Report on Integration: Trapezoidal and Simpson's Rules

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1 Introduction

This report details various numerical integrations on the interval $[-2, 2]$ of the following four functions:

$$\begin{aligned} f(x) &= 1 + x^2 \\ f(x) &= xe^{-x^2} \\ f(x) &= xe^{-x} \\ f(x) &= \sin(x) \end{aligned} \tag{1}$$

For the integrations, two methods were investigated: the trapezoidal rule and Simpson's Rule. In the trapezoidal rule, each pair of adjacent sampling points was used to generate a trapezoid with the x-axis, calculating

$$\int_a^b f(x)\mathrm{d}x = \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i-1})}{2}\Delta x_i \tag{2}$$

In Simpson's Rule, each trio of adjacent sampling points with uniform spacing was used to calculate the area of the parabola through the three points with the x-axis, finding

$$\int_a^b f(x)\mathrm{d}x = \frac{x_1-x_0}{3}\sum_{i=1}^{n}\left(f(x_{2i-2})+4f(x_{2i-1})+f(x_{2i})\right) \tag{3}$$

1.1 Various Notes

In this project, all code was completed in Python and all plots were generated with Matplotlib. All symbolic integrals were calculated using the SymPy package. Because of the use of Python, no issues with floating point arithmetic were detected even for arbitrarily small h in the calculation of the integral.

2 Trapezoidal Rule

For the Trapezoidal Rule, the following RMS Errors were evaluated as shown in Table 1:

Table 1: **RMS Error for Trapezoidal Fit**

		Function Tested			
		$\mathbf{1+x^2}$	$\mathbf{xe^{-x^2}}$	$\mathbf{xe^{-x}}$	$\sin(\mathbf{x})$
# of Samples	**3**	0.7303	0.2663	2.491	0.2780
	7	0.08114	0.04119	0.3599	0.03116
	15	0.01490	0.007933	0.06831	0.005742
	31	0.003246	0.001744	0.01496	0.001251
	63	0.0007599	0.0004092	0.003508	0.0002930
	127	1.840e-4	9.913e-5	8.497e-4	7.094e-5

As seen from the values in Table 1, the RMS Error converges quadratically for the trapezoidal rule. The percent errors are presented in Table 2, where N/A represents the actual value of the integral being 0, preventing computation of the percent error.

Asymptotically, Table 2 shows that the % Error for the Trapezoidal rule also decreases quadratically. Finally, Fig. 1 presents a visualization of the trapezoids the rule creates for 7 samples.

Table 2: **% Error for Trapezoidal Rule**

		Function Tested			
		$\mathbf{1+x^2}$	$\mathbf{xe^{-x^2}}$	$\mathbf{xe^{-x}}$	$\sin(\mathbf{x})$
# of Samples	**3**	28.5714	N/A	86.1107	N/A
	7	3.1746	N/A	10.4690	N/A
	15	0.5831	N/A	1.9420	N/A
	31	0.1270	N/A	0.4237	N/A
	63	0.02973	N/A	0.09923	N/A
	127	0.007199	N/A	0.02403	N/A

