

# Incremental Fixpoint Computation: A Two-Level Architecture

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### Abstract

We observe that the incremental dead code elimination (DCE) algorithm from our reactive DCE work is an instance of a more general pattern: *incremental fixpoint computation*. This note proposes a two-level architecture for incremental fixpoints: (1) a low-level API that assumes user-provided incremental operations, and (2) a potential high-level DSL where these operations are derived automatically from a structured definition of the fixpoint operator. The relationship between these levels is analogous to that between manual gradient computation and automatic differentiation.

## 1 Motivation: DCE as Incremental Fixpoint

In reactive DCE, the live set is defined as the least fixpoint of a monotone operator:

$$F_G(S) = G.\text{roots} \cup \{v \mid \exists u \in S. (u, v) \in G.\text{edges}\}$$

That is,  $\text{liveSet}(G) = \text{lfp}(F_G)$ .

When the graph changes ( $G \rightarrow G' = G \pm f$ ), we want to update the fixpoint incrementally rather than recomputing from scratch. The key observations are:

- **Expansion** ( $G \rightarrow G \oplus f$ ): The operator grows, so  $\text{lfp}(F_G) \subseteq \text{lfp}(F_{G'})$ . The old fixpoint is an underapproximation; we iterate upward.
- **Contraction** ( $G \rightarrow G \ominus f$ ): The operator shrinks, so  $\text{lfp}(F_{G'}) \subseteq \text{lfp}(F_G)$ . The old fixpoint is an overapproximation; we must remove unjustified elements.

This pattern—incremental maintenance of a least fixpoint under changes to the underlying operator—arises in many domains beyond DCE.

## 2 The General Pattern

**Definition 1** (Monotone Fixpoint Problem). *Given a complete lattice  $(L, \sqsubseteq)$  and a monotone operator  $F : L \rightarrow L$ , the least fixpoint is  $\text{lfp}(F) = \bigcap \{x \mid F(x) \sqsubseteq x\}$ .*

For set-based fixpoints (our focus),  $L = \mathcal{P}(A)$  for some element type  $A$ , ordered by  $\subseteq$ , and  $F$  is typically of the form:

$$F(S) = \text{base} \cup \text{step}(S)$$

where **base** provides seed elements and **step** derives new elements from existing ones.

**Definition 2** (Incremental Fixpoint Problem). *Given:*

- A current fixpoint  $S = \text{lfp}(F)$
- A change that transforms  $F$  into  $F'$

Compute  $S' = \text{lfp}(F')$  efficiently, in time proportional to  $|S' \triangle S|$  rather than  $|S'|$ .

## 3 Level 1: Low-Level Incremental Fixpoint API

### 3.1 API Specification

**Remark 1** (Two API Levels). *The API has two levels depending on which operations are needed:*

- **Simple API** (expansion only): *Supports adding elements to base or edges. Requires only base and stepFwd.*
- **Full API** (expansion + contraction): *Also supports removing elements. Additionally requires stepInv and rank.*

*Many applications only need expansion (e.g., monotonically growing graphs). The simple API suffices and is easier to implement.*

#### 3.1.1 Types

$A$	Element type (e.g., graph nodes)
$\text{Set}(A)$	Finite sets of elements
$\text{Map}(A, \mathbb{N})$	Map from elements to natural numbers (ranks)

#### 3.1.2 Configuration (User Provides)

$\text{base} : \text{Set}(A)$	Seed elements (e.g., roots)	required
$\text{stepFwd} : A \rightarrow \text{Set}(A)$	Forward derivation	required
$\text{stepInv} : A \rightarrow \text{Set}(A)$	Inverse derivation	for contraction

Define  $\text{step}(S) = \bigcup_{x \in S} \text{stepFwd}(x)$  and  $F(S) = \text{base} \cup \text{step}(S)$ .

**Note:** If  $\text{stepInv}$  is not provided, the system can build it from  $\text{stepFwd}$ :

$$\text{stepInv}[y] = \{x \in \text{current} \mid y \in \text{stepFwd}(x)\}$$

This is computed once during initialization and maintained incrementally.

