A Laundry List of Theorems in Analysis (Dark Souls Edition)

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Preface

These notes are a loose amalgamation of ideas, concepts, and explanations drawn from various sources, particularly the following books:

1. Real Analysis: A Long-Form Mathematics Textbook by Jay Cummings [Cu19]
2. Understanding Analysis by Stephen Abbott [Ab15]
3. Introduction to Real Analysis by Bartle & Sherbert [BaSh11]
4. Mathematical Analysis by Tom M. Apostol [Ap74]
5. Counterexamples in Analysis by Gelbaum & Olmsted [GeOl03]

They were originally meant for my own understanding and organization of thoughts, and as such, they may be unpolished, incomplete, or even occasionally incorrect.

I share them in the hope that they may serve as a useful reference, but they should not be treated as a primary source of learning. Readers are strongly encouraged to consult original texts and authoritative resources for a more rigorous and accurate treatment of the topics discussed.

Use these notes as a companion to your studies, not as a substitute for the depth and clarity provided by well-established literature.

Warning: these notes constitute the *Dark Souls Edition*, in which the proofs, long lost to time, are left for the reader to uncover.

Contents

1 The Real Numbers

> "It is a 'simple' theorem, simple both in idea and execution, but there is no doubt at all about it being a theorem of the highest class. It is as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on it."
>
> *G. H. Hardy, A Mathematician's Apology*

Theorem 1.1 (The irrationality of $\sqrt{2}$)

There is no rational number whose square is 2.

Proof. [redacted]

Remark 1.2 — So the rationals aren't quite enough. That said, they do have almost every other fundamental property we would want. To the point: They are what we call an *ordered field*. But first, what's a *field*? It's a set that satisfies the classic additive and multiplicative properties we know and love.

Definition 1.3 (Fields)

A **field** is a nonempty set $\mathbb{F}$, along with two binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If $a, b \in \mathbb{F}$, then $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Axiom 2 (Distributive Law).** If $a, b \in \mathbb{F}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
- **Axiom 3 (Associative Law).** If $a, b \in \mathbb{F}$, then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Axiom 4 (Identity Law).** There are special elements $\mathbf{0}, \mathbf{1} \in \mathbb{F}$, where $a + \mathbf{0} = a$ and $a \cdot \mathbf{1} = a$ for all $a \in \mathbb{F}$.
- **Axiom 5 (Inverse Law).** For each $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = \mathbf{0}$. If $a \neq \mathbf{0}$, then there is also an element $a^{-\mathbf{1}} \in \mathbb{F}$ such that $a \times a^{-\mathbf{1}} = \mathbf{1}$.

Example 1.4

Below are some examples and some non-examples of fields.

- The natural number $\mathbb{N}$ do not form a field; they fail the first half of Axiom 4 and both halves of Axiom 5.
- The integers $\mathbb{Z}$ *almost* form a field; they only fail the second half of Axiom 5.
- One can check that the rationals $\mathbb{Q}$ form a field.

Remark 1.5 — Now, let's come up with a definition for an ordered field. Think about $\mathbb{Q}$. What does $\mathbb{Q}$ have that a field does not? There are three main properties we are missing: First, there are infinitely many rationals (and they are "symmetric" about the 0 element.) Second, the rationals have an ordering to them. Lastly, we would like to talk about how big a number is. Beautifully, Definition 1.6 describes a single elegant axiom that we can include to capture *all* of these properties.

Definition 1.6 (Ordered Fields)

An **ordered field** is a field $\mathbb{F}$, along with the following additional axiom.

Axiom 6 (Order Axiom). There is a nonempty subset $P \subseteq \mathbb{F}$, called the *positive elements*, such that

1. If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;
2. If $a \in \mathbb{F}$ and $a \neq \mathbf{0}$, then either $a \in P$ or $-a \in P$, but not both.

Definition 1.7 (Inequalities)

If $\mathbb{F}$ is an ordered field and $a, b \in \mathbb{F}$, then we say that "$a < b$" if $b - a \in P$. Likewise, $a \leq b$ means that either $a = b$ or $a < b$.

We define "$>$" similarly.

Fact 1.8 (Properties of inequalities)

For a, b, c in an ordered field $\mathbb{F}$:

1. If $a < b$, then $a + c < b + c$.
2. Transitivity: If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$, then $ac < bc$ if $c > 0$, and $ac > bc$ if $c < 0$.
4. If $a \neq 0$, then $a^2 > 0$.

Definition 1.9 (The absolute value function)

If $\mathbb{F}$ is an ordered field, define the *absolute value* function $|\cdot| : \mathbb{F} \to \mathbb{F}$ to be

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Fact 1.10 (Properties of absolute values)

For a, b in an ordered field $\mathbb{F}$:

1. $|a| \geq 0$, with equality if and only if $a = 0$.
2. $|a| = |-a|$.
3. $-|a| \leq a \leq |a|$.
4. $|a \cdot b| = |a| \cdot |b|$.
5. $1/|a| = |1/a|$, if $a \neq 0$.
6. $|a/b| = |a|/|b|$, if $b \neq 0$.
7. $|a| \leq b$ if and only if $-b \leq a \leq b$.

