Technical Document: Geometric Linear Pricing Algorithm for FX Basket Options

1. Abstract

This document describes a novel, non-stochastic methodology for pricing basket options, specifically applied to a portfolio of FX options. The "Geometric Linear Pricing" algorithm avoids traditional stochastic calculus and Monte Carlo simulation in favour of a closed-form solution derived from hyperbolic geometry. The primary objective of this model is not absolute precision in line with market consensus but the rapid and robust identification of relative mispricing between a basket of individual options and a theoretical basket option, thereby uncovering arbitrage opportunities.

2. Introduction & Basis

2.1. Problem Statement

Pricing a basket option, which is a single option on a portfolio of underlying assets, is computationally complex. The primary challenge lies in modelling the joint evolution of the correlated assets. In the FX context, this involves multiple currency pairs with non-trivial correlations.

2.2. Traditional Approaches

Standard industry models include:

- **Black-Scholes for Baskets:** An extension of the Black-Scholes model that requires estimating the aggregate volatility and correlation of the basket.
- **Monte Carlo Simulation:** Path-dependent simulation of all underlying assets, which is accurate but computationally intensive and slow.
- **Moment-Matching Methods (e.g., Levy):** Approximates the basket's distribution by matching its first few moments.

2.3. Proposed Basis: Geometric Intuition

The Geometric Linear Pricing model is founded on a geometric interpretation of option pricing. It treats the relationship between the portfolio spot price (`P`) and the strike price (`K`) as a geometric problem in a hyperbolic space. The core intuition is that the value of a basket call option can be expressed as a function of the hyperbolic sine (`sinh`) of the moneyness metric `ln(P/K)`. This approach bypasses the need for direct volatility and correlation inputs, relying instead on the inherent geometric properties of the portfolio's value.

3. Mathematical Formulation

3.1. Core Equation

The value of a basket call option is given by the closed-form formula:

$$C(P, K, T) = e^{-rT}\left[(P+K)\cdot\frac{\sinh(\ln(P/K))}{\ln(P/K)} - K\right]$$

Where:

- C: Theoretical price of the basket call option.
- P: Current spot price of the portfolio (normalized).
- K: Strike price of the basket option (normalized).
- T: Time to expiration (in years).
- r: Risk-free interest rate.
- $\ln$: Natural logarithm.
- $\sinh$: Hyperbolic sine function, defined as $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

3.2. At-The-Money (ATM) Handling

The formula contains a singularity when $P = K$ (i.e., $\ln(P/K) = 0$)..The model handles this edge case by reverting to the intrinsic value, discounted by the risk-free rate:

$$\text{If } |\ln(P/K)| < \epsilon: \quad C(P, K, T) = e^{-rT}(P - K)$$

3.3. Arbitrage Edge Calculation

The algorithm's primary output is the "arbitrage edge," which quantifies the relative mispricing.

1. **Market Price of Components:** The sum of the mid-prices of all individual options in the basket:

   $$S_{\text{market}} = \sum_{i=1}^{n} C_i$$

2. **Theoretical Basket Price:** The price derived from the geometric formula, $C(P,K,T)$.
3. **Arbitrage Edge:** The relative difference:

   $$\text{Edge} = \frac{S_{\text{market}} - C(P,K,T)}{C(P,K,T)}$$

   A positive edge suggests the individual options are overpriced relative to the basket; a negative edge suggests they are underpriced.

4. Algorithm Workflow & Usage

4.1. Inputs

- **FX Basket Definition:** A list of currency pairs with their respective weights and notionals (e.g., EURUSD: 40%, GBPUSD: 30%, USDJPY: 30%).
- **Market Data:**
  - Spot prices for each currency pair.
  - Mid-prices for individual call/put options on each pair.
- **Model Parameters:** Risk-free rate (`r`), time to expiry (`T`), strike price (`K`).

