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The Boundary of the Numerical Range of Matrix Polynomials

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ABSTRACT

Some algebraic properties of the sharp points of the numerical range of matrix polynomials are the main subject of this paper. We also consider isolated points of the numerical range and the location of the numerical range in a circular annulus. © 1997 Elsevier Science Inc.

INTRODUCTION

Let $\mathcal{M}_n$ be the algebra of $n \times n$ complex matrices, and let

$$L(\lambda) = A_m \lambda^m + \cdots + A_1 \lambda + A_0 \tag{1}$$

be a polynomial with $A_i \in \mathcal{M}_n$ for $i = 0, 1, \dots, m$ and $A_m \neq 0$. It is well known [2, 4] that the *numerical range* (NR) of $L(\lambda)$ is defined by

$$\mathrm{NR}[L(\lambda)] = \{\lambda : x^* L(\lambda) x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\}. \tag{2}$$

When $L(\lambda) = \lambda I - A$, the concept reduces to the classical numerical range of A,

$$\mathrm{NR}[L(\lambda)] \equiv \mathrm{NR}(A) = \{\lambda : x^* A x = \lambda,\ x \in \mathbb{C}^n, \text{ and } \|x\| = 1\}.$$

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Points of special interest on the boundary of NR are the sharp points [5]. A point $\lambda_0 \in \mathrm{NR}[L(\lambda)]$ is called a *sharp point* of $\mathrm{NR}[L(\lambda)]$ if for a connected component w_s of $\mathrm{NR}[L(\lambda)]$ there exist a disk $S(\lambda_0, r)$ with $r > 0$ and two angles ϕ_1 and ϕ_2 with $0 \leqslant \phi_1 < \phi_2 \leqslant 2\pi$ and $\varphi_q - \varphi_1 < n$ such that

$$\operatorname{Re}\left(e^{i\theta}\lambda_0\right) = \max\left\{\operatorname{Re} z : e^{-i\theta} z \in w_s \cap S(\lambda_0, r)\right\}$$

for all $\theta \in [\phi_1, \phi_2]$.

Following [5], we investigate analytic and algebraic properties of boundary points of $\mathrm{NR}[L(\lambda)]$. In [5] it has been proved that the sharp points of $\mathrm{NR}(C)$, where

$$C = \begin{bmatrix} & \mathbf{0} & & I_{n(m-1)} \\ \cdots & \cdots & \cdots & \cdots \\ -A_0 & -A_1 & \cdots & -A_{m-1} \end{bmatrix}$$

is the companion matrix of $L(\lambda)$ when $A_m = I$, are also sharp points of $\mathrm{NR}[L(\lambda)]$, but the converse does not hold. Therefore the sharp points of $\mathrm{NR}(C)$ are eigenvalues of $L(\lambda)$, and it is not known if this characteristic property is also true for the sharp points of $\mathrm{NR}[L(\lambda)]$.

In the first section of our paper we attempt to give an answer for this problem, and as a first step we prove that if $\lambda_0 \in \partial\mathrm{NR}(L(\lambda))$, then the NR of the matrix $L(\lambda_0)$ has the origin as a boundary point. Specifically, if λ_0 is a sharp point of $\mathrm{NR}(A\lambda - B)$, then 0 is also a sharp point of $\mathrm{NR}(A\lambda_0 - B)$ and λ_0 belongs to the spectrum $\sigma(A\lambda - B) = \{\lambda : \det(A\lambda - B) = 0\}$. Thereupon, under some weak conditions for $L(\lambda)$, it is proved that if 0 is a sharp point of $\mathrm{NR}[L(\lambda)]$ then $0 \in \sigma(L)$. Therefore, using the equality [2]

$$\mathrm{NR}\left[L(\lambda + \lambda_0)\right] = \mathrm{NR}\left[L(\lambda)\right] - \lambda_0,$$

we establish that a sharp point λ_0 of $\mathrm{NR}[L(\lambda)]$ is also an eigenvalue of $L(\lambda)$.

