ELSEVIER Linear Algebra and its Applications 338 (2001) 105–123

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

On the shape of numerical range of matrix polynomials

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Received 22 February 2001; accepted 3 May 2001

Submitted by C.-K. Li

Abstract

The numerical range of an $n \times n$ matrix polynomial $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ is defined by

$$W(P) = \left\{\lambda \in \mathbb{C}\colon\ x^*P(\lambda)x = 0,\ x \in \mathbb{C}^n,\ x \neq 0\right\}.$$

In this paper, we investigate the shape of $W(P)$ by using the notion of local dimension. The numerical range of first order matrix polynomials is always simply connected. The special cases of diagonal matrix polynomials and 2×2 matrix polynomials are also considered. © 2001 Elsevier Science Inc. All rights reserved.

AMS classification: 15A60; 47A12

Keywords: Matrix polynomial; Numerical range; Boundary; Component

1. Introduction

Consider a *matrix polynomial*

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{1}$$

where $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m)$ and λ is a complex variable. The spectral analysis of matrix polynomials is very important when studying linear systems of ordinary

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0024-3795/01/$ - see front matter
PII: S 0 0 2 4 - 3 7 9 5 (0 1) 0 0 3 7 4 - 3

differential equations of order m with constant coefficients [5,7]. A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $P(\lambda)$ in (1) if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is the *spectrum* of $P(\lambda)$, namely,

$$\sigma(P) = \left\{\lambda \in \mathbb{C}:\ \det P(\lambda) = 0\right\}.$$

The *numerical range* of $P(\lambda)$ in (1) is defined by

$$W(P) = \left\{\lambda \in \mathbb{C}:\ x^*P(\lambda)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\right\}. \tag{2}$$

Clearly, $W(P)$ is always closed and contains $\sigma(P)$. If $P(\lambda) = I\lambda - A$, then $W(P)$ coincides with the classical numerical range of the matrix A,

$$F(A) = \left\{x^*Ax:\ x \in \mathbb{C}^n,\ x^*x = 1\right\}.$$

The last decade, the numerical range of matrix polynomials has been studied systematically, and a number of interesting results have been obtained (see e.g., [2,6,8,10,11,13]). It is known that $W(P)$ in (2) is not always connected, and it is bounded if and only if $0 \notin F(A_m)$. In this case, $W(P)$ has no more than m connected components [8]. Moreover, if μ is a boundary point of $W(P)$, then the origin is also a boundary point of $F(P(\mu))$, and in general, the corners of $W(P)$ are eigenvalues of $P(\lambda)$ [11].

In this paper, we continue the investigation of the numerical range $W(P)$ in (2), and present new results on the boundary and the geometry of $W(P)$. In Section 2, we study the shape of $W(P)$ obtaining necessary and sufficient conditions for the *local dimension* of a point $\lambda_0 \in W(P)$ to be equal to 1 or 2. In Section 3, it is proved that the numerical range of a *linear pencil* $P(\lambda) = A\lambda - B$ is always *simply connected*. The numerical range of a diagonal matrix polynomial is considered in Section 4, and it is proved that its boundary is contained in a finite union of the numerical ranges of 2×2 diagonal matrix polynomials. Finally, in Section 5, we present a method to compute the point equation of the boundary of the numerical range of a 2×2 matrix polynomial. In particular, if the numerical range of a 2×2 matrix polynomial is not the whole complex plane, then its boundary lies on an algebraic curve of total degree at most $4m$, where m is the degree of the polynomial.

It is worth noting that some of the results of this paper are also valid for more general matrix functions than matrix polynomials. It is clear from their proofs, that Theorems 1 and 2 hold for analytic matrix functions. Furthermore, Propositions 12 and 14 are also true for general continuous matrix functions (since Theorem 1.1 in [11] holds for continuous matrix functions).

2. Local dimension

Let Ω be a closed subset of $\mathbb{C}$, and let $\lambda_0 \in \Omega$. The *local dimension* of the point λ_0 in Ω is defined as the limit

$$\lim_{h\to 0^+} \dim\left\{\Omega \cap S(\lambda_0, h)\right\} \quad (h \in \mathbb{R},\ \ h > 0).$$

Notice that any isolated point of Ω has local dimension equal to zero, and any non-isolated point λ_0 of Ω has local dimension 2 if and only if there exists a sequence $\{\mu_k\}_{k\in\mathbb{N}} \in \operatorname{Int}\Omega$ converging to λ_0 (i.e., λ_0 belongs to the closure of $\operatorname{Int}\Omega$).

A (boundary) point $\lambda_0 \in \Omega$ is said to be a *corner* of Ω if there exist three angles $\theta_0, \theta_1, \theta_2 \in [0, 2\pi]$ and a real $\rho > 0$ such that $0 \leqslant \theta_2 - \theta_1 \leqslant \theta_0 < \pi$ and

$$\theta_1 \leqslant \operatorname{Arg}(z - \lambda_0) \leqslant \theta_2$$

for every $z \in \Omega \cap S(\lambda_0, \rho)$ (cf. [6,11]).

For a matrix polynomial $P(\lambda)$ as in (1), the local dimension of any λ_0 in $W(P)$ is closely connected with the local dimension of the origin in $F(P(\lambda_0))$.

Theorem 1. *Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial, and let $\lambda_0 \in W(P)$ such that the origin is not a corner of $F(P(\lambda_0))$ and $0 \notin F(P'(\lambda_0))$. If the local dimension of λ_0 in $W(P)$ is equal to* 1, *then the local dimension of the origin in $F(P(\lambda_0))$ is also equal to* 1.

Proof. Assume that the local dimension of λ_0 in $W(P)$ is 1 and the local dimension of the origin in $F(P(\lambda_0))$ is 2. It is clear that λ_0 belongs to the boundary $\partial W(P)$ and there is a real $r_0 > 0$ such that

$$W(P) \cap S(\lambda_0, r_0) \subseteq \partial W(P).$$

By Theorem 1.1 in [11], the origin is a boundary point of $F(P(\lambda_0))$. Since $F(P(\lambda_0))$ is convex (see [4]) and 0 is a differentiable point of $F(P(\lambda_0))$, there exists a straight line passing through the origin and defining two closed half planes $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $F(P(\lambda_0)) \subset \mathscr{H}_1$.

For every $r \in [0, r_0]$ and $\vartheta \in [0, 2\pi]$, either $\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta} \notin W(P)$, or $\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta} \in \partial W(P)$. Equivalently, for every $r \in [0, r_0]$ and $\vartheta \in [0, 2\pi]$, either $0 \notin F(P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta}))$, or $0 \in \partial F(P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta}))$ (see [6, Theorem 3.1]). Moreover, the origin does not belong to the convex set $F(P'(\lambda_0))$, and $P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta})$ is written

$$P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta}) = P(\lambda_0) + r\mathrm{e}^{\mathrm{i}\vartheta} P'(\lambda_0) + r\mathrm{e}^{\mathrm{i}\vartheta} R(\lambda_0, r, \vartheta),$$

where $\|R(\lambda_0, r, \vartheta)\| = \mathrm{o}(1)$ as $r \to 0$. Hence, for "small enough" r, there exists a cone

$$\mathscr{K}_{r,\lambda_0} = \left\{z \in \mathbb{C}\colon\ \varphi_1 \leqslant \operatorname{Arg} z \leqslant \varphi_2,\ 0 < \varphi_2 - \varphi_1 \leqslant \psi < \pi\right\}$$

such that

$$F(P'(\lambda_0) + R(\lambda_0, r, \vartheta)) \subset \mathscr{K}_{r,\lambda_0} \backslash \{0\}.$$

