

# **A Laundry List of Theorems in Analysis**

Richard Willie

May 7, 2025

# Preface

These notes are a loose amalgamation of ideas, concepts, and explanations drawn from various books, papers, and personal reflections. They were originally meant for my own understanding and organization of thoughts, and as such, they may be unpolished, incomplete, or even occasionally incorrect.

I share them in the hope that they may serve as a useful reference, but they should not be treated as a primary source of learning. Readers are strongly encouraged to consult original texts and authoritative resources for a more rigorous and accurate treatment of the topics discussed.

Use these notes as a companion to your studies, not as a substitute for the depth and clarity provided by well-established literature.

# Contents

<b>1</b>	<b>The Real Numbers</b>	<b>6</b>
	The irrationality of $\sqrt{2}$ . . . . .	6
	Fields . . . . .	7
	Ordered fields. . . . .	7
	The triangle inequality . . . . .	8
	The reverse triangle inequality . . . . .	9
	Cauchy-Schwarz inequality . . . . .	10
	Upper and lower bounds . . . . .	10
	Completeness . . . . .	11
	Existence of $\mathbb{R}$ . . . . .	11
	Dedekind cuts . . . . .	11
	Uniqueness of $\mathbb{R}$ . . . . .	12
	Axioms of $\mathbb{R}$ . . . . .	14
	Suprema are unique . . . . .	14
	Suprema analytically . . . . .	15
	The Archimedean principle. . . . .	16
	$0.999 \dots = 1$ . . . . .	17
	Density . . . . .	17
	$\mathbb{Q}$ is dense in $\mathbb{R}$ . . . . .	18
	$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ . . . . .	19
	Closed and open intervals . . . . .	19
	Characterization of intervals . . . . .	19
	The nested intervals property . . . . .	20
<b>2</b>	<b>Cardinality</b>	<b>22</b>
	Cardinality . . . . .	22
	Schröder-Bernstein theorem . . . . .	22
	$ \mathbb{Z}  =  \mathbb{N} $ . . . . .	24
	$ \mathbb{Q}  =  \mathbb{N} $ . . . . .	24
	$ \mathbb{R}  >  \mathbb{N} $ . . . . .	25
	Countable and uncountable sets . . . . .	26
	$\mathbb{N}$ is countable . . . . .	26
	$\mathbb{Z}$ is countable . . . . .	27
	$\mathbb{Q}$ is countable . . . . .	27
	$\mathbb{R}$ is uncountable . . . . .	27
	Countable infinity is the smallest infinity . . . . .	27
	The winding bijection . . . . .	29
	Cantor's diagonal argument . . . . .	30
	The continuum hypothesis . . . . .	32

	$ A  <  \mathcal{P}(A) $ . . . . .	32
	There exist infinitely many infinities . . . . .	33
<b>3</b>	<b>Sequences</b>	<b>34</b>
	Sequences . . . . .	34
	Bounded sequences . . . . .	34
	Convergent sequences . . . . .	35
	Convergent sequences (topological version) . . . . .	35
	Divergent sequences . . . . .	37
	Limits are unique . . . . .	40
	Convergent sequences are bounded . . . . .	41
	Sequence limit laws . . . . .	42
	Sequence squeeze theorem . . . . .	43
	Monotone sequences . . . . .	43
	The monotone convergence theorem . . . . .	44
	Subsequences . . . . .	46
	A sequence converges iff every subsequence converges . . . . .	47
	Every sequence has a monotone subsequence . . . . .	48
	The Bolzano-Weierstrass theorem . . . . .	50
	Cauchy sequences . . . . .	50
	Cauchy sequences are bounded . . . . .	50
	Cauchy criterion for convergence . . . . .	51
	Stolz-Cesàro theorem . . . . .	52
<b>4</b>	<b>Series</b>	<b>56</b>
	Infinite series . . . . .	56
	Sequence of partial sums . . . . .	56
	Convergent and divergent series . . . . .	56
	Bounded series . . . . .	57
	Monotone series . . . . .	57
	Series limit laws . . . . .	57
	Cauchy criterion for series . . . . .	58
	Necessary condition for convergence . . . . .	58
	Comparison test . . . . .	58
	Ratio comparison test . . . . .	59
	Geometric series . . . . .	60
	Geometric series test . . . . .	60
	Harmonic series diverges . . . . .	61
	The $p$ -series test . . . . .	62
	Ratio test . . . . .	63
	Root test . . . . .	63
	Strength of convergence tests . . . . .	63
	Root test is stronger than ratio test . . . . .	64

	Alternating series test . . . . .	64
	Absolute convergence test . . . . .	64
	Absolute convergence. . . . .	64
	Conditional convergence . . . . .	64
	Raabe's test . . . . .	65
	Logarithmic test . . . . .	66
	Logarithmic test is stronger than Raabe's test . . . . .	67
	Kummer's test . . . . .	67
	Kummer's characterization of convergent series . . . . .	67
	Kummer's characterization of divergent series. . . . .	68
	Bertrand's test . . . . .	68
	Gauss's test. . . . .	68
	Cauchy's condensation test. . . . .	69
	Rates of convergence and divergence. . . . .	70
	Nonexistence of the universal test . . . . .	70
	Abel's summation formula . . . . .	71
	Abel's test . . . . .	71
	Dirichlet's test . . . . .	71
	Comparison test for general series . . . . .	71
	The associative law holds for convergent series . . . . .	72
	Rearrangements of series . . . . .	72
	The commutative law holds for absolutely convergent series . . . . .	72
	Riemann series theorem . . . . .	72
	Finite decimal representations of real numbers . . . . .	72
	Finite decimal approximations of real numbers . . . . .	73
	Infinite decimal representations of real numbers. . . . .	74
	$0.999 \dots = 1$ . . . . .	74
<b>5</b>	<b>Basic Topology of <math>\mathbb{R}</math></b>	<b>76</b>
	Open sets . . . . .	76
	Characterization of open sets . . . . .	77
	Limit points . . . . .	77
	Isolated points . . . . .	78
	Closed sets . . . . .	78
	Characterization of closed sets . . . . .	78
<b>A</b>	<b>Pathological Examples</b>	<b>79</b>
	An infinite field that cannot be ordered . . . . .	79

# 1 The Real Numbers

“It is a ‘simple’ theorem, simple both in idea and execution, but there is no doubt at all about it being a theorem of the highest class. It is as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on it.”

*G. H. Hardy, A Mathematician's Apology*

## Theorem 1.1 (The irrationality of $\sqrt{2}$ )

There is no rational number whose square is 2.

*Proof.* A rational number is any number that can be expressed in the form  $p/q$ , where  $p$  and  $q$  are integers. Thus, what the theorem asserts is that no matter how  $p$  and  $q$  are chosen, it is never the case that  $(p/q)^2 = 2$ . The line of attack is indirect, using a type of argument referred to as a proof by contradiction. The idea is to assume that there is a rational number whose square is 2 and then proceed along logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erroneous assumption that some rational number squared is equal to 2. In short, we will prove that the theorem is true by demonstrating that it cannot be false.

And so assume, for contradiction, that there exist integers  $p$  and  $q$  satisfying

$$\left(\frac{p}{q}\right)^2 = 2. \quad (1)$$

We may also assume that  $p$  and  $q$  have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Now, equation (1) implies

$$p^2 = 2q^2. \quad (2)$$

From this, we can see that the integer  $p^2$  is an even number (it is divisible by 2), and hence  $p$  must be even as well because the square of an odd number is odd. This allows us to write  $p = 2r$ , where  $r$  is also an integer. If we substitute  $2r$  for  $p$  in equation (2), then a little algebra yields the relationship

$$2r^2 = q^2.$$

But now the absurdity is at hand. This last equation implies that  $q^2$  is even, and hence  $q$  must also be even. Thus, we have shown that  $p$  and  $q$  are both even (i.e., divisible by 2) when they were originally assumed to have no common factor. From this logical impasse, we can only conclude that equation (1) cannot hold for any integers  $p$  and  $q$ , and thus the theorem is proved.  $\square$

**Remark 1.2** — So the rationals aren't quite enough. That said, they do have almost every other fundamental property we would want. To the point: They are what we call an *ordered field*. But first, what's a *field*? It's a set that satisfies the classic additive and multiplicative properties we know and love.

**Definition 1.3 (Fields)**

A **field** is a nonempty set  $\mathbb{F}$ , along with two binary operations, addition (+) and multiplication ( $\cdot$ ), satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If  $a, b \in \mathbb{F}$ , then  $a + b = b + a$  and  $a \cdot b = b \cdot a$ .
- **Axiom 2 (Distributive Law).** If  $a, b \in \mathbb{F}$ ,  $a \cdot (b + c) = a \cdot b + a \cdot c$ .
- **Axiom 3 (Associative Law).** If  $a, b \in \mathbb{F}$ , then  $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- **Axiom 4 (Identity Law).** There are special elements  $\mathbf{0}, \mathbf{1} \in \mathbb{F}$ , where  $a + \mathbf{0} = a$  and  $a \cdot \mathbf{1} = a$  for all  $a \in \mathbb{F}$ .
- **Axiom 5 (Inverse Law).** For each  $a \in \mathbb{F}$ , there is an element  $-a \in \mathbb{F}$  such that  $a + (-a) = \mathbf{0}$ . If  $a \neq \mathbf{0}$ , then there is also an element  $a^{-1} \in \mathbb{F}$  such that  $a \times a^{-1} = \mathbf{1}$ .

**Example 1.4**

Below are some examples and some non-examples of fields.

- The natural number  $\mathbb{N}$  do not form a field; they fail the first half of Axiom 4 and both halves of Axiom 5.
- The integers  $\mathbb{Z}$  *almost* form a field; they only fail the second half of Axiom 5.
- One can check that the rationals  $\mathbb{Q}$  form a field.

**Remark 1.5** — Now, let's come up with a definition for an ordered field. Think about  $\mathbb{Q}$ . What does  $\mathbb{Q}$  have that a field does not? There are three main properties we are missing: First, there are infinitely many rationals (and they are “symmetric” about the 0 element.) Second, the rationals have an ordering to them. Lastly, we would like to talk about how big a number is. Beautifully, Definition 1.6 describes a single elegant axiom that we can include to capture *all* of these properties.

**Definition 1.6 (Ordered Fields)**

An **ordered field** is a field  $\mathbb{F}$ , along with the following additional axiom.

**Axiom 6 (Order Axiom).** There is a nonempty subset  $P \subseteq \mathbb{F}$ , called the *positive elements*, such that

1. If  $a, b \in P$ , then  $a + b \in P$  and  $a \cdot b \in P$ ;
2. If  $a \in \mathbb{F}$  and  $a \neq \mathbf{0}$ , then either  $a \in P$  or  $-a \in P$ , but not both.

**Definition 1.7 (Inequalities)**

If  $\mathbb{F}$  is an ordered field and  $a, b \in \mathbb{F}$ , then we say that “ $a < b$ ” if  $b - a \in P$ . Likewise,  $a \leq b$  means that either  $a = b$  or  $a < b$ .

We define “ $>$ ” similarly.

**Fact 1.8** (Properties of inequalities)

For  $a, b, c$  in an ordered field  $\mathbb{F}$ :

1. If  $a < b$ , then  $a + c < b + c$ .
2. Transitivity: If  $a < b$  and  $b < c$ , then  $a < c$ .
3. If  $a < b$ , then  $ac < bc$  if  $c > 0$ , and  $ac > bc$  if  $c < 0$ .
4. If  $a \neq 0$ , then  $a^2 > 0$ .

**Definition 1.9** (The absolute value function)

If  $\mathbb{F}$  is an ordered field, define the *absolute value* function  $|\cdot| : \mathbb{F} \rightarrow \mathbb{F}$  to be

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

**Fact 1.10** (Properties of absolute values)

For  $a, b$  in an ordered field  $\mathbb{F}$ :

1.  $|a| \geq 0$ , with equality if and only if  $a = 0$ .
2.  $|a| = |-a|$ .
3.  $-|a| \leq a \leq |a|$ .
4.  $|a \cdot b| = |a| \cdot |b|$ .
5.  $1/|a| = |1/a|$ , if  $a \neq 0$ .
6.  $|a/b| = |a|/|b|$ , if  $b \neq 0$ .
7.  $|a| \leq b$  if and only if  $-b \leq a \leq b$ .

**Theorem 1.11** (The triangle inequality)

If  $\mathbb{F}$  is an ordered field and if  $x, y \in \mathbb{F}$ , then

$$|x + y| \leq |x| + |y|.$$

*Proof.* For  $x, y \in \mathbb{F}$ , by Fact 1.10 part 3 we have

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|.$$

Adding these two together gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

And so, by Fact 1.10 part 7,

$$|x + y| \leq |x| + |y|.$$

□



**Corollary 1.12** (The reverse triangle inequality)

Assume that  $\mathbb{F}$  is an ordered field and  $x, y \in \mathbb{F}$ . Then,

$$||x| - |y|| \leq |x - y|.$$

*Proof.* By Fact 1.10 part 7, it suffices to show that

$$-|x - y| \leq |x| - |y| \leq |x - y|.$$

We first show the right-hand one (that  $|x| - |y| \leq |x - y|$ ), and we will do so by applying an application of the triangle inequality. Let  $a = x - y$  and  $b = y$ . Then by the triangle inequality,

$$|a + b| \leq |a| + |b|.$$

That is,

$$\begin{aligned} |(x - y) + y| &\leq |x - y| + |y| \\ |x| &\leq |x - y| + |y|. \end{aligned}$$

Rearranging,

$$|x| - |y| \leq |x - y|.$$

We will use a similar approach to show that  $-|x - y| \leq |x| - |y|$ . Let  $c = y - x$  and  $d = x$ . By the triangle inequality,

$$|c + d| \leq |c| + |d|$$

That is,

$$\begin{aligned} |(y - x) + x| &\leq |y - x| + |x| \\ |y| &\leq |y - x| + |x|. \end{aligned}$$

Rearranging,

$$-|y - x| \leq |x| - |y|,$$

which by Fact 1.10 part 2 implies that

$$-|x - y| \leq |x| - |y|,$$

as desired. □

**Corollary 1.13** (Triangle inequality corollaries)

For both of the following, assume that  $\mathbb{F}$  is an ordered field and  $x, y \in \mathbb{F}$ .

1.  $|x - y| \leq |x| + |y|.$
2.  $|x + y| \geq ||x| - |y||.$

*Proof.*

For part 1, replace  $y$  with  $-y$  in the triangle inequality.

For part 2, replace  $y$  with  $-y$  in the reverse triangle inequality. □

**Theorem 1.14** (Cauchy-Schwarz inequality)

If  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are arbitrary real numbers, we have

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right).$$

*Proof.* A sum of squares can never be negative. Hence we have

$$\sum_{i=1}^n (a_i x + b_i)^2 \geq 0$$

for every  $x \in \mathbb{R}$ , with equality if and only if each term is zero. This inequality can be written in the form

$$Ax^2 + 2Bx + C \geq 0$$

where

$$A = \sum_{i=1}^n a_i^2, \quad B = \sum_{i=1}^n a_i b_i, \quad C = \sum_{i=1}^n b_i^2.$$

If  $A > 0$ , put  $x = -B/A$  to obtain  $B^2 - AC \leq 0$ , which is the desired inequality. Otherwise if  $A = 0$ , the rest of the proof is trivial.  $\square$

**Definition 1.15** (Upper and lower bounds)

Let  $S$  be an ordered field and  $A \subseteq S$  be nonempty.

1. The set  $A$  is *bounded above* if there exists some  $b \in S$  such that  $x \leq b$  for all  $x \in A$ ; in this case  $b$  is called an **upper bound** of  $A$ .
2. The **least upper bound** of  $A$ —if it exists—is some  $b_0 \in S$  such that
  - a)  $b_0$  is an upper bound of  $A$ , and
  - b) if  $b$  is any other upper bound of  $A$ , then  $b_0 \leq b$ .

Such a  $b_0$  is also called the **supremum** of  $A$  and is denoted  $\sup(A)$ .

3. Likewise, the set  $A$  is *bounded below* if there exists some  $b \in S$  such that  $x \geq b$  for all  $x \in A$ ; in this case,  $b$  is called a **lower bound** of  $A$ .
4. Again, like above, the **greatest lower bound** of  $A$ —if it exists—is some  $b_0 \in S$  such that
  - a)  $b_0$  is a lower bound of  $A$ , and
  - b) if  $b$  is any other lower bound of  $A$ , then  $b_0 \geq b$ .

Such a  $b_0$  is also called the **infimum** of  $A$  and is denoted  $\inf(A)$ .

5. If a set is both bounded above and bounded below, then it is simply *bounded*.

**Example 1.16**

The propositions below are left without proof.

- The set  $\mathbb{N} = \{1, 2, 3, \dots\}$  has no upper bounds. Lower bounds on  $\mathbb{N}$  include  $-17$ ,  $1$ ,  $0.123$ , and  $-\pi$ . Note that  $\sup(\mathbb{N})$  does not exist, but  $\inf(\mathbb{N}) = 1$ .
- The set  $\mathbb{Q}$  has no upper or lower bounds; consequently,  $\sup(\mathbb{Q})$  and  $\inf(\mathbb{Q})$  do not exist.
- $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$ ;  $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$ . Note that the supremum here is in the set, while the infimum is not in the set.
- $\sup(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = 1$ ;  $\inf(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = \frac{1}{2}$ . Note that the infimum here is in the set, while the supremum is not in the set.
- In  $\mathbb{Q}$  the set  $\{x \in \mathbb{Q} : x^2 < 2\}$  does not have a supremum. In  $\mathbb{R}$  it will—in fact,  $\sup(\{x \in \mathbb{Q} : x^2 < 2\}) = \sqrt{2}$ .

### Definition 1.17 (Completeness)

Let  $S$  be an ordered field. Then  $S$  has the **least upper bound property** if given any nonempty  $A \subseteq S$  where  $A$  is bounded above,  $A$  has a least upper bound in  $S$ . In other words,  $\sup(A) \in S$  for every such  $A$ .

Such a set  $S$  is also called **complete**.

### Theorem 1.18 (Existence of $\mathbb{R}$ )

The real numbers  $\mathbb{R}$  exists, and it is a complete ordered field.

*Proof Sketch.* We have the ordered field of rational numbers, but they aren't complete—there are holes everywhere, and to get to  $\mathbb{R}$  we must fill in these gaps. There are several ways to do this. But the most common method, which we discuss now, uses *Dedekind cuts*.

Each real number is going to be a set; at this point we have the rationals constructed, so each real number is going to be represented by a set of rationals. The way you want to think about it is this: the real number  $x$  is going to be represented by the set of all rational numbers strictly less than  $x$ . These sets are going to be called *cuts*, and while we discuss them you can start convincing yourself that each real number will indeed correspond to a unique cut, and each cut corresponds to a unique real number.

### Definition 1.19 (Dedekind cuts)

A **cut** should be thought of as the set  $(-\infty, b) \cap \mathbb{Q}$ . That is, all rational numbers up to a certain point. Formally, it is defined as any set  $C_b$  satisfying the following three conditions.

1.  $C_b \subseteq \mathbb{Q}$ , but  $C_b \neq \emptyset$  and  $C_b \neq \mathbb{Q}$ ;
2. If  $p \in C_b$  and  $q \notin C_b$ , then  $p < q$ ;
3. If  $p \in C_b$ , then there exists some  $q \in C_b$  where  $q < p$ .

And then  $\mathbb{R}$  is defined as the set of all cuts.

Although we have an intuitive picture of a cut, at the moment it is just a set satisfying the above properties. We are interested in putting an algebraic structure on this collection of cuts. Addition and order work quite smoothly. For a pair of cuts  $C_a$  and  $C_b$ , define the following.

- $C_a + C_b := \{p + q : p \in C_a, q \in C_b\}$
- $C_a < C_b$  if and only if  $C_a \subsetneq C_b$

One can verify that the addition of two cuts is still a cut, and that addition is commutative and associative, that the  $\mathbf{0}$  cut behaves as it should (in turn giving *positive* and *negative* cuts) and that additive inverses exist. The inequality has the property that, given any two cuts  $C_a$  and  $C_b$ , exactly one of the following holds:  $C_a < C_b$ ,  $C_a = C_b$ , or  $C_b < C_a$ .

Defining multiplication is trickier, because if you simply multiply the two sets together you'll have massive negative numbers multiplying against each other, creating massive positive numbers. Intuitively, you want  $C_a \cdot C_b = C_{ab}$ . That is, the cut  $(-\infty, a) \cap \mathbb{Q}$  times the cut  $(-\infty, b) \cap \mathbb{Q}$  should equal the cut  $(-\infty, a \cdot b) \cap \mathbb{Q}$ ; but cuts aren't defined with such  $a$  and  $b$ —they produce  $a$  and  $b$ . One way around this is to first define multiplication for positive cuts. That is, if  $C_a$  and  $C_b$  are both positive (larger than the cut  $\{q \in \mathbb{Q} : q < 0\}$ ), then define

$$C_a \cdot C_b := \{p \cdot q : p \in C_a, q \in C_b \text{ with } p, q \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

With this you can then define a product of two negative cuts by setting the product equal to the product of the two corresponding cuts. To define multiplication between a positive and a negative cut, you know the product should be negative so one approach is to consider the multiplication when both are positive, and then translate the result to the corresponding negative cut. It's a hassle to write out, but that's the idea.

One can then check that the product of two cuts is still a cut, that multiplicative inverses exist, and that multiplication is commutative and associative, as well as the remaining multiplicative/additive distributive and order properties. These would be quite annoying to work out in detail, but you can smile knowing someone carefully checked them.

With our set built, and with the algebraic properties defined and their properties verified, we now know that  $\mathbb{R}$  is an ordered field. All that is left to show is that it is complete. This is done by showing that cuts satisfy the *least upper bound property*. That is, if  $\mathcal{C}$  is a collection of cuts which is bounded above (meaning there exists some cut  $D$  such that  $C \leq D$  for all  $C \in \mathcal{C}$ ), then there exists a least upper bound (meaning there is a cut  $M$  such that  $M \leq D$  for all upper bounds  $D$ ). The proof is this: Let  $M = \bigcup_{C \in \mathcal{C}} C$ , and show that

1.  $M$  is a cut and therefore  $M \in \mathbb{R}$ ;
2.  $M$  is an upper bound of  $\mathcal{C}$ , but is smaller than all other upper bounds.

With those,  $\mathbb{R}$  is a complete ordered field, and hence the real numbers are constructed.  $\square$

### **Theorem 1.20** (Uniqueness of $\mathbb{R}$ )

Any complete ordered field is isomorphic to  $\mathbb{R}$ .

*Proof Sketch.* First, let us establish that every complete ordered field  $R$  is Archimedean. This means that there is no element in  $R$  that is larger than every finite sum  $1 + 1 + \cdots + 1$ . (Why? Refer to Lemma 1.27. If we let  $a = 1$ , the Archimedean principle states that for any  $b$ , there exists an  $n$  such that  $n \cdot 1 > b$ . This  $n \cdot 1$  is precisely a finite sum  $1 + 1 + \cdots + 1$ ). To prove this property, we use contradiction. Suppose there exists such an element, then by completeness (Definition 1.17), there is a least upper bound  $b$  to these all these finite sums. But then  $b - 1$  would also be an upper bound, since adding  $1$  to any sum would still be less than  $b$ . This contradicts  $b$  being the least upper bound. Therefore, no such element exists, and  $R$  must be Archimedean.

Consider two complete ordered fields,  $\mathbb{R}_0$  and  $\mathbb{R}_1$ . We construct their respective prime subfields, that is, their copies of the rationals  $\mathbb{Q}_0$  and  $\mathbb{Q}_1$ . This is done by computing inside them all the finite quotients of the form  $\pm(1 + 1 + \cdots + 1)/(1 + 1 + \cdots + 1)$ , which essentially represent all fractions  $p/q$  where  $p, q \in \mathbb{Z}$  and  $q > 0$ . The fractional representation naturally defines an isomorphism between  $\mathbb{Q}_0$  and  $\mathbb{Q}_1$ . This means that corresponding rational elements in each field map to each other, and this mapping preserves addition, multiplication, and order. This is illustrated in the diagram with the blue dots and arrows connecting corresponding rational elements.

