optimal point is encountered. The simplex algorithm does not require a starting point in the worksheet; to put it another way, it determines its own starting solution. This means that Solver ignores the initial information in the decision variable cells when solving a linear model. Once the solution procedure finds a **feasible solution**, one that satisfies all the constraints, it proceeds from there to an optimal solution. An **optimal solution** must satisfy all constraints and its objective function must equal the best value that can be achieved. Moreover, the simplex method *guarantees* that it will find a global optimum (if there is one), and in that sense, the simplex method is completely reliable. We cannot say the same for nonlinear optimization algorithms (such as GRG), except in special circumstances.

In mathematical terms, linear models are a special case of nonlinear models, and, in principle, the GRG algorithm could be used as a solution procedure for the examples we present here. However, the simplex algorithm is especially suited to linear models, and it avoids numerical problems that sometimes affect the performance of the GRG algorithm. The linear solver is the preferred choice for any linear programming problem.

Linear programming models come in many sizes and shapes, but there are only a few standard types. It is helpful, therefore, to think in terms of a few basic structures when learning how to build and interpret linear programming models. In this chapter and the next, we present four different types. Most linear programming models are, in fact, combinations of these four types, but understanding the building blocks helps to clarify the key modeling concepts. In our framework, the four types are

- *allocation* models,
- *covering* models,
- *blending* models,
- *network* models.

We cover the first three types in this chapter and network models separately in Chapter 12.

11.2 ALLOCATION MODELS

The allocation model calls for maximizing an objective (usually profit) subject to less-than constraints on capacity. Consider the Veerman Furniture Company as an example.

EXAMPLE *Veerman Furniture Company*

Veerman Furniture Company makes three kinds of office furniture: chairs, desks, and tables. Each product requires some labor in the parts fabrication department, the assembly department, and the shipping department. The furniture is sold through a regional distributor, who has estimated the maximum potential sales for each product in the coming quarter. Finally, the accounting department has provided some data showing the profit contributions on each product. The decision problem is to determine the product mix—that is, to maximize Veerman's profit for the quarter by choosing production quantities for the chairs, desks, and tables. The data shown below summarize the parameters of the problem:

	Hours per Unit			
Department	**Chairs**	**Desks**	**Tables**	**Hours Available**
Fabrication	4	6	2	1,850
Assembly	3	5	7	2,400
Shipping	3	2	4	1,500
Demand Potential	360	300	100	
Profit	$15	$24	$18	

■

11.2.1 Formulation

As recommended in the previous chapter, we approach the formulation of the optimization model by asking three basic questions. To determine the decision variables, we ask, ''What must be decided?'' The answer is the product mix, so we define decision variables as the number of chairs, desks, and tables produced. For the purposes of notation, we use C, D, and T to represent the number of chairs, the number of desks, and the number of tables respectively, in the product mix.

Next we ask, ''What measure can we use to compare alternative sets of decision variables?'' To choose between two different product mixes, we would calculate the total profit contribution for each one and choose the higher profit. To calculate profit, we add the profit from chairs, the profit from desks, and the profit from tables. Thus, an algebraic expression for total profit is:

$$\text{Profit} = 15C + 24D + 18T$$

To identify the model's constraints, we ask, ''What restrictions limit our choice of decision variables?'' In this scenario, there are two kinds of limitations: one due to production capacity and the other due to demand potential. In words, a production capacity constraint states that the number of hours *consumed* in the fabrication department must be less than or equal to the number of hours *available*. In symbols, we write:

$$\textit{Fabrication hours consumed} = 4C + 6D + 2T \leq 1{,}850 \text{ (Fabrication hours available)}$$

Similar constraints hold for the assembly and shipping departments:

$$\textit{Assembly hours consumed} = 3C + 5D + 7T \leq 2{,}400 \text{ (Assembly hours available)}$$
$$\textit{Shipping hours consumed} = 3C + 2D + 4T \leq 1{,}500 \text{ (Shipping hours available)}$$

Another type of constraint relates to demands. We require that the number of chairs *produced* must be less than or equal to the estimated demand *potential* for chairs. In symbols, we write:

$$\textit{Chairs produced} = C \leq 360 \text{ (Chair demand potential)}$$

Similar constraints hold for desks and tables:

$$\textit{Desks produced} = D \leq 300 \text{ (Desk demand potential)}$$
$$\textit{Tables produced} = T \leq 100 \text{ (Table demand potential)}$$

We now have six constraints that describe the restrictions limiting our choice of decision variables C, D, and T. The entire model, stated in algebraic terms, reads as follows:

$$\text{Maximize } z = 15C + 24D + 18T$$

subject to

$$\begin{array}{rcrcrcr}
4C & + & 6D & + & 2T & \leq & 1{,}850 \\
3C & + & 5D & + & 7T & \leq & 2{,}400 \\
3C & + & 2D & + & 4T & \leq & 1{,}500 \\
C & & & & & \leq & 360 \\
 & & D & & & \leq & 300 \\
 & & & & T & \leq & 100
\end{array}$$

11.2.2 Spreadsheet Model

This algebraic statement reflects a widely used format for linear programs. Variables appear in columns, constraints appear as rows, and the objective function appears as a special row at the top of the model. We will adopt this layout as a standard for spreadsheet display.