For suitable $\vartheta \in [0, 2\pi]$, $\mathrm{e}^{\mathrm{i}\vartheta} F(P'(\lambda_0) + R(\lambda_0, r, \vartheta))$ lies in the interior of $\mathscr{H}_2$. One can see that for every unit vector $x \in \mathbb{C}^n$,

$$x^* P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta})x = x^* P(\lambda_0)x + r\mathrm{e}^{\mathrm{i}\vartheta} x^*(P'(\lambda_0) + R(\lambda_0, r, \vartheta))x,$$

where $\operatorname{Arg}\{r\mathrm{e}^{\mathrm{i}\vartheta} x^*(P'(\lambda_0) + R(\lambda_0, r, \vartheta))x\} \in [\varphi_1 + \vartheta, \varphi_2 + \vartheta]$. Thus, for every $\rho = x_\rho^* P(\lambda_0)x_\rho \in F(P(\lambda_0))$ and for every $r \in [0, r_0]$ such that $\rho + r\mathrm{e}^{\mathrm{i}(\varphi_1+\vartheta)}, \rho + r\mathrm{e}^{\mathrm{i}(\varphi_2+\vartheta)} \in \mathscr{H}_2$, the point

$$x_\rho^* P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta})x_\rho = \rho + r\mathrm{e}^{\mathrm{i}\vartheta} x_\rho^*\big(P'(\lambda_0) + R(\lambda_0, r, \vartheta)\big)x_\rho$$

also lies in $\mathscr{H}_2$. Consequently, as r takes values from 0 to r_0, the part of $F(P(\lambda_0))$ close to the origin "moves" into the half plane $\mathscr{H}_2$ (note that the numerical range $F(P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\vartheta}))$ depends continuously on r, with respect to the Hausdorff metric). Thus, for suitable $r_\vartheta \in [0, r_0]$, the origin lies in the interior of

$$F\big(P(\lambda_0) + r_\vartheta \mathrm{e}^{\mathrm{i}\vartheta}[P'(\lambda_0) + R(\lambda_0, r, \vartheta)]\big) \equiv F(P(\lambda_0 + r_\vartheta \mathrm{e}^{\mathrm{i}\vartheta})).$$

This is a contradiction and the proof is complete. □

Theorem 2. *Suppose that $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial, and $\lambda_0 \in W(P)$ is not a corner of $W(P)$ or a node point of the boundary $\partial W(P)$. If $0 \notin F(P'(\lambda_0))$, and the local dimension of λ_0 in $W(P)$ is equal to 2, then the local dimension of the origin in $F(P(\lambda_0))$ is also equal to 2.*

(At this point, we comment that an example of a linear pencil $P(\lambda) = A\lambda - B$ with node points on $\partial W(P)$ can be found in [2].)

Proof. If λ_0 is an interior point of $W(P)$, then by Theorem 3.1 in [6], the origin is an interior point of $F(P(\lambda_0))$, and thus with local dimension in $F(P(\lambda_0))$ equal to 2.

If $\lambda_0 \in \partial W(P)$, then since λ_0 is not a corner of $W(P)$ or a node point of $\partial W(P)$, there exists an angle $\varphi_0 \in [0, 2\pi]$ such that for every $\varphi \in (\varphi_0, \varphi_0 + \pi)$, there is a real $r_\varphi > 0$ with

$$\lambda_0 + r_\varphi \in \mathrm{Int}W(P).$$

For the sake of contradiction, assume that the local dimension of the origin in $F(P(\lambda_0))$ is 1. Then by the convexity of $F(P(\lambda_0))$, it follows that $F(P(\lambda_0))$ is a line segment passing through the origin. The line of $F(P(\lambda_0))$ defines two closed half planes $\mathscr{H}_1$ and $\mathscr{H}_2$ in $\mathbb{C}$. As in the previous theorem, $P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\varphi})$ is written

$$P(\lambda_0 + r\mathrm{e}^{\mathrm{i}\varphi}) = P(\lambda_0) + r\mathrm{e}^{\mathrm{i}\varphi}P'(\lambda_0) + r\mathrm{e}^{\mathrm{i}\varphi}R(\lambda_0, r, \varphi),$$

where $\|R(\lambda_0, r, \varphi)\| = \mathrm{o}(1)$ as $r \to 0$. Hence, for "small enough" r, there exists a cone

$$\mathscr{K}_{r,\lambda_0} = \big\{z \in \mathbb{C}\colon \varphi_1 \leqslant \operatorname{Arg} z \leqslant \varphi_2,\ 0 < \varphi_2 - \varphi_1 \leqslant \psi < \pi\big\}$$

such that

$$F\big(P'(\lambda_0) + R(\lambda_0, r, \varphi)\big) \subset \mathscr{K}_{r,\lambda_0}\backslash\{0\}.$$

One can verify that for some $\vartheta \in (\varphi_0, \varphi_0 + \pi)$, $\mathrm{e}^{\mathrm{i}\vartheta}F(P'(\lambda_0) + R(\lambda_0, r, \varphi))$ lies in the interior of the half plane $\mathscr{H}_1$. Since

$$F\big(P(\lambda_0 + r_\vartheta\mathrm{e}^{\mathrm{i}\vartheta})\big) \subseteq F(P(\lambda_0)) + r_\vartheta\mathrm{e}^{\mathrm{i}\vartheta}F\big(P'(\lambda_0) + R(\lambda_0, r, \varphi)\big),$$

it is clear that $F(P(\lambda_0 + r_\vartheta\mathrm{e}^{\mathrm{i}\vartheta}))$ also lies in the interior of $\mathscr{H}_1$, and thus,

$$0 \notin F(P(\lambda_0 + r_\vartheta e^{i\vartheta})).$$

This is a contradiction because $\lambda_0 + r_\vartheta e^{i\vartheta}$ belongs to $W(P)$. Hence, the local dimension of the origin in $F(P(\lambda_0))$ is equal to 2. □

3. Linear pencils

Consider a linear pencil $A\lambda - B$, where A and B are $n \times n$ complex matrices. This special case of matrix polynomials plays an important role in the study of linear dynamical systems (see [1] and the references therein). The last years, the numerical range of linear pencils has attracted the attention (see e.g., [2,9,12]). From the results of the previous section, the next corollary follows immediately.

Corollary 3. *Suppose that $W(A\lambda - B)$ is bounded, and let $\lambda_0 \in W(P)$.*
(i) *If the origin is not a corner of $F(A\lambda_0 - B))$, and the local dimension of λ_0 in $W(A\lambda - B)$ is equal to 1, then the local dimension of the origin in $F(P(\lambda_0))$ is also equal to 1.*
(ii) *If λ_0 is not a corner of $W(A\lambda - B)$ or a node point of $\partial W(A\lambda - B)$, and the local dimension of λ_0 in $W(A\lambda_0 - B)$ is equal to 2, then the local dimension of the origin in $F(A\lambda_0 - B)$ is also equal to 2.*

A bounded connected set $\Omega \subset \mathbb{C}$ is called *simply connected* if $\mathbb{C}\backslash\Omega$ is connected (in particular, it has no "holes"). If $\Omega \subset \mathbb{C}$ is unbounded, then we consider the set $\Omega \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\}$, and we say that $\Omega \cup \{\infty\}$ is *simply connected* if $(\mathbb{C} \cup \{\infty\})\backslash\Omega$ is connected. (Note that the two definitions coincide when Ω is a bounded subset of $\mathbb{C}$.) By [8], it is known that if $W(A\lambda - B)$ is bounded, then it is also connected. Furthermore, we have the following theorem.

