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A Note on the Level Sets of a Matrix Polynomial and Its Numerical Range

Panayiotis J. Psarrakos

Dedicated with admiration to my teacher and friend Peter Lancaster on the occasion of his 70th birthday.

Abstract. Let $P(\lambda)$ be an $n \times n$ matrix polynomial with bounded numerical range $W(P)$ and let $n > 2$. If Ω is a connected subset of $W(P)$, then the set

$$\bigcup_{\omega \in \Omega} \{x \in \mathbb{C}^n : x^* P(\omega) x = 0,\ x^* x = 1\}$$

is also connected. As a consequence, if $P(\lambda)$ is selfadjoint, then every $\omega \in (\overline{W(P) \backslash \mathbb{R}}) \cap \mathbb{R}$ is a multiple root of the equation $x_\omega^* P(\lambda) x_\omega = 0$ for some unit $x_\omega \in \mathbb{C}^n$.

1. Introduction

Consider the $n \times n$ *matrix polynomial*

$$P(\lambda) = A_m \lambda^m + \cdots + A_1 \lambda + A_0, \tag{1.1}$$

where A_j $(j = 0, 1, \ldots, m)$ are $n \times n$ matrices, with $A_m \neq 0$, and λ is a complex variable. If $A_m = I$, then $P(\lambda)$ is called *monic* and if the coefficients A_j $(j = 0, 1, \ldots, m)$ are Hermitian matrices, then $P(\lambda)$ is called *selfadjoint.*

The *numerical range* of $P(\lambda)$ is defined by

$$W(P) = \{\lambda \in \mathbb{C} : x^* P(\lambda) x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\}. \tag{1.2}$$

Evidently, $W(P)$ is always closed and it contains the *spectrum* $\sigma(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$ of $P(\lambda)$. For $P(\lambda) = I\lambda - A$, $W(P)$ coincides with the *classical numerical range (field of values)* of the matrix A,

$$F(A) = \{x^* A x \in \mathbb{C} : x \in \mathcal{S}\},$$

where $\mathcal{S} = \{x \in \mathbb{C}^n : x^* x = 1\}$ is the unit sphere in $\mathbb{C}^n$. It is known that $W(P)$ is bounded if and only if $0 \notin F(A_m)$, [2]. Moreover, if $W(P)$ is bounded, then it has no more than m connected components. If $W(P)$ is unbounded, then it may have as many as $2m$ connected components. The closure and the interior of $W(P)$ are denoted by $\overline{W(P)}$ and $\mathrm{Int}W(P)$, respectively.

If $n > 2$, then for every $\omega \in W(P)$, the corresponding *level set*

$$\mathcal{L}(\omega) = \{x \in \mathcal{S} : x^* P(\omega) x = 0\}$$

is path-connected (see Main Theorem in [4]). In this paper we continue the study of this subject working on connected subsets of $W(P)$. An interesting result on selfadjoint matrix polynomials is also obtained.

2. Connectivity of level sets

Let $P(\lambda)$ be an $n \times n$ matrix polynomial, as in (1.1), and assume that the numerical range $W(P)$ in (1.2) is bounded and $n > 2$. If we consider a connected subset of $W(P)$, then it follows that the union of the corresponding level sets is also connected.

Theorem 2.1. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial with bounded numerical range and $n > 2$. If Ω is a connected subset of $W(P)$, then the set $\cup_{\omega\in\Omega}\mathcal{L}(\omega)$ is a connected subset of $\mathcal{S}$.*

Proof. Assume that $\cup_{\omega\in\Omega}\mathcal{L}(\omega)$ is not connected. Then there exist two disjoint open sets $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$ such that

$$\mathcal{A} \cap [\cup_{\omega\in\Omega}\mathcal{L}(\omega)] \neq \emptyset, \quad \mathcal{B} \cap [\cup_{\omega\in\Omega}\mathcal{L}(\omega)] \neq \emptyset$$

and

$$\cup_{\omega\in\Omega}\mathcal{L}(\omega) \subseteq \mathcal{A} \cup \mathcal{B}.$$

For every $\omega \in \Omega$, the level set $\mathcal{L}(\omega)$ is path-connected and thus,

$$\mathcal{L}(\omega) \subset \mathcal{A} \quad \text{or} \quad \mathcal{L}(\omega) \subset \mathcal{B}.$$

