EUCLIDEAN GEOMETRY

ROHAN RAMCHAND

Up to this point, we have studied vector spaces over an arbitrary field F. In studies of natural phenomena, however, it becomes necessary to specify this field; the natural specification, or at least the naive specification, is the field $F = \mathbb{R}$. The geometry of this vector space is known as **Euclidean geometry**.

In order to study Euclidean geometry somewhat more specifically, we need to define the notion of "distance". We will begin by defining a particular bilinear form $b : V \times V \to \mathbb{R}$.

Definition 1 (Euclidean Inner Product)**.** *Consider a bilinear form $b : V \times V \to \mathbb{R}$. Then b is an* **Euclidean inner product** *iff it satisfies:*

> **Symmetry:** *The inner product of $\mathbf{x}$ with $\mathbf{y}$ is equal to the inner product of $\mathbf{y}$ with $\mathbf{x}$:*
> $$\forall \mathbf{x}, \mathbf{y} : b(\mathbf{x}, \mathbf{y}) = b(\mathbf{y}, \mathbf{x}).$$
>
> **Positivity:** *The inner product of any vector with itself is positive for $\mathbf{v} \neq \mathbf{0}$ and zero otherwise:*
> $$\forall \mathbf{v} \in V : b(\mathbf{v}, \mathbf{v}) > 0 \vee (b(\mathbf{v}, \mathbf{v}) = 0 \leftrightarrow \mathbf{v} = \mathbf{0}).$$

Euclidean inner products are often denoted as
$$b(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle.$$

The inner product is used extensively in linear algebra and in analytic geometry; later we will study the concept of an orthonormal basis, a special type of basis used to define rectangular coordinate systems. For now, however, we will continue with our definition of distance.

Definition 2 (Norm)**.** *Given an inner product $\langle -, - \rangle$, consider a function $\|\mathbf{x}\| : V \to \mathbb{R}_{\geq 0}$ defined as*
$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$
This function is called the **Euclidean norm** *of $\mathbf{x}$ and satisfies:*

> **Positivity:** *The inner product of a vector $\mathbf{v}$ is positive if $\mathbf{v} \neq \mathbf{0}$ or zero otherwise.*

Proof. This follows directly from positivity of the inner product. □

Homogeneity: *For every* $\mathbf{v} \in V$ *and* $\lambda \in \mathbb{R}$,

$$\|\lambda \mathbf{v}\| = |\lambda| \|\mathbf{v}\|.$$

Triangle Inequality: *For every* $\mathbf{x}, \mathbf{y} \in V$,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Proof. Let $s, t \in \mathbb{R}$ and let $\mathbf{x}, \mathbf{y} \in V$ such that $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$. Consider the inner product

$$\langle t\mathbf{x} + s\mathbf{y}, t\mathbf{x} + s\mathbf{y} \rangle.$$

Then we have

$$\begin{aligned} \langle t\mathbf{x} + s\mathbf{y}, t\mathbf{x} + s\mathbf{y} \rangle &= t^2\|\mathbf{x}\| + s^2\|\mathbf{y}\| + 2st\langle \mathbf{x}, \mathbf{y} \rangle \\ &\geq 0 \end{aligned}$$

Let $s = \frac{1}{\|\mathbf{y}\|}$ and $t = \frac{1}{\|\mathbf{x}\|}$. Then

$$\begin{aligned} t^2\|\mathbf{x}\| + s^2\|\mathbf{y}\| + 2st\langle \mathbf{x}, \mathbf{y} \rangle &= 1 + 1 + \frac{2}{\|\mathbf{x}\|\|\mathbf{y}\|}\langle \mathbf{x}, \mathbf{y} \rangle \\ -1 &\leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} \\ \langle \mathbf{x}, \mathbf{y} \rangle &\geq -\|\mathbf{x}\|\|\mathbf{y}\|. \end{aligned}$$

Now consider $s = \frac{1}{\|\mathbf{y}\|}$ and $t = -\frac{1}{\|\mathbf{x}\|}$. Then we have

$$\begin{aligned} t^2\|\mathbf{x}\| + s^2\|\mathbf{y}\| + 2st\langle \mathbf{x}, \mathbf{y} \rangle &= 1 + 1 - \frac{2}{\|\mathbf{x}\|\|\mathbf{y}\|}\langle \mathbf{x}, \mathbf{y} \rangle \\ 1 &\leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} \\ \langle \mathbf{x}, \mathbf{y} \rangle &\leq \|\mathbf{x}\|\|\mathbf{y}\|. \end{aligned}$$

Then we have $-\|\mathbf{x}\|\|\mathbf{y}\| \leq \langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\|\|\mathbf{y}\|$ and therefore

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|.$$

This inequality is the celebrated **Cauchy-Schwarz inequality** and is a fundamental statement about inner product spaces (i.e. vector spaces equipped with an inner product). The proof of this inequality is important for any introductory student of linear algebra to understand.