Theorem 1.11 (The triangle inequality)

If $\mathbb{F}$ is an ordered field and if $x, y \in \mathbb{F}$, then

$$|x + y| \leq |x| + |y|.$$

Proof. [redacted]

Corollary 1.12 (The reverse triangle inequality)
Assume that $\mathbb{F}$ is an ordered field and $x, y \in \mathbb{F}$. Then,

$$||x| - |y|| \leq |x - y|.$$

Proof. [redacted]

Corollary 1.13 (Triangle inequality corollaries)
For both of the following, assume that $\mathbb{F}$ is an ordered field and $x, y \in \mathbb{F}$.

1. $|x - y| \leq |x| + |y|$.
2. $|x + y| \geq ||x| - |y||$.

Proof. [redacted]

Theorem 1.14 (Cauchy-Schwarz inequality)

If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are arbitrary real numbers, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Proof. [redacted]

Definition 1.15 (Upper and lower bounds)

Let S be an ordered field and $A \subseteq S$ be nonempty.

1. The set A is *bounded above* if there exists some $b \in S$ such that $x \leq b$ for all $x \in A$; in this case b is called an **upper bound** of A.
2. The **least upper bound** of A—if it exists—is some $b_0 \in S$ such that
   a) b_0 is an upper bound of A, and
   b) if b is any other upper bound of A, then $b_0 \leq b$.

   Such a b_0 is also called the **supremum** of A and is denoted $\sup(A)$.
3. Likewise, the set A is *bounded below* if there exists some $b \in S$ such that $x \geq b$ for all $x \in A$; in this case, b is called a **lower bound** of A.
4. Again, like above, the **greatest lower bound** of A—if it exists—is some $b_0 \in S$ such that
   a) b_0 is a lower bound of A, and
   b) if b is any other lower bound of A, then $b_0 \geq b$.

   Such a b_0 is also called the **infimum** of A and is denoted $\inf(A)$.
5. If a set is both bounded above and bounded below, then it is simply *bounded*.

Example 1.16

The propositions below are left without proof.

- The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ has no upper bounds. Lower bounds on $\mathbb{N}$ include -17, 1, 0.123, and $-\pi$. Note that $\sup(\mathbb{N})$ does not exist, but $\inf(\mathbb{N}) = 1$.
- The set $\mathbb{Q}$ has no upper or lower bounds; consequently, $\sup(\mathbb{Q})$ and $\inf(\mathbb{Q})$ do not exist.
- $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$; $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$. Note that the supremum here is in the set, while the infimum is not in the set.
- $\sup(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = 1$; $\inf(\{\frac{n}{n+1} : n \in \mathbb{N}\}) = \frac{1}{2}$. Note that the infimum here is in the set, while the supremum is not in the set.
- In $\mathbb{Q}$ the set $\{x \in \mathbb{Q} : x^2 < 2\}$ does not have a supremum. In $\mathbb{R}$ it will—in fact, $\sup(\{x \in \mathbb{Q} : x^2 < 2\}) = \sqrt{2}$.

Definition 1.17 (Completeness)

Let S be an ordered field. Then S has the **least upper bound property** if given any nonempty $A \subseteq S$ where A is bounded above, A has a least supper bound in S. In other words, $\sup(A) \in S$ for every such A.

Such a set S is also called **complete**.

Theorem 1.18 (Existence of $\mathbb{R}$)

The real numbers $\mathbb{R}$ exists, and it is a complete ordered field.

Proof. [redacted]

[redacted]

Theorem 1.20 (Uniqueness of $\mathbb{R}$)

Any complete ordered field is isomorphic to $\mathbb{R}$.

Proof. [redacted]

Corollary 1.21 (Existence and uniqueness of $\mathbb{R}$)

There exists a unique complete ordered field. We call this field *the real numbers* $\mathbb{R}$.

Proposition 1.22 (Axioms of $\mathbb{R}$)

The set $\mathbb{R}$ has two binary operations, addition $(+)$ and multiplication $(\cdot)$, and is the unique set satisfying the following axioms.

- **Axiom 1 (Commutative Law).** If $a, b \in \mathbb{R}$, then $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Axiom 2 (Distributive Law).** If $a, b \in \mathbb{R}$, $a \cdot (b+c) = a \cdot b + a \cdot c$.
- **Axiom 3 (Associative Law).** If $a, b \in \mathbb{R}$, then $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Axiom 4 (Identity Law).** There are special elements $\mathbf{0}, \mathbf{1} \in \mathbb{R}$, where $a + \mathbf{0} = a$ and $a \cdot \mathbf{1} = a$ for all $a \in \mathbb{R}$.
- **Axiom 5 (Inverse Law).** For each $a \in \mathbb{R}$, there is an element $-a \in \mathbb{R}$ such that $a + (-a) = \mathbf{0}$. If $a \neq \mathbf{0}$, then there is also an element $a^{-\mathbf{1}} \in \mathbb{R}$ such that $a \times a^{-\mathbf{1}} = \mathbf{1}$.
- **Axiom 6 (Order Axiom).** There is a nonempty subset $P \subseteq \mathbb{R}$, called the *positive elements*, such that
  1. If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;
  2. If $a \in \mathbb{R}$ and $a \neq \mathbf{0}$, then either $a \in P$ or $-a \in P$, but not both.
- **Axiom 7 (Completeness Axiom).** Given any nonempty $A \subseteq \mathbb{R}$ where A is bounded above, A has a least upper bound. In other words, $\sup(A) \in \mathbb{R}$ for every such A.

