The 2D Helmholtz decomposition of a vector field with open boundaries

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1 The Problem

Given a velocity field $\boldsymbol{u}(x,y) = u(x,y)\hat{\boldsymbol{\imath}} + v(x,y)\hat{\boldsymbol{\jmath}}$ defined on a 2D space, we place a square inside the field and define this as our boundary $\mathcal{B}$ with zonal length L_x and meridional length L_y. The problem is then to **find a ψ and a ϕ such that**

$$\begin{aligned}
\boldsymbol{u}(x,y) &= \nabla\phi + \nabla\psi \times \hat{\boldsymbol{k}} \\
\Longrightarrow u(x,y) &= \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y} \\
\Longrightarrow v(x,y) &= \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}
\end{aligned}$$

within the area A bounded by $\mathcal{B}$. This involves solving the system of poisson equations

$$\nabla^2\phi = \nabla\cdot\boldsymbol{u} \tag{1}$$
$$\nabla^2\psi = -\nabla\times\boldsymbol{u} \tag{2}$$

which are coupled through the boundary conditions

$$\frac{\partial\phi}{\partial x}|_{\mathcal{B}} + \frac{\partial\psi}{\partial y}|_{\mathcal{B}} = u(x,y)|_{\mathcal{B}} \tag{3}$$
$$\frac{\partial\phi}{\partial y}|_{\mathcal{B}} - \frac{\partial\psi}{\partial x}|_{\mathcal{B}} = v(x,y)|_{\mathcal{B}} \tag{4}$$

As the potentials ϕ, ψ are only defined up to a constant, we set

$$\phi(0,0) = 0 = \psi(0,0) \tag{5}$$

Applying the divergence theorem to our boundary $\mathcal{B}$ gives

$$\int_A \nabla\cdot\boldsymbol{u}\,dxdy = \int_{\mathcal{B}} \boldsymbol{u}\cdot\boldsymbol{n}\;dl \tag{6}$$

where $\boldsymbol{n}$ is the outward pointing normal and dl is a line element along the boundary. Substituing in the boundary conditions for the velocity gives a constraint on the velocity potential ϕ

$$\int_{\mathcal{B}} \frac{\partial\phi}{\partial n}\,dl = \int_A \nabla\cdot\boldsymbol{u}\,dxdy \tag{7}$$

Similarly applying Stokes' theorem to our boundary $\mathcal{B}$

$$\int_A \nabla \times \boldsymbol{u}\, dxdy = -\int_{\mathcal{B}} \boldsymbol{u} \cdot \boldsymbol{t}\, dl, \tag{8}$$

where $\boldsymbol{t}$ is the tangent vector to the boundary, gives a constraint on the streamfunction ψ

$$\int_{\mathcal{B}} \frac{\partial \psi}{\partial n}\, dl = -\int_A \nabla \times \boldsymbol{u}\, dxdy \tag{9}$$

These constraints must also be satisfied by the solution.

2 Continous method description

The boundary conditions (3) and (4) are applied only to the velocity and thus we are free to apply any boundary condition to either ϕ or ψ with the boundary condition on the other determined by (3) and (4), providing that our chosen boundary condition does not violate (7) or (9). This is where the non-uniqueness of the solutions comes in. We will get a solution ψ, ϕ to the problem, where the splitting depends on the chosen boundary condition.
In the particular case in which the vector field is the velocity on a horizontal surface in the ocean, we know that this field is mostly

non-divergent (geostrophic balance). Therefore a sensible choice for the boundary condition is one that emphasizes the streamfunction ψ over the vector potential ϕ. We will make the choice

$$\phi|_{\mathcal{B}} = 0 \tag{10}$$

Note that this choice does not contradict any of the boundary conditions or the divergence or Stokes' theorem. Having made this choice, we can solve equation (1) for the velocity potential ϕ. At this stage, we have managed to split the velocity into purely divergent and purely rotational parts

$$\boldsymbol{u}_{div} = \nabla \phi \quad \& \quad \boldsymbol{u}_{rot} = \boldsymbol{u} - \boldsymbol{u}_{div} = \nabla \psi \times \hat{\boldsymbol{k}} \tag{11}$$

