A Laundry List of Theorems in Analysis

Richard Willie

April 3, 2025

Preface

These notes are a loose amalgamation of ideas, concepts, and explanations drawn from various sources, particularly the following books:

1. Real Analysis: A Long-Form Mathematics Textbook by Jay Cummings [Cu19]
2. Understanding Analysis by Stephen Abbott [Ab15]
3. Introduction to Real Analysis by Bartle & Sherbert [BaSh11]
4. Mathematical Analysis by Tom M. Apostol [Ap74]

They were originally meant for my own understanding and organization of thoughts, and as such, they may be unpolished, incomplete, or even occasionally incorrect.

I share them in the hope that they may serve as a useful reference, but they should not be treated as a primary source of learning. Readers are strongly encouraged to consult original texts and authoritative resources for a more rigorous and accurate treatment of the topics discussed.

Use these notes as a companion to your studies, not as a substitute for the depth and clarity provided by well-established literature.

Contents

1 The Real Numbers

> "It is a 'simple' theorem, simple both in idea and execution, but there is no doubt at all about it being a theorem of the highest class. It is as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on it."
>
> *G. H. Hardy, A Mathematician's Apology*

Theorem 1.1 (The irrationality of $\sqrt{2}$)
There is no rational number whose square is 2.

Proof. A rational number is any number that can be expressed in the form p/q, where p and q are integers. Thus, what the theorem asserts is that no matter how p and q are chosen, it is never the case that $(p/q)^2 = 2$. The line of attack is indirect, using a type of argument referred to as a proof by contradiction. The idea is to assume that there is a rational number whose square is 2 and then proceed along logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erroneous assumption that some rational number squared is equal to 2. In short, we will prove that the theorem is true by demonstrating that it cannot be false.

And so assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 2. \tag{1}$$

We may also assume that p and q have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Now, equation (1) implies

$$p^2 = 2q^2. \tag{2}$$

From this, we can see that the integer p^2 is an even number (it is divisible by 2), and hence p must be even as well because the square of an odd number is odd. This allows us to write $p = 2r$, where r is also an integer. If we substitute $2r$ for p in equation (2), then a little algebra yields the relationship

$$2r^2 = q^2.$$

But now the absurdity is at hand. This last equation implies that q^2 is even, and hence q must also be even. Thus, we have shown that p and q are both even (i.e., divisible by 2) when they were originally assumed to have no common factor. From this logical impasse, we can only conclude that equation (1) cannot hold for any integers p and q, and thus the theorem is proved. □

Remark 1.2 — So the rationals aren't quite enough to describe all numbers. That said, they do have almost every other fundamental property we would want. To the point: They are what we call an *ordered field*. But first, what's a *field*? It's a set that satisfies the classic additive and multiplicative properties we know and love.

Definition 1.3 (Fields)

A **field** is a nonempty set $\mathbb{F}$, along with two binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If $a, b \in \mathbb{F}$, then $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Axiom 2 (Distributive Law).** If $a, b \in \mathbb{F}$, $a \cdot (b+c) = a \cdot b + a \cdot c$.
- **Axiom 3 (Associative Law).** If $a, b \in \mathbb{F}$, then $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Axiom 4 (Identity Law).** There are special elements $\mathbf{0}, \mathbf{1} \in \mathbb{F}$, where $a + \mathbf{0} = a$ and $a \cdot \mathbf{1} = a$ for all $a \in \mathbb{F}$.
- **Axiom 5 (Inverse Law).** For each $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = \mathbf{0}$. If $a \neq \mathbf{0}$, then there is also an element $a^{-\mathbf{1}} \in \mathbb{F}$ such that $a \times a^{-\mathbf{1}} = \mathbf{1}$.

Example 1.4

Below are some examples and some non-examples of fields.

- The natural number $\mathbb{N}$ do not form a field; they fail the first half of Axiom 4 and both halves of Axiom 5.
- The integers $\mathbb{Z}$ *almost* form a field; they only fail the second half of Axiom 5.
- One can check that the rationals $\mathbb{Q}$ form a field.

Remark 1.5 — Now, let's come up with a definition for an ordered field. Think about $\mathbb{Q}$. What does $\mathbb{Q}$ have that a field does not? There are three main properties we are missing: First, there are infinitely many rationals (and they are "symmetric" about the 0 element.) Second, the rationals have an ordering to them. Lastly, we would like to talk about how big a number is. Beautifully, Definition 1.6 describes a single elegant axiom that we can include to capture *all* of these properties.

Definition 1.6 (Ordered Fields)

An **ordered field** is a field $\mathbb{F}$, along with the following additional axiom.

Axiom 6 (Order Axiom). There is a nonempty subset $P \subseteq \mathbb{F}$, called the *positive elements*, such that

1. If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;
2. If $a \in \mathbb{F}$ and $a \neq \mathbf{0}$, then either $a \in P$ or $-a \in P$, but not both.

Definition 1.7 (Inequalities)

If $\mathbb{F}$ is an ordered field and $a, b \in \mathbb{F}$, then we say that "$a < b$" if $b - a \in P$. Likewise, $a \leq b$ means that either $a = b$ or $a < b$.

We define "$>$" similarly.

Fact 1.8 (Properties of inequalities)

For a, b, c in an ordered field $\mathbb{F}$:

1. If $a < b$, then $a + c < b + c$.
2. Transitivity: If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$, then $ac < bc$ if $c > 0$, and $ac > bc$ if $c < 0$.
4. If $a \neq 0$, then $a^2 > 0$.

Definition 1.9 (The absolute value function)

If $\mathbb{F}$ is an ordered field, define the *absolute value* function $|\cdot| : \mathbb{F} \to \mathbb{F}$ to be

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Fact 1.10 (Properties of absolute values)

For a, b in an ordered field $\mathbb{F}$:

1. $|a| \geq 0$, with equality if and only if $a = 0$.
2. $|a| = |-a|$.
3. $-|a| \leq a \leq |a|$.
4. $|a \cdot b| = |a| \cdot |b|$.
5. $1/|a| = |1/a|$, if $a \neq 0$.
6. $|a/b| = |a|/|b|$, if $b \neq 0$.
7. $|a| \leq b$ if and only if $-b \leq a \leq b$.

Theorem 1.11 (The triangle inequality)

If $\mathbb{F}$ is an ordered field and if $x, y \in \mathbb{F}$, then

$$|x + y| \leq |x| + |y|.$$

Proof. For $x, y \in \mathbb{F}$, by Fact 1.10 part 3 we have

$$-|x| \leq x \leq |x| \qquad \text{and} \qquad -|y| \leq y \leq |y|.$$

Adding these two together gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

And so, by Fact 1.10 part 7,

$$|x + y| \leq |x| + |y|.$$

□

Corollary 1.12 (The reverse triangle inequality)
Assume that $\mathbb{F}$ is an ordered field and $x, y \in \mathbb{F}$. Then,

$$||x| - |y|| \leq |x - y|.$$

Proof. By Fact 1.10 part 7, it suffices to show that

$$-|x - y| \leq |x| - |y| \leq |x - y|.$$

We first show the right-hand one (that $|x| - |y| \leq |x - y|$), and we will do so by applying an application of the triangle inequality. Let $a = x - y$ and $b = y$. Then by the triangle inequality,

$$|a + b| \leq |a| + |b|.$$

That is,

$$\begin{aligned} |(x - y) + y| &\leq |x - y| + |y| \\ |x| &\leq |x - y| + |y|. \end{aligned}$$

Rearranging,

$$|x| - |y| \leq |x - y|.$$

We will use a similar approach to show that $-|x - y| \leq |x| - |y|$. Let $c = y - x$ and $d = x$. By the triangle inequality,

$$|c + d| \leq |c| + |d|$$

That is,

$$\begin{aligned} |(y - x) + x| &\leq |y - x| + |x| \\ |y| &\leq |y - x| + |x|. \end{aligned}$$

Reaaranging,

$$-|y - x| \leq |x| - |y|,$$

which by Fact 1.10 part 2 implies that

$$-|x - y| \leq |x| - |y|,$$

as desired. □

Corollary 1.13 (Triangle inequality corollaries)
For both of the following, assume that $\mathbb{F}$ is an ordered field and $x, y \in \mathbb{F}$.

1. $|x - y| \leq |x| + |y|$.
2. $|x + y| \geq ||x| - |y||$.

Proof.

For part 1, replace y with $-y$ in the triangle inequality.

For part 2, replace y with $-y$ in the reserve triangle inequality. □

Theorem 1.14 (Cauchy-Schwarz inequality)

If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real numbers, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Proof. A sum of squares can never be negative. Hence we have

$$\sum_{i=1}^{n} (a_i x + b_i)^2 \geq 0$$

for every $x \in \mathbb{R}$, with equality if and only if each term is zero. This inequality can be written in the form

$$Ax^2 + 2Bx + C \geq 0$$

where

$$A = \sum_{i=1}^{n} a_i^2, \qquad B = \sum_{i=1}^{n} a_i b_i, \qquad C = \sum_{i=1}^{n} b_i^2.$$

If $A > 0$, put $x = -B/A$ to obtain $B^2 - AC \leq 0$, which is the desired inequality. Otherwise if $A = 0$, the rest of the proof is trivial. □

Definition 1.15 (Upper and lower bounds)

Let S be an ordered field and $A \subseteq S$ be nonempty.

