The MATSLISE test set

1 Introduction

A collection of more than 50 examples of Sturm-Liouville problems of the form

$$-(p(x)y'(x))' + q(x)y(x) = Ew(x)y(x) \quad x \in (a,b)$$

has been predefined in the MATSLISE package. This MATSLISE test set collects problems which are commonly used to test Sturm-Liouville software, some examples from physical applications and some classical mathematical problems. Many examples have been taken from earlier test sets:

- the list of test problems provided by John D. Pryce in the book *Numerical Solution of Sturm-Liouville Problems, Oxford University Press, 1993,* and the SLTSTPAK package: *A test package for Sturm-Liouville solvers, ACM Trans. Math. Software 25 (1999).*
- the catalogue of Sturm-Liouville problems composed by W.N. Everitt: *A Catalogue of Sturm-Liouville differential equations, in: Sturm-Liouville Theory. Past and Present, W.O. Amrein, A.M. Hinz, D.B. Pearson (eds), Birkhuser-Verlag 2005.*

For some of the examples explicit information on the spectrum of associated boundary value problems is provided against which the program calculated results can be compared.

2 List of Test Problems

1. **Airy equation**. (`Airy.mat`). (SLTSTPAK #27).
   $p(x) = 1 \qquad q(x) = x \qquad w(x) = 1$
   $a = 0$ Regular $y(a) = 0$
   $b = +\infty$ LPN
   Number of eigenvalues: ∞ continuous spectrum: none
   Eigenvalues are the zeros of Airy function $Ai(E) = (J_{1/3} + J_{-1/3})(\frac{2}{3}E^{1/3})$.
   $E_0 = 2.3381074104 \quad E_9 = 12.828776753$
   Reference: E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Oxford University Press, 1946, p.91.

2. **Anharmonic oscillator potential**. (`anharm_oscillator.mat`).
   $p(x) = 1 \qquad q(x) = x^2 + \lambda x^2/(1+gx^2) \qquad w(x) = 1$
   λ, g parameters
   $a = -\infty$ LPN
   $b = +\infty$ LPN
   Number of eigenvalues: ∞ continuous spectrum: none

$\lambda = 0.1$, $g = 0.1$: $\quad E_0 = 1.043173713$
$\lambda = 10.0$, $g = 10.0$: $\quad E_0 = 1.580022327$
Reference: V. Fack and G. Vanden Berghe. (Extended) Numerov method for computing eigenvalues of specific Schrödinger equations. *J. Phys. A: Math. Gen.*, 20 (1987), 4153-4160.

3. **Associated Legendre equation in Liouville normal form**.
(`associated_Legendre_normalform.mat`). (SLTSTPAK #16).
$p(x) = 1 \qquad q(x) = -1/4 + (c - 1/4)\sec(x)^2 \qquad w(x) = 1$
Let $c = \nu^2 \geq 0$
$a = -\pi/2$, $b = +\pi/2$
$\nu = 0$: LCN
$\nu = 1/2$: Regular
$0 < \nu < 1$, $\nu \neq 1/2$: LCN
$\nu \geq 1$: LPN
Number of eigenvalues: ∞ continuous spectrum: none
$E_k = (k + \nu + 1)(k + \nu),\ k = 0, 1, \ldots$
Reference: Dunford and Schwartz 1963, Linear Operators: Part II: Spectral Theory, Self Adjoint Operators in Hilbert Space. Pure and Applied Mathematics. Wiley-Interscience, New York, NY p. 1510.

4. `BaileyEtAl.mat`. (SLTSTPAK #33).
$p(x) = 1 \qquad q(x) = -7x^2 + 0.5x^3 + x^4 \qquad w(x) = 0.5$
$a = -\infty$ LPN
$b = \infty$ LPN
Number of eigenvalues: ∞ continuous spectrum: none
$E_0 = -24.5175977072 \quad E_5 = 8.10470769427$.
Reference: P.B. Bailey, M.K. Gordon and L.F. Shampine. Solving Sturm-Liouville eigenproblems. *Sandia Tech. Rep. SAND76-0560*, (1976).
also: Coll and Melius. Theoretical calculation of Raman scattering cross-sections for use in flame analysis. *Rep. SAND79-8204*, (1976).

5. `Bender_Orszag.mat`. (SLTSTPAK #14).
$p(x) = 1 \qquad q(x) = -m(m+1)/\cosh^2 x \qquad w(x) = 1$
m parameter
$a = -\infty$ LPN/O
$b = +\infty$ LPN/O
Number of eigenvalues: Number of integers in range $0 \leq k < m$
Continuous spectrum: $(0, \infty)$
$E_k = -(m - k)^2, \quad 0 \leq k < m.$
Reference: Bender and Orszag. *Adv. Math. Methods for Scientists and Engineers.* McGraw-Hill, N.Y., 1978, p. 28.