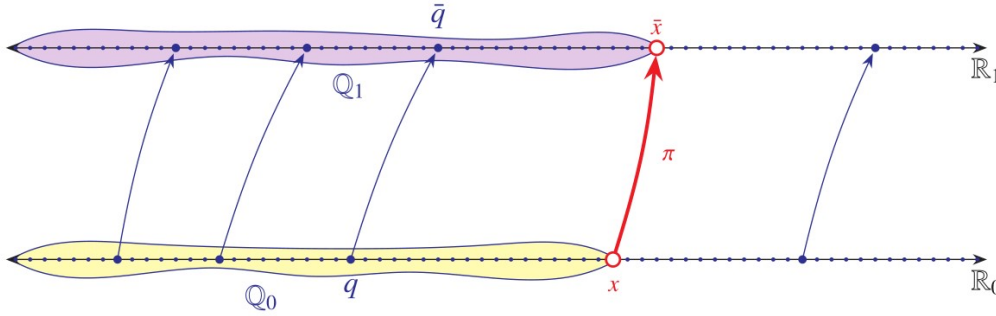


Figure 1.1: An isomorphic mapping  $\pi : \mathbb{R}_0 \rightarrow \mathbb{R}_1$  (courtesy of Hamkins [Ha]).

We now extend this isomorphism to the entire fields,  $\mathbb{R}_0$  and  $\mathbb{R}_1$ , using Dedekind cuts (Definition 1.19). The Archimedean property ensures that every element  $x$  in  $\mathbb{R}_0$  defines a unique cut in  $\mathbb{Q}_0$ , dividing it into two sets: Those rational elements less than  $x$  (shown in yellow), and those greater than or equal to  $x$ . Since we have an isomorphism between  $\mathbb{Q}_0$  and  $\mathbb{Q}_1$ , this cut in  $\mathbb{Q}_0$  corresponds to a similar division in  $\mathbb{Q}_1$ . By the completeness of  $\mathbb{R}_1$ , there must be a unique element  $\bar{x} \in \mathbb{R}_1$  that makes this exact same cut in  $\mathbb{Q}_1$  (shown in violet). This defines our mapping  $\pi : x \mapsto \bar{x}$  from  $\mathbb{R}_0$  to  $\mathbb{R}_1$ .

Finally, we verify that the mapping  $\pi$  is indeed a field isomorphism:

1. **Surjection:** Every  $y \in \mathbb{R}_1$  determines a cut in  $\mathbb{Q}_1$ . By the isomorphism between  $\mathbb{Q}_0$  and  $\mathbb{Q}_1$ , this corresponds to a cut in  $\mathbb{Q}_0$ . By the completeness of  $\mathbb{R}_0$ , there exists an  $x \in \mathbb{R}_0$  that defines this cut. Thus,  $\pi(x) = y$ .
2. **Injection:** If two elements  $x_1, x_2 \in \mathbb{R}_0$  determine the same cut in  $\mathbb{Q}_0$ , then  $x_1 = x_2$  (by the definition of Dedekind cuts). Thus, if  $\pi(x_1) = \pi(x_2)$ , then  $x_1$  and  $x_2$  must define the same cut in  $\mathbb{Q}_0$ , which means  $x_1 = x_2$ .
3. **Field homomorphism:** The mapping  $\pi$  preserves field operations (i.e. addition, multiplication, and order) because it is constructed as a continuous extension of the isomorphism between the rational subfields  $\mathbb{Q}_0$  and  $\mathbb{Q}_1$ .

This completes the proof. □

**Corollary 1.21** (Existence and uniqueness of  $\mathbb{R}$ )

There exists a unique complete ordered field. We call this field *the real numbers*  $\mathbb{R}$ .

**Proposition 1.22** (Axioms of  $\mathbb{R}$ )

The set  $\mathbb{R}$  has two binary operations, addition (+) and multiplication ( $\cdot$ ), and is the unique set satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If  $a, b \in \mathbb{R}$ , then  $a + b = b + a$  and  $a \cdot b = b \cdot a$ .
- **Axiom 2 (Distributive Law).** If  $a, b \in \mathbb{R}$ ,  $a \cdot (b + c) = a \cdot b + a \cdot c$ .
- **Axiom 3 (Associative Law).** If  $a, b \in \mathbb{R}$ , then  $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- **Axiom 4 (Identity Law).** There are special elements  $0, 1 \in \mathbb{R}$ , where  $a + 0 = a$  and  $a \cdot 1 = a$  for all  $a \in \mathbb{R}$ .
- **Axiom 5 (Inverse Law).** For each  $a \in \mathbb{R}$ , there is an element  $-a \in \mathbb{R}$  such that  $a + (-a) = 0$ . If  $a \neq 0$ , then there is also an element  $a^{-1} \in \mathbb{R}$  such that  $a \times a^{-1} = 1$ .
- **Axiom 6 (Order Axiom).** There is a nonempty subset  $P \subseteq \mathbb{R}$ , called the *positive elements*, such that
  1. If  $a, b \in P$ , then  $a + b \in P$  and  $a \cdot b \in P$ ;
  2. If  $a \in \mathbb{R}$  and  $a \neq 0$ , then either  $a \in P$  or  $-a \in P$ , but not both.
- **Axiom 7 (Completeness Axiom).** Given any nonempty  $A \subseteq \mathbb{R}$  where  $A$  is bounded above,  $A$  has a least upper bound. In other words,  $\sup(A) \in \mathbb{R}$  for every such  $A$ .

**Proposition 1.23** (Suprema are unique)

If the supremum or infimum of  $A \subseteq \mathbb{R}$  exists, then it is unique.

We will only prove that suprema are unique. The infima case is analogous.

*Proof.* Assume for a contradiction that  $\alpha$  and  $\beta$  are distinct least upper bounds of  $A$ . In particular, both are upper bounds of  $A$ , while  $\alpha \neq \beta$ . On one hand, since  $\alpha$  is a least upper bound and  $\beta$  is an upper bound, we must have  $\alpha \leq \beta$ . On the other hand, since  $\beta$  is a least upper bound and  $\alpha$  is an upper bound, we must have  $\beta \leq \alpha$ . In summary,

$$\alpha \leq \beta \quad \text{and} \quad \beta \leq \alpha.$$

This implies that  $\alpha = \beta$ , giving our contradiction. □

**Theorem 1.24** (Square roots exist)

If  $a \in \mathbb{R}$  and  $a \geq 0$ , then  $\sqrt{a} \in \mathbb{R}$ .

*Proof Idea.* One can show that  $\sqrt{a} = \sup(\{x \in \mathbb{R} : x^2 < a\})$ , which is in  $\mathbb{R}$  by completeness.  $\square$

**Theorem 1.25 (Suprema analytically)**

Let  $A \subseteq \mathbb{R}$ . Then  $\sup(A) = \alpha$  if and only if

1.  $\alpha$  is an upper bound of  $A$ , and
2. Given any  $\epsilon > 0$ ,  $\alpha - \epsilon$  is *not* an upper bound of  $A$ . That is, there is some  $x \in A$  for which  $x > \alpha - \epsilon$ .

Likewise,  $\inf(A) = \beta$  if and only if

1.  $\beta$  is a lower bound of  $A$ , and
2. Given any  $\epsilon > 0$ ,  $\beta + \epsilon$  is *not* a lower bound of  $A$ . That is, there is some  $x \in A$  for which  $x < \beta + \epsilon$ .

We will only prove the suprema case. The infima case is analogous.

*Proof.*

( $\Rightarrow$ ) First, assume that  $\sup(A) = \alpha$ . We aim to prove part 1 and 2. The first of these is immediate: Since  $\sup(A) = \alpha$ ,  $\alpha$  is the least upper bound of  $A$ , which of course also implies that it is an upper bound of  $A$ .

Now we will show part 2. Let  $\epsilon > 0$ . Since  $\alpha - \epsilon < \alpha$ , we know that  $\alpha - \epsilon$  is not an upper bound of  $A$ , because if so that would contradict  $\alpha$  being the least upper bound of  $A$ . And so, since  $\alpha - \epsilon$  is not an upper bound, there must be some  $x$  who is greater than  $\alpha - \epsilon$ .

( $\Leftarrow$ ) Now assume part 1 and 2. We aim to prove that  $\sup(A) = \alpha$ . That is, we wish to show that  $\alpha$  is an upper bound of  $A$  (which is implied directly by part 1), and for any other upper bound  $\beta$ , we have  $\alpha \leq \beta$ . We have only the latter to prove. Assume that  $\beta$  is some other upper bound of  $A$ , and assume for a contradiction that  $\beta < \alpha$ . Note that  $0 < \alpha - \beta$ . We will use  $(\alpha - \beta)$  as our  $\epsilon$ , and then apply part 2 to contradict  $\beta$  being an upper bound.

Now we will work it out formally. Let  $\epsilon = \alpha - \beta$ . Since  $\epsilon > 0$ , by part 2 there exists some  $x \in A$  such that  $x > \alpha - \epsilon = \alpha - (\alpha - \beta) = \beta$ . But this is a contradiction, because we assumed that  $\beta$  was an upper bound of  $A$ , and yet we found another element  $x \in A$  that is larger than  $\beta$ .  $\square$

**Remark 1.26** — Note that the forward direction of the proof also works well by contrapositive. The contrapositive of

$\sup(A) = \alpha \implies$  For all  $\epsilon > 0$  there is an  $x \in A$  such that  $x > \alpha - \epsilon$   
is

There is an  $\epsilon > 0$  such that for all  $x \in A$  we have  $x \leq \alpha - \epsilon \implies \sup(A) \neq \alpha$ .

And to prove this, just observe that the left-hand side implies that  $\alpha - \epsilon$  is an upper bound of  $A$ , and so  $\sup(A) \leq \alpha - \epsilon$ , which of course implies that  $\sup(A) \neq \alpha$ , as desired.

**Lemma 1.27** (The Archimedean principle)

If  $a$  and  $b$  are real numbers with  $a > 0$ , then there exists a natural number  $n$  such that  $na > b$ .

In particular, for any  $\epsilon > 0$  there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < \epsilon$ .

*Proof.* We aim to show that  $na > b$  for some  $n \in \mathbb{N}$ ; by dividing over the  $a$ , we aim to prove that there is some  $n \in \mathbb{N}$  such that  $n > b/a$ . Now, the number  $b/a$  is just some real number that we know nothing about. In fact, let's just call it  $x$ . So, equivalently, we are trying to prove that given any real number  $x$ , there is some integer  $n$  such that  $n > x$ .

Assume for a contradiction that there is no integer larger than  $x$ . That is, assume that  $x$  is an upper bound on the set  $\mathbb{N}$ . Then  $\mathbb{N}$  is a subset of  $\mathbb{R}$  that is bounded above, and so by the completeness of  $\mathbb{R}$  we deduce that  $\sup(\mathbb{N})$  exists. Call this supremum  $\alpha$ . Since  $\alpha$  is the least upper bound of  $\mathbb{N}$ , we know that  $\alpha - 1$  is not an upper bound. That is, there exists some integer  $m > \alpha - 1$ . Adding 1 to each side,

$$m + 1 > \alpha.$$

But this is a contradiction. If  $\alpha$  is the supremum of  $\mathbb{N}$ , then it is an upper bound on  $\mathbb{N}$ . But we found  $(m + 1) \in \mathbb{N}$  which is larger than  $\alpha$ . This concludes the first statement in the principle.

The second part follows directly from the first by letting  $a = \epsilon$  and  $b = 1$ , and dividing over the  $n$ .  $\square$

**Example 1.28**

Show that  $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$ .

*Proof.* Let  $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ . We will use the analytic definition of suprema (Theorem 1.25). We must then show that 0 is a lower bound of  $A$  and that, for all  $\epsilon > 0$ ,  $0 + \epsilon$  is not a lower bound of  $A$ .

The first of these is almost immediate: Since 1 and  $n$  are positive for each  $n \in \mathbb{N}$ , so is  $1/n$ . So  $1/n > 0$ , and thus 0 is indeed a lower bound for  $A$ .

Working toward the second, let  $\epsilon > 0$ . Then by the Archimedean principle (Lemma 1.27), there exists some  $n \in \mathbb{N}$  such that  $\frac{1}{n} < \epsilon$ . This element,  $\frac{1}{n}$ , is in  $A$  and is less than  $0 + \epsilon$ . So  $0 + \epsilon$  is not a lower bound of  $A$ .  $\square$

**Example 1.29**

Show that  $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$ .

*Proof.* Let  $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ . We will use the analytic definition of suprema (Theorem 1.25). We must then show that 1 is an upper bound of  $A$  and that, for all  $\epsilon > 0$ ,  $1 - \epsilon$  is not an upper bound of  $A$ .

For the first of these, note that since  $n \geq 1$  for all  $n \in \mathbb{N}$ , and by dividing over the  $n$  we have that  $1 \geq \frac{1}{n}$  for all  $n \in \mathbb{N}$ . So 1 is indeed an upper bound for  $A$ .



Working towards the second, let  $\epsilon > 0$ . We need to show that there is some  $x \in A$  such that  $1 - \epsilon < x$ . But this is always accomplished by the number 1: Clearly  $1 \in A$  and  $1 - \epsilon < 1$ .  $\square$

**Theorem 1.30** ( $0.999\dots = 1$ )

$0.999\dots$  is a repeating decimal representation of the real number 1.

*Proof Idea.* If we place  $0.9, 0.99, 0.999, \dots$  on the real number line, we see immediately that all these points are to the left of 1, and that they get closer and closer to 1. For any number  $x$  that is less than 1, the sequence  $0.9, 0.99, 0.999, \dots$  will eventually reach a number larger than  $x$ . So it does not make sense to identify  $0.999\dots$  with any number less than 1. This means that  $0.999\dots$  must be equal to 1.  $\square$

*Proof.* Let  $0.(9)_n$  denote the number  $0.999\dots 9$  with  $n$  nines after the decimal point. For example,  $0.(9)_1 = 0.9$ ,  $0.(9)_2 = 0.99$ ,  $0.(9)_3 = 0.999$ , and so on. Observe that for each  $n \in \mathbb{N}$ , we have

$$1 - 0.(9)_n = \frac{1}{10^n}.$$

Now, let  $x = 0.999\dots$ . Clearly,  $0.(9)_n < x \leq 1$  for every  $n \in \mathbb{N}$ . Subtracting these inequalities from 1 gives us

$$0 \leq 1 - x < 1 - 0.(9)_n = \frac{1}{10^n}.$$

By the Archimedean principle (Lemma 1.27), there is no positive number smaller than

$$\frac{1}{10^n} \quad \text{for all } n \in \mathbb{N}.$$

Therefore,  $1 - x$  must be equal to 0. That is,  $x = 1$ .  $\square$

**Remark 1.31** — So despite the persistent rumors,  $0.999\dots$  is not “almost exactly 1” or “very, very nearly but not quite 1” or “1’s clingy little sibling.” Rather,  $0.999\dots$  and 1 are, in fact, the same number. The next time someone insists otherwise, feel free to hand them a copy of these notes.

**Remark 1.32** — The Archimedean principle used in Theorem 1.30 differs from the one stated in Lemma 1.27. One can show that these two formulations are equivalent.

**Remark 1.33** — We will revisit Theorem 1.30 in future chapters and discuss various different proofs.

**Definition 1.34** (Density)

Suppose  $A$  and  $B$  are ordered field. Then  $A$  is **dense** in  $B$  if, for any  $x, y \in B$ , there exists  $a \in A$  such that  $x < a < y$ .

**Example 1.35**

The propositions below are left without proof.

- $\mathbb{Q}$  is dense in  $\mathbb{Q}$ .
- $\mathbb{R} \setminus \mathbb{Q}$  is dense in  $\mathbb{Q}$ .
- $\mathbb{Z}$  is *not* dense in  $\mathbb{Q}$ .

**Lemma 1.36**

Let  $x, y \in \mathbb{R}$ . If  $y - x > 1$ , then there exists  $z \in \mathbb{Z}$  such that  $x < z < y$ .

*Proof.* First assume that  $x$  and  $y$  are at least 0, and consider the set

$$A = \{n \in \mathbb{N}_0 : n \leq x\}.$$

Since  $x \geq 0$ , this set is non-empty, and since it is a set of nonnegative integers which is bounded above by  $x$ , this set is finite. By induction on the size of  $A$ , we can show that  $\max(A)$  exists and is an element of  $A$ . Call this maximum  $M$ . We claim  $z := M + 1$  works.

Note that since  $M \in \mathbb{N}_0$ , also  $z \in \mathbb{N}_0$ . Furthermore, since  $z$  is larger than the largest element of  $A$ ,  $z$  is not in  $A$ , implying that  $x < z$ . Finally,

$$M \leq x \quad \text{implies that} \quad M + 1 \leq x + 1 \leq y.$$

So  $z < y$ . In summary, we have shown that  $x < z < y$ , as desired.

The cases where  $x$  and  $y$  are not at least 0 are similar. If both are negative, then by considering  $-x$  and  $-y$  the above argument gives an integer  $z$  where  $-y < z < -x$ , showing that  $-z$  works, since  $x < -z < y$ . If one is positive and one is negative, then 0 works.  $\square$

**Theorem 1.37** ( $\mathbb{Q}$  is dense in  $\mathbb{R}$ )

The rational numbers are dense in the real numbers.

*Proof.* Pick any  $x, y \in \mathbb{R}$  where  $x < y$ . We need to show that there exists some  $\frac{m}{n} \in \mathbb{Q}$  (with  $m, n \in \mathbb{Z}$ ) such that

$$x < \frac{m}{n} < y.$$

First note that if  $x < 0 < y$  then we are done, since  $0 \in \mathbb{Q}$ . Furthermore, if we can show that the theorem holds for the case that  $x$  and  $y$  are positive, then it holds when they are negative ( $0 < x < \frac{m}{n} < y$  implies  $-y < \frac{-m}{n} < -x < 0$ ), so we may assume  $x$  and  $y$  are positive.

Since  $y - x > 0$ , by the Archimedean principle (Lemma 1.27) there exists some  $n \in \mathbb{N}$  such that  $n(y - x) > 1$ ; i.e.  $ny - nx > 1$ . And so, by Lemma 1.36, there is some integer  $m$  with

$$nx < m < ny.$$

That is,

$$x < \frac{m}{n} < y,$$

which concludes the proof.  $\square$

**Corollary 1.38** ( $\mathbb{R} \setminus \mathbb{Q}$  is dense in  $\mathbb{R}$ )

The irrational numbers are dense in the real numbers.

*Proof.* Pick any  $x, y \in \mathbb{R}$  where  $x < y$ . By Theorem 1.37, there are rational numbers  $r_1, r_2 \in \mathbb{Q}$  such that  $x < r_1 < y$  and  $r_1 < r_2 < y$ . Setting

$$s = r_1 + (r_2 - r_1)/\sqrt{2}$$

we have  $r_1 < s < r_2$  and also  $x < s < y$ . Finally,  $s$  cannot be a rational number since otherwise

$$\sqrt{2} = (r_2 - r_1)/(s - r_1)$$

would also be rational.  $\square$

**Remark 1.39** — The proof of Lemma 1.36 also implies that, for any  $x \in \mathbb{R}$ , there exists an integer  $M$  such that  $M \leq x \leq M + 1$ . In particular, it implies that the floor and ceiling functions exist.

**Definition 1.40** (Floor and ceiling functions)

Let  $x \in \mathbb{R}$ .

- The *floor* of  $x$ , denoted  $\lfloor x \rfloor$ , is the integer  $n$  such that  $x - 1 < n \leq x$ .
- The *ceiling* of  $x$ , denoted  $\lceil x \rceil$ , is the integer  $n$  such that  $x \leq n < x + 1$ .

**Definition 1.41** (Closed and open intervals)

Define the *closed interval*  $[a, b]$  to be  $\{x \in \mathbb{R} : a \leq x \leq b\}$ . Likewise the *open interval*  $(a, b)$  is defined to be  $\{x \in \mathbb{R} : a < x < b\}$ , and half-open intervals and intervals to  $\pm\infty$  are again exactly as you would expect.

**Theorem 1.42** (Characterization of intervals)

Let  $S$  be a subset of  $\mathbb{R}$  that contains at least two points. If  $S$  has the property such that

$$\text{if } x, y \in S \text{ and } x < y, \text{ then } [x, y] \subseteq S, \quad (1)$$

then  $S$  is an interval.

*Proof.* There are four cases to consider: (1)  $S$  is bounded, (2)  $S$  is bounded above but not below, (3)  $S$  is bounded below but not above, and (4)  $S$  is neither bounded above nor below.

**Case (1).** Let  $a = \inf(S)$  and  $b = \sup(S)$ . Then  $S \subseteq [a, b]$  and we will show that  $(a, b) \subseteq S$ . If  $a < z < b$ , then  $z$  is not a lower bound of  $S$ , so there exists  $x \in S$  with  $x < z$ . Also,  $z$  is not an upper bound of  $S$ , so there exists  $y \in S$  with  $z < y$ . Therefore,  $z \in [x, y]$ , so property (1) implies that  $z \in S$ . Since  $z$  is an arbitrary element of  $(a, b)$ , we conclude that  $(a, b) \subseteq S$ . Now if  $a \in S$  and  $b \in S$ , then  $S = [a, b]$ . If  $a \notin S$  and  $b \notin S$ , then  $S = (a, b)$ . The other possibilities lead to either  $S = (a, b]$  or  $S = [a, b)$ .

**Case (2).** Let  $b = \sup(S)$ . Then  $S \subseteq (-\infty, b]$  and we will show that  $(-\infty, b) \subseteq S$ . If  $z < b$ , then  $z$  is not an upper bound of  $S$ , so there exists  $y \in S$  with  $z < y$ . Also, since  $S$  is not bounded below, there exists  $x \in S$  with  $x < z$ . By property 1,  $z \in [x, y] \subseteq S$ . Since  $z$  is an arbitrary element of  $(-\infty, b)$ , we conclude that  $(-\infty, b) \subseteq S$ . Now if  $b \in S$ , then  $S = (-\infty, b]$ , and if  $b \notin S$ , then  $S = (-\infty, b)$ .

**Case (3).** Let  $a = \inf(S)$ . Then  $S \subseteq [a, \infty)$  and we will show that  $(a, \infty) \subseteq S$ . If  $a < z$ , then  $z$  is not a lower bound of  $S$ , so there exists  $x \in S$  with  $x < z$ . Also, since  $S$  is not bounded above, there exists  $y \in S$  with  $z < y$ . By property 1,  $z \in [x, y] \subseteq S$ . Since  $z$  is an arbitrary element of  $(a, \infty)$ , we conclude that  $(a, \infty) \subseteq S$ . Now if  $a \in S$ , then  $S = [a, \infty)$ , and if  $a \notin S$ , then  $S = (a, \infty)$ .

**Case (4).** We will show that  $S = (-\infty, \infty)$ . Pick any  $x, y \in S$  with  $x < y$ , property 1 implies that  $[x, y] \subseteq S$ . Since  $S$  is neither bounded above nor below, and the choice of  $x, y$  is arbitrary, we conclude that  $(-\infty, \infty) \subseteq S$ . Also, every subset of  $\mathbb{R}$  is a subset of  $(-\infty, \infty)$ , thus  $S = (-\infty, \infty)$ .  $\square$

**Theorem 1.43 (The nested intervals property)**

For each  $n \in \mathbb{N}$ , assume we are given a closed interval  $I_n = [a_n, b_n]$ . Also, assume that each  $I_n$  contains  $I_{n+1}$ . Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection. That is,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

*Proof.* In order to show that  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$  is not empty, we are going to use the completeness axiom to produce a single real number  $x$  satisfying  $x \in I_n$  for every  $n \in \mathbb{N}$ . Now, the completeness axiom is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in \mathbb{N}\}$$

of left-hand endpoints of the interval. Because the intervals are nested, we see that every  $b_n$  serves as an upper bound for  $A$ . Thus, we are justified in setting

$$x = \sup(A).$$

Now, consider a particular  $I_n = [a_n, b_n]$ . Because  $x$  is an upper bound for  $A$ , we have  $a_n \leq x$ . The fact that each  $b_n$  is an upper bound for  $A$  and that  $x$  is the least upper bound implies  $x \leq b_n$ .

