LINEAR TRANSFORMATIONS

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Let V, W be two vector spaces over a field F.

Definition 1. *A* **linear transformation** *is a mapping $T : V \to W$ such that for $u, v \in V$ and $\lambda \in F$,*

$$T(u+v) = T(u) + T(v) \qquad\qquad T(\mathbf{0}) = \mathbf{0}$$
$$T(\lambda u) = \lambda T(u) \qquad\qquad T(-u) = -T(u)$$

Let $T : F^n \to F^m$ be a linear transformation from an n-dimensional field to itself. Then T can be represented as an m-dimensional vector of linear transformations from F^n to F:

$$T = (T_1, T_2, ..., T_m), T_i : F^n \to F.$$

Further, each T_i is a linear transformation:

$$T_i(x_1, x_2, ..., x_n) = a_1^i x_1 + a_2^i x_2 + ... + a_n^i x_n.$$

Then any linear transformation can be represented as an $n \times m$ matrix:

$$T = \begin{bmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \dots & a_n^m \end{bmatrix}$$

Example. *Let $T(x, y) = (x + y, x - y)$ be a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$. Then*

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Example. *Let X, Y be finite sets and $\varphi : X \to Y$ be a map from X to Y. Let $F(X), F(Y)$ be the field of F-valued functions on X and Y, respectively, and let $\varphi^* : F(Y) \to F(X)$ be a mapping of fields such that $\varphi^*(f) = f \circ \varphi$. Then φ^* is a linear transformation.*

Proof.

$$\begin{aligned}
\varphi^*(f+g)(x) &= (f+g)(\varphi(x)) \\
&= f(\varphi(x)) + g(\varphi(x)) \\
&= \varphi^*(f)(x) + \varphi^*(g)(x) \\
\varphi^*(\lambda f) &= (\lambda f)(\varphi(x)) \\
&= \lambda f(\varphi(x)) \\
&= \lambda \varphi^*(f)(x)
\end{aligned}$$

Therefore, φ^* satisfies both properties of a linear transformation and the proof is complete. □

Operations on Linear Transformations

We define two operations on linear transformations, addition and multiplication by a scalar.

Definition 2. *Let $T, S : V \to W$ be linear transformations from a vector space V to a vector space W over the field F. Then*

$$(T+S)(v) = T(v) + S(v).$$

Definition 3. *Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W over the field F. Then for some $\lambda \in F$,*

$$(\lambda \cdot T)(v) = \lambda \cdot T(v).$$

The following theorem is presented without proof.

Theorem 1. *Let $T, S : V \to W$ and $\lambda \in F$ as above. Then both $T+S$ and $\lambda \cdot T$ are linear transformations.*

Definition 4. *Let V and W be two vector spaces over a field F. Then $\mathrm{Hom}(V,W)$ is the set of linear transformations from V to W.*

Theorem 2. *Let $+ : \mathrm{Hom}(V,W) \times \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$ and $\alpha : F \times \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$ the operations on linear transformations as defined above. Then the triple $(\mathrm{Hom}(V,W), +, \alpha)$ is a vector space over F.*

($\mathrm{Hom}(V,W)$ is also referred to as the tensor product of V and W-dual, $V \bigoplus W^+$.)

Proof. $(\mathrm{Hom}(V,W), +)$ is an abelian group by definition. Let $\mathbf{0} : V \to W$ be the identity function on $\mathrm{Hom}(V,W)$; then $\mathbf{0}_V = \mathbf{0}_W \forall v \in V$. Finally, define the additive inverse as $(-T)(v) = T(-v) = -T(v)$. Then $(\mathrm{Hom}(V,W), +, \alpha)$ is a vector space. □

Note that $\mathrm{Hom}(\mathrm{Hom}(V,W),V)$ is a vector space, as is $\mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V,W),V),W)$, etc. This is known as the *self-reducibility of linear algebra.*

Let $T, S : F^n \to F^m$ be linear transformations. Then, as defined above, $T \leftrightarrow \begin{bmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{bmatrix}$ and $S \leftrightarrow \begin{bmatrix} b_1^1 & \dots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^m & \dots & b_n^m \end{bmatrix}$. Then $T + S \leftrightarrow \begin{bmatrix} c_1^1 & \dots & c_n^1 \\ \vdots & \ddots & \vdots \\ c_1^m & \dots & c_n^m \end{bmatrix}$, where $c_i^j = a_i^j + b_i^j$.

Definition 5. *Let* $\mathrm{Mat}_{m\times n}(F)$ *be the set of all* $m \times n$ *matrices with elements in* F*. Let* $+ : \mathrm{Mat}(F) \times \mathrm{Mat}(F) \to \mathrm{Mat}(F)$ *and* $\alpha : F \times \mathrm{Mat}(F) \to \mathrm{Mat}(F)$*;* $(\mathrm{Mat}_{m\times n}(F), +, \alpha)$ *is a vector space.*

With this definition, the following theorem is presented without proof.

Theorem 3. *Let* $\mathrm{Hom}(F^n, F^m)$ *be the set of linear transformations from* F^n *to* F^m*, and let* $\mathrm{Mat}_{m\times n}(F)$ *be the set of* $m \times n$ *matrices with elements in* F*. Then* $\mathrm{Hom}(F^n, F^m) \cong \mathrm{Mat}_{m\times n}(F)$*.*

We now define one final operation on linear transformations, composition of transformations.

Definition 6. *Let* U, V, W *be three vector spaces over a field* F*. Let* $T : U \to V$ *and* $S : V \to W$ *be linear transformations. Then we define* **composition of linear transformations** *as* $S \circ T : U \to W = S(T(u)), u \in U$*.*

Theorem 4. *Let* $S \circ T$ *be the composition of linear transformations* S *and* T*. Then* $S \circ T$ *is a linear transformation.*

Proof.

$$\begin{aligned} (S \circ T)(u_1 + u_2) &= S(T(u_1 + u_2)) \\ &= S(T(u_1) + T(u_2)) \\ &= S(T(u_1)) + S(T(u_2)) \\ (S \circ T)(\lambda \cdot u) &= S(T(\lambda \cdot u)) \\ &= S(\lambda \cdot T(u)) \\ &= \lambda \cdot S(T(u)) \end{aligned}$$

Then $S \circ T$ is a linear transformation and the proof is complete. $\square$

Finally, as above, let $T, S : F^n \to F^n$ be linear transformations. Then, as defined above, $T \leftrightarrow \begin{bmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \dots & a_n^n \end{bmatrix}$ and $S \leftrightarrow \begin{bmatrix} b_1^1 & \dots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^n & \dots & b_n^n \end{bmatrix}$. Then $S \circ T \leftrightarrow \begin{bmatrix} c_1^1 & \dots & c_n^1 \\ \vdots & \ddots & \vdots \\ c_1^n & \dots & c_n^n \end{bmatrix}$, where $c_i^j = b_i^1 a_1^j + b_i^2 a_2^j + ... + b_i^n a_n^j$. Therefore, $S \circ T = TS$. (This is presented without proof.)