In Section 2 we consider isolated points of $\mathrm{NR}[L(\lambda)]$, with applications to the factorization of $L(\lambda)$, and, (Section 3) to the location of $\mathrm{NR}[L(\lambda)]$ in a circular annulus.

1. PROPERTIES OF SHARP POINTS

By the definition of sharp points, it is obvious that they are boundary points. In the following proposition we present a connection of these points with respect to the origin as a boundary point.

THEOREM 1.1. *If $\lambda_0 \in \partial \mathrm{NR}[L(\lambda)]$, then 0 is a boundary point of the numerical range of the matrix $L(\lambda_0)$.*

Proof. Since $\mathrm{NR}[L(\lambda)]$ is closed in $\mathbb{C}$ [2], the point λ_0 belongs to $\mathrm{NR}[L(\lambda)]$ and there exists a unit vector x_0 such that $x_0^* L(\lambda_0) x_0 = 0$. Thus $0 \in \mathrm{NR}[L(\lambda_0)]$, and it is enough to prove that 0 is not an interior point of $\mathrm{NR}[L(\lambda_0)]$.

Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence of elements of $\mathbb{C} \setminus \mathrm{NR}[L(\lambda)]$ converging to λ_0. If there exists a disk $S(0, \varepsilon) \subset \mathrm{NR}[L(\lambda_0)]$, we can find unit vectors x_1, x_2, x_3 such that 0 belongs to the interior of the triangle

$$\mathrm{Conv\ hull}\{x_1^* L(\lambda_0) x_1, x_2^* L(\lambda_0) x_2, x_3^* L(\lambda_0) x_3\} \subset S(0, \varepsilon).$$

Then for ε small enough the vertices of the triangle are close to 0, and consequently, by the convexity of $\mathrm{NR}[L(\lambda_n)]$, the equality

$$\lim_{n \to \infty} x_i^* L(\lambda_n) x_i = x_i^* L(\lambda_0) x_i \qquad (i = 1, 2, 3)$$

implies that $0 \in \mathrm{NR}[L(\lambda_n)]$ for $n \geqslant n_0 \in \mathbb{N}$. Thus, for suitable x we have $x^* L(\lambda_n) x = 0$, i.e. $\lambda_n \in \mathrm{NR}[L(\lambda)]$, which contradicts the assumption. ∎

The converse of Theorem 1.1 is not true, as is illustrated in the following example. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad L(\lambda) = \lambda^2 I - \lambda(I + A) + A = (\lambda - 1)(\lambda I - A).$$

Then, as $L(1) = 0$, we easily have $0 \in \partial \mathrm{NR}[L(1)]$. On the other hand, we have $\mathrm{NR}[L(\lambda)] = \mathrm{NR}(A) = \bar{S}(1, 0.5)$. Hence, 1 is an interior point of $\mathrm{NR}[L(\lambda)]$.

For further investigation of sharp points we need the following lemma:

LEMMA. *Suppose x_0 is a unit vector such that $0 = x_0^* A x_0$ belongs to the numerical range of $A \in \mathscr{M}_n$.*

(a) *If $x^* A x$ has nonnegative real parts for all x of the neighborhood $S(x_0, \epsilon) = \{x \in \mathbb{C}^n : \|x - x_0\|_2 < \varepsilon\}$, then $A + A^*$ is positive semidefinite.*

(b) *If $x^* A x = 0$ for all $x \in S(x_0, \varepsilon)$, then $A = 0$.*

Proof. (a): Denoting by $\lambda_1, \dots, \lambda_n$ the eigenvalues of the Hermitian matrix $A + A^*$, we see that

$$x^*(A + A^*) x = y^* D y,$$

where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, $y = P^*x$, and P is a unitary matrix. For $y_0 = P^*x_0 = [y_1, \dots, y_n]^T$, we have

$$y_0^* D y_0 = \lambda_1 |y_1|^2 + \cdots + \lambda_n |y_n|^2 = 0. \tag{3}$$

Since

$$\|y - y_0\|_2 = \|P^*\|_2 \|x - x_0\|_2 = \|x - x_0\|_2,$$

and $\operatorname{Re}(x^*Ax) \geqslant 0$ for all $x \in S(x_0 \varepsilon)$, there exists a neighborhood $S(y_0, \varepsilon)$ such that $y^*Dy \geqslant 0$ for any $y \in S(y_0, \varepsilon)$.