6. **Bessel equation, order** $1/2$. (`Bessel.mat`). (SLTSTPAK #19).
$p(x) = x \qquad q(x) = \alpha/x \qquad w(x) = x$
$\alpha = \nu^2$, $\nu = \frac{1}{2}$
$a = 0$ LCN
$b = 1$ Regular $\quad y(b) = 0$
$E_k = ((k+1)\pi)^2$, this is $-v'' = Ev$ transformed by $v = x^{1/2}u$.

Reference: E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Oxford University Press, 1946, p.81.

7. **Bessel equation in normal form, order** $1/4$. (`Bessel_normalform.mat`). (SLTSTPAK #13).
   $p(x) = 1 \qquad q(x) = (\alpha - 1/4)/x^2 \qquad w(x) = 1$
   $\alpha = 1/2$
   $a = 0$ LCN
   $b = 1$ Regular $y(b) = 0$
   Number of eigenvalues: ∞ continuous spectrum: none.
   $E_0 = 11.7768123$, $E_9 = 1007.256998$

8. **Bessel equation in normal form, order 0**. (`Bessel_normalform0.mat`). (SLTSTPAK #18).
   $p(x) = 1 \qquad q(x) = (\alpha - 1/4)/x^2 \qquad w(x) = 1$
   $\alpha = 0$
   $a = 0$ LCN
   $b = 1$ Regular $y(b) = 0$
   $E_0 = 5.78318596295 \quad E_{19} = 3850.01252885.$
   Reference: E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Oxford University Press, 1946, p.81.

9. **Bessel equation in normal form, order 0.01**. (`Bessel_normalform0.01.mat`). (SLTSTPAK #43).
   Bessel equation in normal form with $\alpha = 0.01$.
   LCN for small $\alpha \geq 0$
   Number of eigenvalues: ∞ continuous spectrum: none
   $E_0 = 6.540555712 \quad E_{24} = 6070.441468.$

10. **Biswas potential**. (`Biswas.mat`).
    $p(x) = 1 \qquad q(x) = \mu x^2 + \nu x^4 \qquad w(x) = 1$
    μ, ν parameters
    $a = -\infty$ LPN
    $b = +\infty$ LPN
    Number of eigenvalues: ∞ continuous spectrum: none
    $\mu = 0.0, \nu = 1.0: \quad E_0 = 1.06036209$
    $\mu = 1.0, \nu = 1.0: \quad E_0 = 1.39235164.$
    Reference: S.N. Biswas, K. Datta, R.P. Saxena, P.K. Srivastava and V.S. Varma. Eigenvalues of λx^{2m} anharmonic oscillators . *J. Math. Phys.*, 14 (1973), 1190.
    also: V. Fack and G. Vanden Berghe. A finite difference approach for the calculation of perturbed oscillator energies. *J. Phys. A: Math. Gen.*, 18 (1985), 3355-3363.

11. **Border of LPN and LCN**. (`border_LPN_LCN.mat`).(SLTSTPAK #44).
    $p(x) = 1 \qquad q(x) = x^{\alpha-2} \qquad w(x) = 1$
    $a = 0$ LCN ($\alpha > 0$), LPN ($\alpha \leq 0$)
    $b = 1$ Regular $y(b) = 0$
    Number of eigenvalues: ∞ continuous spectrum: none
    $\alpha = +0.01 : E_0 = 15.87305674, E_{24} = 6316.899940$
    $\alpha = -0.01 : E_0 = 15.96808975, E_{24} = 6325.038047.$

12. **Boyd equation**. (`Boyd1.mat`).
$p(x) = 1 \qquad q(x) = -1/x \qquad w(x) = 1$
$a = 0$ LCN
$b = 1$ Regular $y(b) = 0$
Number of eigenvalues: ∞ continuous spectrum: none
$E_0 = 7.373985$, $E_4 = 242.705559$
Reference: J.P. Boyd. Sturm-Liouville eigenvalue problems with an interior pole. *J. Math. Phys.*, 22 (1981), 1575.
also: P.B. Bailey, W.N. Everitt and A. Zettl. Computing eigenvalues of singular Sturm-Liouville problems. *Results in Mathematics*, 20 (1991) 391.

13. **Boyd equation**. (`Boyd2.mat`).
$p(x) = r(x)^2 \qquad q(x) = -r(x)^2 \ln(x)^2 \qquad w(x) = r(x)^2, \quad r(x) = \exp(-(x \ln(x) - x))$
$a = 0$ LCN
$b = 1$ Regular $y(b) = 0$
Number of eigenvalues: ∞ continuous spectrum: none
Reference: J.P. Boyd. Sturm-Liouville eigenvalue problems with an interior pole. *J. Math. Phys.*, 22 (1981), 1575.
also: P.B. Bailey, W.N. Everitt and A. Zettl. Computing eigenvalues of singular Sturm-Liouville problems. *Results in Mathematics*, 20 (1991) 391.