Theorem 4. *If the numerical range $W(A\lambda - B)$ is bounded, then it is simply connected.*

Proof. Suppose that $W(A\lambda - B)$ is not simply connected. Then $W(A\lambda - B)$ has a "hole", i.e., there is a complex number $\omega_0 \notin W(A\lambda - B)$ such that for every $\varphi \in [0, 2\pi]$, there exists a real $r_\varphi > 0$ satisfying

$$\omega_0 + r_\varphi e^{i\varphi} \in W(A\lambda - B).$$

Since $W(A(\lambda + \mu) - B) = W(A\lambda - B) - \mu$ $(\mu \in \mathbb{C})$, without loss of generality, assume that $\omega_0 = 0$. Then we have that

$$0 \notin W(A\lambda - B)$$

and for every $\varphi \in [0, 2\pi]$,

$$r_\varphi e^{i\varphi} \in W(A\lambda - B),$$

or equivalently,

$$0 \notin F(B)$$

and for every $\varphi \in [0, 2\pi]$,

$$0 \in F(Ar_\varphi e^{i\varphi} - B).$$

Since the origin does not belong to the convex sets $F(A)$ and $F(B)$, there exist two cones

$$\mathscr{K}_1 = \left\{z \in \mathbb{C}:\ \vartheta_1 \leqslant \operatorname{Arg} z \leqslant \tilde{\vartheta}_1,\ 0 < \tilde{\vartheta}_1 - \vartheta_1 \leqslant \psi_1 < \pi\right\}$$

and

$$\mathscr{K}_2 = \left\{z \in \mathbb{C}:\ \vartheta_2 \leqslant \operatorname{Arg} z \leqslant \tilde{\vartheta}_2,\ 0 < \tilde{\vartheta}_2 - \vartheta_2 \leqslant \psi_2 < \pi\right\}$$

such that $F(A) \subset \operatorname{Int}\mathscr{K}_1$ and $-F(B) \subset \operatorname{Int}\mathscr{K}_2$. Moreover, there exists an angle $\varphi_0 \in [0, 2\pi]$ such that both $F(r_{\varphi_0} e^{i\varphi_0} A) \equiv r_{\varphi_0} e^{i\varphi_0} F(A)$ and $-F(B)$ belong to the interior of a cone

$$\mathscr{K}_0 = \left\{z \in \mathbb{C}:\ \vartheta_0 \leqslant \operatorname{Arg} z \leqslant \tilde{\vartheta}_0,\ 0 < \tilde{\vartheta}_0 - \vartheta_0 \leqslant \psi_0 < \pi\right\},$$

where $\max\{\psi_1, \psi_2\} \leqslant \psi_0 < \pi$. As a consequence, the numerical range

$$F\left(A(r_{\varphi_0} e^{i\varphi_0}) - B\right) \subseteq r_{\varphi_0} e^{i\varphi_0} F(A) + F(-B) \subset \operatorname{Int}\mathscr{K}_0$$

does not contain the origin; a contradiction. The proof is complete. □

By the proof of the above theorem, it also follows that for every exterior point μ of the bounded numerical range $W(A\lambda - B)$, there is a cone

$$\mathscr{K}_\mu = \left\{z \in \mathbb{C}:\ \vartheta_1 \leqslant \operatorname{Arg}(z - \mu) \leqslant \vartheta_2,\ 0 < \vartheta_2 - \vartheta_1 \leqslant \vartheta_0 < \pi\right\},$$

such that $\mathscr{K}_\mu \cap W(A\lambda - B) = \emptyset$ (see also [12, Theorem 5]).

The numerical ranges $W(A\lambda - B)$ and $W(B\lambda - A)$ satisfy [8]

$$W(B\lambda - A)\backslash\{0\} = \left\{\mu^{-1}:\ \mu \in W(A\lambda - B)\backslash\{0\}\right\}. \tag{3}$$

As a consequence, Theorem 4 yields the following.

Theorem 5. *If the numerical range $W(A\lambda - B)$ is unbounded, then the set $W(A\lambda - B) \cup \{\infty\}$ is simply connected in the extended plane $\mathbb{C} \cup \{\infty\}$ (or the Riemann sphere S^2).*

Proof. Since $\mathbb{C} \cup \{\infty\} \cong S^2$ is simply connected, we have nothing to prove when $W(A\lambda - B) = \mathbb{C}$. Suppose now that $W(A\lambda - B)$ is unbounded, that is $0 \in F(A)$ [8], and let $\lambda_0 \notin W(A\lambda - B)$. Since $W(A(\lambda + \lambda_0) - B) = W(A\lambda - B) - \lambda_0$, $W(A\lambda - B) \cup \{\infty\}$ is homeomorphic to the set $W(A\lambda - (B - A\lambda_0)) \cup \{\infty\}$. Hence, we can assume that $0 \notin W(A\lambda - B)$, or equivalently, $0 \notin F(B)$. Then by (3), we have (in the extended plane)

$$W(B\lambda - A) = \left\{\mu^{-1}:\ \mu \in W(A\lambda - B) \cup \{\infty\}\right\},$$

and the map $\Psi(\mu) = \mu^{-1}$ for $\mu \in W(A\lambda - B)$ and $\Psi(\infty) = 0$ is a homeomorphism of $W(A\lambda - B) \cup \{\infty\}$ onto $W(B\lambda - A)$. By Theorem 4, the bounded range $W(B\lambda - A)$ is simply connected, and since simply connectedness is a topological property, $W(A\lambda - B) \cup \{\infty\}$ is simply connected in the extended plane $\mathbb{C} \cup \{\infty\}$. □

A nonempty subset Ω of $\mathbb{C}$ is said to be *p-convex* if for every pair of points $\mu_1, \mu_2 \in \mathbb{C}$, either

$$\left\{t\mu_1 + (1-t)\mu_2:\ 0 \leqslant t \leqslant 1\right\} \subset \Omega,$$

or

$$\left\{t\mu_1 + (1-t)\mu_2:\ t \leqslant 0 \text{ or } t \geqslant 1\right\} \subset \Omega.$$

In [9], it is proved that if the matrix A is Hermitian, then the numerical range $W(A\lambda - B)$ is always p-convex.

In general, the numerical range of a linear pencil has no isolated points.

Proposition 6. *Let $A\lambda - B$ be an $n \times n$ linear pencil, and suppose that $W(A\lambda - B)$ is not a singleton. Then the numerical range $W(A\lambda - B)$ has no isolated points.*

Proof. If $0 \notin F(A)$, or $0 \in F(A)$ and $F(A)\backslash\{0\}$ is connected, then the closed range $W(A\lambda - B)$ is connected [8], and thus it has no isolated points.

If $0 \in F(A)$ and $F(A)\backslash\{0\}$ is not connected, then there is an angle $\varphi_0 \in [0, 2\pi]$ such that the matrix $e^{i\varphi_0}A$ is Hermitian. Then the numerical range $W(A\lambda - B) = W(e^{i\varphi_0}(A\lambda - B))$ is p-convex completing the proof. □

The case, where $W(A\lambda - B)$ is a singleton is described by Proposition 2(i) in [12]. Moreover, the local dimension of the points in the numerical range of a linear pencil is always constant.

Theorem 7. *Let $A\lambda - B$ be an $n \times n$ linear pencil. Then the local dimension of every point $\mu \in W(A\lambda - B)$ is constant. Furthermore, if every point of the numerical range $W(A\lambda - B)$ has local dimension in $W(A\lambda - B)$ equal to 1, then $W(A\lambda - B)$ lies, either on a straight line, or on a circle.*

Proof. By the above proposition, the numerical range $W(A\lambda - B)$ contains isolated points (i.e., of zero local dimension) if and only if $W(A\lambda - B)$ is a singleton. Consequently, for the first part of the theorem, it is enough to prove that if there is at least one $\lambda_0 \in W(A\lambda - B)$ of local dimension 1, then every point of $W(A\lambda - B)$ has local dimension 1.