Proposition 1.23 (Suprema are unique)

If the supremum or infimum of $A \subseteq \mathbb{R}$ exists, then it is unique.

We will only prove that suprema are unique. The infima case is analogous.

Proof. [redacted]

Theorem 1.24 (Square roots exist)

If $a \in \mathbb{R}$ and $a \geq 0$, then $\sqrt{a} \in \mathbb{R}$.

Proof. [redacted]

Theorem 1.25 (Suprema analytically)
Let $A \subseteq \mathbb{R}$. Then $\sup(A) = \alpha$ if and only if

1. α is an upper bound of A, and
2. Given any $\epsilon > 0$, $\alpha - \epsilon$ is *not* an upper bound of A. That is, there is some $x \in A$ for which $x > \alpha - \epsilon$.

Likewise, $\inf(A) = \beta$ if and only if

1. β is a lower bound of A, and
2. Given any $\epsilon > 0$, $\beta + \epsilon$ is *not* a lower bound of A. That is, there is some $x \in A$ for which $x < \beta + \epsilon$.

We will only prove the suprema case. The infima case is analogous.

Proof. [redacted]

Remark 1.26 — Note that the forward direction of the proof also works well by contrapositive. The contrapositive of

$$\sup(A) = \alpha \implies \text{For all } \epsilon > 0 \text{ there is an } x \in A \text{ such that } x > \alpha - \epsilon$$

is

$$\text{There is an } \epsilon > 0 \text{ such that for all } x \in A \text{ we have } x \leq \alpha - \epsilon \implies \sup(A) \neq \alpha.$$

And to prove this, just observe that the left-hand side implies that $\alpha - \epsilon$ is an upper bound of A, and so $\sup(A) \leq \alpha - \epsilon$, which of course implies that $\sup(A) \neq \alpha$, as desired.

Lemma 1.27 (The Archimedean principle)

If a and b are real numbers with $a > 0$, then there exists a natural number n such that $na > b$.

In particular, for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. [redacted]

Example 1.28

Show that $\inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$.

Proof. [redacted]

Example 1.29

Show that $\sup(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$.

Proof. [redacted]

[redacted]

Theorem 1.30 ($0.999\ldots = 1$)

$0.999\ldots$ is a repeating decimal representation of the real number 1.

Proof. [redacted]

Remark 1.31 — Despite the persistent rumors, $0.999\ldots$ is not “almost exactly 1” or “very, very nearly but not quite 1” or “1’s clingy little sibling.” Rather, $0.999\ldots$ and 1 are, in fact, the same number. So the next time someone insists otherwise, feel free to hand them a copy of these notes.

Remark 1.32 — The Archimedean principle used in Theorem 1.30 differs from the one stated in Lemma 1.27. One can show that these two formulations are equivalent.

Remark 1.33 — We will revisit Theorem 1.30 in future chapters and discuss various different proofs.

Definition 1.34 (Density)

Suppose A and B are ordered field. Then A is **dense** in B if, for any $x, y \in B$, there exists $a \in A$ such that $x < a < y$.

Example 1.35

The propositions below are left without proof.

- $\mathbb{Q}$ is dense in $\mathbb{Q}$.
- $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{Q}$.
- $\mathbb{Z}$ is *not* dense in $\mathbb{Q}$.

Lemma 1.36

Let $x, y \in \mathbb{R}$. If $y - x > 1$, then there exists $z \in \mathbb{Z}$ such that $x < z < y$.

Proof. [redacted]

Theorem 1.37 ($\mathbb{Q}$ is dense in $\mathbb{R}$)

The rational numbers are dense in the real numbers.

Proof. [redacted]

[redacted]

Remark 1.38 — The proof of Lemma 1.36 also implies that, for any $x \in \mathbb{R}$, there exists an integer M such that $M \leq x \leq M + 1$. In particular, it implies that the floor and ceiling functions exist.

Definition 1.39 (Floor and ceiling functions)
Let $x \in R$.

- The *floor* of x, denoted $\lfloor x \rfloor$, is the integer n such that $x - 1 < n \leq x$.
- The *ceiling* of x, denoted $\lceil x \rceil$, is the integer n such that $x \leq n < x + 1$.