1. The set A is *bounded above* if there exists some $b \in S$ such that $x \leq b$ for all $x \in A$; in this case b is called an **upper bound** of A.
2. The **least upper bound** of A—if it exists—is some $b_0 \in S$ such that
   a) b_0 is an upper bound of A, and
   b) if b is any other upper bound of A, then $b_0 \leq b$.

   Such a b_0 is also called the **supremum** of A and is denoted $\sup(A)$.
3. Likewise, the set A is *bounded below* if there exists some $b \in S$ such that $x \geq b$ for all $x \in A$; in this case, b is called a **lower bound** of A.
4. Again, like above, the **greatest lower bound** of A—if it exists—is some $b_0 \in S$ such that
   a) b_0 is a lower bound of A, and
   b) if b is any other lower bound of A, then $b_0 \geq b$.

   Such a b_0 is also called the **infimum** of A and is denoted $\inf(A)$.
5. If a set is both bounded above and bounded below, then it is simply *bounded*.

Example 1.16

The propositions below are left without proof.

- The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ has no upper bounds. Lower bounds on $\mathbb{N}$ include -17, 1, 0.123, and $-\pi$. Note that $\sup(\mathbb{N})$ does not exist, but $\inf(\mathbb{N}) = 1$.
- The set $\mathbb{Q}$ has no upper or lower bounds; consequently, $\sup(\mathbb{Q})$ and $\inf(\mathbb{Q})$ do not exist.
- $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$; $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$. Note that the supremum here is in the set, while the infimum is not in the set.
- $\sup(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = 1$; $\inf(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = \frac{1}{2}$. Note that the infimum here is in the set, while the supremum is not in the set.
- In $\mathbb{Q}$ the set $\{x \in \mathbb{Q} : x^2 < 2\}$ does not have a supremum. In $\mathbb{R}$ it will—in fact, $\sup(\{x \in \mathbb{Q} : x^2 < 2\}) = \sqrt{2}$.

Definition 1.17 (Completeness)

Let S be an ordered field. Then S has the **least upper bound property** if given any nonempty $A \subseteq S$ where A is bounded above, A has a least supper bound in S. In other words, $\sup(A) \in S$ for every such A.

Such a set S is also called **complete**.

Theorem 1.18 (Existence of $\mathbb{R}$)

The real numbers $\mathbb{R}$ exists, and it is a complete ordered field.

Proof Sketch. We have the ordered field of rational numbers, but they aren't complete—there are holes everywhere, and to get to $\mathbb{R}$ we must fill in these gaps. There are several ways to do this. But the most common method, which we discuss now, uses *Dedekind cuts*.

Each real number is going to be a set; at this point we have the rationals constructed, so each real number is going to be represented by a set of rationals. The way you want to think about it is this: the real number x is going to be represented by the set of all rational numbers strictly less than x. These sets are going to be called *cuts*, and while we discuss them you can start convincing yourself that each real number will indeed correspond to a unique cut, and each cut corresponds to a unique real number.

Definition 1.19 (Dedekind cuts)

A **cut** should be thought of as the set $(-\infty, b) \cap \mathbb{Q}$. That is, all rational numbers up to a certain point. Formally, it is defined as any set C_b satisfying the following three conditions.

1. $C_b \subseteq \mathbb{Q}$, but $C_b \neq \varnothing$ and $C_b \neq \mathbb{Q}$;
2. If $p \in C_b$ and $q \notin C_b$, then $p < q$;
3. If $p \in C_b$, then there exists some $q \in C_b$ where $p < q$.

And then $\mathbb{R}$ is defined as the set of all cuts.

Although we have an intuitive picture of a cut, at the moment it is just a set satisfying the above properties. We are interested in putting an algebraic structure on this collection of cuts. Addition and order work quite smoothly. For a pair of cuts C_a and C_b, define the following.

- $C_a + C_b := \{p + q : p \in C_a, q \in C_b\}$
- $C_a < C_b$ if and only if $C_a \subsetneq C_b$

One can verify that the addition of two cuts is still a cut, and that addition is commutative and associative, that the $\mathbf{0}$ cut behaves as it should (in turn giving *positive* and *negative* cuts) and that additive inverses exist. The inequality has the property that, given any two cuts C_a and C_b, exactly one of the following holds: $C_a < C_b$, $C_a = C_b$, or $C_b < C_a$.

Defining multiplication is trickier, because if you simply multiply the two sets together you'll have massive negative numbers multiplying against each other, creating massive positive numbers. Intuitively, you want $C_a \cdot C_b = C_{ab}$. That is, the cut $(-\infty, a) \cap \mathbb{Q}$ times the cut $(-\infty, b) \cap \mathbb{Q}$ should equal the cut $(-\infty, a \cdot b) \cap \mathbb{Q}$; but cuts aren't defined with such a and b—they produce a and b. One way around this is to first define multiplication for positive cuts. That is, if C_a and C_b are both positive (larger than the cut $\{q \in \mathbb{Q} : q < 0\}$), then define

$$C_a \cdot C_b := \{p \cdot q : p \in C_a, q \in C_b \text{ with } p, q \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

With this you can then define a product of two negative cuts by setting the product equal to the product of the two corresponding cuts. To define multiplication between a positive and a negative cut, you know the product should be negative so one approach is to consider the multiplication when both are positive, and then translate the result to the corresponding negative cut. It's a hassle to write out, but that's the idea.

One can then check that the product of two cuts is still a cut, that multiplicative inverses exist, and that multiplication is commutative and associative, as well as the remaining multiplicative/additive distributive and order properties. These would be quite annoying to work out in detail, but you can smile knowing someone carefully checked them.

With our set built, and with the algebraic properties defined and their properties verified, we now know that $\mathbb{R}$ is an ordered field. All that is left to show is that it is complete. This is done by showing that cuts satisfy the *least upper bound property*. That is, if $\mathcal{C}$ is a collection of cuts which is bounded above (meaning there exists some cut D such that $C \leq D$ for all $C \in \mathcal{C}$), then there exists a least upper bound (meaning there is a cut M such that $M \leq D$ for all upper bounds D). The proof is this: Let $M = \bigcup_{C \in \mathcal{C}} C$, and show that

1. M is a cut and therefore $M \in \mathbb{R}$;
2. M is an upper bound of $\mathcal{C}$, but is smaller than all other upper bounds.

With those, $\mathbb{R}$ is a complete ordered field, and hence the real numbers are constructed. □

Theorem 1.20 (Uniqueness of $\mathbb{R}$)
Any complete ordered field is isomorphic to $\mathbb{R}$.

Proof Sketch. First, let us establish that every complete ordered field R is Archimedean. This means that there is no element in R that is larger than every finite sum $\mathbf{1}+\mathbf{1}+\cdots+\mathbf{1}$. (Why? Refer to Lemma 1.27. If we let $a = \mathbf{1}$, the Archimedean principle states that for any b, there exists an n such that $n \cdot \mathbf{1} > b$. This $n \cdot \mathbf{1}$ is precisely a finite sum $\mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}$). To prove this property, we use contradiction. Suppose there exists such an element, then by completeness (Definition 1.17), there is a least upper bound b to these all these finite sums. But then $b - \mathbf{1}$ would also be an upper bound, since adding $\mathbf{1}$ to any sum would still be less than b. This contradicts b being the least upper bound. Therefore, no such element exists, and R must be Archimedean.

Consider two complete ordered fields, $\mathbb{R}_0$ and $\mathbb{R}_1$. We construct their respective prime subfields, that is, their copies of the rationals $\mathbb{Q}_0$ and $\mathbb{Q}_1$. This is done by computing inside them all the finite quotients of the form $\pm(\mathbf{1} + \mathbf{1} + \cdots + \mathbf{1})/(\mathbf{1} + \mathbf{1} + \cdots + \mathbf{1})$, which essentially represent all fractions p/q where $p, q \in \mathbb{Z}$ and $q > 0$. The fractional representation naturally defines an isomorphism between $\mathbb{Q}_0$ and $\mathbb{Q}_1$. This means that corresponding rational elements in each field map to each other, and this mapping preserves addition, multiplication, and order. This is illustrated in the diagram with the blue dots and arrows connecting corresponding rational elements.