In (3), if $\lambda_k < 0$, we consider the vector

$$y_\delta = y_0 + \delta e_k = \begin{bmatrix} y_1 & \cdots & y_k + \delta & \cdots & y_n \end{bmatrix}^T,$$

where $\delta \in \mathbb{C}$ such that $0 < |\delta| < \varepsilon$ and $|y_k + \delta| > |y_k|$. Then, for a vector y_δ of the neighborhood $S(y_0, \varepsilon)$, we have

$$\begin{aligned} y_\delta^* D y_\delta &= \lambda_1 |y_1|^2 + \cdots + \lambda_k |y_k + \delta|^2 + \cdots + \lambda_n |y_n|^2 \\ &= \lambda_k \left(|y_k + \delta|^2 - |y_k|^2 \right) < 0, \end{aligned}$$

which is not true. Therefore $\lambda_k \geqslant 0$ for $k = 1, \dots, n$, and $D \geqslant 0$.

(b): For the matrix A we consider the Hermitian matrices:

$$H(A) = \tfrac{1}{2}(A + A^*), \qquad S(A) = \frac{1}{2i}(A^* - A).$$

Then by the equations $x_0^* A x_0 = 0$ and $x^*Ax = 0$ for any $x \in S(x_0, \varepsilon)$, it is clear that

$$x_0^* H(A) x_0 = 0, \qquad x^* H(A) x = 0 \quad \forall x \in S(x_0, \varepsilon)$$

and

$$x_0^* S(A) x_0 = 0, \qquad x^* S(A) x = 0 \quad \forall x \in S(x_0, \varepsilon).$$

Thus, by statement (a) we have that $H(A)$ and $S(A)$ are both positive and negative semidefinite. Then, $H(A) = S(A) = 0$, i.e., $A = H(A) + iS(A) = 0$. ∎

THEOREM 1.2. *Let $A \in \mathscr{M}_n$, and x_0 be a unit vector such that $0 = x_0^* A x_0$ belongs to* NR(A). *Then* 0 *is a sharp point of* NR(A) *if and only if there exist $\varepsilon > 0$, ϕ_1, and ϕ_2 such that*

$$\phi_1 \leqslant \arg(x^* A x) \leqslant \phi_2 \tag{4}$$

with $\phi_2 - \phi_1 < \pi$ for all $x \in S(x_0, \varepsilon)$.

Proof. Let the relationship (4) holds. If $\omega_1 = \pi/2 - \phi_1$ and $\omega_2 = 3\pi/2 - \phi_2$, then $0 < \omega_2 - \omega_1 < \pi$, and for the matrix $e^{i\omega}A + e^{-i\omega}A^*$ we have

$$x_0^*(e^{i\omega}A + e^{-i\omega}A^*)x_0 = 0$$

and

$$\mathrm{Re}(x^* e^{i\omega}Ax) = \tfrac{1}{2}x^*(e^{i\omega}A + e^{-i\omega}A^*)x \leqslant 0$$

for any $\omega \in [\omega_1, \omega_2]$ and for all $x \in S(x_0, \varepsilon)$. Therefore, by the Lemma, the matrix $e^{i\omega}A + e^{-i\omega}A^*$ is negative semidefinite and

$$\max\{x^*(e^{i\omega}A + e^{-i\omega}A^*)x : \|x\| = 1\} = 0,$$

for any $\omega \in [\omega_1, \omega_2]$. Thus,

$$\max\{\mathrm{Re}\, z : z \in e^{i\omega}\mathrm{NR}(A)\} = 0,$$

i.e., the origin is a sharp point of NR(A).