In order to calculate the streamfunction ψ we move the now known terms involving ϕ in the boundary conditions (3) and (4) to the RHS, which is equivalent to replacing the total velocity in (3) and (4) with the rotational velocity $\boldsymbol{u}_{rot}$. We now have the simpler problem

$$\nabla^2 \psi = -\nabla \times \boldsymbol{u} = \nabla \times \boldsymbol{u}_{rot} \tag{12}$$

with the boundary conditions

$$\frac{\partial \psi}{\partial y}|_{\mathcal{B}} = u_{rot}(x, y)|_{\mathcal{B}} \tag{13}$$

$$-\frac{\partial \psi}{\partial x}|_{\mathcal{B}} = v_{rot}(x, y)|_{\mathcal{B}} \tag{14}$$

This problem can be solved by performing a perimeter integral of the tangent derivative of ψ along the boundary starting with the value $\psi(0,0) = 0$ in the bottom left corner. If the solution obtained for the velocity potential ϕ is correct, then through the divergence theorem this implies that this perimeter integral will give $\psi(0,0) = 0$ upon returning to the bottom left corner. (This is even true if there was divergence on the boundary, because this divergence will have been taken care of by the normal derivative $\frac{\partial \phi}{\partial n}$.) The streamfunction is then obtained as the solution to (12) using the dirichlett boundary conditions obtained through the perimeter integral.

3 Numerical Inplementation

The method for finding a numerical solution used here involves first placing the velocity field on a uniform staggered grid like the Arakawa C-grid in ROMS. This is required for an accurate solution because on a simple grid taking a central different twice (for example on the streamfunction or velocity potential) results in a stencil 5-points wide. For example,

$$\frac{\partial \psi}{\partial x}|_{ij} = \frac{\psi_{(i+1)j} - \psi_{(i-1)j}}{2\Delta x} \tag{15}$$

$$\Longrightarrow \frac{\partial}{\partial x}\frac{\partial \psi}{\partial x}|_{ij} = \frac{\psi_{(i+2)j} - 2\psi_{ij} + \psi_{(i-2)j}}{4(\Delta x)^2} \neq \frac{\psi_{(i+1)j} - 2\psi_{ij} + \psi_{(i-1)j}}{(\Delta x)^2} \tag{16}$$

This makes dealing with the boundary conditions much more difficult. Therefore, we use a natural staggered Arakawa C-grid with central differences which move results from one grid to another grid. This works out naturally for this problem.
Given input data lon, lat, u, v colocated on a curved grid (i.e. output of satellite altimetry or ROMS *GetVar*), the solution method follows the steps:

- create a regular Arakawa C-grid with constant spacing Δx, Δy placed within the outer bounds of the input data.
- Interpolate the velocities u and v onto the appropriate grid points (i.e. LON_u and LON_v)
- Calculate the divergence $\nabla \cdot \boldsymbol{u}$ which will be on the ϕ points.
- set the initial ϕ field to zero.
- Contstruct the Dirichlett Laplacian matrix for the ϕ grid one grid point inside the outer points (i.e. the outer ϕ are set to zero and not included in the solution).
- Construct the RHS vector containing the divergence on the inner grid. Note that with the zero boundary conditions, no additional BC terms need to be added to the RHS as these are zero.
- Solve the matrix inversion problem using \.
- Calculate the divergent and thus rotational velocity through subtraction from the full velocity. Calculate the curl of the rotational (or full) velocity.
- As a result of this process we now have a completely non-divergent field that we can use for boundary conditions for the next poisson equation. However, we only have this field on the interior points and so to use it we decrease the size of our domain (on the rotational velocity) by one grid point on all sides in order to solve for ψ.
- On this new decreased grid, we perform a perimeter integral of the tangent derivative part of the boundary conditions (13) and (14) starting in the bottom left corner ψ point. We now have a dirichlett boundary condition on ψ.
- Construct the Dirichlett Laplacian matrix on the smaller grid.
- Construct the RHS vector being the curl of the rotational velocity. However, we now have a non-zero dirichlett boundary condition determined by the perimeter integral, and thus must add these boundary terms to the RHS vector.
- Solve the matrix inversion problem.