Altogether then, we have  $a_n \leq x \leq b_n$ , which means  $x \in I_n$  for every choice of  $n \in \mathbb{N}$ . Hence,  $x \in \bigcap_{n=1}^{\infty} I_n$ , and the intersection is not empty.  $\square$

**Remark 1.44** — Note that the conclusion of Theorem 1.43 need not hold if each  $I_n$  is allowed to be an open interval.

# 2 Cardinality

“No one shall expel us from the paradise that Cantor has created.”

*David Hilbert, Über das Unendliche*

## Definition 2.1 (Cardinality)

Let  $S$  and  $T$  be sets. Then,  $|S| = |T|$  if and only if there is a bijection from  $S$  to  $T$ .

## Definition 2.2 (Cardinality cont.)

$|S| \leq |T|$  if and only if there is an injection from  $S$  to  $T$ .

**Remark 2.3** — Our definitions above introduce two fundamental relations on cardinality, i.e.  $|S| = |T|$  and  $|S| \leq |T|$ . We need to make sure that these relations have the mathematical properties we expect them to have. That is, we want  $|S| = |T|$  to be an equivalence relation and  $|S| \leq |T|$  to be a partial order.

For the relation  $|S| = |T|$ , it's easy to show that it defines an equivalence relation:

- Reflexivity: Every set has a bijection with itself, i.e.  $|S| = |S|$ .
- Symmetry: If there is a bijection  $f$  from  $S$  to  $T$ , then  $f^{-1}$  is a bijection from  $T$  to  $S$ , and thus  $|T| = |S|$ .
- Transitivity: If there are bijections  $f$  from  $S$  to  $T$  and  $g$  from  $T$  to  $U$ , then their composition  $h = g \circ f$  is a bijection from  $S$  to  $U$ . Hence,  $|S| = |U|$ .

For the relation  $|S| \leq |T|$ , we must establish that it's a partial order:

- Reflexivity: Every set has an injection to itself, i.e.  $|S| \leq |S|$ .
- Transitivity: If there exist injections  $f$  from  $S$  to  $T$  and  $g$  from  $T$  to  $U$ , then their composition  $h = g \circ f$  is an injection from  $S$  to  $U$ . Hence,  $|S| \leq |U|$ .

The remaining property, antisymmetry, is where things get interesting, since it's not immediately obvious. Antisymmetry means that if  $|S| \leq |T|$  and  $|T| \leq |S|$ , then  $|S| = |T|$ . Using our definition, this translates to: If there is an injection from  $S$  to  $T$  and an injection from  $T$  to  $S$ , then there should be a bijection between  $S$  and  $T$ . This is exactly what the Schröder-Bernstein theorem (Theorem 2.4) guarantees.

## Theorem 2.4 (Schröder-Bernstein theorem)

If there exist injections  $f : A \rightarrow B$  and  $g : B \rightarrow A$  between the sets  $A$  and  $B$ , then there exists a bijection  $h : A \rightarrow B$ .

In terms of the cardinality of the two sets, this implies that if  $|A| \leq |B|$  and  $|B| \leq |A|$ , then  $|A| = |B|$ .

*Proof.* The strategy is to partition  $A$  and  $B$  into components

$$A = X \cup X' \quad \text{and} \quad B = Y \cup Y'$$

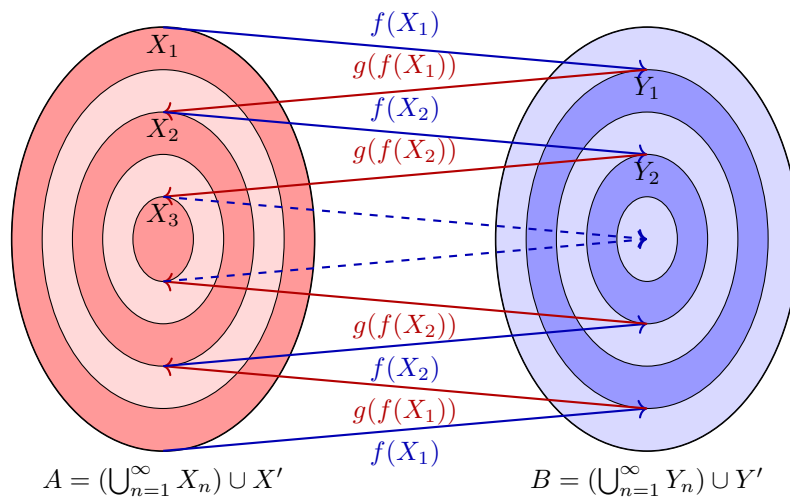
with  $X \cap X' = \emptyset$  and  $Y \cap Y' = \emptyset$ , in such a way that  $f$  is a surjection from  $X$  to  $Y$ , and  $g$  is a surjection from  $Y'$  to  $X'$ . Achieving this would lead to a proof that there is a bijection  $h$  from  $A$  to  $B$ . Why? For all  $x \in X'$ , there exists a unique  $y \in Y'$  satisfying  $g(y) = x$ . This means that there is a well-defined inverse function  $g^{-1}(x) = y$  that maps  $X'$  to  $Y'$ . Setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \\ g^{-1}(x) & \text{if } x \in X' \end{cases}$$

gives the desired bijection from  $A$  to  $B$ .

Now, let  $X_1 = A \setminus g(B)$  and inductively define a sequence of sets by letting  $X_{n+1} = g(f(X_n))$ . Let  $X = \bigcup_{n=1}^{\infty} X_n$  and  $Y = \bigcup_{n=1}^{\infty} f(X_n)$ .

To understand these definitions intuitively, observe that  $X_1$  consists of all elements in  $A$  that are not in the image of  $g$ . Then  $X_2$  consists of elements that are images under  $g \circ f$  of elements in  $X_1$ , and so on. Similarly, each  $Y_n = f(X_n)$  consists of the images of  $X_n$  under  $f$ . The following diagram illustrates this construction and the relationship between these sets.



We show that  $\{X_n : n \in \mathbb{N}\}$  is a pairwise disjoint collection of subsets of  $A$ , while  $\{f(X_n) : n \in \mathbb{N}\}$  is a similar collection in  $B$ . To see that the sets  $X_1, X_2, X_3, \dots$  are pairwise disjoint, note that  $X_1 \cap X_n = \emptyset$  for all  $n \geq 2$  because  $X_1 = A \setminus g(B)$  and  $X_n \subseteq g(B)$ . (Why?  $f$  is a mapping from  $A$  to  $B$ , so we have that  $f(X_n) \subseteq B$ , and thus  $X_{n+1} = g(f(X_n)) \subseteq g(B)$ .) In the general case of  $X_n \cap X_m$  where  $1 < n < m$ , note that if  $x \in X_n \cap X_m$  then  $f^{-1}(g^{-1}(x)) \in X_{n-1} \cap X_{m-1}$ . Continuing in this way, we can show  $X_1 \cap X_{m-n+1}$  is not empty, which is a contradiction. Thus  $X_n \cap X_m = \emptyset$ . Just to be clear, the disjointness of the  $X_n$  sets is not crucial to the overall proof, but it does help paint a clearer picture of how the sets  $X$  and  $X'$  are constructed.

We show that  $f$  is a surjection from  $X$  to  $Y$ . This is straightforward. Each  $x \in X$  comes from some  $X_n$  and so  $f(x) \in f(X_n) \subseteq Y$ . Likewise, each  $y \in Y$  is an element of some  $f(X_n)$  and thus  $y = f(x)$  for some  $x \in X_n \subseteq X$ . Thus  $f : A \rightarrow B$  is a surjection.

Let  $y \in Y'$ . Then  $y \notin f(X_n)$  for all  $n$  (by definition of  $Y'$ ). We also conclude that  $g(y) \notin X_{n+1}$  for all  $n$ . (Why? Suppose if  $g(y) \in X_{n+1}$  for some  $n$ , then  $g(y) \in g(f(X_n))$ .

That is,  $g(y) = g(z)$  for some  $z \in f(X_n)$ , and since  $g$  is injective, we have  $y = z$  and thus  $y \in f(X_n)$ , which is a contradiction.) Clearly,  $g(y) \notin X_1$  either (because  $g(y) \subseteq g(B)$  and  $X_1 = A \setminus g(B)$ ) and so  $g$  is a mapping from  $Y'$  to  $X'$ . To see that  $g$  is a surjection from  $Y'$  to  $X'$ , let  $x \in X'$  be arbitrary. Because  $X' \subseteq g(Y') \subseteq g(B)$ , there exists  $y \in B$  with  $g(y) = x$ . Could  $y$  be an element of  $Y$ ? No, because if  $y \in Y$ ,  $g(y)$  would be in  $g(Y)$ , and since  $g(Y) \subseteq X$  (by definition of  $Y$ ), this would mean  $g(y) \in X$ . But we're considering an  $x \in X'$  with  $g(y) = x$ , so this is a contradiction as  $X \cap X' = \emptyset$ . Hence  $y \in Y'$  and  $g : Y' \rightarrow X'$  is a surjection.  $\square$

**Definition 2.5 (Cardinality cont.)**

$|S| \geq |T|$  if and only if there is a surjection from  $S$  to  $T$ .

**Remark 2.6** — Again, we must establish that  $|S| \geq |T|$  defines a partial order. Reflexivity and transitivity are obvious. Now we will show antisymmetry. Suppose  $|S| \geq |T|$  and  $|T| \geq |S|$ . Then, by definition, there are surjections from  $S$  to  $T$  and from  $T$  to  $S$ . Using the axiom of choice, one can prove that there exists a surjection from  $X$  to  $Y$  if and only if there exists an injection from  $Y$  to  $X$ . We conclude that  $|T| \leq |S|$  and  $|S| \leq |T|$ , which implies  $|S| = |T|$  by Theorem 2.4.

**Theorem 2.7 ( $|\mathbb{Z}| = |\mathbb{N}|$ )**

There are as many integers as there are natural numbers.

*Proof.* Define the function  $f : \mathbb{N} \rightarrow \mathbb{Z}$  with

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even.} \end{cases}$$

Clearly,  $f$  is a bijection. (See the following diagram.)

$$\begin{array}{cccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \cdots \end{array}$$

Hence, by Definition 2.1,  $|\mathbb{N}| = |\mathbb{Z}|$ .  $\square$

**Remark 2.8** — Theorem 2.7 also shows that two sets can have the same cardinality even if one is a proper subset of the other and the “larger” one even has infinitely many more elements than the “smaller” one. Make sure you take a moment to appreciate how remarkably, wonderfully weird this is.

**Theorem 2.9 ( $|\mathbb{Q}| = |\mathbb{N}|$ )**

There are as many rational numbers as there are natural numbers.



*Proof.* Let  $A_1 = \{0\}$  and for each  $n \geq 2$ , let  $A_n$  be the set given by

$$A_n = \left\{ \pm \frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}.$$

The first few of these sets look like

$$\begin{aligned} A_1 &= \{0\}, & A_2 &= \left\{ \frac{1}{1}, \frac{-1}{1} \right\}, & A_3 &= \left\{ \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1} \right\}, \\ A_4 &= \left\{ \frac{1}{3}, \frac{-1}{3}, \frac{3}{1}, \frac{-3}{1} \right\}, & \text{and} & & A_5 &= \left\{ \frac{1}{4}, \frac{-1}{4}, \frac{2}{3}, \frac{-2}{3}, \frac{3}{2}, \frac{-3}{2}, \frac{4}{1}, \frac{-4}{1} \right\}. \end{aligned}$$

The crucial observation is that each  $A_n$  is finite and every rational number appears in exactly one of these sets. A bijection with  $\mathbb{N}$  is then achieved by consecutively listing the elements in each  $A_n$ .

$\mathbb{N} :$	1	2	3	4	5	6	7	8	9	10	11	12	$\dots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Q} :$	0	$\frac{1}{1}$	$-\frac{1}{1}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{2}{1}$	$-\frac{2}{1}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{3}{1}$	$-\frac{3}{1}$	$\frac{1}{4}$	$\dots$
	$\underbrace{\hspace{1.5em}}$	$\underbrace{\hspace{2.5em}}$		$\underbrace{\hspace{4.5em}}$				$\underbrace{\hspace{4.5em}}$					
	$A_1$	$A_2$		$A_3$				$A_4$					

Admittedly, writing an explicit formula for this correspondence would be an awkward task, and attempting to do so is not the best use of time. What matters is that we see why every rational number appears in the correspondence exactly once. Given, say,  $22/7$ , we have that  $22/7 \in A_{29}$ . Because the set of elements in  $A_1, \dots, A_{28}$  is finite, we can be confident that  $22/7$  eventually gets included in the sequence. The fact that this line of reasoning applies to any rational number  $p/q$  is our proof that the correspondence is surjective. To verify that it is injective, we observe that the sets  $A_n$  were constructed to be disjoint so that no rational number appears twice. This completes the proof.  $\square$

**Theorem 2.10** ( $|\mathbb{R}| > |\mathbb{N}|$ )

There are more real numbers than natural numbers.

*Proof.* Assume, for the sake of contradiction, that  $|\mathbb{R}| = |\mathbb{N}|$ , which implies that there is a bijection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . What this suggests is that it is possible to enumerate the elements of  $\mathbb{R}$ . If we let  $x_1 = f(1)$ ,  $x_2 = f(2)$ , and so on, then the fact that  $f$  is bijective (and thus surjective) means that we can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\} \tag{1}$$

and be confident that every real number appears somewhere on the list. We will now use the nested intervals property (Theorem 1.43) to produce a real number that is not there.

Let  $I_1$  be a closed interval that does not contain  $x_1$ . Next, let  $I_2$  be a closed interval, contained in  $I_1$ , which does not contain  $x_2$ . The existence of such  $I_2$  is easy to verify. Certainly  $I_1$  contains two smaller disjoint closed intervals, and  $x_2$  can only be in one of these. In general, given an interval  $I_n$ , construct  $I_{n+1}$  to satisfy

1.  $I_{n+1} \subseteq I_n$  and

2.  $x_{n+1} \notin I_{n+1}$ .

We now consider the intersection  $\bigcap_{n=1}^{\infty} I_n$ . If  $x_k$  is some real number from the list in (1), then we have  $x_k \notin I_k$ , and it follows that

$$x_k \notin \bigcap_{n=1}^{\infty} I_n.$$

Now, we are assuming that the list in (1) contains every real number, and this leads to the conclusion that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

However, the nested intervals property (Theorem 1.43) asserts that  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . So there is at least one  $x \in \bigcap_{n=1}^{\infty} I_n$  that, consequently, cannot be on the list in (1). This contradiction means that such an enumeration of  $\mathbb{R}$  is impossible, and we conclude that  $|\mathbb{R}| \neq |\mathbb{N}|$ .

We are not done yet. We still have to show that  $|\mathbb{R}| \geq |\mathbb{N}|$ . This step is straightforward. Define the function  $g : \mathbb{R} \rightarrow \mathbb{N}$  with

$$g(x) = \begin{cases} n & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

This function maps each real number that is a natural number to itself, and all other real numbers to 0. Since every natural number is the image of at least one real number under  $g$ , the function is surjective. Therefore, by Definition 2.5, we have  $|\mathbb{R}| \geq |\mathbb{N}|$ . Combining this with our previous result that  $|\mathbb{R}| \neq |\mathbb{N}|$ , we conclude that  $|\mathbb{R}| > |\mathbb{N}|$ .  $\square$

### Definition 2.11 (Countable and uncountable sets)

A set  $S$  is **countable** if

1. its cardinality  $|S|$  is less than or equal to  $|\mathbb{N}|$ .
2. there exists an injection from  $S$  to  $\mathbb{N}$ .
3.  $S$  is empty or there exists a surjection  $\mathbb{N}$  to  $S$ .
4. there exists a bijection from  $S$  to a subset of  $\mathbb{N}$ .
5.  $S$  is either finite or *countably infinite*.

All of the definitions above are equivalent.

A set  $S$  is **countably infinite** if its cardinality  $|S|$  is exactly  $\aleph_0$ .

A set  $S$  is **uncountable** if it is not countable. That is, its cardinality  $|S|$  is greater than  $|\mathbb{N}|$ .

### Corollary 2.12 ( $\mathbb{N}$ is countable)

The set of natural numbers is countable.

**Corollary 2.13** ( $\mathbb{Z}$  is countable)

The set of integers is countable.

**Corollary 2.14** ( $\mathbb{Q}$  is countable)

The set of rational numbers is countable.

**Corollary 2.15** ( $\mathbb{R}$  is uncountable)

The set of real numbers is uncountable.

**Theorem 2.16**

An uncountable collection of disjoint open intervals in  $\mathbb{R}$  cannot exist.

*Proof.* We use contradiction. Suppose there exists an uncountable collection of disjoint open intervals in  $\mathbb{R}$ . By the density of  $\mathbb{Q}$  in  $\mathbb{R}$  (Theorem 1.37), every open interval in  $\mathbb{R}$  contains at least one rational number. Therefore, there are uncountably many rational numbers. A contradiction of Corollary 2.14. Hence, such a collection cannot exist.  $\square$

**Theorem 2.17** (Countable infinity is the smallest infinity)

If  $A \subseteq B$  and  $B$  is countable, then  $A$  is either countable or finite.

*Proof.*  $B$  is a countable set. Thus, there exists a bijection  $f : \mathbb{N} \rightarrow B$ . Let  $A \subseteq B$  be an infinite subset of  $B$ . We show that  $A$  is countable. Let  $n_1 \in \min\{n \in \mathbb{N} : f(n) \in A\}$ . As a start to a definition of  $g : \mathbb{N} \rightarrow A$ , let  $g(1) = f(n_1)$ . Next let  $n_2 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1)\}\}$  and let  $g(2) = f(n_2)$ . We inductively continue this process to produce a bijection  $g$  from  $\mathbb{N}$  to  $A$ . In general, assume we have defined  $g(k)$  for  $k < m$ , and let  $g(m) = f(n_m)$  where  $n_m = \min\{n \in \mathbb{N} : A \setminus \{f(n_1), \dots, f(n_{k-1})\}\}$ .

To show that  $g$  is injective, observe that  $m \neq m'$  implies  $n_m \neq n_{m'}$  and it follows that  $g(m) = f(n_m) \neq f(n_{m'}) = g(m')$  because  $f$  is injective. To show that  $g$  is surjective, let  $a \in A$  be arbitrary. Because  $f$  is surjective,  $a = f(n')$  for some  $n' \in \mathbb{N}$ . This means  $n' \in \{n : f(n) \in A\}$  and as we inductively remove the minimal element,  $n'$  must eventually be the minimum by at least the  $(n' - 1)$ -th step.  $\square$

**Corollary 2.18** ( $|\mathbb{N}|$  is the smallest infinity)

If  $A \subseteq \mathbb{N}$ , then either  $A$  is finite or  $|A| = |\mathbb{N}|$ .

**Corollary 2.19** (Sizes of infinity)

There are different sizes of infinity, with countable infinity being the smallest. Moreover,  $\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{Q}$  are countable while  $\mathbb{R}$  is uncountable.

**Theorem 2.20** (Countable union of countable sets is countable)

A countable union of countable sets is countable. More precisely:

1. If  $A_1, A_2, \dots, A_m$  are each countable sets, then  $A_1 \cup A_2 \cup \dots \cup A_m$  is countable.
2. If  $A_n$  is a countable set for each  $n \in \mathbb{N}$ , then the set  $\bigcup_{n=1}^{\infty} A_n$  is also countable.

*Proof.* First, we prove part 1 for two countable sets,  $A_1$  and  $A_2$ . Some technicalities can be avoided by first replacing  $A_2$  with the set  $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$ . The point of this is that the union  $A_1 \cup B_2$  is equal to  $A_1 \cup A_2$  and the sets  $A_1$  and  $B_2$  are disjoint.

Now, because  $A_1$  is countable, there exists a bijection  $f : \mathbb{N} \rightarrow A_1$ . If  $B_2 = \emptyset$ , then  $A_1 \cup A_2 = A_1$  which we already know to be countable. If  $B_2 = \{b_1, b_2, \dots, b_m\}$  has  $m$  elements then define  $h : \mathbb{N} \rightarrow A_1 \cup B_2$  via

$$h(n) = \begin{cases} b_n & \text{if } n \leq m \\ f(n - m) & \text{if } n > m. \end{cases}$$

The fact that  $h$  is bijective follows immediately from the same property of  $f$ . If  $B_2$  is infinite, then by Theorem 2.17 it is countable, and so there exists a bijection  $g : \mathbb{N} \rightarrow B_2$ . In this case we define  $h : \mathbb{N} \rightarrow A_1 \cup B_2$  by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that  $h$  is bijective is derived directly from the fact that  $f$  and  $g$  are both bijections. Graphically, the correspondence takes the form

$$\begin{array}{ccccccc} \mathbb{N} : & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ A_1 \cup B_2 : & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \dots \end{array}$$

To prove the more general statement in part 1, we may use induction. We have just seen that the result holds for two countable sets. Now let's assume that the union of  $m$  countable sets is countable, and show that the union of  $m+1$  countable sets is countable.

Given  $m+1$  countable sets  $A_1, A_2, \dots, A_{m+1}$ , we can write

$$A_1 \cup A_2 \cup \dots \cup A_{m+1} = (A_1 \cup A_2 \cup \dots \cup A_m) \cup A_{m+1}.$$

Then  $C_m = A_1 \cup \dots \cup A_m$  is countable by the induction hypothesis, and  $C_m \cup A_{m+1}$  is just the union of two countable sets which we know to be countable. This completes the proof for part 1.

For part 2, induction cannot be used because we have an infinite number of sets. Instead, we show how arranging  $\mathbb{N}$  into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 5 & 9 & 14 & \dots & \\ 4 & 8 & 13 & \dots & & \\ 7 & 12 & \dots & & & \\ 11 & \dots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof.

Let's first consider the case where the sets  $\{A_n\}$  are disjoint. In order to achieve bijection between  $\mathbb{N}$  and  $\bigcup_{n=1}^{\infty} A_n$ , we first label the elements in each countable set  $A_n$  as

$$A_n = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}.$$

Now arrange the elements of  $\bigcup_{n=1}^{\infty} A_n$  in an array similar to the one for  $\mathbb{N}$ :

$$\begin{array}{rcl} A_1 & = & a_{11} \ a_{12} \ a_{13} \ a_{14} \ a_{15} \ \cdots \\ A_2 & = & a_{21} \ a_{22} \ a_{23} \ a_{24} \ \cdots \\ A_3 & = & a_{31} \ a_{32} \ a_{33} \ \cdots \\ A_4 & = & a_{41} \ a_{42} \ \cdots \\ A_5 & = & a_{51} \ \cdots \\ & \vdots & \end{array}$$

This establishes a bijection  $g : \mathbb{N} \rightarrow \bigcup_{n=1}^{\infty} A_n$  where  $g(n)$  corresponds to the element  $a_{jk}$  where  $(j, k)$  is the row and column location of  $n$  in the array for  $\mathbb{N}$ .