Conversely, assuming that 0 is sharp point, in the converse way we obtain $\arg(x^*Ax) \in [\phi_1, \phi_2]$, where $\phi_2 - \phi_1 = (3\pi/2 - \omega_2) - (\pi/2 - \omega_1) < \pi$. ∎

THEOREM 1.3. *Let λ_0 be a sharp point of the numerical range of the pencil $A\lambda - B$. Then*

(a) 0 *is a sharp point of* NR($A\lambda_0 - B$);
(b) λ_0 *is an eigenvalue of* $A\lambda - B$.

Proof. (a): By the equality $\mathrm{NR}[A(\lambda + \lambda_0) - B] = \mathrm{NR}[A\lambda - B] - \lambda_0$ and the fact that λ_0 is a sharp point of $\mathrm{NR}[A\lambda - B]$, it is implied that 0 is a sharp point of $\mathrm{NR}[A\lambda + (A\lambda_0 - B)]$. Thus, there exists a vector x_0 such that $x_0^*(A\lambda_0 - B)x_0 = 0$ and $x_0^* A x_0 = k \neq 0$. It is not possible to have $x_0^*(A\lambda_0 - B)x_0 = x_0^* A x_0 = 0$, since then $\mathrm{NR}(A\lambda + A\lambda_0 - B) = \mathbb{C}$. Since the point 0 is a sharp point of $\mathrm{NR}(A\lambda + A\lambda_0 - B)$, there exists an $r > 0$ such that for any complex number

$$\mu_x = -\frac{x^*(A\lambda_0 - B)x}{x^* A x} \in S(0, r) \cap \mathrm{NR}(A\lambda + A\lambda_0 - B)$$

it is implied that

$$\phi_1 \leqslant \arg\left(-\frac{x^*(A\lambda_0 - B)x}{x^* A x}\right) \leqslant \phi_2 \tag{4'}$$

with $\phi_2 - \phi_1 < \pi$. Moreover, by the continuity of the functions $F_1(x) = x^* A x$ and $F_2(x) = x^*(A\lambda_0 - B)x$, for any $\varepsilon > 0$ there exists a neighborhood $S(x_0, \delta)$ such that for any $x \in S(x_0, \delta)$,

$$x^* A x \in S(k, \varepsilon) \quad \text{and} \quad \mu_x \in S(0, r).$$

Thus, by the equation

$$\arg(x^* A x) + \arg(\mu_x) = \arg(x^*(A\lambda_0 - B)x),$$

for ε small enough we have $\arg(x^*(A\lambda_0 - B)x) \in [\theta_1, \theta_2]$ for every $x \in S(x_0, \delta)$ and for suitable θ_1, θ_2 with $\theta_2 - \theta_1 < \pi$. Following Theorem 1.2, it is clear that 0 is a sharp point of $\mathrm{NR}(A\lambda_0 - B)$.

(b): Since 0 is a sharp point of $\mathrm{NR}[A\lambda_0 - B]$, it is also an eigenvalue of the matrix $A\lambda_0 - B$ [1, Theorem 1.6.3]. So we have $\det(A\lambda_0 - B) = 0$, which implies that λ_0 is an eigenvalue of $A\lambda - B$. ■

Denoting by $M_{L(\lambda)}(\lambda_0)$ the *comrade set*

$$M_{L(\lambda)}(\lambda_0) = \{x : x^* L(\lambda_0) x = 0,\ x \in \mathbb{C}^n \setminus \{0\}\}, \tag{5}$$

for any $\lambda_0 \in \mathrm{NR}[L(\lambda)]$, and defining $L'(\lambda) = dL(\lambda)/d\lambda$, we state the final result of this section:

THEOREM 1.4. *Let $L(\lambda)$ be the mth degree matrix polynomial in* (1), *and λ_0 be a sharp point of* NR$[L(\lambda)]$. *If there exists a vector $x_0 \in M_{L(\lambda)}(\lambda_0)$ such that $x_0^* L'(\lambda_0)x_0 \neq 0$ and $x_0^* A_m x_0 \neq 0$, then λ_0 is an eigenvalue of $L(\lambda)$.*