4.2. Operational Steps

The provided code implements the following workflow:

1. **Data Acquisition:** Connects to a market data source (e.g., Interactive Brokers API) to stream live spot and option prices.
2. **Data Normalization:** Scales the portfolio's total market value and strike to a standardized base (e.g., 100) for stable numerical computation within the geometric formula.
3. **Theoretical Pricing:** For a given `(P, K, T)`, the geometric formula calculates the fair value of the basket option.
4. **Mispricing Detection:** Compares the theoretical basket price to the sum of the individual option premiums to calculate the arbitrage edge.
5. **Signal Generation:** If the absolute value of the edge exceeds a pre-defined threshold (e.g., 1%), an arbitrage signal is generated.

    - **Positive Edge:** `S_market > C_theoretical` → Signal to **SELL** the individual options and **BUY** the theoretical basket (or a proxy).
    - **Negative Edge:** `S_market < C_theoretical` → Signal to **BUY** the individual options and **SELL** the theoretical basket.
6. **Continuous Monitoring:** The analysis runs in a loop, continuously scanning for new opportunities under different market scenarios (normal, mispriced, efficient).

5. Benefits and Comparative Advantages

Feature	Geometric Linear Pricing	Traditional Models (Black-Scholes, Monte Carlo)
Computational Speed	**Extremely Fast.** Single, closed-form calculation. Ideal for high-frequency arbitrage scanning.	**Slower.** Requires numerical methods, correlation matrices, or thousands of simulations.
Input Simplicity	**Minimal.** Requires only `P`, `K`, `T`, `r`. No volatility surface or correlation matrix.	**Complex.** Requires volatilities, dividends, and a full correlation matrix for all assets.
Calibration	**Not Required.** The model is parametric and does not need fitting to market prices.	**Essential.** Volatilities and correlations must be calibrated to market data, introducing model risk.
Primary Use Case	**Relative Value & Arbitrage.** Excels at identifying *relative* mispricing quickly and robustly.	**Absolute Pricing.** Aims to price an option to its "fair" market value for risk management or OTC trading.

Feature	Geometric Linear Pricing	Traditional Models (Black-Scholes, Monte Carlo)
Model Risk	**Different Profile.** Risk comes from the validity of the geometric analogy, not from miscalibrated Greeks.	**High.** Sensitive to errors in volatility and correlation inputs. "Garbage in, garbage out."

Key Advantages Summarized:

1. **Robustness in Arbitrage Detection:** By eliminating the need for volatile and hard-to-estimate correlation inputs, the model provides a more stable measure of relative value. An arbitrage signal is less likely to be a false positive caused by a faulty correlation estimate.
2. **Speed and Efficiency:** The computational lightness allows for monitoring hundreds of potential baskets simultaneously in real-time.
3. **Conceptual Elegance:** It offers a fundamentally different viewpoint on option pricing, which can serve as an independent check against correlation-dependent models.

6. Limitations and Considerations

- **Absolute Accuracy:** This model is not designed for pinpoint absolute pricing. Its theoretical price may consistently differ from the market price of a traded basket option. Its value is in the *difference* between its internal consistency check and the market.
- **Theoretical Justification:** The model is presented as a heuristic with a geometric intuition. A rigorous derivation from first principles (e.g., a novel no-arbitrage argument) would be required for broader academic acceptance.
- **Implementation Challenges:** Executing the arbitrage may be difficult. A direct, exchange-traded basket option on a custom FX portfolio is rare. Execution would typically involve delta-hedging the individual options to synthetically create the basket, introducing additional complexity and execution risk.
- **Market Scenarios:** The model's effectiveness may vary across different market regimes (high volatility, low volatility).

7. Conclusion

The Geometric Linear Pricing algorithm represents an innovative, heuristic approach to a classic quantitative finance problem. Its strength lies not in displacing traditional models for absolute pricing, but in complementing them as a highly efficient and robust tool for relative value and statistical arbitrage detection. By leveraging a closed-form geometric formula, it provides a fast, simple, and correlation-free method for identifying mispricing in portfolios of options, making it a valuable addition to the arsenal of any systematic or arbitrage-focused trading desk.