Given this inequality, we can now derive the triangle inequality. We will also assume positivity of the inner product and abandon the absolute-value signs. Then

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &\leq \|\mathbf{x}\|\|\mathbf{y}\| \\
\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle &\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| \\
\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle &\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\
\langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle &\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\
\|\mathbf{x} + \mathbf{y}\|^2 &\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\
\rightarrow \|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|
\end{aligned}$$

and the proof is complete. □

The norm of a vector $\mathbf{v}$ is equivalent to the distance from the vector $\mathbf{0}$. We can therefore generalize this function to the distance between *any* two vectors.

Definition 3 (Distance Function)**.** *Given a norm $\|\mathbf{v}\|$ for some $v \in V$, the distance between any two vectors $\mathbf{x}, \mathbf{y} \in V$ as*

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

such that d satisfies:

Positivity: *For any two vectors $\mathbf{x}, \mathbf{y} \in V$,*

$$d(\mathbf{x}, \mathbf{y}) \geq 0$$

with equality iff $\mathbf{x} = \mathbf{y}$.

Proof. This follows directly from positivity of the norm. □

Symmetry: *For any two vectors $\mathbf{x}, \mathbf{y} \in V$*

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}).$$

Proof. This follows directly from symmetry of the inner product. □

Triangle Inequality: *Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. Then*

$$d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$

Proof.

$$\begin{aligned}
d(\mathbf{x}, \mathbf{z}) &= \|\mathbf{x} - \mathbf{z}\| \\
&= \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\| \\
&\leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| \\
&= d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})
\end{aligned}$$

□

We will now explore some examples of the inner product.

Example. *Let $V = \mathbb{R}$ over the field $\mathbb{R}$; in other words, the real line. The standard inner product of V is simply $\langle \mathbf{x}, \mathbf{y} \rangle = xy$ for some $\mathbf{x}, \mathbf{y} \in V$. (The use of vector notation here is simply for consistency; however, $\mathbf{x} = x$ and $\mathbf{y} = y$ for some real numbers x, y.) Then $\|\mathbf{x}\| = |x| = \sqrt{x^2}$.*

Note that for some $a \in \mathbb{R}_{\geq 0}$, the bilinear form $\langle \mathbf{x}, \mathbf{y} \rangle = axy$ is also an inner product; geometrically, this is equivalent to a "rescaling" of the real line by the scaling factor a. For example, for $a = 4$, the norm is $\|\mathbf{x}\| = 2x$ and therefore the distance from, for example, 5 to the origin is not 5 but rather 10.

Example. *Consider the vector space $V = \mathbb{R}^n$ over the field $\mathbb{R}$. Then the canonical inner product is given as $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$, which is just the dot product of two vectors $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$. The norm, furthermore, is $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$.*

Orthogonal Projections

We will now consider the theory of orthogonal projections, a construction that underlies the majority of Euclidean geometry. In particular, we turn our attention to a variety of symmetric bilinear form which obeys the property of non-degeneracy.

Definition 4 (Non-Degeneracy)**.** *Let V be a vector space over a field F, equipped with a symmetric bilinear form $b : V \times V \to F$. Then b is a* **non-degenerate bilinear form** *if*

$$\forall\, \mathbf{u} \in V \;\; b(\mathbf{v}, \mathbf{u}) = 0 \implies \mathbf{v} = \mathbf{0}.$$

This implies that $b(\mathbf{u}, \mathbf{v})$ must satisfy the same conditions by bilinearity. Furthermore, the function $b(\mathbf{v}, -) : V \to F$ is a member of V^*, and if b is nondegenerate, $b(\mathbf{v}, -) = 0 \implies \mathbf{v} = \mathbf{0}$.

Let $U \subseteq V$ and let $U^\perp$ be the orthogonal complement of U. Then we define the orthogonal complement with respect to b as

$$U^{\perp/b} = \{\mathbf{v} \in V \mid \forall\, \mathbf{u} \in U\, b(\mathbf{v}, \mathbf{u}) = 0\} = \{\mathbf{v} \in V \mid b(\mathbf{v}, -) \in U^\perp\}$$

where $U^\perp$ is the standard orthogonal complement.

We now present a fundamental theorem of $U^{\perp/b}$.