#### 3.1.3 State (System Maintains)

$\text{current} : \text{Set}(A)$	Current live set = $\text{lfp}(F)$	always
$\text{rank} : \text{Map}(A, \mathbb{N})$	BFS distance from base	for contraction

#### 3.1.4 Required Properties

1. **stepInv correctness** (if provided):  $y \in \text{stepInv}(x) \Leftrightarrow x \in \text{stepFwd}(y)$
2. **Element-wise** (automatic):  $\text{step}$  decomposes via  $\text{stepFwd}$
3. **Additive** (automatic):  $\text{step}(A \cup B) = \text{step}(A) \cup \text{step}(B)$

**Example 1** (DCE Instance).

$$\begin{aligned} \text{base} &= \text{roots} \\ \text{stepFwd}(u) &= \{v \mid (u, v) \in \text{edges}\} && (\text{successors}) \\ \text{stepInv}(v) &= \{u \mid (u, v) \in \text{edges}\} && (\text{predecessors, optional}) \end{aligned}$$

## 3.2 Algorithms

### 3.2.1 Expansion (BFS)

When the operator grows ( $F \sqsubseteq F'$ : base or edges added), propagate new elements:

```
expand(state, config'):
    frontier = config'.base \ state.current
    r = 0
    while frontier != {}:
        for x in frontier:
            state.current.add(x)
            state.rank[x] = r
        nextFrontier = {}
        for x in frontier:
            for y in config'.stepFwd(x):
                if y not in state.current:
                    nextFrontier.add(y)
        frontier = nextFrontier
        r += 1
```

### 3.2.2 Contraction (Worklist Cascade)

When the operator shrinks ( $F' \sqsubseteq F$ : base or edges removed), remove unsupported elements:

```
contract(state, config'):
    // Initialize: nodes that lost base membership or an incoming edge
    worklist = { x | x lost support }
    dying = {}

    while worklist != {}:
        x = worklist.pop()
        if x in dying or x in config'.base: continue

        // Check for well-founded deriver (strictly lower rank)
        hasSupport = false
        for y in config'.stepInv(x):
            if y in (state.current \ dying) and state.rank[y] < state.rank[x]:
                hasSupport = true
                break

        if not hasSupport:
            dying.add(x)
            // Notify dependents
            for z where x in config'.stepInv(z):
                worklist.add(z)

    state.current -= dying
```

### 3.2.3 Why Ranks Break Cycles

The rank check  $\text{rank}[y] < \text{rank}[x]$  is essential:

- Cycle members have *equal* ranks (same BFS distance from base)
- Therefore, they cannot provide well-founded support to each other
- An unreachable cycle has no well-founded support and is correctly removed

## 3.3 Correctness and Analysis

### 3.3.1 Proven Properties (Lean, all complete)

**Theorem 1** (Expansion Correctness). *If  $F \sqsubseteq F'$  and expansion terminates, then  $\text{current} = \text{lfp}(F')$ .*

**Theorem 2** (Contraction Correctness). *If  $F' \sqsubseteq F$  and contraction terminates, then  $\text{current} = \text{lfp}(F')$ .*

**Theorem 3** (Soundness). *At all intermediate states: expansion gives  $\text{current} \subseteq \text{lfp}(F')$ ; contraction gives  $\text{lfp}(F') \subseteq \text{current}$ .*

All proofs complete in Lean with no `sorry`.<sup>1</sup>

### 3.3.2 Complexity Analysis

Operation	Time	Space
Expansion	$O( \text{new}  +  \text{edges from new} )$	$O( \text{new} )$
Contraction	$O( \text{dying}  +  \text{edges to dying} )$	$O( \text{dying} )$
Rank storage	—	$O( \text{current} )$ integers

For DCE, this matches the complexity of dedicated graph reachability algorithms.

### 3.3.3 Gap Between Proofs and Algorithms

Gap	Description	Status
Refinement	Pseudo-code implements the spec	Visually obvious
Termination	Algorithms halt on finite sets	Clear (monotonic)
stepInv	User provides correct inverse	Assumed

These gaps are mechanical to close. A good engineer can implement with confidence.

## 3.4 Formal Definitions (Reference)

For completeness, the formal definitions used in Lean proofs.

**Definition 3** (Decomposed Operator).  $F(S) = B \cup \text{step}(S)$  where `step` is monotone.

**Definition 4** (Rank).  $\text{rank}(x) = \min\{n \mid x \in F^n(\emptyset)\}$ .

**Definition 5** (Well-Founded Derivation).  $y$  *wf-derives*  $x$  if  $\text{rank}(y) < \text{rank}(x)$  and  $x \in \text{step}(\{y\})$ .

**Definition 6** (Semi-Naive Iteration).  $C_0 = I$ ,  $\Delta_0 = I$ ,  $\Delta_{n+1} = \text{step}(\Delta_n) \setminus C_n$ ,  $C_{n+1} = C_n \cup \Delta_{n+1}$ .

**Definition 7** (Well-Founded Cascade).  $K_0 = I$ ,  $K_{n+1} = K_n \setminus \{x \mid x \notin B \wedge \text{no wf-deriver in } K_n\}$ .

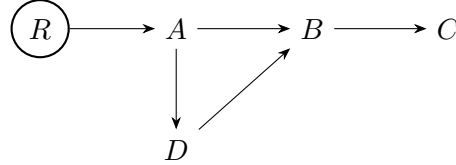
<sup>1</sup>See `lean-formalisation/IncrementalFixpoint.lean`.