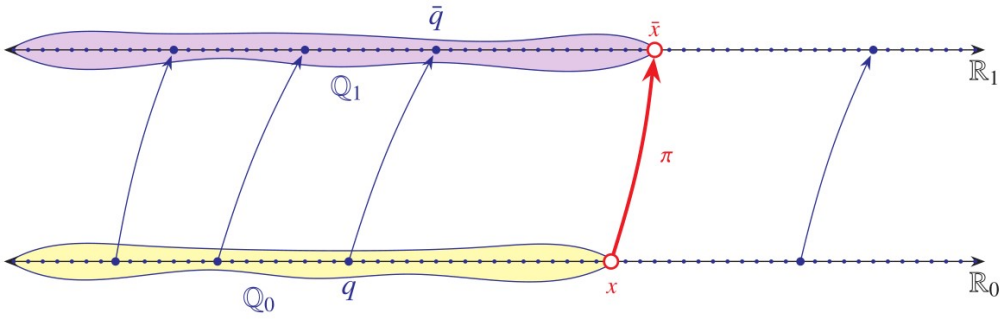


Figure 1.1: An isomorphic mapping $\pi : \mathbb{R}_0 \to \mathbb{R}_1$ (courtesy of Hamkins [Ha]).

We now extend this isomorphism to the entire fields, $\mathbb{R}_0$ and $\mathbb{R}_1$, using Dedekind cuts (Definition 1.19). The Archimedean property ensures that every element x in $\mathbb{R}_0$ defines a unique cut in $\mathbb{Q}_0$, dividing it into two sets: Those rational elements less than x (shown in yellow), and those greater than or equal to x. Since we have an isomorphism between $\mathbb{Q}_0$ and $\mathbb{Q}_1$, this cut in $\mathbb{Q}_0$ corresponds to a similar division in $\mathbb{Q}_1$. By the completeness of $\mathbb{R}_1$, there must be a unique element $\overline{x} \in \mathbb{R}_1$ that makes this exact same cut in $\mathbb{Q}_1$ (shown in violet). This defines our mapping $\pi : x \mapsto \overline{x}$ from $\mathbb{R}_0$ to $\mathbb{R}_1$.

Finally, we verify that the mapping π is indeed a field isomorphism:

1. Surjection: Every $y \in \mathbb{R}_1$ determines a cut in $\mathbb{Q}_1$. By the isomorphism between $\mathbb{Q}_0$ and $\mathbb{Q}_1$, this corresponds to a cut in $\mathbb{Q}_0$. By the completeness of $\mathbb{R}_0$, there exists an $x \in \mathbb{R}_0$ that defines this cut. Thus, $\pi(x) = y$.

2. Injection: If two elements $x_1, x_2 \in \mathbb{R}_0$ determine the same cut in $\mathbb{Q}_0$, then $x_1 = x_2$ (by the definition of Dedekind cuts). Thus, if $\pi(x_1) = \pi(x_2)$, then x_1 and x_2 must define the same cut in $\mathbb{Q}_0$, which means $x_1 = x_2$.

3. Field homomorphism: The mapping π preserves field operations (i.e. addition, multiplication, and order) because it is constructed as a continuous extension of the isomorphism between the rational subfields $\mathbb{Q}_0$ and $\mathbb{Q}_1$.

This completes the proof. □

Corollary 1.21 (Existence and uniqueness of $\mathbb{R}$)

There exists a unique complete ordered field. We call this field *the real numbers* $\mathbb{R}$.

Proposition 1.22 (Axioms of $\mathbb{R}$)

The set $\mathbb{R}$ has two binary operations, addition $(+)$ and multiplication $(\cdot)$, and is the unique set satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If $a, b \in \mathbb{R}$, then $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Axiom 2 (Distributive Law).** If $a, b \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
- **Axiom 3 (Associative Law).** If $a, b \in \mathbb{R}$, then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Axiom 4 (Identity Law).** There are special elements $\mathbf{0}, \mathbf{1} \in \mathbb{R}$, where $a + \mathbf{0} = a$ and $a \cdot \mathbf{1} = a$ for all $a \in \mathbb{R}$.
- **Axiom 5 (Inverse Law).** For each $a \in \mathbb{R}$, there is an element $-a \in \mathbb{R}$ such that $a + (-a) = \mathbf{0}$. If $a \neq \mathbf{0}$, then there is also an element $a^{-\mathbf{1}} \in \mathbb{R}$ such that $a \times a^{-\mathbf{1}} = \mathbf{1}$.
- **Axiom 6 (Order Axiom).** There is a nonempty subset $P \subseteq \mathbb{R}$, called the *positive elements*, such that
  1. If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;
  2. If $a \in \mathbb{R}$ and $a \neq \mathbf{0}$, then either $a \in P$ or $-a \in P$, but not both.
- **Axiom 7 (Completeness Axiom).** Given any nonempty $A \subseteq \mathbb{R}$ where A is bounded above, A has a least upper bound. In other words, $\sup(A) \in \mathbb{R}$ for every such A.

Proposition 1.23 (Suprema are unique)

If the supremum or infimum of $A \subseteq \mathbb{R}$ exists, then it is unique.

We will only prove that suprema are unique. The infima case is analogous.

Proof. Assume for a contradiction that α and β are distinct least upper bounds of A. In particular, both are upper bounds of A, while $\alpha \neq \beta$. One one hand, since α is a least upper bound and β is an upper bound, we must have $\alpha \leq \beta$. On the other hand, since β is a least upper bound and α is an upper bound, we must have $\beta \leq \alpha$. In summary,

$$\alpha \leq \beta \qquad \text{and} \qquad \beta \leq \alpha.$$

This implies that $\alpha = \beta$, giving our contradiction. □

Theorem 1.24 (Square roots exist)

If $a \in \mathbb{R}$ and $a \geq 0$, then $\sqrt{a} \in \mathbb{R}$.

Proof Idea. One can show that $\sqrt{a} = \sup(\{x \in \mathbb{R} : x^2 < a\})$, which is in $\mathbb{R}$ by completeness. □

Theorem 1.25 (Suprema analytically)

Let $A \subseteq \mathbb{R}$. Then $\sup(A) = \alpha$ if and only if

1. α is an upper bound of A, and
2. Given any $\epsilon > 0$, $\alpha - \epsilon$ is *not* an upper bound of A. That is, there is some $x \in A$ for which $x > \alpha - \epsilon$.

Likewise, $\inf(A) = \beta$ if and only if

1. β is a lower bound of A, and
2. Given any $\epsilon > 0$, $\beta + \epsilon$ is *not* a lower bound of A. That is, there is some $x \in A$ for which $x < \beta + \epsilon$.

We will only prove the suprema case. The infima case is analogous.

Proof.

($\Rightarrow$) First, assume that $\sup(A) = \alpha$. We aim to prove part 1 and 2. The first of these is immediate: Since $\sup(A) = \alpha$, α is the least upper bound of A, which of course also implies that it is an upper bound of A.

Now we will show part 2. Let $\epsilon > 0$, Since $\alpha - \epsilon < \alpha$, we know that $\alpha - \epsilon$ is not an upper bound of A, because if so that would contradict α being the least upper bound of A. And so, since $\alpha - \epsilon$ is not an upper bound, there must be some x who is greater than $\alpha - \epsilon$.

($\Leftarrow$) Now assume part 1 and 2. We aim to prove that $\sup(A) = \alpha$. That is, we wish to show that α is an upper bound of A (which is implied directly by part 1), and for any other upper bound β, we have $\alpha \leq \beta$. We have only the latter to prove. Assume that β is some other upper bound of A, and assume for a contradiction that $\beta < \alpha$. Note that $0 < \alpha - \beta$. We will use $(\alpha - \beta)$ as our ϵ, and then apply part 2 to contradict β being an upper bound.

Now we will work it out formally. Let $\epsilon = \alpha - \beta$. Since $\epsilon > 0$, by part 2 there exists some $x \in A$ such that $x > \alpha - \epsilon = \alpha - (\alpha - \beta) = \beta$. But this is a contradiction, because we assumed that β was an upper bound of A, and yet we found another element $x \in A$ that is larger than β. □

Remark 1.26 — Note that the forward direction of the proof also works well by contrapositive. The contrapositive of

$$\sup(A) = \alpha \implies \text{For all } \epsilon > 0 \text{ there is an } x \in A \text{ such that } x > \alpha - \epsilon$$

is

$$\text{There is an } \epsilon > 0 \text{ such that for all } x \in A \text{ we have } x \leq \alpha - \epsilon \implies \sup(A) \neq \alpha.$$

And to prove this, just observe that the left-hand side implies that $\alpha - \epsilon$ is an upper bound of A, and so $\sup(A) \leq \alpha - \epsilon$, which of course implies that $\sup(A) \neq \alpha$, as desired.

Lemma 1.27 (The Archimedean principle)

If a and b are real numbers with $a > 0$, then there exists a natural number n such that $na > b$.

In particular, for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. We aim to show that $na > b$ for some $n \in \mathbb{N}$; by dividing over the a, we aim to prove that there is some $n \in \mathbb{N}$ such that $n > b/a$. Now, the number b/a is just some real number that we know nothing about. In fact, let's just call it x. So, equivalently, we are trying to prove that given any real number x, there is some integer n such that $n > x$.