Suppose that $\lambda_0 \in W(A\lambda - B)$ has local dimension in $W(A\lambda - B)$ equal to 1. If $0 \in F(A)$, then the arguments in the proof of Theorem 5 apply to obtain that the 1-dimensional part of $W(B\lambda - A)$ is nonempty. If $W(B\lambda - A)$ lies on a curve, then $W(A\lambda - B)$ also lies on a curve. Hence, without loss of generality, assume that $0 \notin F(A)$. If $\lambda_0 \in W(A\lambda - B)$ such that the origin is a corner of $F(A\lambda_0 - B)$, then $0 \in \sigma(A\lambda_0 - B)$ [4], and thus λ_0 is an eigenvalue of $A\lambda - B$. Since $W(A\lambda_0 - B) \neq \mathbb{C}$, the linear pencil $A\lambda - B$ has no more than n eigenvalues, and consequently, there is a $\lambda_0 \in W(A\lambda - B)$ of local dimension 1 such that the origin is not a corner of $F(A\lambda_0 - B)$. Since $W(A(\lambda + \lambda_0) - B) = W(A\lambda - B) - \lambda_0$, we can also assume that $\lambda_0 = 0$. Then by Corollary 3(i), the local dimension of the origin in $F(B)$ is equal to 1. The convexity of $F(B)$ implies that $F(B)$ is a line segment passing through the origin, and thus there exists an angle $\varphi_0 \in [0, 2\pi]$ such that the matrix $e^{i\varphi_0}B$ is Hermitian. Moreover,

$$W(A\lambda - B)\backslash\{0\} = \left\{\mu^{-1} :\ \mu \in W(e^{i\varphi_0}(B\lambda - A))\right\},$$

where the numerical range $W(e^{i\varphi_0}(B\lambda - A))$ is p-convex [9], and has an nonempty 1-dimensional part. Hence, either

$$W(B\lambda - A) = \left\{t\alpha + (1-t)\beta :\ 0 \leqslant t \leqslant 1\right\},$$

or

$$W(B\lambda - A) = \left\{t\alpha + (1-t)\beta :\ t \leqslant 0 \ \text{ or } \ t \geqslant 1\right\}$$

for some $\alpha, \beta \in \mathbb{C}$. Since by a Möbius transformation

$$\omega = \frac{az + b}{cz + d},$$

the straight line is transformed, either into a circle, or into a straight line, the proof is complete. □

Next we characterize the linear pencils whose numerical range has no interior and lies on a straight line or a circle.

Theorem 8. *Let $A\lambda - B$ be an $n \times n$ linear pencil. Then the numerical range $W(A\lambda - B)$ has no interior points if and only if there exist two linearly independent Hermitian matrices H_1 and H_2, and complex numbers a, b, c and d such that $0 \notin F(H_1 + iH_2)$ and*

$$A = aH_1 + bH_2 \quad \text{and} \quad B = cH_1 + dH_2. \tag{4}$$

Proof. Suppose that $W(A\lambda - B)$ has no interior points and $\lambda_0 \in W(A\lambda - B)$. Then the origin belongs to $F(B - A\lambda_0) = -F(A\lambda_0 - B)$ and has local dimension in $F(B - A\lambda_0)$ equal to 1. By the convexity of $F(B - A\lambda_0)$, it follows that $F(B - A\lambda_0)$ is a line segment passing through the origin, and thus there exists an angle $\varphi_1 \in [0, 2\pi]$ such that the matrix $H_1 = e^{i\varphi_1}(B - A\lambda_0)$ is Hermitian. Using now the p-convexity of the unbounded (1-dimensional) range $W((B - A\lambda_0)\lambda - A)$ [9], we obtain that there is a $\varphi_2 \in [0, 2\pi]$ for which $W((B - A\lambda_0)\lambda - e^{i\varphi_2}A)$ lies on a line parallel to the real axis. Hence, there exists a complex number γ such that $W((B - A\lambda_0)\lambda - (\mathrm{e}^{\mathrm{i}\varphi_2} A + \gamma B - \gamma A\lambda_0))$ lies on the real axis. It is also clear that the fraction

$$\frac{x^*(\mathrm{e}^{\mathrm{i}\varphi_2} A + \gamma B - \gamma A\lambda_0)x}{x^*(B - A\lambda_0)x} = \frac{\mathrm{e}^{\mathrm{i}\varphi_1} x^*(\mathrm{e}^{\mathrm{i}\varphi_2} A + \gamma B - \gamma A\lambda_0)x}{\mathrm{e}^{\mathrm{i}\varphi_1} x^*(B - A\lambda_0)x}$$

is real for every unit vector $x \in \mathbb{C}^n$ with $x^*(B - A\lambda_0)x \neq 0$. Since (4) is obvious when the matrices A and B are linearly dependent, we assume that A and B are linearly independent. In this case, the set

$$\left\{x \in \mathbb{C}^n :\ x^*(B - A\lambda_0)x \neq 0 \ \text{ and } \ x^*x = 1\right\}$$

is dense in the unit sphere of $\mathbb{C}^n$. Consequently, for every unit $x \in \mathbb{C}^n$, $\mathrm{e}^{\mathrm{i}\varphi_1} x^*(\mathrm{e}^{\mathrm{i}\varphi_2} A + \gamma B - \gamma A\lambda_0)x$ is real, and thus the matrix

$$H_2 = \mathrm{e}^{\mathrm{i}\varphi_1}(\mathrm{e}^{\mathrm{i}\varphi_2} A + \gamma B - \gamma A\lambda_0)$$

is Hermitian [4]. Moreover, the matrices A and B are written as in (4) with

$$a = -\mathrm{e}^{-\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\varphi_2}\gamma, \quad b = \mathrm{e}^{-\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\varphi_2},$$
$$c = \mathrm{e}^{-\mathrm{i}\varphi_1} - \mathrm{e}^{-\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\varphi_2}\lambda_0\gamma, \quad d = \lambda_0\mathrm{e}^{-\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\varphi_2}.$$

Finally, by the condition $W(A\lambda - B) \neq \mathbb{C}$, it follows immediately that for every unit vector $y \in \mathbb{C}^n$, $(y^*H_1y, y^*H_2y) \neq (0,0)$, that is, $0 \notin F(H_1 + \mathrm{i}H_2)$.

Conversely, suppose that the matrices A and B are written as in (4), where the Hermitian matrices H_1 and H_2 satisfy the hypothesis of the theorem. If $ad = bc$, then the range $W(A\lambda - B)$ is a singleton. Assume now that $ad \neq bc$. Since $0 \notin F(H_1 + \mathrm{i}H_2)$, the numerical range $W(H_1\lambda - H_2)$ lies on the real axis [8]. If $a = 0$, then $bc \neq 0$ and the numerical range

$$W(bH_2\lambda - (cH_1 + dH_2)) = b^{-1}(d + cW(H_2\lambda - H_1))$$

has no interior points. If $a \neq 0$, then set $d' = d - (bc)/a \neq 0$ and observe that the range

$$W(d'H_2\lambda - (aH_1 + bH_2)) = (d')^{-1}(b + aW(H_2\lambda - H_1))$$

has no interior points, or equivalently, $W((aH_1 + bH_2)\lambda - d'H_2)$ has no interior points. Hence, the numerical range

$$W((aH_1 + bH_2)\lambda - (c/a)(aH_1 + bH_2) - d'H_2) = W(A\lambda - B)$$

has also no interior points, and the proof is complete. $\square$

Note that if A and B are written in the form

$$A = \mathrm{e}^{\mathrm{i}\vartheta_1} H_1 \quad \text{and} \quad B = \mathrm{e}^{\mathrm{i}\vartheta_2} H_2,$$

where $\vartheta_1, \vartheta_2 \in [0, 2\pi]$ and the matrices H_1 and H_2 are Hermitian, then the numerical range $W(A\lambda - B) = \mathrm{e}^{\mathrm{i}(\vartheta_2 - \vartheta_1)} W(H_1\lambda - H_2)$, either coincides with the whole complex plane, or lies on the line

$$\left\{z \in \mathbb{C}\colon\ \operatorname{Arg} z = \vartheta_2 - \vartheta_1 \text{ or} \operatorname{Arg} z = \pi + \vartheta_2 - \vartheta_1\right\}.$$

By Theorem 1.7.17 in [4], it is easy to see that the matrices H_1 and H_2 in Theorem 8 are simultaneously diagonalizable by congruence.