## 4 Worked Example: DCE in Detail

We illustrate the API and algorithms with Dead Code Elimination (DCE), showing how expansion and contraction work on concrete graphs.

### 4.1 Setup

Consider a program represented as a directed graph where nodes are code units and edges represent dependencies (“ $u \rightarrow v$ ” means  $u$  uses  $v$ ).



- $\text{base} = \{R\}$  ( $R$  is the root/entry point)
- $\text{stepFwd}(R) = \{A\}$ ,  $\text{stepFwd}(A) = \{B, D\}$ ,  $\text{stepFwd}(B) = \{C\}$ ,  $\text{stepFwd}(D) = \{B\}$

**Initial state after BFS expansion:**

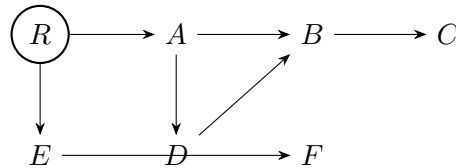
$$\begin{aligned} \text{current} &= \{R, A, B, C, D\} \\ \text{rank} &= \{R \mapsto 0, A \mapsto 1, B \mapsto 2, C \mapsto 3, D \mapsto 2\} \end{aligned}$$

Note:  $B$  and  $D$  have the same rank (both at distance 2 from  $R$ ).

For contraction we also write  $\text{stepInv}(x)$  for the set of predecessors  $y$  with an edge  $y \rightarrow x$ , and maintain a set dying of nodes scheduled for removal.

### 4.2 Example: Expansion (Adding an Edge)

Suppose we add an edge  $R \rightarrow E$  where  $E$  is a new node with  $\text{stepFwd}(E) = \{F\}$ .



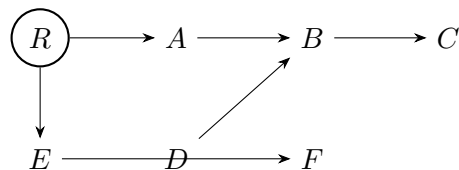
**Expansion algorithm:**

1.  $\text{frontier} = \{E\}$  (new successors of  $R$ ),  $r = 1$
2. Add  $E$  with  $\text{rank}[E] = 1$
3.  $\text{frontier} = \{F\}$ ,  $r = 2$
4. Add  $F$  with  $\text{rank}[F] = 2$
5.  $\text{frontier} = \{\}$ , done

**Result:**  $\text{current} = \{R, A, B, C, D, E, F\}$

### 4.3 Example: Contraction (Removing an Edge)

Now suppose we remove the edge  $A \rightarrow D$ .



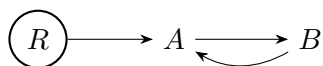
**Contraction algorithm:**

1.  $\text{worklist} = \{D\}$  (lost its incoming edge from  $A$ )
2. Process  $D$ :
  - $D \notin \text{base}$
  - Check for wf-deriver:  $\text{stepInv}(D) = \{A\}$ , but edge  $A \rightarrow D$  removed
  - No wf-deriver found, so  $\text{dying} = \{D\}$
  - All dependents of  $D$  already have another wf-deriver, so no additional nodes are added to the worklist
3.  $\text{worklist} = \{\}$ , done

**Result:**  $\text{current} = \{R, A, B, C, E, F\}$  ( $D$  removed)

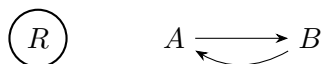
### 4.4 Example: Contraction with Cycles

This example shows why *ranks* are essential. Consider:



- $\text{rank} = \{R \mapsto 0, A \mapsto 1, B \mapsto 2\}$
- $A$  has a wf-deriver:  $R$  with  $\text{rank}[R] = 0 < 1$
- $B$  has a wf-deriver:  $A$  with  $\text{rank}[A] = 1 < 2$

**Now remove edge  $R \rightarrow A$ :**



**Contraction:**

1.  $\text{worklist} = \{A\}$  (lost edge from  $R$ )
2. Process  $A$ :
  - $\text{stepInv}(A) = \{R, B\}$
  - $R \rightarrow A$  removed, so  $R$  doesn't count
  - $B$  is in current, but  $\text{rank}[B] = 2 > 1 = \text{rank}[A]$  — **not a wf-deriver!**
  - No wf-deriver, so  $A$  dies. Add  $B$  to worklist.
3. Process  $B$ :
  - $\text{stepInv}(B) = \{A\}$
  - $A$  is dying, so doesn't count
  - No wf-deriver, so  $B$  dies.
4.  $\text{worklist} = \{\}$ , done

**Result:**  $\text{current} = \{R\}$  (entire cycle removed)

**Key insight:** Without rank checking,  $A$  and  $B$  would keep each other alive (each derives the other). The rank check  $\text{rank}[y] < \text{rank}[x]$  breaks this mutual support because cycle members have equal or increasing ranks along cycle edges.

