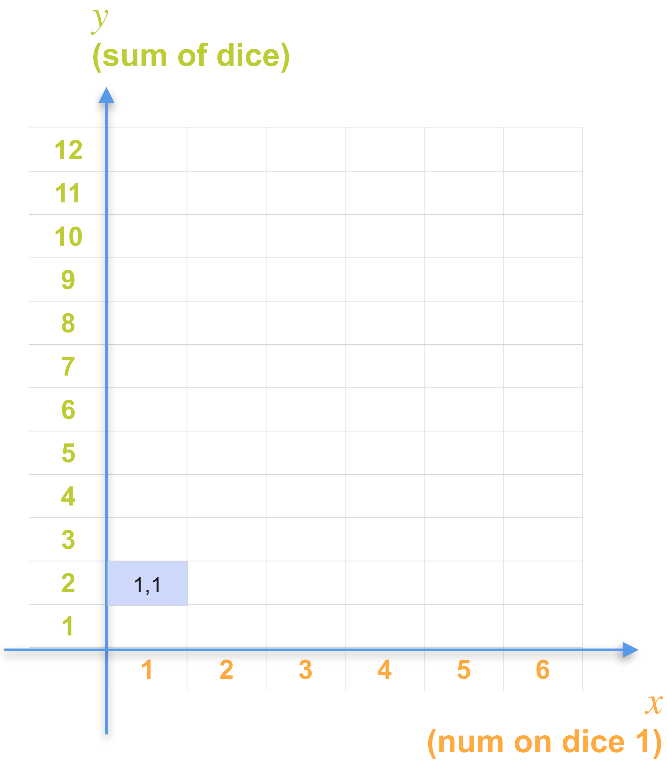
Joint Distributions: Example 3



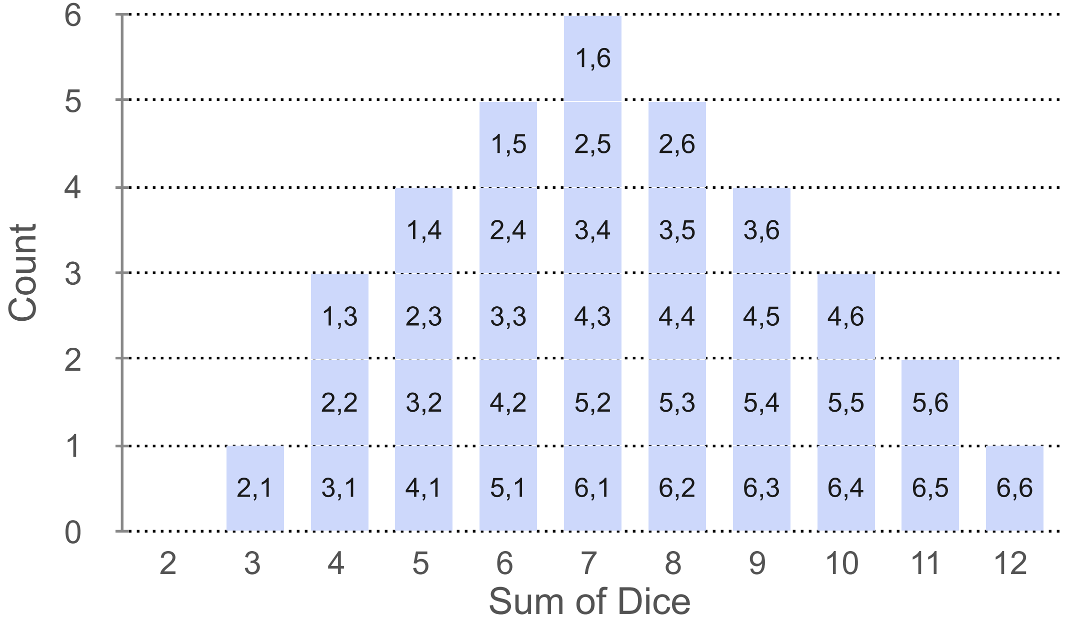


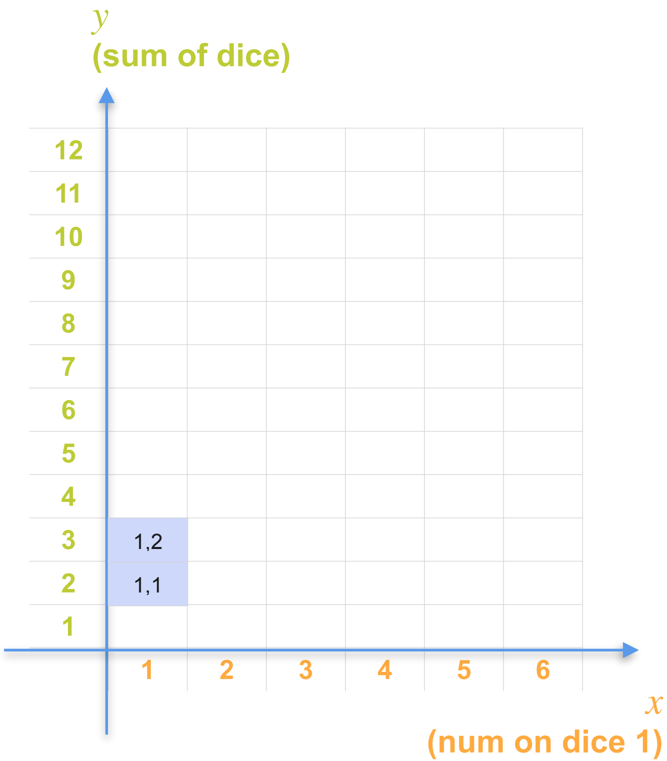
Joint Distributions: Example 3



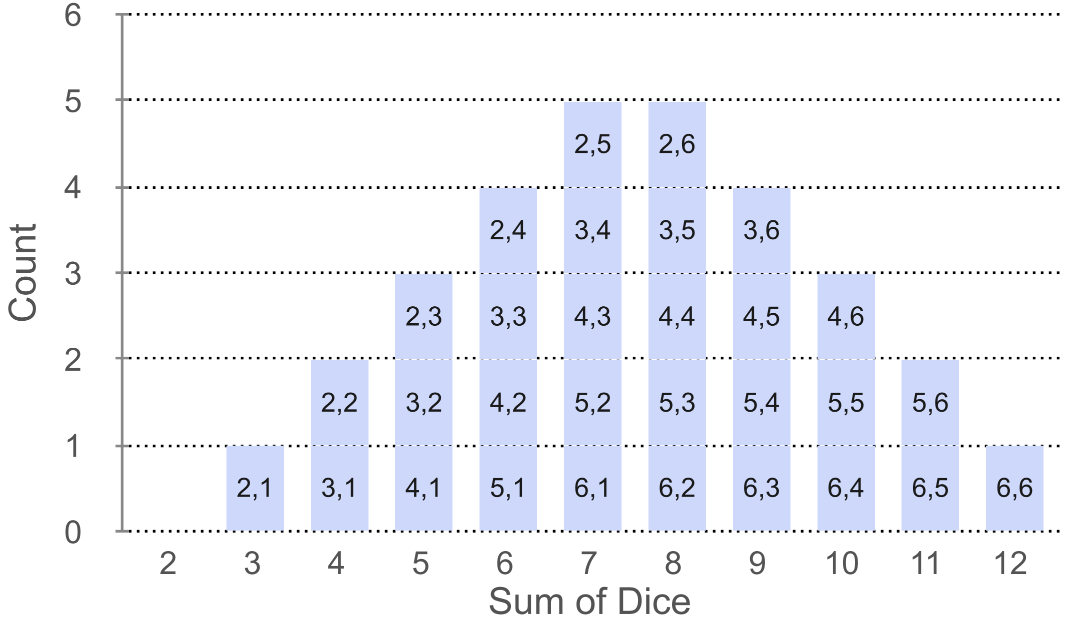


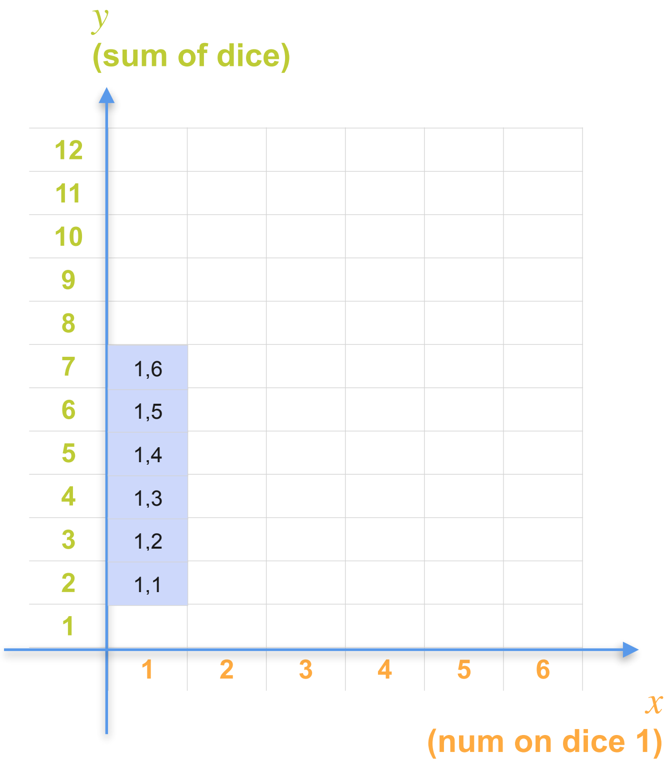
Joint Distributions: Example 3



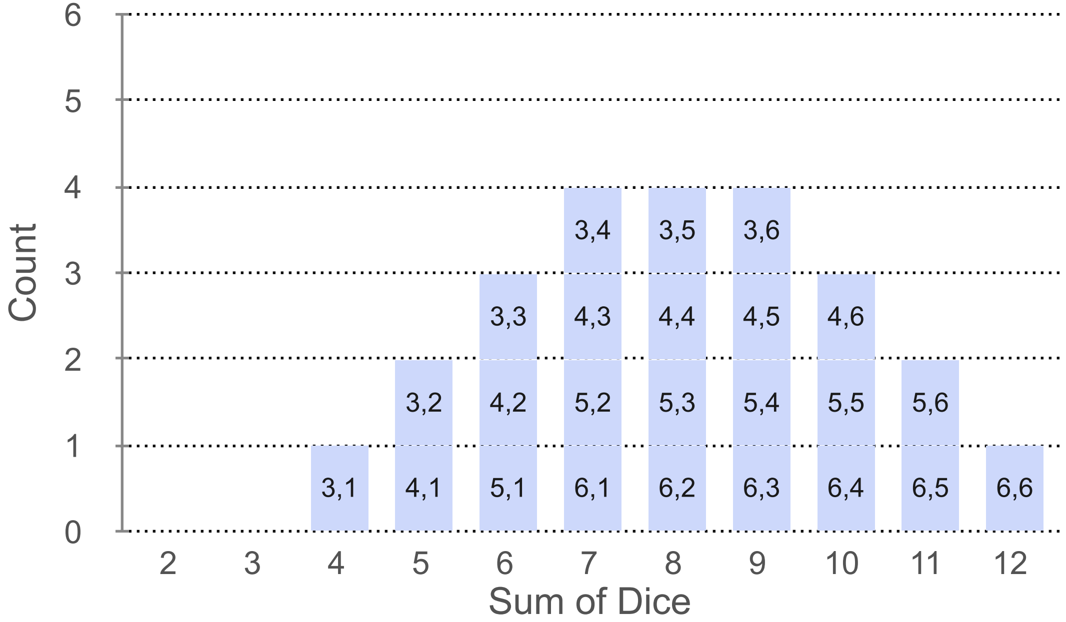


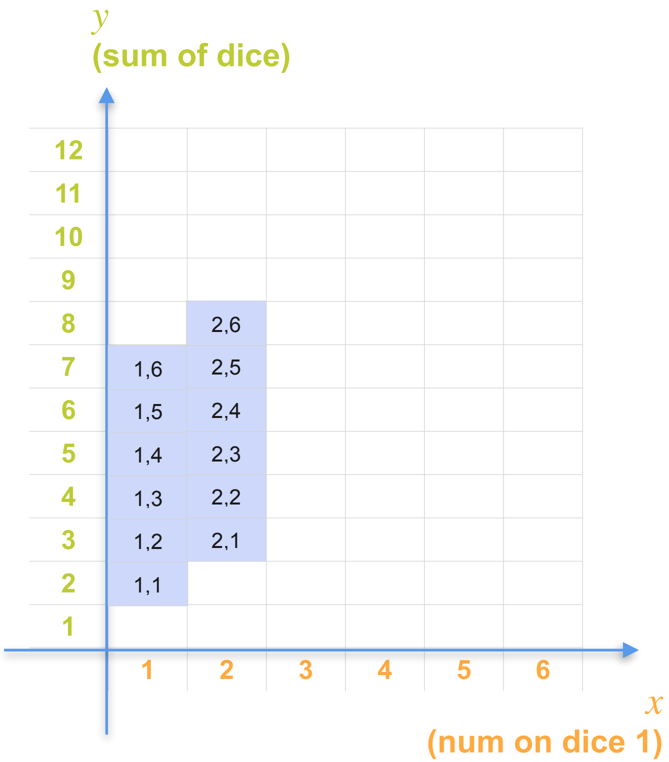
Joint Distributions: Example 3



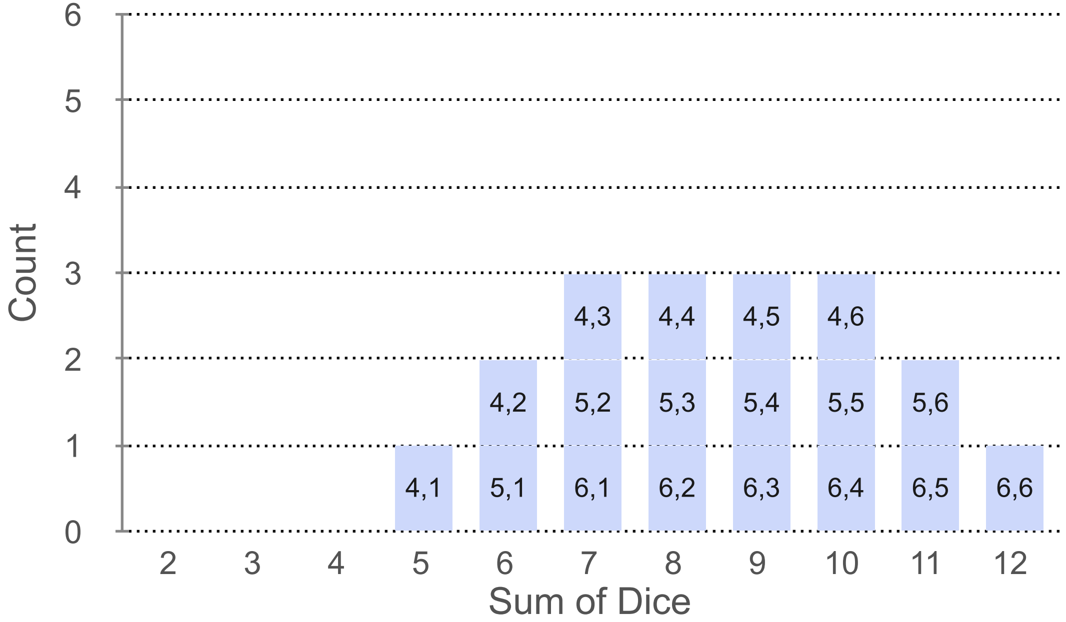


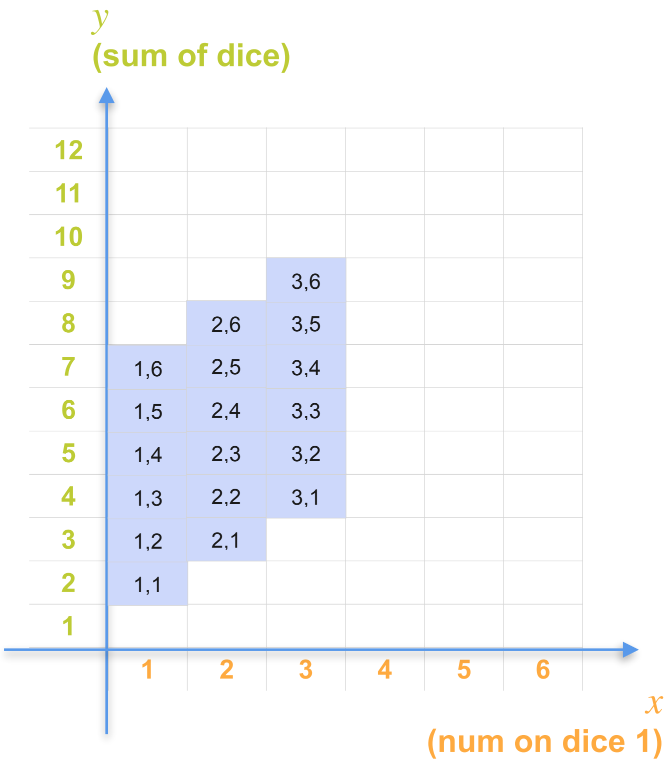
Joint Distributions: Example 3



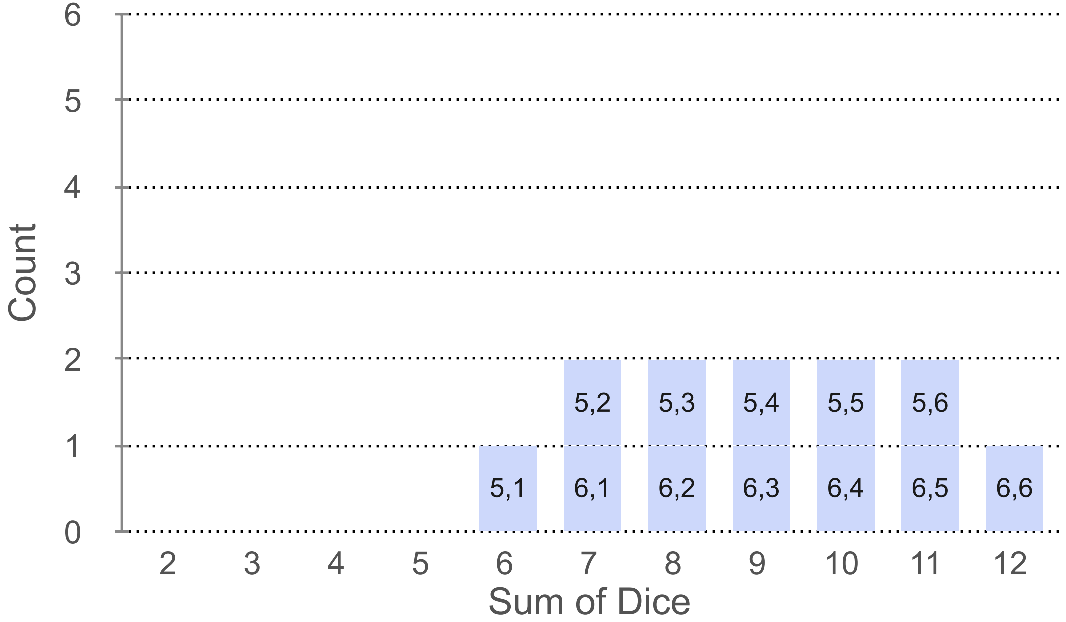


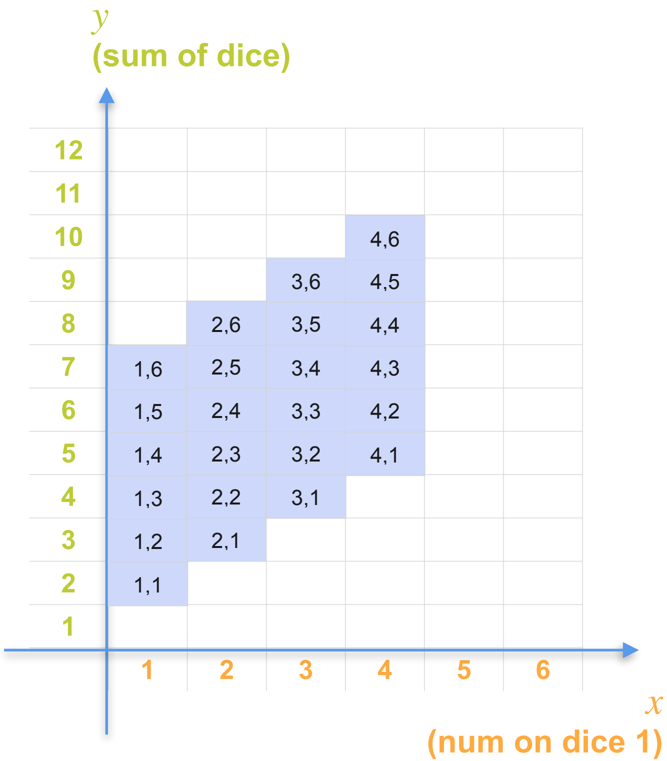
Joint Distributions: Example 3



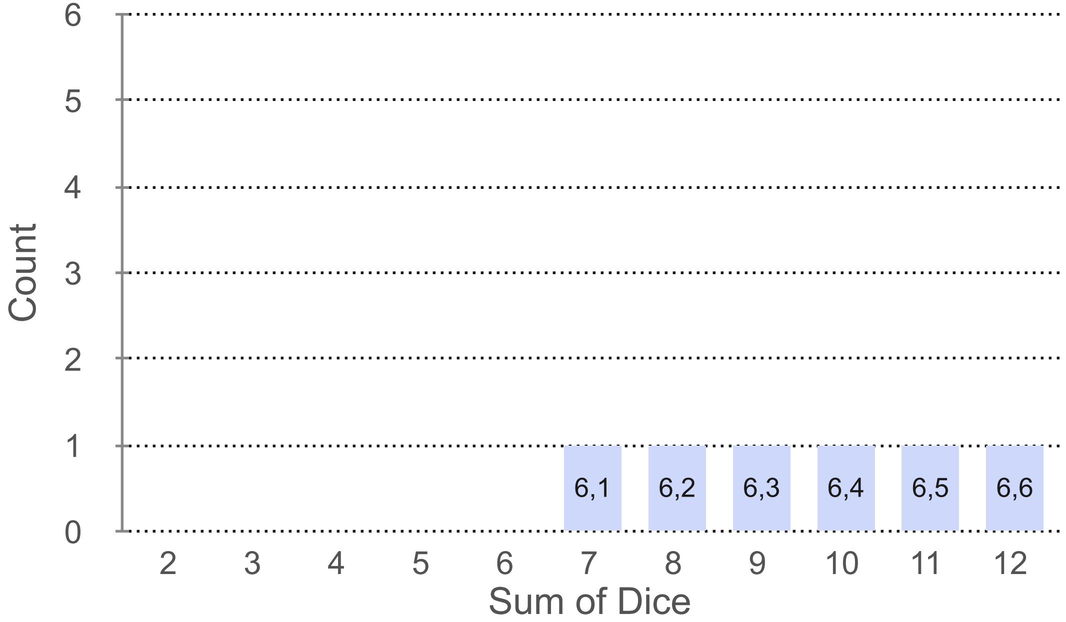


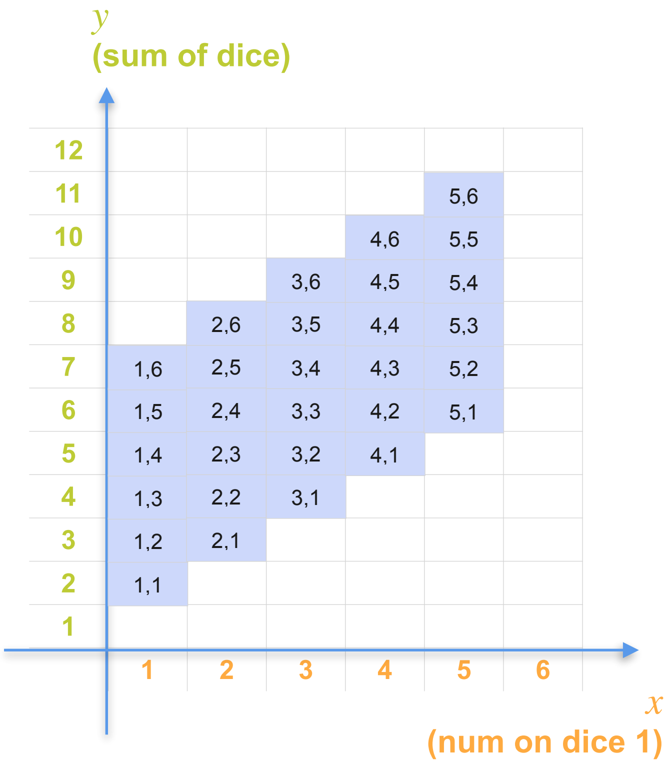
Joint Distributions: Example 3





Joint Distributions: Example 3





Joint Distributions: Example 3

Joint Distribution for X and Y

36 possible outcomes

Divide by sum (36)

y (sum of dice); x (num on dice 1)

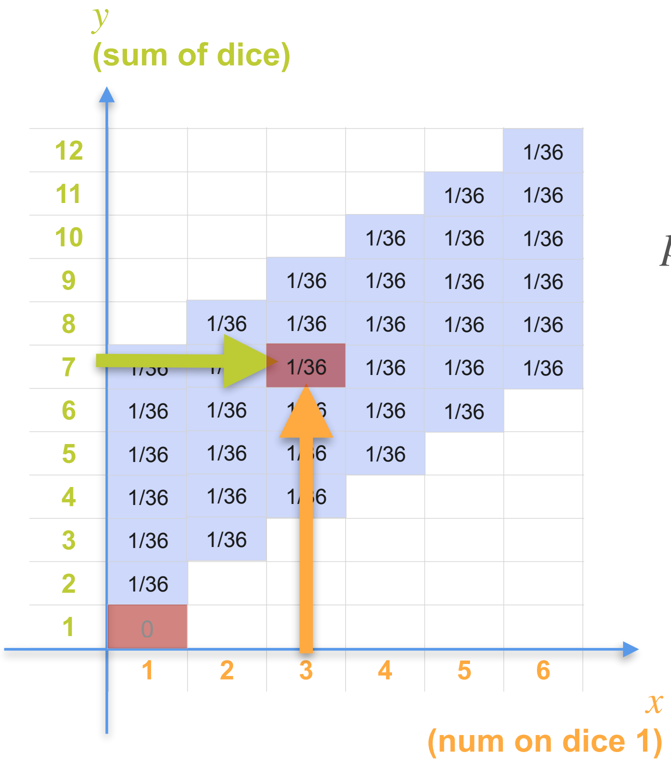
y \ x	1	2	3	4	5	6
12						1/36
11					1/36	1/36
10				1/36	1/36	1/36
9			1/36	1/36	1/36	1/36
8		1/36	1/36	1/36	1/36	1/36
7	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	
5	1/36	1/36	1/36	1/36		
4	1/36	1/36	1/36			
3	1/36	1/36				
2	1/36					
1						

y (sum of dice); x (num on dice 1)

y \ x	1	2	3	4	5	6
12						1
11					1	1
10				1	1	1
9			1	1	1	1
8		1	1	1	1	1
7	1	1	1	1	1	1
6	1	1	1	1	1	
5	1	1	1	1		
4	1	1	1			
3	1	1				
2	1					
1						

Joint Distributions: Example 3

Joint Distribution for
X **and** Y



$$p_{XY}(3, 7) = \mathbf{P}(X = 3, Y = 7) = \frac{1}{36}$$

$$p_{XY}(1, 1) = \mathbf{P}(X = 1, Y = 1) = 0$$

Probability Distributions with Multiple Variables

DeepLearning.AI

Joint Distribution (Continuous)

Joint Continuous Distributions

X : age of a child in year

Y : discrete values of height of child in inches

X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

X and Y are Discrete Random Variables

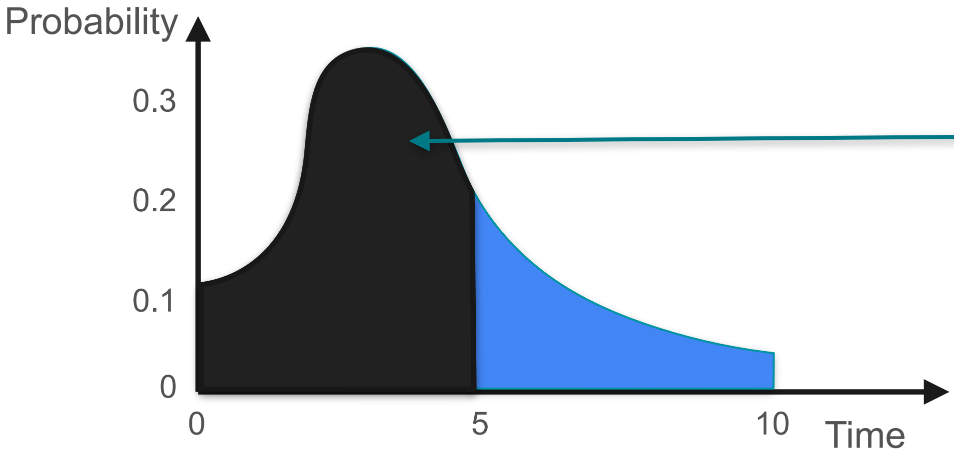
X : the number rolled on the 1st dice

Y : sum of both dice

What about when X and Y are Continuous Random Variables?

