Rocky Mountain North America Regional 2006

Solution Sketches

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Disclaimer *This is an unofficial analysis of possible solutions to the problems of the Rocky Mountain North America Regional 2006. Any error in this text it's my fault. If you find any error or know of other ways to solve this problems, I'd happy to know about it at* `gomez.renzo@gmail.com` *or my blog.*

A. Bridge Bidding

This is a simulation problem. Just follow the rules carefully.

B. Baskets of Gold Coins

Let W be the real weight of the coins. Also, observe that if their weight were w you'd get $\frac{N(N-1)w}{2}$. Then,

$$\frac{N(N-1)w}{2} - W = d \cdot j,$$

where $j \in \{0, 1, \ldots, N-1\}$. If $j = 0$, it means that the N-th basket has the exceptional coin, otherwise j gives the answer.

C. Toothpick Arithmetic

This problem can be solved using dynamic programing. Let

$$M(i) = \text{Minimum number of sticks to get } i \text{ only with multiplications.}$$

Clearly, it's defined by de following recurrence,

$$M(i) = \min(i, \min\{M(j) + M(\tfrac{i}{j}) + 2 : \ j \text{ divides } i\}).$$

Because multiplication has precedence over summation, in an expression using $+$ and $\times$, between $+$ signs there can only be a number or an expression using only $\times$ signs. So, let

$$S(i) = \text{Minimum number of sticks to get } i.$$

By the latter argument, it's defined by the following recurrence

$$S(i) = \min(M(i), \min\{M(j) + S(i-j) + 2 : \ 1 \leqslant j < i\}).$$

Note that, to pass the time limit, we must pre-calculate $S(n)$ for every n, $1 \leqslant n \leqslant 5000$.

D. Human Knot

First of all, cycles are planar graphs so it's always possible to find a planar embedding of them. But, there is a condition imposed on the problem stamens: *"... obtain one n-sided polygon ..."* . Then, if the graph is disconnected there is no solution.

The first observation is that, given an initial configuration, any untangled one with minimum number of moves preserves the relative position of at least one vertex in the circle. Now, consider a vertex k and relabel the vertices, starting in k, with 1 and continue in clockwise order. After that, to detect the crossing lines, choose one orientation of the cycle and let p_i be the order of vertex i in that orientation (beginning from k). Observe that the cycle is untangled if the vector p is a monotone sequence.

Then, to minimize the number of moves, we find the length longest monotone (increasing or decreasing) subsequence of p. Let r be such length, the number of moves will be $n-r-1$. Don't forget to test every vertex k as the fixed vertex on the circle.

E. Permutation Recovery

Consider a set P of free positions, $P = \{1, 2, \dots, n\}$. Then, note that 1 is the minimum number and the rest of elements are greater than it. So, 1 must be on the position $a_1 + 1$ (positions are numbered from left to right). Also, by a similar argument, 2 must be on the position $a_2 + 1$ of $P - \{a_1 + 1\}$. We continue in the same way until all the positions are occupied. This process gives an $O(n^2)$ algorithm, but it can be easily improve to $O(n \lg n)$ using a data structure like segment tree (although it's not necessary for the time limit).

F. Marbles in Three Baskets

Lets represent each possible state of the baskets by a vertex(a_1, a_2, a_3), where a_i is the number of marbles on basket i. Moreover, if we can reach state v_j from v_i in a single move, we add the directed edge $v_i \to v_j$.

Then, the problem reduces to given two states, say v_s and v_e, find a shortest path from v_s to v_e. This can be accomplished by a BFS (Breadth First Search) from v_s.

G. Doors and Penguins

Let

$$\begin{aligned} D &= \{(x, y) : \text{it's a location of a vendor supporting Doors.}\}, \\ P &= \{(x, y) : \text{it's a location of a vendor supporting Penguins.}\}. \end{aligned}$$

The problem asks for the existence of a line L that separates D and P strictly (no point of D or P lies on L). Lets denote by $\mathbf{conv}(D)$ and $\mathbf{conv}(P)$ the convex hull generated by D and P respectively. The key observation is that the existence of such a line is guaranteed $\Leftrightarrow$ $\mathbf{conv}(D)$ and $\mathbf{conv}(P)$ are disjoint (Separation Theorem).

H. Knots

Consider that the women moves are fixed. We focus on the male moves, considering the following events

$$E_k \quad = \quad \text{Choose the } k\text{-th pair of strands correctly}, \quad k = 1, 2, \ldots, \tfrac{N}{2}.$$

Observe that when we choose the first pair, there are $\frac{N}{2}$ forbidden pairs. So

$$\begin{aligned} P(E_1) &= 1 - P({E_1}^c), \\ &= 1 - \frac{N/2}{\binom{N}{2}}, \\ &= \frac{N-2}{N-1} \end{aligned}$$

After choosing the first pair, the problem is reduced to one of $N-2$ strands and $\frac{N-2}{2}$ forbidden pairs. Then, by a similar argument, $P(E_2|E_1) = \frac{N-4}{N-3}$. Let $N = 2n$, generalizing the previous formula

$$\begin{aligned} P(\cap_{i=1}^{n} E_i) &= P(E_1) \times P(E_2|E_1) \times \ldots \times P(E_n| \cap_{j=1}^{n-1} E_j), \\ &= \frac{(N-2)(N-4)\ldots 2}{(N-1)(N-3)\ldots 3}. \end{aligned}$$

I. String Equations

Let $\{S_1, S_2, \ldots, S_n\}$ be the set of strings. We define the following coefficients and variables

$$\begin{aligned} a_{ij} &= \text{number of times } i \text{ appears in string } j, \text{ for } i = A, \ldots, Z, \\ x_j &= \text{number of times string } j \text{ is used.} \end{aligned}$$

In the definition of x_j we impose the convention that if $x_j < 0$, the string j is used on the left side of the equation and if $x > 0$, we use it on the right side. Then, the problem reduces to find a non-trivial solution to the homogeneous linear system $A \cdot x = 0$. An important fact from homogeneous systems is that they have the trivial solution as unique solution $\Leftrightarrow$ $\det(A) \neq 0$.

To determine if that system has a non-trivial solution and obtain one, we reduce the extended matrix $[A|I_n]$ (where I_n denotes the identity matrix of dimension $n \times n$) to row echelon form using the gaussian elimination algorithm. Let $[A'|B]$ denote the extended matrix after the gaussian elimination. It can be proved that for each row A'_i,

$$A'_i \quad = \quad \textstyle\sum_{j=1}^{n} b_{ij} \cdot A_j$$

Then, if A' has a null row, we can obtain a solution to the problem from the B matrix. Otherwise there is no solution.