

Incremental Fixpoint Computation: A Two-Level Architecture

Abstract

We observe that the incremental dead code elimination (DCE) algorithm from our reactive DCE work is an instance of a more general pattern: *incremental fixpoint computation*. This note proposes a two-level architecture for incremental fixpoints: (1) a low-level API that assumes user-provided incremental operations, and (2) a potential high-level DSL where these operations are derived automatically from a structured definition of the fixpoint operator. The relationship between these levels is analogous to that between manual gradient computation and automatic differentiation.

1 Motivation: DCE as Incremental Fixpoint

In reactive DCE, the live set is defined as the least fixpoint of a monotone operator:

$$F_G(S) = G.\text{roots} \cup \{v \mid \exists u \in S. (u, v) \in G.\text{edges}\}$$

That is, $\text{liveSet}(G) = \text{lfp}(F_G)$.

When the graph changes ($G \rightarrow G' = G \pm f$), we want to update the fixpoint incrementally rather than recomputing from scratch. The key observations are:

- **Expansion** ($G \rightarrow G \oplus f$): The operator grows, so $\text{lfp}(F_G) \subseteq \text{lfp}(F_{G'})$. The old fixpoint is an underapproximation; we iterate upward.
- **Contraction** ($G \rightarrow G \ominus f$): The operator shrinks, so $\text{lfp}(F_{G'}) \subseteq \text{lfp}(F_G)$. The old fixpoint is an overapproximation; we must remove unjustified elements.

This pattern—incremental maintenance of a least fixpoint under changes to the underlying operator—arises in many domains beyond DCE.

2 The General Pattern

Definition 1 (Monotone Fixpoint Problem). *Given a complete lattice (L, \sqsubseteq) and a monotone operator $F : L \rightarrow L$, the least fixpoint is $\text{lfp}(F) = \bigcap\{x \mid F(x) \sqsubseteq x\}$.*

For set-based fixpoints (our focus), $L = \mathcal{P}(A)$ for some element type A , ordered by \subseteq , and F is typically of the form:

$$F(S) = \text{base} \cup \text{step}(S)$$

where **base** provides seed elements and **step** derives new elements from existing ones.

Definition 2 (Incremental Fixpoint Problem). *Given:*

- A current fixpoint $S = \text{lfp}(F)$
- A change that transforms F into F'

Compute $S'' = \text{lfp}(F')$ efficiently, in time proportional to $|S' \Delta S|$ rather than $|S'|$.

3 Level 1: Low-Level Incremental Fixpoint API

The low-level API assumes the user provides the necessary incremental operations.

3.1 Required Ingredients

For Semi-Naive Expansion. When $F' \supseteq F$ (the operator grows), we use *semi-naive evaluation*:

- Maintain the “delta” $\Delta S = \text{elements added in the last iteration}$
- Instead of computing $F'(S)$, compute only $\text{stepFromDelta}(\Delta S) \setminus S$

The user provides:

$$\text{stepFromDelta} : \text{Params} \times \mathcal{P}(A) \rightarrow \mathcal{P}(A)$$

Given the current parameters and a delta set, return elements derivable from that delta.

Example 1 (DCE). $\text{stepFromDelta}(G, \Delta) = \{v \mid \exists u \in \Delta. (u, v) \in G.\text{edges}\}$

For Counting-Based Contraction. When $F' \subseteq F$ (the operator shrinks), we use *counting-based deletion*:

- Track how many “derivations” support each element
- When a derivation is removed, decrement the count
- When count reaches zero, remove the element and propagate

The user provides:

$$\text{derivationCount} : \text{Params} \times \mathcal{P}(A) \times A \rightarrow \mathbb{N}$$

Given the current fixpoint, how many ways is element x derived?

Example 2 (DCE). $\text{derivationCount}(G, \text{live}, v) = |\{u \mid (u, v) \in G.\text{edges} \wedge u \in \text{live}\}|$
This is exactly the refcount maintained by the DCE algorithm.