Joint Continuous Distributions

X variable: Waiting time



$\mathbf{P}(X$ between 0 and 5 mins$)$

Area occupied by this region

Joint Continuous Distributions

X

Waiting time
before a call is picked up
[0 - 10 minutes]

2.4 minutes

1.5 minutes

Both variables are continuous

Y

Customer
satisfaction rating
[0 - 10]

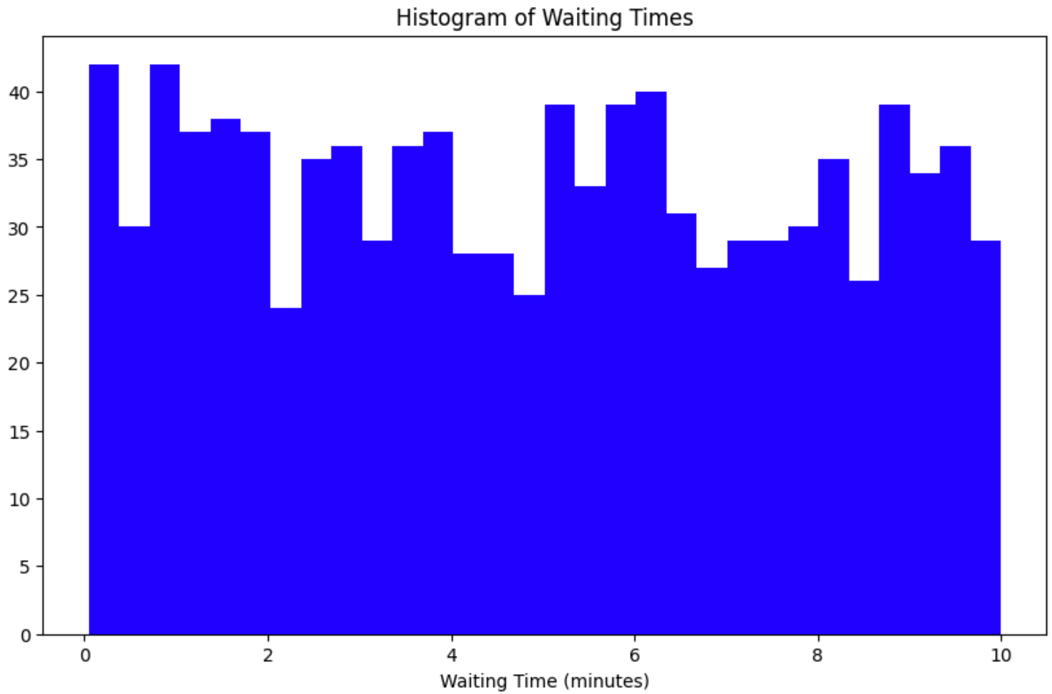
0.0

5.7

Joint Continuous Distributions: Dataset

X variable: Waiting time (mins)
0 - 10 mins

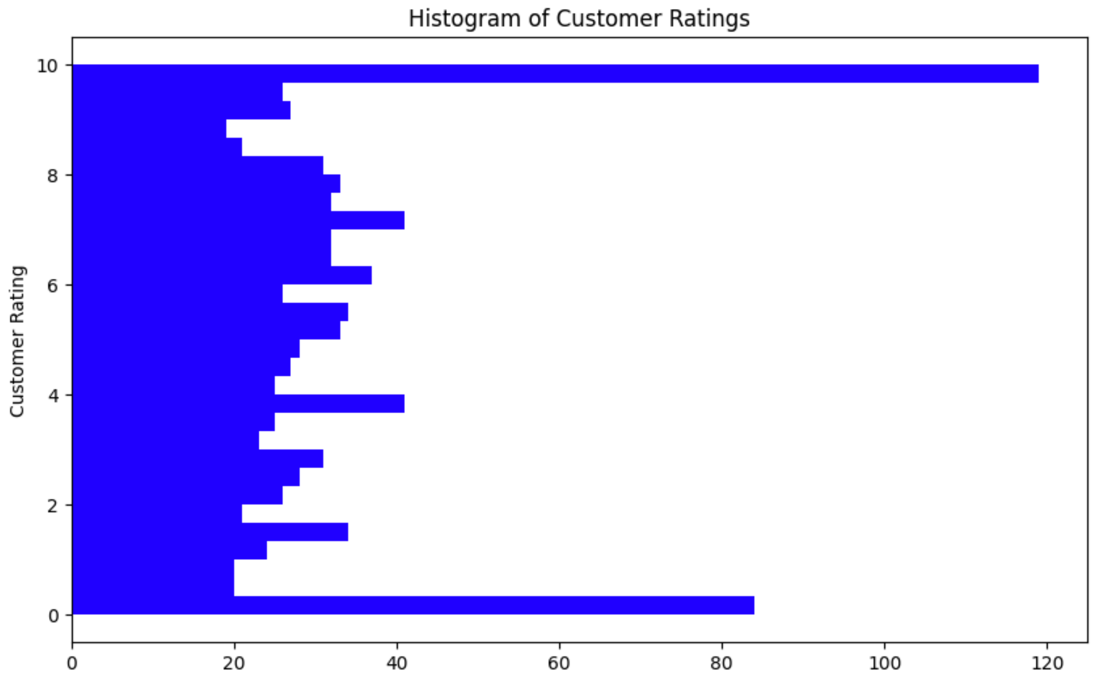
1000 customers



Joint Continuous Distributions: Dataset

Y variable: Satisfaction rating 0 - 10

1000 customers

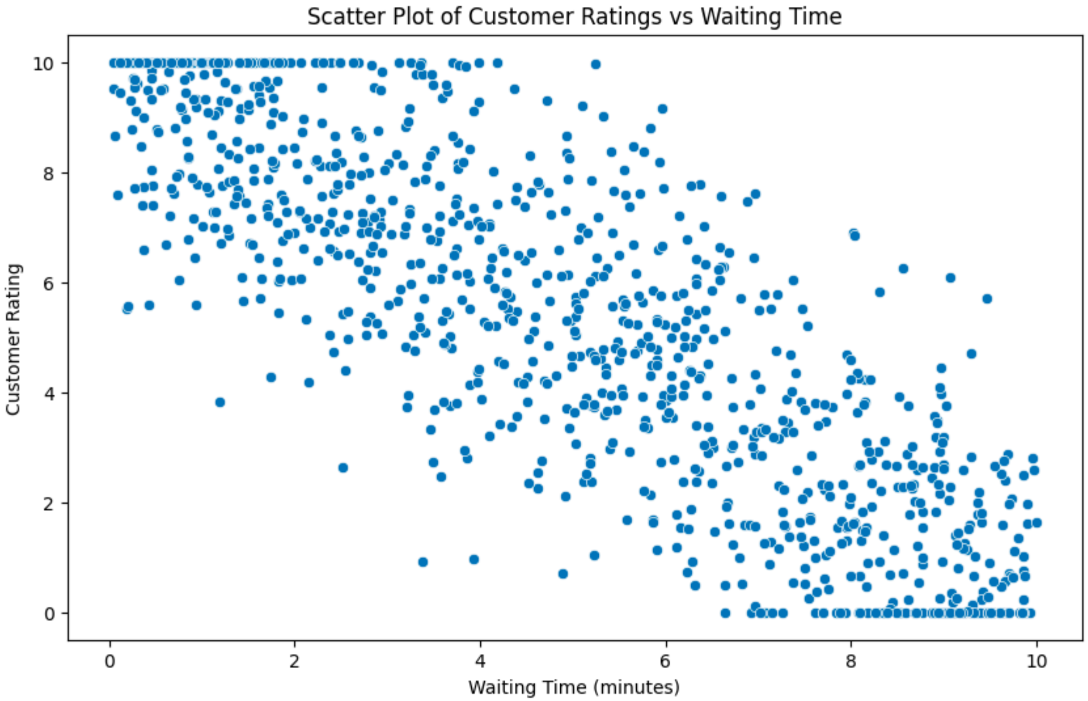


Joint Continuous Distributions: Dataset

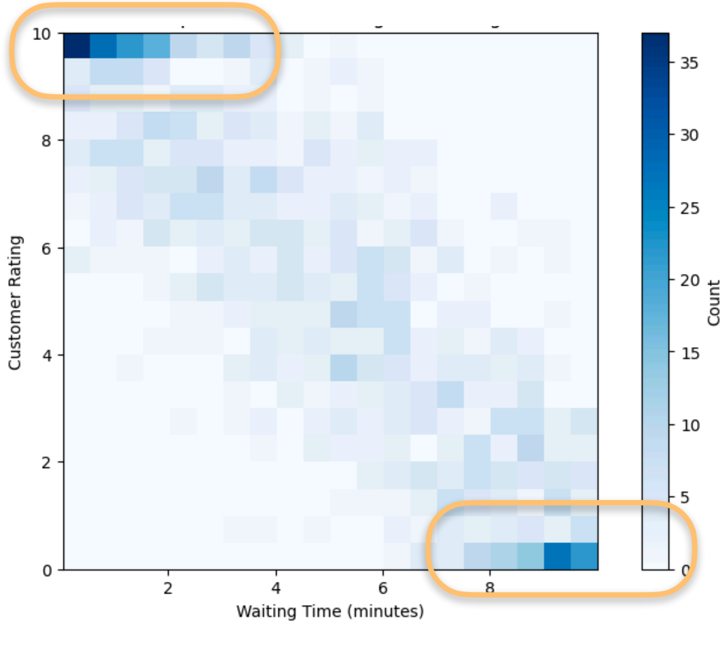
X variable: Waiting time (mins)
0 - 10 mins

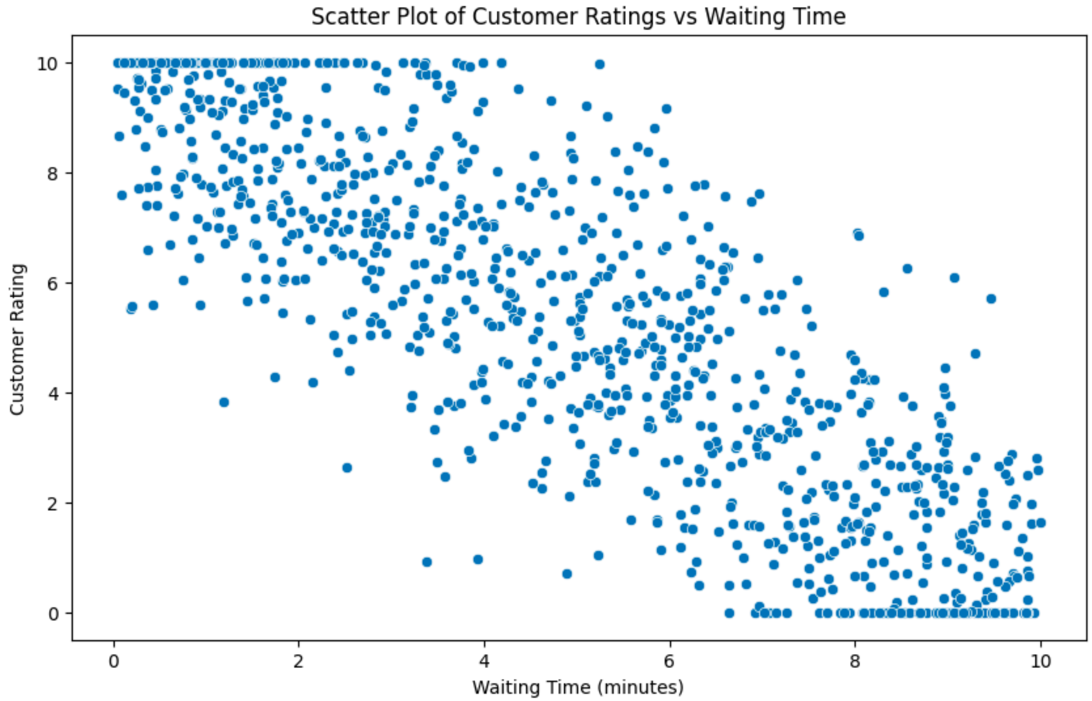
Y variable: Satisfaction rating
0 - 10

1000 customers



Joint Continuous Distributions: Dataset





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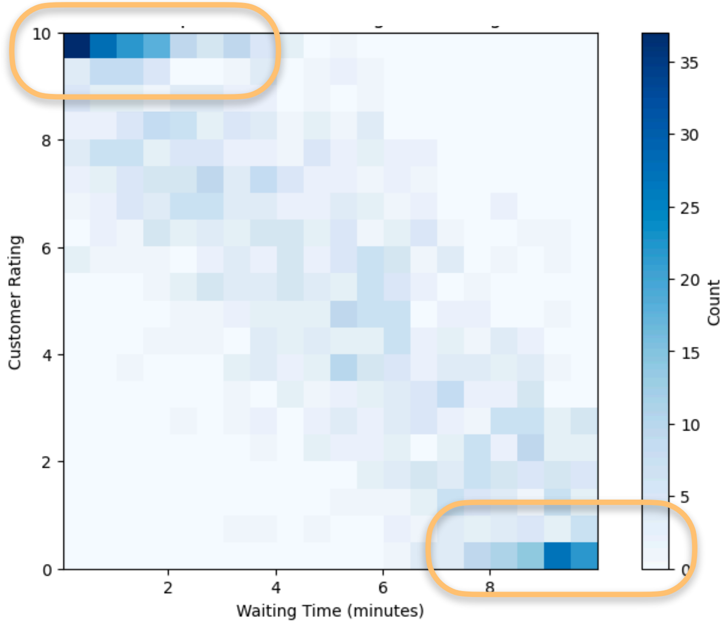
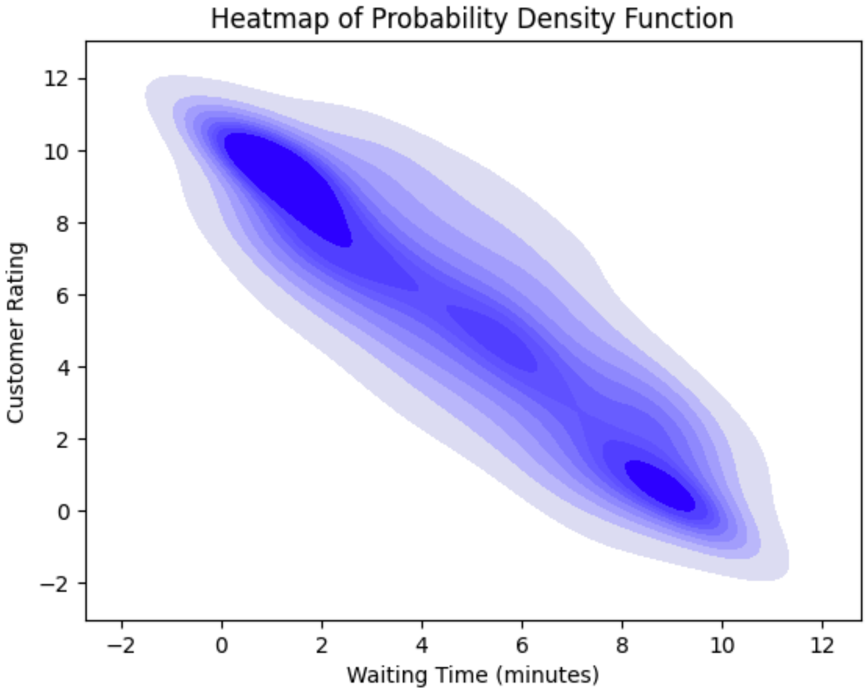
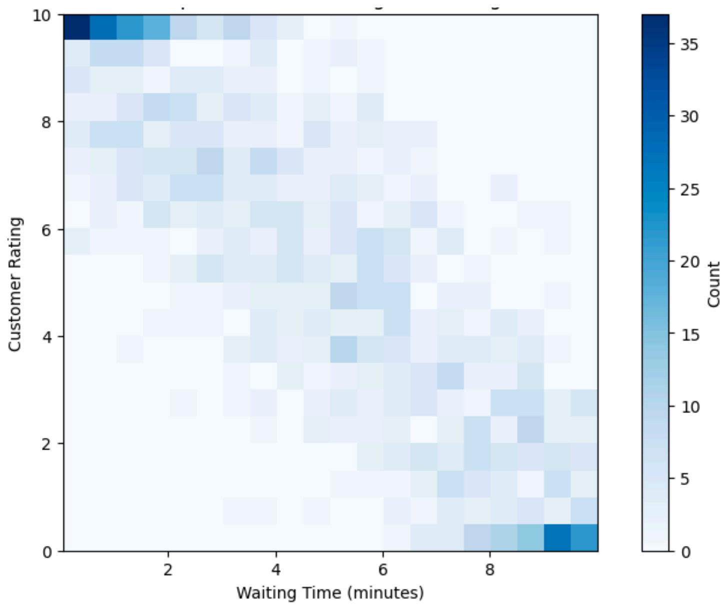
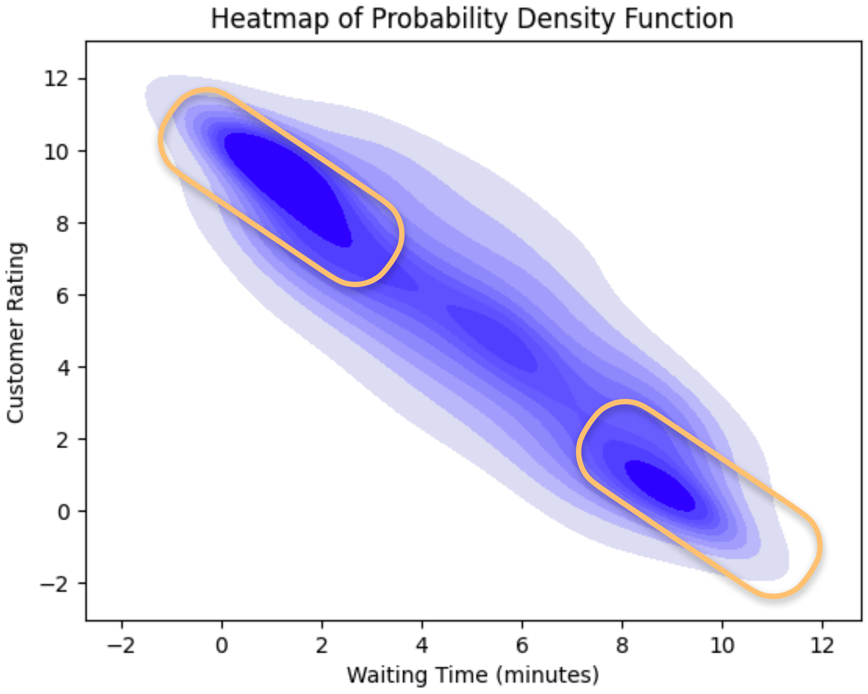
Joint Continuous Distributions: Dataset



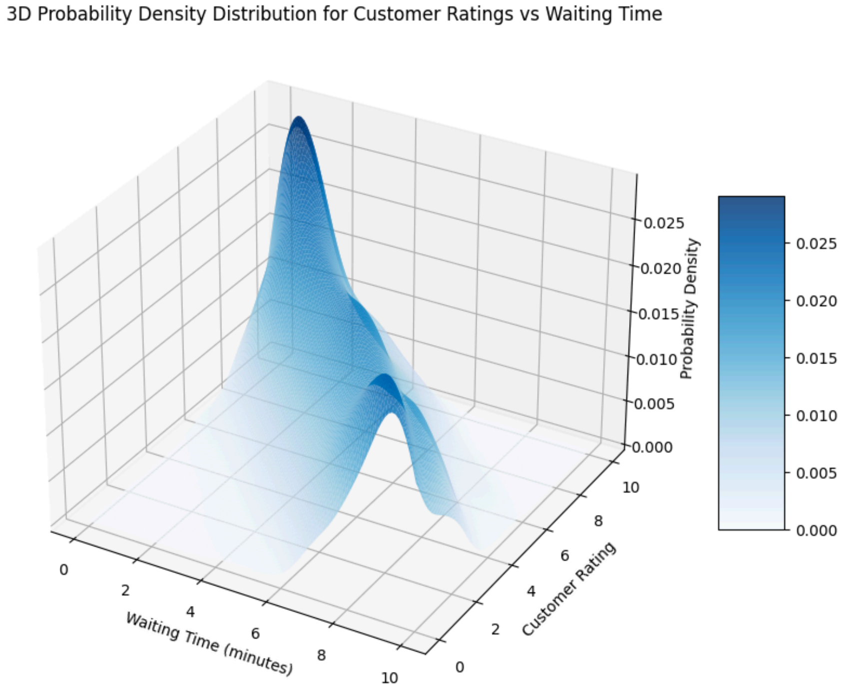


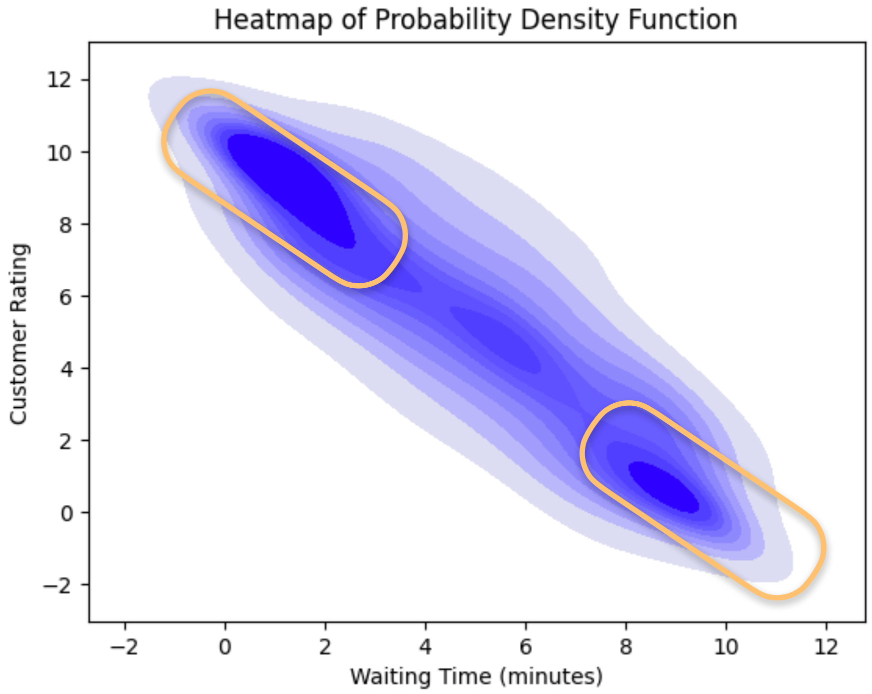
Joint Continuous Distributions: Dataset





Joint Continuous Distributions: Dataset





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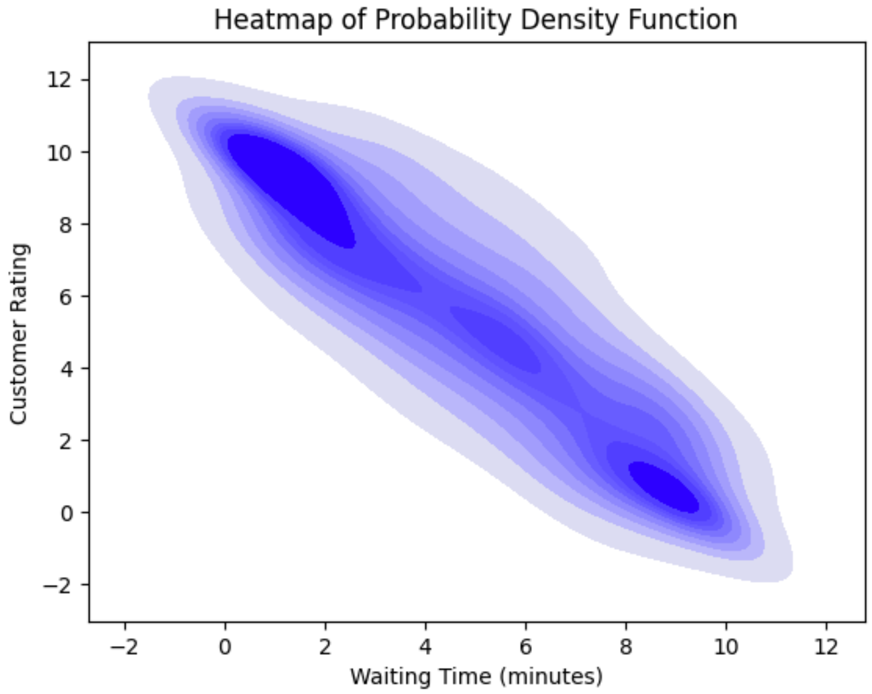
Joint Continuous Distributions: Dataset

