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A novel algorithm for inverting a general k-tridiagonal matrix

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ABSTRACT

In this paper we present a novel algorithm, that will never fail, for inverting a general nonsingular k-tridiagonal matrix. The computational cost of the algorithm is given. Some illustrative examples are introduced.

1. Introduction

The $n \times n$ general tridiagonal matrix T_n takes the form:

$$T_n = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_1 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-2} & d_{n-1} & a_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & d_n \end{bmatrix}. \tag{1}$$

The matrix in (1) frequently appears in many applications. For example, in parallel computing, telecommunication system analysis, solving differential equations using finite differences, heat conduction and fluid flow problems. The interested reader may refer to [1–7] and the references therein. Inverting tridiagonal matrices in (1) has been considered by many authors. See for instance, [8,9,3,10–17]. An $n \times n$ general k-tridiagonal matrix $T_n^{(k)}$ takes the form:

$$T_n^{(k)} = \begin{bmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 & a_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \cdots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & \cdots & \ddots & a_{n-k} \\ b_1 & 0 & \cdots & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & b_2 & \ddots & \cdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \cdots & 0 & d_{n-1} & 0 \\ 0 & \cdots & 0 & b_{n-k} & 0 & \cdots & 0 & d_n \end{bmatrix}, \tag{2}$$

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where $1 \leq k < n$. For $k \geq n$, the matrix $T_n^{(k)}$ is a diagonal matrix and the case $k = 1$ gives the ordinary tridiagonal matrix in (1). The matrix $T_n^{(k)}$ plays an important role in describing generalized k-Fibonacci numbers [18]. Furthermore, the matrix $T_n^{(k)}$ has recently received attention by some authors [19–22]. The nonzero elements of the matrix in (2) can be stored in $3n-2k$ memory locations by using the three vectors $\mathbf{a} = [a_1, a_2, \dots, a_{n-k}]$, $\mathbf{b} = [b_1, b_2, \dots, b_{n-k}]$ and $\mathbf{d} = [d_1, d_2, \dots, d_n]$. This is always a good habit in computation in order to save memory space.

Inversion of k-tridiagonal matrices of the form (2) is only considered very recently for some special cases. The authors in [20] have considered the inversion of k-tridiagonal matrices with Toeplitz structure by imposing some restrictive conditions. The motivation of the current paper is to consider the inversion of any nonsingular k-tridiagonal matrix without imposing any restrictive conditions. To the best of our knowledge, this subject has not been considered yet.

Throughout this paper, the word 'simplify' means simplify the expression under consideration to its simplest rational form.

The organization of the paper is as follows. The main result is given in Section 2. Some illustrative examples are given in Section 3.

2. Main result

In this section we are going to consider the construction of new computational algorithms for inverting any nonsingular k-tridiagonal matrix. For this purpose it is helpful to introduce an n-component vector $\mathbf{c} = [c_1, c_2, \dots, c_n]$ whose components are given by:

$$c_i = \begin{cases} d_i, & \text{for } i = 1, 2, \dots, k \\ d_i - y_{i-k} b_{i-k}, & \text{for } i = k+1, k+2, \dots, n, \end{cases} \tag{3}$$
$$\text{where } y_i = \frac{a_i}{c_i} \text{ for } i = 1, 2, \dots, n-k.$$

With the help of the vector $\mathbf{c}$ in (3) we may formulate the following result.