Moreover, consider the sets

$$\Omega_{\mathcal{A}} = \{\omega \in \Omega : \mathcal{L}(\omega) \subseteq \mathcal{A}\} = \Omega \cap \{\omega \in W(P) : \mathcal{L}(\omega) \subseteq \mathcal{A}\}$$

and

$$\Omega_{\mathcal{B}} = \{\omega \in \Omega : \mathcal{L}(\omega) \subseteq \mathcal{B}\} = \Omega \cap \{\omega \in W(P) : \mathcal{L}(\omega) \subseteq \mathcal{B}\},$$

which are open in the relative topology of Ω. Then it follows that

$$\Omega \cap \Omega_{\mathcal{A}} \neq \emptyset, \quad \Omega \cap \Omega_{\mathcal{B}} \neq \emptyset$$

and

$$\Omega = \Omega_{\mathcal{A}} \cup \Omega_{\mathcal{B}}.$$

Since $\mathcal{A} \cap \mathcal{B} = \emptyset$, we have that $\Omega_{\mathcal{A}} \cap \Omega_{\mathcal{B}} = \emptyset$ and consequently, Ω is not connected, that is a contradiction. $\square$

Assume that Ω is an open subset of $W(P)$. For any $x_0 \in \cup_{\omega\in\Omega}\mathcal{L}(\omega)$, there exists an $\omega_0 \in \Omega$ satisfying the equation $x_0^* P(\omega_0) x_0 = 0$. Since Ω is open, there is a real $\varepsilon > 0$ such that the disk $S(\omega_0, \varepsilon)$, with centre ω_0 and radius ε, belongs to Ω. By the continuous dependence of the roots of polynomials on their coefficients, [6], there exists an $r > 0$ such that for every $x \in S(x_0, r)$, the equation $x^* P(\lambda) x = 0$ has a root in $S(\omega_0, \varepsilon) \subset \Omega$. Thus, $S(x_0, r) \subset \cup_{\omega\in\Omega}\mathcal{L}(\omega)$ and $\cup_{\omega\in\Omega}\mathcal{L}(\omega)$ is an open subset of $\mathcal{S}$. It is also easy to verify that if Ω is a closed subset of $W(P)$, then $\cup_{\omega\in\Omega}\mathcal{L}(\omega)$ is a closed subset of $\mathcal{S}$.

Corollary 2.2. *Suppose that $P(\lambda)$ is an $n \times n$ matrix polynomial with bounded numerical range and $n > 2$. If Ω is an open path-connected subset of $W(P)$, then the set $\cup_{\omega\in\Omega}\mathcal{L}(\omega)$ is an open path-connected subset of $\mathcal{S}$.*

Proof. Since the sets $\cup_{\omega\in\Omega}\mathcal{L}(\omega)$ and Ω are both open (see the discussion above), the notions of connectivity and path-connectivity are equivalent (see Corollary 26.7 in [7]). □

Proposition 2.3. *Let $u(t) : [0,1] \to \mathbb{C}$ be a continuous rectifiable curve in the interior of $W(P)$ and let δ be any positive real number. Then there exist a continuous vector-curve $y_\delta(t) : [0,1] \to \mathcal{S}$ and a continuous curve $u_\delta(t) : [0,1] \to$ Int$W(P)$ such that $u(0) = u_\delta(0)$, $u(1) = u_\delta(1)$ and for every $s \in [0,1]$,*

$$y_\delta(s)^* P(u_\delta(s)) y_\delta(s) = 0 \quad and \quad \min\{|u_\delta(s) - u(t)| : t \in [0,1]\} < \delta.$$

Proof. Let $\Gamma \subset \mathrm{Int}W(P)$ be the image of u(t). For any $\varepsilon > 0$, there exists a finite number of points $\omega_1, \omega_2, \ldots, \omega_k \in \Gamma$ such that

$$\Omega_\varepsilon = \bigcup_{j=1}^{k} \mathrm{Int}S(\omega_j, \varepsilon)$$

is an open covering of Γ. The set Ω_ε is path-connected, and for ε sufficiently small, Ω_ε lies in the interior of $W(P)$. Moreover, by Corollary 2.2, the set $\cup_{\omega\in\Omega_\varepsilon}\mathcal{L}(\omega)$ is also path-connected. Hence, there is a vector-curve $y_\varepsilon(t) : [0,1] \to \cup_{\omega\in\Omega_\varepsilon}\mathcal{L}(\omega)$ such that $y_\varepsilon(0)^* P(u_\varepsilon(0)) y_\varepsilon(0) = y_\varepsilon(1)^* P(u_\varepsilon(1)) y_\varepsilon(1) = 0$. Thus, by the continuous dependence of the roots of the equation $y_\varepsilon(t)^* P(\lambda) y_\varepsilon(t) = 0$ on $t \in [0,1]$, the proof is complete. □

3. Selfadjoint matrix polynomials

Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ selfadjoint matrix polynomial. It is easy to see that the numerical range $W(P)$ is symmetric with respect to the real axis. As a consequence, the points of $(\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$ are of particular interest (see [1] and [5]). The results of the previous section yield a generalization of Proposition 4 in [1] and Theorem 3.1 in [5].