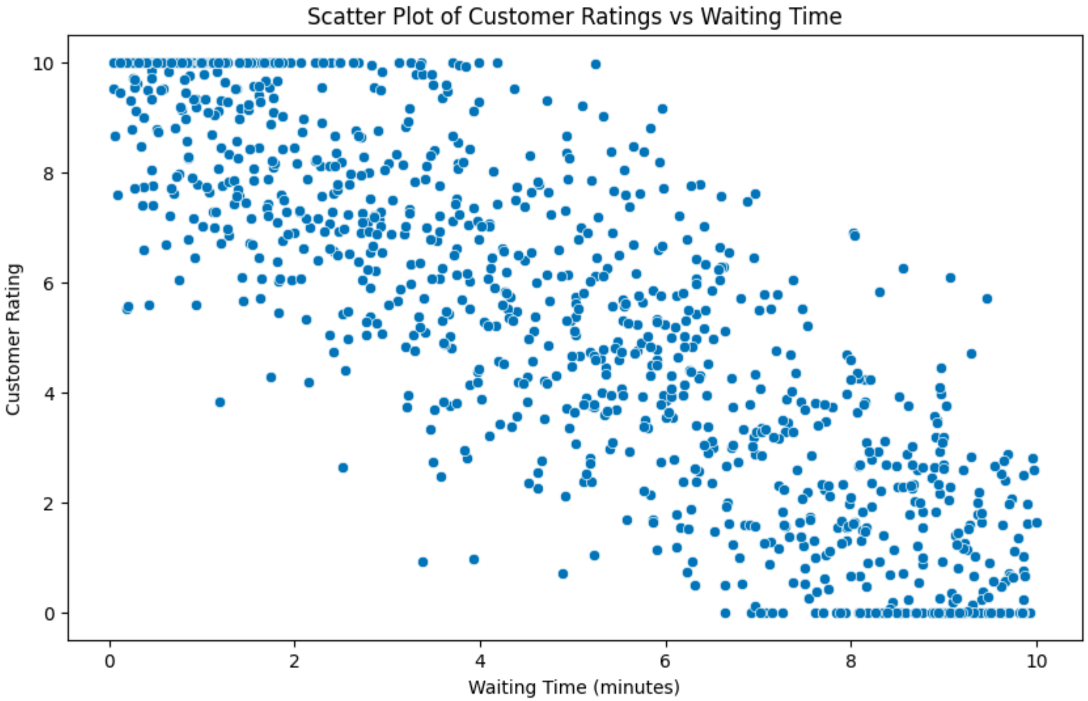


Joint Continuous Distributions: Dataset

X variable: Waiting time (mins)
0 - 10 mins

Y variable: Satisfaction rating
0 - 10

1000 customers



Expected Value

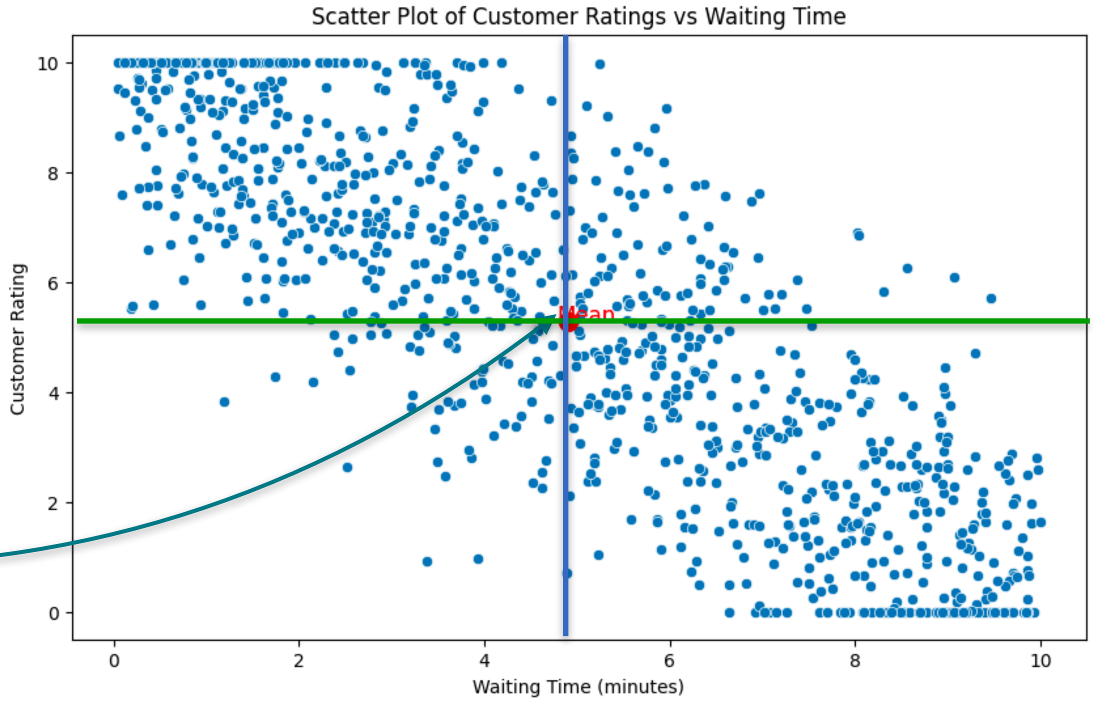
X variable: Waiting time (mins)
0 - 10 mins

Y variable: Satisfaction rating
0 - 10

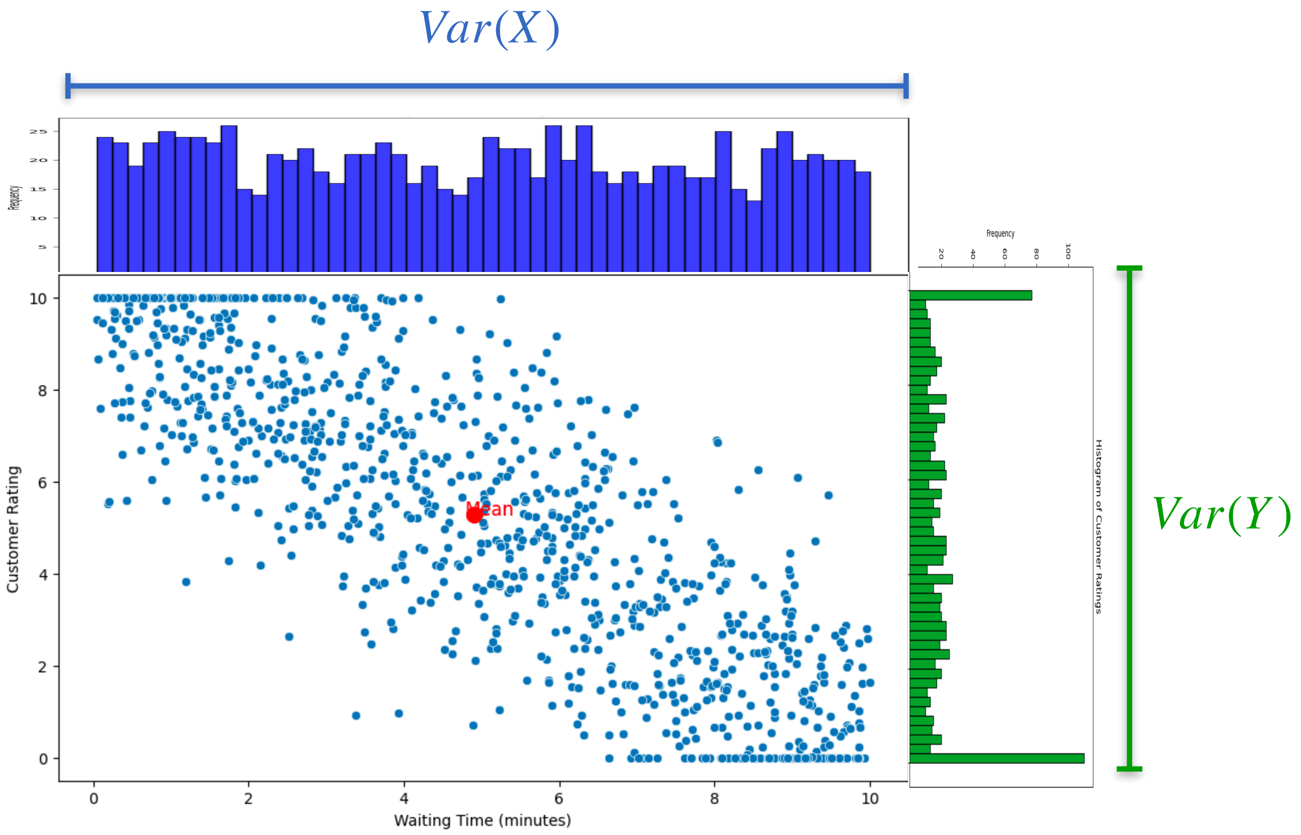
1000 customers

$\mathbb{E}[X] =$ 4.903 minutes

$\mathbb{E}[Y] =$ 5.280



Variances



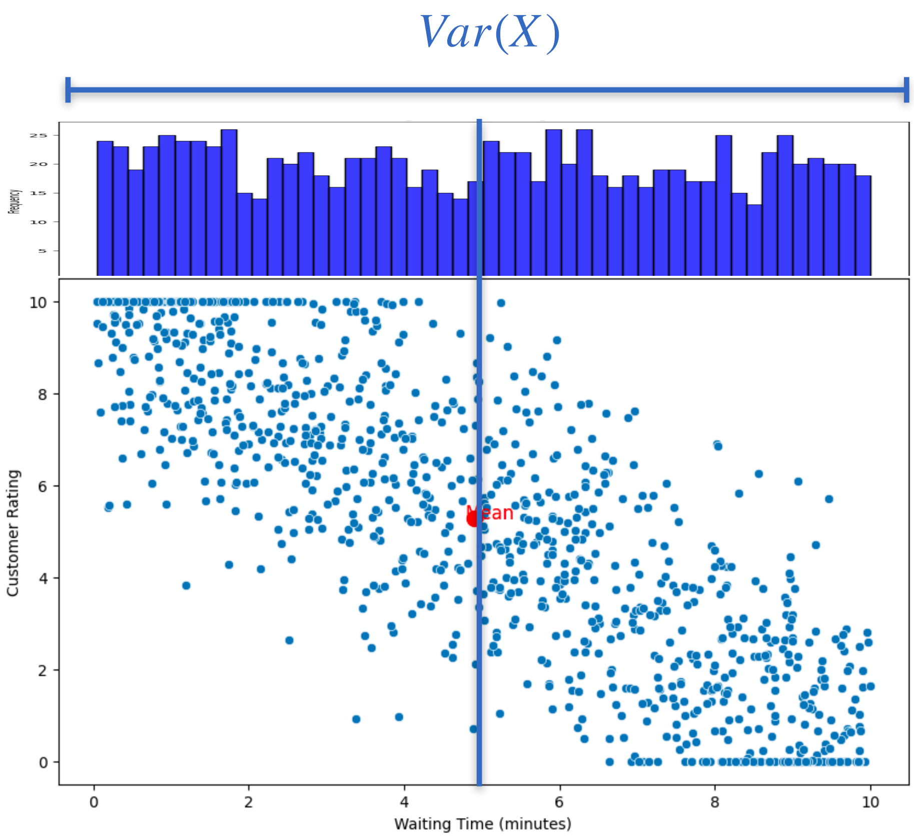
$Var(X)$
Frequency
Customer Rating
Waiting Time (minutes)
Mean
Frequency
Histogram of Customer Ratings
$Var(Y)$

Variances

$\mathbb{E}[X] = 4.903$ minutes

$\mathbb{E}[X^2] = 32.561$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= 32.561 - 4.903^2$$
$$= 8.526$$



Variances

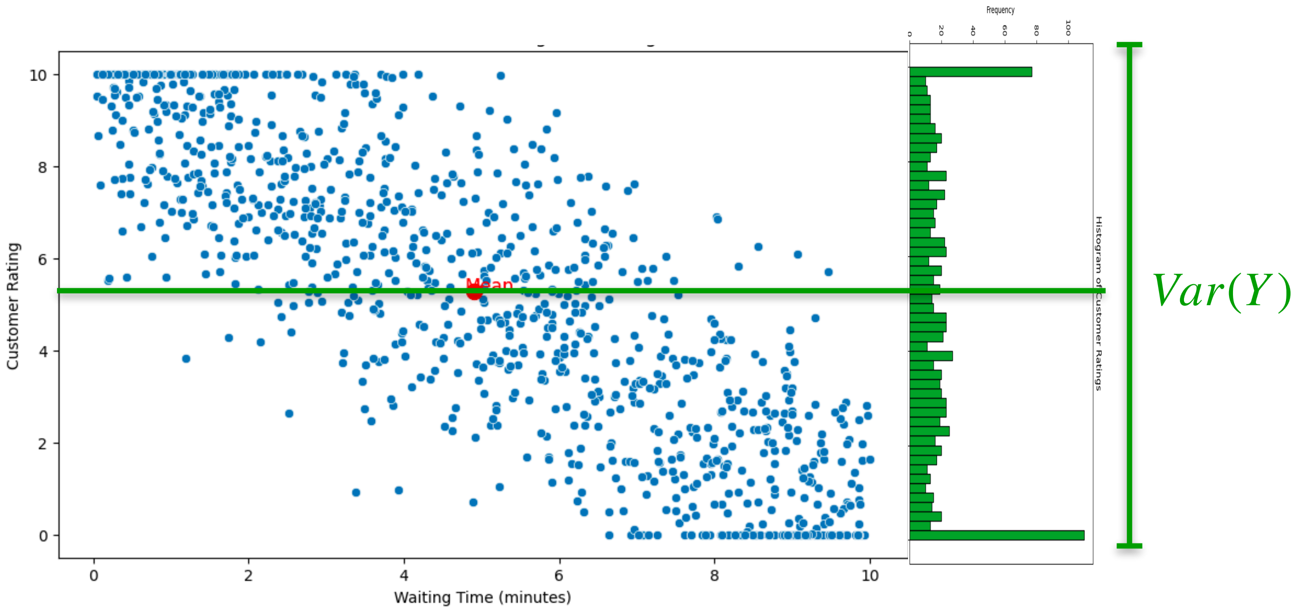
$$\mathbb{E}[Y] = 5.280$$

$$\mathbb{E}[Y^2] = 38.037$$

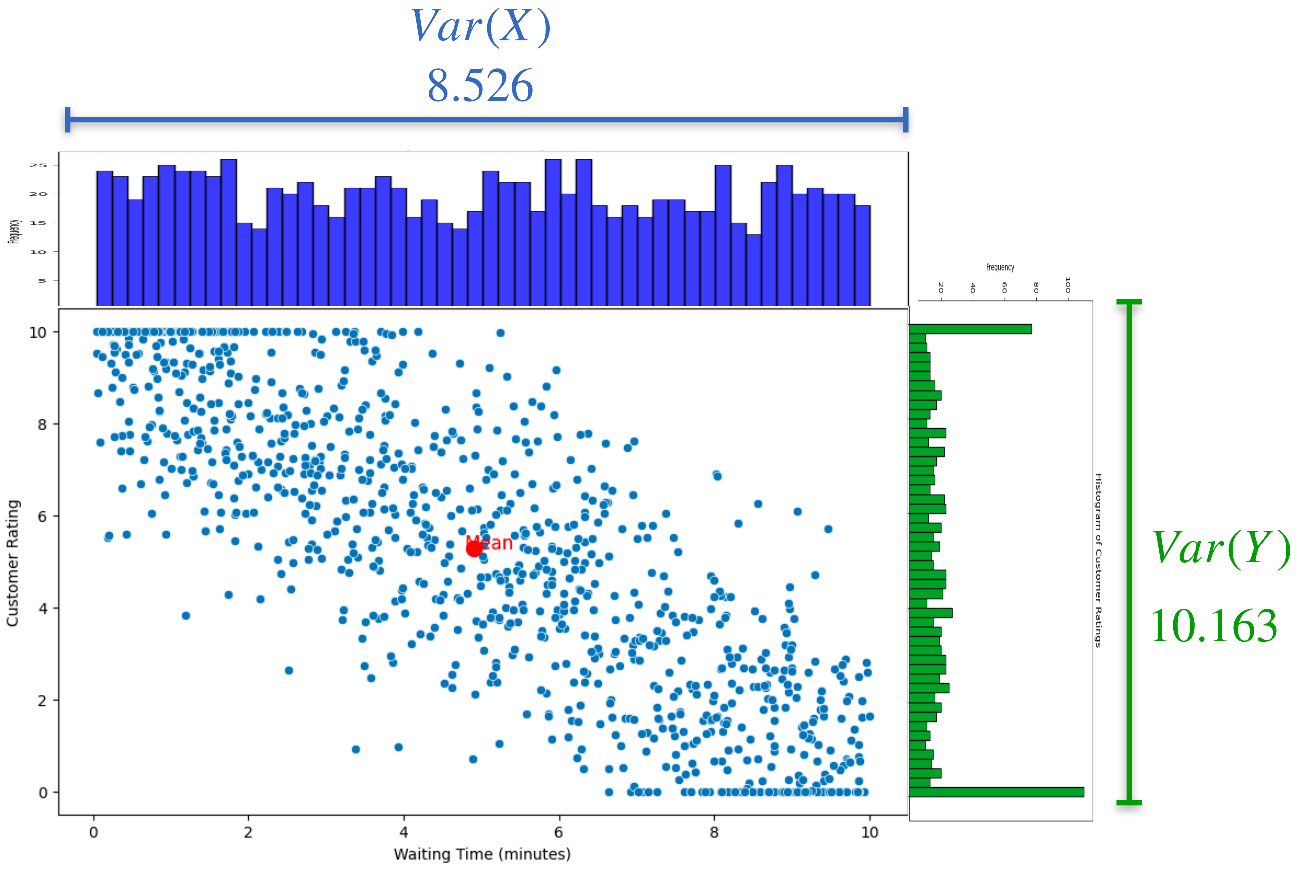
$$Var(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

$$= 38.037 - 5.280^2$$

$$= 10.163$$



Variances



$Var(X)$
8.526
Frequency
25
20
15
10
5
10
8
6
4
2
0
Customer Rating
Mean
0
2
4
6
8
10
Waiting Time (minutes)
Frequency
20
40
60
80
100
Histogram of Customer Ratings
$Var(Y)$
10.163

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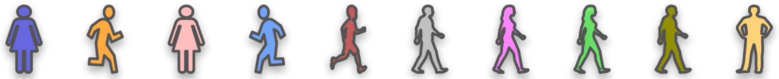
Probability Distributions with Multiple Variables

Marginal and Conditional Distribution

Marginal Distribution: Example 1

	Height (Y)					
Age (X)	45	46	47	48	49	50
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10



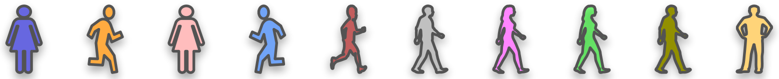
Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

Marginal Distribution

Distribution of one variable while ignoring the other

Marginal Distribution: Example 1

	Height (Y)					
Age (X)	**45**	**46**	**47**	**48**	**49**	**50**
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10



Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

To find the marginal distribution of height:

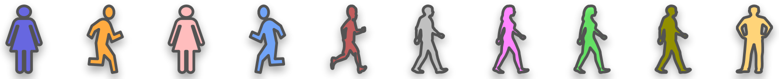
sum the joint probability distribution over all values of age

Marginal Distribution: Example 1

Height (Y)

Age (X)

	45	46	47	48	49	50
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10
	1/10	2/10	2/10	0	3/10	2/10



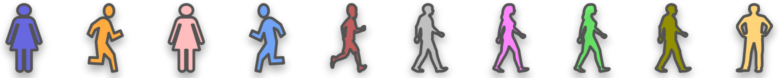
Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j)$$

Marginal Distribution: Example 1

Height (Y)

Age (X)	45	46	47	48	49	50
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10
	1/10	2/10	2/10	0	3/10	2/10

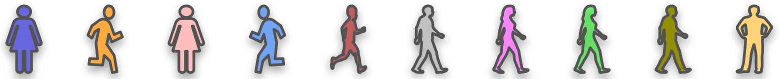


Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j)$$

$$p_Y(50) =$$

Marginal Distribution: Example 1



Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

Height (Y)

Age (X)	**45**	**46**	**47**	**48**	**49**	**50**
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10
	1/10	2/10	2/10	0	3/10	2/10

$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j)$$

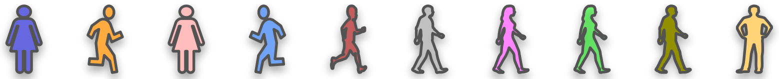
$$p_Y(50) = \sum_i p_{XY}(x_i, 50)$$

$$p_Y(50) = \frac{2}{10}$$

Marginal Distribution: Example 1

Height (Y)

Age (X)	45	46	47	48	49	50	
7	1/10	2/10	0	0	0	0	3/10
8	0	0	2/10	0	0	0	2/10
9	0	0	0	0	3/10	1/10	4/10
10	0	0	0	0	0	1/10	1/10
	1/10	2/10	2/10	0	3/10	2/10	



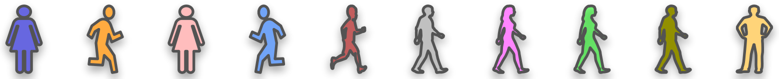
Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

$$p_X(x_i) = \sum_j p_{XY}(x_i, y_j)$$

Marginal Distribution: Example 1

Height (Y)

Age (X)	45	46	47	48	49	50	
7	1/10	2/10	0	0	0	0	3/10
8	0	0	2/10	0	0	0	2/10
9	0	0	0	0	3/10	1/10	4/10
10	0	0	0	0	0	1/10	1/10
	1/10	2/10	2/10	0	3/10	2/10	



Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

$$p_X(x_i) = \sum_j p_{XY}(x_i, y_j)$$

$$p_X(7) = \sum_j p_{XY}(7, y_j)$$

$$p_X(7) = \frac{3}{10}$$

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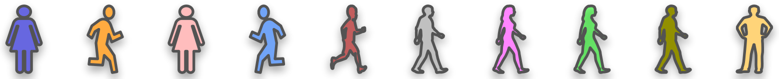
Marginal Distribution: Example 1



Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

Height (Y)

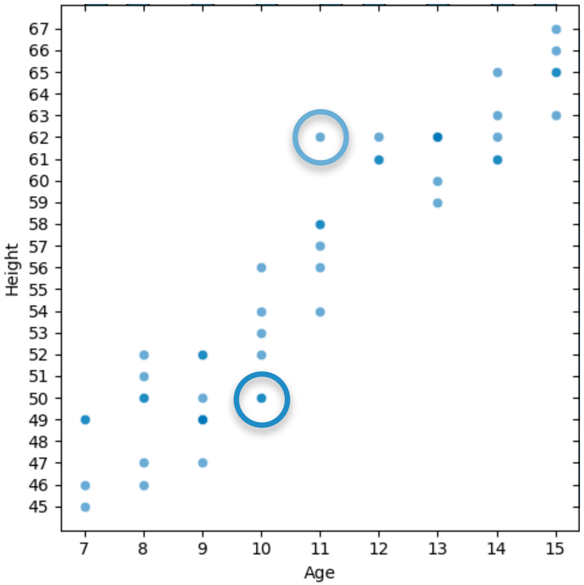
Age (X)		45	46	47	48	49	50	
	7	1/10	2/10	0	0	0	0	3/10
	8	0	0	2/10	0	0	0	2/10
	9	0	0	0	0	3/10	1/10	4/10
	10	0	0	0	0	0	1/10	1/10
		1/10	2/10	2/10	0	3/10	2/10	

Marginal Distribution of Age

Marginal Distribution of Height

Marginal Distribution: Example 1

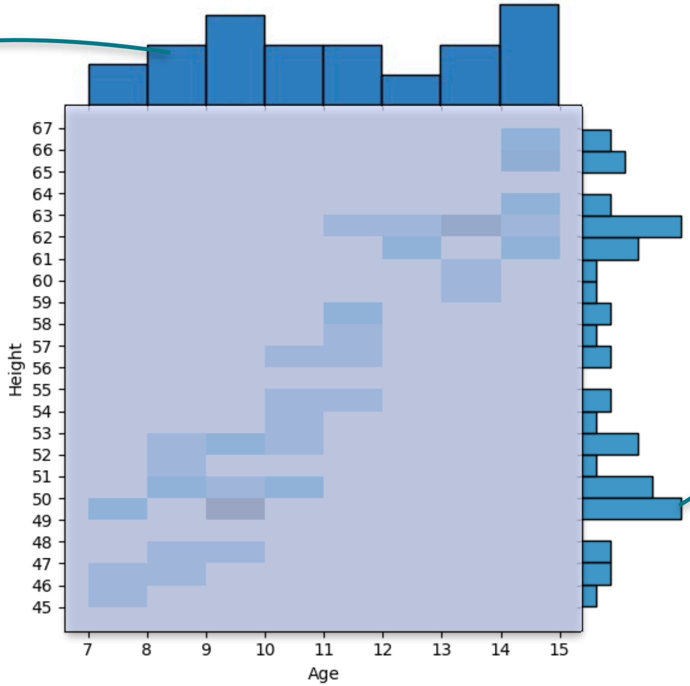
Age and Height Dataset
for 50 children



Marginal Distribution: Example 1

Marginal Distribution of Age

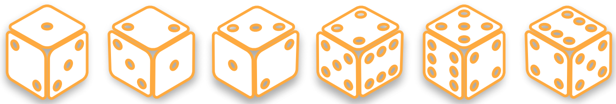
Age and Height Dataset
for 50 children



Marginal Distribution of Height

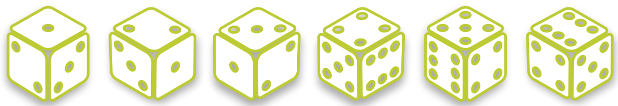
Marginal Distributions: Example 2

X : the number rolled on the 1st dice



$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

Y : the number rolled on the 2nd dice



$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

Marginal Distributions: Example 2

X \ Y	1	2	3	4	5	6	
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	1/6	1/6	1/6	1/6	1/6	1/6	

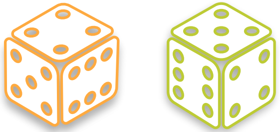
X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

$$p_X(x_i) = \frac{1}{6}$$

$$p_Y(y_j) = \frac{1}{6}$$

Marginal Distributions: Example 3



X : the number rolled on the 1st dice

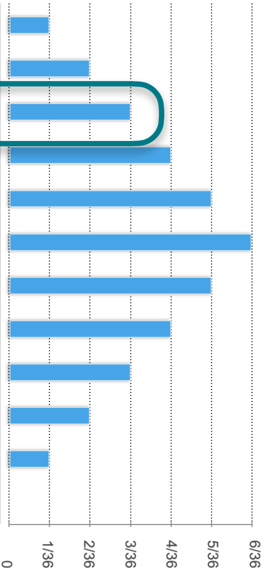
Y : sum of the two dice

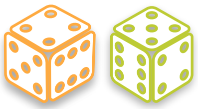
Marginal Distributions: Example 3

y (sum of dice)	1	2	3	4	5	6
12						1/36
11					1/36	1/36
10				1/36	1/36	1/36
9			1/36	1/36	1/36	1/36
8		1/36	1/36	1/36	1/36	1/36
7	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	
5	1/36	1/36	1/36	1/36		
4	1/36	1/36	1/36			
3	1/36	1/36				
2	1/36					
1						

x (num on dice 1)

12	1/36
11	2/36
10	3/36
9	4/36
8	5/36
7	6/36
6	5/36
5	4/36
4	3/36
3	2/36
2	1/36
1	0