If the sets  $\{A_n\}$  are not disjoint then our mapping may not be injective. In this case we could again replace  $A_n$  with  $B_n = A_n \setminus \{A_1 \cup \dots \cup A_{n-1}\}$ . Another approach is to use the previous argument to establish a bijection between  $\bigcup_{n=1}^{\infty} A_n$  and an infinite subset of  $\mathbb{N}$ , and then appeal to Theorem 2.17. This completes the proof for part 2.  $\square$

**Theorem 2.21** ( $\mathbb{R} \setminus \mathbb{Q}$  is uncountable)

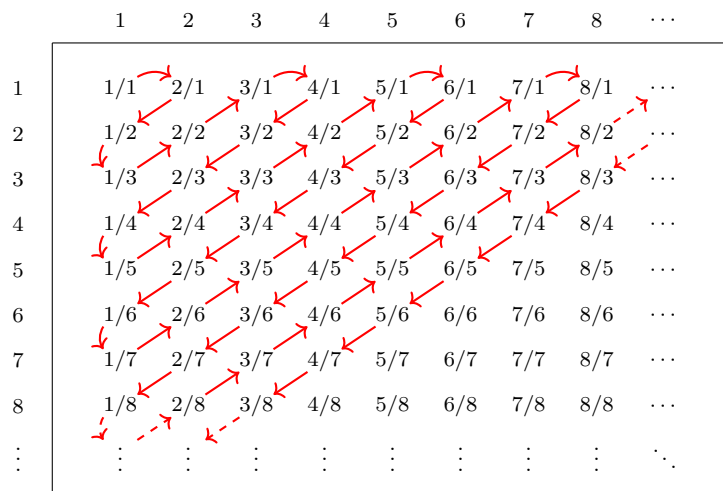
There are uncountably many irrational numbers.

*Proof.* We use contradiction. Suppose  $\mathbb{R} \setminus \mathbb{Q}$  is countable. By Theorem 2.20, we know that a countable union of countable sets is countable, thus  $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$  is countable. A contradiction of Corollary 2.15. Hence,  $\mathbb{R} \setminus \mathbb{Q}$  is uncountable.  $\square$

**Theorem 2.22** ( $|\mathbb{Q}_+| = |\mathbb{N}|$ )

There are as many positive rational numbers as there are natural numbers.

*Proof.* The following diagram arranges the rational numbers (with some repetition) and illustrates our bijection, which we call the *winding bijection*.



Weaving through this chart, you are guaranteed to hit every positive rational number. So if you pair up 1 with the first number you hit, 2 with the second number you hit, 3 with the third, and so on, then every positive rational number is in a pair. Now, there's just one small problem: each rational number is actually hit more than once. The number  $p/q$  will be written in positions  $(p, q), (2p, 2q), (3p, 3q), \dots$ . But the fix is easy: When you come across a number that has already been hit, just skip it. Clearly you won't run out of rational numbers, so this does indeed pair up everything. So

$$f(n) = \text{the } n\text{th new rational number you reach.}$$

And that's it. □

**Theorem 2.23** ( $|(0, 1)| > |\mathbb{N}|$ )

There are more numbers in the open interval  $(0, 1)$  than there are natural numbers.

We will demonstrate a clever argument known as *Cantor's diagonal argument*.

*Proof.* We proceed by contradiction and assume that there does exist a function  $f : \mathbb{N} \rightarrow (0, 1)$  that is bijective. For each  $m \in \mathbb{N}$ ,  $f(m)$  is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\dots$$

What is meant here is that for each  $m, n \in \mathbb{N}$ ,  $a_{mn}$  is the digit from the set  $\{0, 1, 2, \dots, 9\}$  that represents the  $n$ th digit in the decimal expansion of  $f(m)$ . The bijection between  $\mathbb{N}$  and  $(0, 1)$  can be summarized in the doubly indexed array

$\mathbb{N}$	$(0, 1)$									
1	$\longleftrightarrow$	$f(1)$	=	<b>.a<sub>11</sub></b>	$a_{12}$	$a_{13}$	$a_{14}$	$a_{15}$	$a_{16}$	$\dots$
2	$\longleftrightarrow$	$f(2)$	=	$.a_{21}$	<b>a<sub>22</sub></b>	$a_{23}$	$a_{24}$	$a_{25}$	$a_{26}$	$\dots$
3	$\longleftrightarrow$	$f(3)$	=	$.a_{31}$	$a_{32}$	<b>a<sub>33</sub></b>	$a_{34}$	$a_{35}$	$a_{36}$	$\dots$
4	$\longleftrightarrow$	$f(4)$	=	$.a_{41}$	$a_{42}$	$a_{43}$	<b>a<sub>44</sub></b>	$a_{45}$	$a_{46}$	$\dots$
5	$\longleftrightarrow$	$f(5)$	=	$.a_{51}$	$a_{52}$	$a_{53}$	$a_{54}$	<b>a<sub>55</sub></b>	$a_{56}$	$\dots$
6	$\longleftrightarrow$	$f(6)$	=	$.a_{61}$	$a_{62}$	$a_{63}$	$a_{64}$	$a_{65}$	<b>a<sub>66</sub></b>	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The key assumption about this correspondence is that *every* real number in  $(0, 1)$  is assumed to appear somewhere on the list. Now for the pearl of the argument. Define a real number  $x \in (0, 1)$  with the decimal expansion  $x = .b_1b_2b_3b_4\dots$  using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Observe that  $x = .b_1b_2b_3b_4\dots$  cannot be  $f(1)$ .  $x$  also cannot be  $f(2)$ , and in general  $x \neq f(n)$  for any  $n \in \mathbb{N}$ . Therefore, the real number  $x$  is nowhere on the list! This is a contradiction; clearly we were unable to pair up all the reals, if  $x$  got left out. □

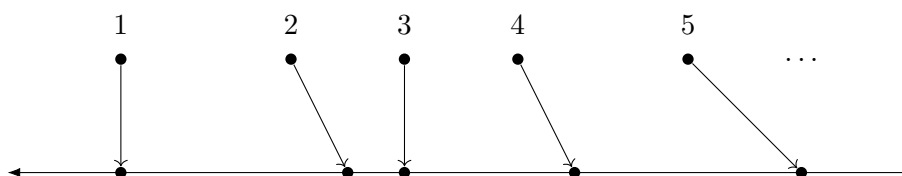
**Remark 2.24** — Consider the following complaints about the proof of Theorem 2.23.

1. Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of  $\mathbb{Q}$  must be countable, the proof of Theorem 2.23 must be flawed.
2. Some numbers have two different decimal representations (see Theorem 1.30). Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance,  $1/2$  can be written as  $.5$  or as  $.4999\dots$ . Doesn't this cause some problems?

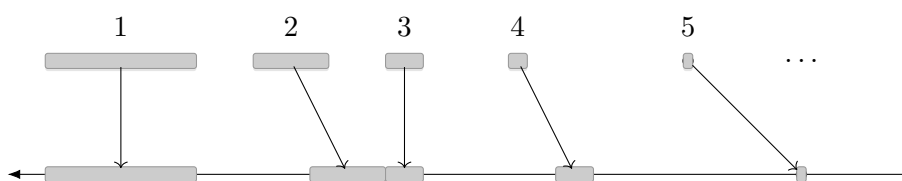
Are these complaints valid?

Here's another proof of Theorem 2.23 that I particular enjoy.

*Proof Sketch.* Imagine you were trying to map  $\mathbb{N}$  on to the real number line (which clearly has  $|\mathbb{R}|$  points).



We aim to show that it is impossible for this map to be a bijection—there must be points on the real line that were missed. But instead of mapping just these points, let's make our job slightly harder. Around 1, let's put a little interval of length  $\frac{1}{2}$ . Around 2, let's put a little interval of length  $\frac{1}{4}$ . Around 3, let's place a little interval of length  $\frac{1}{8}$ . And so on. Now, when you map the points in  $\mathbb{N}$  to the real line, send over the intervals too (possibly some intervals will overlap; this is ok). We'll now prove that not only are there points on the real line that weren't mapped to, but there are even points that these intervals don't cover!



See what happened? Our intervals' lengths add up to  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$  (and with overlaps, their collective length when mapped to the real line may even be smaller than 1). But the whole real line has length  $\infty$ ! So certainly there is no chance that all the points are covered; not only do the points of  $\mathbb{N}$  not cover the line, but even if we fatten them up with these intervals, those intervals don't even cover the real line! And any point that's in the " $\infty - 1$ " portion of the real line that is not covered by an interval was certainly not mapped to. So we have all sorts of points that were missed, and so the mapping is far from being a bijection.

And we could instead pick intervals that add up to 0.0001, or any other tiny number. This proof therefore provides a visualization of how we aren't just missing the one point that Cantor's diagonalization argument finds, but "most" points are missed.  $\square$

**Theorem 2.25** ( $|(0, 1)| = |\mathbb{R}|$ )

There are as many numbers in the open interval  $(0, 1)$  as there are real numbers.

*Proof Idea.* The function  $f(x) = (x - 1/2)/(x - x^2)$  is a bijection from  $(0, 1)$  to  $\mathbb{R}$ . This shows, by Definition 2.1, that  $|(0, 1)| = |\mathbb{R}|$ .  $\square$

**Remark 2.26** — Theorem 2.25 effectively establishes that Theorem 2.10 and Theorem 2.23 are equivalent.

**Remark 2.27** — We now know that  $|\mathbb{N}| < |\mathbb{R}|$ . Here's one natural question: Is there any infinity between these two? An astounding fact is that, based on the axioms of set theory (called ZFC), whether or not there exists such an infinity is *unprovable*. And what I don't mean is that mathematicians are not smart enough to find the answer; no, I mean that they *are* smart enough to have shown that *no proof can possibly exist*. That's right, there are statements in math which are impossible to prove and also impossible to disprove (but we *are* able to prove that they are unprovable, amazingly).

This particular question is among the most famous in mathematical history. It was posed by Georg Cantor and is known as *the continuum hypothesis*. It is the first of Hilbert's 23 problems—an influential list of unsolved problems that David Hilbert presented in 1900 at the International Congress of Mathematicians, setting the mathematical agenda for the 20th century. Decades later, Kurt Gödel's groundbreaking incompleteness theorems revealed that virtually every mathematical theory contains unprovable statements.

ZFC set theory is the foundational framework upon which nearly all of modern mathematics is built. Gödel constructed a model of ZFC where the continuum hypothesis holds, while Paul Cohen later constructed a model where it fails. Together, their results established that the continuum hypothesis is independent of ZFC—it can be neither proved nor disproved within the system. Thus, the continuum hypothesis—which asks what is presumably a basic question about the infinite—is unprovable.

**Hypothesis 2.28** (The continuum hypothesis)

There is no set whose cardinality is strictly between that of the naturals and the reals.

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

**Theorem 2.29** ( $|A| < |\mathcal{P}(A)|$ )

If  $A$  is a set and  $\mathcal{P}(A)$  is the power set of  $A$ , then

$$|A| < |\mathcal{P}(A)|.$$



*Proof.* Assume for a contradiction that  $|A| \geq |\mathcal{P}(A)|$ . That is, assume that there is a surjection  $f$  from  $A$  to  $\mathcal{P}(A)$ . Since  $f$  is a surjection, for every  $T \subseteq A$ , there is some element  $t \in A$  where  $f(t) = T$ . To reach our contradiction, we will construct a set  $B \subsetneq A$  which is not hit.

For each  $a$  there is one special property about the set  $f(a)$  that we are going to care about: Is  $a \in f(a)$  or is  $a \notin f(a)$ ? In general, consider the set of all elements  $a$  such that  $a \notin f(a)$ , and call this set  $B$ :

$$B = \{a \in A : a \notin f(a)\}.$$

By the above, if we can show that there is no  $b$  where  $f(b) = B$ , then we are done; we will have discovered an element of  $\mathcal{P}(A)$  that was not hit by  $f$ , a contradiction.

Claim. There is no  $b \in A$  such that  $f(b) = B$ .

Proof of claim. Assume for a contradiction that there does exist some  $b \in A$  such that  $f(b) = B$ . Note by the definition of  $B$  that

$$b \in B \text{ if and only if } b \notin f(b).$$

But since we assumed that  $f(b) = B$ , this is equivalent to

$$b \in f(b) \text{ if and only if } b \notin f(b),$$

which is clearly a contradiction. □

**Corollary 2.30** (There exist infinitely many infinities)

There exist infinitely many distinct infinite cardinalities.

*Proof.* By Theorem 2.29, the following is a chain of distinct infinite cardinalities

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))))| < \dots$$

□

# 3 Sequences

“Erdős loved epsilons—his word for small children (in mathematics the Greek letter epsilon is used to represent small quantities).”

*Paul Hoffman, The Man Who Loved Only Numbers*

## Definition 3.1 (Sequences)

A **sequence** of real numbers is a function  $a : \mathbb{N} \rightarrow \mathbb{R}$ .

## Definition 3.2 (Bounded sequences)

A sequence  $(a_n)$  is **bounded** if the range  $\{a_n : n \in \mathbb{N}\}$  is bounded. That is, if there exists a lower bound  $L \in \mathbb{R}$  and an upper bound  $U \in \mathbb{R}$  where

$$L \leq a_n \leq U$$

for all  $n$ .

## Proposition 3.3

A sequence  $(a_n)$  is bounded if and only if there exists some  $C \in \mathbb{R}$  for which  $|a_n| \leq C$  for all  $n$ .

*Proof.* Recall that boundedness means that there exists a lower bound  $L \in \mathbb{R}$  and an upper bound  $U \in \mathbb{R}$  where

$$L \leq a_n \leq U$$

for all  $n$ . Now let's prove each direction.

( $\Leftarrow$ ) Assume that there exists a  $C \in \mathbb{R}$  where

$$|a_n| \leq C.$$

Then

$$-C \leq a_n \leq C.$$

And so, by setting  $L = -C$  and  $U = C$ , we have shown that

$$L \leq a_n \leq U,$$

which means that  $(a_n)$  is bounded.

( $\Rightarrow$ ) If  $(a_n)$  is bounded, then there exists such an  $L$  and  $U$ . Let  $C = \max(|L|, U)$ . Note that this implies that  $C \geq U$  and (since  $C \geq |L|$ , that)  $-C \leq -|L|$ . Thus, for all  $n$  we have

$$-C \leq -|L| \leq L \leq a_n \leq U \leq C.$$

So we see that

$$-C \leq a_n \leq C$$

which by Fact 1.10 is the same as  $|a_n| \leq C$ . □

**Definition 3.4** (Convergent sequences)

A sequence  $(a_n)$  **converges** to  $a \in \mathbb{R}$  if for all  $\epsilon > 0$  there exists some  $N \in \mathbb{N}$  such that  $|a_n - a| < \epsilon$  for all  $n > N$ .

When this happens,  $a$  is called the **limit** of  $a_n$ .

**Definition 3.5** ( $\epsilon$ -neighborhood)

Given a real number  $a \in \mathbb{R}$  and a positive number  $\epsilon > 0$ , the set

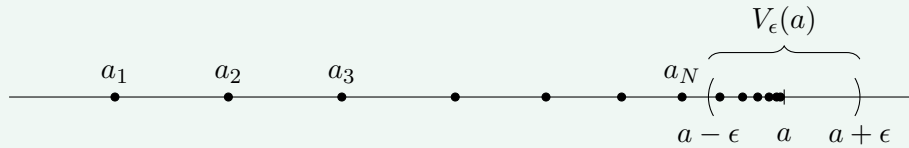
$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the  **$\epsilon$ -neighborhood** of  $a$ .

**Definition 3.6** (Convergent sequences (topological version))

A sequence  $(a_n)$  *converges* to  $a \in \mathbb{R}$  if, given any  $\epsilon$ -neighborhood  $V_\epsilon(a)$  of  $a$ , there exists a point in the sequence after which all of the terms are in  $V_\epsilon(a)$ . In other words, every  $\epsilon$ -neighborhood contains all but finite number of the terms of  $(a_n)$ .

**Remark 3.7** — Definition 3.4 and Definition 3.6 say precisely the same thing: the natural number  $N$  in Definition 3.4 is the point after which the sequence  $(a_n)$  enters  $V_\epsilon(a)$ , never to leave. It should be apparent that the value of  $N$  depends on the choice of  $\epsilon$ . The smaller the  $\epsilon$ -neighborhood, the larger the  $N$  may have to be.



**Remark 3.8** — The following are notationally equivalent:

- The sequence  $(a_n)$  *converges* to  $a$
- $a_n \rightarrow a$  as  $n \rightarrow \infty$
- $a_n \rightarrow a$
- $\lim_{n \rightarrow \infty} a_n = a$

**Example 3.9**

Show that the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$$

converges to 0.

*Proof.* Fix any  $\epsilon > 0$ . By the Archimedean principle (Lemma 1.27), there is an  $N$

for which  $\frac{1}{N} < \epsilon$ . This  $N$  has the property that, for all  $n > N$ , we have  $\frac{1}{n} < \frac{1}{N} < \epsilon$ . That is, for all  $n > N$ ,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon.$$

And so, by Definition 3.4, we may conclude that  $a_n \rightarrow 0$ .  $\square$

**Remark 3.10** — Using the Archimedean principle will not usually work, though. We need a more general approach, which is described in Outline 3.11.

**Outline 3.11.** To show that  $a_n \rightarrow a$ , begin with preliminary work:

0. Scratch work: Start with  $|a_n - a| < \epsilon$  and unravel to solve for  $n$ . This tells you which  $N$  to pick for step 2 below.

Now for your actual proof:

1. Let  $\epsilon > 0$ .
2. Let  $N$  be the final value of  $n$  you got in your scratch work, and let  $n > N$ .
3. Redo scratch work (without  $\epsilon$ 's), but at the end use  $N$  to show that  $|a_n - a| < \epsilon$ .

### Example 3.12

Again, show that the sequence

$$(a_n) = \left( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right)$$

converges to 0. This time, use the method described in Outline 3.11.

*Scratch work.* Given an arbitrary  $\epsilon > 0$ , we will find what specific  $N$  guarantees that, for every  $n > N$ , we have  $|a_n - 0| < \epsilon$ . For example, if  $\epsilon = \frac{1}{2}$ , then  $N = 2$  works. If  $\epsilon = \frac{1}{3}$ , then  $N = 3$  works. You see the pattern, but here is how we might come about it in general. We want the following:

$$\begin{aligned} |a_n - a| &< \epsilon \\ \left| \frac{1}{n} - 0 \right| &< \epsilon \\ \frac{1}{n} &< \epsilon \\ \frac{1}{\epsilon} &< n. \end{aligned}$$

So as long as we choose  $N = \frac{1}{\epsilon}$ , then for any  $n > N$ , we will have  $n > \frac{1}{\epsilon}$ , which by the above will imply that  $\frac{1}{n} \rightarrow 0$ , as desired. The solution below is how we formally solve it.

*Proof.* Fix any  $\epsilon > 0$ . Then for any  $n > N$  (implying  $\frac{1}{n} < \frac{1}{N}$ ),

$$|a_n - a| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\epsilon} = \epsilon.$$

That is,  $|a_n - 0| < \epsilon$ . So by Definition 3.4, we have shown that  $\frac{1}{n} \rightarrow 0$ .  $\square$

### Example 3.13

Let  $a_n = \frac{3n+1}{n+2}$ . Prove that  $\lim_{n \rightarrow \infty} a_n = 3$ .

*Scratch Work.* Again, we first play around. We start with where we want to get to (that  $|a_n - a| < \epsilon$ ), and then do some algebra to figure out which values of  $n$  would give this.

We want the following:

$$\begin{aligned} |a_n - a| &< \epsilon \\ \left| \frac{3n+1}{n+2} - 3 \right| &< \epsilon \\ \left| \frac{-5}{n+2} \right| &< \epsilon \\ \frac{5}{n+2} &< \epsilon \\ \frac{5}{\epsilon} &< n+2 \\ \frac{5}{\epsilon} - 2 &< n \end{aligned}$$

So as long as we choose  $N = \frac{5}{\epsilon} - 2$ , then for any  $n > N$  we will have  $n > \frac{5}{\epsilon} - 2$ , which by the above will imply that  $\frac{3n+1}{n+2} \rightarrow 3$ , as desired.

*Proof.* Fix any  $\epsilon > 0$ . Set  $N = \frac{5}{\epsilon} - 2$ . Then for any  $n > N$ ,

$$\begin{aligned} |a_n - a| &= \left| \frac{3n+1}{n+2} - 3 \right| = \left| \frac{3n+1}{n+2} - \frac{3n+6}{n+2} \right| \\ &= \frac{5}{n+2} < \frac{5}{N+2} = \frac{5}{(\frac{5}{\epsilon} - 2) + 2} \\ &= \frac{5}{5/\epsilon} = \epsilon. \end{aligned}$$

That is,  $|a_n - a| < \epsilon$ . So by Definition 3.4 we have shown that  $\frac{3n+1}{n+2} \rightarrow 3$ .  $\square$

### Definition 3.14 (Divergent Sequences)

If a sequence  $(a_n)$  does not converge, then it **diverges**.

Divergence can come in three forms.

1.  $(a_n)$  *diverges to  $\infty$*  (notation:  $\lim_{n \rightarrow \infty} a_n = \infty$ ) if, for all  $M > 0$ , there exists some  $N \in \mathbb{N}$  such that  $a_n > M$  for all  $n > N$ .
2.  $(a_n)$  *diverges to  $-\infty$*  (notation:  $\lim_{n \rightarrow \infty} a_n = -\infty$ ) if, for all  $M < 0$ , there exists some  $N \in \mathbb{N}$  such that  $a_n < M$  for all  $n > N$ .
3. Otherwise,  $(a_n)$ 's limit *does not exist*.

**Example 3.15**

Let  $a_n = n^2$ . Show that  $\lim_{n \rightarrow \infty} a_n = \infty$ .

*Scratch work.* We want

$$\begin{aligned} a_n &> M \\ n^2 &> M \\ n &> \sqrt{M}. \end{aligned}$$

So setting  $N = \sqrt{M}$  should work.

*Proof.* Fix any  $M > 0$ . Set  $N = \sqrt{M}$ . Then for any  $n > N$ ,

$$a_n = n^2 > N^2 = (\sqrt{M})^2 = M.$$

So we have shown that if  $n > N$ , then  $a_n > M$ . Therefore  $\lim_{n \rightarrow \infty} a_n = \infty$ .  $\square$

**Outline 3.16.** What if the sequence's limit does not exist? Then how do we show the sequence diverges? One way to show that  $a_n$  diverges is to show that  $a_n \not\rightarrow a$  for any  $a$ . Note first, by Definition 3.4, that  $a_n \rightarrow a$  means that

For every  $\epsilon > 0$  there exists some  $N$   
such that for all  $n > N$  we have  $|a_n - a| < \epsilon$ .

So to show that  $a_n \not\rightarrow a$ , we need to show the *negation* of that statement. That is, we must show that

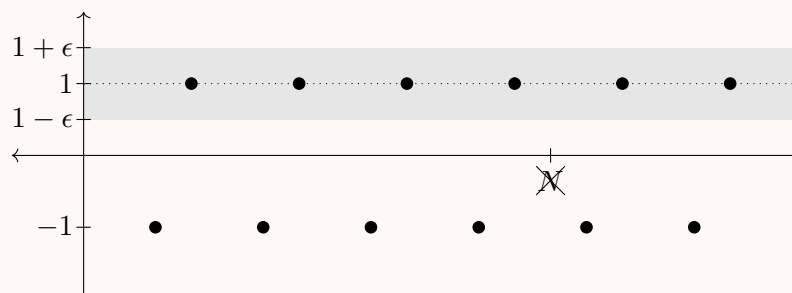
There exists some  $\epsilon > 0$  where for all  $N$   
there exists some  $n > N$  such that  $|a_n - a| \geq \epsilon$ .

In practice, this is usually done with a proof by contradiction. You assume that  $a_n \rightarrow a$  and then you demonstrate a specific  $\epsilon$  where it fails, giving the contradiction.

**Example 3.17**

Let  $a_n = (-1)^n$ . Prove that  $(a_n)$  diverges.