3.2 The API

```
interface IncrementalFixpoint<A, Params, Delta> {
    // User provides:
    base: (params: Params) => Set<A>
    stepFromDelta: (params: Params, delta: Set<A>) => Set<A>
    derivationCount: (params: Params, fp: Set<A>, x: A) => Nat
    applyDelta: (params: Params, delta: Delta) => Params

    // System provides:
    current: Set<A>
```

```

counts: Map<A, Nat>

update(delta: Delta): { added: Set<A>, removed: Set<A> }
}

```

3.3 Update Algorithm

Expansion. When F grows:

1. Compute initial delta: $\Delta_0 = F'(\text{current}) \setminus \text{current}$
2. Semi-naive iterate:

$$\begin{aligned}\Delta_{n+1} &= \text{stepFromDelta}(\Delta_n) \setminus \text{current} \\ \text{current} &\leftarrow \text{current} \cup \Delta_{n+1}\end{aligned}$$

3. Update derivation counts for new elements

Contraction. When F shrinks:

1. Update derivation counts for removed derivations
2. Initialize cascade: $Q = \{x \in \text{current} \mid \text{counts}[x] = 0 \wedge x \notin \text{base}\}$
3. Propagate:
 - Remove x from current
 - Decrement counts of elements derived from x
 - Add newly-zero elements to Q

3.4 Formal Definitions and Correctness

3.4.1 Decomposed Operators

Definition 3 (Decomposed Operator). An operator $F : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is decomposed if $F(S) = B \cup \text{step}(S)$ where B is a fixed base set and step is monotone: $S \subseteq T \Rightarrow \text{step}(S) \subseteq \text{step}(T)$.

Definition 4 (Operator Expansion and Contraction). We say F expands to F' , written $F \sqsubseteq F'$, if $\forall S. F(S) \subseteq F'(S)$. Dually, F contracts to F' if $F' \sqsubseteq F$.

Theorem 1 (Fixpoint Monotonicity).

1. If $F \sqsubseteq F'$ then $\text{lfp}(F) \subseteq \text{lfp}(F')$.
2. If $F' \sqsubseteq F$ then $\text{lfp}(F') \subseteq \text{lfp}(F)$.

3.4.2 Semi-Naive Iteration (Expansion)

Definition 5 (Semi-Naive Iteration). Given a decomposed operator (B, step) and initial set I , define:

$$\begin{aligned}C_0 &= I & \Delta_0 &= I \\ C_{n+1} &= C_n \cup \Delta_{n+1} & \Delta_{n+1} &= \text{step}(\Delta_n) \setminus C_n\end{aligned}$$

Theorem 2 (Semi-Naive Monotonicity). $C_n \subseteq C_{n+1}$ for all $n \geq 0$.

Theorem 3 (Semi-Naive Soundness). If $I \subseteq \text{lfp}(F)$, then $C_n \subseteq \text{lfp}(F)$ for all $n \geq 0$.

3.4.3 Counting-Based Cascade (Contraction)

Definition 6 (Derivation Count). A function $\text{count} : \mathcal{P}(A) \times A \rightarrow \mathbb{N}$ is a valid derivation count for operator (B, step) if:

$$x \in \text{step}(S) \iff \text{count}(S, x) > 0$$

Definition 7 (Cascade Iteration). Given operator (B, step) , count function count , and initial set I :

$$\begin{aligned}\text{shouldDie}(S) &= \{x \in S \mid x \notin B \wedge \text{count}(S, x) = 0\} \\ \text{cascadeStep}(S) &= S \setminus \text{shouldDie}(S) \\ K_0 &= I \\ K_{n+1} &= \text{cascadeStep}(K_n)\end{aligned}$$

Theorem 4 (Cascade Monotonicity). $K_{n+1} \subseteq K_n$ for all $n \geq 0$.

Theorem 5 (Base Preservation). If $B \subseteq I$, then $B \subseteq K_n$ for all $n \geq 0$.

Definition 8 (Cascade Fixpoint). $K^* = \bigcap_{n \geq 0} K_n$.

Definition 9 (Cascade Stability). Cascade is stable at step n if $K_{n+1} = K_n$.

Theorem 6 (Stability Persistence). If cascade is stable at step n , then $K_m = K_n$ for all $m \geq n$.