- resize the ϕ field to that of the ψ field, although for an exact output they are on different grids.

4 Examples

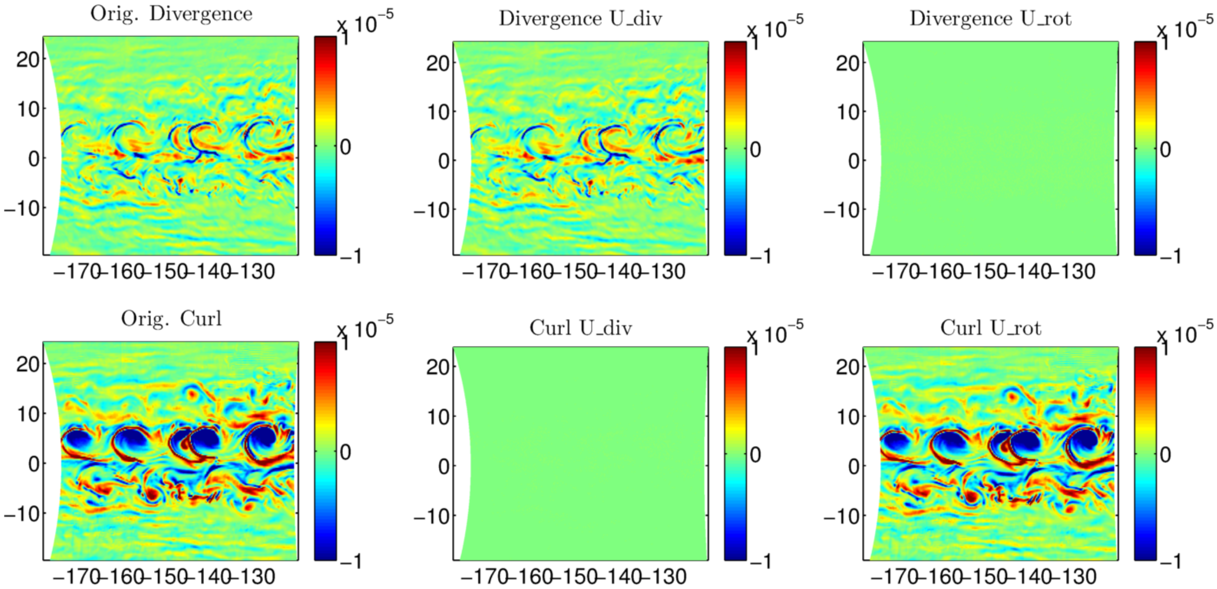

Figure 1: Figure showing an example calculation where (left) full divergence and curl (middle) divergence and curl of $\boldsymbol{u}_{div}$ and (right) divergence and curl of $\boldsymbol{u}_{rot}$

