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The inverse of a tridiagonal matrix

Ranjan K. Mallik*

Department of Electrical Engineering, Indian Institute of Technology, Hauz Khas, New Delhi 110016, India

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Abstract

In this paper, explicit formulae for the elements of the inverse of a general tridiagonal matrix are presented by first extending results on the explicit solution of a second-order linear homogeneous difference equation with variable coefficients to the nonhomogeneous case, and then applying these extended results to a boundary value problem. A formula for the characteristic polynomial is obtained in the process. We also establish a connection between the matrix inverse and orthogonal polynomials. In addition, the case of a cyclic tridiagonal system is discussed. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Tridiagonal matrices [1–4] are connected with different areas of science and engineering, including telecommunication system analysis [5] and finite difference methods for solving partial differential equations [4,6,7]. In many of these areas, inversions of tridiagonal matrices are necessary. Efficient algorithms [8], indirect formulae [1,9–12], and direct expressions in some special cases [4,7] for such inversions are known. Bounds on the elements of the inverses of diagonally dominant tridiagonal matrices have also been obtained [13]. However, explicit formulae for the

* Tel.: +91-11-659-1049; fax: +91-11-658-1606.
E-mail address: rkmallik@ee.iitd.ernet.in (R.K. Mallik).

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elements of a general tridiagonal matrix inverse, which can give a better analytical treatment to a problem, are not available in the open literature [1].

In this paper, we present explicit formulae for the elements of the inverse of a general tridiagonal matrix. The approach is based on linear difference equations [14,15], and is as follows. Results on the *explicit solution of a second-order linear homogeneous difference equation with variable coefficients* [16] are extended to the nonhomogeneous case. Then these extended results are applied to a boundary value problem to obtain the desired formulae. A formula for the characteristic polynomial is obtained in the process. We also establish a connection between the matrix inverse and orthogonal polynomials. In addition, the case of a cyclic tridiagonal system is discussed.

2. Results on the second-order linear homogeneous difference equation with variable coefficients

In this section, we will review some results on the second-order linear homogeneous difference equation with variable coefficients [16]. Let $\mathbb{N}$ denote the set of natural numbers. A set $S_q(L, U)$, where $q, L, U \in \mathbb{N}$, has been defined in [16] as the set of all q-tuples with elements from $\{L, L+1, \dots, U\}$ arranged in ascending order *so that no two consecutive elements are present*, that is,

$$
\begin{aligned}
S_q(L,U) &\triangleq \{L, L+1, \dots, U\} \quad \text{if } U \geqslant L \text{ and } q = 1, && \text{(1a)}\\
&\triangleq \Big\{(k_1, \dots, k_q)\colon\ k_1, \dots, k_q \in \{L, L+1, \dots, U\};\ k_l - k_{l-1} \geqslant 2 \\
&\qquad \text{for } l = 2, \dots, q\ \Big\} \\
&\qquad \text{if } U \geqslant L+2 \text{ and } 2 \leqslant q \leqslant \left\lfloor \frac{U-L+2}{2} \right\rfloor, && \text{(1b)}\\
&\triangleq \emptyset \text{ otherwise.} && \text{(1c)}
\end{aligned}
$$

For example,

$$
\begin{aligned}
S_1(2,6) &= \{2,3,4,5,6\},\\
S_2(2,6) &= \{(2,4),(2,5),(2,6),(3,5),(3,6),(4,6)\},\\
S_3(2,6) &= \{(2,4,6)\},\\
S_4(2,6) &= \emptyset.
\end{aligned}
$$

The following results have been proved in [16]:

- For $U \geqslant L$, $1 \leqslant q \leqslant \lfloor (U-L+2)/2 \rfloor$,

$$
|S_q(L,U)| = \binom{U-L-q+2}{q} = \frac{(U-L-q+2)!}{q!(U-L-2q+2)!}. \tag{2}
$$

- For $U \geqslant L$, $q \geqslant 2$,

$$S_q(L, U+1) = S_q(L, U) \cup \big\{(k_1, \ldots, k_{q-1}, k_q)\colon\ k_q = U+1;\ (k_1, \ldots, k_{q-1}) \in S_{q-1}(L, U-1)\big\}. \tag{3}$$

We add a proposition which is similar to result (3).

Proposition 1. *For $U \geqslant L$, $q \geqslant 2$,*

$$S_q(L-1, U) = S_q(L, U) \cup \big\{(k_1, k_2, \ldots, k_q)\colon\ k_1 = L-1;\ (k_2, \ldots, k_q) \in S_{q-1}(L+1, U)\big\}. \tag{4}$$

Proof. If either $U = L$, $q \geqslant 2$ or $U \geqslant L+1$, $q > \lfloor (U-L+3)/2 \rfloor$, then Eq. (4) holds trivially.

For $U \geqslant L+1$, $2 \leqslant q \leqslant \lfloor (U-L+3)/2 \rfloor$, we get, from (1a)–(1c),

$$S_q(L-1, U) = \big\{(k_1, \ldots, k_q)\colon\ k_1, \ldots, k_q \in \{L-1, L, \ldots, U\};\ k_l - k_{l-1} \geqslant 2 \text{ for } l = 2, \ldots, q\big\}. \tag{5}$$

The right-hand side of (5) can be expressed as the union of two disjoint sets, one containing q-tuples with $k_1 \neq L-1$ and the other containing q-tuples with $k_1 = L-1$, which we denote as $S_{q_0}(L-1, U)$ and $S_{q_1}(L-1, U)$, respectively. Therefore,

$$S_q(L-1, U) = S_{q_0}(L-1, U) \cup S_{q_1}(L-1, U)\,, \tag{6}$$

where

$$S_{q_0}(L-1, U) = \big\{(k_1, \ldots, k_q)\colon\ k_1, \ldots, k_q \in \{L, L+1, \ldots, U\};\ k_l - k_{l-1} \geqslant 2 \text{ for } l = 2, \ldots, q\big\},$$

$$S_{q_1}(L-1, U) = \big\{(k_1, \ldots, k_q)\colon\ k_1 = L-1;\ k_2, \ldots, k_q \in \{L, L+1, \ldots, U\};\ k_l - k_{l-1} \geqslant 2 \text{ for } l = 2, \ldots, q\big\}. \tag{7}$$

It is clear from (1a)–(1c) that

$$S_{q_0}(L-1, U) = S_q(L, U). \tag{8}$$

In the set $S_{q_1}(L-1, U)$, since $k_1 = L-1$, we have $k_2 - (L-1) \geqslant 2$, which implies that $k_2, \ldots, k_q \in \{L+1, L+2, \ldots, U\}$. Therefore,

$$S_{q_1}(L-1, U) = \big\{(k_1, k_2, \ldots, k_q)\colon\ k_1 = L-1;\ (k_2, \ldots, k_q) \in S_{q-1}(L+1, U)\big\}. \tag{9}$$

Combining Eqs. (6), (8) and (9), we get (4). $\square$

Consider the *second-order linear homogeneous difference equation*

$$y_{n+2} = A_n\, y_{n+1} + B_n y_n, \quad n \geqslant 1, \tag{10}$$

with integral index *n*, variable complex coefficients A_n and B_n, $B_n \neq 0$, and complex initial values y_1, y_2. Define, for $k \geqslant 2$,

$$\sigma_k \triangleq \frac{B_k}{A_{k-1}A_k}. \tag{11}$$

It has been shown in [16] that the *explicit solution* of difference equation (10), which is an expression for y_{n+2} in terms of only coefficients $A_1, \ldots, A_n, B_1, \ldots, B_n$, and initial values y_1, y_2, is given by

$$y_{n+2} = C_n y_2 + D_n y_1, \quad n \geqslant 0, \tag{12}$$

where

$$C_0 = 1, \quad C_1 = A_1, \tag{13a}$$

$$D_0 = 0, \quad D_1 = B_1, \quad D_2 = B_1 A_2 \tag{13b}$$

and

$$C_n = (A_1 \cdots A_n)\left(1 + \sum_{q=1}^{\lfloor n/2 \rfloor} \sum_{(k_1,\ldots,k_q)\in S_q(2,n)} (\sigma_{k_1} \cdots \sigma_{k_q})\right) \quad \text{for } n \geqslant 2, \tag{14a}$$

$$D_n = B_1(A_2 \cdots A_n)\left(1 + \sum_{q=1}^{\lfloor (n-1)/2 \rfloor} \sum_{(k_1,\ldots,k_q)\in S_q(3,n)} (\sigma_{k_1} \cdots \sigma_{k_q})\right) \quad \text{for } n \geqslant 3, \tag{14b}$$

where $S_q(L, U)$ is defined by (1a)–(1c), and σ_k is defined by (11).

3. Solution of the second-order linear nonhomogeneous difference equation with variable coefficients

We now focus on the *second-order linear nonhomogeneous difference equation*

$$y_{n+2} = A_n y_{n+1} + B_n y_n + x_{n+2}, \quad n \geqslant 1, \tag{15}$$

with integral index *n*, variable complex coefficients A_n and B_n, $B_n \neq 0$, complex forcing term x_{n+2}, and complex initial values y_1, y_2.

Let $\sigma_k = B_k/(A_{k-1}A_k)$ for $k \geqslant 2$ as in (11). For $n \geqslant 0$, $i \geqslant 1$, define a quantity $E_n(i)$, which is a function of the A_ks and the σ_ks, as

$$\begin{aligned} E_n(i) &\triangleq (A_i \cdots A_n)\left(1 + \sum_{q=1}^{\lfloor (n-i+1)/2 \rfloor} \sum_{(k_1,\ldots,k_q)\in S_q(i+1,n)} (\sigma_{k_1}\cdots\sigma_{k_q})\right) \\ &\qquad \text{if } i = 1,\ldots,n-1,\ n \geqslant 2, \\ &\triangleq A_n \quad \text{if } i = n,\ n \geqslant 1, \\ &\triangleq 1 \quad \text{if } i = n+1,\ n \geqslant 0, \\ &\triangleq 0 \quad \text{otherwise.} \end{aligned} \tag{16}$$

Therefore, $E_n(i)$ is nontrivial only for $n \geqslant i-1$, $i \geqslant 1$. Note that

$$E_n(1) = C_n \quad \text{for } n \geqslant 0, \tag{17}$$

where C_n is given by (13a) and (14a).