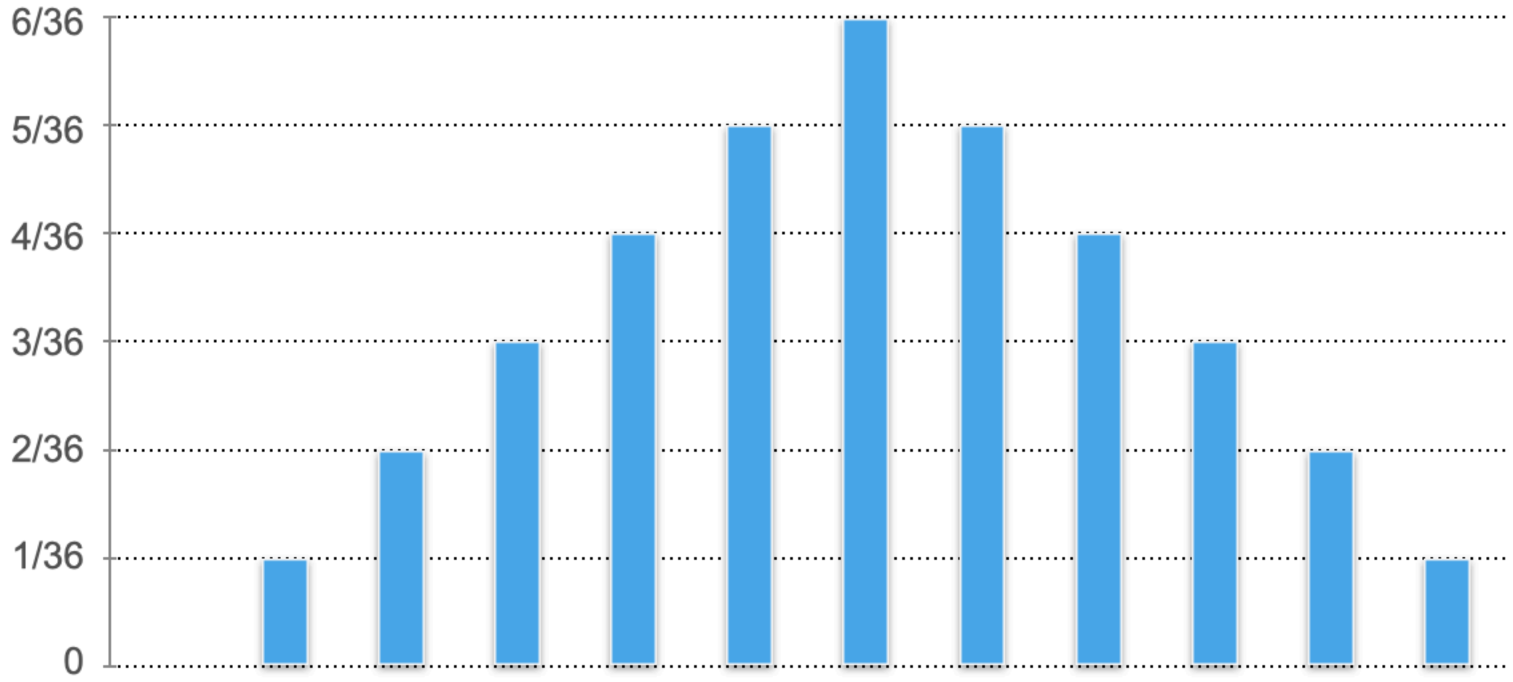
X : the number rolled on the 1st dice

Y : sum of the two dice

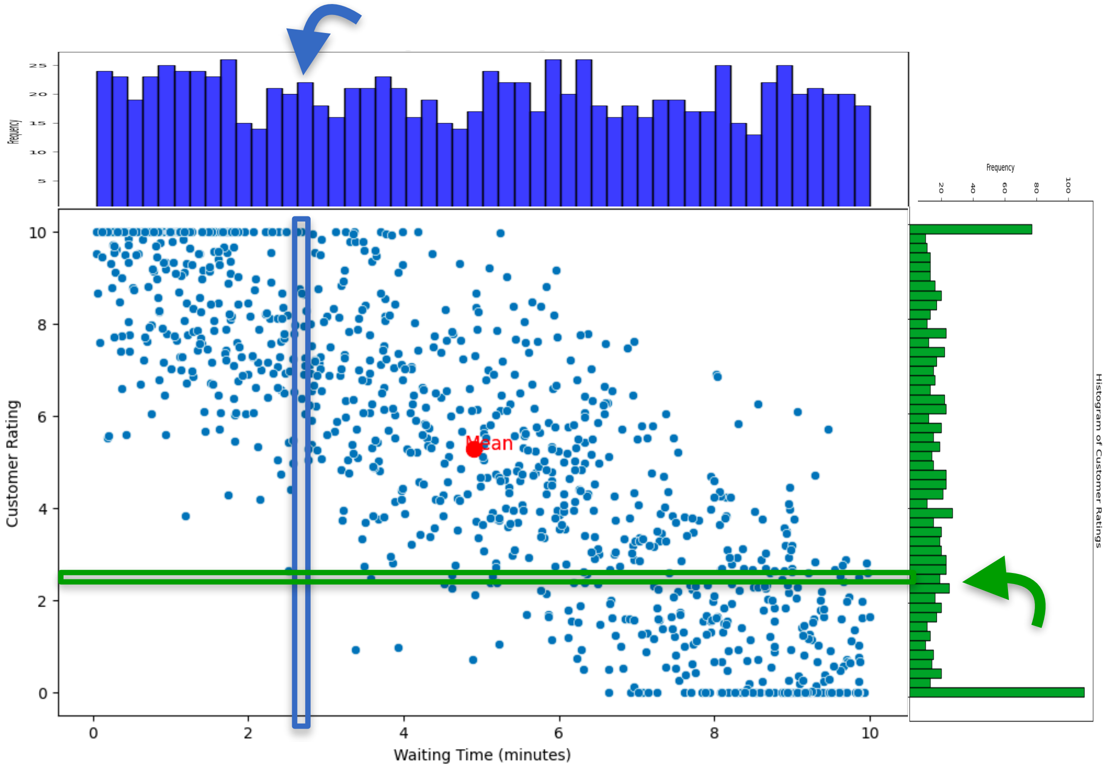
$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j) = ?$$

$$p_Y(10) = ? \quad = \frac{3}{36}$$

Marginal Distributions: Example 2



Marginal Distributions



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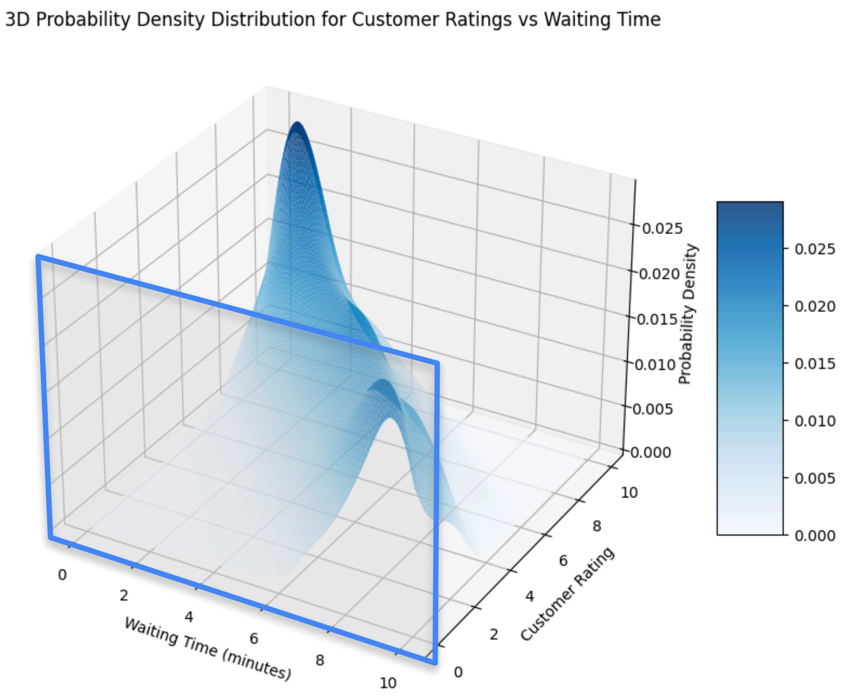
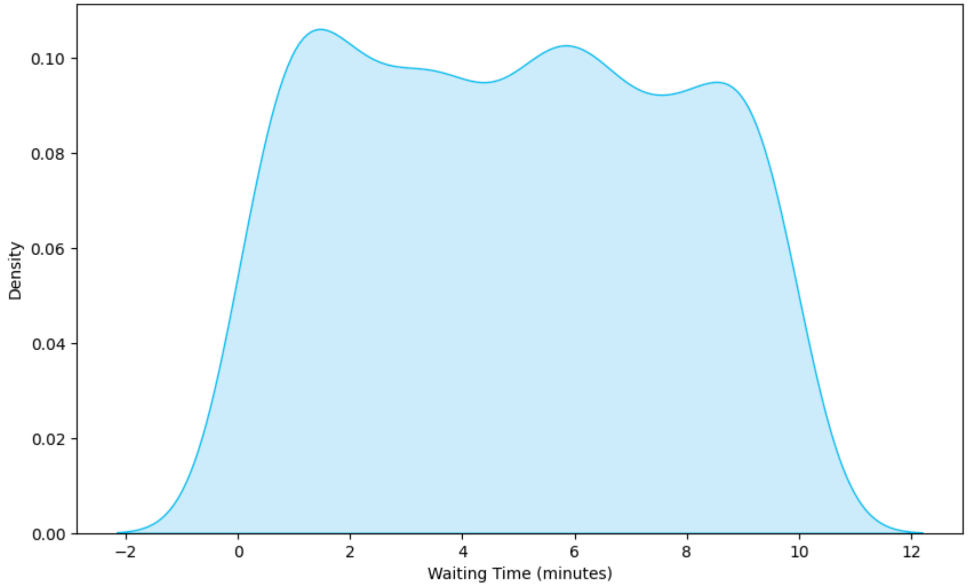
Continuous Marginal Distribution





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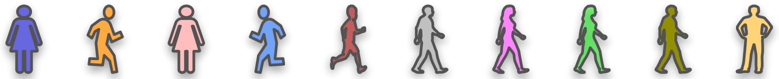
Probability Distributions with Multiple Variables

Conditional Distribution

Conditional Distribution: Example 1



Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

Height (Y)

Age (X)	45	46	47	48	49	50	
7	1/10	2/10	0	0	0	0	3/10
8	0	0	2/10	0	0	0	2/10
9	0	0	0	0	3/10	1/10	4/10
10	0	0	0	0	0	1/10	1/10
	1/10	2/10	2/10	0	3/10	2/10	

Marginal Distribution of Age

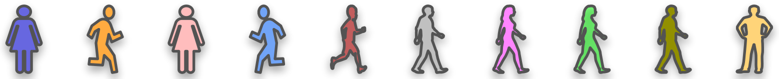
Marginal Distribution of Height

Conditional Distribution: Example 1

Height (Y)

Age (X)

	45	46	47	48	49	50
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10



Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

If age = 9, what is the distribution across the height variable?

Conditional Distribution

Conditional Distribution: Example 1

Height (Y)

Age (X)

	45	46	47	48	49	50
7	1/10	2/10	0	0	0	0
8	0	0	2/10	0	0	0
9	0	0	0	0	3/10	1/10
10	0	0	0	0	0	1/10

Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

If age = 9, what is the distribution across the height variable?

$$p_{Y|X=9}(y) = \mathbf{P}(Y = y \mid X = 9)$$

Conditional Distribution: Example 1

Height (Y) | Sum $\mathbf{P}(X=9)$

Age (X)	45	46	47	48	49	50
9	0	0	0	0	3/10	1/10

Normalize

Divide by row sum

9	0	0	0	0	3/4	1/4

$$\mathbf{P}(Y=49\,|\,X=9)=\frac{3}{4}$$

$$p_{Y|X=9}(y)=\mathbf{P}(Y=y\,|\,X=9)$$

$$\mathbf{P}(A,B)=\mathbf{P}(A)\cdot\mathbf{P}(B\,|\,A)$$

$$\mathbf{P}(X=9,Y=49)=P(X=9)\cdot P(Y=49\,|\,X=9)$$

$$P(Y=49\,|\,X=9)=\frac{\mathbf{P}(X=9,Y=49)}{P(X=9)}$$

Row sum

$$\mathbf{P}(Y=49\,|\,X=9)=\frac{3/10}{4/10}=\frac{3}{4}$$

Conditional Distribution: Example 1

	Height (Y)					Sum $\mathbf{P}(X=9)$
(X)	45	46	47	48	49	50
9	0	0	0	0	3/10	1/10

$$p_{Y|X=x}(y) = \frac{p_{XY}(x,y)}{p_X(x)}$$

$$p_{Y|X=9}(y) = \mathbf{P}(Y=y\,|\,X=9)$$

$$\mathbf{P}(A,B) = \mathbf{P}(A)\cdot\mathbf{P}(B\,|\,A)$$

$$\mathbf{P}(X=9,Y=49) = P(X=9)\cdot P(Y=49\,|\,X=9)$$

$$P(Y=49\,|\,X=9) = \frac{\mathbf{P}(X=9,Y=49)}{P(X=9)}$$

Row sum

$$\mathbf{P}(Y=49\,|\,X=9) = \frac{3/10}{4/10} = \frac{3}{4}$$

Discrete Conditional Distribution: Formula

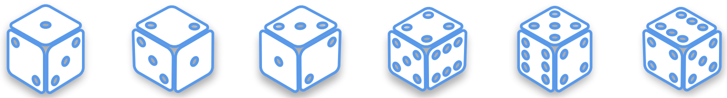
Joint PDF of X and Y

Conditional PDF of Y

$$p_{Y|X=x}(y) = \frac{p_{XY}(x, y)}{p_X(x)}$$

Marginal distribution of X

Conditional Distributions: Example 2



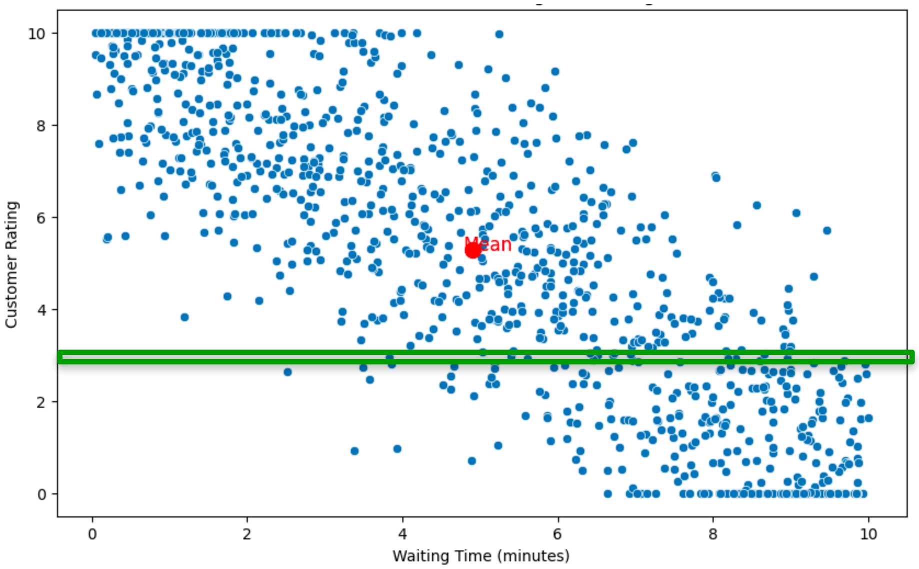
Dice 1:	1/6	1/6	1/6	1/6	1/6	1/6
Dice 2:	1/6	1/6	1/6	1/6	1/6	1/6

$$p_{Y|X=4}(y=1) = \frac{p_{XY}(x=4, y=1)}{p_X(x=4)} = \frac{1/36}{1/6} = \frac{1}{6}$$

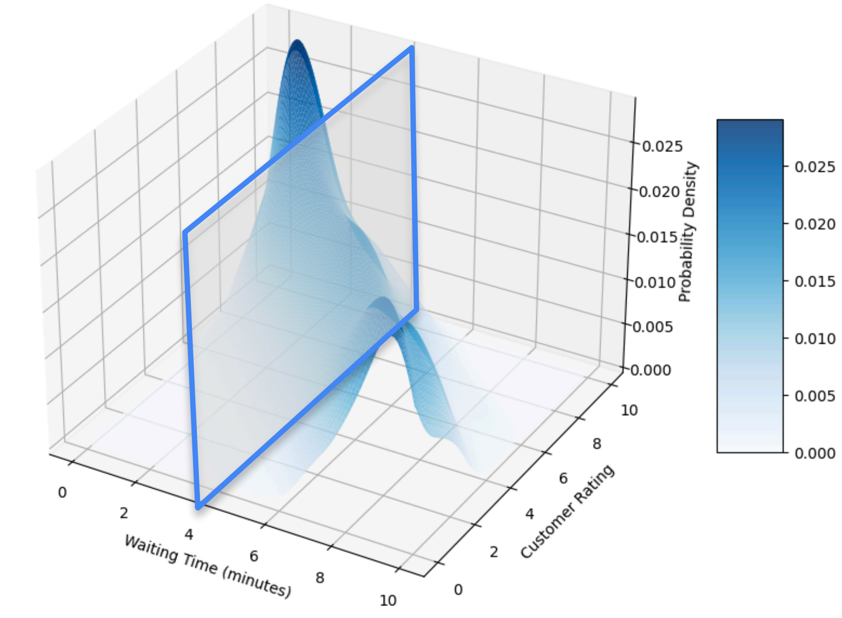
Y (columns), X (rows)

	1	2	3	4	5	6	Sum
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6

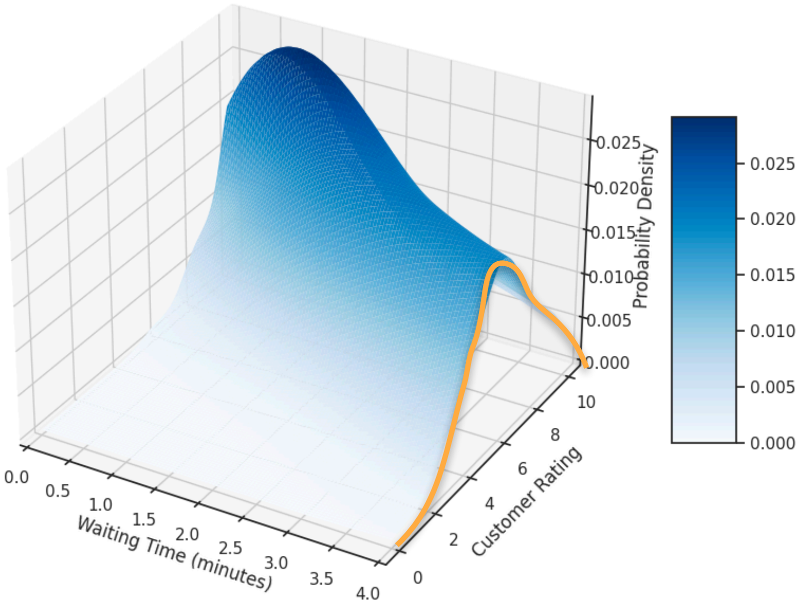
Conditional Distributions: Example 4



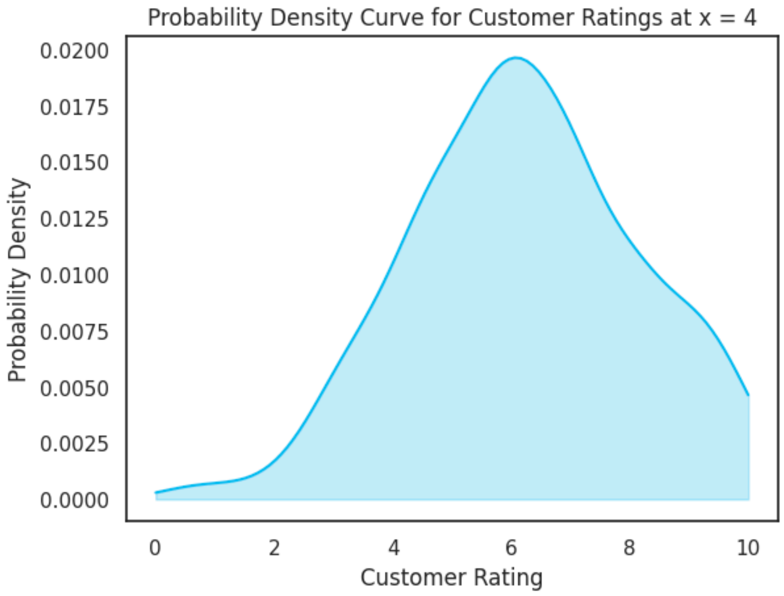
Continuous Conditional Distribution



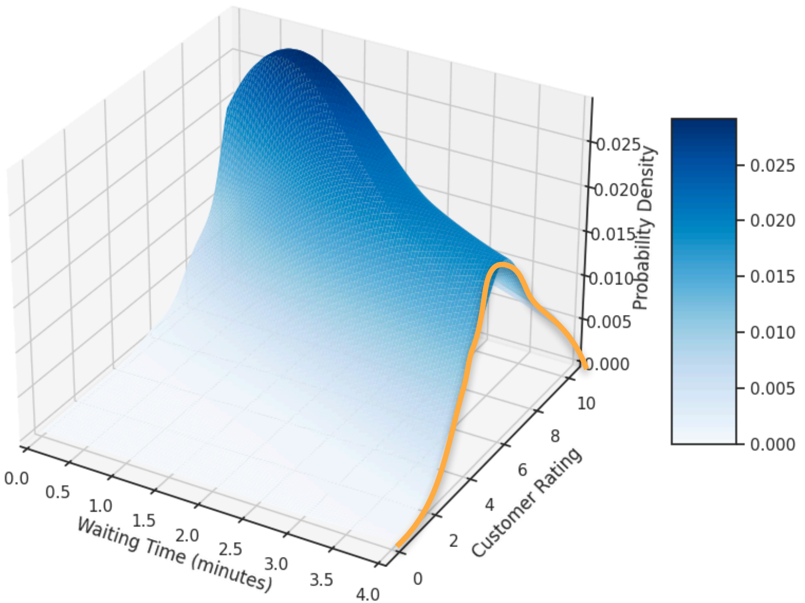
Probability distribution for rating given that waiting time was 4 minutes



Continuous Conditional Distribution



Conditional PDF of y given x = 4



Discrete Conditional Distribution

Joint PDF of X and Y

Conditional PDF of Y

$$p_{Y|X=x}(y) = \frac{p_{XY}(x, y)}{p_X(x)}$$

Marginal distribution of X

Continuous Conditional Distribution: Formula

Joint PDF of X and Y

Conditional PDF of Y

$$f_{Y|X=x}(y) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Marginal distribution of X

Probability Distributions with Multiple Variables

DeepLearning.AI

Covariance of a Dataset

Introduction to Covariance

- **How is variable X related to each of the Y variables?**
- **How do you compare these relations?**

X : age of a child

Y_1: height of the child (in)

Age (X)	Height (Y_1)
6	50
7	52
8	55
9	57
10	60
11	62
12	64
13	65
14	67
15	68

Y_2 : grades in a test

Age (X)	Grades (Y_2)
6	5
7	7
8	8
9	3
10	1
11	1
12	6
13	10
14	2
15	7

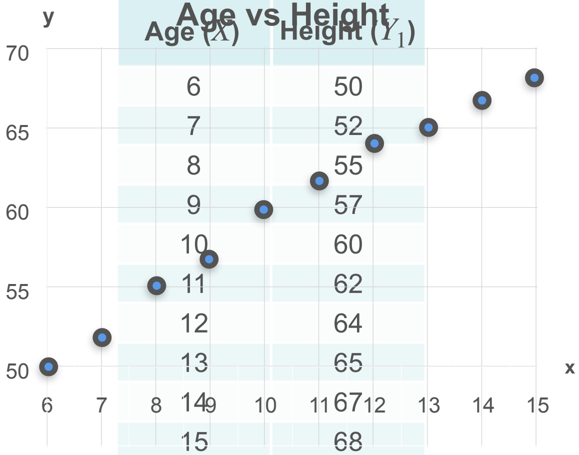
Y_3: number of naps per day

Age (X)	Naps per day (Y_3)
6	8
7	7
8	6
9	5
10	4
11	3
12	2
13	1
14	1
15	0

Introduction to Covariance

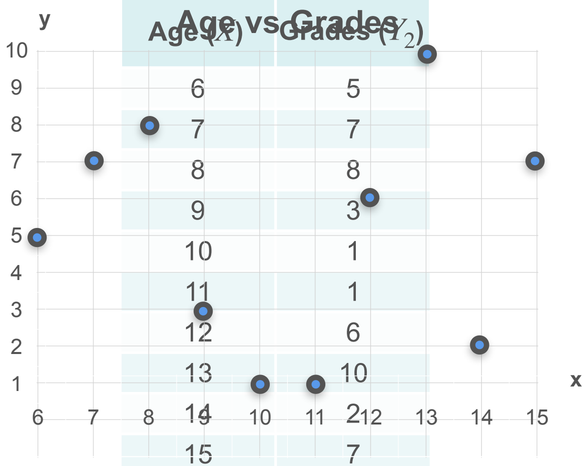
X : age of a child

Y_1: height of the child (in)



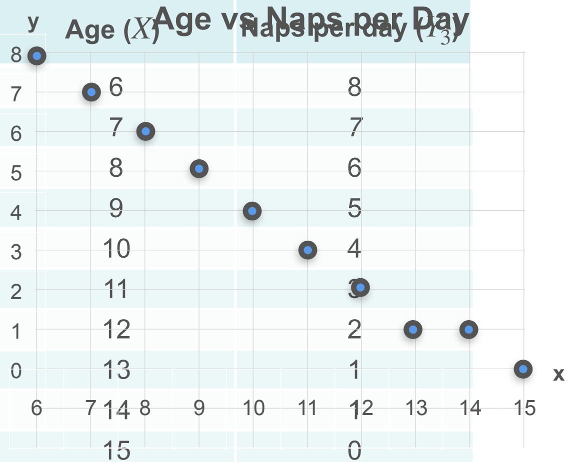
Age (X)	Height (Y_1)
6	50
7	52
8	55
9	57
10	60
11	62
12	64
13	65
14	67
15	68

Y_2 : grades in a test



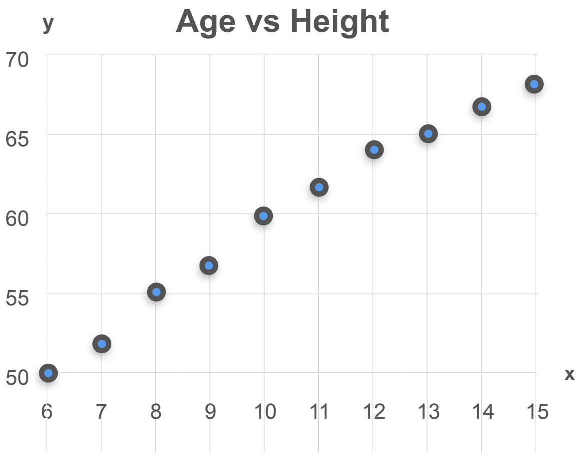
Age (X)	Grades (Y_2)
6	5
7	7
8	8
9	3
10	1
11	1
12	6
13	10
14	2
15	7

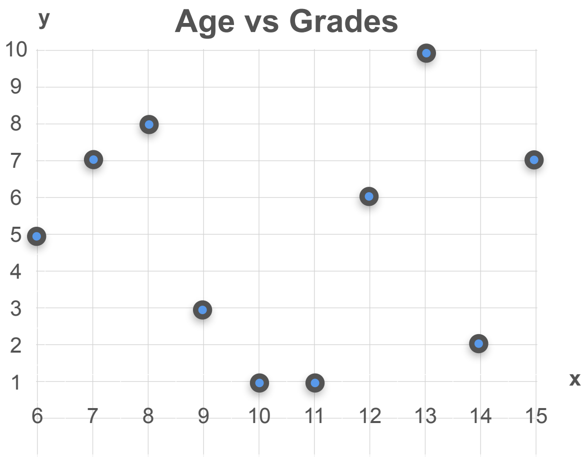
Y_3: number of naps per day

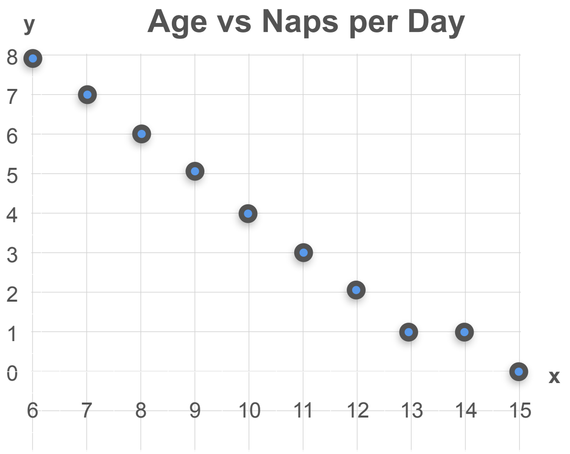


Age (X)	Naps per day (Y_3)
6	8
7	7
8	6
9	5
10	4
11	3
12	2
13	1
14	1
15	0

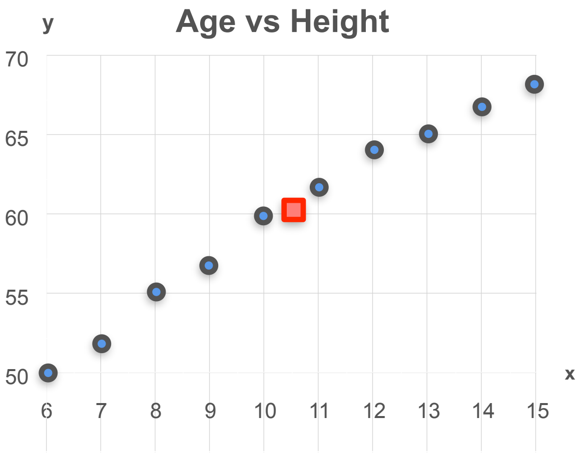
How To Compare These?



