Question

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations represented by

$$\begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mn} \end{pmatrix} \text{ and } \begin{pmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \cdots & S_{mn} \end{pmatrix}$$

respectively. Define the function $(T + S) : \mathbb{R}^n \to \mathbb{R}^m$ to have the rule that

$$(T + S)(x) = T(x) + S(x).$$

Show that

(a) $T + S$ is linear *(optional but encouraged)*

(b) $T + S$ is represented by the matrix

$$\begin{pmatrix} T_{11} + S_{11} & \cdots & T_{1n} + S_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1} + S_{m1} & \cdots & T_{mn} + S_{mn} \end{pmatrix}.$$

Part (a)

To show that $T + S$ is linear, we know from problem 5 of homework 5 that it is enough to show

1. $(T + S)(\alpha x) = \alpha \cdot (T + S)(x)$
2. $(T + S)(x + y) = (T + S)(x) + (T + S)(y)$

for some $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. Observe that

$$\begin{aligned} (T + S)(\alpha x) &= T(\alpha x) + S(\alpha x) && \text{(Definition of } T + S\text{)} \\ &= \alpha T(x) + \alpha S(x) && \text{(Linearity of } T \text{ and } S\text{)} \\ &= \alpha \cdot (T + S)(x) && \text{(Definition of } T + S\text{)} \end{aligned}$$

and

$$\begin{aligned} (T + S)(x + y) &= T(x + y) + S(x + y) && \text{(Definition of } T + S\text{)} \\ &= T(x) + T(y) + S(x) + S(y) && \text{(Linearity of } T \text{ and } S\text{)} \\ &= (T(x) + S(x)) + (T(y) + S(y)) \\ &= (T + S)(x) + (T + S)(y), && \text{(Definition of } T + S\text{)} \end{aligned}$$

so $T + S$ is linear.

Part (b)

To find the matrix representing $T+S$, recall that a linear transformation sends the standard basis vectors $e^1, \ldots, e^n$ to the columns of the matrix. Thus, the matrix representing $T+S$ is

$$\begin{bmatrix} | & & | \\ (T+S)(e^1) & \cdots & (T+S)(e^n) \\ | & & | \end{bmatrix}.$$

For each i between 1 and n, we have

$$(T+S)(e^i) = T(e^i) + S(e^i) = t^i + s^i.$$

Hence the matrix representing $T+S$ is

$$\begin{bmatrix} | & & | \\ (t^1+s^1) & \cdots & (t^n+s^n) \\ | & & | \end{bmatrix},$$

or in coordinates,

$$\begin{bmatrix} T_{11}+S_{11} & \cdots & T_{1n}+S_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1}+S_{m1} & \cdots & T_{mn}+S_{mn} \end{bmatrix}.$$

So to add matrices, we add their respective elements!