Two recurrences for $E_n(i)$ are established by the following two propositions.

Proposition 2. *For $n \geqslant i-1$, $i \geqslant 1$,*

$$E_{n+2}(i) = A_{n+2}E_{n+1}(i) + B_{n+2}E_n(i). \tag{18}$$

Proof. Using the definition of $E_n(i)$ in (16) and σ_i in (11), the right-hand side of (18) can be written for $n = i-1$, $i \geqslant 1$ as

$$\begin{aligned} A_{i+1}E_i(i) + B_{i+1}E_{i-1}(i) &= A_{i+1}A_i + B_{i+1} \\ &= (A_iA_{i+1})(1+\sigma_{i+1}) \\ &= E_{i+1}(i) \end{aligned} \tag{19}$$

and for $n = i$, $i \geqslant 1$ as

$$\begin{aligned} A_{i+2}E_{i+1}(i) + B_{i+2}E_i(i) &= (A_{i+2}A_{i+1}A_i)(1+\sigma_{i+1}) + B_{i+2}A_i \\ &= (A_iA_{i+1}A_{i+2})(1+\sigma_{i+1}+\sigma_{i+2}) \\ &= E_{i+2}(i), \end{aligned} \tag{20}$$

which imply that (18) holds for $n = i-1, i$, $i \geqslant 1$.

Now consider the case when $n \geqslant i+1$, $i \geqslant 1$. Again, using (16) and noting the fact that $B_{n+2} = A_{n+1}A_{n+2}\sigma_{n+2}$, we get

$$\begin{aligned} &A_{n+2}E_{n+1}(i) + B_{n+2}E_n(i) \\ &\quad = (A_i \cdots A_{n+2})\left(1 + \sum_{q=1}^{\lfloor (n-i+2)/2 \rfloor} \sum_{(k_1,\ldots,k_q)\in S_q(i+1,n+1)} (\sigma_{k_1}\cdots\sigma_{k_q})\right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \sigma_{n+2} + \sum_{q=2}^{\lfloor (n-i+3)/2 \rfloor} \sum_{(k_1,\ldots,k_{q-1}) \in S_{q-1}(i+1,n)} (\sigma_{k_1} \cdots \sigma_{k_{q-1}}) \sigma_{n+2} \Bigg) \\
&= (A_i \cdots A_{n+2}) \Bigg(1 + \sum_{k_1=i+1}^{n+2} \sigma_{k_1} \\
&\quad + \sum_{q=2}^{\lfloor (n-i+2)/2 \rfloor} \sum_{(k_1,\ldots,k_q) \in S_q(i+1,n+1)} (\sigma_{k_1} \cdots \sigma_{k_q}) \\
&\quad + \sum_{q=2}^{\lfloor (n-i+3)/2 \rfloor} \sum_{(k_1,\ldots,k_{q-1}) \in S_{q-1}(i+1,n)} (\sigma_{k_1} \cdots \sigma_{k_{q-1}}) \sigma_{n+2} \Bigg). \qquad (21)
\end{aligned}
$$

Now

$$
\left\lfloor \frac{n-i+3}{2} \right\rfloor = \begin{cases} \left\lfloor \dfrac{n-i+2}{2} \right\rfloor & \text{if } (n-i+2) \text{ is even,} \\ \left\lfloor \dfrac{n-i+2}{2} \right\rfloor + 1 & \text{if } (n-i+2) \text{ is odd.} \end{cases} \qquad (22)
$$

If $(n-i+2)$ is odd, by (1a)–(1c),

$$
S_{\lfloor (n-i+3)/2 \rfloor}(i+1, n+1) = S_{\lfloor (n-i+2)/2 \rfloor + 1}(i+1, n+1) = \emptyset.
$$

Then, using (3), we get

$$
\begin{aligned}
S_q(i+1, n+2) = {}& S_q(i+1, n+1) \cup \big\{ (k_1, \ldots, k_q) \colon \; k_q = n+2; \\
& (k_1, \ldots, k_{q-1}) \in S_{q-1}(i+1, n) \big\},
\end{aligned} \qquad (23)
$$

for $q = 2, \ldots, \lfloor (n-i+3)/2 \rfloor$. Putting together the summation terms of (21) using (22) and (23), we obtain

$$
\begin{aligned}
&A_{n+2} E_{n+1}(i) + B_{n+2} E_n(i) \\
&\quad = (A_i \cdots A_{n+2}) \Bigg(1 + \sum_{q=1}^{\lfloor (n-i+3)/2 \rfloor} \sum_{(k_1,\ldots,k_q) \in S_q(i+1,n+2)} (\sigma_{k_1} \cdots \sigma_{k_q}) \Bigg) \\
&\quad = E_{n+2}(i) \qquad (24)
\end{aligned}
$$

for $n \geqslant i+1$, $i \geqslant 1$. A combination of (19), (20) and (24) yields (18). $\square$

Proposition 3. *For* $1 \leqslant i \leqslant n-1$, $n \geqslant 2$,

$$
E_n(i) = A_i E_n(i+1) + B_{i+1} E_n(i+2). \qquad (25)
$$

Proof. From the definition of $E_n(i)$ in (16), the right-hand side of (25) can be written for $i = n-1$ as

$$\begin{aligned}A_{n-1}E_n(n) + B_nE_n(n+1) &= A_{n-1}A_n + B_n\\ &= (A_{n-1}A_n)(1+\sigma_n)\\ &= E_n(n-1)\end{aligned} \tag{26}$$

and for $i = n-2$ as

$$\begin{aligned}A_{n-2}E_n(n-1) + B_{n-1}E_n(n) &= (A_{n-2}A_{n-1}A_n)(1+\sigma_n) + B_{n-1}A_n\\ &= (A_{n-2}A_{n-1}A_n)(1+\sigma_n+\sigma_{n-1})\\ &= E_n(n-2),\end{aligned} \tag{27}$$

which imply that (25) holds for $i = n-2, n-1$.

Now consider the case when $1 \leqslant i \leqslant n-3$. Using (16) and (11) we obtain

$$\begin{aligned}&A_iE_n(i+1) + B_{i+1}E_n(i+2)\\ &\quad= (A_i\cdots A_n)\Bigg(1 + \sum_{q=1}^{\lfloor (n-i)/2\rfloor}\ \sum_{(k_1,\dots,k_q)\in S_q(i+2,n)} (\sigma_{k_1}\cdots\sigma_{k_q}) + \sigma_{i+1}\\ &\qquad + \sum_{q=2}^{\lfloor (n-i+1)/2\rfloor}\ \sum_{(k_2,\dots,k_q)\in S_{q-1}(i+3,n)} \sigma_{i+1}(\sigma_{k_2}\cdots\sigma_{k_q})\Bigg)\\ &\quad= (A_i\cdots A_n)\Bigg(1 + \sum_{k_1=i+1}^{n}\sigma_{k_1} + \sum_{q=2}^{\lfloor (n-i)/2\rfloor}\ \sum_{(k_1,\dots,k_q)\in S_q(i+2,n)} (\sigma_{k_1}\cdots\sigma_{k_q})\\ &\qquad + \sum_{q=2}^{\lfloor (n-i+1)/2\rfloor}\ \sum_{(k_2,\dots,k_q)\in S_{q-1}(i+3,n)} \sigma_{i+1}(\sigma_{k_2}\cdots\sigma_{k_q})\Bigg).\end{aligned} \tag{28}$$

Now

$$\left\lfloor \frac{n-i+1}{2}\right\rfloor = \begin{cases}\left\lfloor \frac{n-i}{2}\right\rfloor & \text{if } (n-i) \text{ is even},\\ \left\lfloor \frac{n-i}{2}\right\rfloor + 1 & \text{if } (n-i) \text{ is odd}.\end{cases} \tag{29}$$

If $(n-i)$ is odd, then by (1a)–(1c),

$$S_{\lfloor (n-i+1)/2\rfloor}(i+2,n) = S_{\lfloor (n-i)/2\rfloor+1}(i+2,n) = \emptyset.$$

Then, using Proposition 1, we get

$$\begin{aligned}S_q(i+1,n) = S_q(i+2,n) \cup \big\{&(k_1,\dots,k_q)\colon\ k_1 = i+1;\\ &(k_2,\dots,k_q)\in S_{q-1}(i+3,n)\big\},\end{aligned} \tag{30}$$

for $q = 2, \ldots, \lfloor (n - i + 1)/2 \rfloor$. Putting together the summation terms of (28) using (29) and (30), we obtain (25). $\square$

The following proposition provides the *explicit solution* of the difference equation (15), which consists of an expression for y_{n+2} in terms of only coefficients $A_1, \ldots, A_n, B_1, \ldots, B_n$, initial values y_1, y_2, and forcing terms $x_3, \ldots, x_{n+2}$.

Proposition 4. *The solution of difference equation* (15) *with initial values* y_1, y_2 *is given by*

$$y_{n+2} = C_n y_2 + D_n y_1 + \sum_{k=3}^{n+2} E_n(k-1)x_k, \quad n \geqslant 1, \tag{31}$$

where $E_n(k)$ *is defined by* (16), *and* C_n *and* D_n *by* (13a), (13b), (14a), *and* (14b).

