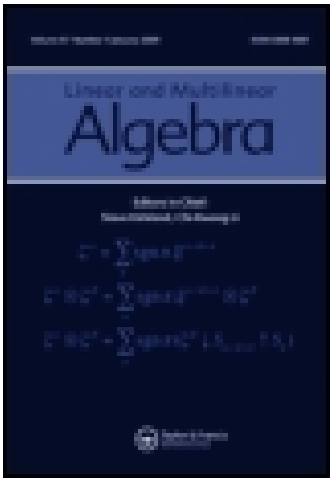

Linear and Multilinear Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/glma20

The q-numerical range of matrix polynomials

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Published online: 30 May 2007.

Linear and Multilinear Algebra, Vol. 47, pp. 1–9
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The q-numerical Range of Matrix Polynomials

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Communicated by C.-K. Li

(Received 4 July 1998)

Let $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a matrix polynomial, where $A_j(j = 0, 1, \ldots, m)$ are $n \times n$ complex matrices and λ is a complex variable. For a q ●[0, 1] the q-numerical range of $P(\lambda)$ is defined as

$$W_q[P(\lambda)] = \{\lambda \in \mathbb{C} : x^*P(\lambda)y = 0,\ x^*x = y^*y = 1 \text{ and } x^*y = q\}.$$

In this paper we study $W_q[P(\lambda)]$ and our emphasis is on the geometrical properties of $W_q[P(\lambda)]$. We consider the location of $W_q[P(\lambda)]$ in the complex plane and a theorem concerning the boundary of $W_q[P(\lambda)]$ is also obtained.

Keywords: Matrix polynomial; q-numerical range; connected component

0. INTRODUCTION

Let M_n be the algebra of all $n \times n$ complex matrices. Suppose that

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 \tag{1}$$

is a matrix polynomial, where $A_j \in M_n$ $(j = 0, 1, \ldots, m)$ and λ is a complex variable. For a real number $q \in [0, 1]$, the *q-numerical range* of $P(\lambda)$

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is defined by

$$W_q[P(\lambda)] = \{\lambda \in \mathbb{C} : x^* P(\lambda) y = 0,\ x, y \in \mathbf{S} \text{ with } x^* y = q\},$$

where $\mathbf{S} = \{x \in \mathbb{C}^n : x^* x = 1\}$ is the unit sphere in $\mathbb{C}^n$.

When $q \in (0, 1]$ and $P(\lambda) = I\lambda - A$, the concept reduces to the q-numerical range of the matrix $q^{-1}A$,

$$W_q[q^{-1}A] = q^{-1} W_q[A] = \{q^{-1} x^* A y : x, y \in \mathbf{S} \text{ with } x^* y = q\}.$$

The q-numerical range $W_q[P(\lambda)]$ is a generalization of the (*classical*) *numerical range* of $P(\lambda)$ namely

$$W[P(\lambda)] \doteq W_1[P(\lambda)] = \{\lambda \in \mathbb{C} : x^* P(\lambda) x = 0,\ x \in \mathbf{S}\}$$

and it always contains the *spectrum* of $P(\lambda)$,

$$\sigma[P(\lambda)] = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}.$$

In the last few years the q-numerical range of matrices and the numerical range of matrix polynomials have attracted the attention of a number of authors and many interesting results have been obtained [2–7]. This paper is a first study of q-numerical range of matrix polynomials. Building on results of [5] and [6] we describe some basic properties of $W_q[P(\lambda)]$. In Section 1 we study the relation of $W_q[P(\lambda)]$ with the q-numerical range of a corresponding linearization. We also consider the location of $W_q[P(\lambda)]$ in a circular annulus. In Section 2 we prove that the q-numerical range of $P(\lambda)$ in (1) has no more than m connected components and we study the boundary of $W_q[P(\lambda)]$.

1. LOCATION

For the q-numerical range of a matrix polynomial, the following properties can be easily verified.

PROPOSITION 1.1 *Let* $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$, *where* $A_m \neq 0$ *and* $q \in [0, 1]$.

