Vrije Universiteit Amsterdam

Bachelor Thesis

Logical Verification of AVL Trees in Lean

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Contents

1 AVL Trees

give a small introduction to the section

1.1 Binary Search Trees

I begin by defining a binary search tree. A binary tree is a tree data structure where each node can have no more than two children. These two children are called the *left child* and *right child* subtree. In a binary search tree, nodes are placed according to their key. The *binary search property* states that given any node in the tree, all keys in the left child tree are smaller than that of the node, and all keys in the right child tree are larger than that of the node reference. When a tree holds this property, it can be called an *ordered* tree. This allows for lookup and insertion to be done in add complexity here time in the worst case, as at any given node half of the tree is skipped.

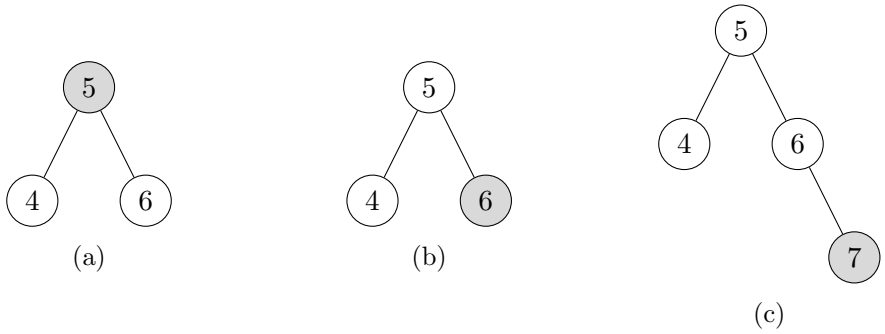

Figure 1: Insertion operation in a binary search tree.

Search, insertion and retrieval can be done recursively. Starting at the root node, the input key and the node key are compared: if the input key is smaller, the operation is done recursively on the left subtree; if the input key is larger, then the operation is done recursively on the right subtree. Figure 1 shows a node with key 7 being inserted into a binary search tree. At 1(a), the new node is compared to the root. As $7 > 5$, the operation continues at the right subtree. At 1(b) the comparison is done again. As $7 > 6$, the operation continues at the right subtree. Because the node 6 doesn't have any children and $7 > 6$, a new right child node is created.

1.2 Balance and rotation

An AVL tree is based on a binary search tree, with one very important distinction - it is *balanced*. To define what it means for a tree to be balanced, I will first define what the *height* of a tree is. get concrete definition of tree height. Balance is reliant on this definition - an AVL tree is only balanced when the heights of any given left and right child subtrees does not differ by more than one [1]. By keeping balance, the structure ensures that there is a high ratio between the number of nodes in the tree and the height. This allows for retrieval and search operations to be done in $O(\log n)$ time in the worst case, with n being the amount of nodes in a tree [2].

During the insertion operation, the tree can become imbalanced, which can be mitigated with either a right rotation or a left rotation.

References

[1] Adelson-Velskiy, G., and Landis, E. An algorithm for the organization of information. In *Soviet Mathematics Doklady* (1962), vol. 3, pp. 1259–1263.

[2] O'Donnel, J., Hall, C., and Page, R. *Discrete Mathematics with a Computer*. Springer-Verlag, 2006, ch. 12, pp. 312–354.

Appendices

A Definitions

```
universe u

inductive btree (α : Type u)
| empty {} : btree
| node (l : btree) (k : nat) (a : α) (r : btree) : btree

namespace btree
variables {α : Type u}

def empty_tree : btree α := btree.empty

def lookup (x : nat) : btree α → option α
| btree.empty := none
| (btree.node l k a r) :=
  if x < k then lookup l
  else if x > k then lookup r
  else a

def bound (x : nat) : btree α → bool
| btree.empty := ff
| (btree.node l k a r) :=
  if x < k then bound l
  else if x > k then bound r
  else tt

def insert (x : nat) (a : α) : btree α → btree α
| btree.empty := btree.node btree.empty x a btree.empty
| (btree.node l k a' r) :=
  if x < k then btree.node (insert l) k a' r
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  else if x > k then btree.node l k a' (insert r)
  else btree.node l x a r

section ordering

def forall_keys (p : nat → nat → Prop) : nat → btree α → Prop
| x btree.empty := tt
| x (btree.node l k a r) :=
  forall_keys x l ∧ (p x k) ∧ forall_keys x r

def ordered : btree α → Prop
| btree.empty := tt
| (btree.node l k a r) := ordered l ∧ ordered r ∧ (forall_keys (>) k l) ∧
    (forall_keys (<) k r)

-- inductive bst {α : Type u} : btree α → Prop
-- | empty {} : bst (btree.empty)
-- | node (l : btree α) (k : nat) (v : α) (r : btree α) :
--   (forall_keys (>) k l) → (forall_keys (<) k r) → bst l → bst r

end ordering

section balancing

def height : btree α → nat
| btree.empty := 0
| (btree.node l k a r) :=
  1 + (max (height l) (height r))

def balanced : btree α → bool
| btree.empty := tt
| (btree.node l k a r) := (height l - height r) ≤ 1

def outLeft : btree α → bool
| btree.empty := ff
| (btree.node (btree.node xL x a xR) z d zR) :=
  (height xL ≥ height xR) ∧ (height xL ≤ height xR + 1) ∧
  (height xR ≥ height zR) ∧ (height xL = height zR + 1)
| (btree.node l k a r) := ff

-- inductive outLeft {α : Type u} : btree α → Prop
-- | empty {} : outLeft (btree.empty)
-- | node (xL xR zR : btree α) (x z : nat) (a d : α) :
--     (height xL ≥ height xR) →
--     (height xL ≤ height xR + 1) →
--     (height xR ≥ height zR) →
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```
--      (height xL = height zR + 1) →
--      outLeft (btree.node (btree.node xL x a xR) z d zR)

def outRight : btree α → bool
| btree.empty := ff
| (btree.node zL z d (btree.node yL y b yR)) :=
  (height yL ≤ height yR) ∧ (height yL ≤ height yR + 1) ∧
  (height yR ≥ height zL) ∧ (height zL + 1 = height yR)
| (btree.node l k a r) := ff

def easyR : btree α → btree α
| btree.empty := btree.empty
| (btree.node (btree.node xL x a xR) z d zR) :=
  (btree.node xL x a (btree.node xR z d zR))
| (btree.node l k a r) := btree.node l k a r

def easyL : btree α → btree α
| btree.empty := btree.empty
| (btree.node zL z d (btree.node yL y b yR)) :=
  (btree.node (btree.node zL z d yL) y b yR)
| (btree.node l k a r) := btree.node l k a r

def rotR : btree α → btree α
| btree.empty := btree.empty
| (btree.node (btree.node xL x a xR) z d zR) :=
  if (height xL < height xR) then easyR (btree.node (easyL (btree.node xL
    x a xR)) z d zR)
  else easyR (btree.node (btree.node xL x a xR) z d zR)
| (btree.node l k a r) := btree.node (rotR l) k a r

def rotL : btree α → btree α
| btree.empty := btree.empty
| (btree.node zL z d (btree.node yL y b yR)) :=
  if (height yR < height yL) then easyL (btree.node zL z d (easyR (btree.
    node yL y b yR)))
  else easyL (btree.node zL z d (btree.node yL y b yR))
| (btree.node l k a r) := btree.node l k a (rotL r)

end balancing

end btree
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