Corollary 9. *If $W(A\lambda - B)$ has no interior points, then there is a nonsingular matrix T such that the pencil $T^*(A\lambda - B)T$ is diagonal.*

It is known in the literature that every square matrix A is written in the form

$$A = H_1(A) + \mathrm{i}H_2(A),$$

where the matrices $H_1(A) = (A + A^*)/2$ and $H_2(A) = (A - A^*)/(2\mathrm{i})$ are Hermitian.

Corollary 10. *Suppose that $A\lambda - B$ is an $n \times n$ linear pencil with $W(A\lambda - B) \neq \mathbb{C}$. Then the following conditions are (mutually) equivalent.*

(i) *The numerical range $W(A\lambda - B)$ has a nonempty interior.*

(ii) *The numerical range $W(A\lambda - B)$ is the closure of its interior.*

(iii) *The real linear space spanned by the Hermitian matrices $H_1(A)$, $H_2(A)$, $H_1(B)$ and $H_2(B)$ has dimension at least* 3.

4. Diagonal matrix polynomials

For an $n \times n$ matrix polynomial $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$, the *joint numerical range* of its coefficients is defined by

$$\operatorname{JNR}(P) = \left\{(x^*A_0x, x^*A_1x, \ldots, x^*A_mx) \in \mathbb{C}^n\colon\ x \in \mathbb{C}^n,\ x^*x = 1\right\}.$$

One can easily see that

$$W(P) = \left\{\lambda \in \mathbb{C}\colon a_m\lambda^m + \cdots + a_1\lambda + a_0 = 0,\ (a_0, a_1, \ldots, a_m) \in \operatorname{JNR}(P)\right\},$$

and if $P(\lambda)$ is diagonal, then $\operatorname{JNR}(P)$ is a convex polyhedron in $\mathbb{C}^{m+1}$. Furthermore, the numerical range of a general matrix polynomial can be approximated by using numerical ranges of diagonal matrix polynomials [13].

Theorem 11 [13, Theorem 4.2]. *Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial. Then*

$$\bigcup_{D_1} W(D_1) = W(P) = \bigcap_{D_2} W(D_2),$$

where the union (intersection) is taken over all diagonal matrix polynomials $D_1(\lambda)$ (respectively, $D_2(\lambda)$) of degree m for which $\operatorname{JNR}(D_1) \subseteq \operatorname{JNR}(P) \subseteq \operatorname{JNR}(D_2)$.

Motivated by the above theorem, next we consider the problem of drawing the numerical range of a diagonal matrix polynomial

$$D(\lambda) = \operatorname{diag}\big\{d_1(\lambda), d_2(\lambda), \dots, d_n(\lambda)\big\}.$$

For any choice of indices $1 \leqslant k_1 < k_2 < \cdots < k_s \leqslant n$, denote

$$D(\lambda\colon\ k_1, k_2, \dots, k_s) = \operatorname{diag}\big\{d_{k_1}(\lambda), d_{k_2}(\lambda), \dots, d_{k_s}(\lambda)\big\}. \tag{5}$$

Notice also that the numerical range of a diagonal matrix $\operatorname{diag}\{a_1, a_2, \dots, a_n\}$, with $n > 3$, is the convex hull of the diagonal elements and consists of a union of convex polygons with s $(3 \leqslant s < n)$ vertices. In particular,

$$F\big(\operatorname{diag}\{a_1, a_2, \dots, a_n\}\big) = \bigcup_{1\leqslant k_1<k_2<\cdots<k_s\leqslant n} F\big(\operatorname{diag}\{a_{k_1}, a_{k_2}, \dots, a_{k_s}\}\big).$$

By using this simple observation, the problem of drawing the numerical range of a diagonal matrix polynomial is easily reduced.

Proposition 12. *Let $D(\lambda)$ be an $n \times n$ diagonal matrix polynomial with $n > 3$, and let $s \in \{3, 4, \dots, n-1\}$. Then*

$$W(D) = \bigcup_{1\leqslant k_1<k_2<\cdots<k_s\leqslant n} W(D(\lambda\colon k_1, k_2, \dots, k_s)).$$

Proof. Consider a diagonal matrix polynomial

$$D(\lambda) = \operatorname{diag}\big\{d_1(\lambda), d_2(\lambda), \dots, d_n(\lambda)\big\} \quad (n > 3)$$

and a positive integer $s \in \{3, 4, \dots, n-1\}$. Then $\lambda_0 \in W(D)$ if and only if

$$0 \in F(D(\lambda_0)) = \bigcup_{1\leqslant k_1<k_2<\cdots<k_s\leqslant n} F\left(\operatorname{diag}\big\{d_{k_1}(\lambda_0), d_{k_2}(\lambda_0), \dots, d_{k_s}(\lambda_0)\big\}\right),$$

or equivalently,

$$\lambda_0 \in W(D(\lambda\colon\ k_1, k_2, \dots, k_s))$$

for some indices $1 \leqslant k_1 < k_2 < \cdots < k_s \leqslant n$. $\square$

Moreover, for an $n \times n$ diagonal matrix polynomial $D(\lambda)$, the boundary $\partial W(D)$ is proved to be a subset of a finite union of numerical ranges of 2×2 diagonal matrix polynomials. This is quite useful since the numerical range of a 2×2 diagonal matrix polynomial has no interior points, and thus, it is easy to be sketched.

Proposition 13. *If $D(\lambda)$ is a 2×2 diagonal matrix polynomial, then $W(D)$ has no interior points, i.e., every point of $W(D)$ has local dimension 1.*

Proof. Let $D(\lambda) = \operatorname{diag}\{d_1(\lambda), d_2(\lambda)\}$ be of mth degree with $d_1(\lambda) = b_m\lambda^m + \cdots + b_1\lambda + b_0$ and $d_2(\lambda) = c_m\lambda^m + \cdots + c_1\lambda + c_0$, and assume that $\operatorname{Int}W(D) \neq \emptyset$. Observe that for every $\mu \in \operatorname{Int}W(D)$, the origin is a boundary point of $F(D(\mu))$. By Theorem 3.1 in [6], it follows that for every $\lambda_0 \in \operatorname{Int}W(D)$,

$$0 \in F(D'(\lambda_0)),$$

or equivalently,

$$\lambda_0 \in W(D').$$

By induction, we have

$$\mathrm{Int} W(D) \subseteq \mathrm{Int} W(D') \subseteq \cdots \subseteq \mathrm{Int} W(D^{(m-1)}).$$

The numerical range of the 2×2 linear pencil

$$D^{(m-1)}(\lambda) = (m-1)! \left(m \operatorname{diag}\{b_m, c_m\}\lambda + \operatorname{diag}\{b_{m-1}, c_{m-1}\}\right),$$

namely,

$$W(D^{(m-1)}) = \frac{1}{m}\left\{-\frac{b_{m-1}t + c_{m-1}(1-t)}{b_m t + c_m(1-t)} : \ t \in [0,1]\right\}$$

has no interior points (cf. Theorems 7 and 8), and the proof is complete. □

Proposition 14. *If $D(\lambda)$ is an $n \times n$ diagonal matrix polynomial, then*

$$\partial W(D) \subseteq \bigcup_{1 \leqslant j < k \leqslant n} W(D(\lambda : j, k)).$$

Proof. Let $D(\lambda) = \operatorname{diag}\{d_1(\lambda), d_2(\lambda), \ldots, d_n(\lambda)\}$ and let $\lambda_0 \in \partial W(D)$. Then by Theorem 1.1 in [11], the origin is a boundary point of $F(D(\lambda_0))$, where $F(D(\lambda_0))$ coincides with the convex hull of $d_1(\lambda_0), d_2(\lambda_0), \ldots, d_n(\lambda_0)$. Hence,

$$0 \in F\left(\operatorname{diag}\{d_j(\lambda_0), d_k(\lambda_0)\}\right)$$

for some $j, k \in \{1, 2, \ldots, n\}$ with $j < k$, and thus $\lambda_0 \in W(D(\lambda : j, k))$. □

The above proposition and the second part of Theorem 6 yield the following corollary.

