A Laundry List of Theorems in Analysis

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April 11, 2025

Preface

These notes are a loose amalgamation of ideas, concepts, and explanations drawn from various sources, particularly the following books:

1. Real Analysis: A Long-Form Mathematics Textbook by Jay Cummings [Cu19]
2. Understanding Analysis by Stephen Abbott [Ab15]
3. Introduction to Real Analysis by Bartle & Sherbert [BaSh11]
4. Mathematical Analysis by Tom M. Apostol [Ap74]
5. Counterexamples in Analysis by Gelbaum & Olmsted [GeOl03]

They were originally meant for my own understanding and organization of thoughts, and as such, they may be unpolished, incomplete, or even occasionally incorrect.

I share them in the hope that they may serve as a useful reference, but they should not be treated as a primary source of learning. Readers are strongly encouraged to consult original texts and authoritative resources for a more rigorous and accurate treatment of the topics discussed.

Use these notes as a companion to your studies, not as a substitute for the depth and clarity provided by well-established literature.

Contents

1 The Real Numbers

> "It is a 'simple' theorem, simple both in idea and execution, but there is no doubt at all about it being a theorem of the highest class. It is as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on it."
>
> *G. H. Hardy, A Mathematician's Apology*

Theorem 1.1 (The irrationality of $\sqrt{2}$)
There is no rational number whose square is 2.

Proof. A rational number is any number that can be expressed in the form p/q, where p and q are integers. Thus, what the theorem asserts is that no matter how p and q are chosen, it is never the case that $(p/q)^2 = 2$. The line of attack is indirect, using a type of argument referred to as a proof by contradiction. The idea is to assume that there is a rational number whose square is 2 and then proceed along logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erroneous assumption that some rational number squared is equal to 2. In short, we will prove that the theorem is true by demonstrating that it cannot be false.

And so assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 2. \tag{1}$$

We may also assume that p and q have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Now, equation (1) implies

$$p^2 = 2q^2. \tag{2}$$

From this, we can see that the integer p^2 is an even number (it is divisible by 2), and hence p must be even as well because the square of an odd number is odd. This allows us to write $p = 2r$, where r is also an integer. If we substitute $2r$ for p in equation (2), then a little algebra yields the relationship

$$2r^2 = q^2.$$

But now the absurdity is at hand. This last equation implies that q^2 is even, and hence q must also be even. Thus, we have shown that p and q are both even (i.e., divisible by 2) when they were originally assumed to have no common factor. From this logical impasse, we can only conclude that equation (1) cannot hold for any integers p and q, and thus the theorem is proved. □

Remark 1.2 — So the rationals aren't quite enough. That said, they do have almost every other fundamental property we would want. To the point: They are what we call an *ordered field*. But first, what's a *field*? It's a set that satisfies the classic additive and multiplicative properties we know and love.

Definition 1.3 (Fields)

A **field** is a nonempty set $\mathbb{F}$, along with two binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If $a, b \in \mathbb{F}$, then $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Axiom 2 (Distributive Law).** If $a, b \in \mathbb{F}$, $a \cdot (b+c) = a \cdot b + a \cdot c$.
- **Axiom 3 (Associative Law).** If $a, b \in \mathbb{F}$, then $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Axiom 4 (Identity Law).** There are special elements $\mathbf{0}, \mathbf{1} \in \mathbb{F}$, where $a + \mathbf{0} = a$ and $a \cdot \mathbf{1} = a$ for all $a \in \mathbb{F}$.
- **Axiom 5 (Inverse Law).** For each $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = \mathbf{0}$. If $a \neq \mathbf{0}$, then there is also an element $a^{-\mathbf{1}} \in \mathbb{F}$ such that $a \times a^{-\mathbf{1}} = \mathbf{1}$.

Example 1.4

Below are some examples and some non-examples of fields.

- The natural number $\mathbb{N}$ do not form a field; they fail the first half of Axiom 4 and both halves of Axiom 5.
- The integers $\mathbb{Z}$ *almost* form a field; they only fail the second half of Axiom 5.
- One can check that the rationals $\mathbb{Q}$ form a field.

Remark 1.5 — Now, let's come up with a definition for an ordered field. Think about $\mathbb{Q}$. What does $\mathbb{Q}$ have that a field does not? There are three main properties we are missing: First, there are infinitely many rationals (and they are "symmetric" about the 0 element.) Second, the rationals have an ordering to them. Lastly, we would like to talk about how big a number is. Beautifully, Definition 1.6 describes a single elegant axiom that we can include to capture *all* of these properties.

Definition 1.6 (Ordered Fields)

An **ordered field** is a field $\mathbb{F}$, along with the following additional axiom.

Axiom 6 (Order Axiom). There is a nonempty subset $P \subseteq \mathbb{F}$, called the *positive elements*, such that

1. If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;
2. If $a \in \mathbb{F}$ and $a \neq \mathbf{0}$, then either $a \in P$ or $-a \in P$, but not both.

Definition 1.7 (Inequalities)

If $\mathbb{F}$ is an ordered field and $a, b \in \mathbb{F}$, then we say that "$a < b$" if $b - a \in P$. Likewise, $a \leq b$ means that either $a = b$ or $a < b$.

We define "$>$" similarly.

Fact 1.8 (Properties of inequalities)
For a, b, c in an ordered field $\mathbb{F}$:

1. If $a < b$, then $a + c < b + c$.
2. Transitivity: If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$, then $ac < bc$ if $c > 0$, and $ac > bc$ if $c < 0$.
4. If $a \neq 0$, then $a^2 > 0$.

Definition 1.9 (The absolute value function)
If $\mathbb{F}$ is an ordered field, define the *absolute value* function $|\cdot| : \mathbb{F} \to \mathbb{F}$ to be

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Fact 1.10 (Properties of absolute values)
For a, b in an ordered field $\mathbb{F}$:

1. $|a| \geq 0$, with equality if and only if $a = 0$.
2. $|a| = |-a|$.
3. $-|a| \leq a \leq |a|$.
4. $|a \cdot b| = |a| \cdot |b|$.
5. $1/|a| = |1/a|$, if $a \neq 0$.
6. $|a/b| = |a|/|b|$, if $b \neq 0$.
7. $|a| \leq b$ if and only if $-b \leq a \leq b$.

Theorem 1.11 (The triangle inequality)
If $\mathbb{F}$ is an ordered field and if $x, y \in \mathbb{F}$, then

$$|x + y| \leq |x| + |y|.$$

Proof. For $x, y \in \mathbb{F}$, by Fact 1.10 part 3 we have

$$-|x| \leq x \leq |x| \qquad \text{and} \qquad -|y| \leq y \leq |y|.$$

Adding these two together gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

And so, by Fact 1.10 part 7,

$$|x + y| \leq |x| + |y|.$$

$\square$

Corollary 1.12 (The reverse triangle inequality)

Assume that $\mathbb{F}$ is an ordered field and $x, y \in \mathbb{F}$. Then,

$$||x| - |y|| \leq |x - y|.$$

Proof. By Fact 1.10 part 7, it suffices to show that

$$-|x - y| \leq |x| - |y| \leq |x - y|.$$

We first show the right-hand one (that $|x| - |y| \leq |x - y|$), and we will do so by applying an application of the triangle inequality. Let $a = x - y$ and $b = y$. Then by the triangle inequality,

$$|a + b| \leq |a| + |b|.$$

That is,

$$\begin{aligned} |(x - y) + y| &\leq |x - y| + |y| \\ |x| &\leq |x - y| + |y|. \end{aligned}$$

Rearranging,

$$|x| - |y| \leq |x - y|.$$

We will use a similar approach to show that $-|x - y| \leq |x| - |y|$. Let $c = y - x$ and $d = x$. By the triangle inequality,

$$|c + d| \leq |c| + |d|$$

That is,

$$\begin{aligned} |(y - x) + x| &\leq |y - x| + |x| \\ |y| &\leq |y - x| + |x|. \end{aligned}$$

Reaaranging,

$$-|y - x| \leq |x| - |y|,$$

which by Fact 1.10 part 2 implies that

$$-|x - y| \leq |x| - |y|,$$

as desired. □

Corollary 1.13 (Triangle inequality corollaries)

For both of the following, assume that $\mathbb{F}$ is an ordered field and $x, y \in \mathbb{F}$.

1. $|x - y| \leq |x| + |y|$.
2. $|x + y| \geq ||x| - |y||$.

Proof.

For part 1, replace y with $-y$ in the triangle inequality.

For part 2, replace y with $-y$ in the reserve triangle inequality. □

Theorem 1.14 (Cauchy-Schwarz inequality)

If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real numbers, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Proof. A sum of squares can never be negative. Hence we have

$$\sum_{i=1}^{n} (a_i x + b_i)^2 \geq 0$$

for every $x \in \mathbb{R}$, with equality if and only if each term is zero. This inequality can be written in the form

$$Ax^2 + 2Bx + C \geq 0$$

where

$$A = \sum_{i=1}^{n} a_i^2, \qquad B = \sum_{i=1}^{n} a_i b_i, \qquad C = \sum_{i=1}^{n} b_i^2.$$

If $A > 0$, put $x = -B/A$ to obtain $B^2 - AC \leq 0$, which is the desired inequality. Otherwise if $A = 0$, the rest of the proof is trivial. $\square$

Definition 1.15 (Upper and lower bounds)

Let S be an ordered field and $A \subseteq S$ be nonempty.

1. The set A is *bounded above* if there exists some $b \in S$ such that $x \leq b$ for all $x \in A$; in this case b is called an **upper bound** of A.
2. The **least upper bound** of A—if it exists—is some $b_0 \in S$ such that
   a) b_0 is an upper bound of A, and
   b) if b is any other upper bound of A, then $b_0 \leq b$.

   Such a b_0 is also called the **supremum** of A and is denoted $\sup(A)$.
3. Likewise, the set A is *bounded below* if there exists some $b \in S$ such that $x \geq b$ for all $x \in A$; in this case, b is called a **lower bound** of A.
4. Again, like above, the **greatest lower bound** of A—if it exists—is some $b_0 \in S$ such that
   a) b_0 is a lower bound of A, and
   b) if b is any other lower bound of A, then $b_0 \geq b$.

   Such a b_0 is also called the **infimum** of A and is denoted $\inf(A)$.
5. If a set is both bounded above and bounded below, then it is simply *bounded*.

Example 1.16

The propositions below are left without proof.

- The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ has no upper bounds. Lower bounds on $\mathbb{N}$ include -17, 1, 0.123, and $-\pi$. Note that $\sup(\mathbb{N})$ does not exist, but $\inf(\mathbb{N}) = 1$.
- The set $\mathbb{Q}$ has no upper or lower bounds; consequently, $\sup(\mathbb{Q})$ and $\inf(\mathbb{Q})$ do not exist.
- $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$; $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$. Note that the supremum here is in the set, while the infimum is not in the set.
- $\sup(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = 1$; $\inf(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = \frac{1}{2}$. Note that the infimum here is in the set, while the supremum is not in the set.
- In $\mathbb{Q}$ the set $\{x \in \mathbb{Q} : x^2 < 2\}$ does not have a supremum. In $\mathbb{R}$ it will—in fact, $\sup(\{x \in \mathbb{Q} : x^2 < 2\}) = \sqrt{2}$.

Definition 1.17 (Completeness)

Let S be an ordered field. Then S has the **least upper bound property** if given any nonempty $A \subseteq S$ where A is bounded above, A has a least supper bound in S. In other words, $\sup(A) \in S$ for every such A.

Such a set S is also called **complete**.

Theorem 1.18 (Existence of $\mathbb{R}$)

The real numbers $\mathbb{R}$ exists, and it is a complete ordered field.

Proof Sketch. We have the ordered field of rational numbers, but they aren't complete—there are holes everywhere, and to get to $\mathbb{R}$ we must fill in these gaps. There are several ways to do this. But the most common method, which we discuss now, uses *Dedekind cuts.*

Each real number is going to be a set; at this point we have the rationals constructed, so each real number is going to be represented by a set of rationals. The way you want to think about it is this: the real number x is going to be represented by the set of all rational numbers strictly less than x. These sets are going to be called *cuts*, and while we discuss them you can start convincing yourself that each real number will indeed correspond to a unique cut, and each cut corresponds to a unique real number.

Definition 1.19 (Dedekind cuts)

A **cut** should be thought of as the set $(-\infty, b) \cap \mathbb{Q}$. That is, all rational numbers up to a certain point. Formally, it is defined as any set C_b satisfying the following three conditions.

1. $C_b \subseteq \mathbb{Q}$, but $C_b \neq \varnothing$ and $C_b \neq \mathbb{Q}$;
2. If $p \in C_b$ and $q \notin C_b$, then $p < q$;
3. If $p \in C_b$, then there exists some $q \in C_b$ where $p < q$.

And then $\mathbb{R}$ is defined as the set of all cuts.

Although we have an intuitive picture of a cut, at the moment it is just a set satisfying the above properties. We are interested in putting an algebraic structure on this collection of cuts. Addition and order work quite smoothly. For a pair of cuts C_a and C_b, define the following.

- $C_a + C_b := \{p + q : p \in C_a, q \in C_b\}$
- $C_a < C_b$ if and only if $C_a \subsetneq C_b$

One can verify that the addition of two cuts is still a cut, and that addition is commutative and associative, that the **0** cut behaves as it should (in turn giving *positive* and *negative* cuts) and that additive inverses exist. The inequality has the property that, given any two cuts C_a and C_b, exactly one of the following holds: $C_a < C_b$, $C_a = C_b$, or $C_b < C_a$.

Defining multiplication is trickier, because if you simply multiply the two sets together you'll have massive negative numbers multiplying against each other, creating massive positive numbers. Intuitively, you want $C_a \cdot C_b = C_{ab}$. That is, the cut $(-\infty, a) \cap \mathbb{Q}$ times the cut $(-\infty, b) \cap \mathbb{Q}$ should equal the cut $(-\infty, a \cdot b) \cap \mathbb{Q}$; but cuts aren't defined with such a and b—they produce a and b. One way around this is to first define multiplication for positive cuts. That is, if C_a and C_b are both positive (larger than the cut $\{q \in \mathbb{Q} : q < 0\}$), then define

$$C_a \cdot C_b := \{p \cdot q : p \in C_a, q \in C_b \text{ with } p, q \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

With this you can then define a product of two negative cuts by setting the product equal to the product of the two corresponding cuts. To define multiplication between a positive and a negative cut, you know the product should be negative so one approach is to consider the multiplication when both are positive, and then translate the result to the corresponding negative cut. It's a hassle to write out, but that's the idea.