Assume for a contradiction that there is no integer larger than x. That is, assume that x is an upper bound on the set $\mathbb{N}$. Then $\mathbb{N}$ is a subset of $\mathbb{R}$ that is bounded above, and so by the completeness of $\mathbb{R}$ we deduce that $\sup(\mathbb{N})$ exists. Call this supremum α. Since α is the least upper bound of $\mathbb{N}$, we know that $\alpha - 1$ is not an upper bound. That is, there exists some integer $m > \alpha - 1$. Adding 1 to each side,

$$m + 1 > \alpha.$$

But this is a contradiction. If α is the supremum of $\mathbb{N}$, then it is an upper bound on $\mathbb{N}$. But we found $(m + 1) \in \mathbb{N}$ which is larger than α. This concludes the first statement in the principle.

The second part follows directly from the first by letting $a = \epsilon$ and $b = 1$, and dividing over the n. □

Example 1.28

Show that $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$.

Proof. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. We will use the analytic definition of suprema (Theorem 1.25). We must then show that 0 is a lower bound of A and that, for all $\epsilon > 0$, $0 + \epsilon$ is not a lower bound of A.

The first of these is almost immediate: Since 1 and n are positive for each $n \in \mathbb{N}$, so is $1/n$. So $1/n > 0$, and thus 0 is indeed a lower bound for A.

Working toward the second, let $\epsilon > 0$. Then by the Archimedean principle (Lemma 1.27), there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. This element, $\frac{1}{n}$, is in A and is less than $0 + \epsilon$. So $0 + \epsilon$ is not a lower bound of A. □

Example 1.29

Show that $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$.

Proof. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. We will use the analytic definition of suprema (Theorem 1.25). We must then show that 1 is an upper bound of A and that, for all $\epsilon > 0$, $1 - \epsilon$ is not an upper bound of A.

For the first of these, note that since $n \geq 1$ for all $n \in \mathbb{N}$, and by dividing over the n we have that $1 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$. So 1 is indeed an upper bound for A.

Working towards the second, let $\epsilon > 0$. We need to show that there is some $x \in A$ such that $1 - \epsilon < x$. But this is always accomplished by the number 1: Clearly $1 \in A$ and $1 - \epsilon < 1$. □

Definition 1.30 (Density)

Suppose A and B are ordered field. Then A is **dense** in B if, for any $x, y \in B$, there exists $a \in A$ such that $x < a < y$.

Example 1.31

The propositions below are left without proof.

- $\mathbb{Q}$ is dense in $\mathbb{Q}$.
- $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{Q}$.
- $\mathbb{Z}$ is *not* dense in $\mathbb{Q}$.

Lemma 1.32

Let $x, y \in \mathbb{R}$. If $y - x > 1$, then there exists $z \in \mathbb{Z}$ such that $x < z < y$.

Proof. First assume that x and y are at least 0, and consider the set

$$A = \{n \in \mathbb{N}_0 : n \leq x\}.$$

Since $x \geq 0$, this set is non-empty, and since it is a set of nonnegative integers which is bounded above by x, this set is finite. By induction on the size of A, we can show that $\max(A)$ exists and is an element of A. Call this maximum M. We claim $z := M + 1$ works.

Note that since $M \in \mathbb{N}_0$, also $z \in \mathbb{N}_0$. Furthermore, since z is larger than the largest element of A, z is not in A, implying that $x < z$. Finally,

$$M \leq x \qquad \text{implies that} \qquad M + 1 \leq x + 1 \leq y.$$

So $z < y$. In summary, we have shown that $x < z < y$, as desired.

The cases where x and y are not at least 0 are similar. If both are negative, then by considering $-x$ and $-y$ the above argument gives an integer z where $-y < z < -x$, showing that $-z$ works, since $x < -z < y$. If one is positive and one is negative, then 0 works. □

Theorem 1.33 ($\mathbb{Q}$ is dense in $\mathbb{R}$)

The rational numbers are dense in the real numbers.

Proof. Pick any $x, y \in \mathbb{R}$ where $x < y$. We need to show that there exists some $\frac{m}{n} \in \mathbb{Q}$ (with $m, n \in \mathbb{Z}$) such that

$$x < \frac{m}{n} < y.$$

First note that if $x < 0 < y$ then we are done, since $0 \in \mathbb{Q}$. Furthermore, if we can show that the theorem holds for the case that x and y are positive, then it holds when they are negative ($0 < x < \frac{m}{n} < y$ implies $-y < \frac{-m}{n} < -x < 0$), so we may assume x and y are positive.

Since $y - x > 0$, by the Archimedean principle (Lemma 1.27) there exists some $n \in \mathbb{N}$ such that $n(y - x) > 1$; i.e. $ny - nx > 1$. And so, by Lemma 1.32, there is some integer m with

$$nx < m < ny.$$

That is,

$$x < \frac{m}{n} < y,$$

which concludes the proof. □

Remark 1.34 — The proof of Lemma 1.32 also implies that, for any $x \in \mathbb{R}$, there exists an integer M such that $M \leq x \leq M + 1$. In particular, it implies that the floor and ceiling functions exist.

Definition 1.35 (Ceiling and floor functions)

Let $x \in R$.

- The *ceiling* of x, denoted $\lceil x \rceil$, is the integer n such that $x \leq n < x + 1$.
- The *floor* of x, denoted $\lfloor x \rfloor$, is the integer n such that $x - 1 < n \leq x$.

Definition 1.36 (Closed and open intervals)

Define the *closed interval* $[a, b]$ to be $\{x \in \mathbb{R} : a \leq x \leq b\}$. Likewise the *open interval* (a, b) is defined to be $\{x \in \mathbb{R} : a < x < b\}$, and half-open intervals and intervals to $\pm\infty$ are again exactly as you would expect.

Theorem 1.37 (Characterization of intervals)

Let S be a subset of $\mathbb{R}$ that contains at least two points. If S has the property such that

$$\text{if } x, y \in S \text{ and } x < y, \text{ then } [x, y] \subseteq S, \tag{1}$$

then S is an interval.

Proof. There are four cases to consider: (1) S is bounded, (2) S is bounded above but not below, (3) S is bounded below but not above, and (4) S is neither bounded above nor below.

Case (1). Let $a = \inf(S)$ and $b = \sup(S)$. Then $S \subseteq [a, b]$ and we will show that $(a, b) \subseteq S$. If $a < z < b$, then z is not a lower bound of S, so there exists $x \in S$ with $x < z$. Also, z is not an upper bound of S, so there exists $y \in S$ with $z < y$. Therefore, $z \in [x, y]$, so property (1) implies that $z \in S$. Since z is an arbitrary element of (a, b), we conclude that $(a, b) \subseteq S$. Now if $a \in S$ and $b \in S$, then $S = [a, b]$. If $a \notin S$ and $b \notin S$, then $S = (a, b)$. The other possibilities lead to either $S = (a, b]$ or $S = [a, b)$.

Case (2). Let $b = \sup(S)$. Then $S \subseteq (-\infty, b]$ and we will show that $(-\infty, b) \subseteq S$. If $z < b$, then z is not an upper bound of S, so there exists $y \in S$ with $z < y$. Also, since S is not bounded below, there exists $x \in S$ with $x < z$. By property 1, $z \in [x, y] \subseteq S$. Since z is an arbitrary element of $(-\infty, b)$, we conclude that $(-\infty, b) \subseteq S$. Now if $b \in S$, then $S = (-\infty, b]$, and if $b \notin S$, then $S = (-\infty, b)$.

Case (3). Let $a = \inf(S)$. Then $S \subseteq [a, \infty)$ and we will show that $(a, \infty) \subseteq S$. If $a < z$, then z is not a lower bound of S, so there exists $x \in S$ with $x < z$. Also, since S is not bounded above, there exists $y \in S$ with $z < y$. By property 1, $z \in [x, y] \subseteq S$. Since z is an arbitrary element of (a, ∞), we conclude that $(a, \infty) \subseteq S$. Now if $a \in S$, then $S = [a, \infty]$, and if $a \notin S$, then $S = (a, \infty)$.

Case (4). We will show that $S = (-\infty, \infty)$. Pick any $x, y \in S$ with $x < y$, property 1 implies that $[x, y] \subseteq S$. Since S is neither bounded above nor below, and the choice of x, y is arbitrary, we conclude that $(-\infty, \infty) \subseteq S$. Also, every subset of $\mathbb{R}$ is a subset of $(-\infty, \infty)$, thus $S = (-\infty, \infty)$. □

Theorem 1.38 (The nested intervals property)

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Also, assume that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection. That is,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof. In order to show that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ is not empty, we are going to use the completeness axiom to produce a single real number x satisfying $x \in I_n$ for every $n \in \mathbb{N}$. Now, the completeness axiom is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in \mathbb{N}\}$$

of left-hand endpoints of the interval. Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = \sup(A).$$

Now, consider a particular $I_n = [a_n, b_b]$. Because x is an upper bound for A, we have $a_n \leq x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \leq b_n$.

Altogether then, we have $a_n \leq x \leq b_n$, which means $x \in I_n$ for every choice of $n \in \mathbb{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is not empty. □

Remark 1.39 — Note that the conclusion of Theorem 1.38 need not hold if each I_n is allowed to be an open interval.

2 Cardinality

"No one shall expel us from the paradise that Cantor has created."