Definition 1.40 (Closed and open intervals)
Define the *closed interval* $[a, b]$ to be $\{x \in \mathbb{R} : a \leq x \leq b\}$. Likewise the *open interval* (a, b) is defined to be $\{x \in \mathbb{R} : a < x < b\}$, and half-open intervals and intervals to $\pm\infty$ are again exactly as you would expect.

Theorem 1.41 (Characterization of intervals)
Let S be a subset of $\mathbb{R}$ that contains at least two points. If S has the property such that

$$\text{if } x, y \in S \text{ and } x < y, \text{ then } [x, y] \subseteq S, \tag{1}$$

then S is an interval.

Proof. [redacted]

[redacted]

Theorem 1.42 (The nested intervals property)

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Also, assume that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection. That is,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof. [redacted]

Remark 1.43 — Note that the conclusion of Theorem 1.42 need not hold if each I_n is allowed to be an open interval.

2 Cardinality

"No one shall expel us from the paradise that Cantor has created."

David Hilbert, Über das Unendliche

Definition 2.1 (Cardinality)

Let S and T be sets. Then, $|S| = |T|$ if and only if there is a bijection from S to T.

Definition 2.2 (Cardinality cont.)

$|S| \leq |T|$ if and only if there is an injection from S to T.

Remark 2.3 — Our definitions above introduce two fundamental relations on cardinality, i.e. $|S| = |T|$ and $|S| \leq |T|$. We need to make sure that these relations have the mathematical properties we expect them to have. That is, we want $|S| = |T|$ to be an equivalence relation and $|S| \leq |T|$ to be a partial order.

For the relation $|S| = |T|$, it's easy to show that it defines an equivalence relation:

- Reflexivity: Every set has a bijection with itself, i.e. $|S| = |S|$.
- Symmetry: If there is a bijection f from S to T, then f^{-1} is a bijection from T to S, and thus $|T| = |S|$.
- Transitivity: If there are bijections f from S to T and g from T to U, then their composition $h = g \circ f$ is a bijection from S to U. Hence, $|S| = |U|$.

For the relation $|S| \leq |T|$, we must establish that it's a partial order:

- Reflexivity: Every set has an injection to itself, i.e. $|S| \leq |S|$.
- Transitivity: If there exist injections f from S to T and g from T to U, then their composition $h = g \circ f$ is an injection from S to U. Hence, $|S| \leq |U|$.

The remaining property, antisymmetry, is where things get interesting, since it's not immediately obvious. Antisymmetry means that if $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$. Using our definition, this translates to: If there is an injection from S to T and an injection from T to S, then there should be a bijection between S and T. This is exactly what the Schröder-Bernstein theorem (Theorem 2.4) guarantees.

Theorem 2.4 (Schröder-Bernstein theorem)

If there exist injections $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijection $h : A \to B$.

In terms of the cardinality of the two sets, this implies that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. [redacted]

[redacted]

Definition 2.5 (Cardinality cont.)
$|S| \geq |T|$ if and only if there is a surjection from S to T.

Remark 2.6 — Again, we must establish that $|S| \geq |T|$ defines a partial order. Reflexivity and transitivity are obvious. Now we will show antisymmetry. Suppose $|S| \geq |T|$ and $|T| \geq |S|$. Then, by definition, there are surjections from S to T and from T to S. Using the axiom of choice, one can prove that there exists a surjection from X to Y if and only if there exists an injection from Y to X. We conclude that $|T| \leq |S|$ and $|S| \leq |T|$, which implies $|S| = |T|$ by Theorem 2.4.

Theorem 2.7 ($|\mathbb{Z}| = |\mathbb{N}|$)
There are as many integers as there are natural numbers.

Proof. [redacted]

Remark 2.8 — Theorem 2.7 also shows that two sets can have the same cardinality even if one is a proper subset of the other and the "larger" one even has infinitely many more elements than the "smaller" one. Make sure you take a moment to appreciate how remarkably, wonderfully weird this is.

Theorem 2.9 ($|\mathbb{Q}| = |\mathbb{N}|$)
There are as many rational numbers as there are natural numbers.

Proof. [redacted]

Theorem 2.10 ($|\mathbb{R}| > |\mathbb{N}|$)
There are more real numbers than natural numbers.

Proof. [redacted]

[redacted]

Definition 2.11 (Countable and uncountable sets)

A set S is **countable** if

1. its cardinality $|S|$ is less than or equal to $|\mathbb{N}|$.
2. there exists an injection from S to $\mathbb{N}$.
3. S is empty or there exists a surjection $\mathbb{N}$ to S.
4. there exists a bijection from S to a subset of $\mathbb{N}$.
5. S is either finite or *countably infinite*.

All of the definitions above are equivalent.

A set S is **countably infinite** if its cardinality $|S|$ is exactly $\mathbb{N}$.

A set S is **uncountable** if it is not countable. That is, its cardinality $|S|$ is greater than $|\mathbb{N}|$.

Corollary 2.12 ($\mathbb{N}$ is countable)

The set of natural numbers is countable.