FIGURE 11.6 Optimal Solution for the Veerman Furniture Company Model

	A	B	C	D	E	F	G	H
1	**Allocation: Furniture Production**							
2								
3	**Decision Variables**							
4		C	D	T				
5	Product mix	0	275	100				
6								
7	**Objective Function**				*Total*			
8	Profit	15	24	18	$8,400			
9								
10	**Constraints**				*LHS*		*RHS*	
11	Fabrication	4	6	2	1850	<=	1850	Binding
12	Assembly	3	5	7	2075	<=	2400	Not Binding
13	Shipping	3	2	4	950	<=	1500	Not Binding
14	Chair market	1	0	0	0	<=	360	Not Binding
15	Desk market	0	1	0	275	<=	300	Not Binding
16	Table market	0	0	1	100	<=	100	Binding
17								

11.1 11.2 11.6

of bringing a limited range of products to the market in order to maximize short-term profits. This is one of the places where the simplifications in a model could be assessed against the realities of the actual situation.) On the other hand, if there is time to adjust the resources available at Veerman Furniture, we should explore the possibility of acquiring more fabrication capacity or expanding the demand potential for tables, as these are the binding constraints. In addition, if there is time to adjust marketing policies, we might also want to look into the possibility of raising the price on chairs.

11.3 COVERING MODELS

The covering model calls for minimizing an objective (usually cost) subject to greater-than constraints on required coverage. Consider Dahlby Outfitters as an example.

EXAMPLE ***Dahlby Outfitters***

Dahlby Outfitters wishes to introduce packaged trail mix as a new product. The ingredients for the trail mix are seeds, raisins, flakes, and two kinds of nuts. Each ingredient contains certain amounts of vitamins, minerals, protein, and calories. The marketing department has specified that the product be designed so that a certain minimum nutritional profile is met. The decision problem is to determine the optimal product composition—that is, to minimize the product cost by choosing the amount for each of the ingredients in the mix. The data shown below summarize the parameters of the problem:

	Grams per Pound					
Component	**Seeds**	**Raisins**	**Flakes**	**Pecans**	**Walnuts**	**Nutritional Requirement**
Vitamins	10	20	10	30	20	20
Minerals	5	7	4	9	2	10
Protein	1	4	10	2	1	15
Calories	500	450	160	300	500	600
Cost/pound	$4	$5	$3	$7	$6	

■

11.3.1 Formulation

What must be decided? Here, the answer is the amount of each ingredient to put into a package of trail mix. For the purposes of notation, we use S, R, F, P, and W to represent the number of pounds of each ingredient in a package.

What measure will we use to compare sets of decision variables? This should be the total cost of a package, and our goal is the lowest possible total cost. To calculate the total cost of a particular composition, we add the cost of each ingredient in the package:

$$Cost = 4S + 5R + 3F + 7P + 6W$$

What restrictions limit our choice of decision variables? In this scenario, the main limitation is the requirement to meet the specified nutritional profile. Each dimension of this profile gives rise to a separate constraint. An example of such a constraint would state, in words, that the number of grams of vitamins *provided* in the package must be greater than or equal to the number of grams *required* by the specified profile. In symbols, we write:

$$Vitamin\ content = 10S + 20R + 10F + 30P + 20W \geq 20 \text{ (Vitamin floor)}$$

Similar constraints must hold for the remainder of the profile:

$$Mineral\ content = 5S + 7R + 4F + 9P + 2W \geq 10 \text{ (Mineral floor)}$$
$$Protein\ content = 1S + 4R + 10F + 2P + 1W \geq 15 \text{ (Protein floor)}$$
$$Calorie\ content = 500S + 450R + 160F + 300P + 500W \geq 600 \text{ (Calorie floor)}$$

In this basic scenario, no other constraints occur, although we could imagine that there might also be limited quantities of the ingredients available, expressed as less-than constraints, or a weight requirement for the package, expressed as an equality constraint. Putting the algebraic statements together in one place, we obtain the following model:

$$\text{Minimize } z = 4S + 5R + 3F + 7P + 6W$$

subject to

$$\begin{aligned}
10S + 20R + 10F + 30P + 20W &\geq 20 \\
5S + 7R + 4F + 9P + 2W &\geq 10 \\
1S + 4R + 10F + 2P + 1W &\geq 15 \\
500S + 450R + 160F + 300P + 500W &\geq 600
\end{aligned}$$

11.3.2 Spreadsheet Model

A worksheet for this model appears in Figure 11.7. Again, we see three modules:

- a highlighted row for the decision variables (B5:F5)
- a highlighted single cell for the objective function value (G8)
- a set of constraint relationships (rows 11–14)

If we were to display the formulas in this worksheet, we would again see that the model is made up of only numbers and cells containing the SUMPRODUCT formula.