Lemma 2.1 ([23]). *Let $T_n^{(k)}$ be a k-tridiagonal matrix in (2) for which $c_i \neq 0$, for $i = 1, 2, \dots, n$ and $z_i = \frac{b_i}{c_i}$, for $i = 1, 2, \dots, n-k$. Then the Doolittle LU factorization of $T_n^{(k)}$ is given by $T_n^{(k)} = L_n^{(k)} U_n^{(k)}$*

$$L_n^{(k)} = \begin{bmatrix} 1 & 0 & \dots & & & & \dots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & & & \\ 0 & \vdots & \ddots & \ddots & \ddots & & & \\ z_1 & \ddots & \vdots & \ddots & \ddots & \ddots & & \\ 0 & z_2 & \ddots & \dots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \dots & 0 & \ddots & 0 \\ 0 & \dots & 0 & z_{n-k} & 0 & \dots & 0 & 1 \end{bmatrix}, \quad U_n^{(k)} = \begin{bmatrix} c_1 & 0 & \dots & 0 & a_1 & 0 & \dots & 0 \\ 0 & c_2 & 0 & \dots & 0 & a_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \dots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & \dots & \ddots & a_{n-k} \\ & & & \ddots & \ddots & \ddots & \dots & 0 \\ & & & & \ddots & \ddots & 0 & \vdots \\ \vdots & & & & & \ddots & c_{n-1} & 0 \\ 0 & \dots & & & & \dots & 0 & c_n \end{bmatrix}, \tag{4}$$

and

$$\det(T_n^{(k)}) = \prod_{i=1}^{n} c_i. \tag{5}$$

Now we give the following result whose proof is available upon request.

Theorem 2.2. *Let $T_n^{(k)}$ be a k-tridiagonal matrix in (2) and $c_i \neq 0$, for $i = 1, 2, \dots, n$. Then $T_n^{(k)}$ is invertible. If $(T_n^{(k)})^{-1} = H = (\alpha_{ij})_{i,j=1}^n$, then for $i, j = 1, 2, \dots, n$, we have:*

$$\alpha_{ii} = \begin{cases} \dfrac{1}{c_i}, & \text{for } i = n, n-1, \dots, n-k+1; \\ \dfrac{1}{c_i} + y_i z_i \alpha_{i+k,i+k}, & \text{for } i = n-k, n-k-1, \dots, 1; \end{cases} \tag{6}$$

and

$$\alpha_{ij} = \begin{cases} -y_i\alpha_{i+k,j}, & \text{for } i < j,\ j \equiv i \bmod(k) \\ -z_j\alpha_{i,j+k}, & \text{for } i > j,\ i \equiv j \bmod(k) \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

At this point it is worth to mention the following two remarks:

Remark 1. Eq. (5) is satisfied as long as $c_i \neq 0$ for all $i = 1, 2, \ldots, n-k$. However, if $c_i = 0$ for some $i \in \{1, 2, \ldots, n-k\}$, then $\det(T_n^{(k)})$ may be computed using a symbolic algorithm such as the k-DETGTRI symbolic algorithm presented in [19].

Remark 2. According to Theorem 2.2, we see that any numeric algorithm for inverting the matrix $T_n^{(k)}$ breaks down if $c_i = 0$ for some $i \in \{1, 2, \ldots, n\}$.

The following is a symbolic algorithm, that will never fail, for inverting $T_n^{(k)}$.

Algorithm 2.1. Symbolic algorithm for inverting a general nonsingular k-tridiagonal matrix.