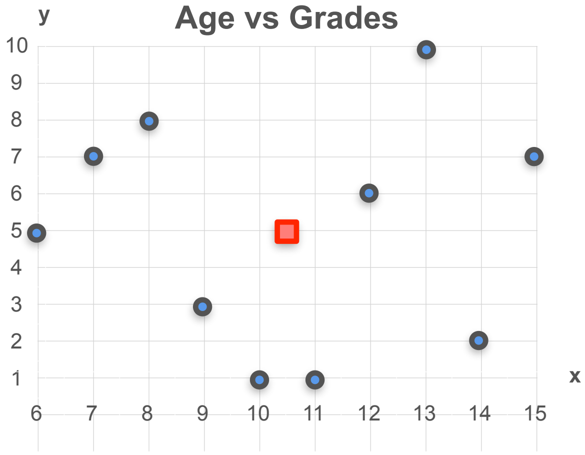




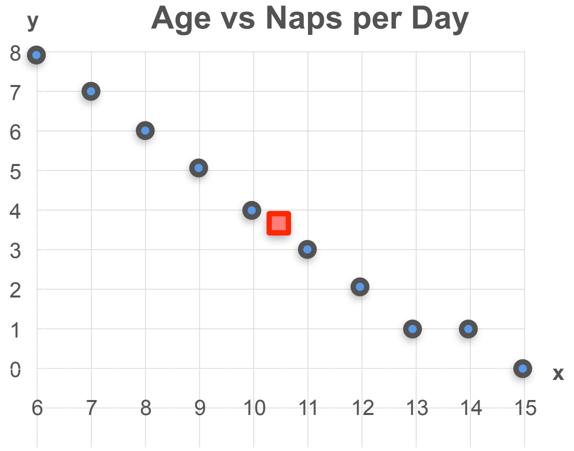
Mean?



$\mu_x = 10.5 \quad \mu_y = 60$

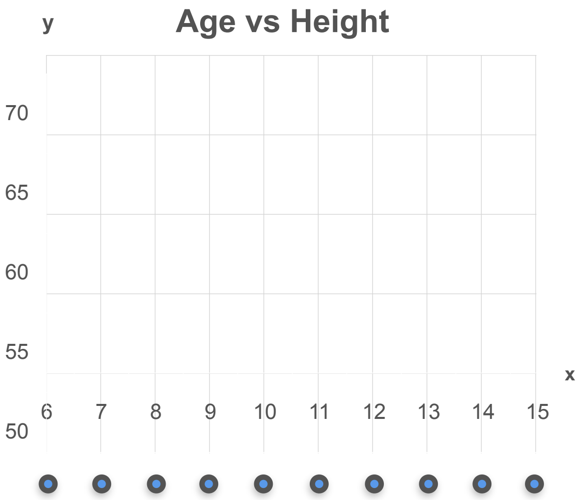


$\mu_x = 10.5 \quad \mu_y = 5$

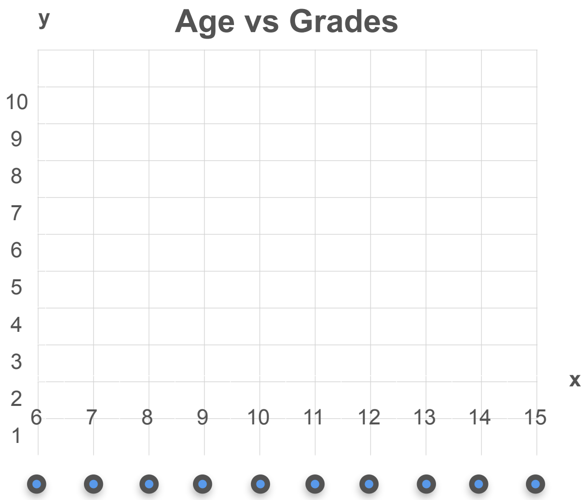


$\mu_x = 10.5 \quad \mu_y = 3.7$

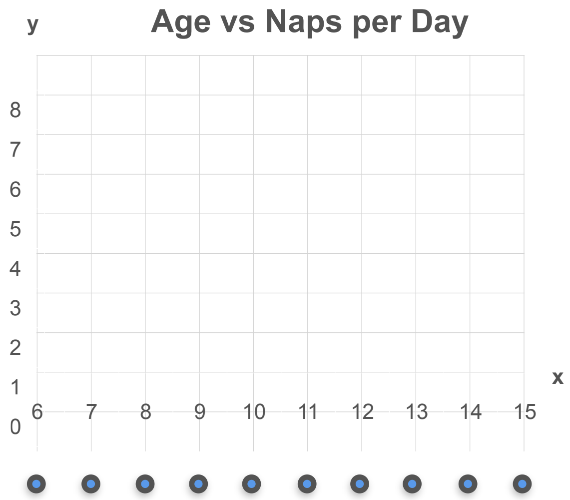
Horizontal (X) Variance



$$Var(X) = 9.17$$

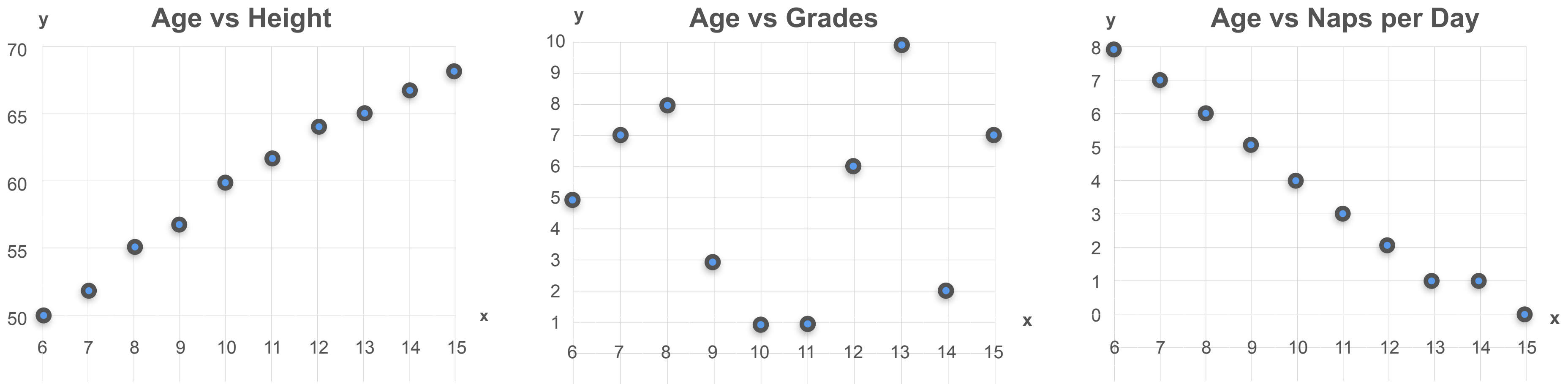


$$Var(X) = 9.17$$

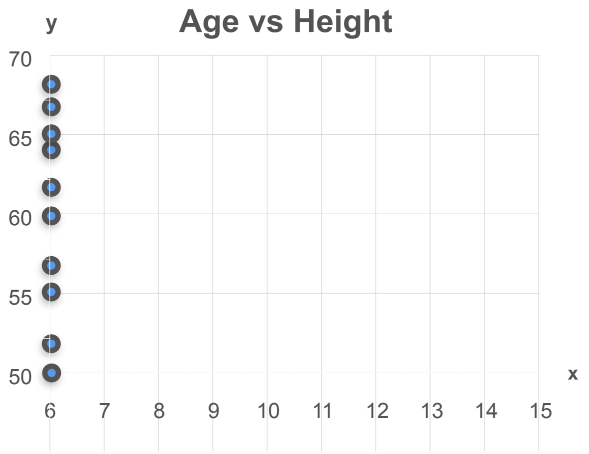


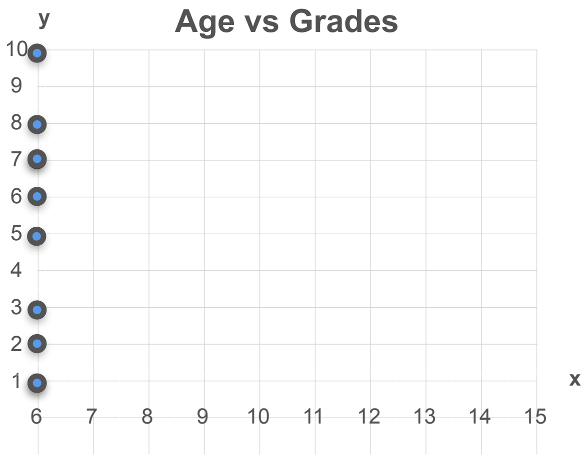
$$Var(X) = 9.17$$

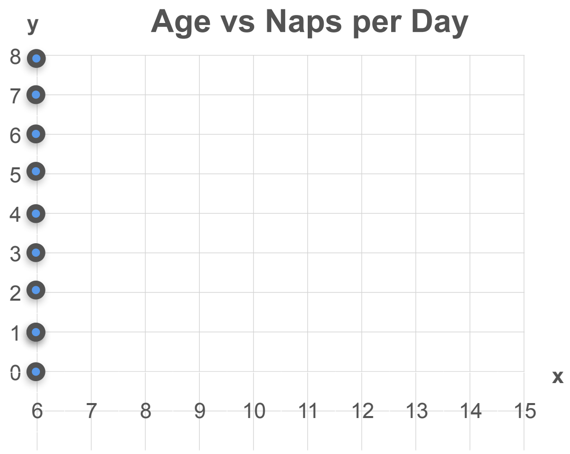
Anything Else?



Vertical (Y) Variance





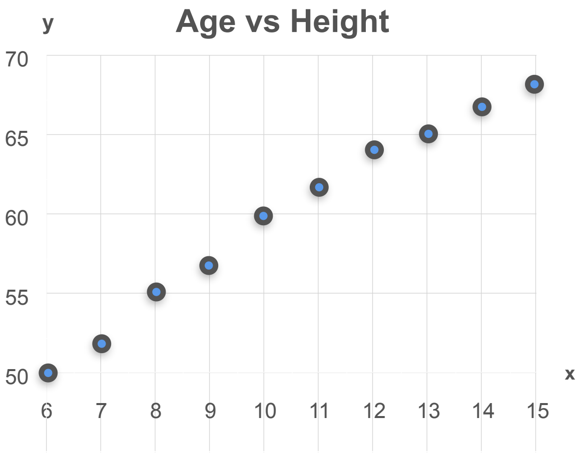


$$Var(Y) = 39.56$$

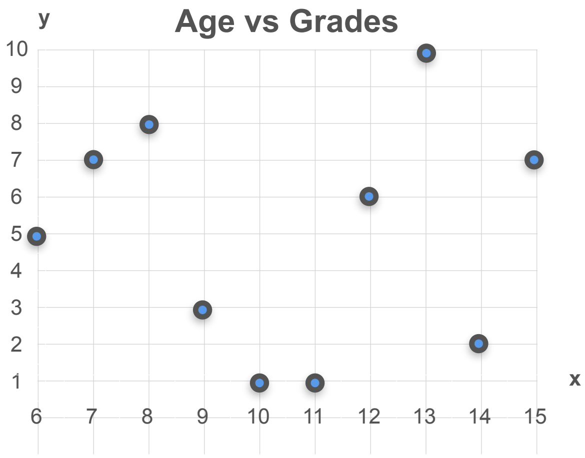
$$Var(Y) = 9.78$$

$$Var(Y) = 7.57$$

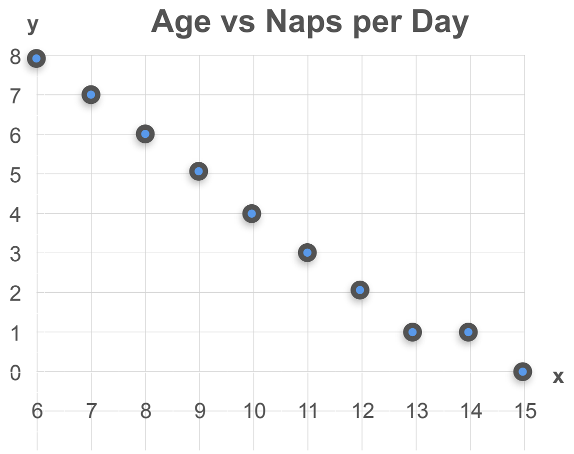
Still no Way To Compare Them



Covariance > 0

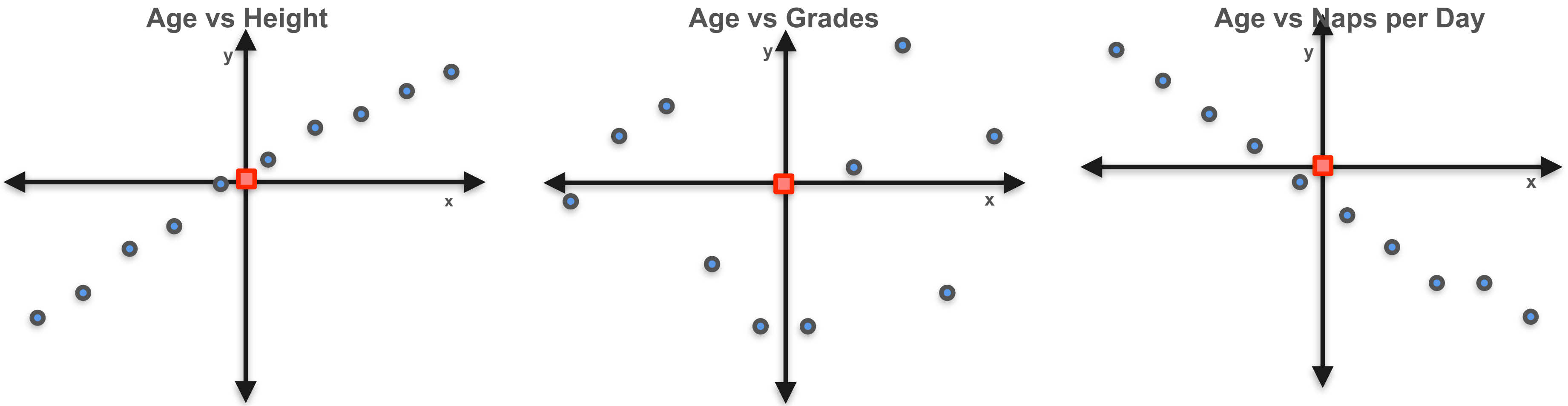


Covariance ≈ 0



Covariance < 0

First Step: Center Them

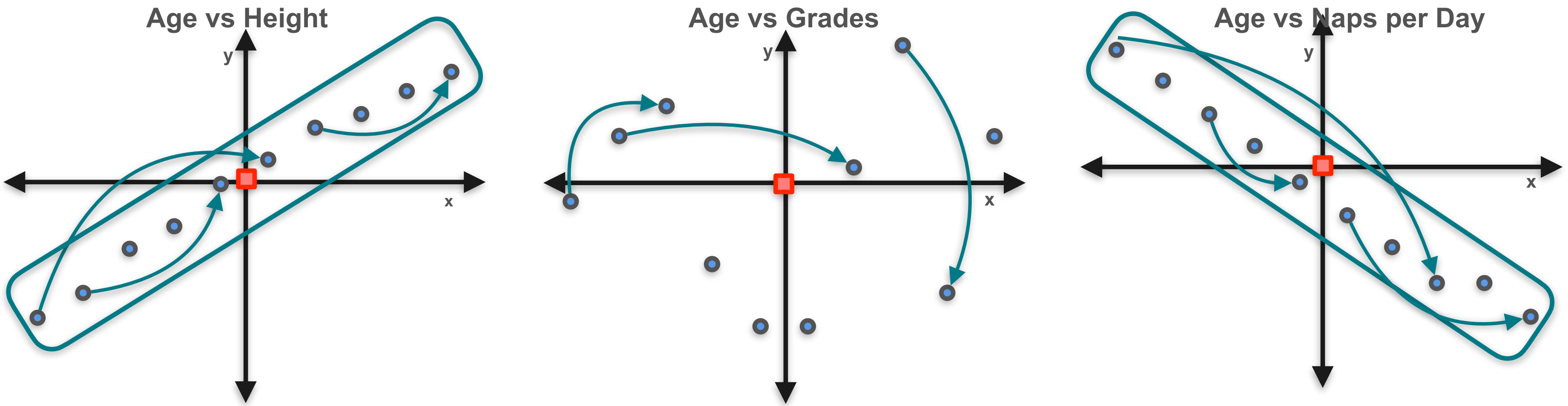


$$\mu_x = 0$$
$$\mu_y = 0$$
$$Var(X) = 1$$
$$Var(Y) = 1$$

$$\mu_x = 0$$
$$\mu_y = 0$$
$$Var(X) = 1$$
$$Var(Y) = 1$$

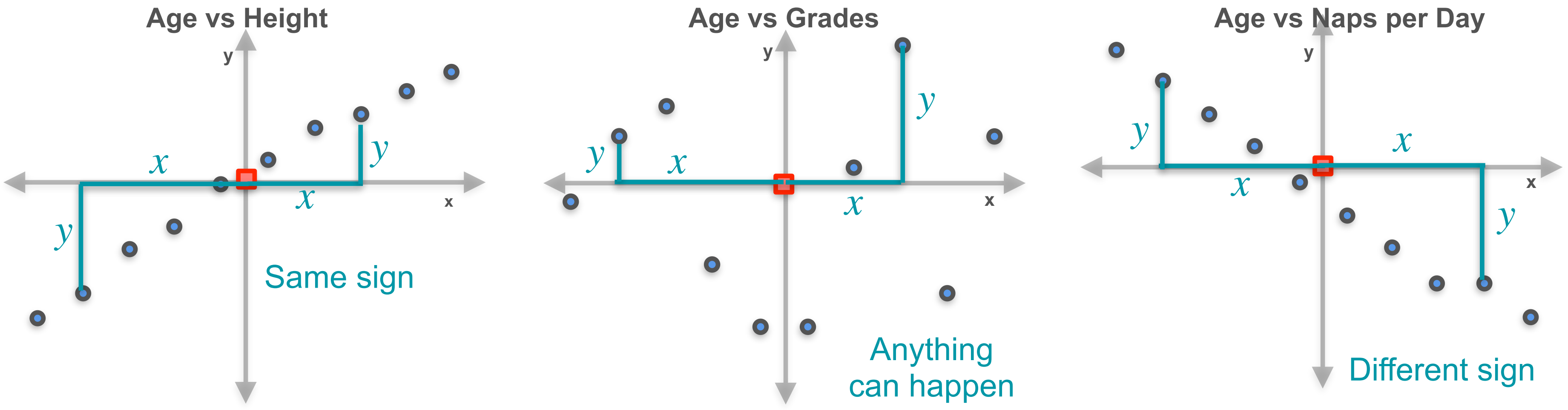
$$\mu_x = 0$$
$$\mu_y = 0$$
$$Var(X) = 1$$
$$Var(Y) = 1$$

Second Step: Notice Trend



Covariance

Positives and Negatives



$\sum xy > 0$

Positive

$\sum xy \approx 0$

Both positive and negative

$\sum xy < 0$

Negative

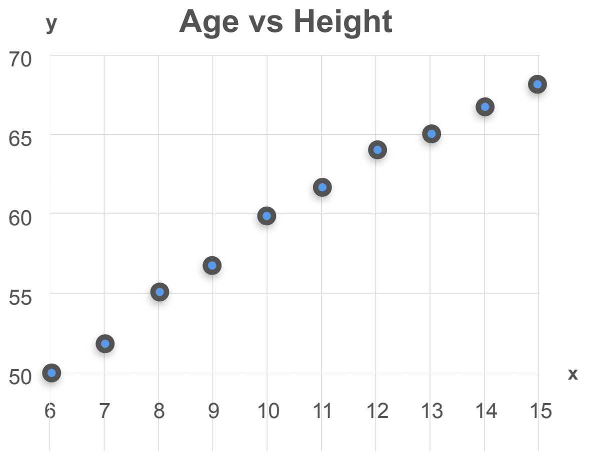
Covariance

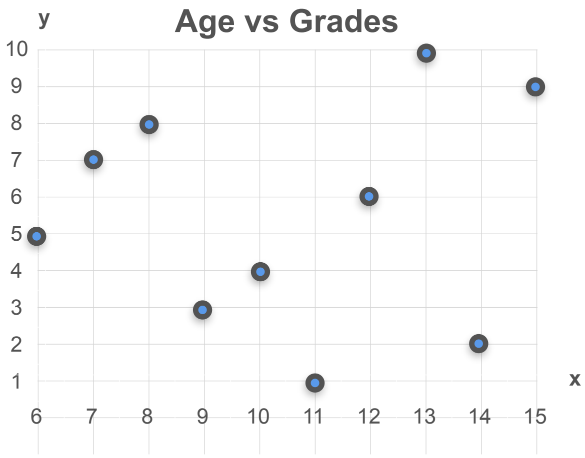
$$Cov(X, Y) = \sum xy$$

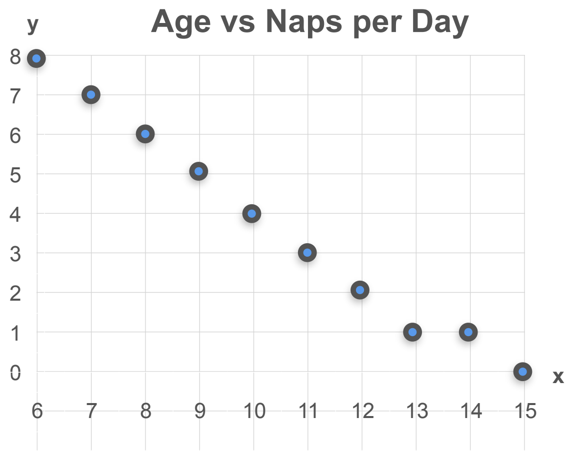
Almost…

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

Covariance







$$Cov(X, Y) > 0$$

$$Cov(X, Y) \approx 0$$

$$Cov(X, Y) < 0$$

Covariance Formula

Age vs Height

Covariance > 0

$\mu_x = 10.5 \quad \mu_y = 60$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

$$Cov(X, Y) = \frac{170}{10} = 17 > 0$$

Age (x)	Height (y)	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
6	50	-4.5	-10	45
7	52	-3.5	-8	28
8	55	-2.5	-5	12.5
9	57	-1.5	-3	4.5
10	60	-0.5	0	0
11	62	0.5	2	1
12	64	1.5	4	6
13	65	2.5	5	12.5
14	67	3.5	7	24.5
15	68	4.5	8	36

$\sum = 170$

Covariance Formula

Age vs Naps per Day

Covariance < 0

$\mu_x = 10.5 \quad \mu_y = 3.7$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

$$Cov(X, Y) = \frac{-74.5}{10} = -7.45 < 0$$

Age (x)	Naps (y)	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
6	8	-4.5	4.3	-19.35
7	7	-3.5	3.3	-11.55
8	6	-2.5	2.3	-5.75
9	5	-1.5	1.3	-1.95
10	4	-0.5	0.3	-0.15
11	3	0.5	-0.7	-0..35
12	2	1.5	-1.7	-2.55
13	1	2.5	-2.7	-6.75
14	1	3.5	-2.7	-9.45
15	0	4.5	-3.7	-16.65

$\sum$ = -74.5

Covariance Formula

Age vs Grades

Covariance ≈ 0

$\mu_x = 10.5 \quad \mu_y = 5$

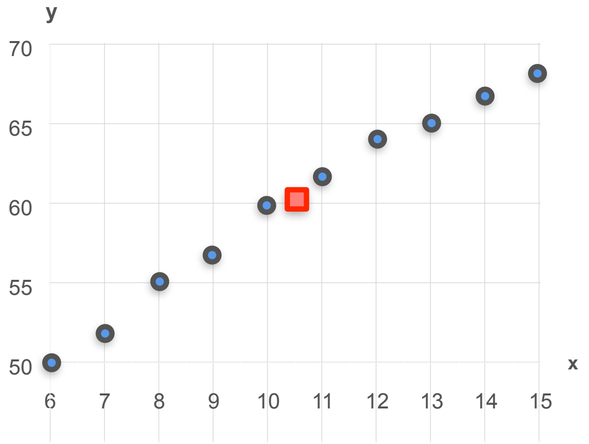
$$Cov(X,Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

$$Cov(X,Y) = \frac{1}{10} = 0.1 \quad \approx 0$$

Age (x)	Grades (y)	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
6	5	-4.5	0	0
7	7	-3.5	2	-7
8	8	-2.5	3	-7.5
9	3	-1.5	-2	3
10	1	-0.5	-4	2
11	1	0.5	-4	-2
12	6	1.5	1	1.5
13	10	2.5	5	12.5
14	2	3.5	-3	-10.5
15	7	4.5	2	9

$\sum = 1$

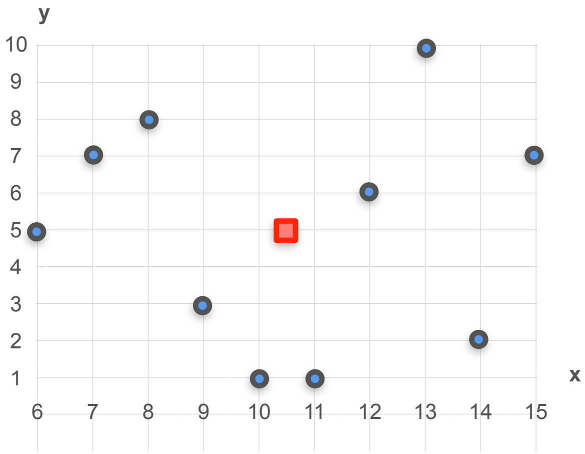
Comparing Correlations



Age vs Height

Covariance > 0

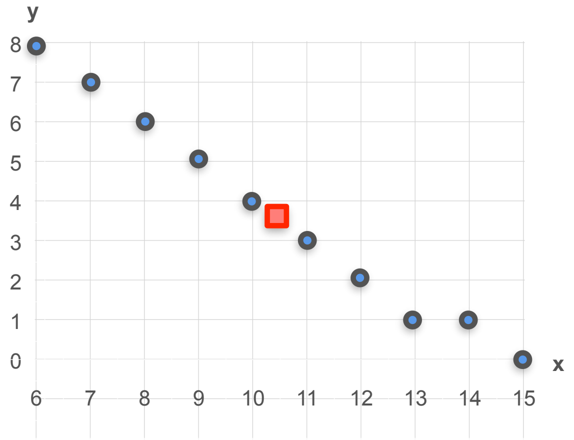
$$Cov(x, y) = 17$$



Age vs Grades

Covariance $\approx$ 0

$$Cov(x, y) = 0.1$$



Age vs Naps per Day

Covariance < 0

$$Cov(x, y) = -7.45$$

DeepLearning.AI

Probability Distributions with Multiple Variables

Covariance of a Probability Distribution

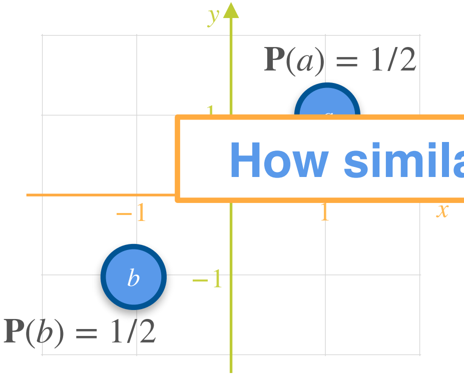
Covariance of a Probability Distribution: Motivation