Proof. Define the matrix polynomial

$$Q(\lambda) = L(\lambda + \lambda_0) = A_m \lambda^m + \cdots + L'(\lambda_0)\lambda + L(\lambda_0).$$

Then

$$\mathrm{NR}[Q(\lambda)] = \mathrm{NR}[L(\lambda)] - \lambda_0,$$

and the origin is a sharp point of NR$[Q(\lambda)]$. Since $x_0^* A_m x_0 \neq 0$ and the function $F(x) = x^* A_m x$ is continuous, there exists a neighborhood $S(x_0, \varepsilon)$ such that $x^* A_m x \neq 0$ for any $x \in S(x_0, \varepsilon)$. Moreover, by $x_0^* Q'(0)x_0 = x_0^* L'(\lambda_0)x_0 \neq 0$, it is implied that zero is a simple root of the equation

$$x_0^* Q(\lambda) x_0 = 0.$$

If $\lambda_1(x), \dots, \lambda_m(x)$ are the roots of $x^* Q(\lambda)x = 0$ and $\lambda_m(x_0) = 0$, then $\lambda_i(x)$ are continuous functions of x in $S(x_0, \varepsilon)$, and the product

$$P_{m-1}(x) = \lambda_1(x) \dots \lambda_{m-1}(x)$$

is nonzero for x_0. Having that 0 is a sharp point of NR$[Q(\lambda)]$, we can define positive numbers $\delta < \epsilon$ and $\eta > 0$ where

$$\lambda_m(x) \in \mathrm{NR}[Q(\lambda)] \cap S(0, \eta)$$

and

$$\arg[\lambda_m(x)] \in [\phi_1, \phi_2], \qquad \phi_2 - \phi_1 < \pi,$$

for any $x \in S(x_0, \delta)$. Moreover, by the continuity of $P_{m-1}(x)$, for suitable $\eta > 0$, we take

$$\arg\left[(-1)^m P_{m-1}(x)\right] \in [\theta_1, \theta_2] \qquad \forall x \in S(x_0, \delta),$$

with $\theta_2 - \theta_1 < \pi - (\phi_2 - \phi_1)$. Thus for the ratio

$$\frac{x^* L(\lambda_0) x}{x^* A_m x} = (-1)^m P_{m-1}(x) \lambda_m(x)$$

we have

$$\arg\left(\frac{x^*L(\lambda_0)x}{x^*A_m x}\right) \in [\omega_1, \omega_2] \qquad \forall x \in S(x_0, \delta),$$

where $\omega_2 - \omega_1 = (\theta_2 - \theta_1) + (\phi_2 - \phi_1) < \pi$, i.e., (4′) is true for the pencil $\lambda A_m - L(\lambda_0)$. Following the statements in the proof of Theorem 1.3, the origin is a sharp point of $\mathrm{NR}[L(\lambda_0)] \equiv \mathrm{NR}[Q(0)]$ and an eigenvalue of $L(\lambda_0)$ [1, Theorem 1.6.3]. By

$$\det L(\lambda_0) = \det Q(0) = 0,$$

we infer directly that $\lambda_0 \in \sigma[L(\lambda)]$. ∎

COROLLARY. *Let $L(\lambda)$ be a monic matrix polynomial ($A_m = I$) and λ_0 is a sharp point of* $\mathrm{NR}[L(\lambda)]$. *If there exists a vector $x_0 \in M_{L(\lambda)}(\lambda_0)$ such that $x_0^* L'(\lambda_0)x_0 \neq 0$, then $\lambda_0 \in \sigma[L(\lambda)]$.*

2. ISOLATED POINTS AND FACTORIZATION

Degenerate cases of sharp points are the isolated points, and we have the following statement:

THEOREM 2.1. *Let $L(\lambda)$ be the matrix polynomial in* (1) *with* $0 \notin \mathrm{NR}(A_m)$. *If λ_0 is an isolated point of* $\mathrm{NR}[L(\lambda)]$, *then*:

(a) $L(\lambda_0) = 0$;
(b) $L(\lambda) = (\lambda - \lambda_0)^k L_k(\lambda)$ *and* $\mathrm{NR}[L_k(\lambda)] = \mathrm{NR}[L(\lambda)] \setminus \{\lambda_0\}$.

