Equations of Motion: Seesaw

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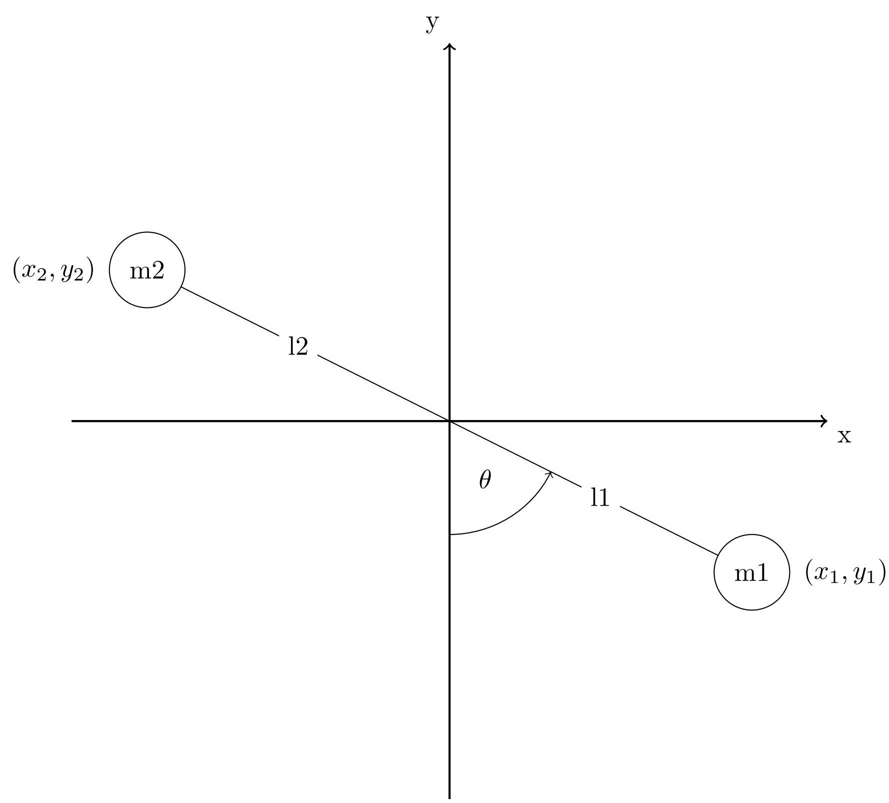
1 Diagram



2 Assumptions

- Point masses
- Massless, rigid rod
- Gravity is present

3 Trigonometric Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{3.1}$$

4 Kinematic Constrains

$$x_1 = l_1 \sin\theta \tag{4.1}$$
$$y_1 = -l_1 \cos\theta \tag{4.2}$$

$$x_2 = -l_2 \sin\theta \tag{4.3}$$
$$y_2 = l_2 \cos\theta \tag{4.4}$$

5 Velocities

$$\dot{x_1} = \dot{\theta} l_1 \cos\theta \tag{5.1}$$
$$\dot{y_1} = \dot{\theta} l_1 \sin\theta \tag{5.2}$$

$$\dot{x_2} = \dot{\theta}(-l_2) \cos\theta \tag{5.3}$$
$$\dot{y_2} = \dot{\theta}(-l_2) \sin\theta \tag{5.4}$$

6 Potential Energy

$$V = m_1 g y_1 + m_2 g y_2 \tag{6.1}$$

Substitute 4.2, 4.4 into 6.1:

$$V = m_1 g(-l_1 \cos\theta) + m_2 g(l_2 \cos\theta) \tag{6.2}$$

Simplify:

$$V = m_2 g l_2 \cos\theta - m_1 g l_1 \cos\theta \tag{6.3}$$

7 Kinetic Energy

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \tag{7.1}$$

Substitute v_1, v_2:

$$T = \frac{1}{2} m_1 (\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2} m_2 (\dot{x_2}^2 + \dot{y_2}^2) \tag{7.2}$$