14. **Close-eigenvalues problem**. (`Close_eigenvalues.mat`). (SLTSTPAK #38).
$p(x) = 1 \qquad q(x) = x^4 - 25x^2 \qquad w(x) = 1$
Double well version of quartic anharmonic oscillator
$a = -\infty$ LPN Trunc. BC.: $y(a) = 0$
$b = +\infty$ LPN Trunc. BC.: $y(b) = 0$
Number of eigenvalues: ∞ continuous spectrum: none
$E_0 = -149.2194561 \quad E_1 = -149.2194561 \quad E_{40} = 75.69072485$

15. **Coffey-Evans equation**. (`Coffey_Evans.mat`). (SLTSTPAK #7).
$p(x) = 1 \qquad q(x) = -2\beta \cos 2x + \beta^2 \sin^2 2x \qquad w(x) = 1$
$a = -\pi/2$ Regular $y(a) = 0$
$b = \pi/2$ Regular $y(b) = 0$
As β increases there are very close eigenvalue triplets $\{E_2, E_3, E_4\}, \{E_6, E_7, E_8\}, ...$ with the other eigenvalues well separated. E_0 is very close to zero.
$\beta = 20 : E_0 = 0.000000000000 \quad E_3 = 151.46322365766$
$\beta = 30 : E_0 = 0.000000000000 \quad E_3 = 231.66492931296$
$\beta = 50 : E_0 = 0.000000000000 \quad E_3 = 391.808191489.$

16. `Collatz.mat`.
$p(x) = 1 \qquad q(x) = 0 \qquad w(x) = 3 + \cos(x)$
$a = -\pi$ Regular $y(a) = 0$
$b = +\pi$ Regular $y(b) = 0$
$E_1 = 0.071250472.$
Reference: L. Collatz. *Differential equations, An Introduction with Applications.* Wiley, Chichester, 1986.

17. **Coulomb potential**. (`Coulomb.mat`). (SLTSTPAK #30).
With $b = 1$, $u(b) = 0$ also called Boyd equation.

$p(x) = 1 \qquad q(x) = -1/x \qquad w(x) = 1$
$a = 0$ LCN
$b = +\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
$E_k = -1/[4(k+1)^2],\ k = 0, 1, ...$
Reference: E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Oxford University Press, 1946, p.98.
Also: P.B. Bailey, W.N. Everitt and A. Zettl. Computing eigenvalues of singular Sturm-Liouville problems. *Results in Mathematics*, 20 (1991), 391–423.

18. **Mysterious exact $E_0 = 7$.** (`exact7.mat`). (SLTSTPAK #26).
$p(x) = x^3 \qquad q(x) = x^3 \qquad w(x) = x^2$
$a = 0$ LPN
$b = 1$ regular $y(b) = 0$
Number of eigenvalues: ∞ continuous spectrum: none
$E_0 = 7.0000000000$, $E_9 = 284.53608972$.
Reference: S. Pruess, C. Fulton and Y. Xie, Performance of the Sturm-Liouville software package SLEDGE, 1991.

19. **Partially screening exponential-cosine potential**. (`Expon_cosine_part_screening.mat`).
$p(x) = 1 \qquad q(x) = l(l+1)/x^2 - 2Z_0 V_{ec}(x, \lambda, \mu) - 2Z_{as}(1/x - V_{ec}(x, \lambda, \mu)) \qquad w(x) = 1$
$V_{ec}(x, E, \mu) = e^{-\lambda x} \cos(\mu x)/x.$
$a = 0$ LCN ($l = 0$), LPN ($l = 5, 10$)
$b = +\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
$l = 0$, $Z_0 = 50$, $Z_{as} = 1$, $\lambda = \mu = 0.025$: E_0=-2497.550000612
$l = 5$, $Z_0 = 50$, $Z_{as} = 1$, $\lambda = \mu = 0.025$: E_0=-66.9947751270
$l = 10$, $Z_0 = 50$, $Z_{as} = 1$, $\lambda = \mu = 0.025$: E_0=-18.2144512404
Reference: L.Gr. Ixaru, H. De Meyer and G. Vanden Berghe. Highly accurate eigenvalues for the distorted Coulomb potential. *Phys. Rev. E*, 61 (2000).

20. **Screening exponential-cosine potential**. (`Expon_cosine_screening.mat`).
$p(x) = 1 \qquad q(x) = l(l+1)/x^2 - 2ZV_{ec}(x, \lambda, \mu) \qquad w(x) = 1$
$V_{ec}(x, \lambda, \mu) = e^{-\lambda x} \cos(\mu x)/x.$
$a = 0$ LCN ($l = 0$), LPN ($l = 5, 10$)
$b = +\infty$ LPN/O
$l = 0$: Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
$l = 5$: Number of eigenvalues: 30 continuous spectrum: $(0, \infty)$
$l = 10$: Number of eigenvalues: 25 continuous spectrum: $(0, \infty)$
Reference: L.Gr. Ixaru, H. De Meyer and G. Vanden Berghe. Highly accurate eigenvalues for the distorted Coulomb potential. *Phys. Rev. E*, 61 (2000).