X and Y are playing 3 games to either win or lose a dollar

GAME 1

a: Both players win \$1 each

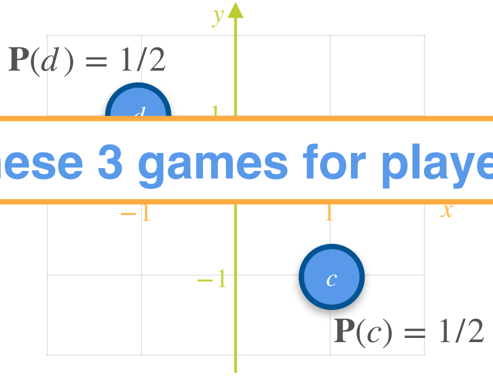
b: Both players lose \$1 each



GAME 2

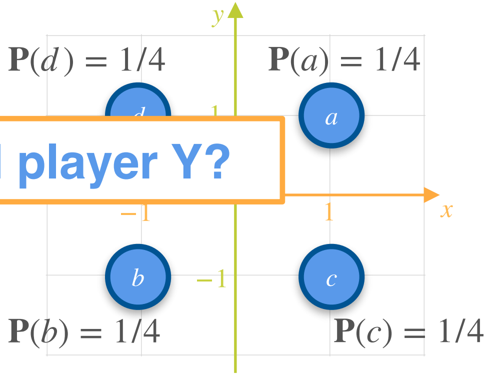
c: X wins \$1 and Y loses \$1

d: X loses \$1 and Y win \$1



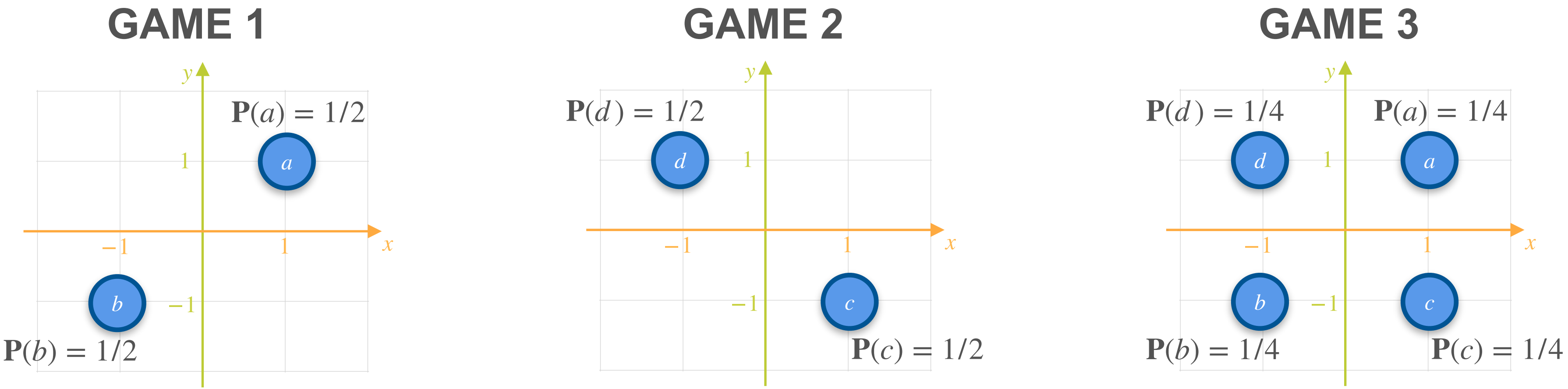
GAME 3

a, b, c or d



How similar are these 3 games for player X and player Y?

Covariance of a Probability Distribution: Motivation



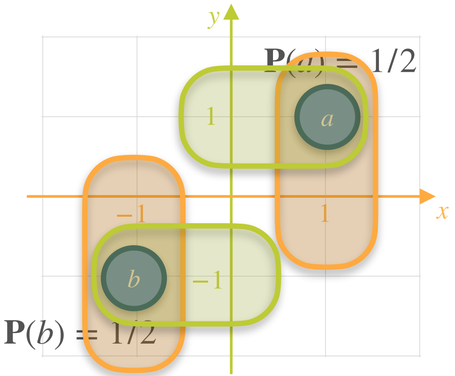
How similar are these 3 games for player X and player Y?

X : how much money in dollars player X wins

Y : how much money in dollars player Y wins

Covariance of a Probability Distribution: Motivation

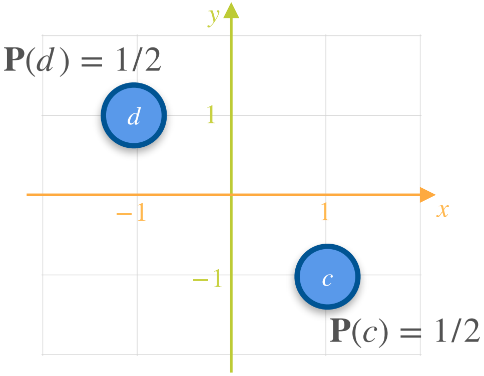
GAME 1



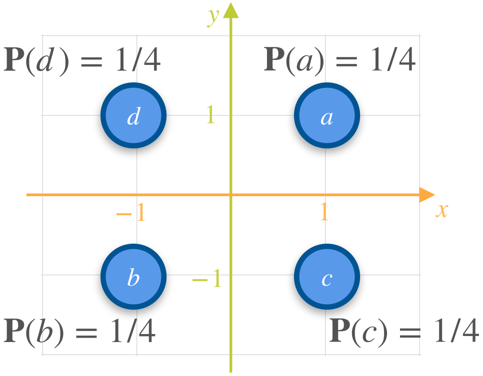
$$\mathbb{E}[X_1] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

$$\mathbb{E}[Y_1] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

GAME 2

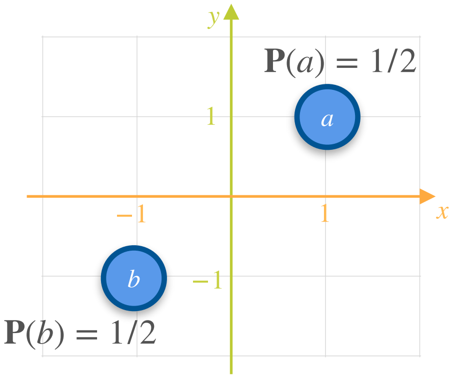


GAME 3

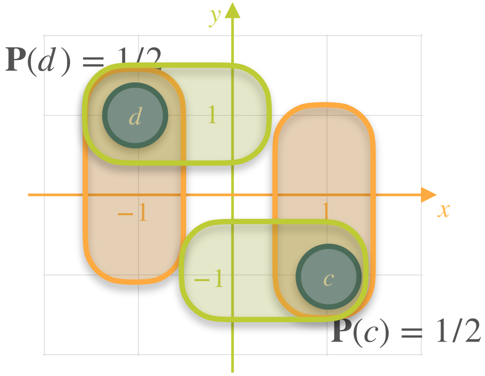


Covariance of a Probability Distribution: Motivation

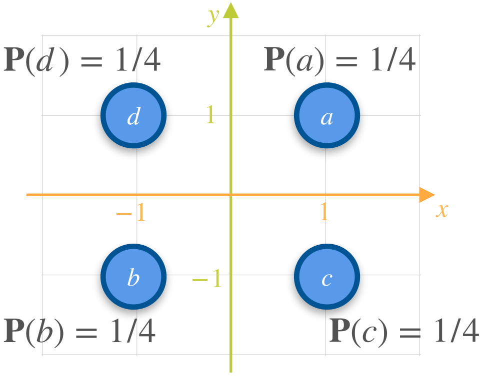
GAME 1



GAME 2



GAME 3



$$\mathbb{E}[X_1] = 0$$

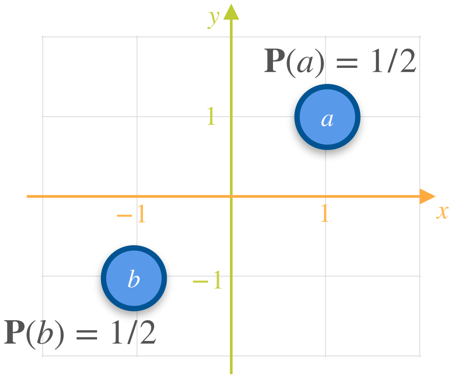
$$\mathbb{E}[Y_1] = 0$$

$$\mathbb{E}[X_2] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

$$\mathbb{E}[Y_2] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

Covariance of a Probability Distribution: Motivation

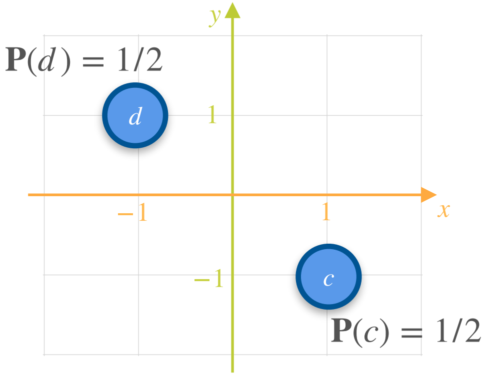
GAME 1



$$\mathbb{E}[X_1] = 0$$

$$\mathbb{E}[Y_1] = 0$$

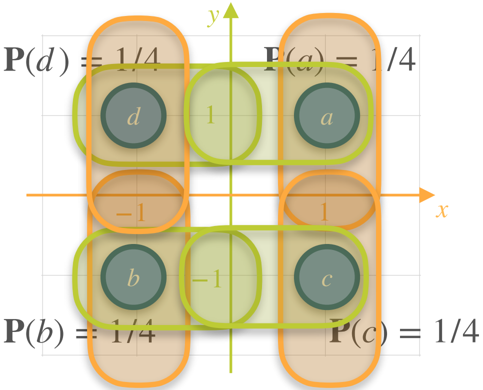
GAME 2



$$\mathbb{E}[X_2] = 0$$

$$\mathbb{E}[Y_2] = 0$$

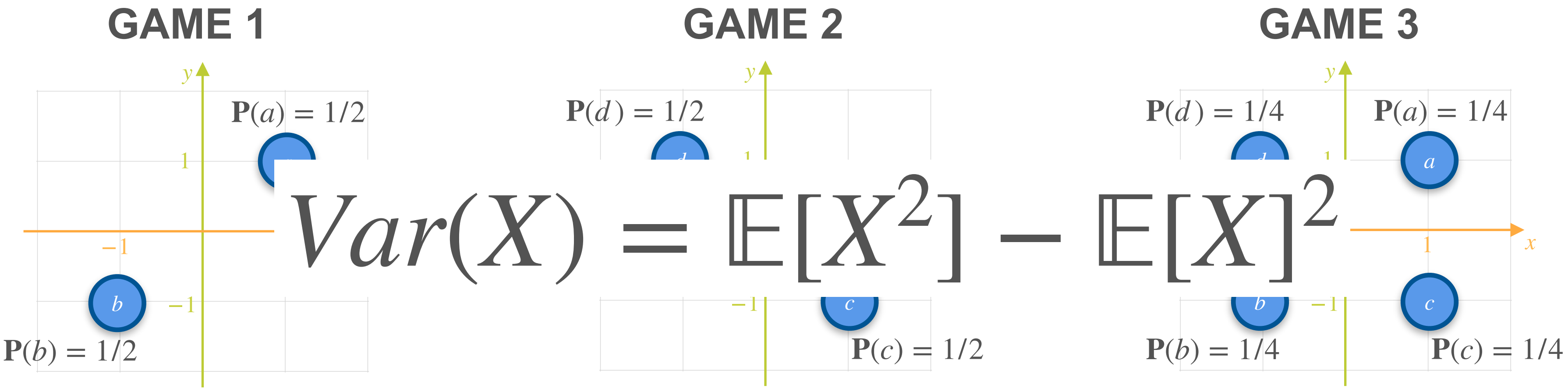
GAME 3



$$\mathbb{E}[X_3] = 2\left(\frac{1}{4}(1)\right) + 2\left(\frac{1}{4}(-1)\right) = 0$$

$$\mathbb{E}[Y_3] = 2\left(\frac{1}{4}(1)\right) + 2\left(\frac{1}{4}(-1)\right) = 0$$

Covariance of a Probability Distribution: Motivation



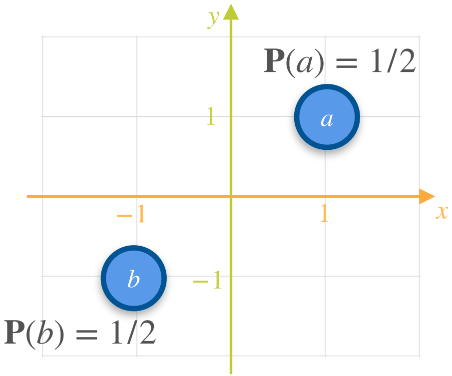
$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$\mathbb{E}[X_1] = 0$ $\mathbb{E}[X_2] = 0$ $\mathbb{E}[X_3] = 0$

$\mathbb{E}[Y_1] = 0$ $\mathbb{E}[Y_2] = 0$ $\mathbb{E}[Y_3] = 0$

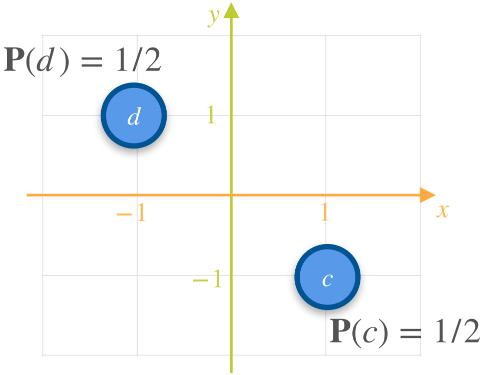
Covariance of a Probability Distribution: Motivation

GAME 1



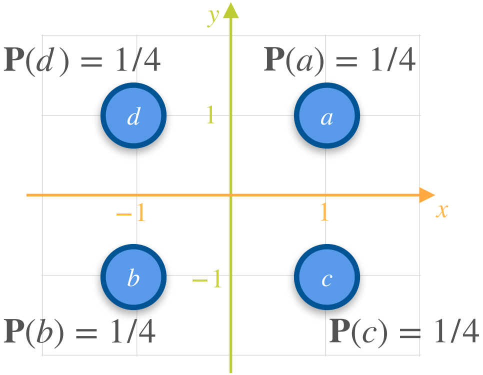
$\mathbb{E}[X_1] = 0 \quad \mathbb{E}[Y_1] = 0$

GAME 2



$\mathbb{E}[X_2] = 0 \quad \mathbb{E}[Y_2] = 0$

GAME 3

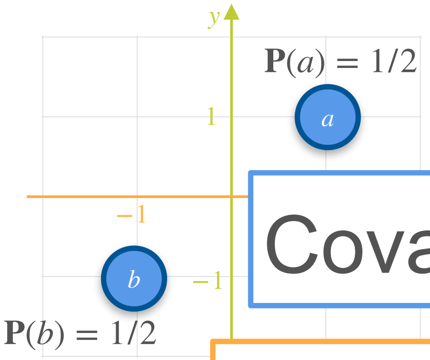


$\mathbb{E}[X_3] = 0 \quad \mathbb{E}[Y_3] = 0$

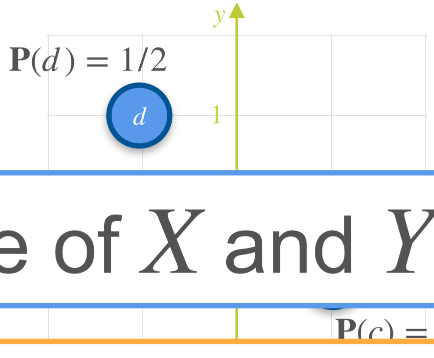
$$Var(X_1) = \mathbb{E}[X_1^2] - E[X_1]^2 = \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 - 0^2 = 1$$

Covariance of a Probability Distribution: Motivation

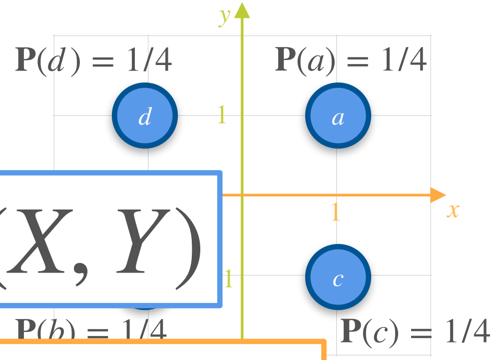
GAME 1



GAME 2



GAME 3



Covariance of X and Y: $Cov(X, Y)$

How similar are these 3 games for player X and player Y?

$\mathbb{E}[X_1] = 0 \quad \mathbb{E}[Y_1] = 0$ $\qquad$ $\mathbb{E}[X_2] = 0 \quad \mathbb{E}[Y_2] = 0$ $\qquad$ $\mathbb{E}[X_3] = 0 \quad \mathbb{E}[Y_3] = 0$

$Var(X_1) = 1$ $\qquad$ $Var(X_2) = 1$ $\qquad$ $Var(X_3) = 1$

$Var(Y_1) = 1$ $\qquad$ $Var(Y_2) = 1$ $\qquad$ $Var(Y_3) = 1$

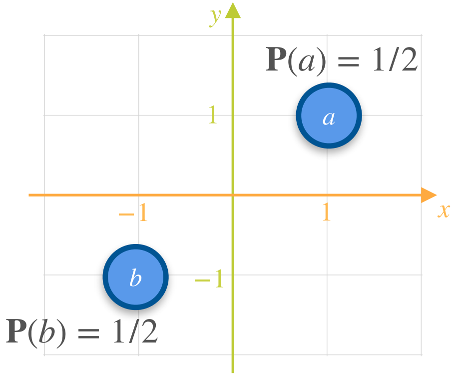
Covariance of a Probability Distribution: Motivation

$\boxed{\text{Covariance of } X \text{ and } Y\text{: } Cov(X, Y)}$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

Covariance of a Probability Distribution: Motivation

GAME 1



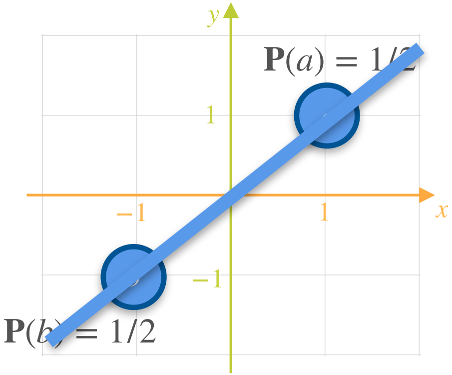
$$Cov(X,Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	1	1	1	1
-1	-1	-1	-1	1

$$Cov(X,Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{2}{2} = 1$$

Covariance of a Probability Distribution: Motivation

GAME 1



$$\mu_x = \mathbb{E}[X_1] = 0$$

$$\mu_y = \mathbb{E}[Y_1] = 0$$

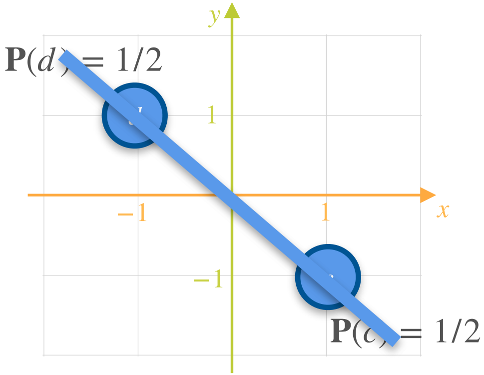
$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	1	1	1	1
-1	-1	-1	-1	1

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{2}{2} = 1$$

Covariance of a Probability Distribution: Motivation

GAME 2



$$\mu_x = \mathbb{E}[X_2] = 0$$

$$\mu_y = \mathbb{E}[Y_2] = 0$$

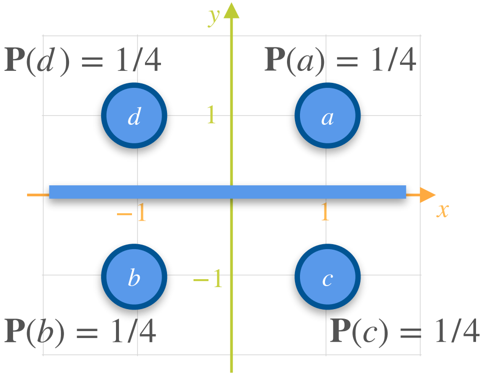
$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	-1	1	-1	-1
-1	1	-1	1	-1

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{-2}{2} = -1$$

Covariance of a Probability Distribution: Motivation

GAME 3



$$\mu_x = \mathbb{E}[X_3] = 0$$

$$\mu_y = \mathbb{E}[Y_3] = 0$$

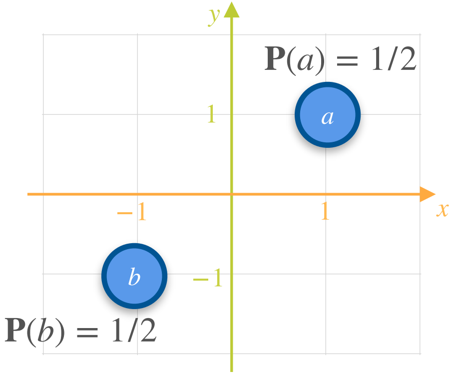
$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	1	1	1	1
1	-1	-1	1	-1
-1	1	1	-1	-1
-1	-1	-1	-1	1

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{0}{4} = 0$$

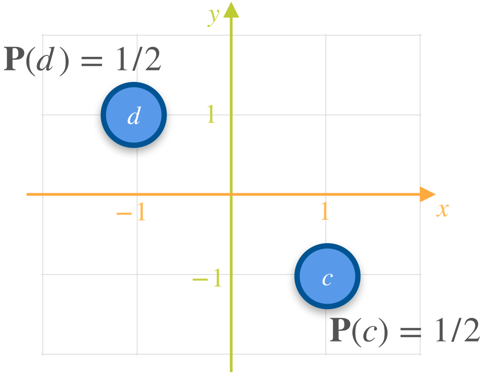
Covariance of a Probability Distribution: Motivation

GAME 1



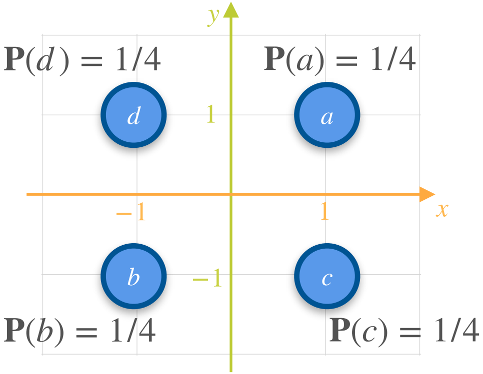
$$Cov(X, Y) = 1$$

GAME 2



$$Cov(X, Y) = -1$$

GAME 3



$$Cov(X, Y) = 0$$

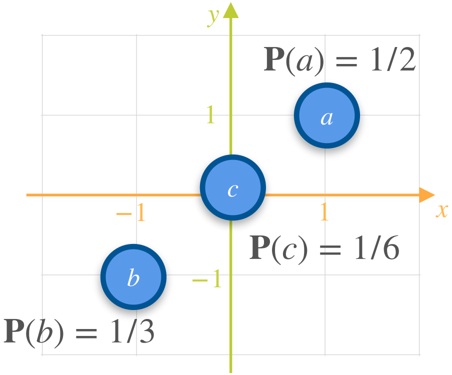
Covariance of a Probability Distribution: Motivation

GAME 4

a: Both players win \$1 each $\mathbf{P}(a) = 1/2$

b: Both players lose \$1 each $\mathbf{P}(b) = 1/3$

c: Neither players wins nor lose anything $\mathbf{P}(c) = 1/6$



Unequal Probabiliites

$$\mathbb{E}[X_4] = \frac{1}{2}(1) + \frac{1}{6}(0) + \frac{1}{3}(-1) = \frac{1}{6}$$

$$\mathbb{E}[Y_4] = \frac{1}{2}(1) + \frac{1}{6}(0) + \frac{1}{3}(-1) = \frac{1}{6}$$

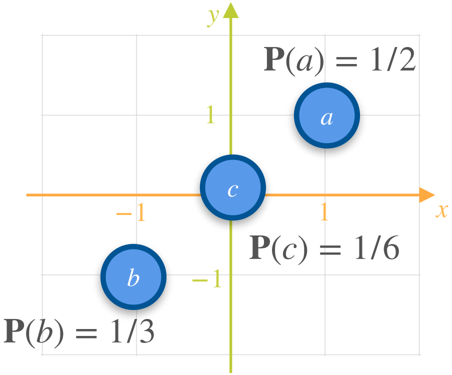
Covariance of a Probability Distribution: Motivation

GAME 4

a: Both players win \$1 each $\mathbf{P}(a) = 1/2$

b: Both players lose \$1 each $\mathbf{P}(b) = 1/3$

c: Neither players wins nor lose anything $\mathbf{P}(c) = 1/6$



$$\mathbb{E}[X_4] = \frac{1}{6}$$

$$\mathbb{E}[Y_4] = \frac{1}{6}$$

Unequal Probabiliites

$$Var(X_4) = \sum_{i=1}^{N} (x_i - \mu_x)^2 \cdot \mathbf{P}(x_i)$$

$$Var(X_4) = \frac{1}{2}(1 - \frac{1}{6})^2 + \frac{1}{6}(0 - \frac{1}{6})^2 + \frac{1}{3}(- 1) - \frac{1}{6})^2$$

$$= \; 0.806$$

Covariance of a Probability Distribution: Motivation

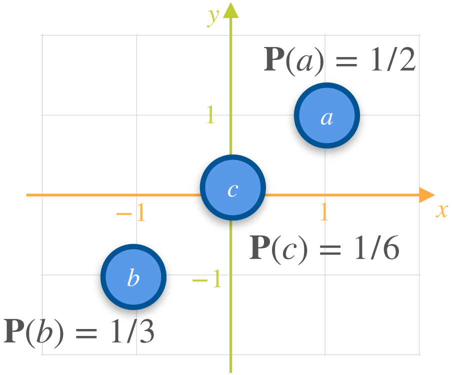
GAME 4

Unequal Probabiliites

a: Both players win \$1 each $\mathbf{P}(a) = 1/2$

b: Both players lose \$1 each $\mathbf{P}(b) = 1/3$

c: Neither players wins nor lose anything $\mathbf{P}(c) = 1/6$



$$\mathbb{E}[X_4] = \frac{1}{6} \qquad Var(X_4) = 0.806$$

$$\mathbb{E}[Y_4] = \frac{1}{6} \qquad Var(Y_4) = 0.806$$

$$\boxed{Cov(X, Y) = ?}$$

Covariance of Probability Distributions

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \boxed{\frac{1}{n}} \sum (x_i - \mu_x)(y_i - \mu_y)$$

equal probabilities

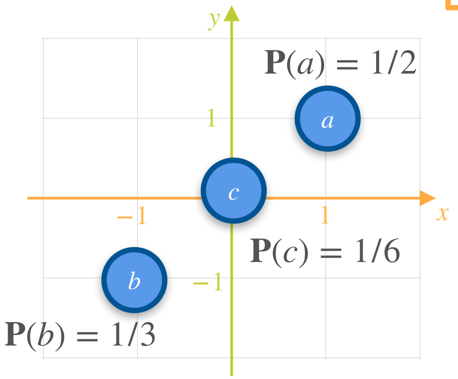
$$Cov(X, Y) = \sum \boxed{p_{XY}(x_i, y_i)} (x_i - \mu_x)(y_i - \mu_y)$$

unequal probabilities

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Covariance of a Probability Distribution: Motivation