David Hilbert, Über das Unendliche

Definition 2.1 (Cardinality)
Let S and T be sets. Then, $|S| = |T|$ if and only if there is a bijection from S to T.

Definition 2.2 (Cardinality cont.)
$|S| \leq |T|$ if and only if there is an injection from S to T.

Remark 2.3 — Our definitions above introduce two fundamental relations on cardinality, i.e. $|S| = |T|$ and $|S| \leq |T|$. We need to make sure that these relations have the mathematical properties we expect them to have. That is, we want $|S| = |T|$ to be an equivalence relation and $|S| \leq |T|$ to be a partial order.

For the relation $|S| = |T|$, it's easy to show that it defines an equivalence relation:

- Reflexivity: Every set has a bijection with itself, i.e. $|S| = |S|$.
- Symmetry: If there is a bijection f from S to T, then f^{-1} is a bijection from T to S, and thus $|T| = |S|$.
- Transitivity: If there are bijections f from S to T and g from T to U, then their composition $h = g \circ f$ is a bijection from S to U. Hence, $|S| = |U|$.

For the relation $|S| \leq |T|$, we must establish that it's a partial order:

- Reflexivity: Every set has an injection to itself, i.e. $|S| \leq |S|$.
- Transitivity: If there exist injections f from S to T and g from T to U, then their composition $h = g \circ f$ is an injection from S to U. Hence, $|S| \leq |U|$.

The remaining property, antisymmetry, is where things get interesting, since it's not immediately obvious. Antisymmetry means that if $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$. Using our definition, this translates to: If there is an injection from S to T and an injection from T to S, then there should be a bijection between S and T. This is exactly what the Schröder-Bernstein theorem (Theorem 2.4) guarantees.

Theorem 2.4 (Schröder-Bernstein theorem)
If there exist injections $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijection $h : A \to B$.

In terms of the cardinality of the two sets, this implies that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. The strategy is to partition A and B into components

$$A = X \cup X' \qquad \text{and} \qquad B = Y \cup Y'$$

with $X \cap X' = \varnothing$ and $Y \cap Y' = \varnothing$, in such a way that f is a surjection from X to Y, and g is a surjection from Y' to X'. Achieving this would lead to a proof that there is a bijection h from A to B. Why? For all $x \in X'$, there exists a unique $y \in Y'$ satisfying $g(y) = x$. This means that there is a well-defined inverse function $g^{-1}(x) = y$ that maps X' to Y'. Setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \\ g^{-1}(x) & \text{if } x \in X' \end{cases}$$

gives the desired bijection from A to B.

Now, let $X_1 = A \setminus g(B)$ and inductively define a sequence of sets by letting $X_{n+1} = g(f(X_n))$. We show that $\{X_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of A, while $\{f(X_n) : n \in \mathbb{N}\}$ is a similar collection in B.

To see that the sets $X_1, X_2, X_3, \ldots$ are pairwise disjoint, note that $X_1 \cap X_n = \varnothing$ for all $n \geq 2$ because $X_1 = A \setminus g(B)$ and $X_n \subseteq g(B)$. (Why? f is a mapping from A to B, so we have that $f(X_n) \subseteq B$, and thus $X_{n+1} = g(f(X_n)) \subseteq g(B)$.) In the general case of $X_n \cap X_m$ where $1 < n < m$, note that if $x \in X_n \cap X_m$ then $f^{-1}(g^{-1}(x)) \in X_{n-1} \cap X_{m-1}$. Continuing in this way, we can show $X_1 \cap X_{m-n+1}$ is not empty, which is a contradiction. Thus $X_n \cap X_m = \varnothing$. Just to be clear, the disjointness of the X_n sets is not crucial to the overall proof, but it does help paint a clearer picture of how the sets X and X' are constructed.

Let $X = \bigcup_{n=1}^{\infty} X_n$ and $Y = \bigcup_{n=1}^{\infty} f(X_n)$. We show that f is a surjection from X to Y. This is straightforward. Each $x \in X$ comes from some X_n and so $f(x) \in f(X_n) \subseteq Y$. Likewise, each $y \in Y$ is an element of some $f(X_n)$ and thus $y = f(x)$ for some $x \in X_n \subseteq X$. Thus $f : A \to B$ is a surjection.

Let $X' = A \setminus X$ and $Y' = B \setminus Y$. Let $y \in Y'$. Then $y \notin f(X_n)$ for all n (by definition of Y'). We also conclude that $g(y) \notin X_{n+1}$ for all n. (Why? Suppose if $g(y) \in X_{n+1}$ for some n, then $g(y) \in g(f(X_n))$. That is, $g(y) = g(z)$ for some $z \in f(X_n)$, and since g is injective, we have $y = z$ and thus $y \in f(X_n)$, which is a contradiction.) Clearly, $g(y) \notin X_1$ either (because $g(y) \subseteq g(B)$ and $X_1 = A \setminus g(B)$) and so g is a mapping from Y' to X'. To see that g is a surjection from Y' to X', let $x \in X'$ be arbitrary. Because $X' \subseteq g(Y') \subseteq g(B)$, there exists $y \in B$ with $g(y) = x$. Could y be an element of Y? No, because if $y \in Y$, $g(y)$ would be in $g(Y)$, and since $g(Y) \subseteq X$ (by definition of Y), this would mean $g(y) \in X$. But we're considering an $x \in X'$ with $g(y) = x$, so this is a contradiction as $X \cap X' = \varnothing$. Hence $y \in Y'$ and $g : Y' \to X'$ is a surjection. □

Definition 2.5 (Cardinality cont.)
$|S| \geq |T|$ if and only if there is a surjection from S to T.

Remark 2.6 — Again, we must establish that $|S| \geq |T|$ defines a partial order. Reflexivity and transitivity are obvious. Now we will show antisymmetry. Suppose $|S| \geq |T|$ and $|T| \geq |S|$. Then, by definition, there are surjections from S to T and from T to S. Using the axiom of choice, one can prove that there exists a surjection

from X to Y if and only if there exists an injection from Y to X. We conclude that $|T| \leq |S|$ and $|S| \leq |T|$, which implies $|S| = |T|$ by Theorem 2.4.

Theorem 2.7 ($|\mathbb{Z}| = |\mathbb{N}|$)
There are as many integers as there are natural numbers.

Proof. Define the function $f : \mathbb{N} \to \mathbb{Z}$ with

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even.} \end{cases}$$

Clearly, f is a bijection. (See the following diagram.)

$$\begin{array}{lcccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \cdots \end{array}$$

Hence, by Definition 2.1, $|\mathbb{N}| = |\mathbb{Z}|$. □

Remark 2.8 — Theorem 2.7 also shows that two sets can have the same cardinality even if one is a proper subset of the other and the "larger" one even has infinitely many more elements than the "smaller" one. Make sure you take a moment to appreciate how remarkably, wonderfully weird this is.

Theorem 2.9 ($|\mathbb{Q}| = |\mathbb{N}|$)
There are as many rational numbers as there are natural numbers.

Proof. Let $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = \text{n}\}.$$

The first few of these sets look like

$$A_1 = \{0\}, \qquad A_2 = \left\{\frac{1}{1}, \frac{-1}{1}\right\}, \qquad A_3 = \left\{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\right\},$$

$$A_4 = \left\{\frac{1}{3}, \frac{-1}{3}, \frac{3}{1}, \frac{-3}{1}\right\}, \qquad \text{and} \qquad A_5 = \left\{\frac{1}{4}, \frac{-1}{4}, \frac{2}{3}, \frac{-2}{3}, \frac{3}{2}, \frac{-3}{2}, \frac{4}{1}, \frac{-4}{1}\right\}.$$

The crucial observation is that each A_n is finite and every rational number appears in exactly one of these sets. A bijection with $\mathbb{N}$ is then achieved by consecutively listing the elements in each A_n.

$$\begin{array}{lcccccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Q}: & 0 & \frac{1}{1} & -\frac{1}{1} & \frac{1}{2} & -\frac{1}{2} & \frac{2}{1} & -\frac{2}{1} & \frac{1}{3} & -\frac{1}{3} & \frac{3}{1} & -\frac{3}{1} & \frac{1}{4} & \cdots \\ & \underbrace{}_{A_1} & \underbrace{\phantom{\frac{1}{1}-\frac{1}{1}}}_{A_2} & & \underbrace{\phantom{\frac{1}{2}-\frac{1}{2}\frac{2}{1}-\frac{2}{1}}}_{A_3} & & & & \underbrace{\phantom{\frac{1}{3}-\frac{1}{3}\frac{3}{1}-\frac{3}{1}}}_{A_4} & & & & & \end{array}$$

Admittedly, writing an explicit formula for this correspondence would be an awkward task, and attempting to do so is not the best use of time. What matters is that we see why every rational number appears in the correspondence exactly once. Given, say, $22/7$, we have that $22/7 \in A_{29}$. Because the set of elements in $A_1, \ldots, A_{28}$ is finite, we can be confident that $22/7$ eventually gets included in the sequence. The fact that this line of reasoning applies to any rational number p/q is our proof that the correspondence is surjective. To verify that it is injective, we observe that the sets A_n were constructed to be disjoint so that no rational number appears twice. This completes the proof. □

Theorem 2.10 ($|\mathbb{R}| > |\mathbb{N}|$)
There are more real numbers than natural numbers.