21. **Fourier equation**. (`Fourier.mat`).
$p(x) = 1 \qquad q(x) = 0 \qquad w(x) = 1$
$a = 0$ Regular $y(a) + y'(a) = 0$
$b = \infty$ Regular $y(b) = 0$
There is an isolated eigenvalue $E_0 = -1$, with eigenfunction $\exp(-x)$ and a continuous spectrum on $[0, \infty)$

22. **Truncated Gelfand-Levitan**. (`Gelfand_Levitan_truncated.mat`). (SLTSTPAK #6).
$p(x) = 1 \qquad q(x) = 2(T \sin 2x + \cos^4 x)/T^2, \quad T = 1 + x/2 + \sin(2x)/4 \qquad w(x) = 1$
$a = 0$ Regular $y(a) - y(a) = 0$
$b = 100$ Regular $y(b) = 0$
Non-uniform oscillations of decreasing size in $q(x)$.
Reference: I. Gelfand and B. Levitan. On the determination of a differential equation from its spectral function. *AMS Translations*, 1 (1955), 253-304.

23. **Harmonic oscillator**. (`Harmonic_oscillator.mat`). (SLTSTPAK #28).
$p(x) = 1 \qquad q(x) = x^2 \qquad w(x) = 1$
$a = -\infty$ LPN
$b = +\infty$ LPN
Number of eigenvalues: ∞ continuous spectrum: none
$E_k = 2k + 1$, $k = 0, 1, \ldots.$
Reference: E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Oxford University Press, 1946, p. 1536.

24. **Half-range anharmonic oscillator**. (`HR_anharm_oscillator.mat`). (SLTSTPAK #17).
$p(x) = 1 \qquad q(x) = x^\alpha \qquad w(x) = 1, \qquad \alpha > 0$
$a = 0$ Regular $y(a) = 0$
$b = +\infty$ LPN
Number of eigenvalues: ∞ continuous spectrum: none
$\alpha = 2$: $E_k = 4k + 3$, $k = 0, 1, 2, \ldots$ (alternate eigenvalues of harmonic oscillator)
$\alpha = 3$: $E_0 = 3.4505626899$ $E_{24} = 228.52088139$
$\alpha = 4$: $E_0 = 3.7996730298$ $E_{24} = 397.14132678$
$\alpha = 5$: $E_0 = 4.0891593149$ $E_{24} = 588.17824969$.
Reference: M. Marletta. Theory and implementation of algorithms for Sturm-Liouville computations. *PhD thesis.* Royal Military College of Science, Cranfield, 1991.

25. **Hulthén partially screening potential**. (`Hulthen_part_screening.mat`).
$p(x) = 1 \qquad q(x) = l(l + 1)/x^2 - 2Z_0 V_H(x, \lambda) - 2Z_{as}(1/x - V_H(x, \lambda)) \qquad w(x) = 1$
$$V_H(x, \lambda) = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda x}} = \frac{e^{-\lambda x/2}}{x\, \eta_0((\lambda x/2)^2)}.$$
$a = 0$
$b = +\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
$l = 0$, $Z_0 = 50$, $Z_{as} = 1$, $\lambda = 0.025$: E_0= -2498.775153125
$l = 5$, $Z_0 = 50$, $Z_{as} = 1$, $\lambda = 0.025$: E_0= -68.2234257245
$l = 10$, $Z_0 = 50$, $Z_{as} = 1$, $\lambda = 0.025$: E_0= -19.4490716959.
Reference: L.Gr. Ixaru, H. De Meyer and G. Vanden Berghe. Highly accurate eigenvalues for the distorted Coulomb potential. *Phys. Rev. E*, 61 (2000).

26. **Hulthén screening potential**. (`Hulthen.mat`).
$p(x) = 1 \qquad q(x) = l(l + 1)/x^2 - 2Z\, V_H(x, \lambda) \qquad w(x) = 1$
$$V_H(x, \lambda) = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda x}}.$$
$a = 0$
$b = +\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
Exact eigenvalues only known for $l = 0$: $E_k = -[2Z - (k + 1)^2\lambda]^2/4(k + 1)^2$, $k = 0, 1, \ldots, k_{max} =$

$\left\lfloor\sqrt{2Z/\lambda}\right\rfloor - 1.$
$Z=50,\ \lambda=0.025,\ l=0$: $E_0=-2498.7501562500,\ E_9=-23.76562500$
$Z=50,\ \lambda=0.025,\ l=5$: $E_0=-68.1985069764,\ E_9=-9.8947071396$
$Z=50,\ \lambda=0.025,\ l=10$: $E_0=-19.42433530452,\ E_9=-5.05679485664$
Reference: L.Gr. Ixaru, H. De Meyer and G. Vanden Berghe. Highly accurate eigenvalues for the distorted Coulomb potential. *Phys. Rev. E*, 61 (2000).