One can then check that the product of two cuts is still a cut, that multiplicative inverses exist, and that multiplication is commutative and associative, as well as the remaining multiplicative/additive distributive and order properties. These would be quite annoying to work out in detail, but you can smile knowing someone carefully checked them.

With our set built, and with the algebraic properties defined and their properties verified, we now know that $\mathbb{R}$ is an ordered field. All that is left to show is that it is complete. This is done by showing that cuts satisfy the *least upper bound property*. That is, if $\mathcal{C}$ is a collection of cuts which is bounded above (meaning there exists some cut D such that $C \leq D$ for all $C \in \mathcal{C}$), then there exists a least upper bound (meaning there is a cut M such that $M \leq D$ for all upper bounds D). The proof is this: Let $M = \bigcup_{C \in \mathcal{C}} C$, and show that

1. M is a cut and therefore $M \in \mathbb{R}$;
2. M is an upper bound of $\mathcal{C}$, but is smaller than all other upper bounds.

With those, $\mathbb{R}$ is a complete ordered field, and hence the real numbers are constructed. $\square$

Theorem 1.20 (Uniqueness of $\mathbb{R}$)

Any complete ordered field is isomorphic to $\mathbb{R}$.

Proof Sketch. First, let us establish that every complete ordered field R is Archimedean. This means that there is no element in R that is larger than every finite sum $\mathbf{1}+\mathbf{1}+\cdots+\mathbf{1}$. (Why? Refer to Lemma 1.27. If we let $a = \mathbf{1}$, the Archimedean principle states that for any b, there exists an n such that $n \cdot \mathbf{1} > b$. This $n \cdot \mathbf{1}$ is precisely a finite sum $\mathbf{1}+\mathbf{1}+\cdots+\mathbf{1}$). To prove this property, we use contradiction. Suppose there exists such an element, then by completeness (Definition 1.17), there is a least upper bound b to these all these finite sums. But then $b-\mathbf{1}$ would also be an upper bound, since adding $\mathbf{1}$ to any sum would still be less than b. This contradicts b being the least upper bound. Therefore, no such element exists, and R must be Archimedean.

Consider two complete ordered fields, $\mathbb{R}_0$ and $\mathbb{R}_1$. We construct their respective prime subfields, that is, their copies of the rationals $\mathbb{Q}_0$ and $\mathbb{Q}_1$. This is done by computing inside them all the finite quotients of the form $\pm(\mathbf{1}+\mathbf{1}+\cdots+\mathbf{1})/(\mathbf{1}+\mathbf{1}+\cdots+\mathbf{1})$, which essentially represent all fractions p/q where $p, q \in \mathbb{Z}$ and $q > 0$. The fractional representation naturally defines an isomorphism between $\mathbb{Q}_0$ and $\mathbb{Q}_1$. This means that corresponding rational elements in each field map to each other, and this mapping preserves addition, multiplication, and order. This is illustrated in the diagram with the blue dots and arrows connecting corresponding rational elements.

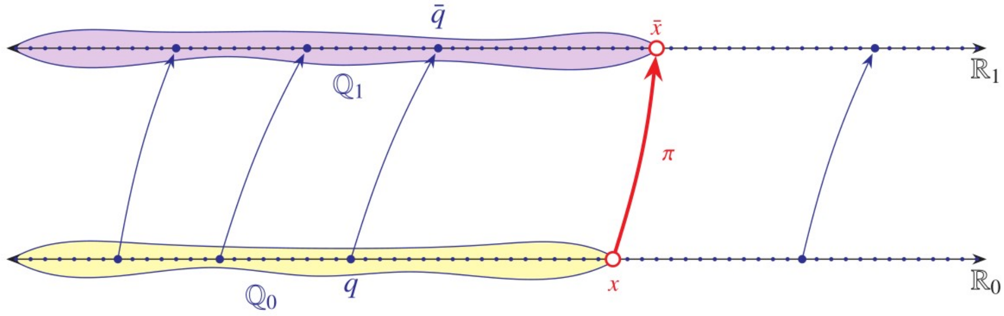


Figure 1.1: An isomorphic mapping $\pi : \mathbb{R}_0 \to \mathbb{R}_1$ (courtesy of Hamkins [Ha]).

We now extend this isomorphism to the entire fields, $\mathbb{R}_0$ and $\mathbb{R}_1$, using Dedekind cuts (Definition 1.19). The Archimedean property ensures that every element x in $\mathbb{R}_0$ defines a unique cut in $\mathbb{Q}_0$, dividing it into two sets: Those rational elements less than x (shown in yellow), and those greater than or equal to x. Since we have an isomorphism between $\mathbb{Q}_0$ and $\mathbb{Q}_1$, this cut in $\mathbb{Q}_0$ corresponds to a similar division in $\mathbb{Q}_1$. By the completeness of $\mathbb{R}_1$, there must be a unique element $\overline{x} \in \mathbb{R}_1$ that makes this exact same cut in $\mathbb{Q}_1$ (shown in violet). This defines our mapping $\pi : x \mapsto \overline{x}$ from $\mathbb{R}_0$ to $\mathbb{R}_1$.

Finally, we verify that the mapping π is indeed a field isomorphism:

1. Surjection: Every $y \in \mathbb{R}_1$ determines a cut in $\mathbb{Q}_1$. By the isomorphism between $\mathbb{Q}_0$ and $\mathbb{Q}_1$, this corresponds to a cut in $\mathbb{Q}_0$. By the completeness of $\mathbb{R}_0$, there exists an $x \in \mathbb{R}_0$ that defines this cut. Thus, $\pi(x) = y$.

2. Injection: If two elements $x_1, x_2 \in \mathbb{R}_0$ determine the same cut in $\mathbb{Q}_0$, then $x_1 = x_2$ (by the definition of Dedekind cuts). Thus, if $\pi(x_1) = \pi(x_2)$, then x_1 and x_2 must define the same cut in $\mathbb{Q}_0$, which means $x_1 = x_2$.

3. Field homomorphism: The mapping π preserves field operations (i.e. addition, multiplication, and order) because it is constructed as a continuous extension of the isomorphism between the rational subfields $\mathbb{Q}_0$ and $\mathbb{Q}_1$.

This completes the proof. $\square$

Corollary 1.21 (Existence and uniqueness of $\mathbb{R}$)

There exists a unique complete ordered field. We call this field *the real numbers* $\mathbb{R}$.

Proposition 1.22 (Axioms of $\mathbb{R}$)

The set $\mathbb{R}$ has two binary operations, addition $(+)$ and multiplication $(\cdot)$, and is the unique set satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If $a, b \in \mathbb{R}$, then $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Axiom 2 (Distributive Law).** If $a, b \in \mathbb{R}$, $a \cdot (b+c) = a \cdot b + a \cdot c$.
- **Axiom 3 (Associative Law).** If $a, b \in \mathbb{R}$, then $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Axiom 4 (Identity Law).** There are special elements $\mathbf{0}, \mathbf{1} \in \mathbb{R}$, where $a + \mathbf{0} = a$ and $a \cdot \mathbf{1} = a$ for all $a \in \mathbb{R}$.
- **Axiom 5 (Inverse Law).** For each $a \in \mathbb{R}$, there is an element $-a \in \mathbb{R}$ such that $a + (-a) = \mathbf{0}$. If $a \neq \mathbf{0}$, then there is also an element $a^{-\mathbf{1}} \in \mathbb{R}$ such that $a \times a^{-\mathbf{1}} = \mathbf{1}$.
- **Axiom 6 (Order Axiom).** There is a nonempty subset $P \subseteq \mathbb{R}$, called the *positive elements*, such that
  1. If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;
  2. If $a \in \mathbb{R}$ and $a \neq \mathbf{0}$, then either $a \in P$ or $-a \in P$, but not both.
- **Axiom 7 (Completeness Axiom).** Given any nonempty $A \subseteq \mathbb{R}$ where A is bounded above, A has a least upper bound. In other words, $\sup(A) \in \mathbb{R}$ for every such A.

Proposition 1.23 (Suprema are unique)

If the supremum or infimum of $A \subseteq \mathbb{R}$ exists, then it is unique.

We will only prove that suprema are unique. The infima case is analogous.

Proof. Assume for a contradiction that α and β are distinct least upper bounds of A. In particular, both are upper bounds of A, while $\alpha \neq \beta$. One one hand, since α is a least upper bound and β is an upper bound, we must have $\alpha \leq \beta$. On the other hand, since β is a least upper bound and α is an upper bound, we must have $\beta \leq \alpha$. In summary,

$$\alpha \leq \beta \qquad \text{and} \qquad \beta \leq \alpha.$$

This implies that $\alpha = \beta$, giving our contradiction. $\square$

Theorem 1.24 (Square roots exist)

If $a \in \mathbb{R}$ and $a \geq 0$, then $\sqrt{a} \in \mathbb{R}$.

Proof Idea. One can show that $\sqrt{a} = \sup(\{x \in \mathbb{R} : x^2 < a\})$, which is in $\mathbb{R}$ by completeness. $\square$

Theorem 1.25 (Suprema analytically)
Let $A \subseteq \mathbb{R}$. Then $\sup(A) = \alpha$ if and only if

1. α is an upper bound of A, and
2. Given any $\epsilon > 0$, $\alpha - \epsilon$ is *not* an upper bound of A. That is, there is some $x \in A$ for which $x > \alpha - \epsilon$.

Likewise, $\inf(A) = \beta$ if and only if

1. β is a lower bound of A, and
2. Given any $\epsilon > 0$, $\beta + \epsilon$ is *not* a lower bound of A. That is, there is some $x \in A$ for which $x < \beta + \epsilon$.

We will only prove the suprema case. The infima case is analogous.

Proof.

($\Rightarrow$) First, assume that $\sup(A) = \alpha$. We aim to prove part 1 and 2. The first of these is immediate: Since $\sup(A) = \alpha$, α is the least upper bound of A, which of course also implies that it is an upper bound of A.

Now we will show part 2. Let $\epsilon > 0$, Since $\alpha - \epsilon < \alpha$, we know that $\alpha - \epsilon$ is not an upper bound of A, because if so that would contradict α being the least upper bound of A. And so, since $\alpha - \epsilon$ is not an upper bound, there must be some x who is greater than $\alpha - \epsilon$.

($\Leftarrow$) Now assume part 1 and 2. We aim to prove that $\sup(A) = \alpha$. That is, we wish to show that α is an upper bound of A (which is implied directly by part 1), and for any other upper bound β, we have $\alpha \leq \beta$. We have only the latter to prove. Assume that β is some other upper bound of A, and assume for a contradiction that $\beta < \alpha$. Note that $0 < \alpha - \beta$. We will use $(\alpha - \beta)$ as our ϵ, and then apply part 2 to contradict β being an upper bound.

Now we will work it out formally. Let $\epsilon = \alpha - \beta$. Since $\epsilon > 0$, by part 2 there exists some $x \in A$ such that $x > \alpha - \epsilon = \alpha - (\alpha - \beta) = \beta$. But this is a contradiction, because we assumed that β was an upper bound of A, and yet we found another element $x \in A$ that is larger than β. $\square$

Remark 1.26 — Note that the forward direction of the proof also works well by contrapositive. The contrapositive of

$$\sup(A) = \alpha \implies \text{For all } \epsilon > 0 \text{ there is an } x \in A \text{ such that } x > \alpha - \epsilon$$

is

$$\text{There is an } \epsilon > 0 \text{ such that for all } x \in A \text{ we have } x \leq \alpha - \epsilon \implies \sup(A) \neq \alpha.$$

And to prove this, just observe that the left-hand side implies that $\alpha - \epsilon$ is an upper bound of A, and so $\sup(A) \leq \alpha - \epsilon$, which of course implies that $\sup(A) \neq \alpha$, as desired.

Lemma 1.27 (The Archimedean principle)

If a and b are real numbers with $a > 0$, then there exists a natural number n such that $na > b$.

In particular, for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. We aim to show that $na > b$ for some $n \in \mathbb{N}$; by dividing over the a, we aim to prove that there is some $n \in \mathbb{N}$ such that $n > b/a$. Now, the number b/a is just some real number that we know nothing about. In fact, let's just call it x. So, equivalently, we are trying to prove that given any real number x, there is some integer n such that $n > x$.

Assume for a contradiction that there is no integer larger than x. That is, assume that x is an upper bound on the set $\mathbb{N}$. Then $\mathbb{N}$ is a subset of $\mathbb{R}$ that is bounded above, and so by the completeness of $\mathbb{R}$ we deduce that $\sup(\mathbb{N})$ exists. Call this supremum α. Since α is the least upper bound of $\mathbb{N}$, we know that $\alpha - 1$ is not an upper bound. That is, there exists some integer $m > \alpha - 1$. Adding 1 to each side,

$$m + 1 > \alpha.$$

But this is a contradiction. If α is the supremum of $\mathbb{N}$, then it is an upper bound on $\mathbb{N}$. But we found $(m + 1) \in \mathbb{N}$ which is larger than α. This concludes the first statement in the principle.

The second part follows directly from the first by letting $a = \epsilon$ and $b = 1$, and dividing over the n. □

Example 1.28

Show that $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$.

Proof. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. We will use the analytic definition of suprema (Theorem 1.25). We must then show that 0 is a lower bound of A and that, for all $\epsilon > 0$, $0 + \epsilon$ is not a lower bound of A.

The first of these is almost immediate: Since 1 and n are positive for each $n \in \mathbb{N}$, so is $1/n$. So $1/n > 0$, and thus 0 is indeed a lower bound for A.

Working toward the second, let $\epsilon > 0$. Then by the Archimedean principle (Lemma 1.27), there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. This element, $\frac{1}{n}$, is in A and is less than $0 + \epsilon$. So $0 + \epsilon$ is not a lower bound of A. □

Example 1.29

Show that $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$.

Proof. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. We will use the analytic definition of suprema (Theorem 1.25). We must then show that 1 is an upper bound of A and that, for all $\epsilon > 0$, $1 - \epsilon$ is not an upper bound of A.

For the first of these, note that since $n \geq 1$ for all $n \in \mathbb{N}$, and by dividing over the n we have that $1 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$. So 1 is indeed an upper bound for A.