Conversely, by the factorization in (b), when $\lambda_0 \notin$ $\mathrm{NR}[L_k(\lambda)]$, *it is implied that λ_0 is an isolated point of* $\mathrm{NR}[L(\lambda)]$.

Proof. (a): Let $x_0 \in \mathbb{C}^n$ be a unit vector such that $x_0^* L(\lambda_0)x_0 = 0$. Since $0 \notin \mathrm{NR}(A_m)$, it has been proved in [2, Theorem 2.2] that for any unit vector $y \in \mathbb{C}^n$ the zeros of the polynomial $y^*L(\lambda)y$ are connected to those of the polynomial $x_0^*L(\lambda)x_0$ by continuous curves in $\mathrm{NR}[L(\lambda)]$. So, for any unit y, the polynomial $y^*L(\lambda)y$ has a zero in the connected component $\{\lambda_0\}$, i.e., $y^*L(\lambda_0)y = 0$, and consequently $L(\lambda_0) = 0$.

(b): By (a), it is clear that

$$L(\lambda) = (\lambda - \lambda_0)L_1(\lambda),$$

where $L_1(\lambda) = A_m\lambda^{m-1} + \cdots$. If $\lambda_0 \notin \mathrm{NR}[L_1(\lambda)]$, the proof is complete; otherwise λ_0 is also an isolated point of $\mathrm{NR}[L_1(\lambda)]$. Thus, we find a positive integer k such that

$$L(\lambda) = (\lambda - \lambda_0)^k L_k(\lambda)$$

with $\lambda_0 \notin \mathrm{NR}[L_k(\lambda)]$.

Conversely, by the factorization in (b), we obtain that $L(\lambda_0) = 0$ and $\mathrm{NR}[L(\lambda)] = \{\lambda_0\} \cup \mathrm{NR}[L_k(\lambda)]$, where $\lambda_0 \notin \mathrm{NR}[L_k(\lambda)]$. So it is obvious that λ_0 is an isolated point of $\mathrm{NR}[L(\lambda)]$. ■

We have to note that the existence of the spectral divisor $L_0(\lambda) = (\lambda - \lambda_0)^k I$ with $\sigma(L_0) \cap \sigma(L) = \varnothing$ and λ_0 as an isolated point is covered by Theorem 26.13 in [4], since there always exists a disk Γ such that

$$x^*L(\lambda)x \neq 0 \qquad \forall\lambda \in \partial\Gamma,\ \|x\| = 1$$

and $\Gamma \cap \mathrm{NR}[L(\lambda)] = \{\lambda_0\}$.

The additional result here is that we know the divisor *a priori*, and thus it leads to the factorization (b) of $L(\lambda)$, generalizing Theorem 5.1 in [2].

COROLLARY. *Let $L(\lambda)$ be the matrix polynomial in* (1) *and $\lambda_1, \dots, \lambda_s$ be isolated points of the bounded numerical range* $\mathrm{NR}[L(\lambda)]$. *Then*

$$L(\lambda) = (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_s)^{k_s} L_0(\lambda), \tag{6}$$

where $\mathrm{NR}[L_0(\lambda)] = \mathrm{NR}[L(\lambda)] \setminus \{\lambda_1, \dots, \lambda_s\}$.

Conversely, by the factorization of $L(\lambda)$ in (6) *it is implied that $\lambda_1, \dots, \lambda_s$ are isolated points of* $\mathrm{NR}[L(\lambda)]$ *when $\lambda_1, \dots, \lambda_s \notin$* $\mathrm{NR}[L_0(\lambda)]$.