27. **Hydrogen atom**. (`hydrogen.mat`). (SLTSTPAK #29).
$p(x)=1 \quad q(x)=-1/x+2/x^2 \quad w(x)=1$
$a=0$ LPN
$b=+\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0,\infty)$
$E_k=-1/(2k+4)^2,\ k=0,1,....$
Reference: E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Oxford University Press, 1946, p.98.

28. **Truncated hydrogen equation**. (`hydrogen_truncated.mat`). (SLTSTPAK #4).
$p(x)=1 \quad q(x)=-1/x+2/x^2 \quad w(x)=1$
$a=0$ LPN
$b=1000$ Regular $y(b)=0$
$E_0=-6.2500000000\ 10^{-2} \quad E_9=-2.066115702478\ 10^{-3}$
$E_{17}=-2.5757359232\ 10^{-4} \quad E_{18}=2.873901310\ 10^{-5}$
The lower eigenvalues approximate those of the infinite problem.

29. **Jörgens equation**. (`Jorgens.mat`).
$p(x)=1 \quad q(x)=\exp(2x)/4-k\exp(x) \quad w(x)=1$
k parameter
$a=-\infty$ LPN
$b=+\infty$ LPN
continuous spectrum: $(0,\infty)$
$k\le 1/2$: no eigenvalues
$h<k-1/2\le h+1$ for $h=0,1,2,3,\ldots$: $E_n=-(k-1/2-n)^2,\ n=0,1,2,\ldots,h.$
Reference: P.B. Bailey, W.N. Everitt and A. Zettl. The SLEIGN2 Sturm-Liouville code.

30. `Klotter.mat`. (SLTSTPAK #3).
$p(x)=1 \quad q(x)=3/(4x^2) \quad w(x)=64\pi^2/(9x^6)$
$a=8/7$ Regular $y(a)=0$
$b=8$ Regular $y(b)=0$
$E_k=(k+1)^2,\ k=0,1,...$
Transformation of $-d^2v/dt^2=Ev,\ v(\pi/48)=0=v(49\pi/48)$ by $t=\frac{4\pi}{3x^2},\ u=x^{3/2}v$. (The original reference had $a=1$, $b=2$ corresponding to $v(\pi/3)=0=v(4\pi/3)$ which is much tamer.
Reference: Klotter. *Technische Schwingungslehre, I.* Heidelberg, 1978, p.12.

31. **Laguerre's equation**. (`Laguerre.mat`). (SLTSTPAK #32).
$p(x)=1 \quad q(x)=x^2+3/(4x^2) \quad w(x)=1$
$a=0$ LPN Trunc. BC: $y(a)=0$
$b=+\infty$ LPN Trunc. BC: $y(b)=0$
Number of eigenvalues: ∞ continuous spectrum: none
$E_k=4(k+1),\ k=0,1,....$

32. **Latzko equation**. (`Latzko.mat`). (SLTSTPAK #24).
$p(x) = 1 - x^7$ $\quad q(x) = 0$ $\quad w(x) = x^7$
$a = 0$ Regular $y(a) = 0$
$b = 1$ LCN
Number of eigenvalues: ∞ continuous spectrum: none
$E_0 = 8.7274703526$, $E_2 = 435.06333218$.

33. **Legendre equation**. (`Legendre.mat`).
$p(x) = (1 - x^2)$ $\quad q(x) = 1/4$ $\quad w(x) = 1$
$a = -1$ LCN
$b = 1$ LCN
Number of eigenvalues: ∞ continuous spectrum: none
$E_k = k(k + 1) + 1/4$, $k = 0, 1, ...$

34. **Legendre equation**. (`Legendre2.mat`). (SLTSTPAK #20).
$p(x) = (1 - x^2)$ $\quad q(x) = 0$ $\quad w(x) = 1$
$a = -1$ LCN
$b = 1$ LCN
Number of eigenvalues: ∞ continuous spectrum: none
$E_k = k(k + 1)$, $k = 0, 1, ...$

35. **Truncated Lennard-Jones LJ(12,6)**. (`LennardJones.mat`). (SLTSTPAK #8).
$p(x) = 1 \qquad q(x) = \dfrac{1.92}{16.858056} D_e \left((R_e/x)^{12} - 2(R_e/x)^6\right) + \dfrac{l(l+1)}{x^2} \qquad w(x) = 1$
$D_e = 62$, $R_e = 3.56$, $l = 7$
$a = 0$ LPN
$b = 38.85$ Regular y(b)=0
This shows that close eigenvalues can happen with highly asymmetric potentials
$E_0 = 0.0899594272$, $E_1 = 0.0899769187$
As b varies from the value 38.85 the splitting $E_1 - E_0$ increases.
$q(x)$ gets very large at 0.