Corollary 2.13 ($\mathbb{Z}$ is countable)
The set of integers is countable.

Corollary 2.14 ($\mathbb{Q}$ is countable)
The set of rational numbers is countable.

Corollary 2.15 ($\mathbb{R}$ is uncountable)
The set of real numbers is uncountable.

Theorem 2.16
An uncountable collection of disjoint open intervals in $\mathbb{R}$ cannot exist.

Proof. [redacted]

Theorem 2.17 (Countable infinity is the smallest infinity)
If $A \subseteq B$ and B is countable, then A is either countable or finite.

Proof. [redacted]

Corollary 2.18 ($|\mathbb{N}|$ is the smallest infinity)
If $A \subseteq \mathbb{N}$, then either A is finite or $|A| = |\mathbb{N}|$.

Corollary 2.19 (Sizes of infinity)
There are different sizes of infinity, with countable infinity being the smallest. Moreover, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable while $\mathbb{R}$ is uncountable.

Theorem 2.20 (Countable union of countable sets is countable)

A countable union of countable sets is countable. More precisely:

1. If $A_1, A_2, \ldots, A_m$ are each countable sets, then $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.
2. If A_n is a countable set for each $n \in \mathbb{N}$, then the set $\bigcup_{n=1}^{\infty} A_n$ is also countable.

Proof. [redacted]

[redacted]

Theorem 2.21 ($\mathbb{R} \setminus \mathbb{Q}$ is uncountable)
There are uncountably many irrational numbers.

Proof. [redacted]

Theorem 2.22 ($|\mathbb{Q}_+| = |\mathbb{N}|$)
There are as many positive rational numbers as there are natural numbers.

Proof. [redacted]

[redacted]

Theorem 2.23 ($|(0, 1)| > |\mathbb{N}|$)
There are more numbers in the open interval $(0, 1)$ than there are natural numbers.

We will demonstrate a clever argument known as *Cantor's diagonal argument*.

Proof. [redacted]

Remark 2.24 — Consider the following complaints about the proof of Theorem 2.23.

1. Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 2.23 must be flawed.

2. Some numbers have two different decimal representations (see Theorem 1.30). Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or as $.4999\ldots$. Doesn't this cause some problems?

Are these complaints valid?

Here's another proof of Theorem 2.23 that I particular enjoy.

Proof. [redacted]

Theorem 2.25 ($|(0,1)| = |\mathbb{R}|$)

There are as many numbers in the open interval $(0,1)$ as there are real numbers.

Proof. [redacted]

Remark 2.26 — Theorem 2.25 effectively establishes that Theorem 2.10 and Theorem 2.23 are equivalent.

Remark 2.27 — We now know that $|\mathbb{N}| < |\mathbb{R}|$. Here's one natural question: Is there any infinity between these two? An astounding fact is that, based on the axioms of set theory (called ZFC), whether or not there exists such an infinity is *unprovable*. And what I don't mean is that mathematicians are not smart enough to find the answer; no, I mean that they *are* smart enough to have shown that *no proof can possibly exist*. That's right, there are statements in math which are impossible to prove and also impossible to disprove (but we *are* able to prove that they are unprovable, amazingly).

This particular question is among the most famous in mathematical history. It was posed by Georg Cantor and is known as *the continuum hypothesis*. It is the first of Hilbert's 23 problems—an influential list of unsolved problems that David Hilbert presented in 1900 at the International Congress of Mathematicians, setting the mathematical agenda for the 20th century. Decades later, Kurt Gödel's groundbreaking incompleteness theorems revealed that virtually every mathematical theory contains unprovable statements.

ZFC set theory is the foundational framework upon which nearly all of modern mathematics is built. Gödel constructed a model of ZFC where the continuum hypothesis holds, while Paul Cohen later constructed a model where it fails. Together, their results established that the continuum hypothesis is independent of ZFC—it can be neither proved nor disproved within the system. Thus, the continuum hypothesis—which asks what is presumably a basic question about the infinite—is unprovable.

Hypothesis 2.28 (The continuum hypothesis)

There is no set whose cardinality is strictly between that of the naturals and the reals.

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

Theorem 2.29 ($|A| < |\mathcal{P}(A)|$)

If A is a set and $\mathcal{P}(A)$ is the power set of A, then

$$|A| < |\mathcal{P}(A)|.$$

Proof. [redacted]

> **Corollary 2.30** (There exist infinitely many infinities)
> There exist infinitely many distinct infinite cardinalities.

Proof. [redacted]

3 Sequences

"Erdős loved epsilons—his word for small children (in mathematics the Greek letter epsilon is used to represent small quantities)."

Paul Hoffman, The Man Who Loved Only Numbers

Definition 3.1 (Sequences)
A **sequence** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$.

Definition 3.2 (Bounded sequences)
A sequence (a_n) is **bounded** if the range $\{a_n : n \in \mathbb{N}\}$ is bounded. That is, if there exists a lower bound $L \in \mathbb{R}$ and an upper bound $U \in \mathbb{R}$ where

$$L \leq a_n \leq U$$

for all n.