(i) $W_q[P(\lambda)]$ *is closed in* $\mathbb{C}$.
(ii) *For any* $k \in \mathbb{C}$, $W_q[P(\lambda + k)] = W_q[P(\lambda)] - k$.
(iii) *If* $Q(\lambda) = A_0\lambda^m + \cdots + A_{m-1}\lambda + A_m$, *then*

$$W_q[Q(\lambda)]\backslash\{0\} = \{\mu^{-1} : \mu \in W_q[P(\lambda)],\ \mu \neq 0\}.$$

(iv) *For any* $n \times s$ *matrix* T *with* $T^*T = I_s (n \geq s)$, $W_q[T^*P(\lambda)T] \subseteq W_q[P(\lambda)]$. *Equality holds if* $n = s$.
(v) *If there exist two unit vectors* x, y *such that* $x^*y = q$ *and* $x^*A_jy = 0$ *for all* $j = 0, 1, \ldots, m$ *then* $W_q[P(\lambda)] = \mathbb{C}$.

It is well known that $0 \in W_0[A]$, for any $A \in M_n$ [3]. So, for any matrix polynomial $P(\lambda)$ (1), we have that $0 \in W_0[P(\mu)]$ for every $\mu \in \mathbb{C}$. Thus, $W_0[P(\lambda)] = \mathbb{C}$.

Taking ideas from [5], we can identify when $W_q[P(\lambda)]$ is bounded.

THEOREM 1.2 *Let* $P(\lambda)$ *be a matrix polynomial and* $q \in (0, 1]$. *Then* $W_q[P(\lambda)]$ *is bounded if and only if* $0 \notin W_q[A_m]$.

Proof If $0 \notin W_q[A_m]$ and $\mu = \min\{|z| : z \in W_q[A_m]\}$, then there exists a $M > 0$ such that

$$\left|(x^*A_my)\lambda^m\right| \geq \left|\mu\lambda^m\right| > \sum_{k=0}^{m-1} \left|(x^*A_ky)\lambda^k\right|,$$

for every $x, y \in \mathbf{S}$ with $x^*y = q$ and $\lambda \in \mathbb{C}$ with $|\lambda| > M$. Obviously, it follows that $W_q[P(\lambda)] \subseteq \{z \in \mathbb{C} : |z| \leq M\}$. For the converse assume that $W_q[P(\lambda)]$ is bounded but $0 \in W_q[A_m]$. Let $x, y \in \mathbf{S}$ with $x^*y = q$ and $x^*A_my = 0$. Since $W_q[(\lambda)] \neq \mathbb{C}$, there exists a coefficient A_s $(s \neq m)$ such that $x^*A_sy \neq 0$. We can find a sequence of unitary matrices $\{U_k\}_{k\in\mathbb{N}}$ converging to the identity matrix I_n, such that $x^*U_k^*A_sU_ky \neq 0$, $k \in \mathbb{N}$. Then $x_k = U_kx \to x$, $y_k = U_ky \to y$ and $x_k^*y_k = q$ for $k \in \mathbb{N}$. For a fixed $\delta > 0$ it is clear that $|x_k^*A_sy_k| > \delta$, for all sufficiently large k. Since $W_q[P(\lambda)]$ is bounded, the elementary symmetric function $\pm(x_k^*A_sy_k)/(x_k^*A_my_k)$ of polynomial $x_k^*P(\lambda)y_k$, is also bounded for all $k \in \mathbb{N}$, which is not true. ■

The $nm \times nm$ matrix polynomial

$$L(\lambda) = \begin{bmatrix} I_n & 0 & \dots & 0 \\ 0 & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{bmatrix} \lambda - \begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_0 & -A_1 & -A_2 & \dots & -A_{m-1} \end{bmatrix} \tag{2}$$

is called *companion linearization* of $P(\lambda)$ and it is well known that the eigenproblems of $L(\lambda)$ and $P(\lambda)$ are closed connected [1]. Moreover we have the next inclusion property.