36. `log.mat`. (SLTSTPAK #11).
$p(x) = 1$ $\quad q(x) = \ln x$ $\quad w(x) = 1$
$a = 0$ Regular $y(a) = 0$
$b = 4$ Regular $y(b) = 0$
$E_0 = 1.1248168097$ $\quad E_{24} = 385.92821596$.
Reference: 11th problem in the Pruess-Fulton test set (S. Pruess, C.T. Fulton and Yuantao Xie. Performance of the Sturm-Liouville software package SLEDGE. *Techical Report MCS-91-19*. Colorado School of Mines, 1991.).

37. **Lohner equation**. (`Lohner.mat`).
$p(x) = 1$ $\quad q(x) = -1000x$ $\quad w(x) = 1$
$a = 0$ Regular $y(a) = 0$
$b = 1$ Regular $y(b) = 0$.
$E_0 = -766.189259$, $E_9 = 508.10800738$, $E_{49} = 24174.8549$.

38. **Marletta equation**. (`Marletta.mat`).
problem with 'pseudo-eigenvalue': some codes report $E = 0$ as a second eigenvalue.
$p(x) = 1$ $\quad q(x) = \frac{3(x-31)}{4(x+1)(x+4)^2}$ $\quad w(x) = 1$
$a = 0$ Regular $5y(a) + 8y'(a) = 0$
$b = +\infty$ LPN

Number of eigenvalues: 1 continuous spectrum: $(0, \infty)$
$E_0 = -1.18521$

39. **Mathieu equation**. (`Mathieu.mat`). (SLTSTPAK #2).
$p(x) = 1 \quad q(x) = 2r\cos(2x) \quad w(x) = 1$
r parameter
$a = 0$ Regular $y(a) = 0$
$b = \pi$ Regular $y(b) = 0$.
$r = 1$: $E_0 = -0.110248816992$, $E_9 = 100.00505067516$
Lower eigenvalues cluster in moderately tight pairs as r becomes large negative.

40. **Version of Mathieu equation**. (`Mathieu_version.mat`). (SLTSTPAK #5).
$p(x) = 1 \quad q(x) = c\cos(x) \quad w(x) = 1$
c parameter
$a = 0$ Regular $y(a) = 0$
$b = 40$ Regular $y(b) = 0$
The lower eigenvalues form clusters of 6; more and tighter clusters as c increases.
$c = 5$: $E_0 = -3.4842389, E_5 = -3.4841397, E_6 = -0.599544$ $E_{11} = -0.595602$, $E_{12} = 1.9329149$, $E_{17} = 1.9954588$.

41. **Morse potential**. (`Morse1.mat`). (SLTSTPAK #35).
$p(x) = 1 \quad q(x) = 9e^{-2x} - 18e^{-x} \quad w(x) = 1$
$a = -\infty$ LPN
$b = +\infty$ LPN/O
Number of eigenvalues: 3 continuous spectrum: $(0, \infty)$
$E_k = -0.25 - (3-k)(2-k)$, $k = 0, 1, 2$.
Reference: Morse, *Phys. Rev.*, 34 (1929), 59-61.

42. **Morse potential**. (`Morse2.mat`). (SLTSTPAK #39).
$p(x) = 1 \quad q(x) = 8000e^{-3x} - 16000e^{-3x/2} \quad w(x) = 1$
$a = -\infty$ LPN
$b = +\infty$ LPN/O
Number of eigenvalues: 60 continuous spectrum: $(0, \infty)$
With this deep well, a large truncated interval seems to be needed to give good approximations to higher eigenvalues.
$E_0 = -7866.398421 \quad E_{57} = -10.193455 \quad E_{58} = -2.865298$.
Reference: M. Marletta. Theory and implementation of algorithms for Sturm-Liouville computations. *PhD thesis.* Royal Military College of Science, Cranfield, 1991, Ch.8.

43. **Morse potential**. (`Morse3.mat`). (SLTSTPAK #36).
$p(x) = 1 \quad q(x) = 2/x^2 - 2000(2e^{-1.7(x-1.3)} - e^{-3.4(x-1.3)}) \quad w(x) = 1$
$a = 0$ LPN
$b = +\infty$ LPN/O
Number of eigenvalues: 26 continuous spectrum: none
$E_0 = -1923.529655 \quad E_1 = -1777.290819 \quad E_{13} = -473.29712549$.
Reference: Secrest et al., *J. Chem. Phys.*, 37 (1962), 830–835.

44. **Nasty p^{-1}, nice q and w**. (`nastyP.mat`). (SLTSTPAK #10).
$p(x) = \sqrt{1 - x^2} \quad q(x) = 0 \quad w(x) = 1$
Regular problem that looks singular

$a = -1$ Regular $py'(a) = 0$
$b = 1$ Regular $y(b) = 0$
$E_0 = 0.385681872027$, $E_{24} = 1031.628249437$.

45. **Nasty w, nice q and p^{-1}**. (`nastyW.mat`). (SLTSTPAK #9).
$p(x) = 1/\sqrt{1-x^2}$ $\quad q(x) = 0$ $\quad w(x) = 1/\sqrt{1-x^2}$
Regular problem that looks singular
$a = -1$ Regular $y(a) = 0$
$b = 1$ Regular $y(b) = 0$
$E_0 = 3.55927996$, $E_9 = 258.8005854$, $E_{24} = 1572.635284$.