Proof. Assume, for the sake of contradiction, that $|\mathbb{R}| = |\mathbb{N}|$, which implies that there is a bijection $f : \mathbb{N} \to \mathbb{R}$. What this suggests is that it is possible to enumerate the elements of $\mathbb{R}$. If we let $x_1 = f(x_1)$, $x_2 = f(x_2)$, and so on, then the fact that f is bijective (and thus surjective) means that we can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\} \tag{1}$$

and be confident that every real number appears somewhere on the list. We will now use the nested intervals property (Theorem 1.38) to produce a real number that is not there.

Let I_1 be a closed interval that does not contain x_1. Next, let I_2 be a closed interval, contained in I_1, which does not contain x_2. The existence of such I_2 is easy to verify. Certainly I_1 contains two smaller disjoint closed intervals, and x_2 can only be in one of these. In general, given an interval I_n, construct I_{n+1} to satisfy

1. $I_{n+1} \subseteq I_n$ and
2. $x_{n+1} \notin I_{n+1}$.

We now consider the intersection $\bigcap_{n=1}^{\infty} I_n$. If x_k is some real number from the list in (1), then we have $x_k \notin I_k$, and it follows that

$$x_k \notin \bigcap_{n=1}^{\infty} I_n.$$

Now, we are assuming that the list in (1) contains every real number, and this leads to the conclusion that

$$\bigcap_{n=1}^{\infty} I_n = \varnothing.$$

However, the nested intervals property (Theorem 1.38) asserts that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$. So there is at least one $x \in \bigcap_{n=1}^{\infty} I_n$ that, consequently, cannot be on the list in (1). This contradiction means that such an enumeration of $\mathbb{R}$ is impossible, and we conclude that $|\mathbb{R}| \neq |\mathbb{N}|$.

We are not done yet. We still have to show that $|\mathbb{R}| \geq |\mathbb{N}|$. This step is straightforward. Define the function $g : \mathbb{R} \to \mathbb{N}$ with

$$g(x) = \begin{cases} n & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

This function maps each real number that is a natural number to itself, and all other real numbers to 0. Since every natural number is the image of at least one real number under g, the function is surjective. Therefore, by Definition 2.5, we have $|\mathbb{R}| \geq |\mathbb{N}|$. Combining this with our previous result that $|\mathbb{R}| \neq |\mathbb{N}|$, we conclude that $|\mathbb{R}| > |\mathbb{N}|$. □

Definition 2.11 (Countable and uncountable sets)

A set S is **countable** if

1. its cardinality $|S|$ is less than or equal to $|\mathbb{N}|$.
2. there exists an injection from S to $\mathbb{N}$.
3. S is empty or there exists a surjection $\mathbb{N}$ to S.
4. there exists a bijection from S to a subset of $\mathbb{N}$.
5. S is either finite or *countably infinite*.

All of the definitions above are equivalent.

A set S is **countably infinite** if its cardinality $|S|$ is exactly $\mathbb{N}$.

A set S is **uncountable** if it is not countable. That is, its cardinality $|S|$ is greater than $|\mathbb{N}|$.

Corollary 2.12 ($\mathbb{N}$ is countable)

The set of natural numbers is countable.

Corollary 2.13 ($\mathbb{Z}$ is countable)

The set of integers is countable.

Corollary 2.14 ($\mathbb{Q}$ is countable)

The set of rational numbers is countable.

Corollary 2.15 ($\mathbb{R}$ is uncountable)

The set of real numbers is uncountable.

Theorem 2.16

An uncountable collection of disjoint open intervals in $\mathbb{R}$ cannot exist.

Proof. We use contradiction. Suppose there exists an uncountable collection of disjoint open intervals in $\mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Theorem 1.33), every open interval in $\mathbb{R}$ contains at least one rational number. Therefore, there are uncountably many rational numbers. A contradiction of Corollary 2.14. Hence, such a collection cannot exist. □

Theorem 2.17 (Countable infinity is the smallest infinity)

If $A \subseteq B$ and B is countable, then A is either countable or finite.

Proof. B is a countable set. Thus, there exists a bijection $f : \mathbb{N} \to B$. Let $A \subseteq B$ be an infinite subset of B. We show that A is countable. Let $n_1 \in \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, let $g(1) = f(n_1)$. Next let $n_2 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1)\}\}$ and let $g(2) = f(n_2)$. We inductively continue this process to produce a bijection g from $\mathbb{N}$ to A. In general, assume we have defined $g(k)$ for $k < m$, and let $g(m) = f(n_m)$ where $n_m = \min\{n \in \mathbb{N} : A \setminus \{f(n_1), \ldots, f(n_{k-1})\}\}$.

To show that g is injective, observe that $m \neq m'$ implies $n_m \neq n_{m'}$ and it follows that $g(m) = f(n_m) \neq f(n_{m'}) = g(m')$ because f is injective. To show that g is surjective, let $a \in A$ be arbitrary. Because f is surjective, $a = f(n')$ for some $n' \in \mathbb{N}$. This means $n' \in \{n : f(n) \in A\}$ and as we inductively remove the minimal element, n' must eventually be the minimum by at least the $(n' - 1)$-th step. □

Corollary 2.18 ($|\mathbb{N}|$ is the smallest infinity)

If $A \subseteq \mathbb{N}$, then either A is finite or $|A| = |\mathbb{N}|$.

Corollary 2.19 (Sizes of infinity)

There are different sizes of infinity, with countable infinity being the smallest. Moreover, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable while $\mathbb{R}$ is uncountable.

Theorem 2.20 (Countable union of countable sets is countable)

A countable union of countable sets is countable. More precisely:

1. If $A_1, A_2, \ldots, A_m$ are each countable sets, then $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.
2. If A_n is a countable set for each $n \in \mathbb{N}$, then the set $\bigcup_{n=1}^{\infty} A_n$ is also countable.

Proof. First, we prove part 1 for two countable sets, A_1 and A_2. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint.

Now, because A_1 is countable, there exists a bijection $f : \mathbb{N} \to A_1$. If $B_2 = \varnothing$, then $A_1 \cup A_2 = A_1$ which we already know to be countable. If $B_2 = \{b_1, b_2, \ldots, b_m\}$ has m elements then define $h : \mathbb{N} \to A_1 \cup B_2$ via

$$h(n) = \begin{cases} b_n & \text{if } n \leq m \\ f(n-m) & \text{if } n > m. \end{cases}$$

The fact that h is bijective follows immediately from the same property of f. If B_2 is infinite, then by Theorem 2.17 it is countable, and so there exists a bijection $g : \mathbb{N} \to B_2$.

In this case we define $h : \mathbb{N} \to A_1 \cup B_2$ by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that h is bijective is derived directly from the fact that f and g are both bijections. Graphically, the correspondence takes the form

$$\begin{array}{rccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2: & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

To prove the more general statement in part 1, we may use induction. We have just seen that the result holds for two countable sets. Now let's assume that the union of m countable sets is countable, and show that the union of $m+1$ countable sets is countable.

Given $m+1$ countable sets $A_1, A_2, \ldots, A_{m+1}$, we can write

$$A_1 \cup A_2 \cup \cdots \cup A_{m+1} = (A_1 \cup A_2 \cup \cdots \cup A_m) \cup A_{m+1}.$$

Then $C_m = A_1 \cup \cdots \cup A_m$ is countable by the induction hypothesis, and $C_m \cup A_{n+1}$ is just the union of two countable sets which we know to be countable. This completes the proof for part 1.

For part 2, induction cannot be used because we have an infinite number of sets. Instead, we show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{ccccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots \\ 4 & 8 & 13 & \cdots \\ 7 & 12 & \cdots \\ 11 & \cdots \\ \vdots \end{array}$$

leads to a proof.

Let's first consider the case where the sets $\{A_n\}$ are disjoint. In order to achieve bijection between $\mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n$, we first label the elements in each countable set A_n as

$$A_n = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}.$$

Now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in an array similar to the one for $\mathbb{N}$:

$$\begin{array}{rcccccc} A_1 & = & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ A_2 & = & a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ A_3 & = & a_{31} & a_{32} & a_{33} & \cdots \\ A_4 & = & a_{41} & a_{42} & \cdots \\ A_5 & = & a_{51} & \cdots \\ & \vdots \end{array}$$

This establishes a bijection $g : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ where $g(n)$ corresponds to the element a_{jk} where (j,k) is the row and column location of n in the array for $\mathbb{N}$.