Theorem 3.1. *Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ selfadjoint matrix polynomial with bounded numerical range $W(P)$ and assume that $n > 2$. For every $\omega \in (\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$, there exists a vector $x_\omega \in \mathcal{S}$ such that ω is a multiple root of the equation $x_\omega^* P(\lambda) x_\omega = 0$.*

Proof. Suppose that $\omega \in (\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$.

If ω is an isolated point of $(\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$, then by Theorem 3.1 (and its proof) in [5], there exists a vector $x_\omega \in \mathcal{S}$ such that ω is a multiple root of the equation $x_\omega^* P(\lambda) x_\omega = 0$.

If ω is not an isolated point of $(\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$, then consider the set

$$T_0 = \{\lambda \in W(P) : x^*P(\lambda)x = x^*P'(\lambda)x = 0 \text{ for some } x \in \mathcal{S}\},$$

where $P'(\lambda)$ is the derivative of $P(\lambda)$. Since the sets $(\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$ and T_0 are both closed, it is enough to show that $T_0 \cap (\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$ is dense in $(\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$. By the symmetry of $W(P)$ with respect to the real axis, it follows that for every real $r > 0$, $S(\omega, r) \cap \mathrm{Int}W(P) \cap \mathbb{R} \neq \emptyset$. Hence, without lost of generality assume that $\omega \in (\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R} \cap \mathrm{Int}W(P)$. Then there is a continuous rectifiable curve

$$u(t) : [0,1] \to \{\lambda \in W(P) : \mathrm{Im}\lambda \geq 0\} \cap \mathrm{Int}W(P)$$

such that $u(1) = \omega$ and $\mathrm{Im}u(t) > 0$ for every $t \in [0,1)$. For any $\delta > 0$, by Proposition 2.3, there exist a continuous vector-curve $y_\delta(t) : [0,1] \to \mathcal{S}$ and a continuous curve $u_\delta(t) : [0,1] \to \mathrm{Int}W(P)$ such that $u(0) = u_\delta(0)$, $u(1) = u_\delta(1)$ and for every $s \in [0,1]$,

$$y_\delta(s)^*P(u_\delta(s))y_\delta(s) = 0$$

and

$$\min\{|u_\delta(s) - u(t)| : t \in [0,1]\} < \delta.$$

Moreover, there exists a $s_0 \in [0,1]$ such that $u_\delta(s_0) \in \mathbb{R}$ and for every $s \in [0, s_0)$, $u_\delta(s) \notin \mathbb{R}$. Thus, for every $s \in [0, s_0)$, the polynomial $y_\delta(s)^*P(\lambda)y_\delta(s)$ is written in the form $y_\delta(s)^*P(\lambda)y_\delta(s) = (\lambda - u_\delta(s))(\lambda - \overline{u_\delta(s)})g_s(\lambda)$, where $g_s(\lambda)$ is a polynomial of $(m-2)$-th degree and its coefficients are continuous on s. By the continuity of the root $u_\delta(s)$ on s, we have

$$\lim_{s \to s_0} u_\delta(s) = \lim_{s \to s_0} \overline{u_\delta(s)} = \omega_\delta,$$

where $|\omega - \omega_\delta| < \delta$. Hence, the set $T_0 \cap (\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$ is dense in $(\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R}$ and the proof is complete. □

Corollary 3.2. *Let $P(\lambda)$ be an $n \times n$ $(n > 2)$ selfadjoint matrix polynomial with bounded numerical range $W(P)$ and suppose that for every unit vector $x \in \mathbb{C}^n$, the equation $x^*P(\lambda)x = 0$ has m distinct roots. Then every connected component of $W(P)$ either has no real points or it is a closed real interval.*

(The above corollary follows also from [3, Theorem 1] and the symmetry of $W(P)$.)

Corollary 3.3. *Suppose that $P(\lambda)$ is an $n \times n$ $(n > 2)$ selfadjoint matrix polynomial with bounded numerical range $W(P)$, and for any $n \times n$ matrix B consider the matrix polynomial $Q_B(\lambda) = P(\lambda) + B$. Then*

$$(\overline{W(Q_B)\backslash\mathbb{R}}) \cap \mathbb{R} \subset W(P') \cap \mathbb{R}.$$

Note that if $P(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ is a quadratic selfadjoint matrix polynomial, such that the numerical range of the derivative $P'(\lambda)$ does not coincide with $\mathbb{C}$, then $W(P') = \{-(x^*A_1x)/(2x^*A_2x) \in \mathbb{C} : x \in \mathbb{C}^n, \text{ with } x^*A_2x \neq 0\} \subseteq \mathbb{R}$. In this case,

$$(\overline{W(P)\backslash\mathbb{R}}) \cap \mathbb{R} \subseteq \overline{\mathrm{Re}(W(P)\backslash\mathbb{R})} \subset W(P').$$

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Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan, S4S 0A2
CANADA
Email: panos@math.uregina.ca

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