FIGURE 11.7 Worksheet for the Dahlby Outfitters Model

	A	B	C	D	E	F	G	H	I	J
1	**Covering: Trail Mix Composition**									
2										
3	**Decision Variables**									
4		S	R	F	P	W				
5	Amounts	0.50	0.40	0.30	0.70	0.20				
6										
7	**Objective Function**						*Total*			
8	Cost	4	5	3	7	6	$11.00			
9										
10	**Constraints**						*LHS*		*RHS*	
11	Vitamins	10	20	10	30	20	41	>=	16	Not Binding
12	Minerals	5	7	4	9	2	13.2	>=	10	Not Binding
13	Protein	1	4	10	2	1	6.7	>=	15	Not Binding
14	Calories	500	450	160	300	500	788	>=	600	Not Binding
15										

11.7

bound constraints and upper bound constraints appear in the Bounds subsection of the model's constraints.

After including the lower bound constraints, a new run of Solver produces the optimal solution shown in Figure 11.9. The lower bounds create an optimal solution that contains all five of the ingredients, as expected. We might have anticipated that nuts would appear at their lower limit, because before we added the lower bound constraints the optimization process kept nuts completely out of the mix. The minimum cost is also \$0.79 higher in the amended model than in the original, at \$8.33. This fact reflects an intuitive principle that complements the one we stated earlier: *When we add constraints to a model, the objective function cannot improve*. In most cases, as in this example, the objective function will get worse when we add a constraint.

11.4 BLENDING MODELS

Blending relationships are very common in linear programming applications, yet they remain difficult for beginners to identify in problem descriptions and to implement in spreadsheet models. Because of this difficulty, we begin with a special case—the representation of proportions. As an example, let's return to the product mix example of Veerman Furniture which was introduced earlier in this chapter. Recall from Figure 11.6 that the optimal product mix consisted of no chairs, 275 desks, and 100 tables. Suppose that this outcome is unacceptable because of the imbalance in volumes. For more balance, the Marketing Department might require that each of the products must make up at least 25 percent of the total units sold.

11.4.1 Blending Constraints

When we describe outcomes in terms of proportions, and when we place a floor (or ceiling) on one or more of those proportions, we are using blending constraints of a special type. Because the total number of units sold is $C + D + T$, a direct statement of the requirement for chairs is the following.

$$\frac{C}{C + D + T} \geq 0.25 \qquad (11.1)$$

This greater-than constraint has a parameter on the right-hand side and all the decision variables on the left-hand side, as is usually the case. Although this is a valid constraint, it is not in *linear* form, because the quantities C, D, and T appear in both the numerator and denominator of the fraction. (In effect, the ratio divides decision variables by decision variables.) However, we can convert the nonlinear inequality to a linear one with a bit of algebra. First, multiply both sides of the inequality by $(C + D + T)$, yielding:

$$C \geq 0.25(C + D + T)$$

Next, collect terms involving the decision variables on the left-hand side, so that we get:

$$0.75C - 0.25D - 0.25T \geq 0 \qquad (11.2)$$

This form conveys the same requirement as the original fractional constraint, and we recognize it immediately as a linear form. The coefficients on the left-hand side of (11.2) turn out to be either the complement of the floor (1 – 0.25) or the floor itself (but with a minus sign). In a similar fashion, the requirement that the other products must respect the floor leads to the following two constraints.

$$-0.25C + 0.75D - 0.25T \geq 0$$
$$-0.25C - 0.25D + 0.75T \geq 0$$

In Step 3, it is not actually necessary to collect terms because Solver allows us to leave a SUMPRODUCT formula on both sides of the inequality. However, we recommend collecting terms so that the variables appear on the left-hand side of the inequality and a constant appears on the right. This format is consistent with allocation and blending constraints and may make it easier to debug a model.

In general, the blending model involves mixing materials with different individual properties and describing the properties of the blend with weighted averages. We might be familiar with the phenomenon of mixing if we have spent time in a chemistry lab mixing fluids with different concentrations, but the concept extends beyond lab work. Consider the Diaz Coffee Company as an example.