Working towards the second, let $\epsilon > 0$. We need to show that there is some $x \in A$ such that $1 - \epsilon < x$. But this is always accomplished by the number 1: Clearly $1 \in A$ and $1 - \epsilon < 1$. □

Theorem 1.30 ($0.999\ldots = 1$)
$0.999\ldots$ is a repeating decimal representation of the real number 1.

Proof Idea. If we place $0.9, 0.99, 0.999, \ldots$ on the real number line, we see immediately that all these points are to the left of 1, and that they get closer and closer to 1. For any number x that is less than 1, the sequence $0.9, 0.99, 0.999, \ldots$ will eventually reach a number larger than x. So it does not make sense to identify $0.999\ldots$ with any number less than 1. This means that $0.999\ldots$ must be equal to 1. □

Proof. Let $0.(9)_n$ denote the number $0.999\ldots9$ with n nines after the decimal point. For example, $0.(9)_1 = 0.9$, $0.(9)_2 = 0.99$, $0.(9)_3 = 0.999$, and so on. Observe that for each $n \in \mathbb{N}$, we have

$$1 - 0.(9)_n = \frac{1}{10^n}.$$

Now, let $x = 0.999\ldots$. Clearly, $0.(9)_n < x \leq 1$ for every $n \in \mathbb{N}$. Subtracting these inequalities from 1 gives us

$$0 \leq 1 - x < 1 - 0.(9)_n = \frac{1}{10^n}.$$

By the Archimedean principle (Lemma 1.27), there is no positive number smaller than

$$\frac{1}{10^n} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $1 - x$ must be equal to 0. That is, $x = 1$. □

Remark 1.31 — Despite the persistent rumors, $0.999\ldots$ is not “almost exactly 1” or “very, very nearly but not quite 1” or “1’s clingy little sibling.” Rather, $0.999\ldots$ and 1 are, in fact, the same number. So the next time someone insists otherwise, feel free to hand them a copy of these notes.

Remark 1.32 — The Archimedean principle used in Theorem 1.30 differs from the one stated in Lemma 1.27. One can show that these two formulations are equivalent.

Remark 1.33 — We will revisit Theorem 1.30 in future chapters and discuss various different proofs.

Definition 1.34 (Density)
Suppose A and B are ordered field. Then A is **dense** in B if, for any $x, y \in B$, there exists $a \in A$ such that $x < a < y$.

Example 1.35

The propositions below are left without proof.

- $\mathbb{Q}$ is dense in $\mathbb{Q}$.
- $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{Q}$.
- $\mathbb{Z}$ is *not* dense in $\mathbb{Q}$.

Lemma 1.36

Let $x, y \in \mathbb{R}$. If $y - x > 1$, then there exists $z \in \mathbb{Z}$ such that $x < z < y$.

Proof. First assume that x and y are at least 0, and consider the set

$$A = \{n \in \mathbb{N}_0 : n \leq x\}.$$

Since $x \geq 0$, this set is non-empty, and since it is a set of nonnegative integers which is bounded above by x, this set is finite. By induction on the size of A, we can show that $\max(A)$ exists and is an element of A. Call this maximum M. We claim $z := M + 1$ works.

Note that since $M \in \mathbb{N}_0$, also $z \in \mathbb{N}_0$. Furthermore, since z is larger than the largest element of A, z is not in A, implying that $x < z$. Finally,

$$M \leq x \qquad \text{implies that} \qquad M + 1 \leq x + 1 \leq y.$$

So $z < y$. In summary, we have shown that $x < z < y$, as desired.

The cases where x and y are not at least 0 are similar. If both are negative, then by considering $-x$ and $-y$ the above argument gives an integer z where $-y < z < -x$, showing that $-z$ works, since $x < -z < y$. If one is positive and one is negative, then 0 works. $\square$

Theorem 1.37 ($\mathbb{Q}$ is dense in $\mathbb{R}$)

The rational numbers are dense in the real numbers.

Proof. Pick any $x, y \in \mathbb{R}$ where $x < y$. We need to show that there exists some $\frac{m}{n} \in \mathbb{Q}$ (with $m, n \in \mathbb{Z}$) such that

$$x < \frac{m}{n} < y.$$

First note that if $x < 0 < y$ then we are done, since $0 \in \mathbb{Q}$. Furthermore, if we can show that the theorem holds for the case that x and y are positive, then it holds when they are negative ($0 < x < \frac{m}{n} < y$ implies $-y < \frac{-m}{n} < -x < 0$), so we may assume x and y are positive.

Since $y - x > 0$, by the Archimedean principle (Lemma 1.27) there exists some $n \in \mathbb{N}$ such that $n(y - x) > 1$; i.e. $ny - nx > 1$. And so, by Lemma 1.36, there is some integer m with

$$nx < m < ny.$$

That is,

$$x < \frac{m}{n} < y,$$

which concludes the proof. $\square$

Remark 1.38 — The proof of Lemma 1.36 also implies that, for any $x \in \mathbb{R}$, there exists an integer M such that $M \leq x \leq M + 1$. In particular, it implies that the floor and ceiling functions exist.

Definition 1.39 (Floor and ceiling functions)
Let $x \in R$.

- The *floor* of x, denoted $\lfloor x \rfloor$, is the integer n such that $x - 1 < n \leq x$.
- The *ceiling* of x, denoted $\lceil x \rceil$, is the integer n such that $x \leq n < x + 1$.

Definition 1.40 (Closed and open intervals)
Define the *closed interval* $[a, b]$ to be $\{x \in \mathbb{R} : a \leq x \leq b\}$. Likewise the *open interval* (a, b) is defined to be $\{x \in \mathbb{R} : a < x < b\}$, and half-open intervals and intervals to $\pm\infty$ are again exactly as you would expect.

Theorem 1.41 (Characterization of intervals)
Let S be a subset of $\mathbb{R}$ that contains at least two points. If S has the property such that

$$\text{if } x, y \in S \text{ and } x < y, \text{ then } [x, y] \subseteq S, \tag{1}$$

then S is an interval.

Proof. There are four cases to consider: (1) S is bounded, (2) S is bounded above but not below, (3) S is bounded below but not above, and (4) S is neither bounded above nor below.

Case (1). Let $a = \inf(S)$ and $b = \sup(S)$. Then $S \subseteq [a, b]$ and we will show that $(a, b) \subseteq S$. If $a < z < b$, then z is not a lower bound of S, so there exists $x \in S$ with $x < z$. Also, z is not an upper bound of S, so there exists $y \in S$ with $z < y$. Therefore, $z \in [x, y]$, so property (1) implies that $z \in S$. Since z is an arbitrary element of (a, b), we conclude that $(a, b) \subseteq S$. Now if $a \in S$ and $b \in S$, then $S = [a, b]$. If $a \notin S$ and $b \notin S$, then $S = (a, b)$. The other possibilities lead to either $S = (a, b]$ or $S = [a, b)$.

Case (2). Let $b = \sup(S)$. Then $S \subseteq (-\infty, b]$ and we will show that $(-\infty, b) \subseteq S$. If $z < b$, then z is not an upper bound of S, so there exists $y \in S$ with $z < y$. Also, since S is not bounded below, there exists $x \in S$ with $x < z$. By property 1, $z \in [x, y] \subseteq S$. Since z is an arbitrary element of $(-\infty, b)$, we conclude that $(-\infty, b) \subseteq S$. Now if $b \in S$, then $S = (-\infty, b]$, and if $b \notin S$, then $S = (-\infty, b)$.

Case (3). Let $a = \inf(S)$. Then $S \subseteq [a, \infty)$ and we will show that $(a, \infty) \subseteq S$. If $a < z$, then z is not a lower bound of S, so there exists $x \in S$ with $x < z$. Also, since S is not bounded above, there exists $y \in S$ with $z < y$. By property 1, $z \in [x, y] \subseteq S$. Since z is an arbitrary element of (a, ∞), we conclude that $(a, \infty) \subseteq S$. Now if $a \in S$, then $S = [a, \infty]$, and if $a \notin S$, then $S = (a, \infty)$.

Case (4). We will show that $S = (-\infty, \infty)$. Pick any $x, y \in S$ with $x < y$, property 1 implies that $[x, y] \subseteq S$. Since S is neither bounded above nor below, and the choice of x, y is arbitrary, we conclude that $(-\infty, \infty) \subseteq S$. Also, every subset of $\mathbb{R}$ is a subset of $(-\infty, \infty)$, thus $S = (-\infty, \infty)$. □

Theorem 1.42 (The nested intervals property)

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Also, assume that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection. That is,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof. In order to show that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ is not empty, we are going to use the completeness axiom to produce a single real number x satisfying $x \in I_n$ for every $n \in \mathbb{N}$. Now, the completeness axiom is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in \mathbb{N}\}$$

of left-hand endpoints of the interval. Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = \sup(A).$$

Now, consider a particular $I_n = [a_n, b_b]$. Because x is an upper bound for A, we have $a_n \leq x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \leq b_n$.

Altogether then, we have $a_n \leq x \leq b_n$, which means $x \in I_n$ for every choice of $n \in \mathbb{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is not empty. □

Remark 1.43 — Note that the conclusion of Theorem 1.42 need not hold if each I_n is allowed to be an open interval.

2 Cardinality

"No one shall expel us from the paradise that Cantor has created."

David Hilbert, Über das Unendliche

Definition 2.1 (Cardinality)

Let S and T be sets. Then, $|S| = |T|$ if and only if there is a bijection from S to T.

Definition 2.2 (Cardinality cont.)

$|S| \leq |T|$ if and only if there is an injection from S to T.

Remark 2.3 — Our definitions above introduce two fundamental relations on cardinality, i.e. $|S| = |T|$ and $|S| \leq |T|$. We need to make sure that these relations have the mathematical properties we expect them to have. That is, we want $|S| = |T|$ to be an equivalence relation and $|S| \leq |T|$ to be a partial order.

For the relation $|S| = |T|$, it's easy to show that it defines an equivalence relation:

- Reflexivity: Every set has a bijection with itself, i.e. $|S| = |S|$.
- Symmetry: If there is a bijection f from S to T, then f^{-1} is a bijection from T to S, and thus $|T| = |S|$.
- Transitivity: If there are bijections f from S to T and g from T to U, then their composition $h = g \circ f$ is a bijection from S to U. Hence, $|S| = |U|$.

For the relation $|S| \leq |T|$, we must establish that it's a partial order:

- Reflexivity: Every set has an injection to itself, i.e. $|S| \leq |S|$.
- Transitivity: If there exist injections f from S to T and g from T to U, then their composition $h = g \circ f$ is an injection from S to U. Hence, $|S| \leq |U|$.

The remaining property, antisymmetry, is where things get interesting, since it's not immediately obvious. Antisymmetry means that if $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$. Using our definition, this translates to: If there is an injection from S to T and an injection from T to S, then there should be a bijection between S and T. This is exactly what the Schröder-Bernstein theorem (Theorem 2.4) guarantees.

Theorem 2.4 (Schröder-Bernstein theorem)

If there exist injections $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijection $h : A \to B$.

In terms of the cardinality of the two sets, this implies that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. The strategy is to partition A and B into components

$$A = X \cup X' \qquad \text{and} \qquad B = Y \cup Y'$$

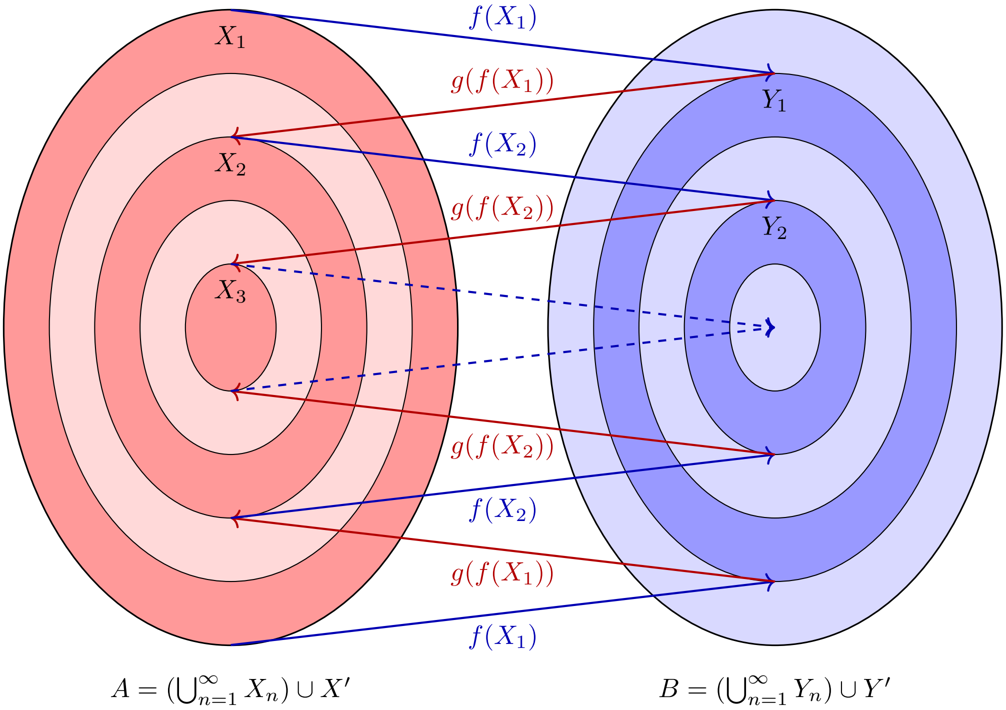
with $X \cap X' = \varnothing$ and $Y \cap Y' = \varnothing$, in such a way that f is a surjection from X to Y, and g is a surjection from Y' to X'. Achieving this would lead to a proof that there is a bijection h from A to B. Why? For all $x \in X'$, there exists a unique $y \in Y'$ satisfying $g(y) = x$. This means that there is a well-defined inverse function $g^{-1}(x) = y$ that maps X' to Y'. Setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \\ g^{-1}(x) & \text{if } x \in X' \end{cases}$$

gives the desired bijection from A to B.

Now, let $X_1 = A \setminus g(B)$ and inductively define a sequence of sets by letting $X_{n+1} = g(f(X_n))$. Let $X = \bigcup_{n=1}^{\infty} X_n$ and $Y = \bigcup_{n=1}^{\infty} f(X_n)$.

To understand these definitions intuitively, observe that X_1 consists of all elements in A that are not in the image of g. Then X_2 consists of elements that are images under $g \circ f$ of elements in X_1, and so on. Similarly, each $Y_n = f(X_n)$ consists of the images of X_n under f. The following diagram illustrates this construction and the relationship between these sets.