THEOREM 1.3 *Let $P(\lambda)$ be a matrix polynomial as in* (1), *$L(\lambda)$ be the corresponding linearization in* (2) *and $q \in (0, 1]$. Then*

$$W_q[P(\lambda)] \cup \{0\} \subseteq W_q[L(\lambda)].$$

Proof For any $\lambda \in \mathbb{C}$ and $x, y \in \mathbf{S}$ with $x^* y = q$, we consider the vectors

$$x_0 = \begin{bmatrix} I_n \\ \lambda I_n \\ \vdots \\ \lambda^{m-1} I_n \end{bmatrix} x \quad \text{and} \quad y_0 = \begin{bmatrix} I_n \\ \lambda I_n \\ \vdots \\ \lambda^{m-1} I_n \end{bmatrix} y. \tag{3}$$

Then we can see that

$$x_0^* L(\lambda) y_0 = \lambda^{m-1} x^* P(\lambda) y,$$

where

$$\frac{x_0^* y_0}{\|x_0\| \|y_0\|} = \frac{q(1 + |\lambda|^2 + \dots + |\lambda^{m-1}|^2)}{\left(\sqrt{1 + |\lambda|^2 + \dots + |\lambda^{m-1}|^2}\right)^2} = q.$$

Thus

$$\begin{aligned} W_q[P(\lambda)] \cup \{0\} &= \left\{ \lambda \in \mathbb{C} : \bar{\lambda}^{m-1} x^* P(\lambda) y = 0,\ \frac{x^* y}{\|x\|\|y\|} = q \right\} \\ &= \{ \lambda \in \mathbb{C} : x_0^* L(\lambda) y_0 = 0,\ \text{with } x_0, y_0 \text{ in (3)} \} \\ &\subseteq \left\{ \lambda \in \mathbb{C} : x_0^* L(\lambda) y_0 = 0,\ \text{with } \frac{x_0^* y_0}{\|x_0\|\|y_0\|} = q \right\} \\ &= W_q[L(\lambda)]. \end{aligned}$$

■

By Theorem 1.3 we have the following corollary.

COROLLARY 1 *If $W_q[L(\lambda)]$ is bounded, then $W_q[P(\lambda)]$ is also bounded.*

The converse statement is not true, as is illustrated in the following example.

Example 1 Let $P(\lambda) = A\lambda^2 + B\lambda + C$ be a 2×2 quadratic matrix polynomial, were $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Then we have

$$W_q[A] = \{-q\}$$

and by Theorem 1.2, $W_q[P(\lambda)]$ is bounded ($0 < q \leq 1$). The leading coefficient matrix M of the companion linearization $L(\lambda)$ of $P(\lambda)$ is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The q-numerical range of M is

$$W_q[M] = \{x + iy \in \mathbb{C} : (x, y) \in \mathbb{R}^2,\ x^2 + y^2/(1 - q^2) \leq 1\}$$

and hence $0 \in W_q[M]$ and $W_q[L(\lambda)]$ is unbounded. ■

By Theorem 1.2, the q-numerical range of a *monic matrix* polynomial (*i.e.*, $A_m = I_n$) is bounded, for every $q \in (0, 1]$. Moreover if we consider the *inner q-numerical radius* of matrix A,

$$\tilde{r}_q(A) = \min\{|\mu| : \mu \in W_q[A]\}$$

and the *outer q-numerical radius*

$$r_q(A) = \max\{|\mu| : \mu \in W_q[A]\},$$

then $W_q[P(\lambda)]$ is located in a circular annulus with $\tilde{r}_q(A)$ and $r_q(A)$ as inner and outer radii.

THEOREM 1.4 *Let* $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ *and* $q \in (0, 1]$. *Then for every* $\mu \in W_q[P(\lambda)]$, *we have*

$$\rho_1 \leq |\mu| \leq 1 + \rho_2,$$

where

$$\rho_1 = \frac{\tilde{r}_q(A_0)}{\tilde{r}_q(A_0) + \max\limits_{k \neq 0} r_q(A_k)} \quad \textit{and} \quad \rho_2 = \max_{k \leq m-1} \frac{r_q(A_k)}{q}.$$