Corollary 15. *The boundary of the numerical range of a diagonal linear pencil coincides with a finite union of line segments and circular arcs.*

Example 1. Let $D(\lambda)$ be the 4×4 diagonal matrix polynomial

$$\begin{aligned} D(\lambda) = {} & I\lambda^3 + \operatorname{diag}\left\{1, -\mathrm{i}, \mathrm{i}, -1+\mathrm{i}\right\}\lambda^2 \\ & + \operatorname{diag}\left\{2\mathrm{i}, 12, \sqrt{5}, 0\right\}\lambda + \operatorname{diag}\left\{\sqrt{13}, -4\mathrm{i}, -5, 4\right\}. \end{aligned}$$

In Fig. 1, we sketch 1000 points of $W(D)$, and in Fig. 2, we add 100 points of each numerical range $W(D(\lambda : j, k))$ $(1 \leqslant j < k \leqslant 4)$. The eigenvalues of $D(\lambda)$ are marked with +'s. The comparison of these two figures shows how helpful is Proposition 14 in studying the shape of the numerical range of a diagonal matrix polynomial.

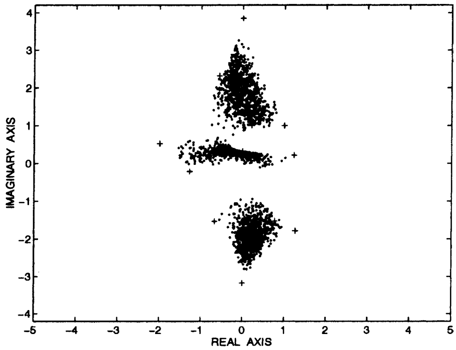

Fig. 1. The numerical range $W(D)$.

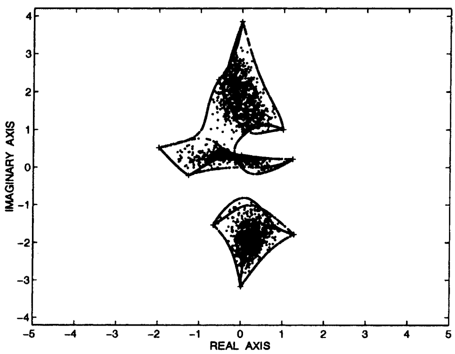

Fig. 2. The numerical range $W(D)$ and its boundary.

5. Computations for $n = 2$

Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial. Then by Theorem 4.1 in [10], $W(P)$ can be approximated by using numerical ranges of 2×2 matrix polynomials. In this section, we investigate the point equation of the boundary of the numerical range of a 2×2 matrix polynomial (cf. [2])

$$Q(\lambda) = B_m\lambda^m + B_{m-1}\lambda^{m-1} + \cdots + B_1\lambda + B_0. \tag{6}$$

Recall that every square matrix A is written $A = H_1(A) + \mathrm{i}H_2(A)$, where the matrices $H_1(A) = (A + A^*)/2$ and $H_2(A) = (A - A^*)/(2\mathrm{i})$ are Hermitian. Moreover, observe that for any 2×2 Hermitian matrix

$$A = \begin{bmatrix} a+d & b+\mathrm{i}c \\ b-\mathrm{i}c & a-d \end{bmatrix},$$

and for any unit vector $y = [\cos\vartheta,\ \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta]^{\mathrm{T}} \in \mathbb{C}^2$, we have

$$y^*Ay = a + d\cos(2\vartheta) + b\sin(2\vartheta)\cos\varphi - c\sin(2\vartheta)\sin\varphi.$$

Consider now y as an element of the complex projective line $\mathbb{CP}^1$, and set

$$X = \sin(2\vartheta)\cos(\varphi), \quad Y = -\sin(2\vartheta)\sin(\varphi) \quad \text{and} \quad Z = \cos(2\vartheta).$$

Then the point $(X, Y, Z) \in \mathbb{R}^3$ satisfies $X^2 + Y^2 + Z^2 = 1$ and we can identify $\mathbb{CP}^1$ with the real 2-dimensional sphere $X^2 + Y^2 + Z^2 = 1$. As a consequence,

$$y^*Ay = a + bX + cY + dZ.$$

The coefficients of $Q(\lambda)$ in (6) can be written in the form

$$B_j = \begin{bmatrix} a_j + d_j & b_j + \mathrm{i}c_j \\ b_j - \mathrm{i}c_j & a_j - d_j \end{bmatrix} + \mathrm{i}\begin{bmatrix} a'_j + d'_j & b'_j + \mathrm{i}c'_j \\ b'_j - \mathrm{i}c'_j & a'_j - d'_j \end{bmatrix} \quad (j = 0, 1, \ldots, m),$$

where $a_j, b_j, c_j, d_j, a'_j, b'_j, c'_j, d'_j \in \mathbb{R}$ $(j = 0, 1, \ldots, m)$, and then

$$\begin{aligned} y^*Q(\lambda)y = &\sum_{j=0}^{m} \lambda^j (a_j + b_jX + c_jY + d_jZ) \\ &+\mathrm{i}\sum_{j=0}^{m} \lambda^j (a'_j + b'_jX + c'_jY + d'_jZ). \end{aligned}$$

For $\lambda = u + \mathrm{i}v$ $(u, v \in \mathbb{R})$, the equation $y^*Q(u + \mathrm{i}v)y = 0$ is rewritten as the system

$$\begin{aligned} \mathrm{Re}(y^*Q(u + \mathrm{i}v)y) = &\ \phi_{1,1}(u, v)X + \phi_{1,2}(u, v)Y \\ &+ \phi_{1,3}(u, v)Z + \phi_{1,0}(u, v) = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} \mathrm{Im}(y^*Q(u + \mathrm{i}v)y) = &\ \phi_{2,1}(u, v)X + \phi_{2,2}(u, v)Y \\ &+ \phi_{2,3}(u, v)Z + \phi_{2,0}(u, v) = 0, \end{aligned} \tag{8}$$

where $\phi_{j,k}(u, v)$ $(j = 1, 2,\ k = 0, 1, 2, 3)$ are real polynomials in u, v of total degree at most m. At this point and for the remainder, we assume that $\phi_{j,k}(u, v) \neq 0$ for some $j = 1, 2,\ k = 1, 2, 3$ since otherwise $Q(\lambda)$ is a scalar polynomial. Furthermore, for every $(u, v) \in \mathbb{R}^2$, consider an affine subspace $\mathscr{L}(u, v)$ of $\mathbb{R}^3$ defined by

$$\mathscr{L}(u, v) = \left\{(X, Y, Z) \in \mathbb{R}^3 :\ (7) \text{ and } (8) \text{ are satisfied} \right\}.$$

Then it is clear that the numerical range $W(Q)$ is the set of the points $\lambda = u + \mathrm{i}v$ $(u, v \in \mathbb{R})$ for which the corresponding affine space $\mathscr{L}(u, v)$ has a common point with the unit sphere $X^2 + Y^2 + Z^2 = 1$.

One of the following three cases occurs:

Case I. The real matrix

$$F_1(u, v) = \begin{bmatrix} \phi_{1,1}(u, v) & \phi_{1,2}(u, v) & \phi_{1,3}(u, v) \\ \phi_{2,1}(u, v) & \phi_{2,2}(u, v) & \phi_{2,3}(u, v) \end{bmatrix} \tag{9}$$

has rank 2 for every $(u, v) \in \mathbb{R}^2$ except for points on an algebraic curve $\mathscr{G}(u, v) = 0$.

Case II. For every $(u, v) \in \mathbb{R}^2$, the real matrix $F_1(u, v)$ in (9) has rank $\leqslant 1$, and the real matrix

$$F_2(u, v) = \begin{bmatrix} \phi_{1,1}(u, v) & \phi_{1,2}(u, v) & \phi_{1,3}(u, v) & \phi_{1,0}(u, v) \\ \phi_{2,1}(u, v) & \phi_{2,2}(u, v) & \phi_{2,3}(u, v) & \phi_{2,0}(u, v) \end{bmatrix} \tag{10}$$

has rank 2 for some $(u, v) \in \mathbb{R}^2$.