We show that $\{X_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of A, while $\{f(X_n) : n \in \mathbb{N}\}$ is a similar collection in B. To see that the sets $X_1, X_2, X_3, \ldots$ are pairwise disjoint, note that $X_1 \cap X_n = \varnothing$ for all $n \geq 2$ because $X_1 = A \setminus g(B)$ and $X_n \subseteq g(B)$. (Why? f is a mapping from A to B, so we have that $f(X_n) \subseteq B$, and thus $X_{n+1} = g(f(X_n)) \subseteq g(B)$.) In the general case of $X_n \cap X_m$ where $1 < n < m$, note that if $x \in X_n \cap X_m$ then $f^{-1}(g^{-1}(x)) \in X_{n-1} \cap X_{m-1}$. Continuing in this way, we can show $X_1 \cap X_{m-n+1}$ is not empty, which is a contradiction. Thus $X_n \cap X_m = \varnothing$. Just to be clear, the disjointness of the X_n sets is not crucial to the overall proof, but it does help paint a clearer picture of how the sets X and X' are constructed.

We show that f is a surjection from X to Y. This is straightforward. Each $x \in X$ comes from some X_n and so $f(x) \in f(X_n) \subseteq Y$. Likewise, each $y \in Y$ is an element of some $f(X_n)$ and thus $y = f(x)$ for some $x \in X_n \subseteq X$. Thus $f : A \to B$ is a surjection.

Let $X' = A \setminus X$ and $Y' = B \setminus Y$. Let $y \in Y'$. Then $y \notin f(X_n)$ for all n (by definition of Y'). We also conclude that $g(y) \notin X_{n+1}$ for all n. (Why? Suppose if $g(y) \in X_{n+1}$ for some n, then $g(y) \in g(f(X_n))$. That is, $g(y) = g(z)$ for some $z \in f(X_n)$, and since g is injective, we have $y = z$ and thus $y \in f(X_n)$, which is a contradiction.) Clearly, $g(y) \notin X_1$ either (because $g(y) \subseteq g(B)$ and $X_1 = A \setminus g(B)$) and so g is a mapping from Y' to X'. To see that g is a surjection from Y' to X', let $x \in X'$ be arbitrary. Because $X' \subseteq g(Y') \subseteq g(B)$, there exists $y \in B$ with $g(y) = x$. Could y be an element of Y? No, because if $y \in Y$, $g(y)$ would be in $g(Y)$, and since $g(Y) \subseteq X$ (by definition of Y), this would mean $g(y) \in X$. But we're considering an $x \in X'$ with $g(y) = x$, so this is a contradiction as $X \cap X' = \varnothing$. Hence $y \in Y'$ and $g : Y' \to X'$ is a surjection. □

Definition 2.5 (Cardinality cont.)
$|S| \geq |T|$ if and only if there is a surjection from S to T.

Remark 2.6 — Again, we must establish that $|S| \geq |T|$ defines a partial order. Reflexivity and transitivity are obvious. Now we will show antisymmetry. Suppose $|S| \geq |T|$ and $|T| \geq |S|$. Then, by definition, there are surjections from S to T and from T to S. Using the axiom of choice, one can prove that there exists a surjection from X to Y if and only if there exists an injection from Y to X. We conclude that $|T| \leq |S|$ and $|S| \leq |T|$, which implies $|S| = |T|$ by Theorem 2.4.

Theorem 2.7 ($|\mathbb{Z}| = |\mathbb{N}|$)
There are as many integers as there are natural numbers.

Proof. Define the function $f : \mathbb{N} \to \mathbb{Z}$ with

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even.} \end{cases}$$

Clearly, f is a bijection. (See the following diagram.)

$$\begin{array}{cccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \cdots \end{array}$$

Hence, by Definition 2.1, $|\mathbb{N}| = |\mathbb{Z}|$. □

Remark 2.8 — Theorem 2.7 also shows that two sets can have the same cardinality even if one is a proper subset of the other and the "larger" one even has infinitely many more elements than the "smaller" one. Make sure you take a moment to appreciate how remarkably, wonderfully weird this is.

Theorem 2.9 ($|\mathbb{Q}| = |\mathbb{N}|$)
There are as many rational numbers as there are natural numbers.

Proof. Let $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = \text{n}\}.$$

The first few of these sets look like

$$A_1 = \{0\}, \qquad A_2 = \left\{\frac{1}{1}, \frac{-1}{1}\right\}, \qquad A_3 = \left\{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\right\},$$

$$A_4 = \left\{\frac{1}{3}, \frac{-1}{3}, \frac{3}{1}, \frac{-3}{1}\right\}, \qquad \text{and} \qquad A_5 = \left\{\frac{1}{4}, \frac{-1}{4}, \frac{2}{3}, \frac{-2}{3}, \frac{3}{2}, \frac{-3}{2}, \frac{4}{1}, \frac{-4}{1}\right\}.$$

The crucial observation is that each A_n is finite and every rational number appears in exactly one of these sets. A bijection with $\mathbb{N}$ is then achieved by consecutively listing the elements in each A_n.

$$\begin{array}{cccccccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{Q}: & \underbrace{0}_{A_1} & \multicolumn{2}{c}{\underbrace{\frac{1}{1} \quad -\frac{1}{1}}_{A_2}} & \multicolumn{4}{c}{\underbrace{\frac{1}{2} \quad -\frac{1}{2} \quad \frac{2}{1} \quad -\frac{2}{1}}_{A_3}} & \multicolumn{4}{c}{\underbrace{\frac{1}{3} \quad -\frac{1}{3} \quad \frac{3}{1} \quad -\frac{3}{1}}_{A_4}} & \frac{1}{4} & \cdots
\end{array}$$

Admittedly, writing an explicit formula for this correspondence would be an awkward task, and attempting to do so is not the best use of time. What matters is that we see why every rational number appears in the correspondence exactly once. Given, say, $22/7$, we have that $22/7 \in A_{29}$. Because the set of elements in $A_1, \ldots, A_{28}$ is finite, we can be confident that $22/7$ eventually gets included in the sequence. The fact that this line of reasoning applies to any rational number p/q is our proof that the correspondence is surjective. To verify that it is injective, we observe that the sets A_n were constructed to be disjoint so that no rational number appears twice. This completes the proof. $\square$

Theorem 2.10 ($|\mathbb{R}| > |\mathbb{N}|$)
There are more real numbers than natural numbers.

Proof. Assume, for the sake of contradiction, that $|\mathbb{R}| = |\mathbb{N}|$, which implies that there is a bijection $f : \mathbb{N} \to \mathbb{R}$. What this suggests is that it is possible to enumerate the elements of $\mathbb{R}$. If we let $x_1 = f(x_1)$, $x_2 = f(x_2)$, and so on, then the fact that f is bijective (and thus surjective) means that we can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\} \tag{1}$$

and be confident that every real number appears somewhere on the list. We will now use the nested intervals property (Theorem 1.42) to produce a real number that is not there.

Let I_1 be a closed interval that does not contain x_1. Next, let I_2 be a closed interval, contained in I_1, which does not contain x_2. The existence of such I_2 is easy to verify. Certainly I_1 contains two smaller disjoint closed intervals, and x_2 can only be in one of these. In general, given an interval I_n, construct I_{n+1} to satisfy

1. $I_{n+1} \subseteq I_n$ and
2. $x_{n+1} \notin I_{n+1}$.

We now consider the intersection $\bigcap_{n=1}^{\infty} I_n$. If x_k is some real number from the list in (1), then we have $x_k \notin I_k$, and it follows that

$$x_k \notin \bigcap_{n=1}^{\infty} I_n.$$

Now, we are assuming that the list in (1) contains every real number, and this leads to the conclusion that

$$\bigcap_{n=1}^{\infty} I_n = \varnothing.$$

However, the nested intervals property (Theorem 1.42) asserts that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$. So there is at least one $x \in \bigcap_{n=1}^{\infty} I_n$ that, consequently, cannot be on the list in (1). This contradiction means that such an enumeration of $\mathbb{R}$ is impossible, and we conclude that $|\mathbb{R}| \neq |\mathbb{N}|$.

We are not done yet. We still have to show that $|\mathbb{R}| \geq |\mathbb{N}|$. This step is straightforward. Define the function $g : \mathbb{R} \to \mathbb{N}$ with

$$g(x) = \begin{cases} n & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

This function maps each real number that is a natural number to itself, and all other real numbers to 0. Since every natural number is the image of at least one real number under g, the function is surjective. Therefore, by Definition 2.5, we have $|\mathbb{R}| \geq |\mathbb{N}|$. Combining this with our previous result that $|\mathbb{R}| \neq |\mathbb{N}|$, we conclude that $|\mathbb{R}| > |\mathbb{N}|$. □

Definition 2.11 (Countable and uncountable sets)

A set S is **countable** if

1. its cardinality $|S|$ is less than or equal to $|\mathbb{N}|$.
2. there exists an injection from S to $\mathbb{N}$.
3. S is empty or there exists a surjection $\mathbb{N}$ to S.
4. there exists a bijection from S to a subset of $\mathbb{N}$.
5. S is either finite or *countably infinite*.

All of the definitions above are equivalent.

A set S is **countably infinite** if its cardinality $|S|$ is exactly $\mathbb{N}$.

A set S is **uncountable** if it is not countable. That is, its cardinality $|S|$ is greater than $|\mathbb{N}|$.

Corollary 2.12 ($\mathbb{N}$ is countable)
The set of natural numbers is countable.

Corollary 2.13 ($\mathbb{Z}$ is countable)
The set of integers is countable.

Corollary 2.14 ($\mathbb{Q}$ is countable)
The set of rational numbers is countable.

Corollary 2.15 ($\mathbb{R}$ is uncountable)
The set of real numbers is uncountable.

Theorem 2.16
An uncountable collection of disjoint open intervals in $\mathbb{R}$ cannot exist.

Proof. We use contradiction. Suppose there exists an uncountable collection of disjoint open intervals in $\mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Theorem 1.37), every open interval in $\mathbb{R}$ contains at least one rational number. Therefore, there are uncountably many rational numbers. A contradiction of Corollary 2.14. Hence, such a collection cannot exist. $\square$

Theorem 2.17 (Countable infinity is the smallest infinity)
If $A \subseteq B$ and B is countable, then A is either countable or finite.

Proof. B is a countable set. Thus, there exists a bijection $f : \mathbb{N} \to B$. Let $A \subseteq B$ be an infinite subset of B. We show that A is countable. Let $n_1 \in \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, let $g(1) = f(n_1)$. Next let $n_2 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1)\}\}$ and let $g(2) = f(n_2)$. We inductively continue this process to produce a bijection g from $\mathbb{N}$ to A. In general, assume we have defined $g(k)$ for $k < m$, and let $g(m) = f(n_m)$ where $n_m = \min\{n \in \mathbb{N} : A \setminus \{f(n_1), \ldots, f(n_{k-1})\}\}$.

To show that g is injective, observe that $m \neq m'$ implies $n_m \neq n_{m'}$ and it follows that $g(m) = f(n_m) \neq f(n_{m'}) = g(m')$ because f is injective. To show that g is surjective, let $a \in A$ be arbitrary. Because f is surjective, $a = f(n')$ for some $n' \in \mathbb{N}$. This means $n' \in \{n : f(n) \in A\}$ and as we inductively remove the minimal element, n' must eventually be the minimum by at least the $(n' - 1)$-th step. $\square$

Corollary 2.18 ($|\mathbb{N}|$ is the smallest infinity)
If $A \subseteq \mathbb{N}$, then either A is finite or $|A| = |\mathbb{N}|$.

Corollary 2.19 (Sizes of infinity)

There are different sizes of infinity, with countable infinity being the smallest. Moreover, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable while $\mathbb{R}$ is uncountable.

Theorem 2.20 (Countable union of countable sets is countable)

A countable union of countable sets is countable. More precisely:

1. If $A_1, A_2, \ldots, A_m$ are each countable sets, then $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.
2. If A_n is a countable set for each $n \in \mathbb{N}$, then the set $\bigcup_{n=1}^{\infty} A_n$ is also countable.

Proof. First, we prove part 1 for two countable sets, A_1 and A_2. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint.

Now, because A_1 is countable, there exists a bijection $f : \mathbb{N} \to A_1$. If $B_2 = \varnothing$, then $A_1 \cup A_2 = A_1$ which we already know to be countable. If $B_2 = \{b_1, b_2, \ldots, b_m\}$ has m elements then define $h : \mathbb{N} \to A_1 \cup B_2$ via

$$h(n) = \begin{cases} b_n & \text{if } n \leq m \\ f(n-m) & \text{if } n > m. \end{cases}$$

The fact that h is bijective follows immediately from the same property of f. If B_2 is infinite, then by Theorem 2.17 it is countable, and so there exists a bijection $g : \mathbb{N} \to B_2$. In this case we define $h : \mathbb{N} \to A_1 \cup B_2$ by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that h is bijective is derived directly from the fact that f and g are both bijections. Graphically, the correspondence takes the form

$$\begin{array}{rccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2: & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

To prove the more general statement in part 1, we may use induction. We have just seen that the result holds for two countable sets. Now let's assume that the union of m countable sets is countable, and show that the union of $m+1$ countable sets is countable.

Given $m+1$ countable sets $A_1, A_2, \ldots, A_{m+1}$, we can write

$$A_1 \cup A_2 \cup \cdots \cup A_{m+1} = (A_1 \cup A_2 \cup \cdots \cup A_m) \cup A_{m+1}.$$

Then $C_m = A_1 \cup \cdots \cup A_m$ is countable by the induction hypothesis, and $C_m \cup A_{n+1}$ is just the union of two countable sets which we know to be countable. This completes the proof for part 1.

For part 2, induction cannot be used because we have an infinite number of sets. Instead, we show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof.

Let's first consider the case where the sets $\{A_n\}$ are disjoint. In order to achieve bijection between $\mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n$, we first label the elements in each countable set A_n as

$$A_n = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}.$$

Now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in an array similar to the one for $\mathbb{N}$:

$$\begin{array}{ccccccc} A_1 & = & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ A_2 & = & a_{21} & a_{22} & a_{23} & a_{24} & \cdots & \\ A_3 & = & a_{31} & a_{32} & a_{33} & \cdots & & \\ A_4 & = & a_{41} & a_{42} & \cdots & & & \\ A_5 & = & a_{51} & \cdots & & & & \\ & \vdots & & & & & & \end{array}$$

This establishes a bijection $g : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ where $g(n)$ corresponds to the element a_{jk} where (j, k) is the row and column location of n in the array for $\mathbb{N}$.

