Consider a vector space V over the real numbers $\mathbb{R}$, equipped with a symmetric, non-degenerate bilinear form $(-,-): V \times V \to \mathbb{R}$. We define the orthogonal projection as $U^{\perp/(-,-)} = \{v \in V \mid (v,u) = \mathbf{0} \forall u \in U\} = U^{\perp}$ (not the same as the orthogonal complement) for some $U \subseteq V$; we further have $\dim U + \dim U^{\perp} = \dim V$ and that $U \subseteq U^{\perp} = \{\mathbf{0}\}$. Furthermore, for some $v \in V$, there exists an unique decomposition of v into two vectors $u \in U$ and $u^{\perp} \in U^{\perp}$ such that $u = P_U(v)$ and $u^{\perp} = P_{U^{\perp}}(v)$, where $P_U : V \to V$ and $P_{U^{\perp}} : V \to V$ and $\operatorname{Im} P_U = U$ and $\operatorname{Im} P_{U^{\perp}} = U^{\perp}$.

P_U and $P_{U^{\perp}}$ satisfy three properties:

1. $P_U^2 = P_U$ and $P_{U^{\perp}}^2 = P_{U^{\perp}}$ (idempotence of P_U and $P_{U^{\perp}}$
2. $P_U + P_{U^{\perp}} = \mathrm{Id}$
3. $P_U \circ P_{U^{\perp}} = P_{U^{\perp}} \circ P_U = \mathbf{0}$

Geometrically, we can interpret $P_U(v)$ as the closest vector to v in U; in other words, $d(v, P_U(v)) \leq d(v, u')$ for some $u' \in U$. Furthermore, we can generalize Pythagoras' theorem into the statement $||u + u^{\perp}||^2 = ||u||^2 + ||u^{\perp}||^2$.

We will explore some examples.

Consider $V = \mathbb{R}^2$, equipped with the inner product $((x,y),(x',y')) = xx' + yy'$. Consider the subspace of scalar multiples of the vector $(1,1)$, denoted $\mathbb{R}(1,1)$. Let $v = (x,y)$; find $P_U(v)$.

By theorem, there exists some unique decomposition $(x,y) = u + u^{\perp}$. In particular, since $\dim V = 2$, we have $\dim U^{\perp} = 1$. Furthermore, we have that $U^{\perp}$ is spanned by $(1,-1)$ (since $((1,1),(1,-1)) = 0$ and therefore $(1,1)$ and $(1,-1)$ are complements. Then we have $(x,y) = a(1,1) + b(1,-1)$ and therefore $(a,b) = (\frac{x+y}{2}, \frac{x-y}{2})$. Then $P_U(v) = a(1,1) = (\frac{x+y}{2}, \frac{x+y}{2})$ and $P_{U^{\perp}}(v) = b(1,-1) = (\frac{x-y}{2}, \frac{y-x}{2})$.

Then $P_U(v) + P_{U^{\perp}}(v) = (x,y) = \mathrm{Id}$, $P_U^2 = P_U(\frac{x+y}{2}, \frac{x+y}{2}) = (\frac{x+y}{2}, \frac{x+y}{2})$ (and the same holds for $P_{U^{\perp}}$, and finally $P_{U^{\perp}} \circ P_U(x,y) = (0,0)$. Then all of the properties defined above hold.

Orthogonal and Orthonormal Sets

Our goal now is to create efficient ways of calculating the orthogonal projection.

Consider, as above, a vector space V over the field $\mathbb{R}$ with an inner product $(-,-): V \times V \to \mathbb{R}$. Furthermore, we specify that $\dim V = n$. Consider the set of vectors $S = \{v_1, v_2, ..., v_l\}$ for some $l \leq n$. Then S is **an orthogonal set** if $\forall\, i \neq j\ (v_i, v_j) = 0$ (pairwise orthogonal). Furthermore, for some orthogonal set S, S is **an orthonormal set** if $(v_i, v_i) = ||v_i||^2 = 1$.

In particular, for some orthogonal set $S = \{v_1, ..., v_l\}$, the set $\tilde{S} = \{fracv_1||v_1||, ..., \frac{v_l}{||v_l||}$ is orthonormal.

Let S be an orthogonal set in V; then S is linearly independent in V.

Proof. If $\sum_{i=1}^{l} a_i v_i = \mathbf{0}$, we want to show that for some i_0, $a_{i_0} = 0$. Then we take the inner product of both sides with respect to v_{i_0}; then we get $\left(\sum_{i=1}^{l} a_i v_i, v_{i_0}\right) = \sum_{i=1}^{l} a_i (v_i, v_{i_0}) = a_{i_0}(v_{i_0}, v_{i_0}) = 0$ and since $v_{i_0} \neq \mathbf{0}$, $a_{i_0} = 0$ and the proof is complete. □

Consider the case where $l = n$. Then if $S = \{v_1, ..., v_n\}$ is orthogonal, S is a basis of V (known as an **orthogonal basis**); if S is orthonormal, S is known as an **orthonormal basis**.

Theorem: every finite-dimensional Euclidean vector space admits an orthonormal vector space. (proof to follow)

Consider $V = \mathbb{R}$ with inner product $(x, y) = xy$. Consider the basis $\{1\}$. Since there is only one element, the basis is trivially orthogonal and orthonormal. Then consider the inner product $(x, y) = axy$ for some $a \in \mathbb{R}$. Then the orthogonal basis with respect to this inner product is $\{\frac{1}{\sqrt{a}}\}$.

Now consider the vector space $V = \mathbb{R}^n$ with inner product $((x_1, ..., x_n), (y_1, ..., y_n)) = \sum x_i y_i$. Then the standard orthonormal basis is $\{e_1, ..., e_n\}$.

Consider $V = \mathbb{R}^2$ with the standard inner product. Then consider the vectors $B_\theta = ((\cos\theta, \sin\theta), (\sin\theta, -\cos\theta))$ for some $\theta \in \mathbb{R}$. Then B_θ is an orthonormal basis on $\mathbb{R}^2$.

Finally, consider $V = \mathbb{R}[X]$ for some set X, equipped with the inner product $(f, g) = \sum_{x \in X} f(x)g(x)$. Then the standard basis is the set of delta functions $\{\delta_x : x \in X\}$. This set is also an orthonormal basis.