Theorem 7 (Stable Fixpoint Characterization). If cascade is stable at step n , then $K^* = K_n$.

Theorem 8 (Cascade Soundness). If cascade is stable at step n , then $K^* \subseteq F(K^*)$. That is, the cascade fixpoint is a prefixpoint of F .

3.4.4 Overall Correctness

Theorem 9 (Expansion Correctness). Let $F \sqsubseteq F'$ (expansion), $S = \text{lfp}(F)$, and $S' = \text{lfp}(F')$. If semi-naive iteration from S with operator F' stabilizes at step n , then:

$$C_n = S'$$

Theorem 10 (Contraction Correctness). Let $F' \sqsubseteq F$ (contraction), $S = \text{lfp}(F)$, and $S' = \text{lfp}(F')$. If cascade from S with operator F' stabilizes at step n , then:

$$K^* = S'$$

These theorems state that the incremental update algorithm is *correct*: starting from the old fixpoint and applying the appropriate algorithm (semi-naive for expansion, cascade for contraction), we obtain exactly the new fixpoint.

Remark 1 (Lean Formalization). All definitions and theorems above are formalized in Lean.¹

Fully proven (no sorry):

- Semi-naive soundness: $C_n \subseteq \text{lfp}(F)$
- Cascade soundness: $K^* \subseteq F(K^*)$ when stable

¹See `lean-formalisation/IncrementalFixpoint.lean`.

- Fixpoint monotonicity under expansion/contraction
- Cascade stability persistence and fixpoint characterization
- Contraction direction: $S' \subseteq K^*$ (new fixpoint survives cascade)

Overall correctness requires additional assumptions:

- **Expansion** ($C_n = S'$): Proven assuming (1) step is additive, i.e., $\text{step}(A \cup B) = \text{step}(A) \cup \text{step}(B)$, and (2) new base \subseteq old fixpoint.
- **Contraction** ($K^* = S'$): The direction $S' \subseteq K^*$ is proven assuming count is monotone in its set argument. The reverse $K^* \subseteq S'$ remains future work for simple counting.

Both additivity and count monotonicity hold for DCE-style operators.

3.4.5 Well-Founded Derivations

Simple counting-based cascade is not complete for cyclic graphs: unreachable cycles keep each other alive with positive counts. The solution is *well-founded derivations*.

Definition 10 (Iterative Construction and Rank). *The least fixpoint is constructed iteratively:*

$$\begin{aligned} F^0(\emptyset) &= \emptyset \\ F^{n+1}(\emptyset) &= F(F^n(\emptyset)) \\ \text{lfp}(F) &= \bigcup_{n \geq 0} F^n(\emptyset) \end{aligned}$$

The rank of $x \in \text{lfp}(F)$ is the minimum n such that $x \in F^n(\emptyset)$. Elements not in $\text{lfp}(F)$ have no finite rank.

Definition 11 (Well-Founded Derivation). *Element y well-foundedly derives x if $\text{rank}(y) < \text{rank}(x)$ and $x \in \text{step}(\{y\})$.*

Definition 12 (Well-Founded Cascade). Well-founded cascade removes elements that:

1. Are not in the base, and
2. Have no well-founded deriver in the current set.

This differs from simple cascade: cycle members have equal (or no) rank, so they don't provide well-founded support to each other.

Theorem 11 (Well-Founded Contraction Correctness). *Let $F' \sqsubseteq F$ (contraction), $S = \text{lfp}(F)$, $S' = \text{lfp}(F')$. Assume:*

1. step is element-wise: $x \in \text{step}(T)$ implies $\exists y \in T. x \in \text{step}(\{y\})$.
2. $S' = \bigcup_n F'^n(\emptyset)$ (the iterative characterization).

Then well-founded cascade from S converges to S' .

Remark 2 (Well-Founded Lean Formalization). *The well-founded cascade and its correctness theorem are fully formalized in Lean:*

- iterF , iterFLimit : iterative construction
- rankLt : well-founded rank comparison
- wfCascadeStep , wfCascadeN , wfCascadeFix : cascade
- $\text{wf_contraction_correctness}$: main theorem (fully proven)
- $\text{wfCascade_removes_non_lfp}'$: non-lfp' elements are removed (fully proven)

The key insight: elements not in $S' = \text{iterFLimit}(F')$ have no finite rank under F' , so nothing has strictly lower rank, hence no well-founded derivars.

