SIR Models for Infectious Disease Dynamics

The goal of this lab is to explore infectious disease dynamics for a population of fixed size. To achieve this, we will employ variations of the SIR model. For your report, include all MATLAB code, figures generated, and answers to included questions.

Problem 1: Simulating SIR model (Demo)

Consider an infectious disease which spreads in a population of size N. In this population, individuals can be considered as susceptible, infectious, or recovered/removed. We assume that $S + I + R = 1$. The dynamics of $S, I,$ and R in terms of fraction of the population can be expressed as

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Given the infection at rate βSI and recovery at rate γI. The speed of disease spread, $r \equiv \beta - \gamma$, can be described as the fate of the disease, where positive speed will suggest an outbreak will occur, while negative speed will suggest an outbreak will dissipate. We can also write this as the basic reproductive number $R_0 \equiv \beta/\gamma$, which corresponds to the average number of new infections caused by a single infectious individual.

A) Write a function in MATLAB that can be passed to an ODE solver to simulate the dynamics of S, I and R. Calculate $\frac{dS}{dt}, \frac{dI}{dt},$ and $\frac{dR}{dt}$ at $t = 0$ for an initial $I_0 = 1$ in a population size $N = 10{,}000$ with parameters $\beta = 0.5\ days^{-1}, \gamma = 0.25\ days^{-1}$.
B) Simulate and plot the dynamics of $S, I,$ and R using $\beta = 0.5\ days^{-1}, \gamma = 0.25\ days^{-1}$ for a time span of $[0\ 100]$ days. Use an initial $I_0 = 1$ and $N = 10{,}000$. Calculate the speed r and reproductive R_0 given β and γ. Do your simulation results and r value suggest that an outbreak will occur or if infectious individuals will recover without increase in the number of cases?
C) Using the simulated values from B), calculate the number of infections to occur each day over the time span of $[0\ 100]$ days. The number of infectious to occur in $[t_0, t_0 + \Delta t] = N\beta S(t_0)I(t_0)\Delta tN$. Plot the daily new infections over $[0\ 100]$ days.
D) Estimate r from the simulated fraction of infectious individuals, $I(t)$. Start by log-transforming $I(t)$ for the first 10 days of the simulation in B) using the built-in `log` function. Fit a line to the log-transformed values using the `polyfit` function. The slope of should estimate r. How does the estimated value of r from the fit compare to the calculated r from part B?
E) Repeat part A-D) for the following β and γ. How do the values of β and γ change the dynamics of the SIR model?

$\beta\ (days^{-1})$	$\gamma\ (days^{-1})$
0.5	0.4
1	0.5
0.25	0.5
0.75	0.25

Problem 2: Estimating COVID-19 Infections with an SIR model.

Using the SIR model from problem 1, fit the infectious component to COVID-19 case data. The dataset provided in `covid_19_data.mat` includes the number of new cases reported in Harris County from September 2021 to March 2022 normalized by the estimate population size of the County. Estimate the parameters β and γ to calculate r and R_0.

A) Plot COVID-19 data provided in `covid_19_data.mat`. You can load .mat files using the `load` function.

B) Write an objective function that calculate the mean squared error between the observed data of infected individuals and the predicted data of infected individuals. Use the values of I(t) to the number of daily new infections as done in problem 1 (i.e., $[t_0, t_0 + \Delta t] = \beta S(t_0) I(t_0) \Delta t$). The mean squared error is given as

$$MSE = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2$$

C) Fit the SIR model using the first 100 days of COVID-19 case data provided and estimate β and γ. Calculate r and R_0 based on the parameter estimates. Do your results suggest that an outbreak will occur or if infectious individuals will recover without increase in the number of cases?

D) Use the fitted parameters to simulate the SIR model over the time span [1 150]. Plot $I(t)$ with the actual COVID-19 data for comparison. How well does the SIR model fit the real data?

Problem 3: Simulating the SIR model with waning immunity.

There are many variations of the SIR model which can be employed depending on the extent to which recovered individuals are immune from later infection or susceptible in the future. Let's consider an SIR model where recovered individuals eventually become susceptible at rate γ. The standard SIR model can then be modified to become

$$\frac{dS}{dt} = -\beta SI + \alpha R$$
$$\frac{dI}{dt} = \beta SI - \gamma I$$
$$\frac{dR}{dt} = \gamma I - \alpha R$$

Where α is the rate of loss of immunity.

A) Update the SIR model from problem 1 to account for waning immunity by including αR to equations for dS/dt and dR/dt. Simulate and plot the dynamics of S, I, and R with waning immunity using the parameters $\beta = 0.5\ days^{-1}$, $\gamma = 0.25\ days^{-1}$ and $\alpha = 0.1\ days^{-1}$ for a time span of $[0\ 100]$ days. Assume $N = 10000$ and $I_0 = 1$.

B) How do the dynamics of S, I, and R change in the case of waning immunity compared to the standard SIR model from problem 1?

C) Repeat part A-B) using $\alpha = 0.25$ and 0.5 $days^{-1}$.

D) How do changes in the rate of loss of immunity affect the dynamics of S, I and R?

Problem 4: Adding an exposed compartment to the SIR model.

In many cases, there is a significant latency period in which individuals have been infected but are not asymptomatic or not infectious themselves. We will extend the basic SIR model to include an exposed compartment, E, to account for individuals in this state. The SEIR model can be expressed as

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dE}{dt} = \beta SI - \eta E$$
$$\frac{dI}{dt} = \eta E - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Where η is the latency rate.

A) Update the SIR model from problem 1 to account for latency of infection by adding the exposed compartment. Simulate and plot the dynamics of $S, I, E,$ and R with waning immunity using the parameters $\beta = 0.5\ days^{-1}$, $\gamma = 0.25\ days^{-1}$ and $\eta = 0.5\ days^{-1}$ for a time span of $[0\ 100]$ days. Assume $N = 10000$ and $I_0 = 1$.

B) How do the dynamics change in the case of including a latency period compared to the standard SIR model?

C) Repeat part A-B) using $\eta = 0.25$ and $0.1\ days^{-1}$.

D) How does changing the latency period affect the dynamics of S, I, E and R?