GAME 4



$$\mu_x = \mu_y = \mathbb{E}[X_4] = \mathbb{E}[Y_4] = 1/6$$

$$Var(X_4) = Var(Y_4) = 0.806$$

Unequal Probabiliites

$$Cov(X, Y) = \sum p_{XY}(x_i, y_i)\boxed{(x_i - \mu_x)(y_i - \mu_y)} = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \frac{1}{2}\boxed{\left(1 - \frac{1}{6}\right)^2} + \frac{1}{6}\left(0 - \frac{1}{6}\right)^2 + \frac{1}{3}\left(-1) - \frac{1}{6}\right)^2$$

$$Cov(X, Y) = 0.806$$

Covariance?

$$\mathbb{E}(X) = 4.903$$

$$\mathbb{E}(Y) = 5.280$$

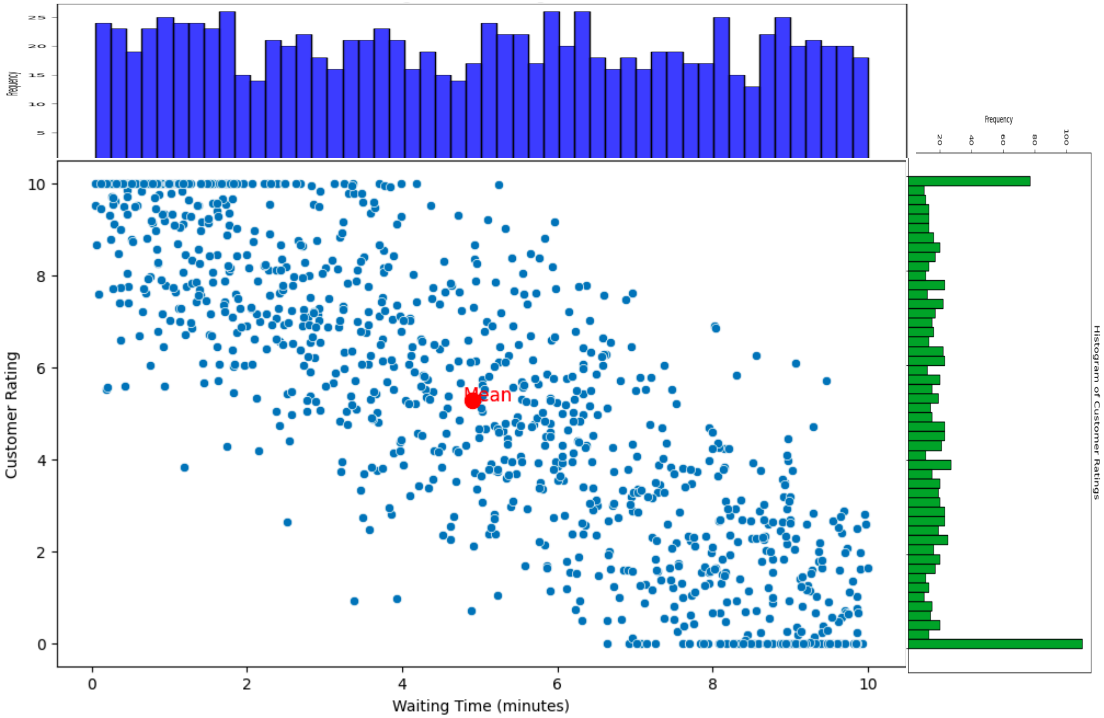
$$\mathbb{E}(XY) = 18.014$$

$$Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$Cov(X, Y) = -7.878$$



Covariance of a Joint Continuous Distribution

$$Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\mathbb{E}(X) = 4.903$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$\mathbb{E}(Y) = 5.280$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$\mathbb{E}(XY) = 18.014$$

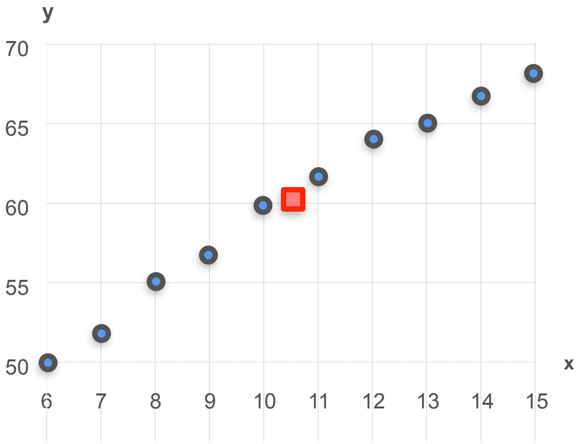
$$Cov(X, Y) = -7.878$$

Probability Distributions with Multiple Variables

DeepLearning.AI

Covariance Matrix

Covariance Matrix

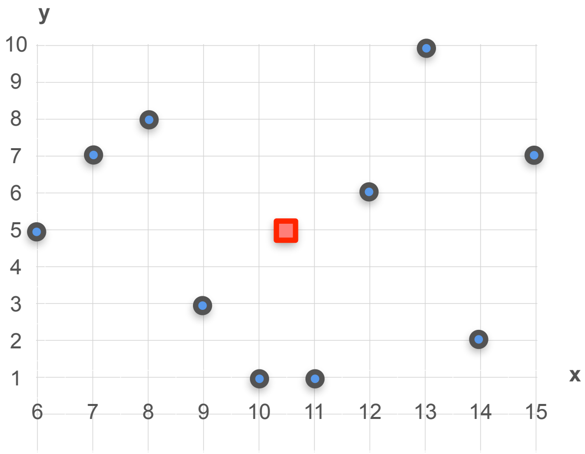


Age vs Height

$$Var(X) = 9.17$$

$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$

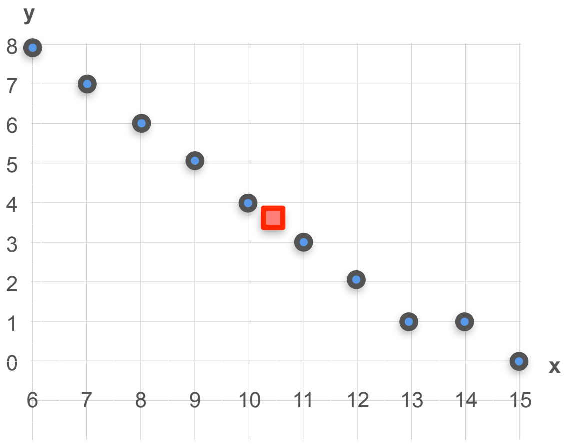


Age vs Grades

$$Var(X) = 9.17$$

$$Var(Y) = 9.78$$

$$Cov(X, Y) = 0.1$$



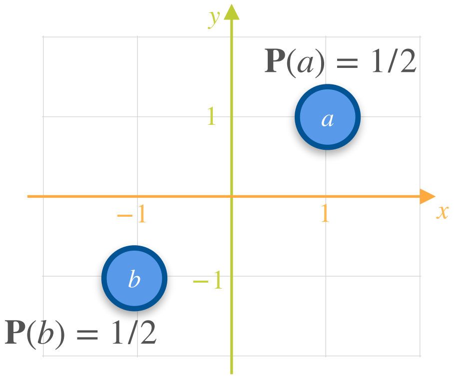
Age vs Naps per Day

$$Var(X) = 9.17$$

$$Var(Y) = 7.57$$

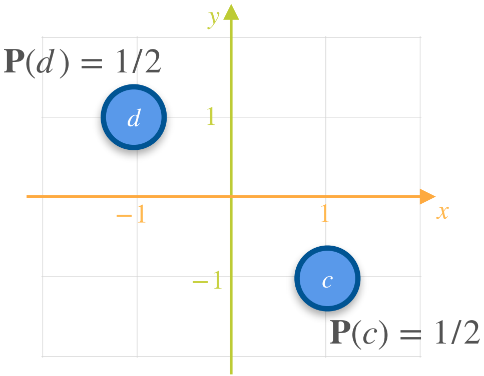
$$Cov(X, Y) = -7.45$$

Covariance Matrix



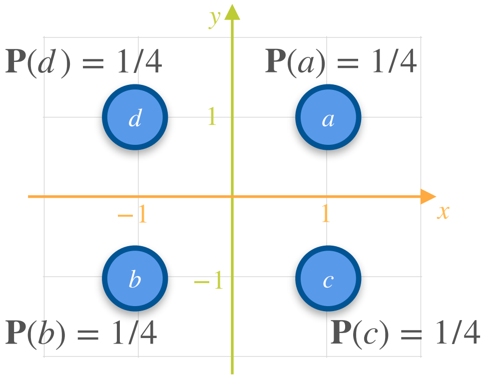
$$Var(X) = Var(Y) = 1$$

$$Cov(X, Y) = 1$$



$$Var(X) = Var(Y) = 1$$

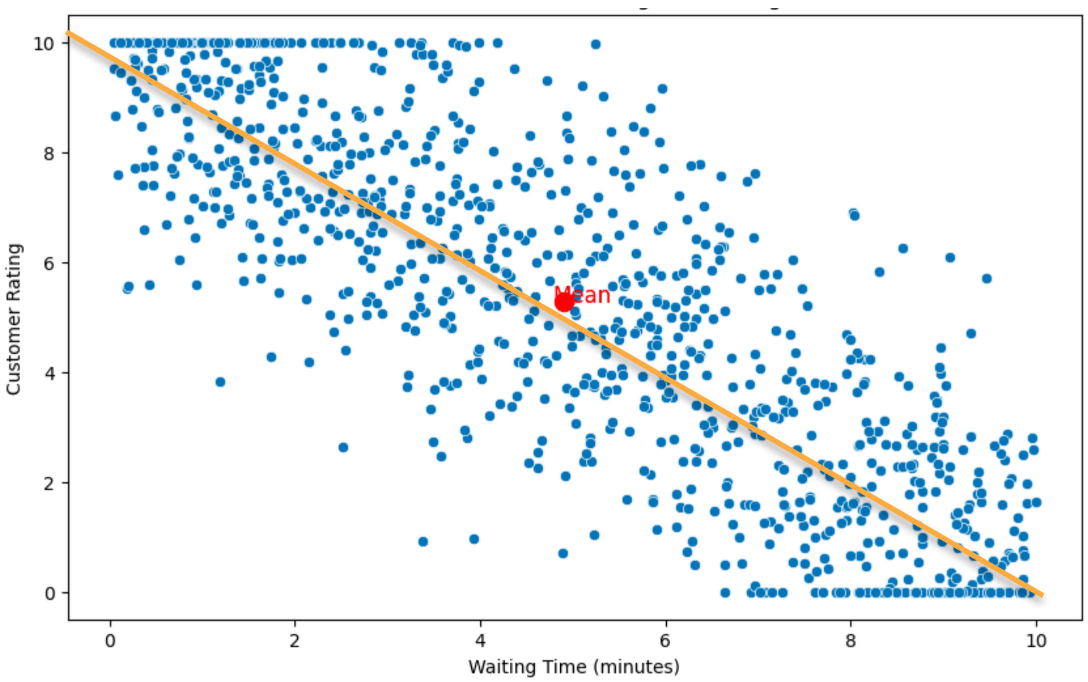
$$Cov(X, Y) = -1$$



$$Var(X) = Var(Y) = 1$$

$$Cov(X, Y) = 0$$

Covariance Matrix

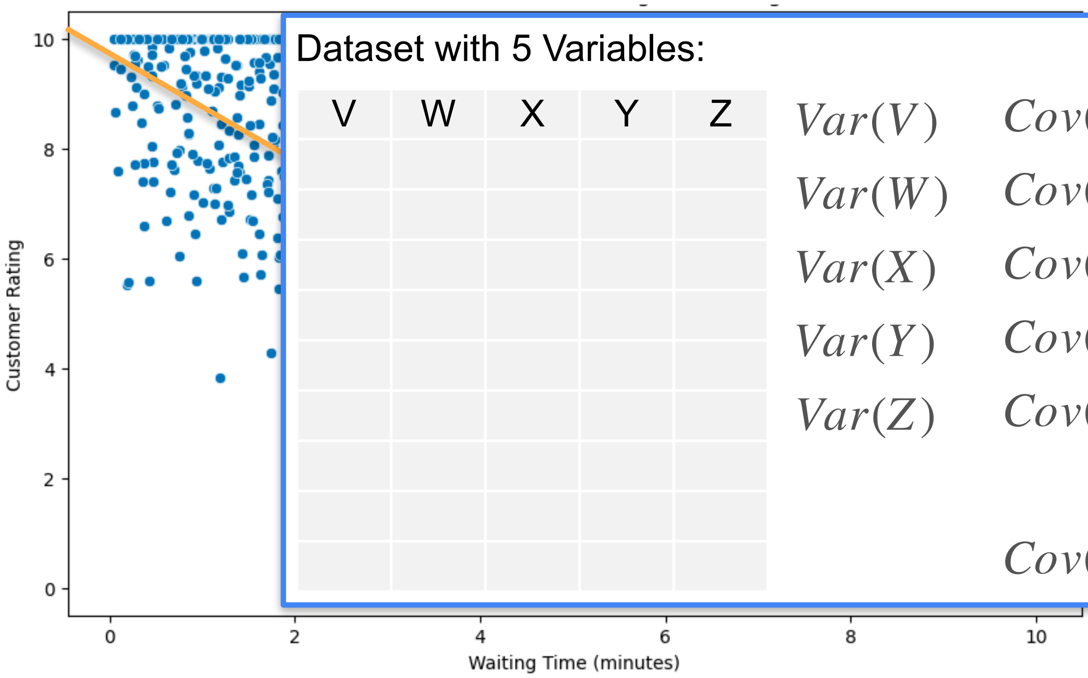


$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

$$Cov(X, Y) = -7.878$$

Covariance Matrix



Dataset with 5 Variables:

V	W	X	Y	Z

$Var(V)$

$Var(W)$

$Var(X)$

$Var(Y)$

$Var(Z)$

$Cov(V, W)$

$Cov(V, X)$

$Cov(V, Y)$

$Cov(V, Z)$

$Cov(W, X)$

$\vdots$

$Cov(Y, Z)$

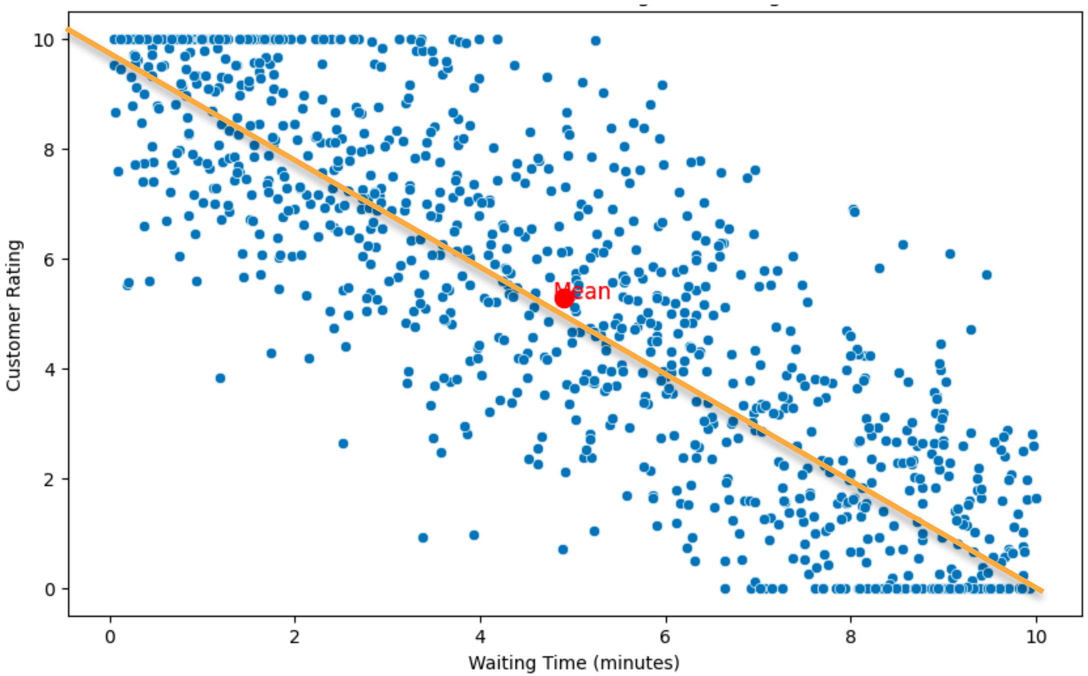
$ar(X) = 8.526$

$ar(Y) = 10.163$

$ov(X, Y) = -7.878$

DeepLearning.AI

Covariance Matrix



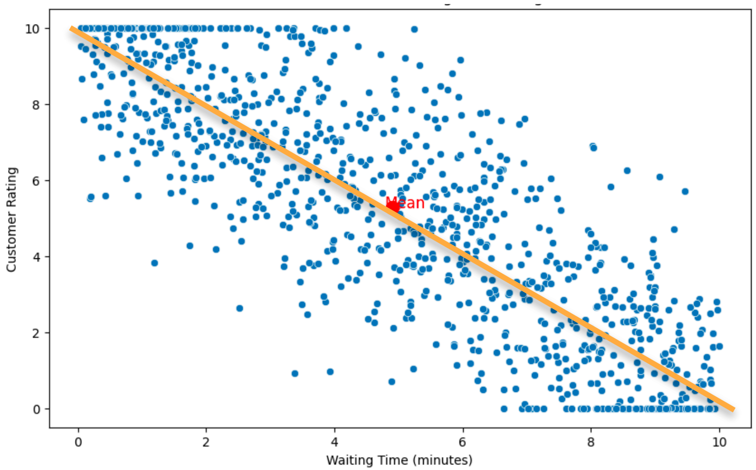
$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

$$Cov(X, Y) = -7.878$$

Covariance Matrix

Covariance Matrix



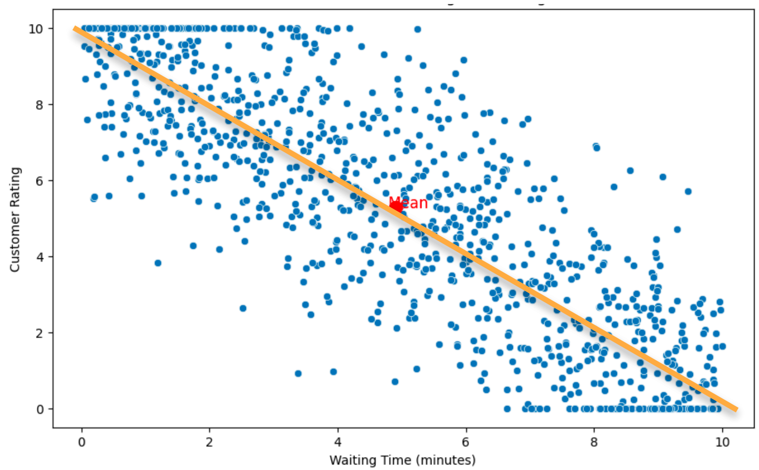
$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

$$Cov(X, Y) = -7.878$$

	X	Y
X	$Var(X)$	$Cov(X, Y)$
Y	$Cov(X, Y)$	$Var(Y)$

Covariance Matrix



$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

$$Cov(X, Y) = -7.878$$

	X	Y
X	$Var(X)$	$Cov(X, Y)$
Y	$Cov(X, Y)$	$Var(Y)$

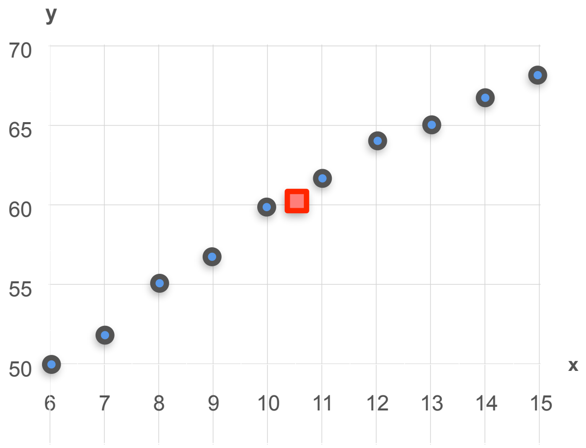
$$\begin{bmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{bmatrix}$$

8.526	-7.878
-7.878	10.163

$$\begin{bmatrix} 8.534 & -7.878 \\ -7.878 & 10.173 \end{bmatrix}$$

Covariance Matrix

Covariance Matrix



Age vs Height

$$Var(X) = 9.17$$

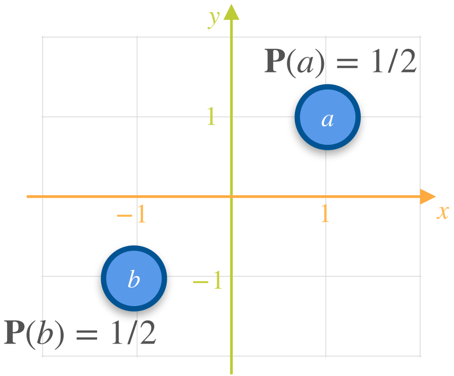
$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$

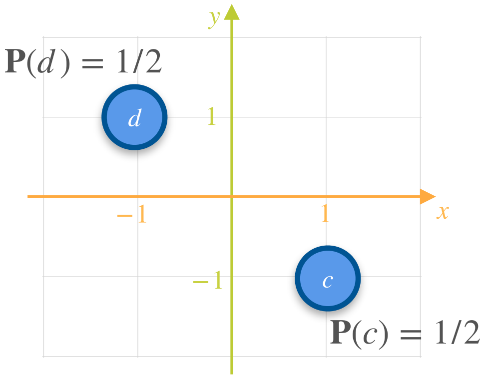
$$\begin{bmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{bmatrix}$$

$$\begin{bmatrix} 9.17 & 17 \\ 17 & 39.56 \end{bmatrix}$$

Covariance Matrix



$$\begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$$



$Var(X) = 1$

$Var(Y) = 1$

$Cov(X,Y) = 1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$Var(X) = 1$

$Var(Y) = 1$

$Cov(X,Y) = -1$

Covariance of a Joint Continuous Distribution

Dataset with 3 Variables:

X	Y	Z

$Var(X)$ $\quad$ $Cov(X, Y)$

$Var(Y)$ $\quad$ $Cov(X, Z)$

$Var(Z)$ $\quad$ $Cov(Y, Z)$

	X	Y	Z
X	$Var(X)$	$Cov(X, Y)$	$Cov(X, Z)$
Y	$Cov(X, Y)$	$Var(Y)$	$Cov(Y, Z)$
Z	$Cov(X, Z)$	$Cov(Y, Z)$	$Var(Z)$

Covariance of a Joint Continuous Distribution

Dataset with 5 Variables:

V	W	X	Y	Z

$Var(V)$ $Cov(V,W)$

$Var(W)$ $Cov(V,X)$

$Var(X)$ $Cov(V,Y)$

$Var(Y)$ $Cov(V,Z)$

$Var(Z)$ $Cov(W,X)$

$\vdots$

$Cov(Y,Z)$

	V	W	X	Y	Z
V	$Var(V)$	$Cov(V,W)$	$Cov(V,X)$	$Cov(V,Y)$	$Cov(V,Z)$
W	$Cov(V,W)$	$Var(W)$	$Cov(W,X)$	$Cov(W,Y)$	$Cov(W,Z)$
X	$Cov(V,X)$	$Cov(W,X)$	$Var(X)$	$Cov(X,Y)$	$Cov(X,Z)$
Y	$Cov(V,Y)$	$Cov(W,Y)$	$Cov(X,Y)$	$Var(Y)$	$Cov(Y,Z)$
Z	$Cov(V,Z)$	$Cov(W,Z)$	$Cov(X,Z)$	$Cov(Y,Z)$	$Var(Z)$

Covariance of a Joint Continuous Distribution

$$\Sigma = \begin{array}{c|ccccc} & V & W & X & Y & Z \\ \hline V & Var(V) & Cov(V,W) & Cov(V,X) & Cov(V,Y) & Cov(V,Z) \\ W & Cov(V,W) & Var(W) & Cov(W,X) & Cov(W,Y) & Cov(W,Z) \\ X & Cov(V,X) & Cov(W,X) & Var(X) & Cov(X,Y) & Cov(X,Z) \\ Y & Cov(V,Y) & Cov(W,Y) & Cov(X,Y) & Var(Y) & Cov(Y,Z) \\ Z & Cov(V,Z) & Cov(W,Z) & Cov(X,Z) & Cov(Y,Z) & Var(Z) \end{array}$$

Covariance Matrix

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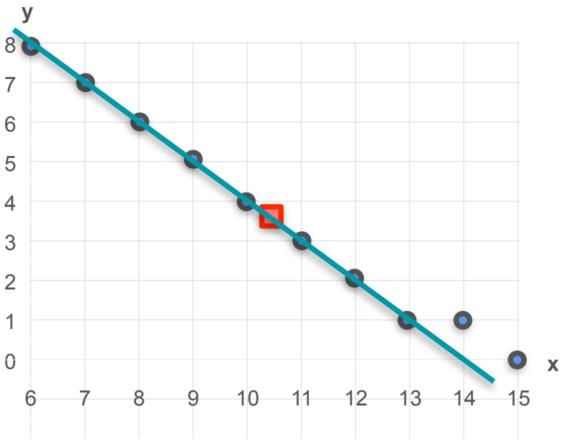
Probability Distributions with Multiple Variables

Correlation Coefficient

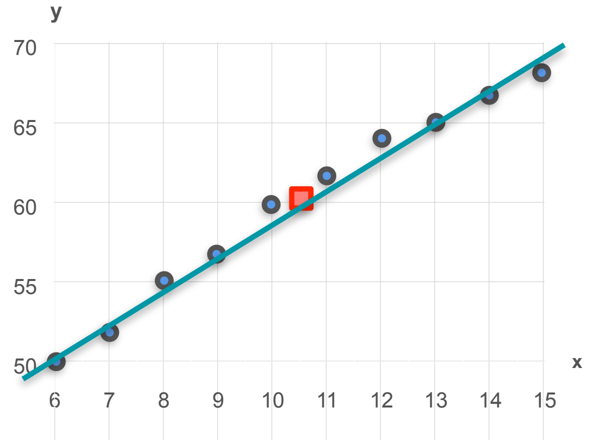
Correlation Coefficient



Age vs Naps per Day

$$Var(X) = 9.17$$
$$Var(Y) = 7.57$$
$$\boxed{Cov(X, Y) = -7.45}$$

Is the correl
here stro ?