Substitute 5.1, 5.2, 5.3, 5.4 into 7.2:

$$\begin{aligned} T = {} & \frac{1}{2} m_1 ((\dot{\theta} l_1 \cos\theta)^2 + (\dot{\theta} l_1 \sin\theta)^2) \\ & + \frac{1}{2} m_2 ((\dot{\theta}(-l_2) \cos\theta)^2 + (\dot{\theta}(-l_2) \sin\theta)^2) \end{aligned} \tag{7.3}$$

Simplify:

$$T = \frac{1}{2}m_1(\dot{\theta}^2 l_1^2(\cos^2\theta + \sin^2\theta)) + \frac{1}{2}m_2(\dot{\theta}^2(-l_2)^2(\cos^2\theta + \sin^2\theta)) \tag{7.4}$$

$$= \frac{1}{2}m_1\dot{\theta}^2 l_1^2(\cos^2\theta + \sin^2\theta) + \frac{1}{2}m_2\dot{\theta}^2(-l_2)^2(\cos^2\theta + \sin^2\theta) \tag{7.5}$$

$$= \frac{1}{2}m_1\dot{\theta}^2 l_1^2(\cos^2\theta + \sin^2\theta) + \frac{1}{2}m_2\dot{\theta}^2 l_2^2(\cos^2\theta + \sin^2\theta) \tag{7.6}$$

Substitute 3.1:

$$T = \frac{1}{2}m_1 l_1^2\dot{\theta}^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}^2 \tag{7.7}$$

8 Lagrangian

$$L = T - V \tag{8.1}$$

Substitute 7.7, 6.3 into 8.1:

$$L = \frac{1}{2}m_1 l_1^2\dot{\theta}^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}^2 + m_1 g l_1\cos\theta - m_2 g l_2\cos\theta \tag{8.2}$$

9 Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \tag{9.1}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m_1 l_1^2 2\dot{\theta} + \frac{1}{2}m_2 * l_2^2 2\dot{\theta} \tag{9.2}$$

$$= m_1 l_1^2\dot{\theta} + m_2 l_2^2\dot{\theta} \tag{9.3}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_1 l_1^2\ddot{\theta} + m_2 l_2^2\ddot{\theta} \tag{9.4}$$

$$\frac{\partial L}{\partial \theta} = m_1 g l_1(-\sin\theta) - m_2 g l_2(-\sin\theta) \tag{9.5}$$

$$= m_2 g l_2\sin\theta - m_1 g l_1\sin\theta \tag{9.6}$$

Substitute 9.4, 9.6 into 9.1:

$$(m_1 l_1^2\ddot{\theta} + m_2 l_2^2\ddot{\theta}) - (m_2 g l_2\sin\theta - m_1 g l_1\sin\theta) = 0 \tag{9.7}$$

Rearrange:

$$m_1 l_1^2\ddot{\theta} + m_2 l_2^2\ddot{\theta} + m_1 g l_1\sin\theta - m_2 g l_2\sin\theta = 0 \tag{9.8}$$

10 State space

extract $\ddot{\theta}$:

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} + m_1 g l_1 \sin\theta - m_2 g l_2 \sin\theta = 0 \tag{10.1}$$

solve for $\ddot{\theta}$:

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} = m_2 g l_2 \sin\theta - m_1 g l_1 \sin\theta \tag{10.2}$$

$$\ddot{\theta} = \frac{m_2 g l_2 \sin\theta - m_1 g l_1 \sin\theta}{m_1 l_1^2 + m_2 l_2^2} \tag{10.3}$$

rewrite as state space:

$$\dot{\vec{y}} = \frac{d}{dt}\left\{ \begin{array}{c} \theta \\ \dot{\theta} \end{array} \right\} = \left\{ \begin{array}{c} \dot{\theta} \\ \frac{m_2 g l_2 \sin\theta - m_1 g l_1 \sin\theta}{m_1 l_1^2 + m_2 l_2^2} \end{array} \right\} \tag{10.4}$$