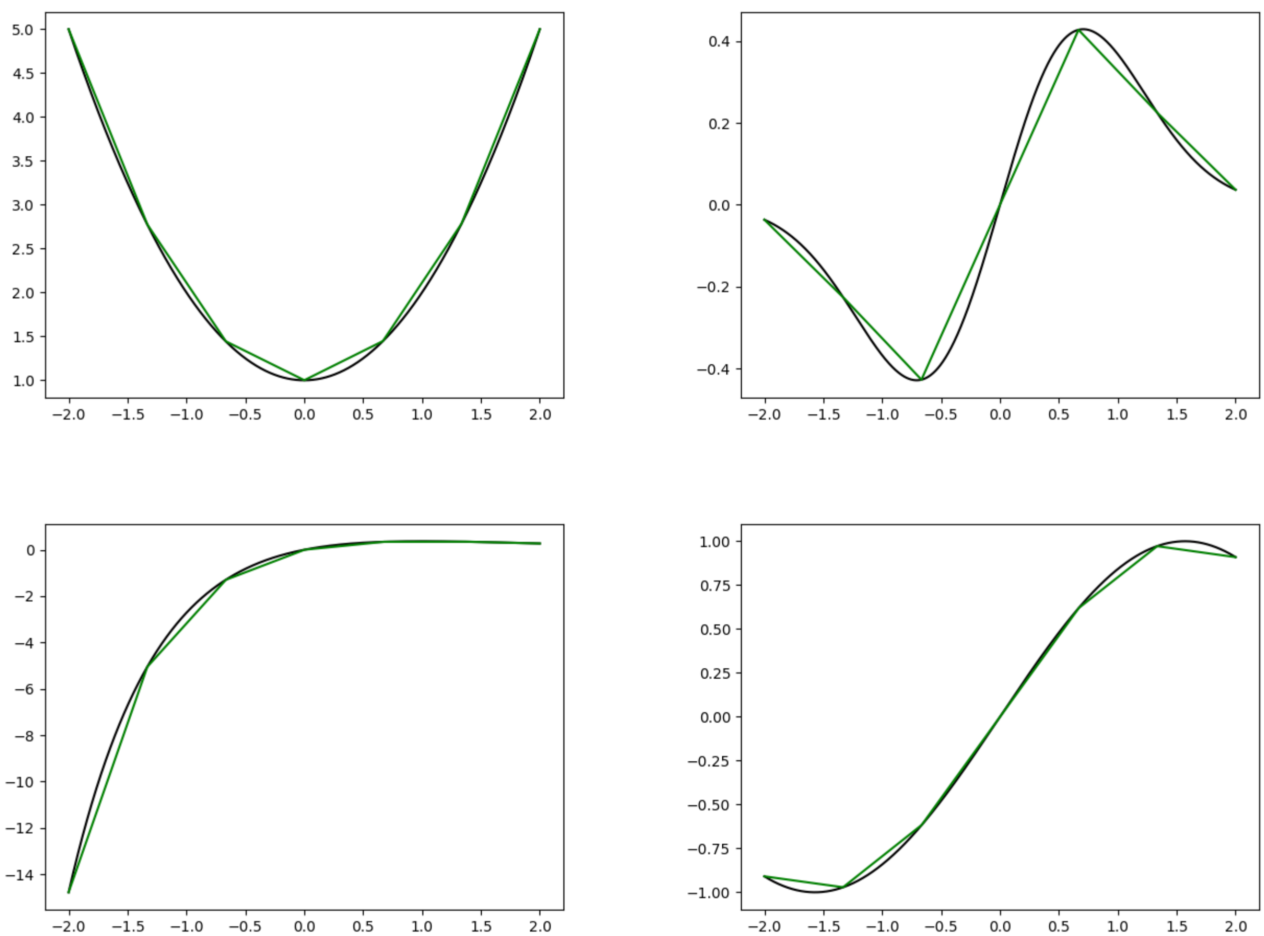

Figure 1: Figure 1 depicts a visualization of the trapezoids created by the trapezoidal rule for $n = 7$ samples.

3 Simpson's Rule

For Simpson's Rule, the following RMS Errors were evaluated as shown in Table 3:

Table 3: **RMS Error for Simpson's Fit**

		Function Tested			
		$\mathbf{1+x^2}$	$\mathbf{xe^{-x^2}}$	$\mathbf{xe^{-x}}$	$\sin(\mathbf{x})$
# of Samples	**3**	3.122e-16	0.2663	1.768	0.2780
	7	3.307e-16	0.04131	0.1383	0.008115
	15	3.217e-16	0.002941	0.01231	6.758e-4
	31	3.261e-16	3.083e-4	0.001281	6.927e-5
	63	3.639e-16	3.516e-5	1.459e-4	7.861e-6
	127	3.796e-16	4.196e-6	1.740e-5	9.370e-7

As seen from the values in Table 3, the RMS Error converges approximately cubically for Simpson's rule. The percent errors are presented in Table 4, where N/A represents the actual value of the integral being 0, preventing computation of the percent error.

Table 4: **% Error for Simpson's Rule**

		Function Tested			
		$\mathbf{1+x^2}$	$\mathbf{xe^{-x^2}}$	$\mathbf{xe^{-x}}$	$\sin(\mathbf{x})$
# of Samples	**3**	1.903e-14	N/A	24.0738	N/A
	7	1.903e-14	N/A	0.4822	N/A
	15	1.903e-14	N/A	0.01725	N/A
	31	3.806e-14	N/A	8.267e-4	N/A
	63	9.516e-14	N/A	4.542e-5	N/A
	127	3.806e-14	N/A	2.664e-6	N/A

Asymptotically, Table 2 shows that the % Error for Simpson's rule decreases approximately quartically. Finally, Fig. 2 presents a visualization of the parabolas the rule creates for 7 samples.