Proof. The solution of (15) with initial values y_1, y_2 can be expressed as

$$y_{n+2} = H_n + P_n, \quad n \geqslant 1, \tag{32}$$

where H_n is the *solution of the homogeneous equation*, and P_n the *particular solution*. From the results of Section 2 (Eqs. (10) and (12)), we have

$$H_n = C_n y_2 + D_n y_1, \quad n \geqslant 1. \tag{33}$$

It is also known that the particular solution P_n satisfies the recurrence

$$P_{n+2} = A_{n+2}P_{n+1} + B_{n+2}P_n + x_{n+4}, \quad n \geqslant 1. \tag{34}$$

Using (15), (13a), (13b), (14a) and (14b), we get

$$\begin{aligned} y_3 &= C_1 y_2 + D_1 y_1 + x_3, \\ y_4 &= C_2 y_2 + D_2 y_1 + (x_4 + A_2 x_3), \end{aligned} \tag{35}$$

where

$$\begin{aligned} C_1 &= A_1, \quad C_2 = A_1 A_2 + B_2, \\ D_1 &= B_1, \quad D_2 = B_1 A_2, \end{aligned} \tag{36}$$

which in turn implies that

$$P_1 = x_3, \quad P_2 = x_4 + A_2 x_3. \tag{37}$$

We now proceed to prove the validity of the expression

$$P_n = \sum_{k=3}^{n+2} E_n(k-1)x_k \tag{38}$$

for $n \geqslant 1$.

It is clear from (37) and the definition of $E_n(k)$ in (16) that expression (38) holds for $n = 1, 2$. Let (38) hold for P_n and P_{n+1} for $n \geqslant 1$. Then, from (34), Proposition 2 and (16), we get

$$
\begin{aligned}
P_{n+2} &= A_{n+2}\sum_{k=3}^{n+3} E_{n+1}(k-1)x_k + B_{n+2}\sum_{k=3}^{n+2} E_n(k-1)x_k + x_{n+4}\\
&= \sum_{k=3}^{n+2}[A_{n+2}E_{n+1}(k-1) + B_{n+2}E_n(k-1)]x_k + A_{n+2}x_{n+3} + x_{n+4}\\
&= \sum_{k=3}^{n+2} E_{n+2}(k-1)x_k + E_{n+2}(n+2)x_{n+3} + E_{n+2}(n+3)x_{n+4}. \qquad (39)
\end{aligned}
$$

Thus,

$$P_{n+2} = \sum_{k=3}^{n+4} E_{n+2}(k-1)x_k, \qquad (40)$$

which agrees with (38). By mathematical induction, expression (38) is valid for all $n \geqslant 1$.

From (32), (33) and (38), we obtain (31). □

3.1. An example

Consider the case when we need to find y_6 in the difference equation (15) in terms of coefficients $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$, initial values y_1, y_2, and forcing terms x_3, x_4, x_5, x_6.

From Proposition 4, we have

$$y_6 = C_4y_2 + D_4y_1 + \sum_{k=3}^{6} E_4(k-1)x_k.$$

Eqs. (14a) and (14b) imply that

$$C_4 = (A_1A_2A_3A_4)\left(1 + \sum_{k_1\in S_1(2,4)} \sigma_{k_1} + \sum_{(k_1,k_2)\in S_2(2,4)} (\sigma_{k_1}\sigma_{k_2})\right),$$

$$D_4 = (B_1A_2A_3A_4)\left(1 + \sum_{k_1\in S_1(3,4)} \sigma_{k_1}\right).$$

Now, from the definition of $S_q(L, U)$ in (1a)–(1c), we get

$$
\begin{aligned}
&S_1(2, 4) = \{2, 3, 4\},\\
&S_2(2, 4) = \{(2, 4)\},\\
&S_1(3, 4) = \{3, 4\}.
\end{aligned}
$$

Therefore,

$$C_4 = (A_1A_2A_3A_4)(1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_2\sigma_4),$$

$$D_4 = (B_1A_2A_3A_4)(1 + \sigma_3 + \sigma_4),$$

where, from (11),

$$\sigma_2 = \frac{B_2}{A_1A_2}, \quad \sigma_3 = \frac{B_3}{A_2A_3}, \quad \sigma_4 = \frac{B_3}{A_3A_4}.$$

In addition, from (16), we obtain

$$E_4(2) = (A_2A_3A_4)(1 + \sigma_3 + \sigma_4),$$

$$E_4(3) = (A_3A_4)(1 + \sigma_4),$$

$$E_4(4) = A_4,$$

$$E_4(5) = 1.$$

Therefore, by substituting the expressions for σ_2, σ_3 and σ_4 in C_4, D_4, $E_4(2)$, $E_4(3)$, $E_4(4)$, $E_4(5)$, we finally have the desired solution, which is

$$\begin{aligned} y_6 = {} & (A_1A_2A_3A_4 + B_2A_3A_4 + B_3A_1A_4 + B_4A_1A_2 + B_2B_4)y_2 \\ & + (B_1A_2A_3A_4 + B_1B_3A_4 + B_1B_4A_2)y_1 \\ & + (A_2A_3A_4 + B_3A_4 + B_4A_2)x_3 \\ & + (A_3A_4 + B_4)x_4 + A_4x_5 + x_6. \end{aligned}$$

4. A boundary value problem and the tridiagonal matrix inverse

Consider the difference equation (15) over the finite index interval $[1, K + 2]$, $K \geqslant 2$, with zero boundary values, that is, the recurrence

$$y_{n+2} = A_n\, y_{n+1} + B_n\, y_n + x_{n+2}, \quad 1 \leqslant n \leqslant K, \tag{41}$$

with boundary values y_1, y_{K+2} such that

$$y_1 = y_{K+2} = 0. \tag{42}$$

The K equations in (41) can be written in matrix form as

$$\begin{bmatrix} -B_1 & -A_1 & 1 & & & & \\ & -B_2 & -A_2 & 1 & & \mathbf{0} & \\ & & \ddots & \ddots & \ddots & & \\ & \mathbf{0} & & -B_{K-1} & -A_{K-1} & 1 & \\ & & & & -B_K & -A_K & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{K+1} \\ y_{K+2} \end{bmatrix}$$

$$= \begin{bmatrix} x_3 \\ \vdots \\ x_{K+2} \end{bmatrix}. \tag{43}$$

Using boundary conditions (42), the solution of the difference equation (41) can be expressed as

$$\begin{bmatrix} y_2 \\ \vdots \\ y_{K+1} \end{bmatrix} = \begin{bmatrix} -A_1 & 1 & & & \\ -B_2 & -A_2 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & -B_{K-1} & -A_{K-1} & 1 \\ & & & -B_K & -A_K \end{bmatrix}^{-1} \begin{bmatrix} x_3 \\ \vdots \\ x_{K+2} \end{bmatrix}. \tag{44}$$

However, from Proposition 4, the quantities $y_3, \ldots, y_{K+2}$ of recurrence (41) can be expressed in terms of y_1, y_2 as

$$y_{i+1} = C_{i-1}y_2 + D_{i-1}y_1 + \sum_{j=1}^{i-1} E_{i-1}(j+1)x_{j+2}, \quad 2 \leqslant i \leqslant K+1. \tag{45}$$

As a result of boundary conditions (42), substitution of $i = K+1$ in (45) gives

$$y_{K+2} = C_K y_2 + \sum_{j=1}^{K} E_K(j+1)x_{j+2} = 0, \tag{46}$$

and therefore, when $C_K \neq 0$,

$$y_2 = \sum_{j=1}^{K} \left[-\frac{E_K(j+1)}{C_K} \right] x_{j+2}. \tag{47}$$

Substituting (47) in (45) and setting $y_1 = 0$ we obtain

$$\begin{aligned} y_{i+1} = &- \sum_{j=1}^{K} \frac{C_{i-1}E_K(j+1)}{C_K} x_{j+2} \\ &+ \sum_{j=1}^{i-1} E_{i-1}(j+1)x_{j+2}, \quad 2 \leqslant i \leqslant K,\ C_K \neq 0. \end{aligned} \tag{48}$$

In other words,

$$\begin{aligned} y_{i+1} = &\sum_{j=1}^{i-1} \left[E_{i-1}(j+1) - \frac{C_{i-1}E_K(j+1)}{C_K} \right] x_{j+2} \\ &+ \sum_{j=i}^{K} \left[-\frac{C_{i-1}E_K(j+1)}{C_K} \right] x_{j+2}, \quad 2 \leqslant i \leqslant K,\ C_K \neq 0. \end{aligned} \tag{49}$$

Using (16) and (17), Eqs. (47) and (49) can be written together as

$$y_{i+1} = \sum_{j=1}^{K} \left[E_{i-1}(j+1) - \frac{E_{i-1}(1)E_K(j+1)}{E_K(1)} \right] x_{j+2},$$
$$1 \leqslant i \leqslant K, \ E_K(1) \neq 0. \tag{50}$$

Thus the *solution* of *boundary value problem* (41), (42) is given by (47) and (49) or alternatively by (50). This solution also gives the inverse of a tridiagonal matrix.

Denoting the $K \times K$ *tridiagonal matrix* derived from the $K \times (K+2)$ matrix on the left-hand side of (43) (by removing the first and $(K+2)$th columns) as

$$\mathbf{T}_K = \begin{bmatrix} -A_1 & 1 & & & \\ -B_2 & -A_2 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & -B_{K-1} & -A_{K-1} & 1 \\ & & & -B_K & -A_K \end{bmatrix}, \quad K \geqslant 2, \tag{51}$$

and its *inverse* as

$$\mathbf{R}_K = \mathbf{T}_K^{-1} = \left[R_{i,j}\right]_{i,j=1}^{K}, \tag{52}$$

we obtain, from (44), (47) and (49), an expression for $R_{i,j}$, the element in the ith row and jth column of $\mathbf{T}_K^{-1}$. It is given by

$$\begin{aligned} R_{i,j} &= E_{i-1}(j+1) - \frac{C_{i-1}E_K(j+1)}{C_K} \\ &\qquad \text{for } j = 1, \ldots, i-1, \ i = 2, \ldots, K, \\ &= -\frac{C_{i-1}E_K(j+1)}{C_K} \\ &\qquad \text{for } j = i, \ldots, K, \ i = 1, \ldots, K, \quad C_K \neq 0, \end{aligned} \tag{53}$$

where $E_i(j)$ is defined in (16), and C_i in (13a) and (14a). Alternatively, from (44) and (50), we get

$$R_{i,j} = E_{i-1}(j+1) - \frac{E_{i-1}(1)E_K(j+1)}{E_K(1)}$$
$$\text{for } i, \ j = 1, \ldots, K, \ E_K(1) \neq 0. \tag{54}$$

The following proposition gives an expression for the determinant of the tridiagonal matrix $\mathbf{T}_K$.