If the sets $\{A_n\}$ are not disjoint then our mapping may not be injective. In this case we could again replace A_n with $B_n = A_n \setminus \{A_1 \cup \cdots \cup A_{n-1}\}$. Another approach is to use the previous argument to establish a bijection between $\bigcup_{n=1}^{\infty} A_n$ and an infinite subset of $\mathbb{N}$, and then appeal to Theorem 2.17. This completes the proof for part 2. $\square$

Theorem 2.21 ($\mathbb{R} \setminus \mathbb{Q}$ is uncountable)
There are uncountably many irrational numbers.

Proof. We use contradiction. Suppose $\mathbb{R} \setminus \mathbb{Q}$ is countable. By Theorem 2.20, we know that a countable union of countable sets is countable, thus $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$ is countable. A contradiction of Corollary 2.15. Hence, $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. $\square$

Theorem 2.22 ($|\mathbb{Q}_+| = |\mathbb{N}|$)
There are as many positive rational numbers as there are natural numbers.

Proof. The following diagram arranges the rational numbers (with some repetition) and illustrates our bijection, which we call the *winding bijection.*

	1	2	3	4	5	6	7	8	$\cdots$
1	1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1	$\cdots$
2	1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	$\cdots$
3	1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3	$\cdots$
4	1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4	$\cdots$
5	1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5	$\cdots$
6	1/6	2/6	3/6	4/6	5/6	6/6	7/6	8/6	$\cdots$
7	1/7	2/7	3/7	4/7	5/7	6/7	7/7	8/7	$\cdots$
8	1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Weaving through this chart, you are guaranteed to hit every positive rational number. So if you pair up 1 with the first number you hit, 2 with the second number you hit, 3 with the third, and so on, then every positive rational number is in a pair. Now, there's just one small problem: each rational number is actually hit more than once. The number p/q will be written in positions $(p, q), (2p, 2q), (3p, 3q), \ldots$. But the fix is easy: When you come across a number that has already been hit, just skip it. Clearly you won't run out of rational numbers, so this does indeed pair up everything. So

$$f(n) = \text{the } n\text{th new rational number you reach.}$$

And that's it. $\square$

Theorem 2.23 ($|(0, 1)| > |\mathbb{N}|$)
There are more numbers in the open interval $(0, 1)$ than there are natural numbers.

We will demonstrate a clever argument known as *Cantor's diagonal argument.*

Proof. We proceed by contradiction and assume that there does exist a function $f : \mathbb{N} \to (0, 1)$ that is bijective. For each $m \in \mathbb{N}$, $f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\ldots.$$

What is meant here is that for each $m, n \in \mathbb{N}$, a_{mn} is the digit from the set $\{0, 1, 2, \ldots, 9\}$ that represents the nth digit in the decimal expansion of $f(m)$. The bijection between $\mathbb{N}$

and $(0,1)$ can be summarized in the doubly indexed array

$$\begin{array}{ccccccccccc}
\mathbb{N} & & (0,1) & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\boldsymbol{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \boldsymbol{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \boldsymbol{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \boldsymbol{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \boldsymbol{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \boldsymbol{a_{66}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The key assumption about this correspondence is that *every* real number in $(0,1)$ is assumed to appear somewhere on the list. Now for the pearl of the argument. Define a real number $x \in (0,1)$ with the decimal expansion $x = .b_1b_2b_3b_4\dots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Observe that $x = .b_1b_2b_3b_4\dots$ cannot be $f(1)$. x also cannot be $f(2)$, and in general $x \neq f(n)$ for any $n \in \mathbb{N}$. Therefore, the real number x is nowhere on the list! This is a contradiction; clearly we were unable to pair up all the reals, if x got left out. □

Remark 2.24 — Consider the following complaints about the proof of Theorem 2.23.

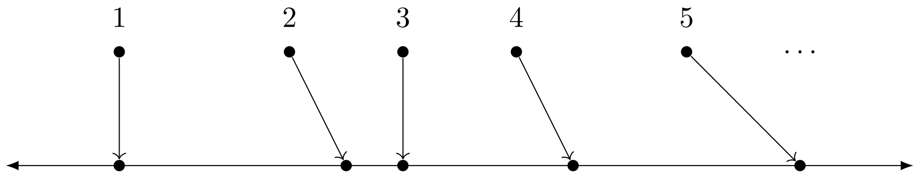
1. Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 2.23 must be flawed.

2. Some numbers have two different decimal representations (see Theorem 1.30). Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or as $.4999\dots$. Doesn't this cause some problems?

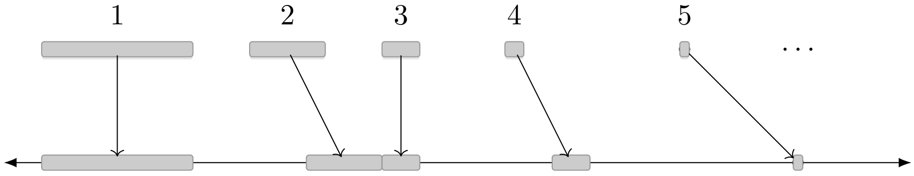
Are these complaints valid?

Here's another proof of Theorem 2.23 that I particular enjoy.

Proof Sketch. Imagine you were trying to map $\mathbb{N}$ on to the real number line (which clearly has $|\mathbb{R}|$ points).



We aim to show that it is impossible for this map to be a bijection—there must be points on the real line that were missed. But instead of mapping just these points, let's make our job slightly harder. Around 1, let's put a little interval of length $\frac{1}{2}$. Around 2, let's put a little interval of length $\frac{1}{4}$. Around 3, let's place a little interval of length $\frac{1}{8}$. And so on. Now, when you map the points in $\mathbb{N}$ to the real line, send over the intervals too (possibly some intervals will overlap; this is ok). We'll now prove that not only are there points on the real line that weren't mapped to, but there are even points that these intervals don't cover!



See what happened? Our intervals' lengths add up to $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$ (and with overlaps, their collective length when mapped to the real line may even be smaller than 1). But the whole real line has length ∞! So certainly there is no chance that all the points are covered; not only do the points of $\mathbb{N}$ not cover the line, but even if we fatten them up with these intervals, those intervals don't even cover the real line! And any point that's in the "$\infty - 1$" portion of the real line that is not covered by an interval was certainly not mapped to. So we have all sorts of points that were missed, and so the mapping is far from being a bijection.

And we could instead pick intervals that add up to 0.0001, or any other tiny number. This proof therefore provides a visualization of how we aren't just missing the one point that Cantor's diagonalization argument finds, but "most" points are missed. $\square$

Theorem 2.25 ($|(0,1)| = |\mathbb{R}|$)

There are as many numbers in the open interval $(0,1)$ as there are real numbers.

Proof Idea. The function $f(x) = (x - 1/2)/(x - x^2)$ is a bijection from $(0,1)$ to $\mathbb{R}$. This shows, by Definition 2.1, that $|(0,1)| = |\mathbb{R}|$. $\square$

Remark 2.26 — Theorem 2.25 effectively establishes that Theorem 2.10 and Theorem 2.23 are equivalent.

Remark 2.27 — We now know that $|\mathbb{N}| < |\mathbb{R}|$. Here's one natural question: Is there any infinity between these two? An astounding fact is that, based on the axioms of set theory (called ZFC), whether or not there exists such an infinity is *unprovable.* And what I don't mean is that mathematicians are not smart enough to find the answer; no, I mean that they *are* smart enough to have shown that *no proof can possibly exist.* That's right, there are statements in math which are impossible to prove and also impossible to disprove (but we *are* able to prove that they are unprovable, amazingly).

This particular question is among the most famous in mathematical history. It was posed by Georg Cantor and is known as *the continuum hypothesis.* It is the first of Hilbert's 23 problems—an influential list of unsolved problems that David Hilbert presented in 1900 at the International Congress of Mathematicians, setting the mathematical agenda for the 20th century. Decades later, Kurt Gödel's groundbreaking incompleteness theorems revealed that virtually every mathematical theory contains unprovable statements.

ZFC set theory is the foundational framework upon which nearly all of modern mathematics is built. Gödel constructed a model of ZFC where the continuum hypothesis holds, while Paul Cohen later constructed a model where it fails. Together, their results established that the continuum hypothesis is independent of ZFC—it can be neither proved nor disproved within the system. Thus, the continuum hypothesis—which asks what is presumably a basic question about the infinite—is unprovable.

Hypothesis 2.28 (The continuum hypothesis)
There is no set whose cardinality is strictly between that of the naturals and the reals.

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

Theorem 2.29 ($|A| < |\mathcal{P}(A)|$)
If A is a set and $\mathcal{P}(A)$ is the power set of A, then

$$|A| < |\mathcal{P}(A)|.$$

Proof. Assume for a contradiction that $|A| \geq |\mathcal{P}(A)|$. That is, assume that there is a surjection f from A to $\mathcal{P}(A)$. Since f is a surjection, for every $T \subseteq A$, there is some element $t \in A$ where $f(t) = T$. To reach our contradiction, we will construct a set $B \subsetneq A$ which is not hit.

For each a there is one special property about the set $f(a)$ that we are going to care about: Is $a \in f(a)$ or is $a \notin f(a)$? In general, consider the set of all elements a such that $a \notin f(a)$, and call this set B:

$$B = \{a \in A : a \notin f(a)\}.$$

By the above, if we can show that that there is no b where $f(b) = B$, then we are done; we will have discovered an element of $\mathcal{P}(A)$ that was not hit by f, a contradiction.

<u>Claim.</u> There is no $b \in A$ such that $f(b) = B$.

<u>Proof of claim.</u> Assume for a contradiction that there does exist some $b \in A$ such that $f(b) = B$. Note by the definition of B that

$$b \in B \text{ if and only if } b \notin f(b).$$

But since we assumed that $f(b) = B$, this is equivalent to

$$b \in f(b) \text{ if and only if } b \notin f(b),$$

which is clearly a contradiction. □

Corollary 2.30 (There exist infinitely many infinities)
There exist infinitely many distinct infinite cardinalities.

Proof. By Theorem 2.29, the following is a chain of distinct infinite cardinalities

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))))| < \dots .$$

□

3 Sequences

"Erdős loved epsilons—his word for small children (in mathematics the Greek letter epsilon is used to represent small quantities)."

Paul Hoffman, The Man Who Loved Only Numbers

Definition 3.1 (Sequences)

A **sequence** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$.

Definition 3.2 (Bounded sequences)

A sequence (a_n) is **bounded** if the range $\{a_n : n \in \mathbb{N}\}$ is bounded. That is, if there exists a lower bound $L \in \mathbb{R}$ and an upper bound $U \in \mathbb{R}$ where

$$L \leq a_n \leq U$$

for all n.

Proposition 3.3

A sequence (a_n) is bounded if and only if there exists some $C \in \mathbb{R}$ for which $|a_n| \leq C$ for all n.

Proof. Recall that boundedness means that there exists a lower bound $L \in \mathbb{R}$ and an upper bound $U \in \mathbb{R}$ where

$$L \leq a_n \leq U$$

for all n. Now let's prove each direction.

($\Leftarrow$) Assume that there exists a $C \in \mathbb{R}$ where

$$|a_n| \leq C.$$

Then

$$-C \leq a_n \leq C.$$

And so, by setting $L = -C$ and $U = C$, we have shown that

$$L \leq a_n \leq U,$$

which means that (a_n) is bounded.

($\Rightarrow$) If (a_n) is bounded, then there exists such an L and U. Let $C = \max(|L|, U)$. Note that this implies that $C \geq U$ and (since $C \geq |L|$, that) $-C \leq -|L|$. Thus, for all n we have

$$-C \leq -|L| \leq L \leq a_n \leq U \leq C.$$

So we see that

$$-C \leq a_n \leq C$$

which by Fact 1.10 is the same as $|a_n| \leq C$. $\square$

Definition 3.4 (Convergent sequences)

A sequence (a_n) **converges** to $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n > N$.

When this happens, a is called the **limit** of a_n.

Definition 3.5 (ϵ-neighborhood)

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

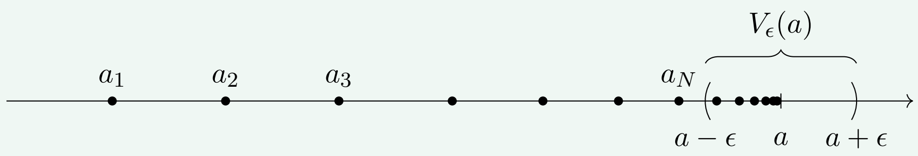
$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the **ϵ-neighborhood** of a.

Definition 3.6 (Convergent sequences (topological version))

A sequence (a_n) *converges* to $a \in \mathbb{R}$ if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but finite number of the terms of (a_n).

Remark 3.7 — Definition 3.4 and Definition 3.6 say precisely the same thing: the natural number N in Definition 3.4 is the point after which the sequence (a_n) enters $V_\epsilon(a)$, never to leave. It should be apparent that the value of N depends on the choice of ϵ. The smaller the ϵ-neighborhood, the larger the N may have to be.



Example 3.8

Show that the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$$

converges to 0.

Proof. Fix any $\epsilon > 0$. By the Archimedean principle (Lemma 1.27), there is an N for which $\frac{1}{N} < \epsilon$. This N has the property that, for all $n > N$, we have $\frac{1}{n} < \frac{1}{N} < \epsilon$. That is, for all $n > N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon.$$

And so, by Definition 3.4, we may conclude that $a_n \to 0$. $\square$

Remark 3.9 — Using the Archimedean principle will not usually work, though. We need a more general approach, which is described in Outline 3.10.

Outline 3.10. To show that $a_n \to a$, begin with preliminary work:

0. Scratch work: Start with $|a_n - a| < \epsilon$ and unravel to solve for n. This tells you which N to pick for step 2 below.

Now for your actual proof:

1. Let $\epsilon > 0$.
2. Let N be the final value of n you got in your scratch work, and let $n > N$.
3. Redo scratch work (without ϵ's), but at the end use N to show that $|a_n - a| < \epsilon$.

Example 3.11

Again, show that the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$$

converges to 0. This time, use the method described in Outline 3.10.