Case III. For every $(u, v) \in \mathbb{R}^2$, the real matrix $F_2(u, v)$ in (10) has rank $\leqslant 1$.

First, we consider Case I. Without loss of generality, assume that

$$\det \begin{bmatrix} \phi_{1,1}(u, v) & \phi_{1,2}(u, v) \\ \phi_{2,1}(u, v) & \phi_{2,2}(u, v) \end{bmatrix}$$

does not vanish on an open dense subset of $\mathbb{R}^2$. On this open set the affine subspace $\mathscr{L}(u, v)$ is 1-dimensional. A parametric representation of the straight line $\mathscr{L}(u, v)$ is obtained by solving the Eqs. (7) and (8) in X, Y,

$$\begin{aligned} X = {} & \frac{-\phi_{1,3}(u, v)\phi_{2,2}(u, v) + \phi_{2,3}(u, v)\phi_{1,2}(u, v)}{\phi_{1,1}(u, v)\phi_{2,2}(u, v) - \phi_{1,2}(u, v)\phi_{2,1}(u, v)} Z \\ & + \frac{-\phi_{1,0}(u, v)\phi_{2,2}(u, v) + \phi_{2,0}(u, v)\phi_{1,2}(u, v)}{\phi_{1,1}(u, v)\phi_{2,2}(u, v) - \phi_{1,2}(u, v)\phi_{2,1}(u, v)}, \\ Y = {} & \frac{\phi_{1,3}(u, v)\phi_{2,1}(u, v) - \phi_{2,3}(u, v)\phi_{1,1}(u, v)}{\phi_{1,1}(u, v)\phi_{2,2}(u, v) - \phi_{1,2}(u, v)\phi_{2,1}(u, v)} Z \\ & + \frac{\phi_{1,0}(u, v)\phi_{2,1}(u, v) - \phi_{2,0}(u, v)\phi_{1,1}(u, v)}{\phi_{1,1}(u, v)\phi_{2,2}(u, v) - \phi_{1,2}(u, v)\phi_{2,1}(u, v)}. \end{aligned}$$

Substituting these relations into the equation $X^2 + Y^2 + Z^2 - 1 = 0$, we have a quadratic equation with discriminant

$$\begin{aligned} \mathscr{D}(u, v) = {} & (\phi_{1,1}\phi_{2,2} - \phi_{1,2}\phi_{2,1})^2 \\ & + (\phi_{1,1}\phi_{2,3} - \phi_{1,3}\phi_{2,1})^2 \\ & + (\phi_{1,2}\phi_{2,3} - \phi_{1,3}\phi_{2,2})^2 \\ & - \left\| \phi_{2,0} \left[\phi_{1,1}, \phi_{1,2}, \phi_{1,3}\right]^{\mathrm{T}} - \phi_{1,0} \left[\phi_{2,1}, \phi_{2,2}, \phi_{2,3}\right]^{\mathrm{T}} \right\|_2^2, \end{aligned}$$

where $\| \cdot \|_2$ is the Euclidean norm. Obviously, $\mathscr{D}(u, v)$ is a real polynomial in u, v of total degree at most $4m$.

If $\lambda_0 = u_0 + \mathrm{i}v_0$ ($u_0, v_0 \in \mathbb{R}$) is an interior point of $W(P)$, then the discriminant $\mathscr{D}(u, v)$ is nonnegative "near" the point (u_0, v_0), and if $\lambda_0 = u_0 + \mathrm{i}v_0$ is an exterior point of $W(P)$, then $\mathscr{D}(u_0, v_0) < 0$. Hence, every boundary point $\lambda_0 = u_0 + \mathrm{i}v_0$ ($u_0, v_0 \in \mathbb{R}$) of $W(P)$ (as a limit point of the interior of $W(P)$) satisfies the equation

$$\mathscr{D}(u_0, v_0) = 0.$$

Note that the points $(u, v) \in \mathbb{R}^2$ for which the matrix $F_1(u, v)$ in (9) has rank $\leqslant 1$ lie on the algebraic curve

$$\begin{aligned}\mathscr{G}(u, v) = {} & (\phi_{1,1}\phi_{2,2} - \phi_{1,2}\phi_{2,1})^2 + (\phi_{1,1}\phi_{2,3} - \phi_{1,3}\phi_{2,1})^2 \\ & + (\phi_{1,2}\phi_{2,3} - \phi_{1,3}\phi_{2,1})^2 = 0.\end{aligned}$$

Remark. For the straight line, in the 3-dimensional Euclidean space,

$$\phi_{1,1}X + \phi_{1,2}Y + \phi_{1,3}Z + \phi_{1,0} = 0,$$

$$\phi_{2,1}X + \phi_{2,2}Y + \phi_{2,3}Z + \phi_{2,0} = 0,$$

the distance d between the origin and the line is given by

$$d^2 = \frac{\left\| \phi_{2,0}\left[\phi_{1,1}, \phi_{1,2}, \phi_{1,3}\right]^{\mathrm{T}} - \phi_{1,0}\left[\phi_{2,1}, \phi_{2,2}, \phi_{2,3}\right]^{\mathrm{T}} \right\|_2^2}{(\phi_{1,1}\phi_{2,2} - \phi_{1,2}\phi_{2,1})^2 + (\phi_{1,1}\phi_{2,3} - \phi_{1,3}\phi_{2,1})^2 + (\phi_{1,2}\phi_{2,3} - \phi_{1,3}\phi_{2,2})^2}.$$

Let us now consider Case II. In this case, for every $u + \mathrm{i}v \in W(P)$, the point $(u, v) \in \mathbb{R}^2$ satisfies the equations:

$$\phi_{1,0}(u, v)\phi_{2,j}(u, v) - \phi_{2,0}(u, v)\phi_{1,j}(u, v) = 0 \quad (j = 1, 2, 3).$$

Notice that at least one of the polynomials

$$\phi_{1,0}(u, v)\phi_{2,j}(u, v) - \phi_{2,0}(u, v)\phi_{1,j}(u, v) \quad (j = 1, 2, 3)$$

does not vanish at some $(u_0, v_0) \in \mathbb{R}^2$. Thus, the numerical range $W(P)$ is contained in an algebraic curve

$$\Gamma(u, v) = \phi_{1,0}(u, v)\phi_{2,j}(u, v) - \phi_{2,0}(u, v)\phi_{1,j}(u, v) = 0$$

for some $j = 1, 2, 3$, and every point of $W(P)$ has local dimension 1 (i.e., $W(P)$ has no interior points).

Finally, we consider Case III. The following lemma is necessary.

Lemma 16. *Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial, and let $P(\mu)$ be normal for every $\mu \in \mathbb{C}$. If there is a $\lambda_0 \in \mathbb{C}$ such that the matrix $P(\lambda_0)$ has n distinct eigenvalues, then there exists an $n \times n$ unitary matrix U such that the matrix polynomial $U^*P(\lambda)U$ is diagonal.* (*In particular, the coefficients $A_0, A_1, \ldots, A_m$ are simultaneously diagonalizable by unitary similarity.*)

Proof. Consider the matrix polynomial $\tilde{P}(\lambda) = P(\lambda - \lambda_0)$. Then it is obvious that $\sigma(\tilde{P}) = \sigma(P) + \lambda_0$ and the matrix $\tilde{P}(\mu)$ is normal for every $\mu \in \mathbb{C}$. Hence, without loss of generality, assume that $P(0) = A_0$ has n distinct eigenvalues. By the normality hypothesis, we have, for real parameter t,

$$P(t)P(t)^* = P(t)^*P(t), \tag{11}$$

$$P(t\mathrm{e}^{\mathrm{i}\vartheta})P(t\mathrm{e}^{\mathrm{i}\vartheta})^* = P(t\mathrm{e}^{\mathrm{i}\vartheta})^*P(t\mathrm{e}^{\mathrm{i}\vartheta}), \quad \vartheta \in [0, 2\pi]. \tag{12}$$

We differentiate these equations, with respect to t, for $\vartheta = \pi/2$. Taking the derivatives at $t = 0$ yields

$$A_1 A_0^* + A_0 A_1^* = A_0^* A_1 + A_1^* A_0,$$

$$\mathrm{i}A_1 A_0^* - \mathrm{i}A_0 A_1^* = \mathrm{i}A_0^* A_1 - \mathrm{i}A_1^* A_0,$$

and thus $A_0 A_1^* = A_1^* A_0$. By hypothesis, A_0 is written $A_0 = U^* D_0 U$, where U is an $n \times n$ unitary matrix and D_0 is an $n \times n$ diagonal matrix with distinct diagonal elements. Then it is straightforward that A_1 is also a normal matrix of the form $A_1 = U^* D_1 U$, where D_1 is diagonal (see [3, pp. 186–187]). Clearly, A_0 and A_1 commute.