46. **Paine problem 1**. (`Paine1.mat`).
$p(x) = 1$ $\quad q(x) = e^x$ $\quad w(x) = 1$
$a = 0$ Regular $y(a) = 0$
$b = \pi$ Regular $y(b) = 0$
$E_0 = 4.8966693800$ $\quad E_1 = 10.045189893$.
Reference: Reference: J.W. Paine, F.R. de Hoog, and R.S. Anderssen. On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems. *Computing*, 26 (1981) 123-139.

47. **Paine problem 2**. (`Paine2.mat`). (SLTSTPAK #1).
$p(x) = 1$ $\quad q(x) = \dfrac{1}{(x+0.1)^2}$ $\quad w(x) = 1$
$a = 0$ Regular $y(a) = 0$
$b = \pi$ Regular $y(b) = 0$
$E_0 = 1.5198658211$ $\quad E_1 = 4.9433098221$.
Reference: Reference: J.W. Paine, F.R. de Hoog, and R.S. Anderssen. On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems. *Computing*, 26 (1981) 123-139.

48. `PaineSLP.mat`.
$p(x) = (u+x)^3$ $\quad q(x) = 4(u+x)$ $\quad w(x) = (u+x)^5$ $\quad u = \sqrt{0.2}$
$a = 0$ Regular $y(a) = 0$
$b = -u + \sqrt{u^2 + 2\pi}$ Regular $y(b) = 0$.
Using Liouville's transformation, this problem becomes a Schrodinger equation with $q(x) = 1/(x+0.1)^2$, i.e. Paine problem 2.
Reference: L.Gr. Ixaru, H. De Meyer and G. Vanden Berghe. SLCPM12 - A program for solving regular Sturm-Liouville problems. *Comp. Phys. Comm.*, 118 (1999).

49. `Pruess_Fulton19.mat`. (SLTSTPAK #25).
$p(x) = x^4$ $\quad q(x) = -2x^2$ $\quad w(x) = x^4$
$a = 0$ LCN
$b = 1$ Regular $y(b) = 0$
$E_k = ((k+1)\pi)^2$, $k = 0, 1, \ldots.$
Reference: 19th problem in the Pruess-Fulton test set.

50. **Pure attractive Coulomb potential**. (`pure_Coulomb.mat`).
$p(x) = 1$ $\quad q(x) = l(l+1)/x^2 - 2Z/x$ $\quad w(x) = 1$
$a = 0$ LCN
$b = +\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
$E_k = -Z^2/(n+l+1)^2$, $k = 0, 1, \ldots.$

51. **Quartic anharmonic oscillator**. (`quartic_anharm_osc.mat`). (SLTSTPAK #37).
$p(x) = 1 \qquad q(x) = x^4 + x^2 \qquad w(x) = 1$
$a = -\infty$ LPN
$b = +\infty$ LPN
Number of eigenvalues: ∞ continuous spectrum: none
$E_0 = 1.3923516415 \qquad E_9 = 46.965069501.$
Reference: Scott, Shampine and Wing. *Computing*, 4 (1969), 10-23.

52. **The Razavy potential**. (`Razavy.mat`).
$p(x) = 1 \qquad q(x) = (1/8)m^2(\cosh(4x) - 1) - m(n+1)\cosh(2x) \qquad w(x) = 1$
n, m parameters
$a = -\infty$ LPN
$b = +\infty$ LPN
Number of eigenvalues: ∞ continuous spectrum: none
$n = 1, m = 1 : E_0 = -2,\ E_1 = 0$
$n = 2, m = 1 : E_0 = -2(1 + \sqrt{2}),\ E_1 = -4,\ E_2 = 2(\sqrt{2} - 1)$
$n = 1, m = 10 : E_0 = -11,\ E_1 = 9$
$n = 2, m = 10 : E_0 = -2(1 + \sqrt{101}),\ E_1 = -4,\ E_2 = 2(\sqrt{101} - 1).$
Reference: V. Fack, G. Vanden Berghe. (Extended) Numerov method for computing eigenvalues of specific Schrödinger equations. *J. Phys. A: Math. Gen.*, 20 (1987), 4153-4160.
also: M. Razavy. *Am. J. Phys.*, 48 (1980), 285.