*Scratch work.* This is the sequence  $-1, 1, -1, 1, -1, 1, \dots$ . It makes sense that there is no  $a$  for which  $a_n \rightarrow a$ . It certainly doesn't converge to 1, since half the time it is at  $-1$  which is far away from 1. (If we let  $\epsilon = 1/2$ , then there is no  $N$  for which, for every  $n > N$ ,  $a_n$  is inside of the shaded band; see below.)



It likewise can't converge to  $-1$ . One might guess  $0$ , since that is halfway between  $-1$  and  $1$ , but that also doesn't make sense since  $a_n$  is always a distance of  $1$  away from  $0$ , so it's certainly not getting "closer and closer" to  $0$ . (Or,  $\epsilon = 1/2$  works again.) Ok, so we believe that it doesn't converge to anything, and we will use Outline 3.16 to show it.

*Proof.* Assume for a contradiction that there is some  $a$  for which  $a_n \rightarrow a$ . Let  $\epsilon = \frac{1}{2}$ . Since we assumed that  $a_n \rightarrow a$ , there must be some  $N$  for which, for all  $n > N$ , we have  $|a_n - a| < \frac{1}{2}$ . That is,  $|(-1)^n - a| < \frac{1}{2}$  for all  $n > N$ . We proceed by cases.

Even  $n$ . If  $n$  is even and  $n > N$ , then we have that

$$|1 - a| < \frac{1}{2}.$$

Unwinding this:

$$\begin{aligned} -\frac{1}{2} &< 1 - a < \frac{1}{2} \\ -\frac{3}{2} &< -a < -\frac{1}{2} \\ \frac{1}{2} &< a < \frac{3}{2}. \end{aligned}$$

Odd  $n$ . If  $n$  is odd and  $n > N$ , then we have that

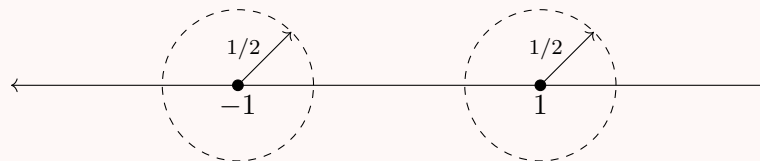
$$|-1 - a| < \frac{1}{2}.$$

Unwinding this:

$$\begin{aligned} -\frac{1}{2} &< -1 - a < \frac{1}{2} \\ \frac{1}{2} &< -a < \frac{3}{2} \\ -\frac{3}{2} &< a < -\frac{1}{2}. \end{aligned}$$

But this is a contradiction; clearly no  $a$  can be inside of both  $(\frac{1}{2}, \frac{3}{2})$  and  $(-\frac{3}{2}, -\frac{1}{2})$ . And so we must have  $a_n \not\rightarrow a$ .  $\square$

Essentially, focusing on the even case created a ball around  $1$ , of radius  $\frac{1}{2}$ , which  $a_n$  would have to live within for all  $n > N$ . The odd case created a ball around  $-1$ , of radius  $\frac{1}{2}$ , which  $a_n$  would have to live within for all  $n > N$ . But these two balls are disjoint, so  $a_n$  can't live in both, creating a contradiction.



Now here's another proof using the triangle inequality (Theorem 1.11).

*Proof.* Again, let  $\epsilon = \frac{1}{2}$ . The even case still gives  $|1 - a| < \frac{1}{2}$ , and the odd case still gives  $|-1 - a| < \frac{1}{2}$ ; although in the odd case we will rewrite  $|-1 - a|$  as  $|1 + a|$ . Then, by the triangle inequality:

$$2 = |(1 - a) + (1 + a)| \leq |1 - a| + |1 + a|.$$

But each of the absolute values on the right are assumed to be less than  $\frac{1}{2}$ . So,

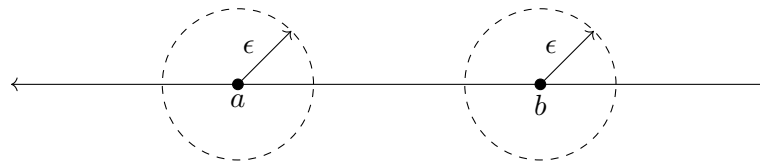
$$2 = |(1 - a) + (1 + a)| \leq |1 - a| + |1 + a| < \frac{1}{2} + \frac{1}{2} = 1.$$

So  $2 < 1$ ? Preposterous! We have our contradiction.  $\square$

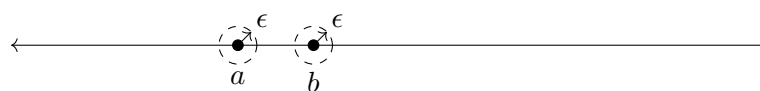
### Proposition 3.18 (Limits are unique)

A sequence cannot have more than one limit.

*Proof Idea.* The idea is to assume that you have a sequence  $a_n$  that converges to some  $a$  and also converges to some other number  $b$ . Since it converges to  $a$ , after some point all the sequence points are within  $\epsilon$  of  $a$ , for some really tiny  $\epsilon$ . But since it also converges to  $b$ , the same thing must happen there too. So if  $\epsilon$  is small enough that those two regions are mutually exclusive, we will have a contradiction.



And this is how we will reach a contradiction: there is no way for  $a_n (n > N)$  to be in both circles at the same time. Of course, if  $a$  and  $b$  are really close together we will need to choose a smaller  $\epsilon$  to make sure the intervals remain disjoint, but that's the only difference.



For instance, if the radius of those balls is a third of the distance between  $a$  and  $b$ , it should work out. With that intuition in mind, here's the proof.  $\square$

*Proof.* Suppose for a contradiction that  $a_n \rightarrow a$  and  $a_n \rightarrow b$ , where  $a \neq b$ ; moreover, without loss of generality let's assume  $a < b$ . Let  $\epsilon = (b - a)/3 > 0$ . Since  $a_n \rightarrow a$  there exists some  $N_1$  such that for  $n > N_1$  we have  $|a_n - a| < (b - a)/3$ . Likewise, since  $a_n \rightarrow b$  there exists some  $N_2$  such that for  $n > N_2$  we have  $|a_n - b| < (b - a)/3$ . Let  $N = \max\{N_1, N_2\}$ . Then for  $n > N$  (implying  $n > N_1$  and  $n > N_2$ ), we have both  $|a_n - a| < (b - a)/3$  and  $|a_n - b| < (b - a)/3$ . That is, for such  $n$  we have

$$-\frac{b-a}{3} < a_n - a < \frac{b-a}{3} \quad \text{and} \quad -\frac{b-a}{3} < a_n - b < \frac{b-a}{3}.$$

i.e.,

$$a - \frac{b-a}{3} < a_n < a + \frac{b-a}{3} \quad \text{and} \quad b - \frac{b-a}{3} < a_n < b + \frac{b-a}{3}.$$



In particular,

$$a_n < \frac{b+2a}{3} \quad \text{and} \quad \frac{2b+a}{3} < a_n.$$

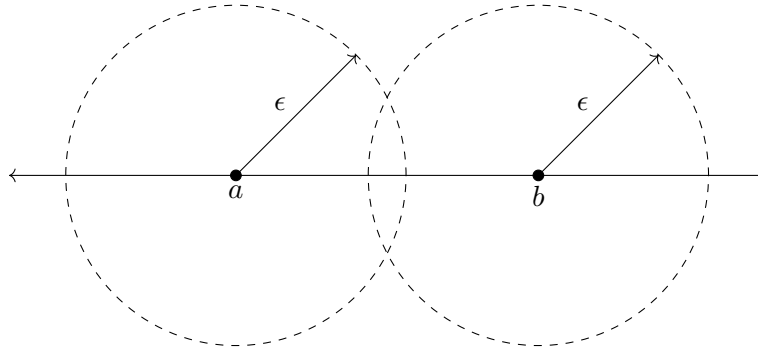
But since  $a < b$ , we must have  $b+2a < 2b+a$ . But this then implies that

$$a_n < \frac{b+2a}{3} < \frac{2b+a}{3} < a_n,$$

which is clearly a contradiction.  $\square$

There is a second proof that is even a little shorter.

*Proof Idea.* The idea is to show that  $|a-b| < \epsilon$  for all  $\epsilon > 0$ , and therefore  $|a-b| = 0$ , implying that  $a = b$ . Intuitively, the way to show this is to say that if  $a_n$  is getting really close to both  $a$  and  $b$ , then that forces  $a$  and  $b$  to be close to each other. If we demand that  $a_n$  is within  $\epsilon/2$  of both  $a$  and  $b$ , then the distance between  $a$  and  $b$  can't be more than  $\epsilon$ :



Using the triangle inequality we can then bound the distance from  $a$  to  $b$  by finding the distance from  $a$  to  $a_n$ , plus the distance from  $a_n$  to  $b$ .  $\square$

*Proof.* Let  $\epsilon > 0$ . Since  $\epsilon/2 > 0$  and  $a_n \rightarrow a$ , there exists some  $N_1$  such that for  $n > N_1$  we have  $|a_n - a| < \epsilon/2$ . Since  $\epsilon/2 > 0$  and  $a_n \rightarrow b$ , there exists some  $N_2$  such that for  $n > N_2$  we have  $|a_n - b| < \epsilon/2$ . Let  $N = \max(N_1, N_2)$  and pick any  $n > N$ . Then

$$\begin{aligned} |a-b| &= |a-a_n+a_n-b| \\ &\leq |a-a_n| + |a_n-b| && \text{(triangle inequality)} \\ &= |a_n-a| + |a_n-b| \\ &< \epsilon/2 + \epsilon/2 && (n > N \text{ implies } n > N_1 \text{ and } n > N_2) \\ &= \epsilon. \end{aligned}$$

Since this holds for any  $\epsilon > 0$ , we have shown that  $|a-b| < \epsilon$  for all  $\epsilon > 0$ , which implies that  $|a-b| = 0$ , and hence  $a = b$ . So indeed,  $a_n$  can converge to only a single point.  $\square$

### Proposition 3.19

If  $(a_n)$  is a convergent sequence, then  $(a_n)$  is bounded.

*Proof.* Since  $(a_n)$  is convergent, let  $a$  be the value it is converging to. By the definition of convergence (with  $\epsilon = 1$ ), there is some  $N$  where

$$|a_n - a| < 1$$

for all  $n > N$ . That is,  $a - 1 < a_n < a + 1$  for all  $n > N$ . Let

$$U = \max\{a_1, a_2, a_3, \dots, a_N, a + 1\}$$

and

$$L = \min\{a_1, a_2, a_3, \dots, a_N, a - 1\}.$$

Note that if  $n \leq N$ , then  $L \leq a_n \leq U$ , since each such  $a_n$  is included in the sets which we are taking the minimum and maximum of. And if  $n > N$ , then we already noted that  $a - 1 < a_n < a + 1$ , which implies that

$$L \leq a - 1 < a_n < a + 1 \leq U,$$

and hence  $L < a_n < U$ . Combining these cases, we have that

$$L \leq a_n \leq U$$

for all  $n$ . And thus, by definition,  $(a_n)$  is bounded.  $\square$

### Corollary 3.20

If  $(a_n)$  is an unbounded sequence, then  $(a_n)$  is divergent.

### Theorem 3.21 (Sequence limit laws)

Assume that  $(a_n)$  and  $(b_n)$  are convergent sequences of real numbers such that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Also assume that  $c \in \mathbb{R}$ . Then,

1.  $(a_n + b_n) \rightarrow a + b$ .
2.  $(a_n - b_n) \rightarrow a - b$ .
3.  $(a_n \cdot b_n) \rightarrow a \cdot b$ .
4.  $\left(\frac{a_n}{b_n}\right) \rightarrow \frac{a}{b}$ , provided each  $b \neq 0$  and each  $b_n \neq 0$ .
5.  $(c \cdot a_n) \rightarrow c \cdot a$ .

### Example 3.22

What is

$$\lim_{n \rightarrow \infty} \frac{1}{2} \cdot \left( \frac{\frac{1}{n} + \frac{1}{n^2} + 4}{5 - \frac{1}{n^2}} \right) \cdot \left( \frac{3n+1}{n+2} + \frac{1}{\sqrt{n}} \right)?$$

By the limit laws, we have

$$\lim_{n \rightarrow \infty} \frac{1}{2} \cdot \left( \frac{\frac{1}{n} + \frac{1}{n^2} + 4}{5 - \frac{1}{n^2}} \right) \cdot \left( \frac{3n+1}{n+2} + \frac{1}{\sqrt{n}} \right) = \frac{1}{2} \cdot \left( \frac{0+0+4}{5} \right) \cdot (3+0) = \frac{6}{5}.$$

**Theorem 3.23** (Sequence squeeze theorem)

Assume  $a_n \leq x_n \leq b_n$  for all  $n$ . Furthermore, assume that

$$a_n \rightarrow L \quad \text{and} \quad b_n \rightarrow L.$$

Then,

$$x_n \rightarrow L.$$

*Proof.* Let  $\epsilon > 0$ .

- Since  $a_n \rightarrow L$ , there exists some  $N_1$  such that  $n > N_1$  implies  $|a_n - L| < \epsilon$ . That is,  $-\epsilon < a_n - L < \epsilon$ . Or,

$$L - \epsilon < a_n < L + \epsilon. \quad (1)$$

- Since  $b_n \rightarrow L$ , there exists some  $N_2$  such that  $n > N_2$  implies  $|b_n - L| < \epsilon$ . That is,  $-\epsilon < b_n - L < \epsilon$ . Or,

$$L - \epsilon < b_n < L + \epsilon. \quad (2)$$

Let  $N = \max(N_1, N_2)$ , and let  $n > N$ . Combining the inequality  $a_n \leq x_n \leq b_n$  with the left half of (1) and the right half of (2), we get

$$\begin{aligned} L - \epsilon &< a_n \leq x_n \leq b_n < L + \epsilon \\ L - \epsilon &< x_n < L + \epsilon \\ -\epsilon &< x_n - L < \epsilon \\ |x_n - L| &< \epsilon. \end{aligned}$$

□

**Example 3.24**

Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2.7} + \sqrt{n} + \pi} = 0.$$

*Proof.* Since

$$0 \leq \frac{1}{n^{2.7} + \sqrt{n} + \pi} \leq \frac{1}{n}, \quad \lim_{n \rightarrow \infty} 0 = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0,$$

by the sequence squeeze theorem (Theorem 3.23),

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2.7} + \sqrt{n} + \pi} = 0.$$

□

**Definition 3.25** (Monotone sequences)

A sequence  $(a_n)$  is **monotone increasing** if  $a_n \leq a_{n+1}$  for all  $n$ . Likewise, a sequence  $(a_n)$  is **monotone decreasing** if  $a_n \geq a_{n+1}$  for all  $n$ . If it is either monotone increasing or monotone decreasing, it is **monotone**.

**Theorem 3.26 (The monotone convergence theorem)**

Suppose  $(a_n)$  is monotone. Then  $(a_n)$  converges if and only if it is bounded. Moreover,

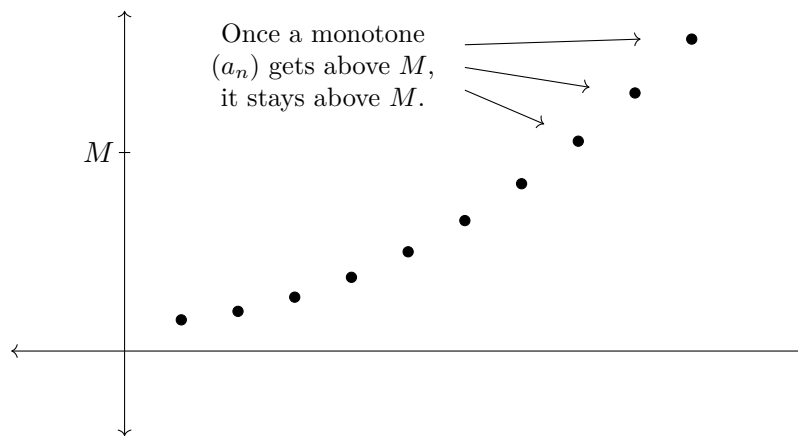
- If  $(a_n)$  is increasing, then either  $(a_n)$  diverges to  $\infty$  or

$$\lim_{n \rightarrow \infty} a_n = \sup(\{a_n : n \in \mathbb{N}\}).$$

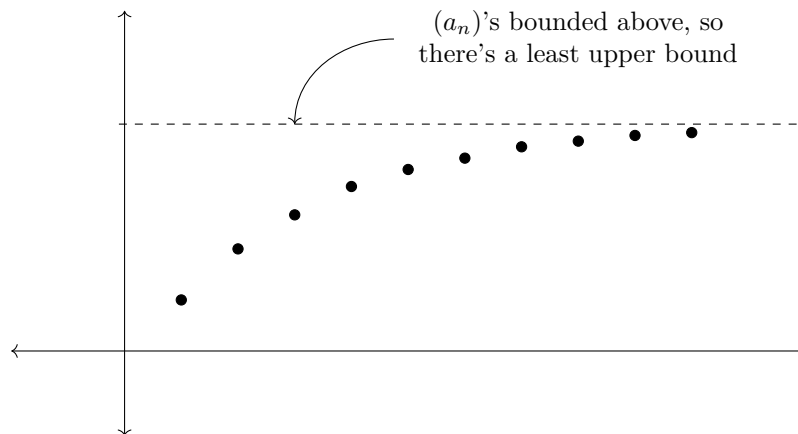
- If  $(a_n)$  is decreasing, then either  $(a_n)$  diverges to  $-\infty$  or

$$\lim_{n \rightarrow \infty} a_n = \inf(\{a_n : n \in \mathbb{N}\}).$$

*Proof Idea.* Suppose  $(a_n)$  is monotonically increasing. If it's not bounded, then given any  $M > 0$ , this  $M$  is not an upper bound and so eventually  $(a_n)$  will get above it. And since it's monotonically increasing, once it gets above a number, it *stays* above that number.



If, on the other hand,  $(a_n)$  is bounded, then the sequence must be leveling out (it's monotone, so it can't go up and down).



□

*Proof.* Assume that  $(a_n)$  is monotonically increasing, and let's suppose first that  $a_n$  is not bounded. Then for any  $M > 0$  there exists some  $N$  such that  $a_N > M$ . But since  $(a_n)$  is monotonically increasing, for  $n > N$  we have  $a_n \geq a_N > M$ . And so, by Definition 3.14,  $(a_n)$  diverges to  $\infty$ .

Next suppose that  $(a_n)$  is bounded. Then we have that  $\{a_n : n \in \mathbb{N}\}$  is a subset of  $\mathbb{R}$  which is bounded above, which by the completeness of  $\mathbb{R}$  implies that  $\sup(\{a_n : n \in \mathbb{N}\})$  exists. Call this supremum  $\alpha$ . We want to show that  $\lim_{n \rightarrow \infty} a_n = \alpha$ ; that is, we want to show that for any  $\epsilon > 0$  there exists some  $N$  such that  $n > N$  implies  $|a_n - \alpha| < \epsilon$ .

To show this, first let  $\epsilon > 0$ . Since  $\sup(\{a_n : n \in \mathbb{N}\}) = \alpha$ , by the analytic definition of suprema (Theorem 1.25) there exists some  $a_N > \alpha - \epsilon$ . And since  $(a_n)$  is monotonically increasing, we see that for any  $n > N$  we have that  $a_n \geq a_N > \alpha - \epsilon$ . And of course,  $a_n < \alpha$  due to the fact that  $\alpha$  is the supremum (and hence an upper bound) of  $\{a_n : n \in \mathbb{N}\}$ . And so, for  $n > N$ , we have  $\alpha - \epsilon < a_n < \alpha$ . This implies  $\alpha - \epsilon < a_n < \alpha + \epsilon$ , and hence  $-\epsilon < a_n - \alpha < \epsilon$ , which, at last, gives

$$|a_n - \alpha| < \epsilon$$

completing the proof in the case when  $(a_n)$  is monotonically increasing.

The case where  $(a_n)$  is monotonically decreasing is very similar. □

**Remark 3.27** — This theorem is nice for several reasons, but one of which is the fact that you do not have to know what the limit of  $(a_n)$  is in order to show that  $(a_n)$  is convergent. This is notable since in the definition of sequence convergence—which, until now, we have heavily relied on to show a sequence converges—requires that you already know (and can write down) what the limit is going to be. Below is an example where it would be quite a challenge to write down the limit of the sequence in a beneficial way. Yet by the monotone convergence theorem, we will be able to conclude that the limit exists.

### Example 3.28

Let  $(a_n)$  be the sequence where  $a_1 = 0.1$ ,  $a_2 = 0.12$ ,  $a_3 = 0.123$ ,  $a_4 = 0.1234$ , and so on. (And, to be clear, this pattern does not change when you reach double digits. For example,  $a_{12} = 0.123456789101112$ .) Prove that  $(a_n)$  converges.

*Proof.* Note that  $a_{n+1}$  and  $a_n$  match exactly until the last digits of  $a_{n+1}$ , which are the digits of  $n + 1$ . Therefore,  $a_{n+1} - a_n$  is a number with a bunch of zeros followed the digits of  $n + 1$ . In particular,

$$a_{n+1} - a_n > 0$$

for all  $n$ . We have shown that  $a_{n+1} > a_n$  for all  $n$ , proving that  $(a_n)$  is monotone increasing. Furthermore note that since all the terms begin with 0.1, we have that  $a_n \leq 1$  for all  $n$ .

We have shown that  $(a_n)$  is monotone increasing and bounded above, therefore by the monotone convergence theorem (Theorem 3.26) the sequence converges, completing the proof. □

**Proposition 3.29**

Suppose  $S \subseteq \mathbb{R}$  is bounded above. Then there exists a sequence  $(a_n)$  where  $a_n \in S$  for each  $n$  and

$$\lim_{n \rightarrow \infty} a_n = \sup(S).$$

Likewise, if  $S$  is bounded below, then there exists a sequence  $(b_n)$  where  $b_n \in S$  for each  $n$  and

$$\lim_{n \rightarrow \infty} b_n = \inf(S).$$

*Proof Idea.* We need to find elements of  $S$  which are getting closer and closer to  $\sup(S)$ . What was our *super* useful theorem for doing just that? The analytic definition of suprema (Theorem 1.25)! That was the theorem that said for any  $\epsilon > 0$ , there exists some  $a \in S$  such that  $a > \sup(S) - \epsilon$ . And since it works for any  $\epsilon > 0$ , we can choose a sequence of  $\epsilon$  values that are getting closer and closer to 0, which in turn produce a sequence of  $a$  values that are getting closer and closer to  $\sup(S)$ .

And once we have found a sequence of these elements which are getting closer and closer to  $\sup(S)$ , how do we formally show they converge to  $\sup(S)$ ? The sequence squeeze theorem (Theorem 3.23) sure sounds like it's up to the job.  $\square$

*Proof.* Suppose  $S \subseteq \mathbb{R}$  is bounded above, implying that  $\sup(S)$  exists. Let  $\sup(S) = \alpha$  (implying  $a_n \leq \alpha$  for all  $n$ ). By the analytic definition of suprema (Theorem 1.25), for any  $\epsilon > 0$  there exists some  $x \in S$  such that  $x > \alpha - \epsilon$ . In particular, there exists some  $a_1 \in S$  such that  $a_1 > \alpha - 1$ ; and there exists some  $a_2 \in S$  such that  $a_2 > \alpha - \frac{1}{2}$ ; and there exists some  $a_3 \in S$  such that  $a_3 > \alpha - \frac{1}{3}$ ; and, in general, for each  $n \in \mathbb{N}$  there exists some  $a_n \in S$  such that  $a_n > \alpha - \frac{1}{n}$ .