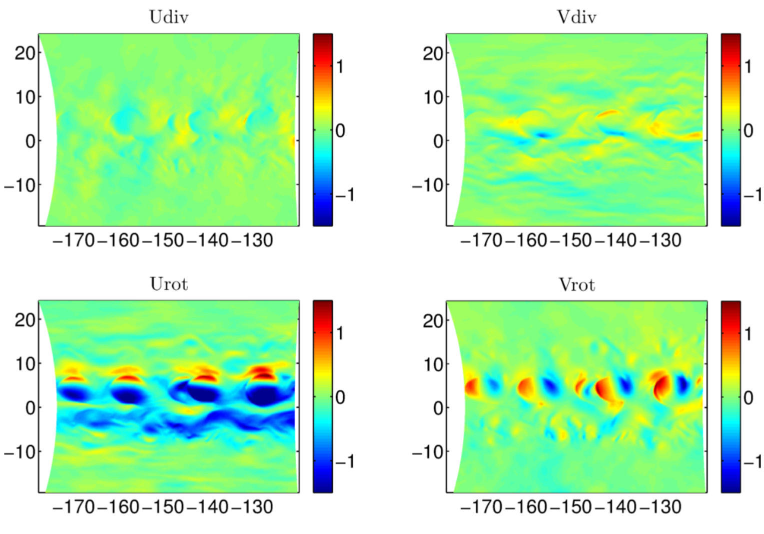

Figure 2: Figure showing $\boldsymbol{u}_{rot}$ and $\boldsymbol{u}_{div}$ for the same fields as in Fig. 1

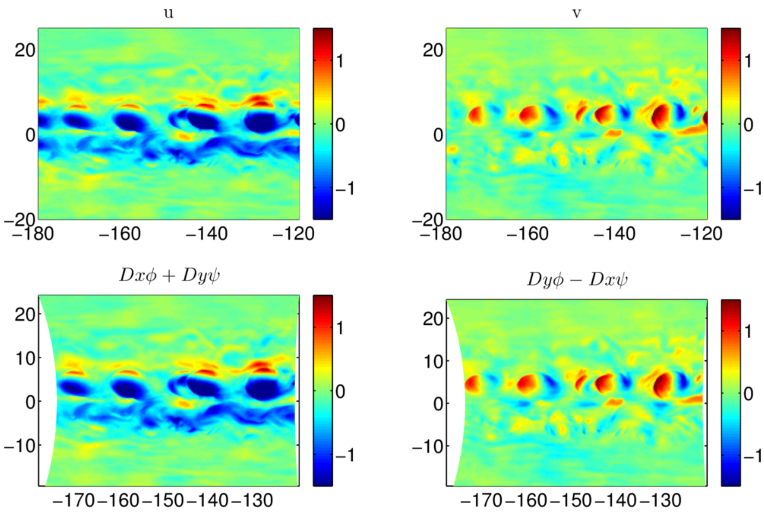

Figure 3: Figure comparing the full velocities and the velocities calculated using the potentials ϕ and ψ

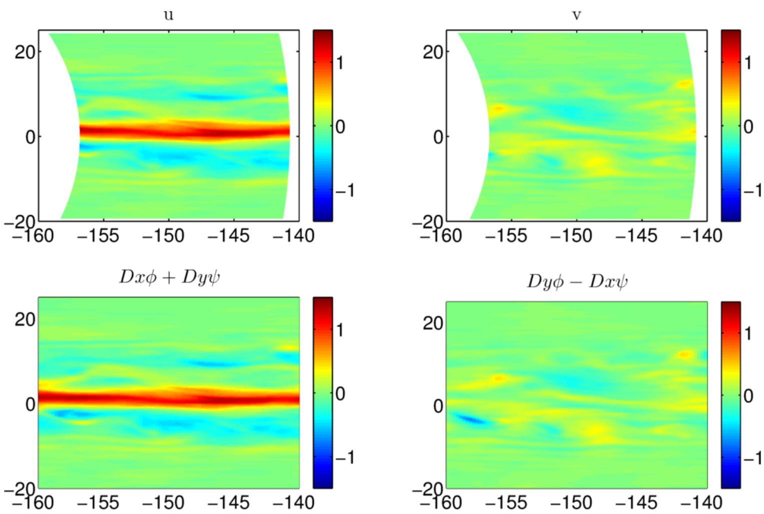

Figure 4: Figure comparing the full velocities and the velocities calculated using the potentials ϕ and ψ for a deeper slice.