Proposition 3.3
A sequence (a_n) is bounded if and only if there exists some $C \in \mathbb{R}$ for which $|a_n| \leq C$ for all n.

Proof. [redacted]

Definition 3.4 (Convergent sequences)

A sequence (a_n) **converges** to $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n > N$.

When this happens, a is called the **limit** of a_n.

Definition 3.5 (ϵ-neighborhood)

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

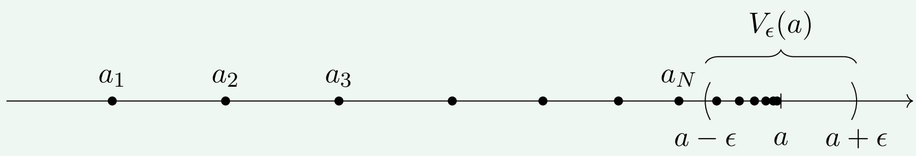
$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the **ϵ-neighborhood** of a.

Definition 3.6 (Convergent sequences (topological version))

A sequence (a_n) *converges* to $a \in \mathbb{R}$ if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but finite number of the terms of (a_n).

Remark 3.7 — Definition 3.4 and Definition 3.6 say precisely the same thing: the natural number N in Definition 3.4 is the point after which the sequence (a_n) enters $V_\epsilon(a)$, never to leave. It should be apparent that the value of N depends on the choice of ϵ. The smaller the ϵ-neighborhood, the larger the N may have to be.



Example 3.8

Show that the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$$

converges to 0.

Proof. [redacted]

Remark 3.9 — Using the Archimedean principle will not usually work, though. We need a more general approach, which is described in Outline 3.10.

Outline 3.10. To show that $a_n \to a$, begin with preliminary work:

0. Scratch work: Start with $|a_n - a| < \epsilon$ and unravel to solve for n. This tells you which N to pick for step 2 below.

Now for your actual proof:

1. Let $\epsilon > 0$.
2. Let N be the final value of n you got in your scratch work, and let $n > N$.
3. Redo scratch work (without ϵ's), but at the end use N to show that $|a_n - a| < \epsilon$.

Example 3.11

Again, show that the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$$

converges to 0. This time, use the method described in Outline 3.10.

Scratch work. Given an arbitrary $\epsilon > 0$, we will find what specifc N guarantees that, for every $n > N$, we have $|a_n - 0| < \epsilon$. For example, if $\epsilon = \frac{1}{2}$, then $N = 2$ works. If $\epsilon = \frac{1}{3}$, then $N = 3$ works. You see the pattern, but here is how we might come about it in general. We want the following:

$$\begin{aligned} |a_n - a| &< \epsilon \\ \left|\frac{1}{n} - 0\right| &< \epsilon \\ \frac{1}{n} &< \epsilon \\ \frac{1}{\epsilon} &< n. \end{aligned}$$

So as long as we choose $N = \frac{1}{\epsilon}$, then for any $n > N$, we will have $n > \frac{1}{\epsilon}$, which by the above will imply that $\frac{1}{n} \to 0$, as desired. The solution below is how we formally solve it.

Proof. [redacted]

Example 3.12

Let $a_n = \frac{3n+1}{n+2}$. Prove that $\lim_{n\to\infty} a_n = 3$.

Scratch Work. Again, we first play around. We start with where we want to get to (that $|a_n - a| < \epsilon$), and then do some algebra to figure out which values of n would give this.

We want the following:

$$\begin{aligned} |a_n - a| &< \epsilon \\ \left|\frac{3n+1}{n+2} - 3\right| &< \epsilon \\ \left|\frac{3n+1}{n+2} - \frac{3(n+2)}{n+2}\right| &< \epsilon \\ \left|\frac{3n+1-3n-6}{n+2}\right| &< \epsilon \\ \left|\frac{-5}{n+2}\right| &< \epsilon \\ \frac{5}{n+2} &< \epsilon \\ \frac{5}{\epsilon} &< n+2 \\ \frac{5}{\epsilon} - 2 &< n \end{aligned}$$

So as long as we choose $N = \frac{5}{\epsilon} - 2$, then for any $n > N$ we will have $n > \frac{5}{\epsilon} - 2$, which by the above will imply that $\frac{3n+1}{n+2} \to 3$, as desired.

Proof. [redacted]

Definition 3.13 (Divergent Sequences)

If a sequence (a_n) does not converge, then it **diverges**.

Divergence can come in three forms.

1. (a_n) *diverges to* ∞ (notation: $\lim_{n\to\infty} a_n = \infty$) if, for all $M > 0$, there exists some $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$.
2. (a_n) *diverges to* $-\infty$ (notation: $\lim_{n\to\infty} a_n = -\infty$) if, for all $M < 0$, there exists some $N \in \mathbb{N}$ such that $a_n < M$ for all $n > N$.
3. Otherwise, (a_n)'s limit *does not exist.*

Example 3.14

Let $a_n = n^2$. Show that $\lim_{n\to\infty} a_n = \infty$.