Thus we obtain a sequence  $(a_n)$  from  $S$ . Moreover, this sequence does indeed converge to  $\alpha$ . Therefore, by the limit laws (Theorem 3.21) we have that

$$\alpha - \frac{1}{n} \rightarrow \alpha - 0 = \alpha.$$

So we have an inequality

$$\alpha - \frac{1}{n} \leq a_n \leq \alpha,$$

where both the upper and lower bounds converge to  $\alpha$ . By the sequence squeeze theorem (Theorem 3.23) we see that  $\lim_{n \rightarrow \infty} a_n = \alpha$ , as desired.

The infimum case is *very* similar.  $\square$

**Definition 3.30**

(Subsequences) Let  $(a_n)$  be a sequence of real numbers and let

$$n_1 < n_2 < n_3 < \dots$$

be an increasing sequence of integers. Then,

$$a_{n_1}, a_{n_2}, a_{n_3}, \dots$$

is called a **subsequence** of  $(a_n)$ , and is denoted  $(a_{n_k})$ .

**Proposition 3.31**

A sequence  $(a_n)$  converges to  $a$  if and only if every subsequence of  $(a_n)$  also converges to  $a$ .

*Proof.*

( $\Rightarrow$ ) Assume that  $(a_n)$  converges to  $a$  and  $(a_{n_k})$  is a subsequence of  $(a_n)$ . Let  $\epsilon > 0$ . Then there exists some  $N$  such that

$$|a_n - a| < \epsilon \quad (1)$$

for all  $n > N$ .

We want to show that  $a_{n_k} \rightarrow a$ . That is, we want to show that there exists some  $N_1$  such that  $|a_{n_k} - a| < \epsilon$  for all  $k > N_1$ . Notice that since each  $n_i \in \mathbb{N}$  and

$$n_1 < n_2 < n_3 < \dots,$$

that we have  $n_k \geq k$ . Therefore by letting  $N_1 = N$ , for any  $k > N_1$  we then know that  $n_k > N$ , so by (1) we have

$$|a_{n_k} - a| < \epsilon.$$

( $\Leftarrow$ ) Assume that every subsequence of  $(a_n)$  converges to  $a$ . Suppose, for contradiction, that  $(a_n)$  does not converge to  $a$ . By Outline 3.16, this means that there exists some  $\epsilon_0 > 0$  such that for every  $N \in \mathbb{N}$ , there is some  $n > N$  with  $|a_n - a| \geq \epsilon_0$ . Using this fact, we can inductively construct a subsequence of  $(a_n)$  as follows:

1. Let  $n_1$  be a natural number with  $|a_{n_1} - a| \geq \epsilon_0$ .
2. Having chosen  $n_1, \dots, n_k$ , let  $n_{k+1} > n_k$  be a natural number with  $|a_{n_{k+1}} - a| \geq \epsilon_0$ .

This process yields a subsequence  $(a_{n_k})$  with the property that  $|a_{n_k} - a| \geq \epsilon_0$  for all  $k \in \mathbb{N}$ . But this means that  $(a_{n_k})$  cannot converge to  $a$ , which contradicts our assumption that every subsequence of  $(a_n)$  converges to  $a$ . Therefore,  $(a_n)$  must converge to  $a$ .  $\square$

**Corollary 3.32**

If  $(a_n)$  has a pair of subsequences converging to different limits, then  $(a_n)$  diverges.

*Proof.* Assume for a contradiction that  $(a_n)$  converges. Say,  $a_n \rightarrow L$ . By Proposition 3.31, this implies that every subsequence also converges to  $L$ . This contradicts the assumption that there are subsequences converging to different limits.  $\square$

**Proposition 3.33**

If a monotone sequence  $(a_n)$  has a convergent subsequence, then  $(a_n)$  converges too, and has the same limit.

*Proof Idea.* First recall that by Proposition 3.31, a convergent sequence always has the same limit as any of its subsequences. So by this, our task reduces to showing that  $(a_n)$  converges.

A monotone sequence is convergent if and only if it is bounded (by the monotone convergence theorem). We are told that  $(a_n)$  is monotone and to show that it's bounded, we use the assumption that it has a convergent subsequence,  $(a_{n_k})$ . Here's the progression:  $(a_n)$  being monotone implies that its subsequence is too, and  $(a_{n_k})$  being convergent (and now monotone) will mean that  $(a_n)$  is bounded too.  $\square$

*Proof.* Assume that  $(a_n)$  is monotone increasing; the case where  $(a_n)$  is monotone decreasing is *very* similar.

Let  $(a_{n_k})$  be a convergent subsequence of  $(a_n)$  and observe that since  $(a_n)$  is monotone increasing, being a subsequence means that  $(a_{n_k})$ 's terms come from  $(a_n)$  and do so in the same order as they appear in  $(a_n)$ , implying that  $(a_{n_k})$  is also monotone increasing. That is,  $(a_{n_k})$  was assumed to be convergent and was shown to be monotone—by the monotone convergence theorem (Theorem 3.26) this means  $(a_{n_k})$  converges, and in particular converges to  $\sup(\{a_{n_k} : k \in \mathbb{N}\})$ .

We now show that  $(a_n)$  is convergent, again by using the monotone convergence theorem. Since  $(a_n)$  is monotone increasing by assumption, we need only show that  $(a_n)$  is bounded above. To see this, observe that since the subsequence  $(a_{n_k})$  is an infinite list of elements from  $(a_n)$ , any particular element  $a_n$  from the sequence has an element  $a_{n_k}$  from this subsequence that comes after it. And since both  $a_n$  and  $a_{n_k}$  are terms of  $(a_n)$ , which is monotone increasing, we have  $a_n \leq a_{n_k}$ . So we have

$$a_n \leq a_{n_k} \leq \sup(\{a_{n_k} : k \in \mathbb{N}\}),$$

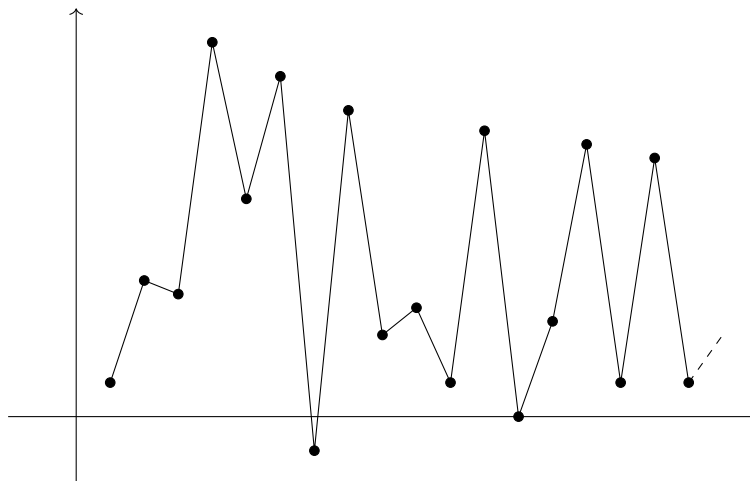
which proves that  $(a_n)$  is bounded above (by this supremum), and so by the monotone convergence theorem we have proven that  $(a_n)$  converges.

Finally, since we have shown that the sequence  $(a_n)$  is convergent, by Proposition 3.32 it has the same limit as any of its subsequences. This means that  $(a_n)$  must also be converging to  $\sup(\{a_{n_k} : k \in \mathbb{N}\})$ , completing the proof.  $\square$

### Lemma 3.34

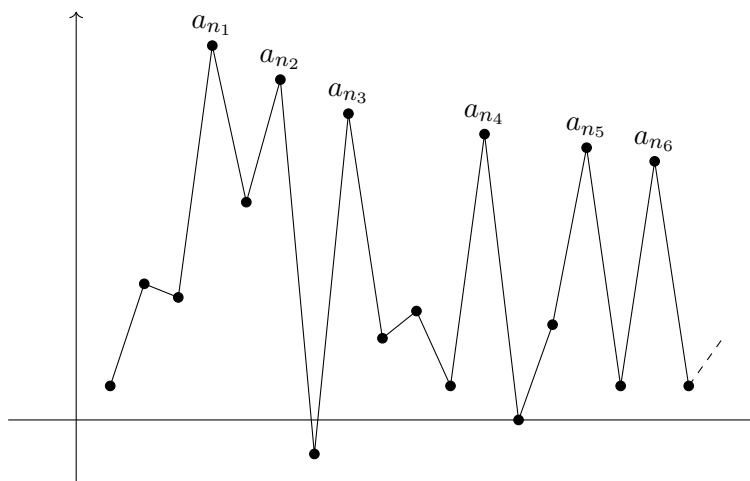
Every sequence has a monotone subsequence.

*Proof Idea.* Here's the idea behind it. We draw a sequence of in the  $xy$ -plane by just plotting the points, and we will connect the dots to make a zig-zagged line:



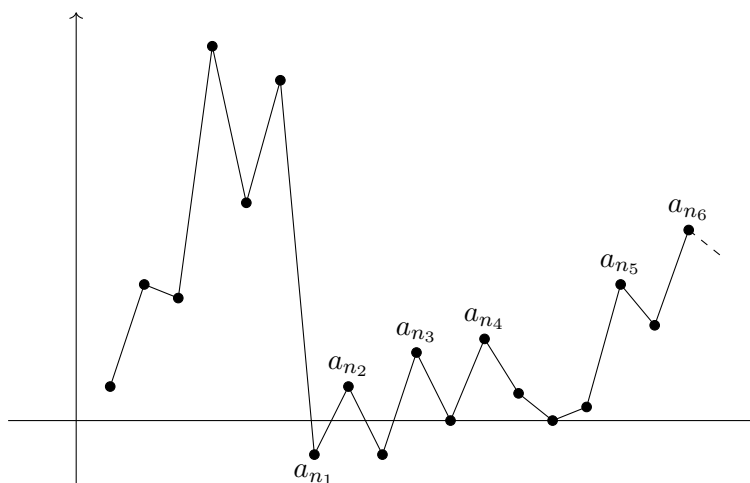


From this you can maybe spot a nice *decreasing* sequence, which is of course monotone:



The magical definition that will solve everything is that of a *peak*; we want those labeled points to form *peaks*, and we want to be able to say that if we have a(n infinite) sequence of peaks, then we do indeed have a decreasing sequence. The definition that does this is this: define a *peak* to be a point  $a_n$  which is larger than every later point; that is,  $a_n$  is a peak if  $a_n \geq a_m$  for all  $m > n$ .

So if we have infinitely many peaks, then we obtain a sequence like the one above which will be decreasing. So what if we don't? Then we only have finitely many peaks. In this case, we can find an *increasing* sequence, which is again monotone. To see how, just note that if you're past the last peak, then any point you pick is not a peak, which means there is some point after it which is larger. So one at a time you can pick larger and larger points, giving an increasing sequence.



□

*Proof.* Call  $a_n$  a *peak* if  $a_n$  is larger than every later point. That is, if  $a_n \geq a_m$  for all  $m > n$ .

Either there are infinitely many peaks or finitely many peaks. Assume first that  $(a_n)$  has infinitely many peaks. Then let  $a_{n_k}$  be the  $k$ th peak. Then, by the definition of a peak,  $a_{n_k} \geq a_{n_{k+1}}$ , implying that  $(a_{n_k})$  is a decreasing subsequence.

Now assume that  $(a_n)$  has finitely many peaks, and let  $a_N$  be the last one. Then let  $a_{n_1} = a_{N+1}$ . Since  $a_N$  was the last peak,  $a_{N+1}$  is *not* a peak, implying that there is some later point  $a_{n_2}$  that is larger than it. Likewise, since  $a_{n_2}$  is after the last peak, it is also not a peak, and so there must be some later point  $a_{n_3}$  that is larger than it. Continuing in this way we construct an increasing subsequence  $(a_{n_k})$ .

In either case we found a monotone subsequence, so we are done.  $\square$

**Theorem 3.35 (The Bolzano-Weierstrass theorem)**

Every bounded sequence has a convergent subsequence.

*Proof.* Assume  $(a_n)$  is a bounded sequence. Then by Lemma 3.34 it has a monotone subsequence,  $(a_{n_k})$ . Also, since  $(a_n)$  is bounded, so is  $(a_{n_k})$ . Since  $(a_{n_k})$  is both bounded and monotone, by the monotone convergence theorem it converges.  $\square$

**Definition 3.36**

A sequence  $(a_n)$  is **Cauchy** if for all  $\epsilon > 0$  there exists some  $N \in \mathbb{N}$  such that

$$|a_m - a_n| < \epsilon$$

for all  $m, n > N$ .

**Lemma 3.37**

If  $(a_n)$  is Cauchy, then  $(a_n)$  is bounded.

*Proof.* Assume  $(a_n)$  is Cauchy. Then (for  $\epsilon = 1$ ) there exists some  $N \in \mathbb{N}$  such that

$$|a_m - a_n| < 1$$

for all  $m, n > N$ . In particular, for all  $n > N$ ,

$$|a_n - a_{n+1}| < 1.$$

Therefore for  $n > N$ , we know that the terms are bounded:

$$a_{N+1} - 1 < a_n < a_{N+1} + 1.$$

And there are only finitely many points before  $a_N$ , so these are certainly bounded. Consequently, we can find a general bound on all  $a_n$ . Indeed, if we let

$$L = \min(\{a_1, a_2, a_3, \dots, a_N, a_{N+1} - 1\})$$

and

$$U = \max(\{a_1, a_2, a_3, \dots, a_N, a_{N+1} + 1\}),$$

then

$$L \leq a_n \leq U$$

for all  $n \in \mathbb{N}$ . So  $(a_n)$  is bounded.  $\square$

**Theorem 3.38** (Cauchy criterion for convergence)

A sequence converges if and only if it is Cauchy.

*Proof Sketch.* For the forward direction, suppose  $a_n$  converges to  $a$ . If  $a_n$  and  $a_m$  are both within  $\epsilon/2$  of  $a$ , then they must be within  $\epsilon$  of each other. We can formally show this using the triangle inequality.

For the backwards direction, the idea is this: It will be helpful to identify what the Cauchy sequence is converging to. And by Bolzano-Weierstrass we can find a subsequence that is converging to some  $a$ . Our goal will then be to prove that  $a$  is in fact the sequence's limit. How? Well, that subsequence gets super close to  $a$ . What about the terms in  $(a_n)$  which are not in the subsequence? By the Cauchy criterion, they get super close to the elements of the subsequence! And if a term in the sequence is super close to a term of the subsequence, which is in turn super close to the limit... then the sequence must be close to the limit, too.  $\square$

*Proof.*

( $\Rightarrow$ ) Assume that  $(a_n)$  converges to some  $a \in \mathbb{R}$ . Let  $\epsilon > 0$ . Since  $\epsilon/2 > 0$ , there exists some  $N \in \mathbb{N}$  such that for any  $n > N$  we have

$$|a_n - a| < \frac{\epsilon}{2}.$$

Then, for any  $n, m > N$ ,

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

( $\Leftarrow$ ) Assume that  $(a_n)$  is Cauchy, and note that by Lemma 3.37,  $(a_n)$  is bounded. Combining these two facts, and applying the Bolzano-Weierstrass theorem (Theorem 3.35), we conclude that some subsequence of  $(a_n)$  converges. Say,  $(a_{n_j})$  converges to  $a$ . Our goal is to show that  $(a_n)$  converges; we will in fact prove that  $a_n \rightarrow a$ .

Let  $\epsilon > 0$ . Since  $\epsilon/2 > 0$  and  $(a_n)$  is Cauchy, there exists some  $N_1$  such that

$$|a_n - a_m| < \frac{\epsilon}{2} \tag{1}$$

for all  $n, m > N_1$ . Since  $\epsilon/2 > 0$  and  $(a_{n_j})$  converges to  $a$ , there exists some  $N_2$  such that  $j > N_2$  implies

$$|a_{n_j} - a| < \frac{\epsilon}{2}. \tag{2}$$

We want to choose some  $J$  so that, for subscripts past this point, both (1) and (2) hold. Choose  $J = \max(N_1, N_2)$ . Notice, from the definition of a subsequence, that  $n_j \geq j$ . In particular,  $n_{J+1} > J$ . And so, for any  $j > J$ ,

$$|a_j - a| = |a_j - a_{n_{J+1}} + a_{n_{J+1}} - a| \leq |a_j - a_{n_{J+1}}| + |a_{n_{J+1}} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

**Theorem 3.39** (Stolz-Cesàro theorem)

Let  $(a_n)$  and  $(b_n)$  be two sequences.

1. Assume that  $a_n \rightarrow 0$  and  $b_n \rightarrow 0$ . Suppose  $b_n$  is strictly decreasing and

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = a,$$

where  $a$  is finite or  $+\infty$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = a.$$

2. Assume that  $b_n$  is a strictly monotone and divergent sequence, i.e. strictly increasing and approaching  $+\infty$ , or strictly decreasing and approaching  $-\infty$ . Suppose

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = a,$$

where  $a$  is finite or  $+\infty$  if  $b_n$  increases, or  $a$  is finite or  $-\infty$  if  $b_n$  decreases. Then,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = a.$$

*Proof of part 2.*

**Case 1.** Suppose  $(b_n)$  is strictly increasing and diverging to  $+\infty$ , and  $a$  is finite. By hypothesis, we have that for all  $\epsilon/2 > 0$  there exists an integer  $N_1$  such that

$$\left| \frac{a_{n+1} - a_n}{b_{n+1} - b_n} - a \right| < \frac{\epsilon}{2} \quad \text{for all } n > N_1,$$

which is to say

$$a - \epsilon/2 < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < a + \epsilon/2 \quad \text{for all } n > N_1.$$

Since  $(b_n)$  is strictly increasing, we have that  $b_{n+1} - b_n > 0$ , and the following holds

$$(a - \epsilon/2)(b_{n+1} - b_n) < a_{n+1} - a_n < (a + \epsilon/2)(b_{n+1} - b_n) \quad \text{for all } n > N_1.$$

Next we notice that

$$a_n = [(a_n - a_{n-1}) + \cdots + (a_{N_1+2} - a_{N_1+1})] + a_{N_1+1}$$

thus, by applying the above inequality to each of the terms in the square brackets, we obtain

$$(a - \epsilon/2)(b_n - b_{N_1+1}) + a_{N_1+1} = (a - \epsilon/2)[(b_n - b_{n-1}) + \cdots + (b_{N_1+2} - b_{N_1+1})] + a_{N_1+1} < a_n$$

and

$$a_n < (a + \epsilon/2)[(b_n - b_{n-1}) + \cdots + (b_{N_1+2} - b_{N_1+1})] + a_{N_1+1} = (a + \epsilon/2)(b_n - b_{N_1+1}) + a_{N_1+1}.$$

Now, since  $b_n \rightarrow +\infty$  as  $n \rightarrow \infty$ , there is an  $N_2$  such that  $b_n > 0$  for all  $n > N_2$ , and we can divide the two inequalities by  $b_n$  for all  $n > \max(N_1, N_2)$

$$(a - \epsilon/2) + \frac{a_{N_1+1} - b_{N_1+1}(a - \epsilon/2)}{b_n} < \frac{a_n}{b_n} < (a + \epsilon/2) + \frac{a_{N_1+1} - b_{N_1+1}(a + \epsilon/2)}{b_n}.$$

Define

$$c_n^+ := \frac{a_{N_1+1} - b_{N_1+1}(a + \epsilon/2)}{b_n} \quad \text{and} \quad c_n^- := \frac{a_{N_1+1} - b_{N_1+1}(a - \epsilon/2)}{b_n}.$$

Since  $b_n \rightarrow +\infty$  and the numerator is a constant number, hence for all  $\epsilon/2 > 0$ , there exist  $N_{\pm} > N_2 > 0$  such that

$$\begin{aligned} |c_n^+| &< \epsilon/2 & \text{for all } n > N_+, \\ |c_n^-| &< \epsilon/2 & \text{for all } n > N_-, \end{aligned}$$

therefore

$$a - \epsilon < a - \epsilon/2 + c_n^- < \frac{a_n}{b_n} < a + \epsilon/2 + c_n^+ < l + \epsilon \quad \text{for all } n > \max(N_1, N_{\pm}),$$

which concludes the proof. The case with  $(b_n)$  strictly decreasing and diverging to  $-\infty$ , and  $a$  is finite, is similar.

**Case 2.** Suppose  $(b_n)$  is strictly increasing and diverging to  $+\infty$ , and  $a = +\infty$ . Proceeding as before, for all  $2M > 0$  there exists  $N_1 > 0$  such that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} > 2M \quad \text{for all } n > N_1.$$

Again, by applying the above inequality to each of the terms inside the square brackets we obtain

$$a_n > 2M(b_n - b_{N_1+1}) + a_{N_1+1} \quad \text{for all } n > N_1,$$

and

$$\frac{a_n}{b_n} > 2M + \frac{a_{N_1+1} - 2Mb_{N_1+1}}{b_n} \quad \text{for all } n > \max(N_1, N_2).$$

Define

$$c_n := \frac{a_{N_1+1} - 2Mb_{N_1+1}}{b_n}.$$

Since  $b_n \rightarrow +\infty$  and the numerator is a constant number, hence for all  $M > 0$ , there exists  $\bar{N} > N_2 > 0$  such that

$$|c_n| < M \quad \text{for all } n > \bar{N},$$

combining this inequality with the previous one we conclude that

$$\frac{a_n}{b_n} > 2M + c_n > M \quad \text{for all } n > \max(N_1, \bar{N}).$$

The proofs of the other cases with  $(b_n)$  strictly increasing and approaching  $+\infty$  or  $-\infty$  respectively and  $a = \pm\infty$  all proceed in the same way.  $\square$

*Proof of part 1.*

**Case 1.** We first consider the case with finite  $a$  and  $(b_n)$  strictly decreasing. This time, for each  $N_1 > 0$ , we can write

$$a_n = (a_n - a_{n+1}) + \cdots + (a_{n+N_1-1} - a_{n+N_1}) + a_{n+N_1},$$

and for any  $\epsilon/2 > 0$ , there exists  $N_2$  such that for all  $n > N_2$  we have

$$(a - \epsilon/2)(b_n - b_{n+N_1}) + a_{n+N_1} = (a - \epsilon/2)[(b_n - b_{n+1}) + \cdots + (b_{n+N_1-1} - b_{n+N_1})] + a_{n+N_1} < a_n$$

and

$$a_n < (a + \epsilon/2)[(b_n - b_{n+1}) + \cdots + (b_{n+N_1-1} - b_{n+N_1})] + a_{n+N_1} = (a + \epsilon/2)(b_n - b_{n+N_1}) + a_{n+N_1}.$$

Define

$$c_{N_1}^+ := \frac{a_{n+N_1} - b_{n+N_1}(a + \epsilon/2)}{b_n} \quad \text{and} \quad c_{N_1}^- = \frac{a_{n+N_1} - b_{n+N_1}(a - \epsilon/2)}{b_n}.$$

Since by hypothesis,  $a_{n+N_1} \rightarrow 0$  and  $b_{n+N_1} \rightarrow 0$  as  $N_1 \rightarrow \infty$ , thus for all  $\epsilon/2 > 0$  there exist  $N_{\pm} > 0$  such that

$$\begin{aligned} |c_{N_1}^+| &< \epsilon/2 & \text{for all } N_1 > N_+, \\ |c_{N_1}^-| &< \epsilon/2 & \text{for all } N_1 > N_-, \end{aligned}$$

thus, choosing  $N_1$  appropriately (which is to say, taking the limit with respect to  $N_1$ ) we obtain

$$a - \epsilon < a - \epsilon/2 + c_{N_1}^- < \frac{a_n}{b_n} < a + \epsilon/2 + c_{N_1}^+ < a + \epsilon \quad \text{for all } n > N_2,$$

which concludes the proof.