4 Conclusion

Overall, throughout the test of the trapezoidal rule and Simpson's rule, Simpson's rule achieved less error and also converged faster, cementing Simpson's rule and other higher order approximations as better than the naive trapezoidal rule for approximating definite integrals in these conditions.

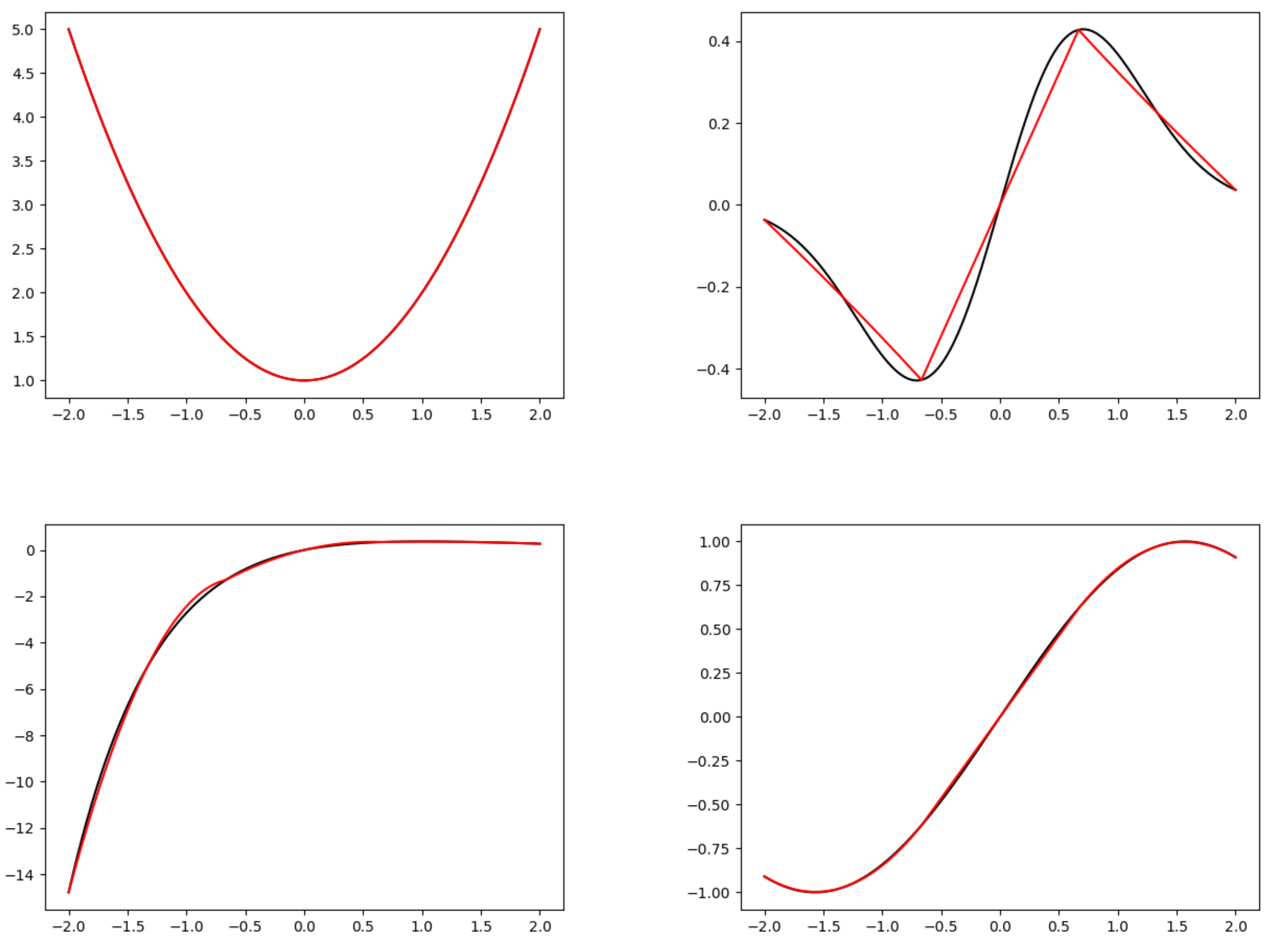

Figure 2: Figure 2 depicts a visualization of the parabolas created by Simpson's rule for $n = 7$ samples.