Theorem 1. *Let V be a vector space over a field F and let $b(\mathbf{v}, \mathbf{u})$ be a symmetric, non-degenerate bilinear form on V. Then*

$$\dim U + \dim U^{\perp/b} = \dim V.$$

Proof. Recall the fundamental theorem of orthogonal projections:

$$\dim U + \dim U^\perp = \dim V.$$

Using this theorem, all we have to do is show that $\dim U^{\perp} = \dim U^{\perp/b}$.

A keen reader might notice a significant problem here: $U^{\perp/b} \subseteq V$, but $U^{\perp} \subseteq V^*$. We will therefore need an isomorphism $\phi : U^{\perp/b} \to U^{\perp}$, which we will construct. Let $\tilde{b}$ be a linear transformation from V to V^* such that $\tilde{b}(\mathbf{v})(\mathbf{u}) = b(\mathbf{v}, \mathbf{u})$. Then $\tilde{b} = b(\mathbf{v}, -)$. To prove our theorem, we must therefore prove the following statements:

(1) $\tilde{b}$ is a linear transformation.

Proof.

$$\begin{aligned}\tilde{b}(\mathbf{u} + \mathbf{v}) &= b(\mathbf{u} + \mathbf{v}, -) \\ &= b(\mathbf{u}, -) + b(\mathbf{v}, -) \\ &= \tilde{b}(\mathbf{u}) + \tilde{b}(\mathbf{v})\end{aligned}$$

The proof that $\tilde{b}(\lambda\mathbf{v}) = \lambda\tilde{b}(\mathbf{v})$ is left as an exercise. □

(2) $\tilde{b}$ is an isomorphism.

Proof. Assume $\tilde{b}$ is non-degenerate. Since $\tilde{b} : V \to V^*$ and $\dim V = \dim V^*$, it is sufficient to prove that $\tilde{b}$ is an injection to prove bijectivity. Then let $\mathbf{v} \in V$ such that $\tilde{b}(\mathbf{v}) = b(\mathbf{v}, -) = 0$. Then by non-degeneracy, $\mathbf{v} = \mathbf{0}$ and therefore $\ker \tilde{b} = \{\mathbf{0}\}$; then $\tilde{b}$ is an injection and therefore an isomorphism.

The other direction (i.e. assuming that $\tilde{b}$ is an isomorphism and proving it's non-degenerate) is left as an exercise. □

(3) $\tilde{b}(U^{\perp/b}) = U^{\perp}$, and therefore $\tilde{b}$ is an isomorphism from $U^{\perp/b}$ to $U^{\perp}$.

Proof. Let $\mathbf{v} \in U^{\perp/b}$. Then $\tilde{b}(\mathbf{v}) = 0$ and therefore $b(\mathbf{v}, -) = 0$; then $\mathbf{v} \in U$. Then $U^{\perp/b} \subseteq U^{\perp}$.

Since $\tilde{b}$ is an isomorphism, there exists an unique $\mathbf{v}$ such that $\alpha = \tilde{b}(\mathbf{v})$ for every $\alpha \in V^*$. Then, letting $\mathbf{u} \in U$, $b(\mathbf{v}, \mathbf{u}) = \alpha(\mathbf{u}) = 0$; then $\mathbf{v} \in U^{\perp/b}$. Then $U^{\perp} \subseteq U^{\perp/b}$.

Since $U^{\perp/b} \subseteq U^{\perp}$ and $U^{\perp} \subseteq U^{\perp/b}$, we have

$$\tilde{b}(U^{\perp/b}) = U^{\perp}$$

and the proof is complete. □

Since we have that $\tilde{b}$ is an isomorphism, the dimensions of $U^{\perp}$ and $U^{\perp/b}$ must be equal; then the proof is complete.

□

Now that we have this definition, we return to our study of Euclidean geometry. In the proof above, we required that our bilinear transformation b be nondegenerate to prove that it was an isomorphism. While this seems like a particularly restrictive condition, we have by the following theorem that any Euclidean inner product is nondegenerate.

Theorem 2. *Any positive bilinear form $b(\mathbf{v}, \mathbf{u})$ is non-degenerate; in particular, all Euclidean inner products are non-degenerate.*

Proof. Let $\mathbf{v} \in V$ such that for some Euclidean inner product $\langle \cdot, \cdot \rangle$ we have $\langle \mathbf{v}, \mathbf{u} \rangle = 0$. Then by positivity, we have $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \implies \mathbf{v} = \mathbf{0}$ and therefore $\langle \cdot, \cdot \rangle$ is non-degenerate. □

Lemma 1. *Let $U^{\perp/\langle \cdot, \cdot \rangle}$ be the orthogonal complement of U with respect to $\langle \cdot, \cdot \rangle$. Then $U \cap U^{\perp/\langle \cdot, \cdot \rangle} = \{\mathbf{0}\}$.*