4 Level 2: DSL with Automatic Derivation (Future)

The low-level API requires the user to provide `stepFromDelta` and `derivationCount`. A higher-level approach would let users define F in a structured DSL, from which these operations are derived automatically.

4.1 Analogy: Automatic Differentiation

	Differentiation	Incremental Fixpoint
Low-level	User provides $f(x)$ and $\frac{df}{dx}$	User provides F , <code>stepFromDelta</code> , <code>derivationCount</code>
High-level (DSL)	User writes expression; system derives gradient	User writes F in DSL; system derives incremental ops
Requirement	f given as expression tree	F given as composition of primitives
Black-box	Finite differences (slow)	Full recomputation (slow)

Just as automatic differentiation requires f to be expressed as a composition of differentiable primitives, automatic incrementalization requires F to be expressed as a composition of “incrementalizable” primitives.

4.2 Potential DSL Primitives

A DSL for fixpoint operators might include:

- $\text{const}(B)$: constant base set
- $\text{union}(F_1, F_2)$: union of two operators
- $\text{join}(R, S, \pi)$: join S with relation R , project via π
- $\text{filter}(P, S)$: filter S by predicate P
- $\text{lfp}(\lambda S.F(S))$: least fixpoint

Each primitive would come with:

- Its incremental step function (for semi-naive)
- Its derivation counting semantics (for deletion)

```

Example 3 (DCE in DSL). live = lfp(S =>
    union(
        const(roots),
        join(edges, S, (u, v) => v)
    )
)

```

The system derives:

- $\text{stepFromDelta}(\Delta) = \text{join}(\text{edges}, \Delta, (u, v) \mapsto v)$
- $\text{derivationCount}(v) = |\{u \mid (u, v) \in \text{edges} \wedge u \in \text{live}\}|$

4.3 Connection to Datalog

Datalog engines already perform this derivation:

- Rules are the structured representation of F
- Semi-naive evaluation is derived from rule structure
- Counting-based deletion (DRed) handles retraction

A general incremental fixpoint DSL would extend this beyond Horn clauses to richer operators (aggregation, negation, etc.).

5 Examples Beyond DCE

The incremental fixpoint pattern applies to many problems:

Problem	Base	Step	Derivation Count
DCE/Reachability	roots	successors	in-degree from live
Type Inference	base types	constraint propagation	# constraints implying type
Points-to Analysis	direct assignments	transitive flow	# flow paths
Call Graph	entry points	callees of reachable	# callers
Datalog	base facts	rule application	# rule firings

6 Relationship to Reactive Systems

In a reactive system like Skip:

- **Layer 1** (reactive aggregation) handles changes to the *parameters* of F (e.g., the graph structure).
- **Layer 2** (incremental fixpoint) maintains the fixpoint as those parameters change.

The two layers compose: reactive propagation delivers deltas to the fixpoint maintainer, which incrementally updates its state and emits its own deltas (added/removed elements) for downstream consumers.

7 Future Work

1. **Design Level 2 DSL:** Define a language of composable fixpoint operators with automatic incrementalization.
2. **Integrate with Skip:** Implement the incremental fixpoint abstraction as a reusable component in the Skip reactive framework.
3. **Explore stratification:** Extend to stratified fixpoints (with negation) where layers must be processed in order.
4. **Benchmark:** Compare incremental vs. recompute performance on realistic workloads.

8 Conclusion

The incremental DCE algorithm is an instance of a general pattern: maintaining least fixpoints incrementally under changes to the underlying operator. We propose a two-level architecture:

1. A **low-level API** where users provide the incremental operations (`stepFromDelta`, `derivationCount`).
2. A **high-level DSL** (future work) where these operations are derived automatically from a structured definition of F , analogous to how automatic differentiation derives gradients from expression structure.

This abstraction unifies incremental algorithms across domains (program analysis, databases, reactive systems) and provides a foundation for building efficient, correct incremental computations.