Proposition 5. *For* $K \geqslant 2$,

$$\det(\mathbf{T}_K) = (-1)^K C_K, \tag{55}$$

where $\mathbf{T}_K$ *is given by* (51) *and* C_K *by* (14a).

Proof. From (51), it is clear that if we find $\det(\mathbf{T}_K)$ by using the Kth row of $\mathbf{T}_K$ followed by its Kth column, we obtain

$$\det(\mathbf{T}_K) = -A_K \det(\mathbf{T}_{K-1}) + B_K \det(\mathbf{T}_{K-2}), \quad K \geqslant 4. \tag{56}$$

It is simple to show that

$$\det(\mathbf{T}_2) = \det\left(\begin{bmatrix} -A_1 & 1 \\ -B_2 & -A_2 \end{bmatrix}\right) = A_1 A_2 (1 + \sigma_2) = C_2, \tag{57a}$$

$$\begin{aligned} \det(\mathbf{T}_3) &= \det\left(\begin{bmatrix} -A_1 & 1 & 0 \\ -B_2 & -A_2 & 1 \\ 0 & -B_3 & -A_3 \end{bmatrix}\right) \\ &= -A_1 A_2 A_3 (1 + \sigma_2 + \sigma_3) \\ &= -C_3. \end{aligned} \tag{57b}$$

Since $C_K = E_K(1)$, it is governed by the second-order recursion

$$C_K = A_K C_{K-1} + B_K C_{K-2}, \quad K \geqslant 4 \tag{58}$$

by Proposition 2. Therefore,

$$(-1)^K C_K = -A_K (-1)^{K-1} C_{K-1} + B_K (-1)^{K-2} C_{K-2}, \quad K \geqslant 4. \tag{59}$$

Recursions (56) and (59) for $\det(\mathbf{T}_K)$ and $(-1)^K C_K$, respectively, are identical, and from (57a) and (57b) we find that they have the same initial values $(-1)^2 C_2$ and $(-1)^3 C_3$. Hence, $\det(\mathbf{T}_K) = (-1)^K C_K$ for $K \geqslant 2$. $\square$

If it so happens that $\det(\mathbf{T}_K) = 0$, then Proposition 5 implies that $C_K = 0$. In that case, the matrix $\mathbf{T}_K$ is not invertible, and Eq. (53) for the elements of $\mathbf{T}_K^{-1}$ does not hold.

Let $\Delta_{j,i}$ denote the *minor* corresponding to the jth row and ith column of $\mathbf{T}_K$. Then, from (53) and Proposition 5, we have

$$\begin{aligned} \Delta_{j,i} &= (-1)^{(K-i-j)} [C_K E_{i-1}(j+1) - C_{i-1} E_K(j+1)] \\ &\qquad \text{for } j = 1, \ldots, i-1,\ i = 2, \ldots, K, \\ &= -(-1)^{(K-i-j)} C_{i-1} E_K(j+1) \\ &\qquad \text{for } j = i, \ldots, K,\ i = 1, \ldots, K, \end{aligned} \tag{60}$$

or, alternatively, from (54) and Proposition 5,

$$\begin{aligned} \Delta_{j,i} &= (-1)^{(K-i-j)} [E_K(1) E_{i-1}(j+1) - E_{i-1}(1) E_K(j+1)] \\ &\quad \text{for } i, j = 1, \ldots, K. \end{aligned} \tag{61}$$

4.1. The tridiagonal matrix inverse

We now consider a nonsingular $K \times K$ tridiagonal matrix Φ_K $(K \geqslant 2)$, with complex entries, given by

$$\Phi_K = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \gamma_2 & \alpha_2 & \beta_2 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & \gamma_{K-1} & \alpha_{K-1} & \beta_{K-1} \\ & & & \gamma_K & \alpha_K \end{bmatrix},$$

$$\beta_1, \ldots, \beta_{K-1} \neq 0, \ \gamma_2, \ldots, \gamma_K \neq 0. \tag{62}$$

Without loss of generality, let β_K be defined as

$$\beta_K \triangleq 1, \tag{63a}$$

and

$$\begin{aligned} A_i &\triangleq -\frac{\alpha_i}{\beta_i}, \quad 1 \leqslant i \leqslant K, \\ B_i &\triangleq -\frac{\gamma_i}{\beta_i}, \quad 2 \leqslant i \leqslant K, \end{aligned} \tag{63b}$$

such that $A_1, \ldots, A_K$, $B_2, \ldots, B_K$ are entries of $\mathbf{T}_K$ in (51). We can express Φ_K as

$$\Phi_K = \operatorname{diag}(\beta_1, \ldots, \beta_K)\mathbf{T}_K,$$

and therefore, the inverse Ψ_K of Φ_K can be expressed as

$$\Psi_K = \Phi_K^{-1} = \left[\psi_{i,j}\right]_{i,j=1}^{K} = \mathbf{T}_K^{-1} \operatorname{diag}\left(\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_K}\right), \tag{64}$$

where, from (52) and (54),

$$\begin{aligned} \psi_{i,j} &= \frac{R_{i,j}}{\beta_j} \\ &= \frac{1}{\beta_j}\left[E_{i-1}(j+1) - \frac{E_{i-1}(1)E_K(j+1)}{E_K(1)}\right], \quad E_K(1) \neq 0, \end{aligned} \tag{65a}$$

such that (using (16) and (63b))

$$\begin{aligned} E_n(m) = &(-1)^{n-m+1}\left(\frac{\alpha_m}{\beta_m} \cdots \frac{\alpha_n}{\beta_n}\right) \\ &\times \left(1 + \sum_{q=1}^{\lfloor (n-m+1)/2 \rfloor} \sum_{(k_1,\ldots,k_q) \in S_q(m+1,n)} (\sigma_{k_1} \cdots \sigma_{k_q})\right) \\ &\text{if } m = 1, \ldots, n-1, \ n = 2, \ldots, K, \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\alpha_n}{\beta_n} \quad \text{if } m = n,\ n = 1, \ldots, K,\\
&= 1 \quad \text{if } m = n + 1,\ n = 0, \ldots, K,\\
&= 0 \quad \text{otherwise}
\end{aligned} \tag{65b}
$$

and (using (11) and (63b))

$$
\sigma_k = -\frac{\beta_{k-1}\gamma_k}{\alpha_{k-1}\alpha_k}. \tag{65c}
$$

The set $S_q(m+1, n)$ in (65b) is defined by (1a)–(1c).

It is known that for the matrix Ψ_K in (64), there exist four sequences $\{u_i\}_{i=1}^K$, $\{v_i\}_{i=1}^K$, $\{r_i\}_{i=1}^K$, $\{s_i\}_{i=1}^K$ [9,10,17] such that

$$
\psi_{i,j} = \begin{cases} u_i v_j & \text{if } i > j, \\ r_i s_j & \text{if } i \leqslant j. \end{cases} \tag{66}
$$

We can express Ψ_K as

$$
\Psi_K = \left(\left(\Phi_K^{\mathrm{T}} \right)^{-1} \right)^{\mathrm{T}}, \tag{67}
$$

where Φ_K^{T} can be obtained from Φ_K by interchanging the positions of β_j and γ_{j+1} in (62) for $j = 1, \ldots, K-1$. Defining, without loss of generality, γ_{K+1} as

$$
\gamma_{K+1} \triangleq 1, \tag{68}
$$

and interchanging β_j and γ_{j+1} for $j = 1, \ldots, K$ in formula (65b) for $E_n(m)$, we obtain a quantity $F_n(m)$ expressed as

$$
\begin{aligned}
F_n(m) &= (-1)^{n-m+1} \left(\frac{\alpha_m}{\gamma_{m+1}} \cdots \frac{\alpha_n}{\gamma_{n+1}} \right)\\
&\quad \times \left(1 + \sum_{q=1}^{\lfloor (n-m+1)/2 \rfloor} \sum_{(k_1, \ldots, k_q) \in S_q(m+1, n)} (\sigma_{k_1} \cdots \sigma_{k_q}) \right)\\
&\qquad \text{if } m = 1, \ldots, n-1,\ n = 2, \ldots, K,\\
&= -\frac{\alpha_n}{\gamma_{n+1}} \quad \text{if } m = n,\ n = 1, \ldots, K,\\
&= 1 \quad \text{if } m = n+1,\ n = 0, \ldots, K,\\
&= 0 \quad \text{otherwise,}
\end{aligned} \tag{69}
$$

where σ_k, which is invariant under the interchange of β_{k-1} and γ_k, is given by (65c). The elements of Ψ_K can alternatively be written in terms of $F_n(m)$ as

$$
\psi_{i,j} = \frac{1}{\gamma_{i+1}} \left[F_{j-1}(i+1) - \frac{F_{j-1}(1) F_K(i+1)}{F_K(1)} \right], \quad F_K(1) \neq 0. \tag{70}
$$

Observe from (65a) and (69) that, for $m \leqslant n$,

$$F_n(m) = E_n(m)\frac{(\beta_m \cdots \beta_n)}{(\gamma_{m+1} \cdots \gamma_{n+1})}. \tag{71}$$

Therefore, combining (65a), (70) and (71), we obtain

$$\psi_{i,j} = \begin{cases} -\dfrac{F_{j-1}(1)F_K(i+1)}{\gamma_{i+1}F_K(1)} \\ \quad = -\dfrac{E_{j-1}(1)E_K(i+1)}{\beta_i E_K(1)}\dfrac{(\gamma_{j+1}\cdots\gamma_i)}{(\beta_j \cdots \beta_{i-1})} & \text{if } i > j, \\ & \quad F_K(1) \neq 0, \\ -\dfrac{E_{i-1}(1)E_K(j+1)}{\beta_j E_K(1)} & \text{if } i \leqslant j,\ E_K(1) \neq 0. \end{cases} \tag{72}$$

Thus, (72), along with (65b) and (69), gives an *explicit formula for the element in the* ith *row and* jth *column of the inverse of the tridiagonal matrix* Φ_K in (62).