If the sets $\{A_n\}$ are not disjoint then our mapping may not be injective. In this case we could again replace A_n with $B_n = A_n \setminus \{A_1 \cup \cdots \cup A_{n-1}\}$. Another approach is to use the previous argument to establish a bijection between $\bigcup_{n=1}^{\infty} A_n$ and an infinite subset of $\mathbb{N}$, and then appeal to Theorem 2.17. This completes the proof for part 2. □

Theorem 2.21 ($\mathbb{R} \setminus \mathbb{Q}$ is uncountable)

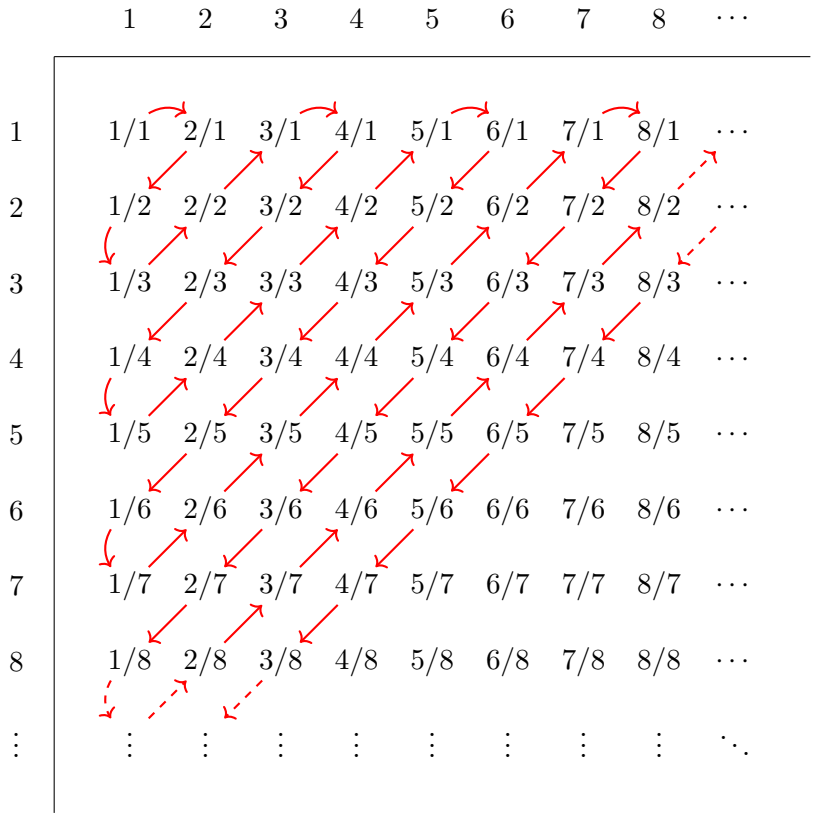
There are uncountably many irrational numbers.

Proof. We use contradiction. Suppose $\mathbb{R} \setminus \mathbb{Q}$ is countable. By Theorem 2.20, we know that a countable union of countable sets is countable, thus $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$ is countable. A contradiction of Corollary 2.15. Hence, $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. □

Theorem 2.22 ($|\mathbb{Q}_+| = |\mathbb{N}|$)

There are as many positive rational numbers as there are natural numbers.

Proof. The following diagram arranges the rational numbers (with some repetition) and illustrates our bijection, which we call the *winding bijection*.



Weaving through this chart, you are guaranteed to hit every positive rational number. So if you pair up 1 with the first number you hit, 2 with the second number you hit, 3 with the third, and so on, then every positive rational number is in a pair. Now, there's just one small problem: each rational number is actually hit more than once. The number p/q will be written in positions $(p, q), (2p, 2q), (3p, 3q), \ldots$. But the fix is easy: When you come across a number that has already been hit, just skip it. Clearly you won't run out of rational numbers, so this does indeed pair up everything. So

$$f(n) = \text{the } n\text{th new rational number you reach.}$$

And that's it. □

Theorem 2.23 ($|(0,1)| > |\mathbb{N}|$)
There are more numbers in the open interval $(0,1)$ than there are natural numbers.

We will demonstrate a clever argument known as *Cantor's diagonal argument*.

Proof. We proceed by contradiction and assume that there does exist a function $f : \mathbb{N} \to (0,1)$ that is bijective. For each $m \in \mathbb{N}$, $f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\dots.$$

What is meant here is that for each $m, n \in \mathbb{N}$, a_{mn} is the digit from the set $\{0, 1, 2, \dots, 9\}$ that represents the nth digit in the decimal expansion of $f(m)$. The bijection between $\mathbb{N}$ and $(0,1)$ can be summarized in the doubly indexed array

$\mathbb{N}$		$(0,1)$								
1	$\longleftrightarrow$	$f(1)$	$=$	$.\boldsymbol{a_{11}}$	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
2	$\longleftrightarrow$	$f(2)$	$=$	$.a_{21}$	$\boldsymbol{a_{22}}$	a_{23}	a_{24}	a_{25}	a_{26}	$\cdots$
3	$\longleftrightarrow$	$f(3)$	$=$	$.a_{31}$	a_{32}	$\boldsymbol{a_{33}}$	a_{34}	a_{35}	a_{36}	$\cdots$
4	$\longleftrightarrow$	$f(4)$	$=$	$.a_{41}$	a_{42}	a_{43}	$\boldsymbol{a_{44}}$	a_{45}	a_{46}	$\cdots$
5	$\longleftrightarrow$	$f(5)$	$=$	$.a_{51}$	a_{52}	a_{53}	a_{54}	$\boldsymbol{a_{55}}$	a_{56}	$\cdots$
6	$\longleftrightarrow$	$f(6)$	$=$	$.a_{61}$	a_{62}	a_{63}	a_{64}	a_{65}	$\boldsymbol{a_{66}}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The key assumption about this correspondence is that *every* real number in $(0,1)$ is assumed to appear somewhere on the list. Now for the pearl of the argument. Define a real number $x \in (0,1)$ with the decimal expansion $x = .b_1b_2b_3b_4\dots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Observe that $x = .b_1b_2b_3b_4\dots$ cannot be $f(1)$. x also cannot be $f(2)$, and in general $x \neq f(n)$ for any $n \in \mathbb{N}$. Therefore, the real number x is nowhere on the list! This is a contradiction; clearly we were unable to pair up all the reals, if x got left out. □

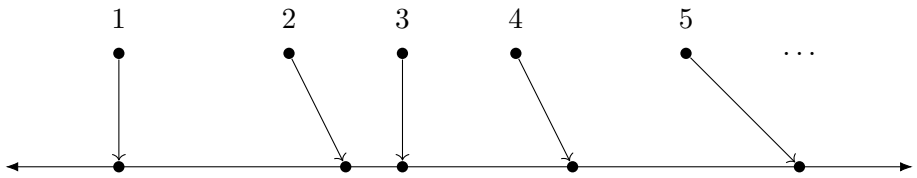
Remark 2.24 — Consider the following complaints about the proof of Theorem 2.23.

1. Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 2.23 must be flawed.
2. Some numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as .5 or as $.4999\dots$. Doesn't this cause some problems?

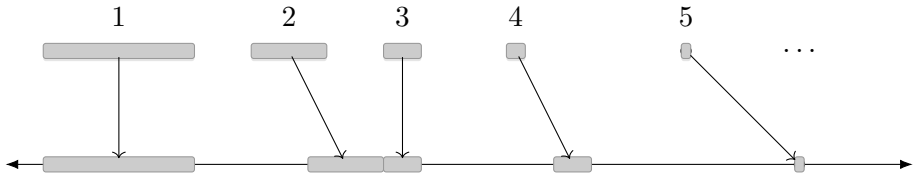
Are these complaints valid?

Here's another proof of Theorem 2.23 that I particular enjoy.

Proof Sketch. Imagine you were trying to map $\mathbb{N}$ on to the real number line (which clearly has $|\mathbb{R}|$ points).



We aim to show that it is impossible for this map to be a bijection—there must be points on the real line that were missed. But instead of mapping just these points, let's make our job slightly harder. Around 1, let's put a little interval of length $\frac{1}{2}$. Around 2, let's put a little interval of length $\frac{1}{4}$. Around 3, let's place a little interval of length $\frac{1}{8}$. And so on. Now, when you map the points in $\mathbb{N}$ to the real line, send over the intervals too (possibly some intervals will overlap; this is ok). We'll now prove that not only are there points on the real line that weren't mapped to, but there are even points that these intervals don't cover!



See what happened? Our intervals' lengths add up to $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$ (and with overlaps, their collective length when mapped to the real line may even be smaller than 1). But the whole real line has length ∞! So certainly there is no chance that all the points are covered; not only do the points of $\mathbb{N}$ not cover the line, but even if we fatten them up with these intervals, those intervals don't even cover the real line! And any point that's in the "$\infty - 1$" portion of the real line that is not covered by an interval was certainly not mapped to. So we have all sorts of points that were missed, and so the mapping is far from being a bijection.

And we could instead pick intervals that add up to 0.0001, or any other tiny number. This proof therefore provides a visualization of how we aren't just missing the one point that Cantor's diagonalization argument finds, but "most" points are missed. □

Theorem 2.25 ($|(0,1)| = |\mathbb{R}|$)
There are as many numbers in the open interval $(0,1)$ as there are real numbers.