**Case 2.** Suppose  $a = +\infty$  and  $(b_n)$  strictly decreasing. For all  $2M > 0$  there exists  $N_1 > 0$  such that for all  $n > N_1$ ,

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} > 2M \implies a_n - a_{n+1} > 2M(b_n - b_{n+1}).$$

Therefore, for each  $N_2 > 0$ ,

$$\frac{a_n}{b_n} > 2M + \frac{a_{n+N_2} - 2Mb_{n+N_2}}{b_n} \quad \text{for all } n > N_1.$$

The sequence

$$c_{N_2} := \frac{a_{n+N_2} - 2Mb_{n+N_2}}{b_n}$$

converges to 0 (keeping  $n$  fixed). Hence, for all  $M > 0$ , there exists  $\bar{N} > 0$  such that

$$|c_{N_2}| < M \quad \text{for all } N_2 > \bar{N},$$

and chooging  $N_2$  conveniently, we conclude the proof

$$\frac{a_n}{b_n} > 2M + c_{N_2} > M \quad \text{for all } n > N_1.$$

□

**Remark 3.40** — The converse of this theorem does not hold, i.e.  $\lim_{n \rightarrow \infty} a_n/b_n = a$  does not necessarily imply

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = a.$$

Here is a counterexample:

$$a_n = 3n - (-1)^n \quad \text{and} \quad b_n = 3n + (-1)^n.$$

**Remark 3.41** — Stolz-Cesàro theorem provides a powerful tool to evaluate limits of sequences, and is viewed as a discrete form of L'Hôpital's rule.

# 4 Series

“Divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever.”

*Niels Henrik Abel, in a letter to Holmboe (1826)*

## Definition 4.1 (Infinite series)

A sum of infinitely many numbers is called an **infinite series** and is denoted by

$$\sum_{k=1}^{\infty} a_k \quad \text{or} \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

## Definition 4.2 (Terms)

Given an infinite series  $\sum_{k=1}^{\infty} a_k$ , the numbers  $a_k$  are the **terms** of the series.

## Definition 4.3 (Sequence of partial sums)

Given an infinite series  $\sum_{k=1}^{\infty} a_k$ , the **sequence of partial sums** is the sequence

$$\left( \sum_{k=1}^n a_k \right)_{n=1}^{\infty}.$$

That is, it's the sequence  $(s_n)$  where:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \end{aligned}$$

## Definition 4.4 (Convergent and divergent series)

Given an infinite series  $\sum_{k=1}^{\infty} a_k$ , the series **converges** to  $L \in \mathbb{R}$ , denoted by

$$\sum_{k=1}^{\infty} a_k = L,$$

if the sequence of partial sums  $(s_n)$  converges to  $L$ .

The series **diverges** (to  $\infty$ , to  $-\infty$ , or does not exist) if  $(s_n)$  does.



**Definition 4.5 (Bounded series)**

Given an infinite series  $\sum_{k=1}^{\infty} a_k$ , the series is **bounded** if the sequence of partial sums  $(s_n)$  is bounded.

**Definition 4.6 (Monotone series)**

Given an infinite series  $\sum_{k=1}^{\infty} a_k$ , the series is **monotone** if the sequence of partial sums  $(s_n)$  is monotone.

**Theorem 4.7 (Series limit laws)**

Assume that  $\sum_{k=1}^{\infty} a_k = \alpha$  and  $\sum_{k=1}^{\infty} b_k = \beta$ . Also assume that  $c \in \mathbb{R}$ . Then,

1.  $\sum_{k=1}^{\infty} (a_k + b_k) = \alpha + \beta$ ,
2.  $\sum_{k=1}^{\infty} (a_k - b_k) = \alpha - \beta$ , and
3.  $\sum_{k=1}^{\infty} (c \cdot a_k) = c \cdot \alpha$ .

*Proof.* We proceed with a proof of part 1. Let

$$s_n = \sum_{k=1}^n a_k \quad \text{and} \quad t_n = \sum_{k=1}^n b_k.$$

Now,  $(s_n)$  is the sequence of partial sums of the series  $\sum_{k=1}^{\infty} a_k$  and  $(t_n)$  is the sequence of partial sums of the series  $\sum_{k=1}^{\infty} b_k$ . Therefore, by assumption and Definition 4.4, we have that

$$s_n \rightarrow \alpha \quad \text{and} \quad t_n \rightarrow \beta.$$

Now let  $v_n = \sum_{k=1}^n (a_k + b_k)$ , and since we can rearrange finite sums,

$$\begin{aligned} s_n + t_n &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) \\ &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= v_n. \end{aligned}$$

Collecting everything,

$$\begin{aligned} \sum_{k=1}^{\infty} (a_k + b_k) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n (a_k + b_k) && \text{(by Definition 4.1)} \\ &= \lim_{n \rightarrow \infty} (v_n) && \text{(by definition of } v_n) \\ &= \lim_{n \rightarrow \infty} (s_n + t_n) && \text{(showed above)} \\ &= \alpha + \beta && \text{(by Theorem 3.21)} \end{aligned}$$

as desired.

In almost exactly the same way we can prove part 2 and part 3. □

**Remark 4.8** — Define  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  as in Theorem 4.7. We cannot conclude that  $\sum_{k=1}^{\infty} (a_k \cdot b_k) = \alpha \cdot \beta$  since, in general,

$$\left( \sum_{k=1}^n a_k \right) \left( \sum_{k=1}^n b_k \right) \neq \sum_{k=1}^n (a_k \cdot b_k).$$

**Theorem 4.9 (Cauchy criterion for series)**

The series  $\sum_{k=1}^{\infty} a_k$  converges if and only if for every  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that

$$\left| \sum_{k=n}^m a_k \right| < \epsilon \quad \text{whenever } m \geq n > N.$$

*Proof.* Observe that

$$|s_m - s_{n-1}| = \left| \sum_{k=1}^m a_k - \sum_{k=1}^{n-1} a_k \right| = \left| \sum_{k=n}^m a_k \right|$$

and apply the Cauchy criterion for convergence (Theorem 3.38) to conclude that  $(s_n)$  converges, which by Definition 4.4 means that  $\sum_{k=1}^{\infty} a_k$  converges.  $\square$

**Theorem 4.10 (Necessary condition for convergence)**

If  $\sum_{k=1}^{\infty} a_k$  converges, then  $a_k \rightarrow 0$ .

*Proof.* Consider the special case  $m = n + 1$  in Theorem 4.9.  $\square$

**Lemma 4.11**

If  $a_k \geq 0$  for all  $k \in \mathbb{N}$ , then  $\sum_{k=1}^{\infty} a_k$  either converges or it diverges to  $\infty$ .

*Proof.* Since each  $a_k$  is nonnegative, observe that the sequence of partial sums is monotone increasing. Applying the monotone convergence theorem (Theorem 3.26) then gives the result.  $\square$

**Proposition 4.12 (Comparison test)**

Assume  $0 \leq a_k \leq b_k$  for all  $k \in \mathbb{N}$ .

1. If  $\sum_{k=1}^{\infty} b_k$  converges, then  $\sum_{k=1}^{\infty} a_k$  converges.
2. If  $\sum_{k=1}^{\infty} a_k$  diverges, then  $\sum_{k=1}^{\infty} b_k$  diverges.

*Proof.* Let  $(s_n)$  be the sequence of partial sums of  $\sum_{k=1}^{\infty} a_k$ , and let  $(t_n)$  be the sequence of partial sums of  $\sum_{k=1}^{\infty} b_k$ . And here is an observation that will be useful: Note that since  $a_k \leq b_k$  for all  $k$ ,

$$s_n = a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n = t_n$$

for all  $n$ . That is,

$$s_n \leq t_n \quad \text{for all } n. \quad (\star)$$

We proceed with a proof of part 1, but first note that by Lemma 4.11 that all we need to show is that  $(s_n)$  is bounded above. If we can do this we know that  $s_n \not\rightarrow \infty$ , which by that proposition implies that it must converge.

Recall that in Proposition 3.19 we showed that if a sequence converges, then it is bounded. And so, since by assumption  $\sum_{k=1}^{\infty} b_k$  converges (meaning that  $(t_n)$  converges), we deduce that this series is bounded above by some value  $M$ . That is,  $t_n \leq M$  for all  $n$  (this is Definition 3.2). And so by  $(\star)$ ,  $s_n \leq t_n \leq M$  for all  $n$ , showing that  $(s_n)$  is also bounded above by  $M$ , completing the proof of part 1.

The proof of part 2 is similar. Let  $M > 0$ . Assume that  $\sum_{k=1}^{\infty} a_k$  diverges, meaning that  $(s_n)$  diverges (and hence by Lemma 4.11 this means it diverges to  $\infty$ ). And so by Definition 3.14, there exists some  $N$  for which  $M < s_n$  for all  $n > N$ . And so by  $(\star)$ ,  $M < s_n \leq t_n$  for all  $n > N$ . Thus, again by Definition 3.14,  $(t_n)$  also diverges to  $\infty$ .  $\square$

Here's an alternative proof that is much shorter.

*Proof.* Both part 1 and part 2 follows immediately from Theorem 4.9 and the observation that

$$\left| \sum_{k=n}^m a_k \right| \leq \left| \sum_{k=n}^m b_k \right|$$

for all  $m, n \in \mathbb{N}$ .  $\square$

**Remark 4.13** — Observe that if, say,  $0 \leq a_k \leq b_k$  for all but the first 10 terms, then the conclusion of the theorem would still hold. Based on the Cauchy criterion, observe that the behavior of a series is determined only by its tails. Thus, in the comparison test, the requirement that  $a_k \leq b_k$  does not really need to hold for all  $n \in \mathbb{N}$  but just needs to be eventually true, i.e. as long as  $a_k \leq b_k$  holds for all but finitely many terms.

**Remark 4.14** — When the ratio between consecutive terms is simpler than the terms themselves, which is especially true when the terms are in product form, the following test is often more convenient to apply.

**Proposition 4.15 (Ratio comparison test)**

Assume  $0 \leq a_{k+1}/a_k \leq b_{k+1}/b_k$  for all  $k \in \mathbb{N}$ .

1. If  $\sum_{k=1}^{\infty} b_k$  converges, then  $\sum_{k=1}^{\infty} a_k$  converges.
2. If  $\sum_{k=1}^{\infty} a_k$  diverges, then  $\sum_{k=1}^{\infty} b_k$  diverges.

*Proof.* By multiplying the inequalities  $a_{i+1}/a_i \leq b_{i+1}/b_i$  from  $i = 1$  to  $i = k - 1$ , we obtain

$$\frac{a_k}{a_1} \leq \frac{b_k}{b_1} \quad \text{or} \quad a_k \leq \frac{a_1}{b_1} b_k.$$

Since the series  $\sum_{k=1}^{\infty} \frac{a_1}{b_1} b_k$  and  $\sum_{k=1}^{\infty} b_k$  have the same behavior, the result follows directly from Proposition 4.12.  $\square$

**Definition 4.16** (Geometric series)

A **geometric series** is a series of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

where  $a, r \in \mathbb{R}$ .

**Lemma 4.17**

The sequence  $(r^n)$  converges to 0 if  $r \in (-1, 1)$ , converges to 1 if  $r = 1$ , and diverges otherwise.

*Proof.*

**Case 1.** For  $r > 1$ , the sequence  $(r^n)$  is strictly increasing and unbounded. By monotone convergence (Theorem 3.26), the sequence diverges to  $\infty$ .

**Case 2.** For  $r = 1$ , we have  $r^n = 1$  for all  $n \in \mathbb{N}$ . Thus, the sequence converges to 1.

**Case 3.** For  $0 < r < 1$ , the sequence  $(r^n)$  is strictly decreasing and bounded below by 0. By monotone convergence, the sequence converges to 0.

**Case 4.** For  $r = 0$ , we have  $r^n = 0$  for all  $n \in \mathbb{N}$ . Thus, the sequence converges to 0.

**Case 5.** For  $-1 < r < 0$ , we can bound the sequence as follows:

$$-|r|^n \leq r^n \leq |r|^n.$$

Since  $0 < |r| < 1$ , by Case 3, we have  $\lim_{n \rightarrow \infty} |r|^n = 0$ . Thus,  $\lim_{n \rightarrow \infty} -|r|^n = 0$  as well. By the sequence squeeze theorem (Theorem 3.23), the sequence  $(r^n)$  converges to 0.

**Case 6.** For  $r \leq -1$ , the proof similar to the one presented in Example 3.17.  $\square$

**Proposition 4.18** (Geometric series test)

Assume  $a$  and  $r$  are non-zero real numbers. Then

$$\sum_{k=0}^{\infty} ar^k = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

*Proof.* The case where  $|r| > 1$  follows from Lemma 4.17 and Theorem 4.10. When  $r = 1$ , the series  $a + a + a + \dots$  which clearly diverges, and when  $r = -1$  the series  $a - a + a - a + a - \dots$  whose sequence of partial sums,  $(a, 0, a, 0, \dots)$ , is clearly not converging. We therefore turn to the case that  $|r| < 1$ . Note that

$$\begin{aligned} (1-r)(1+r+r^2+r^3+\dots+r^n) &= 1+r+r^2+r^3+\dots+r^n \\ &\quad -r-r^2-r^3-\dots-r^{n-1}-r^{n+1} \\ &= 1+0+0+\dots+0-r^{n+1} \\ &= 1-r^{n+1}, \end{aligned}$$

which, by dividing over the  $1 - r$ , shows that

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r},$$

and hence

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^n = \frac{a(1 - r^{n+1})}{1 - r},$$

where  $s_n$  is the  $n$ -th partial sum of  $\sum_{k=0}^{\infty} ar^k$ . Thus,

$$\begin{aligned} \sum_{k=0}^{\infty} ar^k &= \lim_{n \rightarrow \infty} s_n \\ &= \lim_{n \rightarrow \infty} \frac{a(1 - r^{n+1})}{1 - r} \\ &= \frac{a(1 - 0)}{1 - r} && \text{(by Lemma 4.17)} \\ &= \frac{a}{1 - r}. \end{aligned}$$

□

**Theorem 4.19 (Harmonic series diverges)**

The harmonic series  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges.

*Proof.* Observe that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &\geq 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \cdots \end{aligned}$$

In particular, if  $s_n$  is the  $n$ -th partial sum of the harmonic series, then  $(s_n)$  is monotonically increasing and, by the above,

$$s_{2n} \geq 1 + n \cdot \frac{1}{2}.$$

And since  $(1 + n \cdot \frac{1}{2})$  diverges to  $\infty$ , the comparison test (Proposition 4.12) the subsequence  $(s_{2n})$  diverges to  $\infty$ . And for a monotonically increasing sequence, if a subsequence diverges to  $\infty$ , implying that the entire sequence  $(s_n)$  is unbounded, then  $(s_n)$  is also diverging to  $\infty$  by the monotone convergence theorem (Theorem 3.26). That is, the harmonic series diverges to  $\infty$ . □

Here's another proof that I particularly enjoy.

*Proof.* Suppose for a contradiction that the harmonic series diverges converges to  $S$ . Then

$$\begin{aligned}
 S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\
 &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \dots \\
 &> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \dots \\
 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\
 &= S.
 \end{aligned}$$

We've shown  $S > S$ , a contradiction.  $\square$

**Proposition 4.20** (The  $p$ -series test)

The series  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  converges if and only if  $p > 1$ .

*Proof.*

( $\Rightarrow$ ) If  $p \leq 1$ , then

$$\frac{1}{k} \leq \frac{1}{k^p} \quad \text{for all } k \in \mathbb{N}.$$

And since  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges, by the comparison test (Proposition 4.12),  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  diverges too.

( $\Leftarrow$ ) Now assume that  $p > 1$ . Then

$$\begin{aligned}
 \sum_{k=1}^{\infty} \frac{1}{k^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \dots \\
 &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots \\
 &< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \dots \\
 &= 1 + \left(\frac{2}{2^p}\right) + \left(\frac{4}{4^p}\right) + \dots \\
 &= 1 + \left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{4^{p-1}}\right) + \dots \\
 &= 1 + \left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{2^{2(p-1)}}\right) + \dots \\
 &= 1 + \left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{2^{p-1}}\right)^2 + \dots \\
 &= 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2^p}\right)^k.
 \end{aligned}$$

Finally, note that  $\sum_{k=0}^{\infty} \left(\frac{1}{2^p}\right)^k$  is a geometric series with  $|r| = \frac{1}{2^p} < 1$ , so by the geometric series test (Proposition 4.18), this series converges. And the above also shows that, term-by-term, this geometric series is larger than  $\sum_{k=1}^{\infty} \frac{1}{k^p}$ . So by the comparison test (Proposition 4.12),  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  converges too.  $\square$

**Example 4.21**

The series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

**Remark 4.22** — The ratio comparison test yields two simple and powerful tests that are closely related. Both tests work exclusively with the terms of the given series—they require neither an initial guess about convergence nor the discovery of a series for comparison.

**Proposition 4.23 (Ratio test)**

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ . Define

$$r = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

1. If  $r < 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $r > 1$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges.
3. If  $r = 1$ , the test is inconclusive.

**Proposition 4.24 (Root test)**

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ . Define

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}.$$

1. If  $\rho < 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\rho > 1$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges.
3. If  $\rho = 1$ , the test is inconclusive.

**Example 4.25**

Let  $a$  and  $b$  be distinct positive numbers. Determine the convergence of

$$1 + a + ab + a^2b + a^2b^2 + \cdots + a^n b^{n-1} + a^n b^n + \cdots$$

**Definition 4.26 (Strength of convergence tests)**

We consider the strength of convergence tests to be its ability to yield a result. That is, for one convergence test  $A$  to be **stronger than** another test  $B$ , the weaker test  $B$  giving a result implies that the stronger test  $A$  also yields a result. It must also be true that there exists a case where the stronger convergence test  $A$  works but the weaker test  $B$  does not.

**Proposition 4.27** (Root test is stronger than ratio test)

Let  $(a_n)$  be a positive sequence. Suppose that  $r$  and  $\rho$  are given by Proposition 4.23 and Proposition 4.24, respectively. If  $r$  exists, then  $\rho$  exists and

$$r = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho.$$

The converse, however, is not necessarily true.

**Proposition 4.28** (Alternating series test)

Assume that  $(a_k)$  is a monotonically decreasing sequence and  $a_k \rightarrow 0$ . Then  $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$  converges.

**Example 4.29** (Hardy [Ha49])

The alternating series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n}$$

diverges.

**Proposition 4.30** (Absolute convergence test)

If  $\sum_{k=1}^{\infty} |a_k|$  converges, then  $\sum_{k=1}^{\infty} a_k$  converges too.

**Definition 4.31** (Absolute convergence)

A series  $\sum_{k=1}^{\infty} a_k$  is said to **converge absolutely** if  $\sum_{k=1}^{\infty} |a_k|$  converges.

**Definition 4.32** (Conditional convergence)

A series  $\sum_{k=1}^{\infty} a_k$  is said to **converge conditionally** if it converges but does not converge absolutely.

**Example 4.33**

Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}.$$

converges.



**Remark 4.34** — Historically, the most interesting series mathematicians were encountering in the early 19th Century all fell into the inconclusive category of the above tests. To overcome the principal drawback of these tests, various sophisticated and finer tests were developed. For example, Gauss’s test (Proposition 4.50) completely sets the convergence for all hypergeometric series. Kummer’s test (Proposition 4.44) gives characterizations for convergence or divergence of all positive series. Cauchy’s condensation test (Proposition 4.53) provides an approach to study the convergence or divergence of a class of series via a rather “thin” subsequence. To demonstrate the progression of these sophisticated and finer tests, we will first introduce the Raabe (Proposition 4.35) and logarithmic (Proposition 4.41) tests. Then, we present the Kummer (Proposition 4.44) and Gauss (Proposition 4.50) tests. For series with monotonically decreasing terms, we formulate a few simple and interesting tests based on Cauchy’s condensation test (Proposition 4.53). Finally, we show that no single test can determine convergence or divergence for all positive series.

**Proposition 4.35 (Raabe’s test)**

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ . Define

$$\mathcal{R}_n = n \left( 1 - \frac{a_{n+1}}{a_n} \right).$$

1. If  $\mathcal{R}_n \geq r > 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\mathcal{R}_n \leq 1$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges.

**Proposition 4.36 (Raabe’s limit test)**

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ . Define

$$\mathcal{R} = \lim_{n \rightarrow \infty} \mathcal{R}_n = \lim_{n \rightarrow \infty} n \left( 1 - \frac{a_{n+1}}{a_n} \right).$$

1. If  $\mathcal{R} > 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\mathcal{R} < 1$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges.
3. If  $\mathcal{R} = 1$ , the test is inconclusive.

**Example 4.37**

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}}}$$

converges.

**Example 4.38**

Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{n}{e} \right)^n.$$

**Proposition 4.39** (Raabe's limit test corollary)

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ . Define

$$\mathcal{R}^* = \lim_{n \rightarrow \infty} \left( \frac{a_{n+1}}{a_n} \right)^n.$$

1. If  $\mathcal{R}^* < 1/e$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\mathcal{R}^* > 1/e$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges.
3. If  $\mathcal{R}^* = 1/e$ , the test is inconclusive.

**Example 4.40**

Using Proposition 4.39, show that the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}}}$$

converges.

**Proposition 4.41** (Logarithmic test)

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ . Define

$$\mathcal{L}_n = \frac{\ln(1/a_n)}{\ln n}.$$

1. If  $\mathcal{L}_n \geq 1 > 1$ , the series converges.
2. If  $\mathcal{L}_n \leq 1$ , the series diverges.

**Example 4.42**

Let  $p > 1$ . Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}.$$

**Proposition 4.43** (Logarithmic test is stronger than Raabe's test)

Let  $(a_n)$  be a positive sequence. Suppose that  $\mathcal{R}_n$  and  $\mathcal{L}_n$  are given by Proposition 4.35 and Proposition 4.41, respectively. If  $\lim_{n \rightarrow \infty} \mathcal{R}_n$  exists, finite or infinite, then  $\lim_{n \rightarrow \infty} \mathcal{L}_n$  exists and

$$\lim_{n \rightarrow \infty} \mathcal{R}_n = \lim_{n \rightarrow \infty} \mathcal{L}_n.$$

The converse, however, is not necessarily true.

**Proposition 4.44** (Kummer's test)

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ .

1. The series  $\sum_{n=1}^{\infty} a_n$  converges if there is a positive sequence  $(c_n)$  and some constant  $k > 0$  such that

$$\mathcal{K}_n = c_n - \frac{a_{n+1}}{a_n} c_{n+1} \geq k.$$

2. The series  $\sum_{n=1}^{\infty} a_n$  diverges if there is a positive sequence  $(c_n)$  such that  $\sum_{n=1}^{\infty} 1/c_n$  diverges and  $\mathcal{K}_n \leq 0$ .

**Proposition 4.45** (Kummer's limit test)

Assume  $a_n > 0$  for all  $n \in \mathbb{N}$ . For a positive sequence  $(a_n)$ , define

$$\kappa = \lim_{n \rightarrow \infty} \left( c_n - \frac{a_{n+1}}{a_n} c_{n+1} \right).$$

1. If  $\kappa > 0$ , then the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\kappa < 0$  and  $\sum_{n=1}^{\infty} 1/c_n$  diverges, then the series  $\sum_{n=1}^{\infty} a_n$  diverges.
3. If  $\kappa = 0$ , the test is inconclusive.

**Remark 4.46** — The drawback of Kummer's test is the choice of  $c_n$ , which has to come from the user, and usually there is no particular motivation. Thus, using Kummer's test is equally as difficult as using the comparison test.

However, in contrast to all tests we have studied, Kummer's test does not merely give very powerful sufficient conditions for convergence or divergence of a positive series. Surprisingly, the conditions (1) and (2) in the test are indeed necessary. This fact is not well publicized.