## 4.5 Summary

Operation	Algorithm	Key Property
Add edge/root	BFS expansion	Assigns increasing ranks
Remove edge/root	Worklist cascade	Rank check breaks cycles

## 5 Level 2: DSL with Automatic Derivation (Future)

The low-level API requires the user to provide `stepFromDelta` and prove that `step` is element-wise and additive. A higher-level approach would let users define  $F$  in a structured DSL, from which these properties are derived automatically.

### 5.1 Analogy: Automatic Differentiation

	Differentiation	Incremental Fixpoint
Low-level	User provides $f(x)$ and $\frac{df}{dx}$	User provides $F$ , <code>stepFromDelta</code> , proofs
High-level (DSL)	User writes expression; system derives gradient	User writes $F$ in DSL; system derives incremental ops
Requirement	$f$ given as expression tree	$F$ given as composition of primitives
Black-box	Finite differences (slow)	Full recomputation (slow)

Just as automatic differentiation requires  $f$  to be expressed as a composition of differentiable primitives, automatic incrementalization requires  $F$  to be expressed as a composition of “incrementalizable” primitives.



## 5.2 Potential DSL Primitives

A DSL for fixpoint operators might include:

- $\text{const}(B)$ : constant base set
- $\text{union}(F_1, F_2)$ : union of two operators
- $\text{join}(R, S, \pi)$ : join  $S$  with relation  $R$ , project via  $\pi$
- $\text{filter}(P, S)$ : filter  $S$  by predicate  $P$
- $\text{lfp}(\lambda S.F(S))$ : least fixpoint

Each primitive would come with:

- Its incremental step function (for semi-naive)
- Its derivation counting semantics (for deletion)

**Example 2** (DCE in DSL). `live = lfp(S =>`

```
union(
  const(roots),
  join(edges, S, (u, v) => v)
)
```

*The system derives:*

- $\text{stepFromDelta}(\Delta) = \text{join}(\text{edges}, \Delta, (u, v) \mapsto v)$
- *Proof that step is element-wise (each edge provides a single derivation)*
- *Proof that step is additive (union distributes over step)*

## 5.3 Connection to Datalog

Datalog engines already perform similar derivations:

- Rules are the structured representation of  $F$
- Semi-naive evaluation is derived from rule structure
- Well-founded cascade generalizes deletion handling

A general incremental fixpoint DSL would extend this beyond Horn clauses to richer operators (aggregation, negation, etc.).

## 6 Examples Beyond DCE

The incremental fixpoint pattern applies to many problems:

Problem	Base	Step	Derivation Count
DCE/Reachability	roots	successors	in-degree from live
Type Inference	base types	constraint propagation	# constraints implying type
Points-to Analysis	direct assignments	transitive flow	# flow paths
Call Graph	entry points	callees of reachable	# callers
Datalog	base facts	rule application	# rule firings

## 7 Relationship to Reactive Systems

In a reactive system like Skip:

- **Layer 1** (reactive aggregation) handles changes to the *parameters* of  $F$  (e.g., the graph structure).
- **Layer 2** (incremental fixpoint) maintains the fixpoint as those parameters change.

The two layers compose: reactive propagation delivers deltas to the fixpoint maintainer, which incrementally updates its state and emits its own deltas (added/removed elements) for downstream consumers.

## 8 Future Work

1. **Design Level 2 DSL:** Define a language of composable fixpoint operators with automatic incrementalization.
2. **Integrate with Skip:** Implement the incremental fixpoint abstraction as a reusable component in the Skip reactive framework.
3. **Explore stratification:** Extend to stratified fixpoints (with negation) where layers must be processed in order.
4. **Benchmark:** Compare incremental vs. recompute performance on realistic workloads.

## 9 Conclusion

The incremental DCE algorithm is an instance of a general pattern: maintaining least fixpoints incrementally under changes to the underlying operator. We propose a two-level architecture:

1. A **low-level API** where users provide `stepFromDelta` and prove `step` is element-wise and additive.
2. A **high-level DSL** (future work) where these proofs are derived automatically from a structured definition of  $F$ , analogous to how automatic differentiation derives gradients from expression structure.

The key contribution is *well-founded cascade*: using the iterative construction rank to handle cycles correctly. Elements not in the new fixpoint have no finite rank, so they have no well-founded derivers and are removed.

This abstraction unifies incremental algorithms across domains (program analysis, databases, reactive systems) and provides a foundation for building efficient, correct incremental computations.