Age vs Height

$$Var(X) = 9.17$$
$$Var(Y) = 39.56$$
$$\boxed{Cov(X, Y) = 17}$$

Correlation Coefficient

Age vs Naps per Day

$Var(X) = 9.17$

$Var(Y) = 7.57$

$Cov(X, Y) = -7.45$

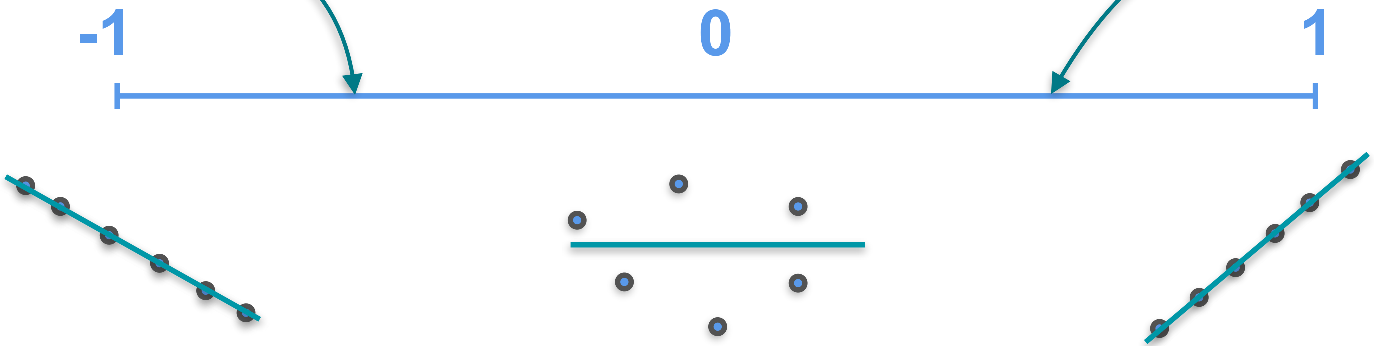
Correlation Coefficient

Age vs Height

$Var(X) = 9.17$

$Var(Y) = 39.56$

$Cov(X, Y) = 17$



Correlation Coefficient

Age vs Naps per Day

$Var(X) = 9.17$

$Var(Y) = 7.57$

$Cov(X, Y) = -7.45$

Age vs Height

$Var(X) = 9.17$

$Var(Y) = 39.56$

$Cov(X, Y) = 17$

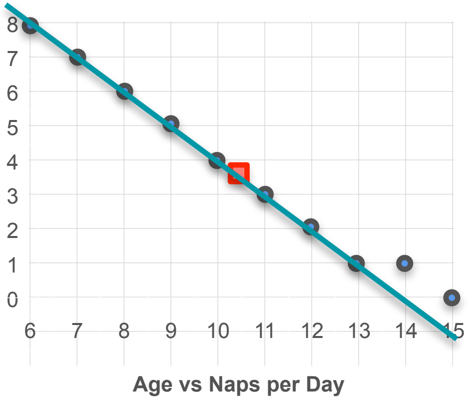
$$\textbf{Correlation Coefficient} = \frac{Cov(X, Y)}{\sigma_x \cdot \sigma_y} = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

Correlation Coefficient

Age vs Naps per Day

$$Var(X) = 9.17$$
$$Var(Y) = 7.57$$

$$Cov(X, Y) = -7.45$$



Correlation Coefficient $= \dfrac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$

$$= \frac{-7.45}{\sqrt{9.17} \cdot \sqrt{7.57}}$$

$$\approx -0.894$$

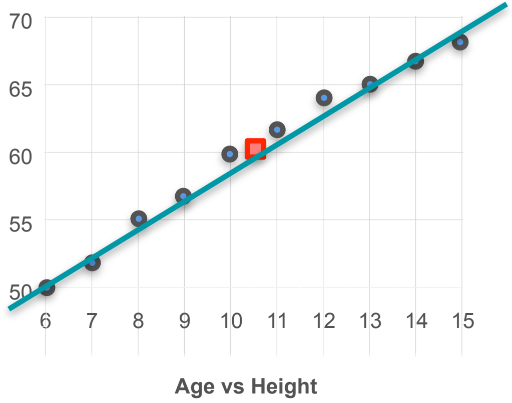
Correlation Coefficient

Age vs Height

$$Var(X) = 9.17$$

$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$



$$\text{Correlation Coefficient} = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

$$= \frac{17}{\sqrt{9.17} \cdot \sqrt{39.56}}$$

$$\approx 0.893$$

Correlation Coefficient

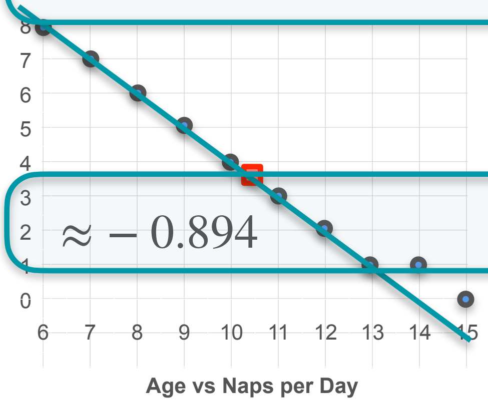
Age vs Naps per Day

$$Var(X) = 9.17$$
$$Var(Y) = 7.57$$

Age vs Height

$$Var(X) = 9.17$$
$$Var(Y) = 39.56$$

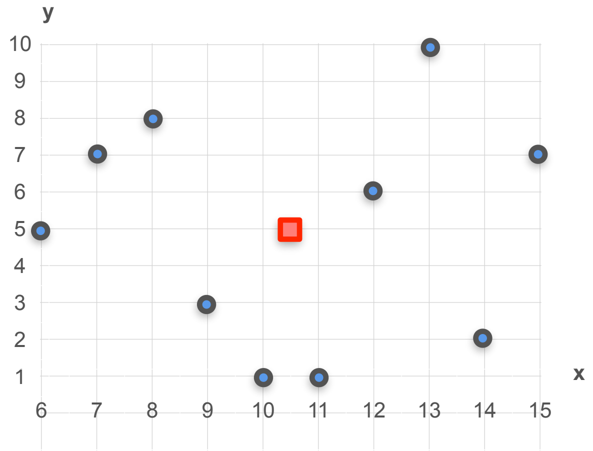
$$Cov(X, Y) = -7.45 \qquad Cov(X, Y) = 17$$



$$\approx -0.894$$

$$\approx 0.893$$

Correlation Coefficient



Age vs Grades

$$Var(X) = 9.17$$

$$Var(Y) = 9.78$$

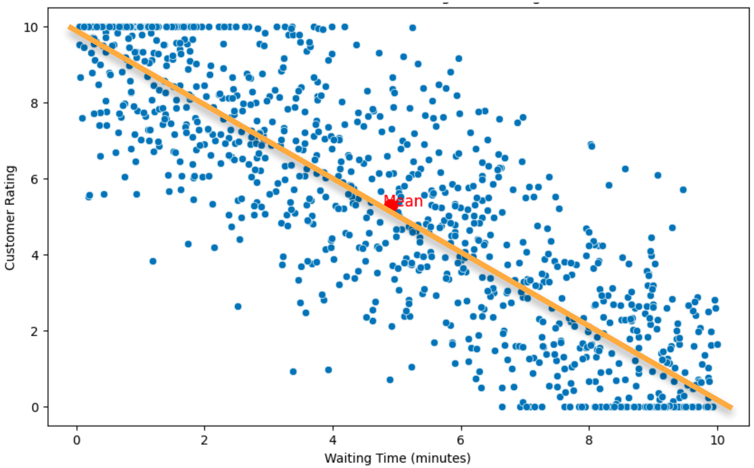
$$Cov(X, Y) = 0.1$$

$$\text{Correlation Coefficient} = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

$$= \frac{0.1}{\sqrt{9.17} \cdot \sqrt{9.78}}$$

$$\approx 0.01$$

Correlation Coefficient



$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

$$Cov(X, Y) = -7.878$$

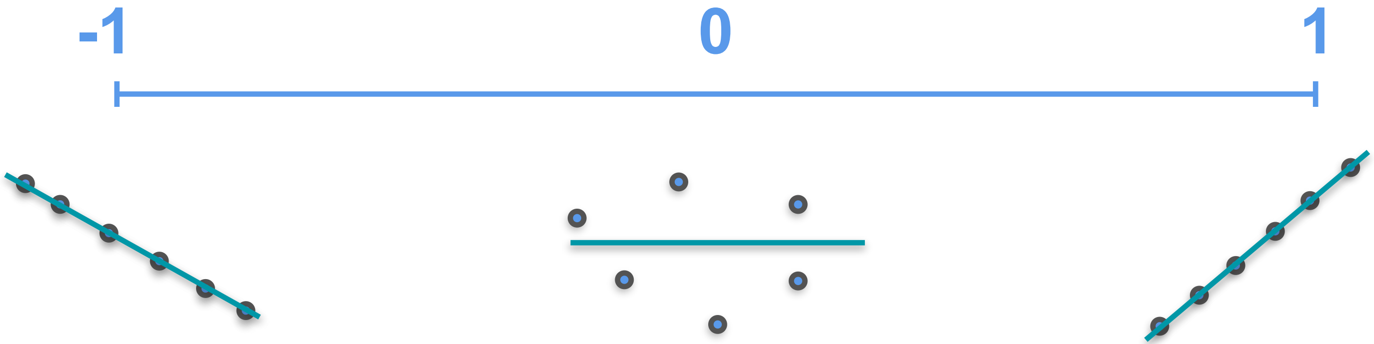
Correlation Coefficient $= \dfrac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$

$$= \frac{-7.878}{\sqrt{8.562} \cdot \sqrt{10.163}}$$

$$\approx -0.845$$

Correlation Coefficient

$$\text{Correlation Coefficient} = \frac{Cov(X,Y)}{\sigma_x \cdot \sigma_y} = \frac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$



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Probability Distributions with Multiple Variables

Multivariate Gaussian Distribution

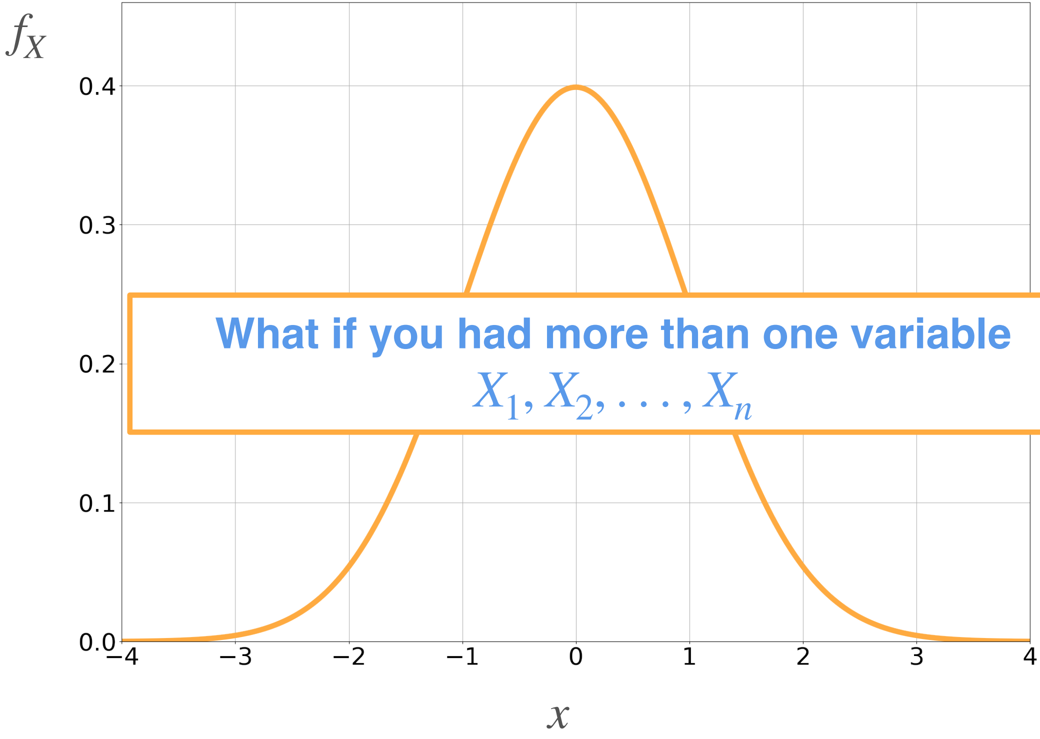
Multivariate Gaussian Distribution

For a single variable, X

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Parameters:

- μ: center of the bell
- σ: spread of the bell



What if you had more than one variable

$X_1, X_2, \ldots, X_n$

Multivariate Gaussian Distribution: an Example

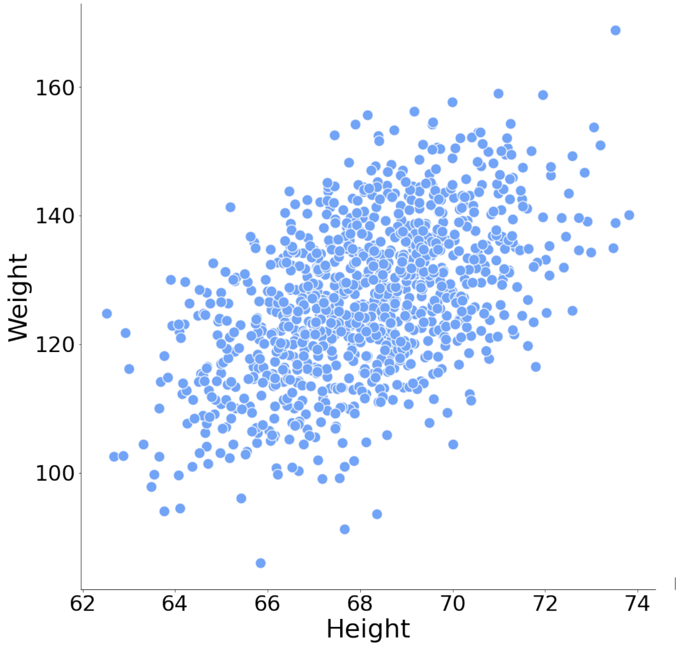
Two variables

H : Height of an adult in inches

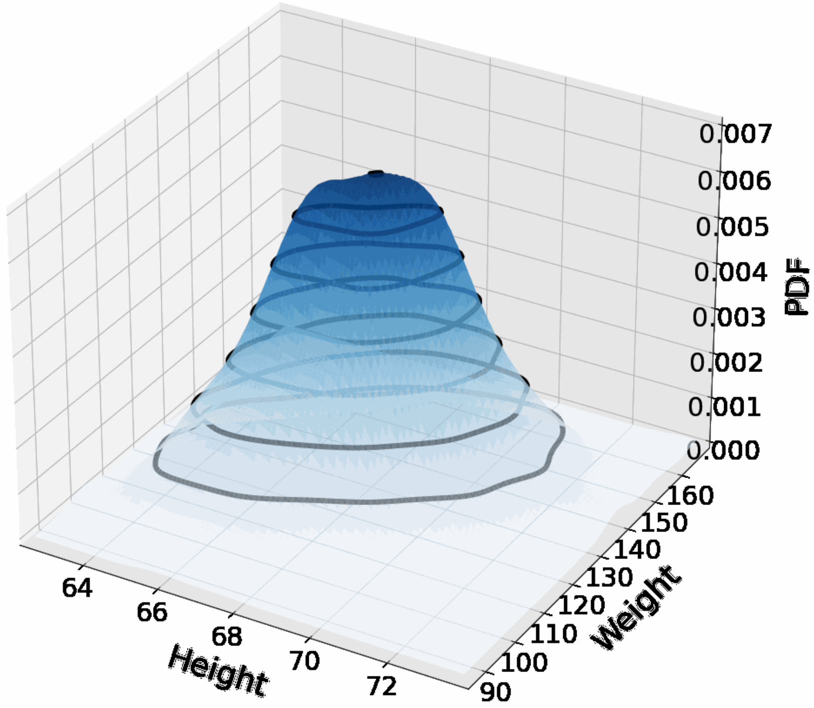
$$H \sim \mathcal{N}(\mu_H, \sigma_H)$$

W : Weight of an adult in pounds

$$W \sim \mathcal{N}(\mu_W, \sigma_W)$$



Multivariate Gaussian Distribution: an Example



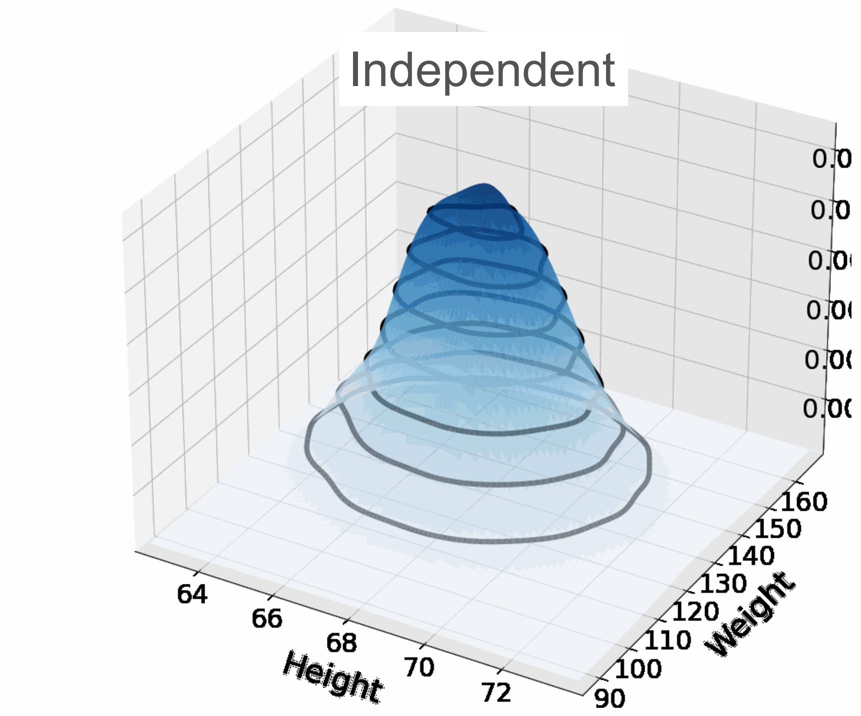
If W, H were independent

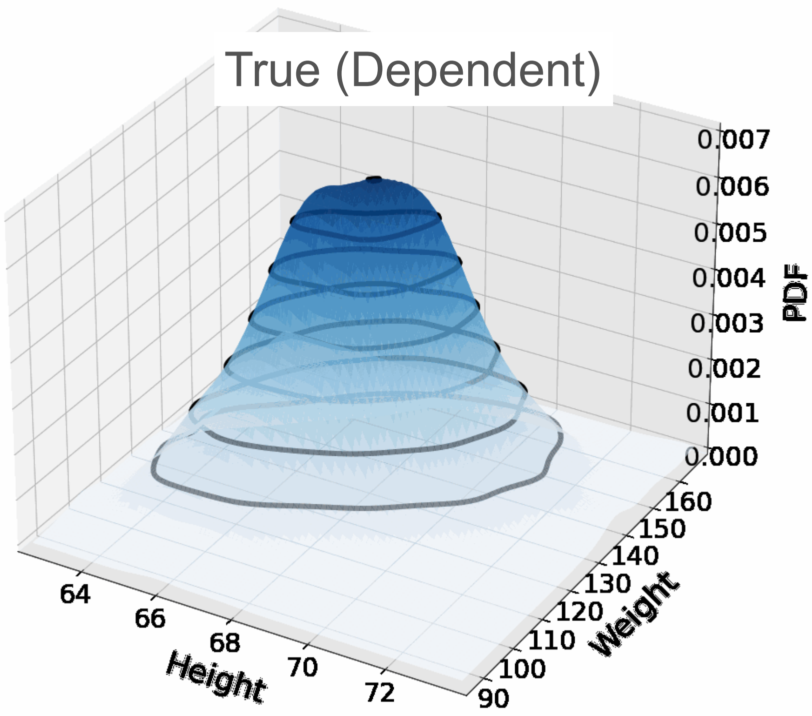
$$f_{HW}(h, w) = f_H(h) f_W(w)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_H} e^{-\frac{1}{2}\frac{(h-\mu_H)^2}{\sigma_H^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{1}{2}\frac{(w-\mu_W)^2}{\sigma_W^2}}$$

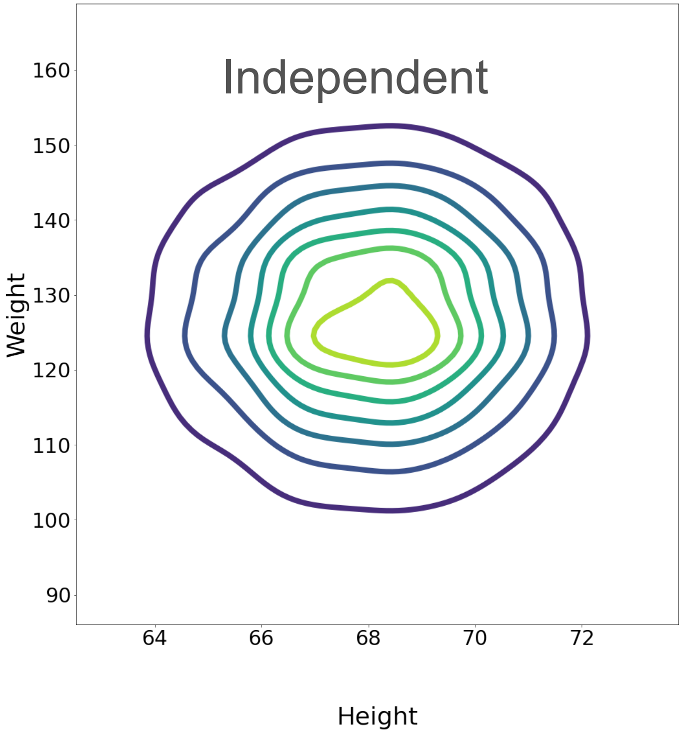
$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

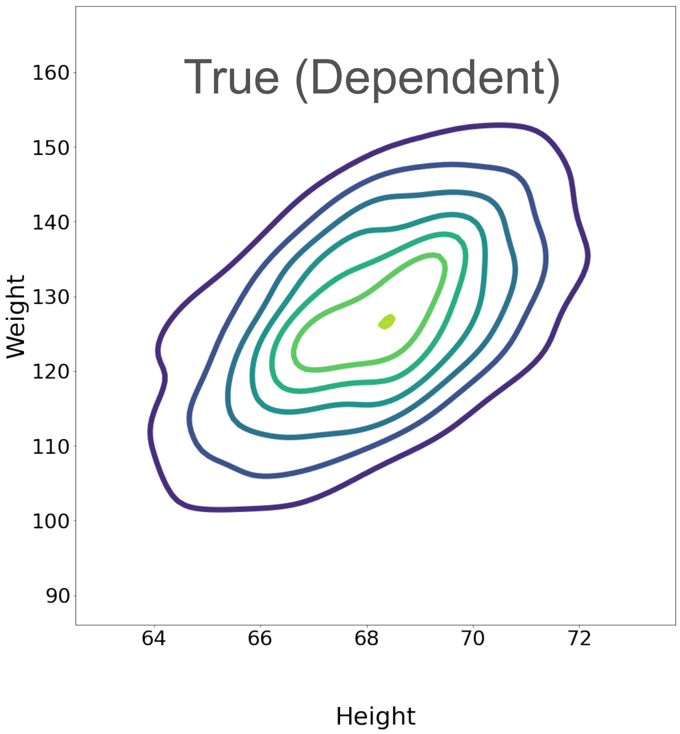
Multivariate Gaussian Distribution: an Example





Multivariate Gaussian Distribution: an Example





Multivariate Gaussian Distribution: an Example

If W, H were **independent**

$$f_{HW}(h,w) = f_H(h) f_W(w)$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\boxed{\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}}\right)}$$

$$\left\| \begin{bmatrix} \dfrac{h-\mu_H}{\sigma_H} \\ \dfrac{w-\mu_W}{\sigma_W} \end{bmatrix} \right\|^2 = \begin{bmatrix} \boxed{\dfrac{h-\mu_H}{\sigma_H} \quad \dfrac{w-\mu_W}{\sigma_W}} \end{bmatrix} \begin{bmatrix} \dfrac{h-\mu_H}{\sigma_H} \\ \dfrac{w-\mu_W}{\sigma_W} \end{bmatrix}$$

$$\begin{bmatrix} h-\mu_h \\ w-\mu_w \end{bmatrix} = \begin{bmatrix} h \\ w \end{bmatrix} - \begin{bmatrix} \mu_h \\ \mu_w \end{bmatrix}$$

Multiply by diagonal matrix

Multivariate Gaussian Distribution: an Example

If W, H were **independent**

$$f_{HW}(h,w) = f_H(h) f_W(w)$$

$$= \frac{1}{2\pi \boxed{\sigma_H \sigma_W}} e^{-\frac{1}{2}\left(\boxed{\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}}\right)}$$

$\det(\Sigma)^{1/2}$

$$\left\| \begin{bmatrix} \frac{h-\mu_H}{\sigma_H} \\ \frac{w-\mu_W}{\sigma_W} \end{bmatrix} \right\|_2^2 = \begin{bmatrix} \frac{h-\mu_H}{\sigma_H} & \frac{w-\mu_W}{\sigma_W} \end{bmatrix} \begin{bmatrix} \frac{h-\mu_H}{\sigma_H} \\ \frac{w-\mu_W}{\sigma_W} \end{bmatrix}$$

$$= ([h \quad w] - [\mu_H \quad \mu_W]) \begin{bmatrix} \frac{1}{\sigma_H^2} & 0 \\ 0 & \frac{1}{\sigma_W^2} \end{bmatrix} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \begin{bmatrix} \mu_H \\ \mu_W \end{bmatrix} \right)$$

Covariance matrix!

(Σ)

$$= \left(\begin{bmatrix} h \\ w \end{bmatrix} - \begin{bmatrix} \mu_H \\ \mu_W \end{bmatrix} \right)^T \boxed{\begin{bmatrix} \sigma_H^2 & 0 \\ 0 & \sigma_W^2 \end{bmatrix}}^{-1} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \boxed{\begin{bmatrix} \mu_H \\ \mu_W \end{bmatrix}} \right)$$

μ

Multivariate Gaussian Distribution: an Example

If W, H were **independent**

$$f_{HW}(h, w) = f_H(h) f_W(w)$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} \exp\left(-\frac{1}{2}\left(\begin{bmatrix} h \\ w \end{bmatrix} - \begin{bmatrix} \mu_H \\ \mu_W \end{bmatrix}\right)^T \begin{bmatrix} \sigma_H^2 & 0 \\ 0 & \sigma_W^2 \end{bmatrix}^{-1} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \begin{bmatrix} \mu_H \\ \mu_W \end{bmatrix}\right)\right)$$

Multivariate Gaussian Distribution: an Example

If W, H were **independent**

$$f_{HW}(h, w) = f_H(h) f_W(w)$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

$$= \frac{1}{2\pi \det \Sigma^{1/2}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix} h \\ w \end{bmatrix} - \mu\right)^T \Sigma^{-1} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \mu\right)\right)$$

Multivariate Gaussian Distribution: an Example

Dependent case:

$$f_{HW}(h, w) = \cancel{f_H(h) f_W(w)}$$

$$\cancel{= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}}$$

$$\Sigma = \begin{bmatrix} \sigma_H^2 & Cov(H, W) \\ Cov(H, W) & \sigma_W^2 \end{bmatrix}$$

$$= \frac{1}{2\pi \det \Sigma^{1/2}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix} h \\ w \end{bmatrix} - \mu\right)^T \Sigma^{-1} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \mu\right)\right)$$

Multivariate Gaussian Distribution: General Definition

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$\boldsymbol{x} = [x_1 \quad x_2 \quad \dots \quad x_n]^T$

Mean vector

$\boldsymbol{\mu} = [\mu_1, \mu_2, \dots, \mu_n]^T$

$f_X(x_1, x_2$ ... $^{1}(\boldsymbol{x}-\boldsymbol{\mu})$

random ...

$X = [X_1 \quad X_2 \quad \dots \quad X_n]$

... / ...ion

- **For univariate, we work with scalar values and variances**
- **For multivariate, we work with vectors and the covariance matrix**

covariance matrix
spread of the bell

$|\Sigma|$ determinant of the covariance matrix

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Probability Distributions with Multiple Variables

Conclusion

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