Scratch work. Given an arbitrary $\epsilon > 0$, we will find what specifc N guarantees that, for every $n > N$, we have $|a_n - 0| < \epsilon$. For example, if $\epsilon = \frac{1}{2}$, then $N = 2$ works. If $\epsilon = \frac{1}{3}$, then $N = 3$ works. You see the pattern, but here is how we might come about it in general. We want the following:

$$\begin{aligned} |a_n - a| &< \epsilon \\ \left|\frac{1}{n} - 0\right| &< \epsilon \\ \frac{1}{n} &< \epsilon \\ \frac{1}{\epsilon} &< n. \end{aligned}$$

So as long as we choose $N = \frac{1}{\epsilon}$, then for any $n > N$, we will have $n > \frac{1}{\epsilon}$, which by the above will imply that $\frac{1}{n} \to 0$, as desired. The solution below is how we formally solve it.

Proof. Fix any $\epsilon > 0$. Then for any $n > N$ (implying $\frac{1}{n} < \frac{1}{N}$),

$$|a_n - a| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\epsilon} = \epsilon.$$

That is, $|a_n - 0| < \epsilon$. So by Definition 3.4, we have shown that $\frac{1}{n} \to 0$. $\square$

Example 3.12

Let $a_n = \frac{3n+1}{n+2}$. Prove that $\lim_{n\to\infty} a_n = 3$.

Scratch Work. Again, we first play around. We start with where we want to get to (that $|a_n - a| < \epsilon$), and then do some algebra to figure out which values of n would give this.

We want the following:

$$\begin{aligned} |a_n - a| &< \epsilon \\ \left|\frac{3n+1}{n+2} - 3\right| &< \epsilon \\ \left|\frac{3n+1}{n+2} - \frac{3(n+2)}{n+2}\right| &< \epsilon \\ \left|\frac{3n+1-3n-6}{n+2}\right| &< \epsilon \\ \left|\frac{-5}{n+2}\right| &< \epsilon \\ \frac{5}{n+2} &< \epsilon \\ \frac{5}{\epsilon} &< n+2 \\ \frac{5}{\epsilon} - 2 &< n \end{aligned}$$

So as long as we choose $N = \frac{5}{\epsilon} - 2$, then for any $n > N$ we will have $n > \frac{5}{\epsilon} - 2$, which by the above will imply that $\frac{3n+1}{n+2} \to 3$, as desired.

Proof. Fix any $\epsilon > 0$. Set $N = \frac{5}{\epsilon} - 2$. Then for any $n > N$,

$$\begin{aligned} |a_n - a| = \left|\frac{3n+1}{n+2} - 3\right| &= \left|\frac{3n+1}{n+2} - \frac{3n+6}{n+2}\right| \\ &= \frac{5}{n+2} < \frac{5}{N+2} = \frac{5}{(\frac{5}{\epsilon} - 2) + 2} \\ &= \frac{5}{5/\epsilon} = \epsilon. \end{aligned}$$

That is, $|a_n - a| < \epsilon$. So by Definition 3.4 we have shown that $\frac{3n+1}{n+2} \to 3$. □

Definition 3.13 (Divergent Sequences)

If a sequence (a_n) does not converge, then it **diverges**.

Divergence can come in three forms.

1. (a_n) *diverges to* ∞ (notation: $\lim_{n\to\infty} a_n = \infty$) if, for all $M > 0$, there exists some $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$.
2. (a_n) *diverges to* $-\infty$ (notation: $\lim_{n\to\infty} a_n = -\infty$) if, for all $M < 0$, there exists some $N \in \mathbb{N}$ such that $a_n < M$ for all $n > N$.
3. Otherwise, (a_n)'s limit *does not exist*.

Example 3.14

Let $a_n = n^2$. Show that $\lim_{n\to\infty} a_n = \infty$.

Scratch work. We want

$$\begin{aligned} a_n &> M \\ n^2 &> M \\ n &> \sqrt{M}. \end{aligned}$$

So setting $N = \sqrt{M}$ should work.

Proof. Fix any $M > 0$. Set $N = \sqrt{M}$. Then for any $n > N$,

$$a_n = n^2 > N^2 = (\sqrt{M})^2 = M.$$

So we have shown that if $n > N$, then $a_n > M$. Therefore $\lim_{n\to\infty} a_n = \infty$. □

Outline 3.15. What if the sequence's limit does not exist? Then how do we show the sequence diverges? One way to show that a_n diverges is to show that $a_n \not\to a$ for any a. Note first, by Definition 3.4, that $a_n \to a$ means that

For <u>every</u> $\epsilon > 0$ there exists <u>some</u> N
such that <u>for all</u> $n > N$ we have $|a_n - a| < \epsilon$.

So to show that $a_n \not\to a$, we need to show the *negation* of that statement. That is, we must show that

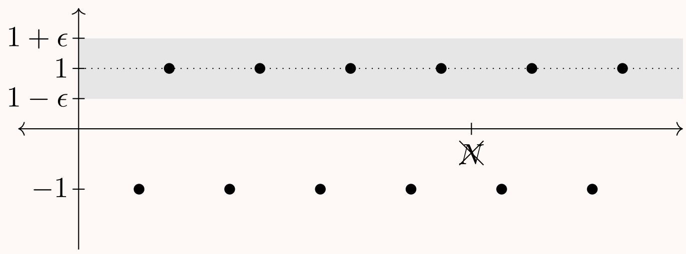
There exists <u>some</u> $\epsilon > 0$ where <u>for all</u> N
there exists <u>some</u> $n > N$ such that $|a_n - a| \geq \epsilon$.

In practice, this is usually done with a proof by contradiction. You assume that $a_n \to a$ and then you demonstrate a specific ϵ where it fails, giving the contradiction.

Example 3.16

Let $a_n = (-1)^n$. Prove that (a_n) diverges.

Scratch work. This is the sequence $-1, 1, -1, 1, -1, 1, \ldots$. It makes sense that there is no a for which $a_n \to a$. It certainly doesn't converge to 1, since half the time it is at -1 which is far away from 1. (If we let $\epsilon = 1/$, then there is no N for which, for every $n > N$, a_n is inside of the shaded band; see below.)



It likewise can't converge to -1. One might guess 0, since that is halfway between -1 and 1, but that also doesn't make sense since a_n is always a distance of 1 away from 0, so it's certainly not getting "closer and closer" to 0. (Or, $\epsilon = 1/2$ works again.) Ok, so we believe that it doesn't converge to anything, and we will use Outline 3.15 to show it.

Proof. Assume for a contradiction that there is some a for which $a_n \to a$. Let $\epsilon = \frac{1}{2}$. Since we assumed that $a_n \to a$, there must be some N for which, for all $n > N$, we have $|a_n - a| < \frac{1}{2}$. That is, $|(-1)^n - a| < \frac{1}{2}$ for all $n > N$. We proceed by cases.

Even n. If n is even and $n > N$, then we have that

$$|1 - a| < \frac{1}{2}.$$

Unwinding this:

$$\begin{aligned} -\frac{1}{2} &< 1 - a < \frac{1}{2} \\ -\frac{3}{2} &< -a < -\frac{1}{2} \\ \frac{1}{2} &< a < \frac{3}{2}. \end{aligned}$$

Odd n. If n is odd and $n > N$, then we have that

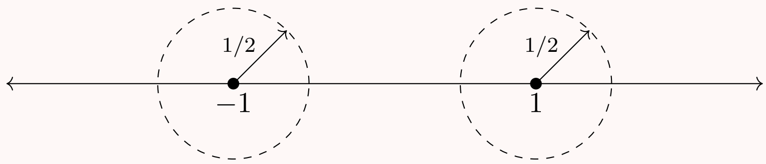
$$|-1 - a| < \frac{1}{2}.$$

Unwinding this:

$$\begin{aligned} -\frac{1}{2} &< -1 - a < \frac{1}{2} \\ \frac{1}{2} &< -a < \frac{3}{2} \\ -\frac{3}{2} &< a < -\frac{1}{2}. \end{aligned}$$

But this is a contradiction; clearly no a can be inside of both $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(-\frac{3}{2}, -\frac{1}{2}\right)$. And so we must have $a_n \not\to a$. □

Essentially, focusing on the even case created a ball around 1, of radius $\frac{1}{2}$, which a_n would have to live within for all $n > N$. The odd case created a ball around -1, of radius $\frac{1}{2}$, which a_n would have to live within for all $n > N$. But these two balls are disjoint, so a_n can't live in both, creating a contradiction.



Now here's another proof using the triangle inequality (Theorem 1.11).

Proof. Again, let $\epsilon = \frac{1}{2}$. The even case still gives $|1-a| < \frac{1}{2}$, and the odd case still gives $|-1-a| < \frac{1}{2}$; although in the odd case we will rewrite $|-1-a|$ as $|1+a|$. Then, by the triangle inequality:

$$2 = |(1-a)+(1+a)| \leq |1-a| + |1+a|.$$

But each of the absolute values on the right are assumed to be less than $\frac{1}{2}$. So,

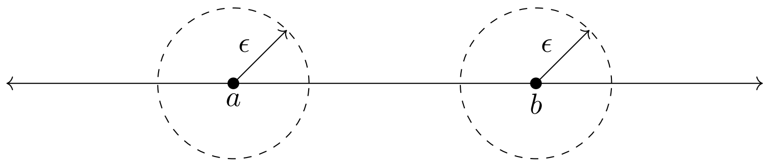
$$2 = |(1-a)+(1+a)| \leq |1-a| + |1+a| < \frac{1}{2} + \frac{1}{2} = 1.$$

So $2 < 1$? Preposterous! We have our contradiction. □

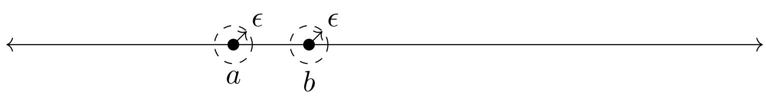
Proposition 3.17 (Limits are unique)

A sequence cannot have more than one limit.

Proof Idea. The idea is to assume that you have a sequence a, that converges to some a and also converges to some other number 6. Since it converges to a, after some point all the sequence points are within ϵ of a, for some really tiny ϵ. But since it also converges to b, the same thing must happen there too. So if ϵ is small enough that those two regions are mutually exclusive, we will have a contradiction.



And this is how we will reach a contradiction: there is no way for $a_n (n > N)$ to be in both circles at the same time. Of course, if a and b are really close together we will need to choose a smaller ϵ to make sure the intervals remain disjoint, but that's the only difference.



For instance, if the radius of those balls is a third of the distance between a and b, it should work out. With that intuition in mind, here's the proof. □

Proof. Suppose for a contradiction that $a_n \to a$ and $a_n \to b$, where $a \neq b$; moreover, without loss of generality let's assume $a < b$. Let $\epsilon = (b-a)/3 > 0$. Since $a_n \to a$ there exists some N_1 such that for $n > N_1$ we have $|a_n - a| < (b-a)/3$. Likewise, since $a_n \to b$ there exists some N_2 such that for $n > N_2$ we have $|a_n - b| < (b-a)/3$. Let $N = \max\{N_1, N_2\}$. Then for $n > N$ (implying $n > N_1$ and $n > N_2$), we have both $|a_n - a| < (b-a)/3$ and $|a_n - b| < (b-a)/3$. That is, for such n we have

$$-\frac{b-a}{3} < a_n - a < \frac{b-a}{3} \qquad \text{and} \qquad -\frac{b-a}{3} < a_n - b < \frac{b-a}{3}.$$

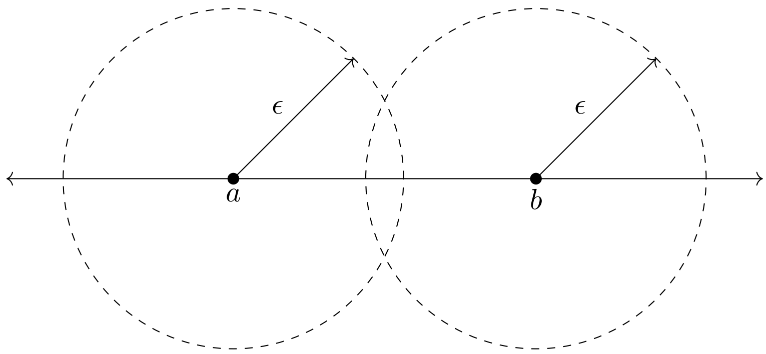
i.e.,

$$a - \frac{b-a}{3} < a_n < a + \frac{b-a}{3} \qquad \text{and} \qquad b - \frac{b-a}{3} < a_n < b + \frac{b-a}{3}.$$

In particular,
$$a_n < \frac{b+2a}{3} \qquad \text{and} \qquad \frac{2b+a}{3} < a_n.$$
But since $a < b$, we must have $b + 2a < 2b + a$. But this then implies that
$$a_n < \frac{b+2a}{3} < \frac{2b+a}{3} < a_n,$$
which is clearly a contradiction. □

There is a second proof that is even a little shorter.

Proof Idea. The idea is to show that $|a - b| < \epsilon$ for all $\epsilon > 0$, and therefore $|a - b| = 0$, implying that $a = b$. Intuitively, the way to show this is to say that if a, is getting really close to both a and b, then that forces a and b to be close to each other. If we demand that a_n, is within $\epsilon/2$ of both a and b, then the distance between a and b can't be more than ϵ:



Using the triangle inequality we can then bound the distance from a to b by finding the distance from a to a_n, plus the distance from a_n to b. □

Proof. Let $\epsilon > 0$. Since $\epsilon/2 > 0$ and $a_n \to a$, there exists some N_1 such that for $n > N_1$ we have $|a_n - a| < \epsilon/2$. Since $\epsilon/2 > 0$ and $a_n \to b$, there exists some N_2 such that for $n > N_2$ we have $|a_n - b| < \epsilon/2$. Let $N = \max(N_1, N_2)$ and pick any $n > N$. Then

$$\begin{aligned} |a-b| &= |a - a_n + a_n - b| \\ &\le |a - a_n| + |a_n - b| && \text{(triangle inequality)} \\ &= |a_n - a| + |a_n - b| \\ &< \epsilon/2 + \epsilon/2 && (n > N \text{ implies } n > N_1 \text{ and } n > N_2) \\ &= \epsilon. \end{aligned}$$

Since this holds for any $\epsilon > 0$, we have shown that $|a - b| < \epsilon$ for all $\epsilon > 0$, which implies that $|a - b| = 0$, and hence $a = b$. So indeed, a_n can converge to only a single point. □

Proposition 3.18

If (a_n) is a convergent sequence, then (a_n) is bounded.