Scratch work. We want

$$\begin{aligned} a_n &> M \\ n^2 &> M \\ n &> \sqrt{M}. \end{aligned}$$

So setting $N = \sqrt{M}$ should work.

Proof. [redacted]

Outline 3.15. What if the sequence's limit does not exist? Then how do we show the sequence diverges? One way to show that a_n diverges is to show that $a_n \not\to a$ for any a. Note first, by Definition 3.4, that $a_n \to a$ means that

For every $\epsilon > 0$ there exists some N
such that for all $n > N$ we have $|a_n - a| < \epsilon$.

So to show that $a_n \not\to a$, we need to show the *negation* of that statement. That is, we must show that

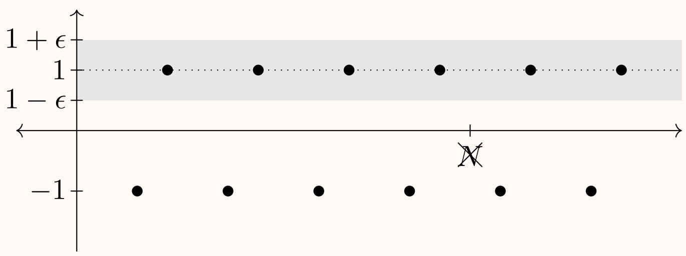
There exists some $\epsilon > 0$ where for all N
there exists some $n > N$ such that $|a_n - a| \geq \epsilon$.

In practice, this is usually done with a proof by contradiction. You assume that $a_n \to a$ and then you demonstrate a specific ϵ where it fails, giving the contradiction.

Example 3.16

Let $a_n = (-1)^n$. Prove that (a_n) diverges.

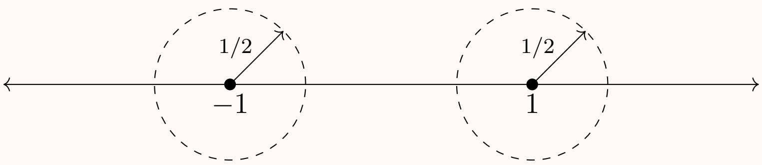
Scratch work. This is the sequence $-1, 1, -1, 1, -1, 1, \ldots$. It makes sense that there is no a for which $a_n \to a$. It certainly doesn't converge to 1, since half the time it is at -1 which is far away from 1. (If we let $\epsilon = 1/$, then there is no N for which, for every $n > N$, a_n is inside of the shaded band; see below.)



It likewise can't converge to -1. One might guess 0, since that is halfway between -1 and 1, but that also doesn't make sense since a_n is always a distance of 1 away from 0, so it's certainly not getting "closer and closer" to 0. (Or, $\epsilon = 1/2$ works again.) Ok, so we believe that it doesn't converge to anything, and we will use Outline 3.15 to show it.

Proof. [redacted]

Essentially, focusing on the even case created a ball around 1, of radius $\frac{1}{2}$, which a_n would have to live within for all $n > N$. The odd case created a ball around -1, of radius $\frac{1}{2}$, which a_n would have to live within for all $n > N$. But these two balls are disjoint, so a_n can't live in both, creating a contradiction.



Now here's another proof using the triangle inequality (Theorem 1.11).

Proof. [redacted]

Proposition 3.17 (Limits are unique)

A sequence cannot have more than one limit.

Proof. [redacted]

[redacted]

There is a second proof that is even a little shorter.

Proof. [redacted]

Proposition 3.18

If (a_n) is a convergent sequence, then (a_n) is bounded.

Proof. [redacted]

Corollary 3.19

If (a_n) is an unbounded sequence, then (a_n) is divergent.

Theorem 3.20 (Sequence limit laws)

Assume that (a_n) and (b_n) are convergent sequences of real numbers such that $a_n \to a$ and $b_n \to b$. Also assume that $c \in \mathbb{R}$. Then,

1. $(a_n + b_n) \to a + b$.
2. $(a_n - b_n) \to a - b$.
3. $(a_n \cdot b_n) \to a \cdot b$.
4. $(\frac{a_n}{b_n}) \to \frac{a}{b}$, provided each $b \neq 0$ and each $b_n \neq 0$.
5. $(c \cdot a_n) \to c \cdot a$.

Example 3.21

What is

$$\lim_{n\to\infty} \frac{1}{2} \cdot \left(\frac{\frac{1}{n} + \frac{1}{n^2} + 4}{5 - \frac{1}{n^2}} \right) \cdot \left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}} \right)?$$

By the limit laws, we have

$$\lim_{n\to\infty} \frac{1}{2} \cdot \left(\frac{\frac{1}{n} + \frac{1}{n^2} + 4}{5 - \frac{1}{n^2}} \right) \cdot \left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}} \right) = \frac{1}{2} \cdot \left(\frac{0+0+4}{5} \right) \cdot (3+0) = \frac{6}{5}.$$

Theorem 3.22 (Sequence squeeze theorem)

Assume $a_n \le x_n \le b_n$ for all n. Furthermore, assume that

$$a_n \to L \qquad \text{and} \qquad b_n \to L.$$

Then,

$$x_n \to L.$$

Proof. [redacted]

Example 3.23

Prove that

$$\lim_{n\to\infty} \frac{1}{n^{2.7} + \sqrt{n} + \pi} = 0.$$

Proof. [redacted]

Definition 3.24

A sequence (a_n) is **monotone increasing** if $a_n \le a_{n+1}$ for all n. Likewise, a sequence (a_n) is **monotone decreasing** if $a_n \ge a_{n+1}$ for all n. If it is either monotone increasing or monotone decreasing, it is monotone.