We see that (72) is consistent with the structure (66) of Ψ_K. Without loss of generality, substituting

$$\begin{aligned} r_i &= E_{i-1}(1), \\ s_j &= -\frac{E_K(j+1)}{\beta_j E_K(1)}, \\ u_i &= -\frac{F_K(i+1)}{\gamma_{i+1}F_K(1)} = s_i\frac{(\gamma_2 \cdots \gamma_i)}{(\beta_1 \cdots \beta_{i-1})}, \\ v_j &= F_{j-1}(1) = r_j\frac{(\beta_1 \cdots \beta_{j-1})}{(\gamma_2 \cdots \gamma_j)}, \end{aligned} \tag{73}$$

we obtain from Proposition 2 the recurrences

$$r_i = -\frac{1}{\beta_{i-1}}(\alpha_{i-1}r_{i-1} + \gamma_{i-1}r_{i-2}), \tag{74a}$$

$$v_j = -\frac{1}{\gamma_j}(\alpha_{j-1}v_{j-1} + \beta_{j-2}v_{j-2}) \tag{74b}$$

for sequences $\{r_i\}$ and $\{v_j\}$, respectively, and from Proposition 3 the recurrences

$$s_j = -\frac{1}{\beta_j}(\alpha_{j+1}s_{j+1} + \gamma_{j+2}s_{j+2}), \tag{75a}$$

$$u_i = -\frac{1}{\gamma_{i+1}}(\alpha_{i+1}u_{i+1} + \beta_{i+1}u_{i+2}) \tag{75b}$$

for sequences $\{s_j\}$ and $\{u_i\}$, respectively. Recurrences (74a), (74b), (75a) and (75b) agree with those in Theorem 2 of [9]. Note that (66) combined with (73) can also be used to obtain the tridiagonal matrix inverse.

The quantities β_K and γ_{K+1} which have been defined to be 1 cancel out due to multiplication in the expression for $\psi_{i,j}$ in (72) as well as in the expressions for u_i, s_j in (73). Hence, *we can also consider β_K and γ_{K+1} to be arbitrary nonzero complex quantities* in the definition of $E_n(m)$ in (65b) and $F_n(m)$ in (69).

4.2. An example

Consider the 3×3 tridiagonal matrix

$$\Phi_3 = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 \\ 0 & \gamma_3 & \alpha_3 \end{bmatrix}, \quad \beta_1, \beta_2 \neq 0,\ \gamma_2, \gamma_3 \neq 0,$$

which is the matrix in (62) with $K = 3$. To obtain the inverse of this matrix, we first obtain the sequences $\{u_i\}_{i=1}^3$, $\{v_i\}_{i=1}^3$, $\{r_i\}_{i=1}^3$, $\{s_i\}_{i=1}^3$ using (73), and then substitute their expressions in (66).

We need the expressions for σ_2 and σ_3 which, from (65c), are given by

$$\sigma_2 = -\frac{\beta_1\gamma_2}{\alpha_1\alpha_2}, \qquad \sigma_3 = -\frac{\beta_2\gamma_3}{\alpha_2\alpha_3}.$$

Now, from (73) and (65b), we have

$$r_1 = E_0(1) = 1,$$

$$r_2 = E_1(1) = -\frac{\alpha_1}{\beta_1},$$

$$r_3 = E_2(1) = \frac{\alpha_1\alpha_2}{\beta_1\beta_2}(1 + \sigma_2) = \frac{1}{\beta_1\beta_2}(\alpha_1\alpha_2 - \beta_1\gamma_2),$$

and

$$\begin{aligned} s_1 &= -\frac{E_3(2)}{\beta_1 E_3(1)} \\ &= -\frac{\dfrac{\alpha_2\alpha_3}{\beta_2\beta_3}(1+\sigma_3)}{\beta_1\left(-\dfrac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2\beta_3}\right)(1+\sigma_2+\sigma_3)} \\ &= \frac{(\alpha_2\alpha_3 - \beta_2\gamma_3)}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)}, \end{aligned}$$

$$s_2 = -\frac{E_3(3)}{\beta_2 E_3(1)}$$

$$
\begin{aligned}
&= -\frac{\left(-\dfrac{\alpha_3}{\beta_3}\right)}{\beta_2\left(-\dfrac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2\beta_3}\right)(1+\sigma_2+\sigma_3)} \\
&= -\frac{\beta_1\alpha_3}{(\alpha_1\alpha_2\alpha_3-\beta_1\gamma_2\alpha_3-\beta_2\gamma_3\alpha_1)},
\end{aligned}
$$

$$
\begin{aligned}
s_3 &= -\frac{E_3(4)}{\beta_3 E_3(1)} \\
&= -\frac{1}{\beta_3\left(-\dfrac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2\beta_3}\right)(1+\sigma_2+\sigma_3)} \\
&= \frac{\beta_1\beta_2}{(\alpha_1\alpha_2\alpha_3-\beta_1\gamma_2\alpha_3-\beta_2\gamma_3\alpha_1)}.
\end{aligned}
$$

By interchanging the positions of β_{k-1} and γ_k, $k = 2, 3, 4$ in r_i, we obtain v_i, and by doing the same in s_i, we obtain u_i. Thus, from (73),

$$v_1 = F_0(1) = 1,$$

$$v_2 = F_1(1) = -\frac{\alpha_1}{\gamma_2},$$

$$v_3 = F_2(1) = \frac{\alpha_1\alpha_2}{\gamma_2\gamma_3}(1+\sigma_2) = \frac{1}{\gamma_2\gamma_3}(\alpha_1\alpha_2-\beta_1\gamma_2),$$

and

$$
\begin{aligned}
u_1 &= -\frac{F_3(2)}{\gamma_2 F_3(1)} \\
&= -\frac{\dfrac{\alpha_2\alpha_3}{\gamma_3\gamma_4}(1+\sigma_3)}{\beta_1\left(-\dfrac{\alpha_1\alpha_2\alpha_3}{\gamma_2\gamma_3\gamma_4}\right)(1+\sigma_2+\sigma_3)} \\
&= \frac{(\alpha_2\alpha_3-\beta_2\gamma_3)}{(\alpha_1\alpha_2\alpha_3-\beta_1\gamma_2\alpha_3-\beta_2\gamma_3\alpha_1)},
\end{aligned}
$$

$$u_2 = -\frac{F_3(3)}{\gamma_3 F_3(1)}$$

$$= -\frac{\left(-\frac{\alpha_3}{\gamma_4}\right)}{\gamma_3\left(-\frac{\alpha_1\alpha_2\alpha_3}{\gamma_2\gamma_3\gamma_4}\right)(1+\sigma_2+\sigma_3)}$$

$$= -\frac{\gamma_2\alpha_3}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)},$$

$$u_3 = -\frac{F_3(4)}{\gamma_4 F_3(1)}$$

$$= -\frac{1}{\gamma_4\left(-\frac{\alpha_1\alpha_2\alpha_3}{\gamma_2\gamma_3\gamma_4}\right)(1+\sigma_2+\sigma_3)}$$

$$= \frac{\gamma_2\gamma_3}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)}.$$

Note that the expressions for $r_1, r_2, r_3, s_1, s_2, s_3, u_1, u_2, u_3, v_1, v_2, v_3$ do not contain β_3 or γ_4.

From (66), the diagonal elements of the inverse of Φ_3 are then given by

$$\psi_{1,1} = u_1v_1 = \frac{(\alpha_2\alpha_3 - \beta_2\gamma_3)}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)},$$

$$\psi_{2,2} = u_2v_2 = \frac{\alpha_1\alpha_3}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)},$$

$$\psi_{3,3} = u_3v_3 = \frac{(\alpha_1\alpha_2 - \beta_1\gamma_2)}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)}.$$

The upper diagonal elements are given by

$$\psi_{1,2} = r_1s_2 = -\frac{\beta_1\alpha_3}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)},$$

$$\psi_{1,3} = r_1s_3 = \frac{\beta_1\beta_2}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)},$$

$$\psi_{2,3} = r_2s_3 = -\frac{\alpha_1\beta_2}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)}.$$

Each of the lower diagonal elements $\psi_{i,j}$ is obtained by interchanging β_{k-1} and γ_k, $k = 2, 3$ in the corresponding upper diagonal element $\psi_{j,i}$. Therefore,

$$\psi_{2,1} = u_2v_1 = -\frac{\gamma_2\alpha_3}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)},$$

$$\psi_{3,1} = u_3v_1 = \frac{\gamma_2\gamma_3}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)},$$

$$\psi_{3,2} = u_3 v_2 = -\frac{\alpha_1\gamma_3}{(\alpha_1\alpha_2\alpha_3 - \beta_1\gamma_2\alpha_3 - \beta_2\gamma_3\alpha_1)}.$$

5. Characteristic polynomial

The tridiagonal matrix of (62) satisfies

$$\Phi_K = \mathrm{diag}(\beta_1, \ldots, \beta_K)\mathbf{T}_K, \tag{76}$$

where $\mathbf{T}_K$ is given by (51) with

$$A_i = -\frac{\alpha_i}{\beta_i}, \quad 1 \leqslant i \leqslant K, \qquad B_i = -\frac{\gamma_i}{\beta_i}, \quad 2 \leqslant i \leqslant K, \qquad \beta_K = 1,$$

as in (63a), (63b). From (76) and Proposition 5, we get

$$\det(\Phi_K) = (-1)^K(\beta_1 \cdots \beta_K)C_K = (-1)^K(\beta_1 \cdots \beta_K)E_K(1). \tag{77}$$

Using (65b), this can be rewritten as

$$\det(\Phi_K) = (\alpha_1 \cdots \alpha_K)\left(1+\sum_{q=1}^{\lfloor K/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,K)}(\sigma_{k_1}\cdots\sigma_{k_q})\right), \tag{78}$$

where $\sigma_k = -((\beta_{k-1}\gamma_k)/(\alpha_{k-1}\alpha_k))$ as in (65c).