Proof. Since (a_n) is convergent, let a be the value it is converging to. By the definition of convergence (with $\epsilon = 1$), there is some N where

$$|a_n - a| < 1$$

for all $n > N$. That is, $a - 1 < a_n < a + 1$ for all $n > N$. Let

$$U = \max\{a_1, a_2, a_3, \dots, a_N, a + 1\}$$

and

$$L = \min\{a_1, a_2, a_3, \dots, a_N, a - 1\}.$$

Note that if $n \leq N$, then $L \leq a_n \leq U$, since each such a_n is included in the sets which we are taking the minimum and maximum of. And if $n > N$, then we already noted that $a - 1 < a_n < a + 1$, which implies that

$$L \leq a - 1 < a_n < a + 1 \leq U,$$

and hence $L < a_n < U$. Combining these cases, we have that

$$L \leq a_n \leq U$$

for all n. And thus, by definition, (a_n) is bounded. □

Corollary 3.19

If (a_n) is an unbounded sequence, then (a_n) is divergent.

Theorem 3.20 (Sequence limit laws)

Assume that (a_n) and (b_n) are convergent sequences of real numbers such that $a_n \to a$ and $b_n \to b$. Also assume that $c \in \mathbb{R}$. Then,

1. $(a_n + b_n) \to a + b$.
2. $(a_n - b_n) \to a - b$.
3. $(a_n \cdot b_n) \to a \cdot b$.
4. $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$, provided each $b \neq 0$ and each $b_n \neq 0$.
5. $(c \cdot a_n) \to c \cdot a$.

Example 3.21

What is

$$\lim_{n\to\infty} \frac{1}{2} \cdot \left(\frac{\frac{1}{n} + \frac{1}{n^2} + 4}{5 - \frac{1}{n^2}} \right) \cdot \left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}} \right)?$$

By the limit laws, we have

$$\lim_{n\to\infty} \frac{1}{2} \cdot \left(\frac{\frac{1}{n} + \frac{1}{n^2} + 4}{5 - \frac{1}{n^2}} \right) \cdot \left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}} \right) = \frac{1}{2} \cdot \left(\frac{0+0+4}{5} \right) \cdot (3+0) = \frac{6}{5}.$$

Theorem 3.22 (Sequence squeeze theorem)

Assume $a_n \le x_n \le b_n$ for all n. Furthermore, assume that

$$a_n \to L \qquad \text{and} \qquad b_n \to L.$$

Then,

$$x_n \to L.$$

Proof. Let $\epsilon > 0$.

- Since $a_n \to L$, there exists some N_1 such that $n > N_1$ implies $|a_n - L| < \epsilon$. That is, $-\epsilon < a_n - L < \epsilon$. Or,
$$L - \epsilon < a_n < L + \epsilon. \tag{1}$$

- Since $b_n \to L$, there exists some N_2 such that $n > N_2$ implies $|b_n - L| < \epsilon$. That is, $-\epsilon < b_n - L < \epsilon$. Or,
$$L - \epsilon < b_n < L + \epsilon. \tag{2}$$

Let $N = \max(N_1, N_2)$, and let $n > N$. Combining the inequality $a_n \le x_n \le b_n$ with the left half of (1) and the right half of (2), we get

$$\begin{gathered}
L - \epsilon < a_n \le x_n \le b_n < L + \epsilon \\
L - \epsilon < x_n < L + \epsilon \\
-\epsilon < x_n - L < \epsilon \\
|x_n - L| < \epsilon.
\end{gathered}$$

□

Example 3.23

Prove that

$$\lim_{n\to\infty} \frac{1}{n^{2.7} + \sqrt{n} + \pi} = 0.$$

Proof. Since

$$0 \le \frac{1}{n^{2.7} + \sqrt{n} + \pi} \le \frac{1}{n}, \qquad \lim_{n\to\infty} 0 = 0, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n} = 0,$$

by the sequence squeeze theorem (Theorem 3.22),

$$\lim_{n\to\infty} \frac{1}{n^{2.7} + \sqrt{n} + \pi} = 0.$$

□

Definition 3.24

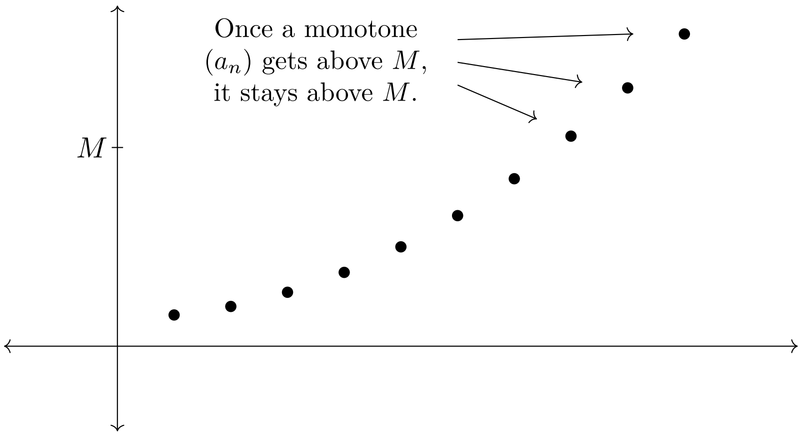
A sequence (a_n) is **monotone increasing** if $a_n \le a_{n+1}$ for all n. Likewise, a sequence (a_n) is **monotone decreasing** if $a_n \ge a_{n+1}$ for all n. If it is either monotone increasing or monotone decreasing, it is monotone.

Theorem 3.25 (The monotone convergence theorem)

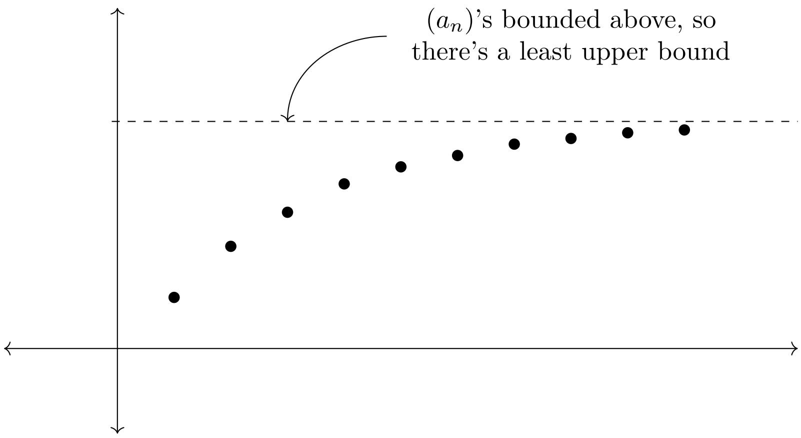
Suppose (a_n) is monotone. Then (a_n) converges if and only if it is bounded. Moreover,

- If (a_n) is increasing, then either (a_n) diverges to ∞ or
$$\lim_{n\to\infty} a_n = \sup(\{a_n : n \in \mathbb{N}\}).$$
- If (a_n) is decreasing, then either (a_n) diverges to $-\infty$ or
$$\lim_{n\to\infty} a_n = \inf(\{a_n : n \in \mathbb{N}\}).$$

Proof Idea. Suppose (a_n) is monotonically increasing. If it's not bounded, then given any $M > 0$, this M is not an upper bound and so eventually (a_n) will get above it. And since it's monotonically increasing, once it gets above a number, it *stays* above that number.



If, on the other hand, (a_n) is bounded, then the sequence must be leveling out (it's monotone, so it can't go up and down).



□

Proof. Assume that (a_n) is monotonically increasing, and let's suppose first that a_n is not bounded. Then for any $M > 0$ there exists some N such that $a_N > M$. But since (a_N) is monotonically increasing, for $n > N$ we have $a_n \geq a_N > M$. And so, by Definition 3.13, (a_n) diverges to ∞.

Next suppose that (a_n) is bounded. Then we have that $\{a_n : n \in \mathbb{N}\}$ is a subset of $\mathbb{R}$ which is bounded above, which by the completeness of $\mathbb{R}$ implies that $\sup(\{a_n : n \in \mathbb{N}\})$ exists. Call this supremum α. We want to show that $\lim_{n\to\infty} a_n = \alpha$; that is, we want to show that for any $\epsilon > 0$ there exists some N such that $n > N$ implies $|a_n {}^{-} \alpha| < \epsilon$.

To show this, first let $\epsilon > 0$. Since $\sup(\{a_n : n \in \mathbb{N}\}) = \alpha$, by the analytic definition of suprema (Theorem 1.25) there exists some $a_N > a - \epsilon$. And since (a_n) is monotonically increasing, we see that for any $n > N$ we have that $a_n \geq a_N > \alpha - \epsilon$. And of course, $a_n < \alpha$ due to the fact that α is the supremum (and hence an upper bound) of $\{a_n : n \in \mathbb{N}\}$. And so, for $n > N$, we have $\alpha - \epsilon < a_n < \alpha$. This implies $\alpha - \epsilon < a_n < \alpha + \epsilon$, and hence $-\epsilon < a_n - \alpha < \epsilon$, which, at last, gives

$$|a_n - \alpha| < \epsilon$$

completing the proof in the case when (a_n) is monotonically increasing.

The case where (a_n) is monotonically decreasing is very similar. □

Remark 3.26 — This theorem is nice for several reasons, but one of which is the fact that you do not have to know what the limit of (a_n) is in order to show that (a_n) is convergent. This is notable since in the definition of sequence convergence—which, until now, we have heavily relied on to show a sequence converges—requires that you already know (and can write down) what the limit is going to be. Below is an example where it would be quite a challenge to write down the limit of the sequence in a beneficial way. Yet by the monotone convergence theorem, we will be able to conclude that the limit exists.

Example 3.27

Let (a_n) be the sequence where $a_1 = 0.1$, $a_2 = 0.12$, $a_3 = 0.123$, $a_4 = 0.1234$, and so on. (And, to be clear, this pattern does not change when you reach double digits. For example, $a_{12} = 0.123456789101112$.) Prove that (a_n) converges.

Proof. Note that a_{n+1} and a_n match exactly until the last digits of a_{n+1}, which are the digits of $n+1$. Therefore, $a_{n+1} - a_n$ is a number with a bunch of zeros followed the digits of $n+1$. In particular,

$$a_{n+1} - a_n > 0$$

for all n. We have shown that $a_{n+1} > a_n$ for all n, proving that (a_n) is monotone increasing. Furthermore note that since all the terms begin with 0.1, we have that $a_n \leq 1$ for all n.

We have shown that (a_n) is monotone increasing and bounded above, therefore by the monotone convergence theorem (Theorem 3.25) the sequence converges, completing the proof. □

Proposition 3.28

Suppose $S \subseteq \mathbb{R}$ is bounded above. Then there exists a sequence (a_n) where $a_n \in S$ for each n and

$$\lim_{n\to\infty} a_n = \sup(S).$$

Likewise, if S is bounded below, then there exists a sequence (b_n) where $b_n \in S$ for each n and

$$\lim_{n\to\infty} b_n = \inf(S).$$

Proof Idea. We need to find elements of S which are getting closer and closer to $\sup(S)$. What was our *super* useful theorem for doing just that? The analytic definition of suprema (Theorem 1.25)! That was the theorem that said for any $\epsilon > 0$, there exists some $a \in S$ such that $a > \sup(S) - \epsilon$. And since it works for any $\epsilon > 0$, we can choose a sequence of ϵ values that are getting closer and closer to 0, which in turn produce a sequence of a values that are getting closer and closer to $\sup(S)$.

And once we have found a sequence of these elements which are getting closer and closer to $\sup(S)$, how do we formally show they converge to $\sup(S)$? The sequence squeeze theorem (Theorem 3.22) sure sounds like it's up to the job. $\square$

Proof. Suppose $S \subseteq \mathbb{R}$ is bounded above, implying that $\sup(S)$ exists. Let $\sup(S) = \alpha$ (implying $a_n \leq \alpha$ for all n). By the analytic definition of suprema (Theorem 1.25), for any $\epsilon > 0$ there exists some $x \in S$ such that $x > \alpha - \epsilon$. In particular, there exists some $a_1 \in S$ such that $a_1 > \alpha - 1$; and there exists some $a_2 \in S$ such that $a_2 > \alpha - \frac{1}{2}$; and there exists some $a_3 \in S$ such that $a_3 > \alpha - \frac{1}{3}$; and, in general, for each $n \in \mathbb{N}$ there exists some $a_n \in S$ such that $a_n > \alpha - \frac{1}{n}$.

Thus we obtain a sequence (a_n) from S. Moreover, this sequence does indeed converge to α. Therefore, by the limit laws (Theorem 3.20) we have that

$$\alpha - \frac{1}{n} \to \alpha - 0 = \alpha.$$

So we have an inequality

$$\alpha - \frac{1}{n} \leq a_n \leq \alpha,$$

where both the upper and lower bounds converge to α. By the sequence squeeze theorem (Theorem 3.22) we see that $\lim_{n\to\infty} a_n = \alpha$, as desired.

The infimum case is *very* similar. $\square$

Definition 3.29

Let (a_n) be a sequence of real numbers and let

$$n_1 < n_2 < n_3 < \dots$$

be an increasing sequence of integers. Then,

$$a_{n_1}, a_{n_2}, a_{n_3}, \dots$$

is called a **subsequence** of (a_n), and is denoted (a_{n_k}).

Proposition 3.30

A sequence (a_n) converges to a if and only if every subsequence of (a_n) also converges to a.

Proof.

($\Rightarrow$) Assume that (a_n) converges to a and (a_{n_k}) is a subsequence of (a_n). Let $\epsilon > 0$. Then there exists some N such that

$$|a_n - a| < \epsilon \tag{1}$$

for all $n > N$.