Theorem 3.25 (The monotone convergence theorem)

Suppose (a_n) is monotone. Then (a_n) converges if and only if it is bounded. Moreover,

- If (a_n) is increasing, then either (a_n) diverges to ∞ or

$$\lim_{n\to\infty} a_n = \sup(\{a_n : n \in \mathbb{N}\}).$$

- If (a_n) is decreasing, then either (a_n) diverges to $-\infty$ or

$$\lim_{n\to\infty} a_n = \inf(\{a_n : n \in \mathbb{N}\}).$$

Proof. [redacted]

[redacted]

Remark 3.26 — This theorem is nice for several reasons, but one of which is the fact that you do not have to know what the limit of (a_n) is in order to show that (a_n) is convergent. This is notable since in the definition of sequence convergence—which, until now, we have heavily relied on to show a sequence converges—requires that you already know (and can write down) what the limit is going to be. Below is an example where it would be quite a challenge to write down the limit of the sequence in a beneficial way. Yet by the monotone convergence theorem, we will be able to conclude that the limit exists.

Example 3.27

Let (a_n) be the sequence where $a_1 = 0.1$, $a_2 = 0.12$, $a_3 = 0.123$, $a_4 = 0.1234$, and so on. (And, to be clear, this pattern does not change when you reach double digits. For example, $a_{12} = 0.123456789101112$.) Prove that (a_n) converges.

Proof. [redacted]

Proposition 3.28

Suppose $S \subseteq \mathbb{R}$ is bounded above. Then there exists a sequence (a_n) where $a_n \in S$ for each n and

$$\lim_{n\to\infty} a_n = \sup(S).$$

Likewise, if S is bounded below, then there exists a sequence (b_n) where $b_n \in S$ for each n and

$$\lim_{n\to\infty} b_n = \inf(S).$$

Proof. [redacted]

Definition 3.29

Let (a_n) be a sequence of real numbers and let

$$n_1 < n_2 < n_3 < \dots$$

be an increasing sequence of integers. Then,

$$a_{n_1}, a_{n_2}, a_{n_3}, \dots$$

is called a **subsequence** of (a_n), and is denoted (a_{n_k}).

Proposition 3.30

A sequence (a_n) converges to a if and only if every subsequence of (a_n) also converges to a.

Proof. [redacted]

Corollary 3.31

If (a_n) has a pair of subsequences converging to different limits, then (a_n) diverges.

Proof. [redacted]

Proposition 3.32

If a monotone sequence (a_n) has a convergent subsequence, then (a_n) converges too, and has the same limit.

Proof. [redacted]

[redacted]

Lemma 3.33

Every sequence has a monotone subsequence.

Proof. [redacted]

[redacted]

[redacted]

Theorem 3.34 (The Bolzano-Weierstrass theorem)
Every bounded sequence has a convergent subsequence.

Proof. [redacted]

Definition 3.35
A sequence (a_n) is **Cauchy** if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$|a_m - a_n| < \epsilon$$

for all $m, n > N$.

Lemma 3.36
If (a_n) is Cauchy, then (a_n) is bounded.

Proof. [redacted]

Theorem 3.37 (Cauchy criterion for convergence)
A sequence converges if and only if it is Cauchy.

Proof. [redacted]

4 Series

"Divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever."

Niels Henrik Abel, in a letter to Holmboe (1826)

A Pathological Examples

> "One of my favorite memories of studying was one night when analysis crept into my dreams. I woke up in a panicky cold sweat. In my dream I was being chased by some analysis monster. My only defense was to use the *blancmange function* as a boomerang. I took it as a good sign at the time that analysis concepts were finding their way into my subconscious."
>
> *Tina Rapke, Confronting Analysis*

It has been said that one of the most important goals of learning real analysis is to collect as many bizarre examples as you can, and to keep them in your back pocket. From a practical standpoint they will inform your conjectures and guide your proofs, but they will also help to demonstrate why real analysis is such a great subject.

§A.1 An infinite field that cannot be ordered

To say that a field F cannot be ordered is to say that it possesses no positive subset P satisfying the order axiom (Definition 1.6). A preliminary comment is that since every ordered field is infinite, no finite field can be ordered.

An example of an *infinite* field that cannot be ordered is the field $\mathbb{C}$ of complex numbers.

Proof. [redacted]

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