As usual, we begin by considering a vector space V over the field $\mathbb{R}$, equipped with an inner product $(-,-) : V \times V \to \mathbb{R}$ which is symmetric and positive (and therefore nondegenerate).

Furthermore, we consider the subspace $U \subseteq V$ and its orthogonal complement $U^\perp \subseteq V$, such that for any $v \in V$ there exists a unique decomposition of v into some vectors $u \in U$ and $u^\perp \in U^\perp$ such that $u = P_U(v)$ and $u^\perp = P_{U^\perp}(v)$.

We consider an orthonormal basis of U $v_1, ..., v_r \in U$, with $||v_i||^2 = 1$ and $(v_i, v_j) = 0, i \neq j$. Then $P_U(v) = \sum_{i=1}^r (v, v_i)v_i$.

Finally, we state the main theorem of orthonormal bases – that any vector space admits an orthonormal basis. This is proven constructively by starting with some basis of V, $v_1, ..., v_n$, and using the Gram-Schmidt Orthogonalization Procedure to end up with some orthonormal basis $u_1, ..., u_n$, where $u_i = \frac{P_{V_{i-1}^\perp}(v_i)}{||P_{V_{i-1}^\perp}||}$ and $V_i = \mathbb{R}\langle v_1, ..., v_i \rangle$ being the vector space spanned by the first i vectors in the basis of V. Then

$$P_{V_{i-1}^\perp}(v_i) = v_i - \sum_{j=1}^{i-1} (v_i, u_j)u_j.$$

Example. *Consider $V = \mathbb{R}^3$ equipped with the standard inner product. Let $U = \mathbb{R}\langle (1,1,0), (1,0,1) \rangle$. Then the orthogonal projection of $x = (x_1, x_2, x_3)$ onto the plane spanned by U is given by the formula $P_U(v) = \sum_{i=1}^r (v, v_i)v_i$, so we need an orthonomral basis. The basis of U is $\{(1,1,0), (1,0,1)\}$, so $u_1 = \frac{1}{||v_1||}v_1 = \frac{1}{\sqrt{2}}(1,1,0)$, and $\tilde{v}_2 = v_2 - (v_2, u_1)u_1 = (1,0,1) - \frac{1}{2}(1,1,0) = \frac{1}{2}(1,-1,2)$ and therefore $u_2 = \frac{\sqrt{2}}{\sqrt{3}}(\frac{1}{2}, -\frac{1}{2}, 1) = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$. Then $P_U(v) = (x, u_1)u_1 + (x, u_2)u_2 = \frac{x_1+x_2}{2}(1,1,0) + \left(\frac{x_1}{\sqrt{6}} - \frac{x_2}{\sqrt{6}} + \frac{\sqrt{2}x_3}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$.*

The General Determinant

We now present a generalized definition of the determinant.

Consider a vector space V over a field F, with $\dim V = n$. Then for some $T : V \to V$ and some basis $(v_1, ..., v_n)$. Then there is some matrix $A = M_T^B$.

Now consider two bases $B_1 = (v_1, ..., v_n)$ and $B_2 = (v_1', ..., v_n')$. Then $\det M_T^{B_1} = \det M_T^{B_2}$, since for some U $A_2 = U \cdot A_1 \cdot U^{-1}$ and

$$\begin{aligned} \det A_2 = \det U \cdot A_1 \cdot U^{-1} \\ &= \det U \det A_1 \det U^{-1} \\ &= \det U (\det U)^{-1} \det A_1 \\ &= \det A_1 \end{aligned}$$

Then we wish to compute the basis of a linear transformation with no specification of basis, since the determinant with respect to any basis is equal.

Consider a vector space V over a field F with $\dim V = n$. Let r be a natural number. Then an **r-form** on V is a map

$$b : V^r \to F$$

which is linear in each coordinate; then $B(v_1, ..., v_{i-1}, av_i' + bv_i'', v_{i+1}, ..., v_r)$ with fixed $v_r, r \neq i$ is a linear transformation in the ith vector:

$$B = aB(v_1, ..., v_{i-1}, v_i', ..., v_r) + bB(v_1, ..., v_{i-1}, v_i'', ..., v_r)$$

.

An r-form is **symmetric** if

$$\forall\, v_1, ..., v_r, \sigma \in \Sigma_r : B(v_1, ..., v_r) = B(v_{\sigma(1)}, ..., v_{\sigma(r)})$$

where $\sigma : \{v_1, ..., v_r\} \to \{v_1, ..., v_r\}$ is permutation of $v_1, ..., v_r$. An r-form is **antisymmetric** if

$$\forall\, v_1, ..., v_r, \sigma \in \Sigma_r : B(v_{\sigma(1)}, ..., v_{\sigma(r)}) = \operatorname{sgn}(\sigma) B(v_1, ..., v_r)$$

where $\operatorname{sgn}(\sigma)$ is the signature map of σ.

Let $M_r(V) = \{B : V^r \to F : B \text{ is an } r\text{-form}\}$, with $S_r(V)$ being the set of all symmetric r-forms and $A_r(V)$ the set of all antisymmetric r-forms. All of these sets are vector spaces, with $(B_1 + B_2)(v_1, ..., v_r) = B_1(v_1, ..., v_r) + B_2(v_1, ..., v_r)$ and $(\lambda \cdot B)(v_1, ..., v_r) = \lambda \cdot B(v_1, ..., v_r)$; then the triple $(M_r(V), +, \alpha)$ with S_r, A_r subspaces. $\dim M_r(V) = n^r$, and the dimensions of S_r and A_r are left as exercises.

Consider $r = n$. Then

$$\dim A_n(V) = 1.$$