Now the characteristic polynomial of Φ_K is given by

$$\det(\lambda\mathbf{I}_K - \Phi_K) = (-1)^K \det(\Phi_K - \lambda\mathbf{I}_K), \tag{79}$$

where $\mathbf{I}_K$ denotes the $K \times K$ identity matrix. By replacing α_k by $-(\lambda - \alpha_k)$ for $k = 1, \ldots, K$ in (79), we get

$$\begin{aligned}\det(\lambda\mathbf{I}_K - \Phi_K) = {} & (\lambda-\alpha_1)\cdots(\lambda-\alpha_K)\\ & \times\left[1+\sum_{q=1}^{\lfloor K/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,K)}\eta_{k_1}(\lambda)\cdots\eta_{k_q}(\lambda)\right],\end{aligned} \tag{80a}$$

where

$$\eta_k(\lambda) = -\frac{\beta_{k-1}\,\gamma_k}{(\lambda-\alpha_{k-1})(\lambda-\alpha_k)}, \tag{80b}$$

and $S_q(2, K)$ is given by (1a)–(1c). Thus, (80a) and (80b) give an *explicit formula for the characteristic polynomial of a tridiagonal matrix*.

For example, when $K = 4$ we get

$$\begin{aligned}\det(\lambda\mathbf{I}_4 - \Phi_4) = {} & (\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)(\lambda-\alpha_4)\\ & \times[1+\eta_2(\lambda)+\eta_3(\lambda)+\eta_4(\lambda)+\eta_2(\lambda)\eta_4(\lambda)]\\ = {} & (\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)(\lambda-\alpha_4)\\ & -\beta_1\gamma_2(\lambda-\alpha_3)(\lambda-\alpha_4)-\beta_2\gamma_3(\lambda-\alpha_1)(\lambda-\alpha_4)\\ & -\beta_3\gamma_4(\lambda-\alpha_1)(\lambda-\alpha_2)+\beta_1\gamma_2\beta_3\gamma_4.\end{aligned}$$

6. Orthogonal polynomials

Consider a set $\{p_n(t)\}_{n\geqslant 0}$ of polynomials that are orthogonal on the interval $[a, b]$ with respect to some nonnegative weight function $w(t)$ [18]. The polynomial $p_n(t)$ has degree *n*, and satisfies the second-order homogeneous difference equation [18,19]

$$\beta_n p_n(t) = (t-\alpha_n)p_{n-1}(t) - \gamma_n p_{n-2}(t), \quad n \geqslant 1, \tag{81}$$

where β_n, $p_n \neq 0$, $p_0(t)$ *is some nonzero constant*, and $p_{-1}(t)$ is a function of *t*, which may be identically zero. The difference equation (81) can be rewritten as

$$p_n(t) = \frac{(t-\alpha_n)}{\beta_n} p_{n-1}(t) - \frac{\gamma_n}{\beta_n} p_{n-2}(t), \quad n \geqslant 1, \tag{82}$$

with initial values $p_{-1}(t)$ and $p_0(t)$. Comparing (82) with the homogeneous difference equation (10), we can write

$$y_{n+2} = p_n(t), \quad n = 1, 2, \ldots, \tag{83a}$$

$$A_n = \frac{(t-\alpha_n)}{\beta_n}, \qquad B_n = -\frac{\gamma_n}{\beta_n}. \tag{83b}$$

From (12)–(14b), (83a) and (83b), the solution of (82) is given by

$$p_n(t) = \mathscr{C}_n(t)p_0(t) + \mathscr{D}_n(t)p_{-1}(t), \quad n \geqslant 0, \tag{84}$$

where

$$\mathscr{C}_0(t) = 1, \qquad \mathscr{C}_1(t) = \frac{(t-\alpha_1)}{\beta_1}, \tag{85a}$$

$$\mathscr{D}_0(t) = 0, \quad \mathscr{D}_1(t) = -\frac{\gamma_1}{\beta_1}, \quad \mathscr{D}_2(t) = -\frac{\gamma_1(t-\alpha_2)}{\beta_1\beta_2} \tag{85b}$$

and

$$\mathscr{C}_n(t) = \frac{(t-\alpha_1)\cdots(t-\alpha_n)}{\beta_1\cdots\beta_n} \times \left[1 + \sum_{q=1}^{\lfloor n/2\rfloor} \sum_{(k_1,\ldots,k_q)\in S_q(2,n)} \eta_{k_1}(t)\cdots\eta_{k_q}(t)\right] \quad \text{for } n \geqslant 2, \tag{86a}$$

$$\mathscr{D}_n(t) = -\frac{\gamma_1(t-\alpha_2)\cdots(t-\alpha_n)}{\beta_1\cdots\beta_n} \times \left[1 + \sum_{q=1}^{\lfloor (n-1)/2\rfloor} \sum_{(k_1,\ldots,k_q)\in S_q(3,n)} \eta_{k_1}(t)\cdots\eta_{k_q}(t)\right] \quad \text{for } n \geqslant 3, \tag{86b}$$

such that

$$\eta_k(t) = -\frac{\beta_{k-1}\gamma_k}{(t-\alpha_{k-1})(t-\alpha_k)}$$

as in (80b).

Note that $\mathscr{C}_n(t)$ is a polynomial of degree n for $n \geqslant 0$, while $\mathscr{D}_n(t)$ is a polynomial of degree $n-1$ for $n \geqslant 1$. Thus, (84) is *an expression for* $p_n(t)$*,* $n \geqslant 0$*, in terms of the recurrence coefficients* $\alpha_1, \ldots, \alpha_n,\ \beta_1, \ldots, \beta_n,\ \gamma_1, \ldots, \gamma_n$ *and initial values* $p_{-1}(t),\ p_0(t)$*.*

Let

$$\mathbf{p}(t) = \begin{bmatrix} p_0(t) \\ \vdots \\ p_{K-1}(t) \end{bmatrix} \tag{87}$$

denote the $K \times 1$ vector of orthogonal polynomials, and Φ_K the tridiagonal matrix in (62) of coefficients $\alpha_1, \ldots, \alpha_K, \beta_1, \ldots, \beta_{K-1}, \gamma_2, \ldots, \gamma_K$ of the difference equation (81) for $n = 1, \ldots, K$. We can rewrite (81) for $n = 1, \ldots, K$ as [19,20]

$$(t\mathbf{I}_K - \Phi_K)\mathbf{p}(t) = \beta_K p_K(t)\mathbf{e}_K + \gamma_1 p_{-1}(t)\mathbf{e}_1, \tag{88}$$

where $\mathbf{e}_1$ and $\mathbf{e}_K$ are the first and last columns, respectively, of $\mathbf{I}_K$. If t is not an eigenvalue of Φ_K, we get from (88)

$$\mathbf{p}(t) = -\beta_K p_K(t)(\Phi_K - t\mathbf{I}_K)^{-1}\mathbf{e}_K - \gamma_1 p_{-1}(t)(\Phi_K - t\mathbf{I}_K)^{-1}\mathbf{e}_1. \tag{89}$$

Replacing each diagonal element α_k by $\alpha_k - t$ in Φ_K, we obtain the matrix $\Phi_K - t\mathbf{I}_K$. Thus, each of the polynomials $p_0(t), \ldots, p_{K-1}(t)$ can be expressed in terms of the inverse of the tridiagonal matrix $\Phi_K - t\mathbf{I}_K$ and the functions $p_K(t)$ and $p_{-1}(t)$.