Proof For any $x, y \in \mathbf{S}$ with $x^*y = q$, every root μ of equation

$$x^*P(\lambda)y = q\lambda^m + (x^*A_{m-1}y)\lambda^{m-1} + \cdots + (x^*A_1y)\lambda + x^*A_0y = 0$$

satisfies the inequality

$$\min_k \frac{\frac{|x^*A_0y|}{q}}{\frac{|x^*A_0y|}{q} + \frac{|x^*A_ky|}{q}} \leq |\mu| \leq 1 + \max\left\{\frac{|x^*A_ky|}{q} : k = 0, 1, \ldots, m-1\right\}.$$

Moreover

$$\begin{aligned} \min_k \frac{|x^*A_0y|}{|x^*A_0y| + |x^*A_ky|} &= \left\{1 + \max_k \frac{|x^*A_ky|}{|x^*A_0y|}\right\}^{-1} \\ &\geq \left\{1 + \frac{\max_k r_q(A_k)}{\tilde{r}_q(A_0)}\right\}^{-1}. \end{aligned}$$

Hence

$$\frac{\tilde{r}_q(A_0)}{\tilde{r}_q(A_0) + \max\limits_{k \neq 0} r_q(A_k)} \leq |\mu| \leq 1 + \max_{k \leq m-1} \frac{r_q(A_k)}{q}$$

(see also [6, Th. 3.1]). ■

Note that the (outer) q-numerical radius can be replaced by the Euclidean norm since $r_q(A) \leq \|A\|_2$, for every $A \in M_n$ and $q \in [0,1]$ [3]. Then the circular annulus is dilated but it is easier to compute.

2. GEOMETRY

In [5] is appeared a very interesting result concerning the number of connected components of the (classical) numerical range $W_1[P(\lambda)]$. We will try to generalize that result.

The next lemma follows from the result in [3] that all eigenvalues of A are interior points of $W_q[A]$ unless A is scalar. We give a short proof for the sake of completeness.

LEMMA *For any $A \in M_n$ and $q \in (0,1)$, $W_q[A]$ has interior points or it is a singleton.*

Proof By Propositiiton 1.1 (iv), we consider the 2×2 case without lost of generality.

Let $A \in M_n$ and $q \in (0,1)$ and suppose that $W_q[A]$ is not a singleton and it has no interior points. By the convexity of $W_q[A]$ [7] and Shcur Theorem we can assume that $W_q[A] = [\mu, \nu] \subset \mathbb{R}$ and $A = \left[\begin{smallmatrix} \alpha & \beta \\ 0 & c \end{smallmatrix}\right]$, where (α, β, c) is not of the form $(k, 0, k)$, $k \in \mathbb{C}$. For all unit vectors $x = [x_1\, x_2]^T$, $y = [y_1\, y_2]^T$ with $x^*y = q$ we have

$$x^*Ay = (\bar{x}_1 y_1)\alpha + (\bar{x}_2 y_2)c + (\bar{x}_1 y_2)\beta = (\bar{x}_1 y_1)(\alpha - c) + qc + (\bar{x}_1 y_2)\beta.$$

It is obvious that we can choose vectors x, y such that $x^*Ay \in \mathbb{C} \backslash \mathbb{R}$. Thus, $W_q[A]$ can not be a closed interval in $\mathbb{R}$. ∎

THEOREM 2.1 *Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ and $q \in (0,1)$. Then $W_q[P(\lambda)]$ has no more than m connected components.*

Proof Suppose $x_1, x_2, y_1, y_2 \in \mathbf{S}$ such that $x_1^* y_1 = x_2^* y_2 = q$, $x_i^* A_m y_i \neq 0 (i = 1,2)$ and let $z_1 = x_1^* A_m y_1$ and $z_2 = x_2^* A_m y_2$.

(i) If $z_1 \neq z_2$ and the line segment $[z_1, z_2]$ does not contain the origin, then there exist two continuous curves $u_i = [0,1] \to \mathbf{S}$ $(i = 1,2)$ such that $u_1(0) = x_1$, $u_1(1) = x_2$, $u_2(0) = y_1$, $u_2(1) = y_2$ and $u_1(t)^* u_2(t) = q$, $u_1(t)^* A_m u_2(t) \in [z_1, z_2]$ for every $t \in [0,1]$ (see the proof of convexity of $W_q[A]$ in [7]).