To find the inverse of a general k-tridiagonal matrix in (2), we may proceed as follows:
INPUT: Order of the matrix n, value of k and the values, $a_i,\ b_i,\ i = 1, 2, \ldots, n-k$,
$d_i,\ i = 1, 2, \ldots, n$.
OUTPUT: The inverse matrix, $H = (\alpha_{ij})_{i,j=1}^n$.
Step 1: For $i = 1, 2, \ldots, k$ **do**
Set: $c_i = d_i$. **If** $c_i = 0$ **then** $c_i = x$ **end if**.
end do.
For $i = k+1, k+2, \ldots, n$ **do**
Compute and simplify:
$y_{i-k} = \frac{a_{i-k}}{c_{i-k}}$
$c_i = d_i - b_{i-k}\, y_{i-k}$ **If** $c_i = 0$ **then** $c_i = x$ **end if**.
$z_i = \frac{b_{i-k}}{c_{i-k}}$
end do.
Step 2: Use the **k-DETGTRI** algorithm [19] to check the non-singularity of the matrix in (2).
Step 3: Calculate the main diagonal entries, $\alpha_{ii},\ i = 1, 2, \ldots, n$ of the inverse matrix
$H = (\alpha_{ij})_{i,j=1}^n$:
For $i = n, n-1, \ldots, n-k+1$ **do**
Compute and simplify: $\alpha_{ii} = \frac{1}{c_i}$
end do.
For $i = n-k, n-k-1, \ldots, 1$ **do**
Compute and simplify:
$\alpha_{ii} = \frac{1}{c_i} + y_i\, z_i\, \alpha_{i+k,i+k}$
end do.
Step 4: Calculate the entries above the main diagonal, α_{ij}, $i < j$:
For $j = n, n-1, \ldots, 2$ **do**
For $i = j-k, j-2k, \ldots, 1$ **do**
Compute and simplify:
$\alpha_{ij} = -y_i\, \alpha_{i+k,j}$
end do.
end do.
Step 5: Calculate the entries below the main diagonal, α_{ij}, $i > j$:
For $i = n, n-1, \ldots, 2$ **do**
For $j = i-k, i-2k, \ldots, 1$ **do**
Compute and simplify:
$\alpha_{ij} = -z_j\, \alpha_{i,j+k}$
end do.
end do.
Step 6: The inverse matrix is: $H|_{x=0}$.

Algorithm 2.1 will be referred to as **KTRINV**. The computational cost of this algorithm is $\frac{n(n-1)}{k} + 6n - 5k$ multiplications/divisions and $2n - 2k$ additions/subtractions. Based on **KTRINV** algorithm, a MAPLE procedure for inverting a general nonsingular k-tridiagonal matrix $T_n^{(k)}$ is listed as an Appendix. Note that the procedure alters the contents of the vectors **d**, **a** and **b**. Eventually, the contents of the vectors **c**, **y** and **z** are stored in **d**, **a** and **b**, respectively.

3. Illustrative examples

In this section we are going to consider two illustrative examples.

Example 3.1. Find the inverse of the 10×10 matrix:

$$T_{10}^{(4)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Solution:

We have: $n = 10$, $k = 4$, $\mathbf{a} = [1, 1, 1, 1, 1, 1]$, $\mathbf{d} = [-1, -2, -2, -2, -2, -2, -2, -2, -2, -2]$ and $\mathbf{b} = [1, 1, 1, 1, 1, 1]$.

By applying **KTRINV** algorithm, we get:

- $\mathbf{c} = [-1, -2, -2, -2, -1, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, -1, -\frac{4}{3}]$.
- $\det(T_{10}^{(4)}) = \prod_{i=1}^{10} c_i = 36$.
- The inverse matrix is:

$$(T_{10}^{(4)})^{-1} = \begin{bmatrix} -3 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{3}{4} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\ -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{3}{4} \end{bmatrix}.$$

Example 3.2. Find the inverse of the 10×10 matrix:

$$T_{10}^{(3)} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \end{bmatrix}.$$

Solution:

We have: $n = 10$, $k = 3$, $\mathbf{a} = [1, -1, 2, 4, 1, 3, 1]$, $\mathbf{d} = [2, 1, -1, 3, 1, -2, 5, 3, -1, 3]$ and $\mathbf{b} = [2, -1, 3, 2, 1, 5, 1]$.

Applying the algorithm **KTRINV**, yields:

- $\mathbf{c} = [2, 1, -1, 2, x, 4, 1, \frac{(3x-1)}{x}, -\frac{19}{4}, 2]$.
- $\det(T_{10}^{(3)}) = (\prod_{i=1}^{10} c_i)_{x=0} = (456x - 152)_{x=0} = -152$.
- The inverse matrix is:

$$
\begin{aligned}
(T_{10}^{(3)})^{-1} &= H|_{x=0} \\
&= \begin{bmatrix}
\frac{9}{4} & 0 & 0 & -\frac{7}{4} & 0 & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\
0 & \frac{3x+2}{3x-1} & 0 & 0 & \frac{3}{(3x-1)} & 0 & 0 & -\frac{1}{(3x-1)} & 0 & 0 \\
0 & 0 & -\frac{13}{19} & 0 & 0 & \frac{2}{19} & 0 & 0 & \frac{6}{19} & 0 \\
-\frac{7}{2} & 0 & 0 & \frac{7}{2} & 0 & 0 & -3 & 0 & 0 & 1 \\
0 & \frac{3}{(3x-1)} & 0 & 0 & \frac{3}{(3x-1)} & 0 & 0 & -\frac{1}{(3x-1)} & 0 & 0 \\
0 & 0 & \frac{3}{19} & 0 & 0 & \frac{1}{19} & 0 & 0 & \frac{3}{19} & 0 \\
\frac{3}{2} & 0 & 0 & -\frac{3}{2} & 0 & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{(3x-1)} & 0 & 0 & -\frac{1}{(3x-1)} & 0 & 0 & \frac{x}{3x-1} & 0 & 0 \\
0 & 0 & \frac{15}{19} & 0 & 0 & \frac{5}{19} & 0 & 0 & -\frac{4}{19} & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{bmatrix}_{x=0} \\
&= \begin{bmatrix}
\frac{9}{4} & 0 & 0 & -\frac{7}{4} & 0 & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -2 & 0 & 0 & -3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\frac{13}{19} & 0 & 0 & \frac{2}{19} & 0 & 0 & \frac{6}{19} & 0 \\
-\frac{7}{2} & 0 & 0 & \frac{7}{2} & 0 & 0 & -3 & 0 & 0 & 1 \\
0 & -3 & 0 & 0 & -3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{3}{19} & 0 & 0 & \frac{1}{19} & 0 & 0 & \frac{3}{19} & 0 \\
\frac{3}{2} & 0 & 0 & -\frac{3}{2} & 0 & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{15}{19} & 0 & 0 & \frac{5}{19} & 0 & 0 & -\frac{4}{19} & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{bmatrix}.
\end{aligned}
$$

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Appendix. A MAPLE procedure for inverting a general nonsingular k-tridiagonal matrix

```
> restart:
  ktrinverse:= proc(n::posint,k::posint, d::vector, a::vector, b::vector)
      local i,j:
      global H,T:
      H:=array(1..n,1..n,sparse):
         # Step 1:
         for i to k do
            if d[i] = 0 then d[i]:=x fi:
         od:
         for i from k+1 to n do
            a[i-k]:=simplify(a[i-k]/d[i-k]):
            d[i] := simplify(d[i]-a[i-k]*b[i-k]):
            if d[i] = 0 then d[i] := x; fi:
            b[i-k] := simplify((b[i-k])/d[i-k]);
         od;
      # Step 2: To compute the determinant of the k-tridiagonal matrix#
      T:= subs(x =0,simplify(product(d[r],r= 1..n))):
      if T = 0 then
         error("Singular Matrix")
      else
         # Step 3: To compute the main diagonal entries of the inverse matrix.
         for i from n by -1 to n-k+1 do
            H[i,i]:=1/d[i]:
         od:
         for i from n-k by -1 to 1 do
            H[i,i]:=simplify((1/d[i]) + a[i]*b[i]*H[i+k,i+k]):
         od:
         # Step 4: To compute the entries above the main diagonal of the inverse matrix.
         for j from n by -1 to 2 do
            for i from j-k by -k to 1 do
               H[i,j]:=simplify(-a[i]*H[i+k,j]):
            od:
         od:
         # Step 5: To compute the entries below the main diagonal of the inverse matrix.
         for i from n by -1 to 2 do
            for j from i-k by -k to 1 do
               H[i,j]:=simplify(-b[j]*H[i,j+k]):
            od:
         od:
         H:=op(H):
      fi:
  end proc:
```