The vector $(\Phi_K - t\mathbf{I}_K)^{-1}\mathbf{e}_K$ is the Kth column of $(\Phi_K - t\mathbf{I}_K)^{-1}$. Replacing α_k by $-(t-\alpha_k)$, $k = 1, \ldots, K$ in (65b), we obtain from (72) the following expression for $\psi_{n+1,K}(t)$, the element in the $(n+1)$th row and Kth column of $(\Phi_K - t\mathbf{I}_K)^{-1}$:

$$\psi_{n+1,K}(t) = -\frac{\dfrac{(t-\alpha_1)\cdots(t-\alpha_n)}{\beta_1\cdots\beta_n}\left[1+\displaystyle\sum_{q=1}^{\lfloor n/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,n)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}{\beta_K\dfrac{(t-\alpha_1)\cdots(t-\alpha_K)}{\beta_1\cdots\beta_K}\left[1+\displaystyle\sum_{q=1}^{\lfloor K/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,K)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}. \tag{90}$$

Similarly, by replacing α_k by $-(t-\alpha_k)$, $k = 1, \ldots, K$ in (69), we obtain from (72) the following expression for $\psi_{n+1,1}(t)$, the element in the $(n+1)$th row and first column of $(\Phi_K - t\mathbf{I}_K)^{-1}$:

$$\psi_{n+1,1}(t)$$
$$= -\frac{\dfrac{(t-\alpha_{n+2})\cdots(t-\alpha_K)}{\gamma_{n+3}\cdots\gamma_{K+1}}\left[1+\sum_{q=1}^{\lfloor (K-n-1)/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(n+3,K)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}{\gamma_{n+2}\dfrac{(t-\alpha_1)\cdots(t-\alpha_K)}{\gamma_2\cdots\gamma_{K+1}}\left[1+\sum_{q=1}^{\lfloor K/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,K)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}. \quad (91)$$

Now $p_n(t)$ is the $(n+1)$th row of $\mathbf{p}$ in (89). Therefore, combining (89), (90) and (91), we obtain, for $n = 0, \ldots, K-1$,

$$p_n(t) = \frac{(\beta_{n+1}\cdots\beta_K)\left[1+\sum_{q=1}^{\lfloor n/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,n)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}{(t-\alpha_{n+1})\cdots(t-\alpha_K)\left[1+\sum_{q=1}^{\lfloor K/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,K)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}p_K(t)$$
$$+\frac{(\gamma_1\cdots\gamma_{n+1})\left[1+\sum_{q=1}^{\lfloor (K-n-1)/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(n+3,K)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}{(t-\alpha_1)\cdots(t-\alpha_{n+1})\left[1+\sum_{q=1}^{\lfloor K/2\rfloor}\sum_{(k_1,\ldots,k_q)\in S_q(2,K)}\eta_{k_1}(t)\cdots\eta_{k_q}(t)\right]}p_{-1}(t), \quad (92)$$

where $\eta_k(t)$ is given by (80b). Thus, (92) is *an expression for the K intermediate polynomials* $p_n(t)$, $0 \leqslant n \leqslant K-1$, *in terms of the recurrence coefficients* $\alpha_1, \ldots, \alpha_K, \beta_1, \ldots, \beta_K, \gamma_1, \ldots, \gamma_K$ *and boundary values* $p_{-1}(t)$, $p_K(t)$.

We now show how the general formula for $p_n(t)$ in (84) can be used to obtain expressions for Legendre, Hermite and Chebyshev polynomials.

6.1. Legendre polynomials

Legendre polynomials can be generated by the difference equation

$$p_n(t) = \frac{(2n-1)}{n}tp_{n-1}(t) - \frac{(n-1)}{n}p_{n-2}(t), \quad n \geqslant 1, \quad (93)$$

with $p_0(t) = 1$. The interval of orthogonality is $[-1, 1]$ and the weight function $w(t) = 1$. Comparing (93) with (82), we obtain

$$\alpha_n = 0, \quad \beta_n = \frac{n}{(2n-1)}, \quad \gamma_n = \frac{(n-1)}{(2n-1)}, \qquad n \geqslant 1. \quad (94)$$

Since $\gamma_1 = 0$, we have $\mathscr{D}_n(t) = 0$. Therefore, $p_{-1}(t)$ does not affect the solution of (93).

Now, from (80b),

$$\eta_k(t) = -\frac{(k-1)^2}{(2k-3)(2k-1)t^2}. \tag{95}$$

Using (86a), we can write the solution of (93) as

$$\begin{aligned} p_n(t) &= \mathscr{C}_n(t) \\ &= \frac{(2n)!}{2^n(n!)^2} \\ &\quad \times \left[t^n + \sum_{q=1}^{\lfloor n/2 \rfloor} (-1)^q t^{n-2q} \sum_{(k_1,\ldots,k_q)\in S_q(2,n)} \prod_{i=1}^{q} \frac{(k_i-1)^2}{(2k_i-3)(2k_i-1)} \right]. \end{aligned} \tag{96}$$

Denote $G_{n,q}$ as

$$G_{n,q} = \sum_{(k_1,\ldots,k_q)\in S_q(2,n)} \prod_{i=1}^{q} \frac{(k_i-1)^2}{(2k_i-3)(2k_i-1)}. \tag{97}$$

Applying property (3) on $S_q(2,n)$ we get

$$\begin{aligned} S_q(2,n) = S_q(2,n-1) \cup \big\{ &(k_1,\ldots,k_{q-1},k_q):\ k_q = n; \\ &(k_1,\ldots,k_{q-1}) \in S_{q-1}(2,n-2) \big\}, \end{aligned} \tag{98}$$

which implies that $G_{n,q}$ in (97) follows the recurrence

$$G_{n,q} = G_{n-1,q} + \frac{(n-1)^2}{(2n-3)(2n-1)} G_{n-2,q-1}. \tag{99}$$

From (99), it can be shown by mathematical induction that

$$G_{n,q} = \frac{(n!)^2(2n-2q)!}{(2n)!q!(n-q)!(n-2q)!}. \tag{100}$$

Combining (96), (97) and (100), we finally obtain

$$p_n(t) = \sum_{q=1}^{\lfloor n/2 \rfloor} \frac{(-1)^q(2n-2q)!}{2^n q!(n-q)!(n-2q)!} t^{n-2q}, \tag{101}$$

which is the expression for the Legendre polynomial of degree n. This can alternatively be written as

$$p_n(t) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\left(t^2-1\right)^n \right].$$

6.2. Hermite polynomials

Hermite polynomials can be generated by the recurrence

$$p_n(t) = tp_{n-1}(t) - (n-1)p_{n-2}(t), \quad n \geqslant 1, \tag{102}$$

with $p_0(t) = 1$. The interval of orthogonality is $(-\infty, \infty)$ and the weight function $w(t) = \mathrm{e}^{-(t^2/2)}$. Comparing (102) with (82), we get

$$\alpha_n = 0, \quad \beta_n = 1, \quad \gamma_n = (n-1), \qquad n \geqslant 1. \tag{103}$$

Since $\gamma_1 = 0$, we have $\mathscr{D}_n(t) = 0$; hence, $p_{-1}(t)$ does not affect the solution of (102).

From (80b),

$$\eta_k(t) = -\frac{(k-1)}{t^2}. \tag{104}$$

We can then write the solution of (102) as

$$p_n(t) = \mathscr{C}_n(t) = t^n + \sum_{q=1}^{\lfloor n/2 \rfloor} (-1)^q t^{n-2q} \sum_{(k_1,\ldots,k_q)\in S_q(2,n)} \prod_{i=1}^{q} (k_i - 1). \tag{105}$$

Denoting $G_{n,q}$ as

$$G_{n,q} = \sum_{(k_1,\ldots,k_q)\in S_q(2,n)} \prod_{i=1}^{q} (k_i - 1), \tag{106}$$

and applying property (3) on $S_q(2, n)$ we obtain the recurrence

$$G_{n,q} = G_{n-1,q} + (n-1)G_{n-2,q-1}. \tag{107}$$

It can be shown by mathematical induction using this recurrence that

$$G_{n,q} = \frac{n!}{2^q q!(n-2q)!}. \tag{108}$$

Combining (105), (106) and (108), we get

$$p_n(t) = \sum_{q=0}^{\lfloor n/2 \rfloor} \frac{(-1)^q n!}{2^q q!(n-2q)!} t^{n-2q}, \tag{109}$$

which is the expression for the Hermite polynomial of degree n. This can alternatively be written as

$$p_n(t) = (-1)^n \,\mathrm{e}^{-(t^2/2)} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\mathrm{e}^{-(t^2/2)}\right].$$

6.3. Chebyshev polynomials

Chebyshev polynomials can be generated by the difference equation

$$p_n(t) = 2tp_{n-1}(t) - p_{n-2}(t), \quad n \geqslant 1, \tag{110}$$

whose coefficients do not vary with n. The interval of orthogonality is $[-1, 1]$ and the weight function $w(t) = 1/\sqrt{1-t^2}$.

Here

$$\alpha_n = 0, \quad \beta_n = \frac{1}{2} = \gamma_n, \qquad n \geqslant 1, \tag{111}$$

and

$$\eta_k(t) = -\frac{1}{4t^2}. \tag{112}$$

Since $\eta_k(t)$ does not depend on k, we can apply property (2) on $S_q(2, n)$ and $S_q(3, n)$ in (86a) and (86b). This results in

$$|S_q(2, n)| = \binom{n-q}{q}, \qquad |S_q(3, n)| = \binom{n-q-1}{q}, \tag{113}$$

followed by

$$\mathscr{C}_n(t) = (2t)^n + \sum_{q=1}^{\lfloor n/2 \rfloor} (-1)^q \binom{n-q}{q} (2t)^{n-2q}, \tag{114a}$$

$$\mathscr{D}_n(t) = -\mathscr{C}_{n-1}(t). \tag{114b}$$

For the Chebyshev polynomials of the *first kind*, we have

$$p_0(t) = 1, \qquad p_{-1}(t) = t,$$

and therefore

$$\begin{aligned} p_n(t) &= \mathscr{C}_n(t) + t\mathscr{D}_n(t) \\ &= \mathscr{C}_n(t) - t\mathscr{C}_{n-1}(t) \\ &= \frac{1}{2}\left\{(2t)^n + \sum_{q=1}^{\lfloor n/2 \rfloor} (-1)^q \left[\binom{n-q}{q} + \binom{n-q-1}{q}\right] (2t)^{n-2q}\right\}. \end{aligned} \tag{115}$$

This can also be expressed as

$$p_n(t) = \cos\big(n \cos^{-1}(t)\big).$$

For the Chebyshev polynomials of the *second kind*, we have

$$p_0(t) = 1, \qquad p_{-1}(t) = 0,$$

and therefore $p_n(t) = \mathscr{C}_n(t)$, where $\mathscr{C}_n(t)$ is given by (114a). This can also be expressed as

$$p_n(t) = \frac{\sin\big((n+1)\ \cos^{-1}(t)\big)}{\sin\big(\cos^{-1}(t)\big)} = \frac{\sin\big((n+1)\ \cos^{-1}(t)\big)}{\sqrt{1-t^2}}.$$