We want to show that $a_{n_k} \to a$. That is, we want to show that there exists some N_1 such that $|a_{n_k} - a| < \epsilon$ for all $k > N_1$. Notice that since each $n_i \in \mathbb{N}$ and

$$n_1 < n_2 < n_3 < \dots,$$

that we have $n_k \geq k$. Therefore by letting $N_1 = N$, for any $k > N_1$ we then know that $n_k > N$, so by (1) we have

$$|a_{n_k} - a| < \epsilon.$$

($\Leftarrow$) Assume that every subsequence of (a_n) converges to a. Suppose, for contradiction, that (a_n) does not converge to a. By Outline 3.15, this means that there exists some $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there is some $n > N$ with $|a_n - a| \geq \epsilon_0$. Using this fact, we can inductively construct a subsequence of (a_n) as follows:

1. Let n_1 be a natural number with $|a_{n_1} - a| \geq \epsilon_0$.
2. Having chosen $n_1, \dots, n_k$, let $n_{k+1} > n_k$ be a natural number with $|a_{n_{k+1}} - a| \geq \epsilon_0$.

This process yields a subsequence (a_{n_k}) with the property that $|a_{n_k} - a| \geq \epsilon_0$ for all $k \in \mathbb{N}$. But this means that (a_{n_k}) cannot converge to a, which contradicts our assumption that every subsequence of (a_n) converges to a. Therefore, (a_n) must converge to a. $\square$

Corollary 3.31

If (a_n) has a pair of subsequences converging to different limits, then (a_n) diverges.

Proof. Assume for a contradiction that (a_n) converges. Say, $a_n \to L$. By Proposition 3.30, this implies that every subsequence also converges to L. This contradicts the assumption that there are subsequences converging to different limits. $\square$

Proposition 3.32

If a monotone sequence (a_n) has a convergent subsequence, then (a_n) converges too, and has the same limit.

Proof Idea. First recall that by Proposition 3.30, a convergent sequence always has the same limit as any of its subsequences. So by this, our task reduces to showing that (a_n) converges.

A monotone sequence is convergent if and only if it is bounded (by the monotone convergence theorem). We are told that (a_n) is monotone and to show that it's bounded, we use the assumption that it has a convergent subsequence, (a_{n_k}). Here's the progression: (a_n) being monotone implies that its subsequence is too, and (a_{n_k}) being convergent (and now monotone) will mean that (a_n) is bounded too. □

Proof. Assume that (a_n) is monotone increasing; the case where (a_n) is monotone decreasing is *very* similar.

Let (a_{n_k}) be a convergent subsequence of (a_n) and observe that since (a_n) is monotone increasing, being a subsequence means that (a_{n_k})'s terms come from (a_n) and do so in the same order as they appear in (a_n), implying that (a_{n_k}) is also monotone increasing. That is, (a_{n_k}) was assumed to be convergent and was shown to be monotone—by the monotone convergence theorem (Theorem 3.25) this means (a_{n_k}) converges, and in particular converges to $\sup(\{a_{n_k} : k \in \mathbb{N}\})$.

We now show that (a_n) is convergent, again by using the monotone convergence theorem. Since (a_n) is monotone increasing by assumption, we need only show that (a_n) is bounded above. To see this, observe that since the subsequence (a_{n_k}) is an infinite list of elements from (a_n), any particular element a_n from the sequence has an element a_{n_k} from this subsequence that comes after it. And since both a_n and a_{n_k} are terms of (a_n), which is monotone increasing, we have $a_n \leq a_{n_k}$. So we have

$$a_n \leq a_{n_k} \leq \sup(\{a_{n_k} : k \in \mathbb{N}\}),$$

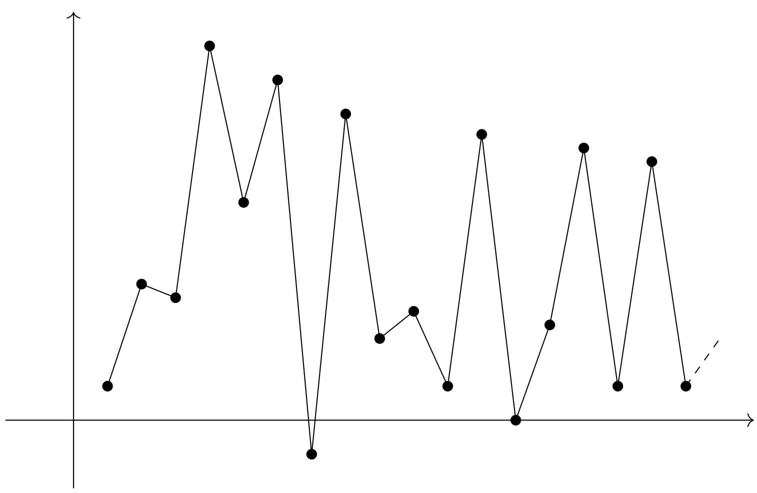
which proves that (a_n) is bounded above (by this supremum), and so by the monotone convergence theorem we have proven that (a_n) converges.

Finally, since we have shown that the sequence (a_n) is convergent, by Proposition 3.32 it has the same limit as any of its subsequences. This means that (a_n) must also be converging to $\sup(\{a_{n_k} : k \in \mathbb{N}\})$, completing the proof. □

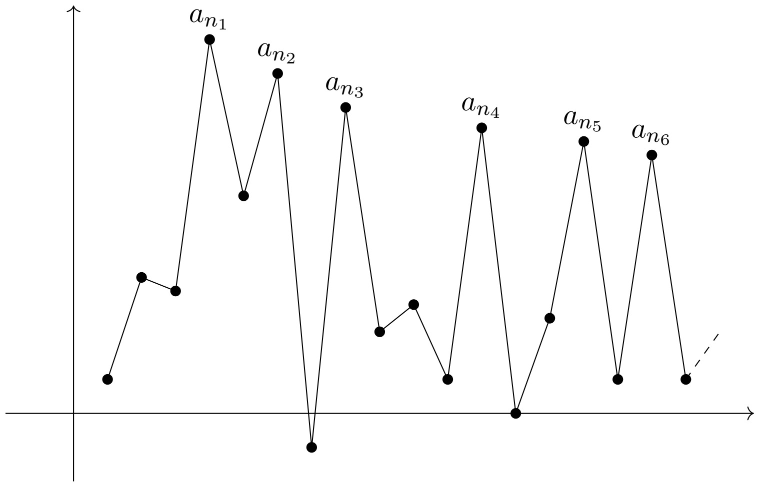
Lemma 3.33

Every sequence has a monotone subsequence.

Proof Idea. Here's the idea behind it. We draw a sequence of in the xy-plane by just plotting the points, and we will connect the dots to make a zig-zagged line:

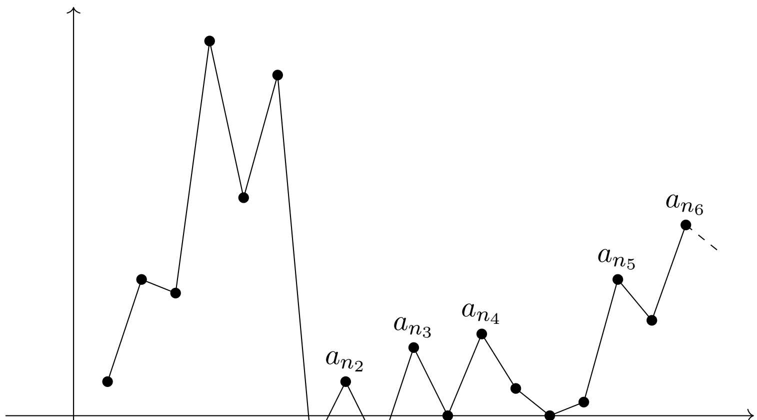


From this you can maybe spot a nice *decreasing* sequence, which is of course monotone:



The magical definition that will solve everything is that of a *peak*; we want those labeled points to form *peaks*, and we want to be able to say that if we have a(n infinite) sequence of peaks, then we do indeed have a decreasing sequence. The definition that does this is this: define a *peak* to be a point a_n which is larger than every later point; that is, a_n is a peak if $a_n \geq a_m$ for all $m > n$.

So if we have infinitely many peaks, then we obtain a sequence like the one above which will be decreasing. So what if we don't? Then we only have finitely many peaks. In this case, we can find an *increasing* sequence, which is again monotone. To see how, just note that if you're past the last peak, then any point you pick is not a peak, which means there is some point after it which is larger. So one at a time you can pick larger and larger points, giving an increasing sequence.



□

Proof. Call a_n a *peak* if a_n is larger than every later point. That is, if $a_n \geq a_m$ for all $m > n$.

Either there are infinitely many peaks or finitely many peaks. Assume first that (a_n) has infinitely many peaks. Then let a_{n_k} be the kth peak. Then, by the definition of a peak, $a_{n_k} \geq a_{n_{k+1}}$, implying that (a_{n_k}) is a decreasing subsequence.

Now assume that (a_n) has finitely many peaks, and let a_N be the last one. Then let $a_{n_1} = a_{N+1}$. Since a_N was the last peak, a_{N+1} is *not* a peak, implying that there is some later point a_{n_2} that is larger than it. Likewise, since a_{n_2} is after the last peak, it is also not a peak, and so there must be some later point a_{n_3} that is larger than it. Continuing in this way we construct an increasing subsequence (a_{n_k}).

In either case we found a monotone subsequence, so we are done. □

Theorem 3.34 (The Bolzano-Weierstrass theorem)
Every bounded sequence has a convergent subsequence.

Proof. Assume (a_n) is a bounded sequence. Then by Lemma 3.33 it has a monotone subsequence, (a_{n_k}). Also, since (a_n) is bounded, so is (a_{n_k}). Since (a_{n_k}) is both bounded and monotone, by the monotone convergence theorem it converges. □

Definition 3.35
A sequence (a_n) is **Cauchy** if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$|a_m - a_n| < \epsilon$$

for all $m, n > N$.

Lemma 3.36
If (a_n) is Cauchy, then (a_n) is bounded.

Proof. Assume (a_n) is Cauchy. Then (for $\epsilon = 1$) there exists some $N \in \mathbb{N}$ such that

$$|a_m - a_n| < 1$$

for all $m, n > N$. In particular, for all $n > N$,

$$|a_n - a_{n+1}| < 1.$$

Therefore for $n > N$, we know that the terms are bounded:

$$a_{N+1} - 1 < a_n < a_{N+1} + 1.$$

And there are only finitely many points before a_N, so these are certainly bounded. Consequently, we can find a general bound on all a_n. Indeed, if we let

$$L = \min(\{a_1, a_2, a_3, \ldots, a_N, a_{N+1} - 1\})$$

and

$$U = \max(\{a_1, a_2, a_3, \ldots, a_N, a_{N+1} + 1\}),$$

then

$$L \leq a_n \leq U$$

for all $n \in \mathbb{N}$. So (a_n) is bounded. □

Theorem 3.37 (Cauchy criterion for convergence)
A sequence converges if and only if it is Cauchy.

Proof Sketch. For the forward direction, suppose a_n converges to a. If a_n and a_m are both within $\epsilon/2$ of a, then they must be within ϵ of each other. We can formally show this using the triangle inequality.

For the backwards direction, the idea is this: It will be helpful to identify what the Cauchy sequence is converging to. And by Bolzano-Weierstrass we can find a subsequence that is converging to some a. Our goal will then be to prove that a is in fact the sequence's limit. How? Well, that subsequence gets super close to a. What about the terms in (a_n) which are not in the subsequence? By the Cauchy criterion, they get super close to the elements of the subsequence! And if a term in the sequence is super close to a term of the subsequence, which is in turn super close to the limit... then the sequence must be close to the limit, too. □

Proof.

(⇒) Assume that (a_n) converges to some $a \in \mathbb{R}$. Let $\epsilon > 0$. Since $\epsilon/2 > 0$, there exists some $N \in \mathbb{N}$ such that for any $n > N$ we have

$$|a_n - a| < \frac{\epsilon}{2}.$$

Then, for any $n, m > N$,

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(⇐) Assume that (a_n) is Cauchy, and note that by Lemma 3.36, (a_n) is bounded. Combining these two facts, and applying the Bolzano-Weierstrass theorem (Theorem 3.34), we conclude that some subsequence of (a_n) converges. Say, (a_{n_j}) converges to a. Our goal is to show that (a_n) converges; we will in fact prove that $a_n \to a$.

Let $\epsilon > 0$. Since $\epsilon/2 > 0$ and (a_n) is Cauchy, there exists some N_1 such that

$$|a_n - a_m| < \frac{\epsilon}{2} \tag{1}$$

for all $n, m > N_1$. Since $\epsilon/2 > 0$ and (a_{n_j}) converges to a, there exists some N_2 such that $j > N_2$ implies

$$|a_{n_j} - a| < \frac{\epsilon}{2}. \tag{2}$$

We want to choose some J so that, for subscripts past this point, both (1) and (2) hold. Choose $J = \max(N_1, N_2)$. Notice, from the definition of a subsequence, that $n_j \geq j$. In particular, $n_{J+1} > J$. And so, for any $j > J$,

$$|a_j - a| = |a_j - a_{n_{J+1}} + a_{n_{J+1}} - a| \leq |a_j - a_{n_{J+1}}| + |a_{n_{J+1}} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

4 Series

"Divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever."

Niels Henrik Abel, in a letter to Holmboe (1826)

A Pathological Examples

> "One of my favorite memories of studying was one night when analysis crept into my dreams. I woke up in a panicky cold sweat. In my dream I was being chased by some analysis monster. My only defense was to use the *blancmange function* as a boomerang. I took it as a good sign at the time that analysis concepts were finding their way into my subconscious."
>
> *Tina Rapke, Confronting Analysis*

It has been said that one of the most important goals of learning real analysis is to collect as many bizarre examples as you can, and to keep them in your back pocket. From a practical standpoint they will inform your conjectures and guide your proofs, but they will also help to demonstrate why real analysis is such a great subject.

§A.1 An infinite field that cannot be ordered

To say that a field F cannot be ordered is to say that it possesses no positive subset P satisfying the order axiom (Definition 1.6). A preliminary comment is that since every ordered field is infinite, no finite field can be ordered.

An example of an *infinite* field that cannot be ordered is the field $\mathbb{C}$ of complex numbers. To show that this is the case, assume that there does exist a positive subset P of $\mathbb{C}$. Consider the element $i \in \mathbb{C}$, where i is the imaginary unit. Since $i \neq 0$, there are two alternative possibilities. The first is $i \in P$ (i.e. i is a positive element), in which case $i^2 = -1 \in P$ and $i^4 = 1 \in P$. Since i^2 and i^4 are additive inverses of each other, and since it is impossible for two additive inverses both to belong to P, we have obtained a contradiction. The other option is $-i \in P$, in which case $(-i)^2 = -1 \in P$ and $(-i)^4 = 1 \in P$. We get the same contradiction as before.

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