53. **Sears-Titchmarsh**. (`Sears_Titchmarsh.mat`).
$p(x) = 1 \qquad q(x) = -\exp(2x) \qquad w(x) = 1$
$a = 0$ regular
$b = +\infty$ LCO

54. `simple_slp1.mat`.
$p(x) = 1 \qquad q(x) = 0 \qquad w(x) = 1$
$a = 0$ Regular $y(a) = 0$
$b = 1$ Regular $y(b) = 0$
Number of eigenvalues: ∞ continuous spectrum: none
$E_k = ((k+1)\pi)^2.$

55. `simple_slp2.mat`.
$p(x) = 1 \qquad q(x) = 0 \qquad w(x) = 1/x^2$
$a = 1$ Regular $y(a) = 0$
$b = e$ Regular $y(b) = 0$
Number of eigenvalues: ∞ continuous spectrum: none
$E_k = ((k+1)\pi)^2 + 1/4.$

56. **Symmetric double-well potential**. (`symm_double_well.mat`).
$p(x) = 1 \qquad q(x) = x^6 - Bx^2 \qquad w(x) = 1$
B parameter
$a = -\infty$ LPN
$b = +\infty$ LPN
Number of eigenvalues: ∞ continuous spectrum: none
$B = 11$: known exact eigenvalues $=\{-8, 0, 8\}$
$B = 13$: known exact eigenvalues $=\{-11.3137085, 0, 11.3137085\}$

$B = 15$: known exact eigenvalues $=\{-15.077508510, -3.559316943, 3.559316943, 15.077508510\}$.
Reference: V. Fack, G. Vanden Berghe. (Extended) Numerov method for computing eigenvalues of specific Schrödinger equations. *J. Phys. A: Math. Gen.*, 20 (1987), 4153-4160.

57. **Transformed hydrogen equation**. (`transformedHydrogen.mat`). (SLTSTPAK #31)
$p(x) = x^2 \quad q(x) = l(l+1) - x \quad w(x) = x^2$
$a = 0 \quad l = 0$: LCN, $l > 0$: LPN
$b = +\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
$E_k = -1/[4(k + l + 1)^2]$, $k = 0, 1, \ldots$

58. **(Slightly) Tricky** $q(x)$. (`tricky_q.mat`). (SLTSTPAK #21).
$p(x) = 1 \quad q(x) = -10/x^{1.5}, \quad w(x) = 1$
$a = 0$ LCN
$b = 1$ Regular $y(b) = 0$
Number of eigenvalues: ∞ continuous spectrum: none

59. `VandenBerghe.mat`. (SLTSTPAK #42).
$p(x) = 1 \quad q(x) = l(l+1)/x^2 + (-1 + 5e^{-2x})/x \quad w(x) = 1$
$a = 0$ LCN($l = 0$), LPN($l > 0$)
$b = +\infty$ LPN/O
Number of eigenvalues: ∞ continuous spectrum: $(0, \infty)$
$l = 0 : E_0 = -0.156358880971 \quad E_2 = -0.023484895664$
$l = 1 : E_0 = -0.061681846633 \quad E_2 = -0.015501561691$.
Coulomb-type decay at $x = \infty$ similar to hydrogen atom. Need to truncate very far out to obtain higher eigenvalues accurately.
Reference: G. Vanden Berghe, V. Fack and H. de Meyer. *J. Comp. Appl. Math.*, 28(1989), 391–401.

60. `Wicke_Harris.mat`. (SLTSTPAK #40).
$p(x) = 1 \quad q(x) = 1250e^{-83.363(x-2.47826)^2} + 3906.25(1 - e^{2.3237-x})^2 \quad w(x) = 1$
$a = 0$ Regular $y(a) = 0$
$b = +\infty$ LPN/O
Number of eigenvalues: 62 continuous spectrum: $(3906.25, \infty)$
$E_0 = 163.2238872 \quad E_9 = 1277.53684$
This has a spike at the bottom of the well.
Reference: Wicke and Harris, Comparison of three numerical techniques for calculating eigenvalues of an unsymmetrical double minimum oscillator. *J. Chem. Phys.*, 64 (1976), 5236.

61. **Woods-Saxon potential**. (`Woods_Saxon.mat`). (SLTSTPAK#41).
$p(x) = 1 \quad q(x) = l(l+1)/x^2 - 50(1 - 5t/(3(1+t)))/(1+t) \quad w(x) = 1$
$t = e^{(x-7)/0.6}$
$a = 0$ LPN $y(a) = 0$
$b = +\infty$ LPN/O
$l = 0$: Number of eigenvalues: 14 continuous spectrum: $(0, \infty)$
$E_0 = -49.457788728 \quad E_{10} = -18.094688282$
$l = 2$: Number of eigenvalues: 13 continuous spectrum: $(0, \infty)$
$E_0 = -48.349481052 \quad E_{10} = -13.522303353$.
Reference: G. Vanden Berghe, V. Fack and H. de Meyer, *J. Comp. Appl. Math.*, 28(1989), 391–401.

3 Problems with Discontinuous Coefficients

1. **Approximate Harmonic oscillator**. (`approximateHarmonicOscillator.m`). (SLTST-PAK#53).
   $p(x) = 1$ $q(x) =$piecewise linear function joining the points (x, x^2) for x an integer $w(x) = 1$
   $a = -\infty$ LPN
   $b = +\infty$ LPN