Next, for $\vartheta = \pi/4$, we take the second order derivative of Eqs. (11) and (12) at $t = 0$. Then

$$A_2 A_0^* + A_0 A_2^* + A_1 A_1^* = A_0^* A_2 + A_2^* A_0 + A_1^* A_1,$$

$$\mathrm{i}A_2 A_0^* - \mathrm{i}A_0 A_2^* + A_1 A_1^* = \mathrm{i}A_0^* A_2 - \mathrm{i}A_2^* A_0 + A_1^* A_1,$$

which implies that $A_0 A_2^* = A_2^* A_0$. Hence, A_2 is also a normal matrix commuting with A_0, and there exists an $n \times n$ diagonal matrix D_1 such that $A_2 = U^* D_2 U$. Continuing this process for $\vartheta = \pi/6, \pi/8, \ldots, \pi/(2m)$, we conclude that $A_0, A_1, \ldots, A_m$ are commuting normal matrices, and they are simultaneously diagonalizable by unitary similarity. The proof is complete. $\square$

By the assumptions of Case III, it follows that for every $(u, v) \in \mathbb{R}^2$, the left-hand sides of (7) and (8) are proportional. Hence, for every unit vector $y \in \mathbb{C}^2$,

$$y^* Q(\lambda) y = \Phi(\lambda) g(\lambda, y)$$

for some complex valued continuous function $\Phi(\lambda)$ and a real valued function $g(\lambda, y)$. This implies that for every $\mu \in \mathbb{C}$, the matrix $Q(\mu)$ is normal and its numerical range, $F(Q(\mu))$, is contained in a straight line passing through the origin. By the above lemma, there exists a 2×2 unitary matrix U such that

$$U Q(\lambda) U^* = \operatorname{diag}\big\{q_1(\lambda), q_2(\lambda)\big\}$$

for two scalar polynomials $q_1(\lambda)$ and $q_2(\lambda)$, and thus,

$$W(Q) = W\big(\operatorname{diag}\{q_1(\lambda), q_2(\lambda)\}\big).$$

(Note that if the matrix $Q(\mu)$ has a double eigenvalue for every $\mu \in \mathbb{C}$, then $Q(\mu)$ is a scalar matrix for every $\mu \in \mathbb{C}$, and the conclusions of Lemma 16 hold.)

If $q_2(\lambda) \equiv 0$, then $W(Q) = \mathbb{C}$, and we have nothing to prove. If $q_2(\lambda) \neq 0$, then since the real matrix $F_2(u, v)$ in (10) always has rank $\leqslant 1$, it follows that for every $\mu \in \mathbb{C}$, the real matrix

$$\begin{bmatrix} \operatorname{Re} q_1(\mu) & \operatorname{Re} q_2(\mu) \\ \operatorname{Im} q_1(\mu) & \operatorname{Im} q_2(\mu) \end{bmatrix}$$

is singular. Consequently, for every $\mu \in \mathbb{C}$, there exists a pair $(\alpha_\mu, \beta_\mu) \in \mathbb{R}^2\backslash\{(0,0)\}$ such that $\alpha_\mu q_1(\mu) + \beta_\mu q_2(\mu) = 0$. This is true only when there is a pair $(\alpha, \beta) \in \mathbb{R}^2\backslash\{(0,0)\}$ such that $\alpha q_1(\lambda) + \beta q_2(\lambda) \equiv 0$. Hence, either $W(Q) = \mathbb{C}$ (when $\alpha\beta \geqslant 0$), or $W(Q)$ coincides with the set of the roots of $q_2(\lambda)$ (when $\alpha\beta < 0$).

So we proved the main result of this section.

Theorem 17. *Let $Q(\lambda) = B_m\lambda^m + \cdots + B_1\lambda + B_0$ be a 2×2 matrix polynomial with numerical range $W(Q) \neq \mathbb{C}$. If $W(Q)$ has no interior points, then $W(Q)$ lies on an algebraic curve of degree at most $2m$. If $W(Q)$ has interior points, then $W(Q)$ coincides with the union of two closed sets W_1 and W_2 such that W_1 lies on an algebraic curve of degree at most $2m$ and the boundary ∂W_2 lies on an algebraic curve of degree at most $4m$.*

Motivated by the results of the previous section, we consider the point equation of the numerical range of the matrix polynomial

$$Q(\lambda) = \operatorname{diag}\{q_1(\lambda), q_2(\lambda)\}. \tag{13}$$

Corollary 18. *Let $Q(\lambda)$ be a 2×2 diagonal matrix polynomial as in* (13) *such that $W(Q) \neq \sigma(Q), \mathbb{C}$. Then $W(Q)$ lies on the curve*

$$\operatorname{Re} q_1(\lambda)\operatorname{Im} q_2(\lambda) - \operatorname{Re} q_2(\lambda)\operatorname{Im} q_1(\lambda) = 0$$

(*recall that by Proposition* 13, *$W(Q)$ has no interior points*).

Example 2. Consider the 2×2 diagonal matrix polynomial

$$Q(\lambda) = \operatorname{diag}\{\lambda^2 + \lambda + 1,\ \lambda^2 + 2\lambda + 2\}.$$

The numerical range $W(Q)$ (in $\mathbb{C} \cong \mathbb{R}^2$), in Fig. 3, is the union of two arcs of the circle $S(-1, 1)$ with centre at $-1 \cong (-1, 0)$ and radius 1. The endpoints of these arcs

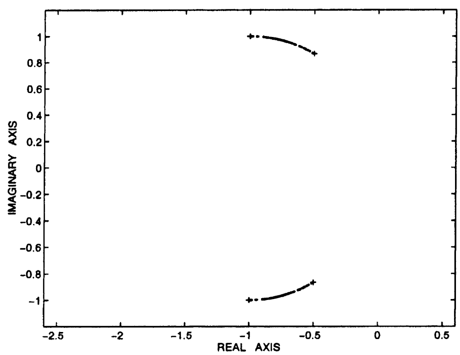

Fig. 3. $W(Q)$ consists of two circular arcs.

are the eigenvalues $-0.5 \pm \mathrm{i}\sqrt{0.75},\ -1 \pm \mathrm{i}$ of $Q(\lambda)$. Furthermore, it is easy to see, writing $\lambda = u + \mathrm{i}v$ $(u, v \in \mathbb{R})$, that the algebraic curve (in $\mathbb{R}^2$)

$$\begin{aligned}&\operatorname{Re} q_1(u+\mathrm{i}v)\operatorname{Im} q_2(u+\mathrm{i}v) - \operatorname{Re} q_2(u+\mathrm{i}v)\operatorname{Im} q_1(u+\mathrm{i}v)\\&\quad = v(u^2+2u+v^2) = 0\end{aligned}$$

coincides with the union of the axis $v = 0$ and the circle $S((-1, 0), 1)$. The above corollary is verified.

Acknowledgment

The authors wish to thank an anonymous referee for his (or her) useful suggestions.

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