(ii) If $z_1 \neq z_2$ and $0 \in [z_1, z_2]$, then we can choose a $z_3 \in W_q[A_m]$ such that $0 \notin [z_1, z_3] \cup [z_3, z_2]$ since $W_q[A_m]$ is convex with interior points. Consequently there exist two continuous curves $u_i(t)$ $(i = 1, 2)$ as in (i), with $u_1(t)^* A_m u_2(t) \in [z_1, z_3] \cup [z_3, z_2]$ for every $t \in [0, 1]$.

(iii) Finally, suppose $z_1 = z_2$. If $W_q[A_m]$ is a singleton, then A_m is a scalar matrix and we can easily take two continuous curves as above joining x_1 with x_2 and y_1 with y_2 respectively. If $W_q[A_m]$ is not a singleton, then there exists a nonzero $z_0 \in W_q[A_m]$ and we work as in case (ii).

In all cases we obtain two continuous curves $u_i : [0, 1] \to \mathbf{S}$ $(i = 1, 2)$ with $u_1(0) = x_1$, $u_1(1) = x_2$, $u_2(0) = y_1$, $u_2(1) = y_2$ and $u_1(t)^* u_2(t) = q$, $u_1(t)^* A_m u_2(t) \neq 0$ for every $t \in [0, 1]$. Since the solutions $\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t)$ of equation

$$u_1(t)^* P(\lambda) u_2(t) = 0$$

are continuous functions of t, the zeros of polynomial $x_1^* P(\lambda) y_1$, are connected to those of $x_2^* P(\lambda) y_2$ by continuous curves in $W_q[P(\lambda)]$. Thus any root function $\lambda_i(t)$ lies in exactly one connected component of $W_q[P(\lambda)]$. So, $W_q[P(\lambda)]$ has no more than m connected components. ■

It is well known that if $\lambda_0 \in \mathbb{C}$ is a boundary point of $W_1[P(\lambda)]$ then the origin is also a boundary point of $W_1[P(\lambda_0)]$ [6]. This result can be generalized easily, for q-numerical ranges.

THEOREM 2.2 *Let* $P(\lambda) = A_m \lambda^m + \cdots + A_1 \lambda + A_0$, $q \in (0, 1]$ *and* $\lambda_0 \in \mathbb{C}$ *be a boundary point of* $W_q[P(\lambda)]$. *Then* 0 *is a boundary point of* $W_q[P(\lambda_0)]$.

Proof The q-numerical range $W_q[P(\lambda)]$ is closed and if $\lambda_0 \in \partial W_q[P(\lambda)]$ then there exist $x, y \in \mathbf{S}$ where $x^* y = q$, such that $x^* P(\lambda_0) y = 0$. Therefore, the origin belongs to $W_q[P(\lambda_0)]$ and we will prove that it is not an interior point of $W_q[P(\lambda_0)]$.

Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be a sequence of points in $\mathbb{C} \backslash W_q[P(\lambda)]$ converging to λ_0. If there exists a disk $S(0, \varepsilon) \subset W_q[P(\lambda_0)]$, then we can find $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbf{S}$, where $x_i^* y_i = q (i = 1, 2, 3)$ such that

$$0 \in \text{Conv. hull}\{x_1^* P(\lambda_0) y_1,\ x_2^* P(\lambda_0) y_2,\ x_3^* P(\lambda_0) y_3\} \subset S(0, \varepsilon).$$

For small enough ε the vertices of the triangle are close to the origin. By the convexity of $W_q[P(\lambda)]$ the equalities

$$\lim_{k\to\infty} x_i^* P(\lambda_k) y_i = x_i^* P(\lambda) y_i; \quad i = 1, 2, 3$$

imply that $0 \in W_q[P(\lambda_k)]$, for large enough k. Consequently $\lambda_k \in W_q[P(\lambda)]$, which contradicts the definition of the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$. ■

Acknowledgment

The authors wish to express their gratitude to Professor Chi-Kwong Li for his usefull remarks and to a referee for providing Example 1.

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