Example proof

In this example, we will prove why multiplying a linear transformation by a scalar implies multiplying each coordinate of its associated matrix by the scalar.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation represented by the matrix

$$\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mn} \end{bmatrix} = \begin{bmatrix} | & & | \\ t^1 & \cdots & t^n \\ | & & | \end{bmatrix}.$$

Given some $\alpha \in \mathbb{R}$, define the function $\alpha T : \mathbb{R}^n \to \mathbb{R}^m$ to have the rule that

$$(\alpha T)(x) = \alpha \cdot T(x).$$

Show that

(a) αT is linear

(b) αT is represented by the matrix

$$\begin{bmatrix} \alpha T_{11} & \cdots & \alpha T_{1n} \\ \vdots & \ddots & \vdots \\ \alpha T_{m1} & \cdots & \alpha T_{mn} \end{bmatrix} = \begin{bmatrix} | & & | \\ \alpha t^1 & \cdots & \alpha t^n \\ | & & | \end{bmatrix}.$$

Part (a)

To show that αT is linear, we know from problem 5 of homework 5 that it is enough to show

1. $(\alpha T)(\beta x) = \beta \cdot (\alpha T)(x)$
2. $(\alpha T)(x + y) = (\alpha T)(x) + (\alpha T)(y)$

for some $\beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. Observe that

$$\begin{aligned} (\alpha T)(\beta x) &= \alpha(T(\beta x)) && \text{(Definition of } \alpha T\text{)} \\ &= \alpha(\beta T(x)) && \text{(Linearity of } T\text{)} \\ &= \beta \alpha T(x) && \text{(Scalar multiplication commutes)} \\ &= \beta \cdot (\alpha T)(x) && \text{(Definiton of } \alpha T\text{)} \end{aligned}$$

and

$$\begin{aligned} (\alpha T)(x + y) &= \alpha(T(x + y)) && \text{(Definition of } \alpha T\text{)} \\ &= \alpha(T(x) + T(y)) && \text{(Linearity of } T\text{)} \\ &= \alpha T(x) + \alpha T(y) && \text{(Distribute } \alpha\text{)} \\ &= (\alpha T)(x) + (\alpha T)(y), && \text{(Definition of } \alpha T\text{)} \end{aligned}$$

so αT is linear.

Part (b)

To find the matrix representing αT, recall that a linear transformation sends the standard basis vectors $e^1, \dots, e^n$ to the columns of the matrix. Thus, the matrix representing αT is

$$\begin{bmatrix} | & & | \\ (\alpha T)(e^i) & \cdots & (\alpha T)(e^n) \\ | & & | \end{bmatrix}.$$

For each i between 1 and n, we have

$$(\alpha T)(e^i) = \alpha T(e^i) = \alpha t^i.$$

Hence the matrix representing αT is

$$\begin{bmatrix} | & & | \\ \alpha t^i & \cdots & \alpha t^i \\ | & & | \end{bmatrix},$$

or in coordinates,

$$\begin{bmatrix} \alpha T_{11} & \cdots & \alpha T_{1n} \\ \vdots & \ddots & \vdots \\ \alpha T_{m1} & \cdots & \alpha T_{mn} \end{bmatrix}.$$