An additional algebraic property for some boundary points is the following:

THEOREM 2.2. *Let $L(\lambda)$ be as in* (1) *and $\lambda_0 \in \partial\mathrm{NR}[L(\lambda)] \cap \sigma(L)$. If $\lambda_0 \notin \sigma(L')$ and the matrices $L(\lambda_0)$ and $L'(\lambda_0)^{-1}$ commute, then $L(\lambda)$ has no generalized eigenvectors for λ_0.*

Proof. Let $x_0 \neq 0$ be such that $L(\lambda_0)x_0 = 0$. Since $\lambda_0 \in \partial\mathrm{NR}[L(\lambda)] \cap \sigma(L)$, by Theorem 1.1 we have that 0 is a boundary point of $\mathrm{NR}[L(\lambda_0)]$, and also it belongs to the spectrum of the matrix $L(\lambda_0)$. Thus, $L(\lambda_0)$ has no generalized eigenvector u corresponding to 0 [1, Theorem 1.6.6]. In fact, if we assume that there exists a vector $y \neq 0$ such that

$$L(\lambda_0)y + L'(\lambda_0)x_0 = 0,$$

then

$$x_0 = -L'(\lambda_0)^{-1}L(\lambda_0)y = L(\lambda_0)u,$$

where $u = -[L'(\lambda_0)]^{-1}y$, and this contradicts the previous step. ■

Note that for $L(\lambda) = \lambda I - A$, the condition of commutation is degenerate, and thus by Theorem 2.2, the first part of Theorem 1.6.6 in [1] is recovered.

3. LOCATION OF $NR[L(\lambda)]$

For the coefficients A_i of $L(\lambda)$ we consider the inner numerical radius $\tilde{r}(A_0) = \min_{\|x\|=1}|x^*A_0x|$ and the outer numerical radius $r(A_i) = \max_{\|x\|=1}|x^*A_ix|$. In the following proposition a circular annulus for the location of the $\mathrm{NR}[L(\lambda)]$ is defined:

THEOREM 3.1. *If $\lambda \in \mathrm{NR}[L(\lambda)]$ for a monic matrix polynomial $L(\lambda)$, then*

$$r_1 \leqslant |\lambda| \leqslant 1 + r_2,$$

where

$$r_1 = \frac{\tilde{r}(A_0)}{r(A_0) + \max_{k\neq 0} r(A_k)}, \qquad r_2 = \max\{r(A_k) : k = 0, 1, \dots, m-1\}.$$

Proof. For any unit vector $x \in \mathbb{C}^n$ the roots of the polynomial

$$l(\lambda) = x^*L(\lambda)x = \lambda^m + (x^*A_{m-1}x)\lambda^{m-1} + \cdots + x^*A_0x$$

lie in the disk $S(0, 1 + \varrho_x)$ [3], where $\varrho_x = \max\{|x^*A_kx| : k = 0, 1, \dots, m-1\} \leqslant \max_k r(A_k)$. Then obviously

$$\mathrm{NR}[L(\lambda)] \subset S(0, 1 + r_2).$$

Moreover, all the roots of $l(\lambda)$ satisfy the inequality [3]

$$|\lambda| \geqslant \min_{k=0,1,\ldots,m} \frac{|x^* A_0 x|}{|x^* A_0 x| + |x^* A_k x|}, \qquad A_m = I,$$

and consequently we take

$$|\lambda| \geqslant \frac{\min|x^* A_0 x|}{\max|x^* A_0 x| + \max_k |x^* A_k x|} = r_1. \qquad \blacksquare \quad (7)$$

Note that in Theorem 3.1 the numerical radius $r(*)$ can be substituted by the Euclidean norm $\|*\|_2$, but then the annulus is dilated.

In (7), if $L(\lambda)$ is monic self-adjoint matrix polynomial, it is clear that $\tilde{r}(A_0)$ is equal to the minimum measure of eigenvalues of A_0, and $\max_{k \neq 0} r(A_k)$ is identified with the maximum measure of eigenvalues of matrices $A_1, \ldots, A_{m-1}, A_m = I$.

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