7. A cyclic tridiagonal system

Consider the second-order linear difference equation over the finite index interval $[1, K+2]$, $K \geqslant 2$, given by

$$\beta_n y_{n+2} = -\alpha_n y_{n+1} - \gamma_n y_n + \beta_n x_{n+2}, \quad 1 \leqslant n \leqslant K, \tag{116}$$

with variable complex coefficients $\beta_n, \alpha_n, \gamma_n$, where $\beta_n, \gamma_n \neq 0$, complex forcing term x_{n+2}, and complex boundary values y_1, y_{K+2}. The boundary values are periodic with period K, that is, we have the boundary conditions

$$y_1 = y_{K+1}, \qquad y_{K+2} = y_2. \tag{117}$$

The K equations in (116) can be written in matrix form after applying the boundary conditions as

$$\Lambda_K \begin{bmatrix} y_2 \\ \vdots \\ y_{K+1} \end{bmatrix} = \mathrm{diag}(\beta_1, \ldots, \beta_K) \begin{bmatrix} x_3 \\ \vdots \\ x_{K+2} \end{bmatrix}, \tag{118a}$$

where

$$\Lambda_K = \begin{bmatrix} \alpha_1 & \beta_1 & & & \gamma_1 \\ \gamma_2 & \alpha_2 & \beta_2 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & \gamma_{K-1} & \alpha_{K-1} & \beta_{K-1} \\ \beta_K & & & \gamma_K & \alpha_K \end{bmatrix},$$

$$\beta_1, \ldots, \beta_K \neq 0, \ \gamma_1, \ldots, \gamma_K \neq 0. \tag{118b}$$

The system ((118a), (118b)) of equations is a *cyclic linear tridiagonal system* [21], and the matrix Λ_K in (118b) a *cyclic tridiagonal matrix*. The solution of system ((118a), (118b)), or alternatively, of the difference equation (116) with boundary conditions (117) for $n = 1, \ldots, K$ can be expressed as

$$\begin{bmatrix} y_2 \\ \vdots \\ y_{K+1} \end{bmatrix} = \Lambda_K^{-1} \begin{bmatrix} \beta_1 x_3 \\ \vdots \\ \beta_K x_{K+2} \end{bmatrix}. \tag{119}$$

Let

$$\Lambda_K^{-1} = \left[\omega_{i,j}\right]_{i,j=1}^{K}. \tag{120}$$

The difference equation (116) can be rewritten as

$$y_{n+2} = A_n y_{n+1} + B_n y_n + x_{n+2}, \quad 1 \leqslant n \leqslant K, \tag{121a}$$

as in (41) with

$$A_n = -\frac{\alpha_n}{\beta_n}, \qquad B_n = -\frac{\gamma_n}{\beta_n}. \tag{121b}$$

From Proposition 4, the quantities $y_3, \ldots, y_{K+2}$ of recurrence (121a) can be expressed in terms of y_1, y_2 as

$$y_{i+1} = C_{i-1} y_2 + D_{i-1} y_1 + \sum_{j=1}^{i-1} E_{i-1}(j+1) x_{j+2}, \quad 2 \leqslant i \leqslant K+1, \tag{122}$$

where $E_n(m)$ is given by (65b), and from (13a), (13b), (14a), (14b) and (121b),

$$C_1 = -\frac{\alpha_1}{\beta_1}, \quad D_1 = -\frac{\gamma_1}{\beta_1}, \quad D_2 = \frac{\gamma_1 \alpha_2}{\beta_1 \beta_2}, \tag{123a}$$

$$C_n = (-1)^n \left(\frac{\alpha_1}{\beta_1} \cdots \frac{\alpha_n}{\beta_n}\right) \left(1 + \sum_{q=1}^{\lfloor n/2 \rfloor} \sum_{(k_1,\ldots,k_q) \in S_q(2,n)} (\sigma_{k_1} \cdots \sigma_{k_q})\right)$$
$$\text{for } n \geqslant 2, \tag{123b}$$

$$D_n = (-1)^n \frac{\gamma_1}{\beta_1} \left(\frac{\alpha_2}{\beta_2} \cdots \frac{\alpha_n}{\beta_n}\right) \left(1 + \sum_{q=1}^{\lfloor (n-1)/2 \rfloor} \sum_{(k_1,\ldots,k_q) \in S_q(3,n)} (\sigma_{k_1} \cdots \sigma_{k_q})\right)$$
$$\text{for } n \geqslant 3, \tag{123c}$$

such that $S_q(L, U)$ is defined by (1a)–(1c), and $\sigma_k = -((\beta_{k-1}\gamma_k)/(\alpha_{k-1}\alpha_k))$ as in (65c).

Putting $i = K$ in (122) and applying the boundary condition $y_{K+1} = y_1$ in the resulting equation, we get

$$(1 - D_{K-1}) y_1 - C_{K-1} y_2 = \sum_{j=1}^{K-1} E_{K-1}(j+1) x_{j+2}.$$

Similarly, putting $i = K+1$ and $y_{K+2} = y_2$ in (122) results in

$$-D_K y_1 + (1 - C_K) y_2 = \sum_{j=1}^{K} E_K(j+1) x_{j+2}.$$

We can then express y_1 and y_2 in terms of $x_3, \ldots, x_{K+2}$ as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 - D_{K-1} & -C_{K-1} \\ -D_K & 1 - C_K \end{bmatrix}^{-1} \begin{bmatrix} \sum_{j=1}^{K-1} E_{K-1}(j+1) x_{j+2} \\ \sum_{j=1}^{K} E_K(j+1) x_{j+2} \end{bmatrix}. \tag{124}$$

Substituting (124) in (122), we can express y_{i+1} as

$$y_{i+1} = \sum_{i=1}^{K} \omega_{i,j} \beta_j x_{j+2}, \tag{125}$$

which implies from (119) and (120) that $\omega_{i,j}$ is the element in the *i*th row and *j*th column of Λ^{-1}. Thus, we can obtain an explicit formula for the inverse of a cyclic tridiagonal matrix by solving a boundary value problem with periodic boundary conditions.

8. Case of constant diagonals

When

$$\alpha_n = \alpha, \quad \beta_n = \beta, \quad \gamma_n = \gamma, \qquad n = 1, \ldots, K, \tag{126}$$

the tridiagonal matrix Φ_K in (62) is *Toeplitz*, while the cyclic tridiagonal matrix Λ_K in (118b) is *circulant*, which also implies that it is Toeplitz. The quantities C_n, D_n and $E_n(m)$ in (65b) and (123a)–(123c) can now be expressed in terms of the Chebyshev polynomials of the second kind discussed in Section 6.3 as

$$E_n(m) = (-1)^{n-m+1} \left(\sqrt{\frac{\gamma}{\beta}}\right)^{n-m+1} p_{n-m+1}\left(\frac{\alpha}{2\sqrt{\gamma\beta}}\right),$$

$$C_n = E_n(1) = (-1)^n \left(\sqrt{\frac{\gamma}{\beta}}\right)^n p_n\left(\frac{\alpha}{2\sqrt{\gamma\beta}}\right),$$

$$D_n = (-1)^n \left(\sqrt{\frac{\gamma}{\beta}}\right)^{n+1} p_{n-1}\left(\frac{\alpha}{2\sqrt{\gamma\beta}}\right),$$

where, using (114a),

$$p_n\left(\frac{\alpha}{2\sqrt{\gamma\beta}}\right) = \left(\frac{\alpha}{\sqrt{\gamma\beta}}\right)^n \left[1 + \sum_{q=1}^{\lfloor n/2\rfloor} (-1)^q \binom{n-q}{q} \left(\frac{\gamma\beta}{\alpha^2}\right)^q\right].$$

From (72), the elements of the inverse of the tridiagonal matrix Φ_K in (62) under condition (126) can be expressed as

$$\psi_{i,j} = \begin{cases} \dfrac{(-1)^{i-j}}{\sqrt{\gamma\beta}} \left(\sqrt{\dfrac{\gamma}{\beta}}\right)^{i-j} \dfrac{p_{j-1}\left(\dfrac{\alpha}{2\sqrt{\gamma\beta}}\right) p_{K-i}\left(\dfrac{\alpha}{2\sqrt{\gamma\beta}}\right)}{p_K\left(\dfrac{\alpha}{2\sqrt{\gamma\beta}}\right)} & \text{if } i > j, \\[2ex] \dfrac{(-1)^{j-i}}{\sqrt{\gamma\beta}} \left(\sqrt{\dfrac{\beta}{\gamma}}\right)^{j-i} \dfrac{p_{i-1}\left(\dfrac{\alpha}{2\sqrt{\gamma\beta}}\right) p_{K-j}\left(\dfrac{\alpha}{2\sqrt{\gamma\beta}}\right)}{p_K\left(\dfrac{\alpha}{2\sqrt{\gamma\beta}}\right)} & \text{if } i \leqslant j, \\ E_K(1) \neq 0. \end{cases}$$

The inverse of the cyclic tridiagonal matrix Λ_K in (118b) under condition (126) can also be obtained from $E_n(m)$, C_n, D_n using the approach discussed in Section 7.

9. Conclusions

We have obtained explicit formulae for the elements of the inverse of a general tridiagonal matrix by deriving the explicit solution of a second-order linear nonhomogeneous difference equation with variable coefficients, and then applying the solution to a boundary value problem with zero boundary values. Using the formula for the determinant, we have got an expression for the characteristic polynomial. A connection between the matrix inverse and orthogonal polynomials has also been established. It has also been shown that how an application of the solution of a second-order linear difference equation to a boundary value problem with periodic boundary conditions can yield the inverse of a cyclic tridiagonal matrix. In the simple case of a tridiagonal or a cyclic tridiagonal matrix with constant diagonals, the elements of the inverse can be expressed in terms of the Chebyshev polynomials of the second kind.

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