SUBSPACES

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Let V be a vector space over a field F.

Definition 1. *A vector space U is a* **sub-vector space** *or* **subspace** *of another vector space V, denoted $U \leq V$, iff the following properties hold:*

$$u_1 + u_2 \in U, u_1, u_2 \in U \qquad \mathbf{0} \in U$$
$$\lambda u \in U, \lambda \in F \qquad u \in U \leftrightarrow u^{-1} \in U$$

This concept is illustrated in the following examples.

Example. *Let X be a set over a field F, and let $Y \subseteq X$. Then let $F(X) = \{f : X \to F\}$ be the space of all functions that map X to F. Then $F_Y(X) = \{f : X \to F | f(y) = 0 \forall\, y \in Y\}$ is a subspace of $F(X)$.*

Example. *Let $\mathbb{R}[X]$ be the set of all real-valued polynomials. Further, for some polynomial $p \in \mathbb{R}[X]$, define the* **degree** *of p as*

$$\deg(p) = \max(\{n \geq 0 | a_n \neq 0\}).$$

Finally, define $\mathbb{R}_{\leq d}[X] = \{p \in \mathbb{R}[X] | \deg(p) \leq d\}$. This is referred to as the **subspace of polynomials**.

Kernel and Image

We begin by defining two sets related to any linear transformation, its kernel and its image.

Definition 2. *Let V, W be spaces over a field F, and let $T : V \to W$ be a linear transformation. Then the* **image of** *T, denoted $\mathrm{Im}(T) \in W$, is defined as*

$$\mathrm{Im}(T) = \{w \in W | \exists\, v \in V : T(v) = w\}.$$

Further, the **kernel of** *T, denoted $\ker(T) \in V$, is defined as*

$$\ker(T) = \{v \in V | T(v) = \mathbf{0}\}.$$

We now state two theorems of kernels and images.

Theorem 1. *Let $T : V \to W$ be a vector space. Then $\mathrm{Im}(T) \leq W$.*

Proof. The proof is left as an exercise to the reader. $\square$

Theorem 2. *Let $T : V \to W$ be a vector space. Then* $\ker(T) \leq V$.

Proof. Let $v_1, v_2 \in \ker(T)$ and $\lambda \in F$. Then

$$\begin{aligned} T(v_1 + v_2) &= T(v_1) + T(v_2) \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0} \\ \therefore v_1 + v_2 &\in \ker(T) \\ T(\lambda v) &= \lambda T(v) \\ &= \lambda \cdot \mathbf{0} \\ &= \mathbf{0} \\ \therefore \lambda v_1 &\in \ker(T) \end{aligned}$$

Therefore, both properties of a subspace are satisfied and the proof is complete. □

Let V be a vector space. We define two canonical subspaces (i.e. subspaces of any arbitrary vector field), along with a corresponding linear transformation.

$$\begin{aligned} \{\mathbf{0}\} &\leq V & V &\leq V \\ T : V &\to V & T : V &\to V \\ \ker(T) &= \{\mathbf{0}\} & \ker(T) &= V \\ T &= \mathrm{Id}_V & T &= \mathbf{0} \end{aligned}$$

A slightly more nuanced example of the kernel and image is presented below. Notice that the definition of the kernel below relies on the definition of the image.

Example. *Let X, Y be sets, and let φ be map from X to Y. As before, define, $\varphi^*(f)(x)$ as a map from $F(Y)$ to $F(X)$, the space of F-valued functions on Y and X, respectively. It has been proven that φ^* is a linear transformation; then the kernel of φ^* is*

$$\ker(\varphi^*) = \{f : Y \to F | f \circ \varphi = 0\} = \{f : Y \to F | \forall\, y \in \mathrm{Im}(\varphi) : f(y) = 0\}.$$

By the example above, this can also be written $F_{\mathrm{Im}(\varphi)}$. The **pullback image**, $\mathrm{Im}(\varphi^*)$, *is left as an exercise to the reader.*

We are now equipped to state two theorems of kernels and images.

Theorem 3. *Let $T : V \to W$ be a linear transformation between two spaces over a field F.*

(1) *T is surjective iff* $\mathrm{Im}(T) = W$.

(2) *T is injective iff* $\ker(T) = \{\mathbf{0}\}$.

Proof. (1) Left as an exercise to the reader.

(2) Let $v_1, v_2 \in V$ and $T(v_1) = T(v_2)$. Then

$$\begin{aligned} T(v_1) - T(v_2) &= \mathbf{0} \\ T(v_1 - v_2) &= \mathbf{0} \\ \rightarrow v_1 - v_2 &\in \ker(T) \\ \rightarrow v_1 - v_2 &= \mathbf{0} \\ v_1 &= v_2. \end{aligned}$$

Then $T(v_1) = T(v_2) \implies v_1 = v_2$ and T is injective.

In the other direction, if T is injective, $\ker(T) = \{\mathbf{0}\}$ by definition, since not more than one element of V can equal 0 by definition of injectivity.

$\square$