References

[1] R.L. Burden, J.D. Faires, Numerical Analysis, seventh ed., Books & Cole Publishing, Pacific Grove, CA, 2001.
[2] C.M. da Fonseca, J. Petronilho, Explicit inverses of some tridiagonal matrices, Linear Algebra Appl. 325 (2001) 7–21.
[3] M.E.A. El-Mikkawy, A. Karawia, Inversion of general tridiagonal matrices, Appl. Math. Lett. 19 (2006) 712–720.
[4] I. Mazilu, D.A. Mazilu, H.T. Williams, Applications of tridiagonal matrices in non-equilibrium statistical physics, Electron. J. Linear Algebra 24 (2012) 7–17.
[5] A. Kavcic, J.M.F. Moura, Matrices with banded inverses: inversion algorithms and factorization of Gauss–Markov processes, IEEE Trans. Inform. Theory 46 (4) (2000) 1495–1509.
[6] H.-B. Li, T.-Z. Huang, X.-P. Liu, H. Li, On the inverses of general tridiagonal matrices, Linear Algebra Appl. 433 (2010) 965–983.
[7] E. Olcayto, Recursive formulae for ladder network optimization, Electron. Lett. 15 (9) (1979) 249–250.
[8] J. Abderram Marrero, M. Rachidi, V. Tomeo, Non-symbolic algorithms for the inversion of tridiagonal matrices, J. Appl. Math. Comput. 252 (2013) 3–11.
[9] M.E.A. El-Mikkawy, On the inverse of a general tridiagonal matrix, Appl. Math. Comput. 150 (2004) 669–679.
[10] M.E.A. El-Mikkawy, El-Desouky Rahmo, A new recursive algorithm for inverting general tridiagonal and anti-tridiagonal matrices, Appl. Math. Comput. 204 (2008) 368–372.
[11] Y. Huang, W.F. McColl, Analytic inversion of general tridiagonal matrices, J. Phys. A 30 (1997) 7919–7933.
[12] G.Y. Hu, R.F. O'Connell, Analytical inversion of symmetric tridiagonal matrices, J. Phys. A 29 (1996) 1511–1513.
[13] R.K. Mallik, The inverse of a tridiagonal matrix, Linear Algebra Appl. 325 (2001) 109–139.
[14] R.-S. Ran, T.-Z. Huang, X.-P. Liu, T.-X. Gu, An inversion algorithm for general tridiagonal matrix, Appl. Math. Mech. (English Ed.) 30 (2009) 247–253.
[15] T. Sugimoto, On an inverse formula of a tridiagonal matrix, Oper. Matrices 6 (3) (2012) 465–480. http://dx.doi.org/10.7153/oam-06-30.
[16] R. Usmani, Inversion of a tridiagonal Jacobi matrix, Linear Algebra Appl. 212/213 (1994) 413–414.

[17] T. Yamamoto, Y. Ikebe, Inversion of band matrices, Linear Algebra Appl. 24 (1979) 105–111.
[18] M. El-Mikkawy, T. Sogabe, A new family of *k*-Fibonacci numbers, Appl. Math. Comput. 215 (12) (2010) 4456–4461.
[19] M.E.A. El-Mikkawy, A generalized symbolic Thomas algorithm, Appl. Math. 3 (4) (2012) 342–345.
[20] J. Jia, T. Sogabe, M. El-Mikkawy, Inversion of *k*-tridiagonal matrices with Toeplitz structure, Comput. Math. Appl. 65 (2013) 116–125.
[21] T. Sogabe, M. El-Mikkawy, Fast block diagonalization of *k*-tridiagonal matrices, Appl. Math. Comput. 218 (6) (2011) 2740–2743.
[22] S.L. Yang, On the *k*-generalized Fibonacci numbers and high-order linear recurence relations, Appl. Math. Comput. 196 (2008) 850–857.
[23] A. Yaliner, The LU factorization and determinants of the *k*-tridiagonal matrices, Asian-Eur. J. Math. 4 (2011) 187–197.