EXAMPLE ***The Diaz Coffee Company***

The Diaz Coffee Company blends three types of coffee beans (Brazilian, Colombian, and Peruvian) into ground coffee to be sold at retail. Suppose that each kind of bean has a distinctive aroma and strength, and the company has a chief taster who can rate these features on a scale of 1 to 100. The features of the beans are tabulated below:

Bean	Aroma Rating	Strength Rating	Cost/lb.	Pounds Available
Brazilian	75	15	\$0.50	1,500,000
Colombian	60	20	\$0.60	1,200,000
Peruvian	85	18	\$0.70	2,000,000

The company would like to create a blend that has an aroma rating of at least 78 and a strength rating of at least 16. Its supplies of the various beans are limited, however. The available quantities are specified above. All beans are delivered under a previously arranged purchase agreement. Diaz wants to make four million pounds of the blend at the lowest possible cost. ■

11.4.2 Formulation

Suppose, for example, that we blend Brazilian and Peruvian beans in equal quantities of 25 pounds each. Then we should expect the blend to have an aroma rating of 80, just halfway between the two pure ratings of 75 and 85. Mathematically, we take the weighted average of the two ratings:

$$\text{Aroma rating} = \frac{25(75) + 25(85)}{25 + 25} = 80$$

Now suppose that we blend the beans in amounts B, C, and P. The blend will have an aroma rating calculated by a weighted average of the three ratings, as follows:

$$\text{Aroma rating} = \frac{B(75) + C(60) + P(80)}{B + C + P}$$

To impose a constraint that requires an aroma rating of at least 78, we write:

$$\frac{B(75) + C(60) + P(80)}{B + C + P} \geq 78 \tag{11.3}$$

This greater-than constraint has a parameter on the right-hand side of (11.3) and all the decision variables on the left-hand side, as is usually the case. Although this is a valid constraint, it is not in *linear* form, because the quantities B, C, and P appear in both the numerator and denominator of the fraction. If we were to include this form of the constraint in Solver, we would be forced to use the nonlinear solver to get a solution. However, we can convert the nonlinear inequality to a linear one by following the steps listed earlier and thus satisfy the linear solver. First, multiply both sides of the inequality by $(B + C + P)$, yielding:

$$75B + 60C + 85P \geq 78(B + C + P)$$

Next, collect terms on the left-hand side, so that we get:

$$-3B - 18C + 7P \geq 0 \tag{11.4}$$

This form conveys the same requirement as the original fractional constraint in (11.3), and we recognize it immediately as a linear constraint. The coefficients on the left-hand side turn out to be just the *differences* between the individual aroma ratings (75, 60, 85) and the requirement of 78, with the signs indicating whether the individual rating is above or below the target. In a similar fashion, a requirement that the strength of the blend must be at least 16 leads to the constraint

$$-1B + 4C + 2P \geq 0$$

Thus, blending requirements are stated initially as fractions in (11.1) and (11.3), and in that form, they lead to nonlinear constraints. We are interested in converting these to linear constraints, as in (11.2) and (11.4), because with a linear model, we can harness the power of the linear solver. As discussed earlier, this means that we can find a global optimum reliably.

Now, knowing how to incorporate the blending requirements, we return to our scenario.

- What must be decided? The decision variables are the quantities to purchase, which we can continue to represent as B, C, and P. However, due to the scale of the model, it is convenient to take the dimensions of these three quantities to be thousands of pounds.
- What measure will we use? Evidently, it is the total purchase cost of meeting our four-million-pound requirement.
- What restrictions must we meet? In addition to the blending constraints, we need a constraint that generates a four-million-pound blend, along with three constraints that limit the supplies of the different beans.

11.4.3 Spreadsheet Model

Figure 11.12 shows the spreadsheet for our model, which contains a greater-than constraint and three less-than constraints, in addition to the blending constraints. In addition, the model has been scaled by taking the supply and output constraints, as well as the decision variables, to be in thousands of pounds. The three decision variables have been set arbitrarily to 100,000 in the figure. In a sense, the model has two key blending constraints, and it also has what we might think of as covering and allocation constraints. Each of the constraints takes the same form: a SUMPRODUCT formula on the left-hand side and a parameter on the right-hand

FIGURE 11.12 Worksheet for the Diaz Coffee Company Model

	A	B	C	D	E	F	G	H
1	**Blending: Coffee beans**							
2								
3	**Decision Variables**							
4		B	C	P				
5	Inputs	100	100	100			in '000	
6								
7	**Objective Function**				*Total*			
8	Cost	0.50	0.60	0.70	$180		in '000	
9								
10	**Constraints**				*LHS*		*RHS*	
11	Blend aroma	-3	-18	7	-1400	>=	0	Not Binding
12	Blend strength	-1	4	2	500	>=	0	Not Binding
13	Output	1	1	1	300	>=	4000	Not Binding
14	B-supply	1	0	0	100	<=	1500	Not Binding
15	C-supply	0	1	0	100	<=	1200	Not Binding
16	P-supply	0	0	1	100	<=	2000	Not Binding
17	*Actual aroma*	75	60	85	73.3		78	
18	*Actual strength*	15	20	18	17.7		16	
19								

11.12