Proof Idea. The function $f(x) = (x - 1/2)/(x - x^2)$ is a bijection from $(0,1)$ to $\mathbb{R}$. This shows, by Definition 2.1, that $|(0,1)| = |\mathbb{R}|$. □

Remark 2.26 — Theorem 2.25 effectively establishes that Theorem 2.10 and Theorem 2.23 are equivalent.

Remark 2.27 — We now know that $|\mathbb{N}| < |\mathbb{R}|$. Here's one natural question: Is there any infinity between these two? An astounding fact is that, based on the axioms of set theory (called ZFC), whether or not there exists such an infinity is

unprovable. And what I don't mean is that mathematicians are not smart enough to find the answer; no, I mean that they *are* smart enough to have shown that *no proof can possibly exist*. That's right, there are statements in math which are impossible to prove and also impossible to disprove (but we *are* able to prove that they are unprovable, amazingly).

This particular question is among the most famous in mathematical history. It was posed by Georg Cantor and is known as *the continuum hypothesis*. It is the first of Hilbert's 23 problems—an influential list of unsolved problems that David Hilbert presented in 1900 at the International Congress of Mathematicians, setting the mathematical agenda for the 20th century. Decades later, Kurt Gödel's groundbreaking incompleteness theorems revealed that virtually every mathematical theory contains unprovable statements.

ZFC set theory is the foundational framework upon which nearly all of modern mathematics is built. Gödel constructed a model of ZFC where the continuum hypothesis holds, while Paul Cohen later constructed a model where it fails. Together, their results established that the continuum hypothesis is independent of ZFC—it can be neither proved nor disproved within the system. Thus, the continuum hypothesis—which asks what is presumably a basic question about the infinite—is unprovable.

Hypothesis 2.28 (The continuum hypothesis)

There is no set whose cardinality is strictly between that of the naturals and the reals.

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

Theorem 2.29 ($|A| < |\mathcal{P}(A)|$)

If A is a set and $\mathcal{P}(A)$ is the power set of A, then

$$|A| < |\mathcal{P}(A)|.$$

Proof. Assume for a contradiction that $|A| \geq |\mathcal{P}(A)|$. That is, assume that there is a surjection f from A to $\mathcal{P}(A)$. Since f is a surjection, for every $T \subseteq A$, there is some element $t \in A$ where $f(t) = T$. To reach our contradiction, we will construct a set $B \subsetneq A$ which is not hit.

For each a there is one special property about the set $f(a)$ that we are going to care about: Is $a \in f(a)$ or is $a \notin f(a)$? In general, consider the set of all elements a such that $a \notin f(a)$, and call this set B:

$$B = \{a \in A : a \notin f(a)\}.$$

By the above, if we can show that that there is no b where $f(b) = B$, then we are done; we will have discovered an element of $\mathcal{P}(A)$ that was not hit by f, a contradiction.

Claim. There is no $b \in A$ such that $f(b) = B$.

Proof of claim. Assume for a contradiction that there does exist some $b \in A$ such that $f(b) = B$. Note by the definition of B that

$$b \in B \text{ if and only if } b \notin f(b).$$

But since we assumed that $f(b) = B$, this is equivalent to

$$b \in f(b) \text{ if and only if } b \notin f(b),$$

which is clearly a contradiction. □

Corollary 2.30 (There exist infinitely many infinities)
There exist infinitely many distinct infinite cardinalities.

Proof. By Theorem 2.29, the following is a chain of distinct infinite cardinalities

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))))| < \dots.$$

□

3 Sequences

"Erdős loved epsilons—his word for small children (in mathematics the Greek letter epsilon is used to represent small quantities)."

Paul Hoffman, The Man Who Loved Only Numbers

Definition 3.1

A **sequence** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$.

Definition 3.2

A sequence (a_n) is **bounded** if the range $\{a_n : n \in \mathbb{N}\}$ is bounded. That is, if there exists a lower bound $L \in \mathbb{R}$ and an upper bound $U \in \mathbb{R}$ where

$$L \leq a_n \leq U$$

for all n.

Proposition 3.3

A sequence (a_n) is bounded if and only if there exists some $C \in \mathbb{R}$ for which $|a_n| \leq C$ for all n.

Definition 3.4

A sequence (a_n) **converges** to $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n > N$.

When this happens, a is called the **limit** of a_n.

Definition 3.5

Let $\epsilon > 0$. The **ϵ-neighborhood** of a point a is the interval

$$(a - \epsilon, a + \epsilon).$$

Definition 3.6

A sequence (a_n) *converges* to $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that a_n is in the ϵ-neighborhood of a for all $n > N$.

Definition 3.7

If a sequence (a_n) does not converge, then it **diverges**.

Divergence can come in three forms.

1. (a_n) *diverges to* ∞ (notation: $\lim_{n\to\infty} a_n = \infty$) if, for all $M > 0$, there exists some $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$.
2. (a_n) *diverges to* $-\infty$ (notation: $\lim_{n\to\infty} a_n = -\infty$) if, for all $M < 0$, there exists some $N \in \mathbb{N}$ such that $a_n < M$ for all $n > N$.
3. Otherwise, (a_n)'s limit *does not exist*.

Proposition 3.8 (Limits are unique)

A sequence cannot have more than one limit.

Proposition 3.9

If (a_n) is a convergent sequence, then (a_n) is bounded.

Theorem 3.10 (Sequence limit laws)

Assume that (a_n) and (b_n) are convergent sequences of real numbers such that $a_n \to a$ and $b_n \to b$. Also assume that $c \in \mathbb{R}$. Then,

1. $(a_n + b_n) \to a + b$.
2. $(a_n - b_n) \to a - b$.
3. $(a_n \cdot b_n) \to a \cdot b$.
4. $(\frac{a_n}{b_n}) \to \frac{a}{b}$, provided each $b \neq 0$ and each $b_n \neq 0$.
5. $(c \cdot a_n) \to c \cdot a$.

Theorem 3.11 (Sequence squeeze theorem)

Assume $a_n \leq x_n \leq b_n$ for all n. Furthermore, assume that

$$a_n \to L \qquad \text{and} \qquad b_n \to L.$$

Then,

$$x_n \to L.$$

Definition 3.12

A sequence (a_n) is **monotone increasing** if $a_n \leq a_{n+1}$ for all n. Likewise, a sequence (a_n) is **monotone decreasing** if $a_n \geq a_{n+1}$ for all n. If it is either monotone increasing or monotone decreasing, it is monotone.

Theorem 3.13 (The monotone convergence theorem)

Suppose (a_n) is monotone. Then (a_n) converges if and only if it is bounded. Moreover,

- If (a_n) is increasing, then either (a_n) diverges to ∞ or

$$\lim_{n\to\infty} a_n = \sup(\{a_n : n \in \mathbb{N}\}).$$

- If (a_n) is decreasing, then either (a_n) diverges to $-\infty$ or

$$\lim_{n\to\infty} a_n = \inf(\{a_n : n \in \mathbb{N}\}).$$

Proposition 3.14

Suppose $S \subseteq \mathbb{R}$ is bounded above. Then there exists a sequence (a_n) where $a_n \in S$ for each n and

$$\lim_{n\to\infty} a_n = \sup(S).$$

Likewise, if S is bounded below, then there exists a sequence (b_n) where $b_n \in S$ for each n and

$$\lim_{n\to\infty} b_n = \inf(S).$$

Definition 3.15

Let (a_n) be a sequence of real numbers and let

$$n_1 < n_2 < n_3 < \dots$$

be an increasing sequence of integers. Then,

$$a_{n_1}, a_{n_2}, a_{n_3}, \dots$$

is called a **subsequence** of (a_n), and is denoted (a_{n_k}).

Proposition 3.16

A sequence (a_n) converges to a if and only if every subsequence of (a_n) also converges to a.

Corollary 3.17

If (a_n) has a pair of subsequences converging to different limits, then (a_n) diverges.

Proposition 3.18

If a monotone sequence (a_n) has a convergent subsequence, then (a_n) converges too, and has the same limit.

Lemma 3.19

Every sequence has a monotone subsequence.

Theorem 3.20 (The Bolzano-Weierstrass theorem)

Every bounded sequence has a convergent subsequence.

Definition 3.21

A sequence (a_n) is **Cauchy** if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$|a_m - a_n| < \epsilon$$

for all $m, n > N$.

Lemma 3.22

If (a_n) is Cauchy, then (a_n) is bounded.

Theorem 3.23 (Cauchy criterion for convergence)

A sequence converges if and only if it is Cauchy.

Bibliography

[**Ab15**] Stephen Abbott. *Understanding analysis*. 2nd ed. Springer, 2015 (cited p. 2)

[**Ap74**] Tom M Apostol. *Mathematical analysis*. 1974 (cited p. 2)

[**BaSh11**] Robert G Bartle and Donald R Sherbert. *Introduction to real analysis*. 4th ed. John Wiley & Sons, Inc., 2011 (cited p. 2)

[**Cu19**] Jay Cummings. *Real analysis: a long-form mathematics textbook*. 2nd ed. CreateSpace Independent Publishing Platform, 2019 (cited p. 2)