**Theorem 4.47** (Kummer's characterization of convergent series)

If the series  $\sum_{n=1}^{\infty} a_n$  converges, then there exists a positive sequence  $(c_n)$  for which

$$\mathcal{K}_n = c_n - \frac{a_{n+1}}{a_n} c_{n+1} \geq 1.$$

**Theorem 4.48** (Kummer's characterization of divergent series)

If the series  $\sum_{n=1}^{\infty} a_n$  diverges, then there exists a positive sequence  $(c_n)$  for which  $\sum_{n=1}^{\infty} 1/c_n$  diverges and

$$\mathcal{K}_n = c_n - \frac{a_{n+1}}{a_n} c_{n+1} \leq 0.$$

**Proposition 4.49** (Bertrand's test)

Define

$$\mathcal{B}_n = \ln n(\mathcal{R}_n - 1) = \ln n \left( n \left( 1 - \frac{a_{n+1}}{a_n} \right) - 1 \right).$$

Suppose that

$$\mathcal{B} = \lim_{n \rightarrow \infty} \mathcal{B}_n.$$

1. If  $\mathcal{B} > 1$ , then the positive series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\mathcal{B} < 1$ , then the positive series  $\sum_{n=1}^{\infty} a_n$  diverges.

**Proposition 4.50** (Gauss's test)

Let  $\sum_{n=1}^{\infty} a_n$  be a positive series. Suppose

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{\theta_n}{n^k}$$

where  $\theta_n$  is a bounded sequence and  $k > 1$ .

1. If  $\mu > 1$ , the series converges.
2. If  $\mu \leq 1$ , the series diverges.

**Example 4.51**

Check the convergence of the *hypergeometric series*

$$F(\alpha, \beta, \gamma) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1) \dots (\alpha+n-1) \beta(\beta+1) \dots (\beta+n-1)}{n! \gamma(\gamma+1) \dots (\gamma+n-1)}.$$

**Example 4.52**

Consider the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

using Gauss's test.

**Proposition 4.53** (Cauchy's condensation test)

Let  $(a_n)$  be a monotonically decreasing sequence where  $a_n > 0$  for all  $n \in \mathbb{N}$ . Then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} 2^n a_{2^n}$  converges.

**Proposition 4.54** (Generalization of Cauchy's condensation test)

Let  $(a_n)$  be a monotonically decreasing sequence where  $a_n > 0$  for all  $n \in \mathbb{N}$ . If there is a strictly increasing sequence  $1 \leq n_1 < n_2 < \dots$  such that  $(n_{k+1} - n_k)/(n_k - n_{k-1})$  is bounded as a function of  $k$ , then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{k=1}^{\infty} (n_{k+1} - n_k) a_{n_k}$  converges.

**Example 4.55**

Consider the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

using Cauchy's condensation test.

**Proposition 4.56**

Let  $(a_n)$  be a monotonically decreasing sequence where  $a_n > 0$  for all  $n \in \mathbb{N}$ . Define

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{2n}}{a_n}.$$

1. If  $\rho < 1/2$ , then the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\rho > 1/2$ , then the series  $\sum_{n=1}^{\infty} a_n$  diverges.
3. If  $\rho = 1/2$ , the test is inconclusive.

**Example 4.57**

Let  $p > 0$ . Study the convergence of

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}.$$

**Definition 4.58** (Rates of convergence and divergence)

Consider two positive series

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^*.$$

1. Suppose both series converge. Let

$$R_n = \sum_{i=n+1}^{\infty} a_i \quad \text{and} \quad R_n^* = \sum_{i=n+1}^{\infty} a_i^*.$$

We say that  $\sum_{n=1}^{\infty} a_n^*$  **converges more slowly** than  $\sum_{n=1}^{\infty} a_n$  if

$$\lim_{n \rightarrow \infty} \frac{R_n}{R_n^*} = 0.$$

2. Suppose both series diverge. Let

$$S_n = \sum_{i=1}^n a_i \quad \text{and} \quad S_n^* = \sum_{i=1}^n a_i^*.$$

We say that  $\sum_{n=1}^{\infty} a_n^*$  **diverges more slowly** than  $\sum_{n=1}^{\infty} a_n$  if

$$\lim_{n \rightarrow \infty} \frac{S_n^*}{S_n} = 0.$$

**Lemma 4.59**

Let  $\sum_{n=1}^{\infty} a_n$  be a positive series.

1. If  $\sum_{n=1}^{\infty} a_n$  converges, then there is a monotone sequence  $(b_n)$  such that  $\lim_{n \rightarrow \infty} b_n = \infty$  and the series  $\sum_{n=1}^{\infty} a_n b_n$  converges.
2. If  $\sum_{n=1}^{\infty} a_n$  diverges, then there is a monotone sequence  $(b_n)$  such that  $\lim_{n \rightarrow \infty} b_n = 0$  and the series  $\sum_{n=1}^{\infty} a_n b_n$  diverges.

**Theorem 4.60** (Nonexistence of the universal test)

There is no universal comparison test that can decide the convergence or divergence of all positive series.

**Remark 4.61** — Under the absolute convergence test (Proposition 4.30), all of the preceding tests for positive series become tests for absolute convergence of general series. But none of the convergence tests that we have examined so far are applicable to a series that is not alternating and does not converge absolutely. We now present

two tests that are applicable to the general series of the form

$$\sum_{n=1}^{\infty} a_n b_n = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots$$

Both tests are based on the following algebraic manipulation, which is often called the *Abel's summation formula*.

**Theorem 4.62** (Abel's summation formula)

Let  $(a_n)$  and  $(b_n)$  be arbitrary sequences. Define  $(s_n)$  to be the sequence of partial sums of  $(a_n)$ . That is,  $s_n = \sum_{k=1}^n a_k$ . Then

$$\sum_{k=1}^n a_k b_k = \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}) + s_n b_n.$$

**Proposition 4.63** (Abel's test)

Let  $\sum_{n=1}^{\infty} a_n$  be a convergent series, and let  $(b_n)$  be a bounded and monotone sequence. Then  $\sum_{n=1}^{\infty} a_n b_n$  converges.

**Proposition 4.64** (Dirichlet's test)

Let  $\sum_{n=1}^{\infty} a_n$  be a bounded series, and let  $(b_n)$  be a positive and monotonically decreasing sequence with  $\lim_{n \rightarrow \infty} b_n = 0$ . Then  $\sum_{n=1}^{\infty} a_n b_n$  converges.

**Example 4.65**

Determine the convergence of

$$\sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \cos(nx)$$

where  $(b_n)$  is positive, monotonically decreasing, and converges to zero.

**Proposition 4.66** (Comparison test for general series)

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  satisfy

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c_n \text{ and } \frac{a_n}{b_n} \text{ is monotone.}$$

Then

1. If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $c \neq 0$ , then both series are either both divergent, both conditionally convergent, or both absolutely convergent.

**Theorem 4.67**

The associative law holds for convergent series.

**Definition 4.68 (Rearrangements of series)**

A **rearrangement** of a series  $\sum_{n=1}^{\infty} a_n$  is a series  $\sum_{n=1}^{\infty} b_n$  for which there is a bijection  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $a_n = b_{f(n)}$ .

**Theorem 4.69**

If the series  $\sum_{n=1}^{\infty} a_n$  converges absolutely, then any rearrangement of this series converges to the same limit.

**Corollary 4.70**

The commutative law holds for absolutely convergent series.

**Theorem 4.71 (Riemann series theorem)**

If the series  $\sum_{n=1}^{\infty} a_n$  converges conditionally, then there exists a rearrangement of this series that either converges to any arbitrary real number or diverges to  $\pm\infty$ .

**Definition 4.72 (Finite decimal representations of real numbers)**

A real number of the form

$$x = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n},$$

where  $a_0$  is a nonnegative integer and  $a_1, \dots, a_n$  are integers satisfying  $0 \leq a_i \leq 9$  and is usually written more briefly as

$$x = a_0.a_1a_2 \dots a_n.$$

This is said to be the **finite decimal representation** of  $x$ .

**Proposition 4.73**

Real numbers with a finite decimal representation are necessary rational.

*Proof.* Trivial. □



**Proposition 4.74** (Finite decimal approximations of real numbers)

Assume  $x$  is a nonnegative real number.

1. Then there is a finite decimal  $r_n = a_0.a_1a_2 \dots a_n$  such that

$$r_n \leq x < r_n + \frac{1}{10^n} \quad \text{for all } n \in \mathbb{N}.$$

2. Then there is a finite decimal  $s_n = b_0.b_1b_2 \dots b_n$  such that

$$s_n < x \leq s_n + \frac{1}{10^n} \quad \text{for all } n \in \mathbb{N}.$$

*Proof.* We proceed with a proof of part 1. Let  $S$  be the set of nonnegative integers less than or equal to  $x$ . Then  $S$  is nonempty since  $0 \in S$ , and  $S$  is bounded above by  $x$ . By the completeness of  $\mathbb{R}$ ,  $S$  has a supremum, which we denote as  $a_0 = \sup(S)$ . Since  $S$  contains only integers and has  $a_0$  as its least upper bound,  $a_0$  must itself be an integer in  $S$ . That is,  $a_0 = \lfloor x \rfloor$ . This gives us our first approximation:

$$a_0 \leq x < a_0 + 1. \quad (1)$$

For the first decimal digit, we define  $a_1 = \lfloor 10x - 10a_0 \rfloor$ . This gives us

$$a_1 \leq 10x - 10a_0 < a_1 + 1. \quad (2)$$

From (1) and (2), we can deduce that  $0 \leq a_1 \leq 9$ , confirming that  $a_1$  is indeed a single decimal digit. Rearranging (2) gives us our second approximation:

$$a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1}{10} + \frac{1}{10}.$$

This process extends naturally to subsequent decimal places. After selecting digits  $a_1, a_2, \dots, a_{n-1}$  with  $0 \leq a_i \leq 9$  for each  $i$ , we define

$$a_n = \lfloor 10^n x - (10^n a_0 + 10^{n-1} a_1 + \dots + 10 a_{n-1}) \rfloor.$$

This gives us

$$a_n \leq 10^n x - (10^n a_0 + 10^{n-1} a_1 + \dots + 10 a_{n-1}) < a_n + 1. \quad (3)$$

Rearranging (3) gives us the following approximation:

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_n}{10^n} \leq x < a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_n}{10^n} + \frac{1}{10^n}.$$

By construction, observe that  $0 \leq a_n \leq 9$  for each  $n$ , and so we have

$$r_n \leq x < r_n + \frac{1}{10^n},$$

where  $r_n = a_0.a_1a_2 \dots a_n$  represents our finite decimal approximation of  $x$  with  $n$  decimal places. This completes the proof of part 1.

The proof of part 2 is quite similar. □

**Theorem 4.75** (Infinite decimal representations of real numbers)

Every real number admits at least one infinite decimal representation.

*Proof.* First, let's consider any nonnegative real number  $x$ . By Proposition 4.74, for each  $n \in \mathbb{N}$ , there exists a finite decimal  $r_n = a_0.a_1a_2 \dots a_n$  such that

$$r_n \leq x < r_n + \frac{1}{10^n},$$

where  $a_0 = \lfloor x \rfloor$  and each  $a_i$  ( $1 \leq i \leq n$ ) is a digit between 0 and 9.

Let's define the infinite decimal representation  $r = a_0.a_1a_2a_3 \dots$  as the series

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \dots = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{10^i}.$$

We claim that  $r = x$ . To prove this, we need to show that  $|r - x| = 0$ .

Choose an arbitrary natural number  $N$ . By our construction, we have

$$r_N \leq x < r_N + \frac{1}{10^N}. \quad (1)$$

Since all digits  $a_i$  are between 0 and 9, we have

$$\begin{aligned} r - r_N &= \sum_{i=N+1}^{\infty} \frac{a_i}{10^i} \leq \sum_{i=N+1}^{\infty} \frac{9}{10^i} \\ &= \frac{9}{10^{N+1}} \cdot \frac{1}{1 - \frac{1}{10}} \quad (\text{by Proposition 4.18}) \\ &= \frac{1}{10^N} \end{aligned}$$

which implies that

$$r_N \leq r \leq r_N + \frac{1}{10^N}. \quad (2)$$

Combining (1) and (2) gives us

$$-\frac{1}{10^N} \leq r - x \leq \frac{1}{10^N} \quad \text{or} \quad |r - x| \leq \frac{1}{10^N}.$$

Since  $N$  was arbitrary, by the Archimedean principle (Lemma 1.27), we must have  $|r - x| = 0$ , which means that  $r = x$ .

For negative real numbers, we simply prepend a negative sign to obtain  $-a_0.a_1a_2a_3 \dots$  where the  $a_i$  are determined from  $|x|$  as above.  $\square$

**Theorem 4.76** ( $0.999 \dots = 1$ )

$0.999 \dots$  is an infinite decimal representation of the real number 1.

*Proof.* Consider the real number  $x = 1$ . By Proposition 4.74, for each  $n \in \mathbb{N}$ , there exists a finite decimal  $s_n = b_0.b_1b_2 \dots b_n$  such that

$$s_n < x \leq s_n + \frac{1}{10^n}$$

For  $x = 1$ , we can identify that  $b_0 = 0$  and  $b_i = 9$  for all  $i \geq 1$ . Thus,

$$s_n = 0.\underbrace{99 \dots 9}_{n \text{ times}} < 1 \leq 0.\underbrace{99 \dots 9}_{n \text{ times}} + \frac{1}{10^n}.$$

Define the infinite decimal  $s = 0.999 \dots$  as the series

$$s = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots = \sum_{i=1}^{\infty} \frac{9}{10^i}.$$

Choose an arbitrary natural number  $N$ . We have

$$s_N < 1 \leq s_N + \frac{1}{10^N} \tag{1}$$

and

$$s - s_N = \sum_{i=N+1}^{\infty} \frac{9}{10^i} = \frac{9}{10^{N+1}} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{1}{10^N}. \tag{2}$$

Combining (1) and (2), we get

$$s_N < 1 \leq s_N + \frac{1}{10^N} = s.$$

Since  $N$  was arbitrary, we have  $s \geq 1$ . Obviously,  $s \leq 1$  as well, hence  $s = 1$ .  $\square$

**Remark 4.77** — Many struggle to accept that  $0.999 \dots = 1$ . Some believe numbers must have unique decimal representations, making two different-looking expressions for the same value seem contradictory. Others misinterpret the notion of limits, viewing  $0.999 \dots$  as an ongoing process that forever approaches 1 without reaching it, rather than as the fixed value that is the limit. These misconceptions stem from intuitive but imprecise understandings of infinity and limits. Within the standard definition of real numbers,  $0.999 \dots$  represents the limit of some sequence and equals exactly 1—not approximately, not infinitely close, but identical in value.

# 5 Basic Topology of $\mathbb{R}$

“If geometry is dressed in a suit coat, topology dons jeans and a T-shirt.”

David S. Richeson, *Euler's Gem: The Polyhedron Formula and the Birth of Topology*

## Definition 5.1 (Open sets)

A set  $O \subseteq \mathbb{R}$  is **open** if for all points  $x \in O$  there exists an  $\epsilon$ -neighborhood  $V_\epsilon(x) \subseteq O$ .

## Example 5.2

Below are some examples and some non-examples of open sets.

1. The set  $\mathbb{R}$  is open.
2. The empty set  $\emptyset$  is open.
3. The open interval  $(a, b)$  is open.
4. The intervals  $(a, \infty)$  and  $(-\infty, b)$  are open.
5. The closed interval  $[a, b]$  is *not* open.

## Proposition 5.3

1. The union of an arbitrary collection of open sets is open.
2. The intersection of a finite collection of open sets is open.

*Proof.* To prove part 1, we let  $\{O_\lambda : \lambda \in \Lambda\}$  be a collection of open sets and let  $O = \bigcup_{\lambda \in \Lambda} O_\lambda$ . Let  $x$  be an arbitrary element of  $O$ . In order to show that  $O$  is open, Definition 5.1 insists that we produce an  $\epsilon$ -neighborhood of  $x$  completely contained in  $O$ . Since  $x \in O$ , we know that  $x \in O_\lambda$  for some  $\lambda \in \Lambda$ . By definition of  $O_\lambda$  being open, this implies that there is an  $\epsilon$ -neighborhood  $V_\epsilon(x) \subseteq O_\lambda$ . The fact that  $O_\lambda \subseteq O$  allows us to conclude that  $V_\epsilon(x) \subseteq O$ . This completes the proof of part 1.

For part 2, let  $\{O_1, O_2, \dots, O_N\}$  be a finite collection of open sets. Now, if  $x \in \bigcup_{k=1}^N O_k$ , then  $x$  is an element of each of the open sets. By the definition of an open set, we know that, for each  $1 \leq k \leq N$ , there exists  $V_{\epsilon_k}(x) \subseteq O_k$ . We are in search of a single  $\epsilon$ -neighborhood of  $x$  that is contained in every  $O_k$ , so the trick is to take the smallest one. Letting  $\epsilon = \min(\epsilon_1, \epsilon_2, \dots, \epsilon_N)$ , it follows that  $V_\epsilon(x) \subseteq V_{\epsilon_k}(x)$  for all  $k$ , and hence  $V_\epsilon(x) \subseteq \bigcap_{k=1}^N O_k$ , as desired.  $\square$

## Example 5.4

Some more examples of open sets.

1. All intervals  $(a, b)$  are open, so by Proposition 5.3,  $(a, b) \cup (c, d)$  is open,  $(a, b) \cup (c, d) \cup (e, f)$  is open,  $(a_1, b_1) \cup (a_2, b_2) \cup \dots$  is open, and any other union of open intervals is open.
2. Likewise,  $(a, b) \cap (c, d)$  is open.

### Theorem 5.5 (Characterization of open sets)

Every open set is a countable union of disjoint open intervals.

*Proof.* Let  $O$  be an open set. Then by Definition 5.1, each  $x \in O$  is contained inside some open interval  $(x - \epsilon, x + \epsilon) \subseteq O$ . There may be a larger open interval that contains  $x$ , though, so let  $I_x \subseteq O$  be the largest open interval containing  $x$ . Formally,  $I_x = (\alpha, \beta)$  where

$$\alpha = \inf\{a : (a, x) \subseteq O\} \quad \text{and} \quad \beta = \sup\{b : (x, b) \subseteq O\}.$$

Note that, given any pair  $x$  and  $y$ , either  $I_x = I_y$  or  $I_x \cap I_y = \emptyset$ . The reason for this is that, if instead you had two intervals overlapping but not equal, then you could expand at least one of them, contradicting the fact that we had chosen the largest intervals.

We claim that these  $I_x$  intervals are the ones we are looking for. To see this, first note that since each  $x \in O$  is in  $I_x \subseteq O$ , the union of all  $I_x$  intervals does indeed equal to  $O$ . Moreover, there are only countably many such disjoint intervals due to Theorem 2.16. This completes the proof.  $\square$

### Definition 5.6 (Limit points)

A point  $x$  is a **limit point** of a set  $A$  if and only if every  $\epsilon$ -neighborhood  $V_\epsilon(x)$  of  $x$  intersects the set  $A$  at some point other than  $x$ .

### Theorem 5.7

A point  $x$  is a limit point of a set  $A$  if and only if there is a sequence  $(a_n)$  contained in  $A \setminus \{x\}$  such that  $a_n \rightarrow x$ .

*Proof.*

( $\Rightarrow$ ) Assume  $x$  is a limit point of  $A$ . In order to produce a sequence  $(a_n)$  converging to  $x$ , we are going to consider the particular  $\epsilon$ -neighborhoods obtained using  $\epsilon = 1/n$ . By Definition 5.6, every neighborhood of  $x$  intersects  $A$  in some point other than  $x$ . This means that, for each  $n \in \mathbb{N}$ , we are justified in picking a point

$$a_n \in V_{1/n}(x) \cap A$$

with the stipulation that  $a_n \neq x$ . It should not be too difficult to see why  $a_n \rightarrow x$ . Given an arbitrary  $\epsilon > 0$ , choose  $N$  such that  $1/N \leq \epsilon$ . It follows that  $|a_n - x| < \epsilon$  for all  $n > N$ .

( $\Leftarrow$ ) We assume  $a_n \rightarrow x$  where  $a_n \in A \setminus \{x\}$ , and let  $V_\epsilon(x)$  be an arbitrary  $\epsilon$ -neighborhood. The definition of convergence assures us that there exists a term  $a_N$  in the sequence satisfying  $a_N \in V_\epsilon(x)$ , and the proof is complete.  $\square$

**Remark 5.8** — The restriction that an  $a_n \neq x$  in Theorem 5.7 deserves a comment. Given a point  $a \in A$ , it is always the case that  $a$  is the limit of a sequence in  $A$  if we are allowed to consider the constant sequence  $(a, a, a, \dots)$ . There will be occasions where we will want to avoid this somewhat uninteresting situation, so it is important to have a vocabulary that can distinguish limit points of a set from *isolated points*.

**Definition 5.9** (Isolated points)

A point  $a \in A$  is an **isolated point** of  $A$  if it is not a limit point of  $A$ .

**Definition 5.10** (Closed sets)

A set  $F \subseteq \mathbb{R}$  is **closed** if it contains its limit points.

**Theorem 5.11** (Characterization of closed sets)

A set  $F \subseteq \mathbb{R}$  is closed if and only if every Cauchy sequence contained in  $F$  has a limit that is also an element of  $F$ .

# A Pathological Examples

“One of my favorite memories of studying was one night when analysis crept into my dreams. I woke up in a panicky cold sweat. In my dream I was being chased by some analysis monster. My only defense was to use the *blancmange function* as a boomerang. I took it as a good sign at the time that analysis concepts were finding their way into my subconscious.”

---

*Tina Rapke, The View from Here: Confronting Analysis*

It has been said that one of the most important goals of learning real analysis is to collect as many bizarre examples as you can, and to keep them in your back pocket. From a practical standpoint they will inform your conjectures and guide your proofs, but they will also help to demonstrate why real analysis is such a great subject.

## §A.1 An infinite field that cannot be ordered

To say that a field  $F$  cannot be ordered is to say that it possesses no positive subset  $P$  satisfying the order axiom (Definition 1.6). A preliminary comment is that since every ordered field is infinite, no finite field can be ordered.

An example of an *infinite* field that cannot be ordered is the field  $\mathbb{C}$  of complex numbers.

*Proof.* To show that this is the case, assume that there does exist a positive subset  $P$  of  $\mathbb{C}$ . Consider the element  $i \in \mathbb{C}$ , where  $i$  is the imaginary unit. Since  $i \neq 0$ , there are two alternative possibilities. The first is  $i \in P$  (i.e.  $i$  is a positive element), in which case  $i^2 = -1 \in P$  and  $i^4 = 1 \in P$ . Since  $i^2$  and  $i^4$  are additive inverses of each other, and since it is impossible for two additive inverses both to belong to  $P$ , we have obtained a contradiction. The other option is  $-i \in P$ , in which case  $(-i)^2 = -1 \in P$  and  $(-i)^4 = 1 \in P$ . We get the same contradiction as before.  $\square$

# Bibliography

- [Ab15]       STEPHEN ABBOTT. *Understanding analysis*. 2nd ed. Springer, 2015
- [AmEsBr05]   HERBERT AMANN, JOACHIM ESCHER, and GARY BROOKFIELD. *Analysis I*. Springer, 2005
- [Ap74]       TOM M APOSTOL. *Mathematical analysis*. 2nd ed. Addison-Wesley, 1974
- [BaSh11]     ROBERT G BARTLE and DONALD R SHERBERT. *Introduction to real analysis*. 4th ed. John Wiley & Sons, Inc., 2011
- [Ch22]       HONGWEI CHEN. *Classical Analysis: An Approach Through Problems*. Chapman and Hall/CRC, 2022
- [Cu19]       JAY CUMMINGS. *Real analysis: a long-form mathematics textbook*. 2nd ed. CreateSpace Independent Publishing Platform, 2019
- [GeOl03]     BERNARD R GELBAUM and JOHN MH OLMSTED. *Counterexamples in analysis*. Dover Publications, Inc., 2003
- [Ha49]       GODFREY H HARDY. *Divergent series*. Clarendon Press, 1949 (cited p. 64)