Proof. Let $\mathbf{v} \in U \cap U^{\perp/\langle \cdot, \cdot \rangle}$. Then, since $v \in U^{\perp/\langle \cdot, \cdot \rangle}$, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and therefore $\mathbf{v} = \mathbf{v}$; then the proof is complete. □

For the rest of this section, we will denote the orthogonal complement of U with respect to the inner product $\langle \cdot, \cdot \rangle$ as $U^{\perp/\langle \cdot, \cdot \rangle}$. Consider the subspace $U \times U^{\perp/\langle \cdot, \cdot \rangle}$. Let $\psi : U \times U^{\perp/\langle \cdot, \cdot \rangle} \to V$ such that for some $\mathbf{u} \in U$ and $\mathbf{u}^{\perp/\langle \cdot, \cdot \rangle} \in U^{\perp/\langle \cdot, \cdot \rangle}$, $\psi(\mathbf{u}, \mathbf{u}^{\perp/\langle \cdot, \cdot \rangle}) = \mathbf{u} + \mathbf{u}^{\perp/\langle \cdot, \cdot \rangle}$.

Theorem 3. *ψ is an isomorphism.*

Proof. The proof of the linearity of ψ is left as an exercise.

Since $\dim U \times U^{\perp/\langle \cdot, \cdot \rangle} = \dim U + \dim U^{\perp/\langle \cdot, \cdot \rangle} = \dim V$, it is sufficient to show that ψ is injective to show that ψ is a bijection. Then assume $\mathbf{u} + \mathbf{u}^{\perp/\langle \cdot, \cdot \rangle} = \mathbf{0}$; then $\mathbf{u} = -\mathbf{u}^{\perp/\langle \cdot, \cdot \rangle}$, and since $\mathbf{u} \in U$, $\mathbf{u}, \mathbf{u}^{\perp/\langle \cdot, \cdot \rangle} \in U \cap U^{\perp/\langle \cdot, \cdot \rangle}$ and therefore $\mathbf{u} = \mathbf{u}^{\perp/\langle \cdot, \cdot \rangle} = \mathbf{0}$. Then ψ is injective and is therefore an isomorphism. □

Consider the natural projections $P_1 : U \times U^{\perp/\langle \cdot, \cdot \rangle} \to U$ and $P_2 : U \times U^{\perp/\langle \cdot, \cdot \rangle} \to U^{\perp/\langle \cdot, \cdot \rangle}$. Since ψ is an isomorphism, there exists a $\psi^{-1} : V \to U \times U^{\perp/\langle \cdot, \cdot \rangle}$. Then, since $U \subseteq V$, we have $P_1 \circ \psi^{-1} : V \to V$ and $P_2 \circ \psi^{-1} : V \to V$. Then the **orthogonal projection onto** U is defined as $P_U = P_1 \circ \psi^{-1}$ and the **orthogonal projection onto** $U^{\perp/\langle \cdot, \cdot \rangle}$ is defined as $P_{U^{\perp/\langle \cdot, \cdot \rangle}} = P_2 \circ \psi^{-1}$.

Let $\mathbf{v} \in V$. Then

$$\mathbf{v} = P_U(\mathbf{v}) + P_{U^{\perp/\langle \cdot, \cdot \rangle}}(\mathbf{v}),$$

and moreover, since ψ is an isomorphism, this decomposition is unique. Furthermore,

$$P_U + P_{U^{\perp/\langle \cdot, \cdot \rangle}} = \mathrm{Id}$$

and therefore

$$P_U = \mathrm{Id} - P_{U^{\perp/\langle \cdot, \cdot \rangle}}.$$

Finally,

$$P_U^2 = P_U \circ P_U = P_U,$$

and therefore P_U is **idempotent**; the same property holds true for $P_{U^{\perp/\langle\cdot,\cdot\rangle}}$.

We now present the main theorem of this section.

Theorem 4. *Let $d(\mathbf{v}, \mathbf{v'}) = \|\mathbf{v} - \mathbf{v'}\|$ be the distance between two vectors $\mathbf{v}, \mathbf{v'} \in V$. Let $U \subseteq V$, and let $P_U(\mathbf{v})$ be the standard orthogonal projection onto U. Then for all $\mathbf{u'} \in U$,*

$$d(\mathbf{v}, \mathbf{u}) \leq d(\mathbf{v}, \mathbf{u'}),$$

with equality iff $\mathbf{u} = \mathbf{u'}$.

Proof. Let $\mathbf{u} \in U$ and $\mathbf{u}^{\perp/\langle\cdot,\cdot\rangle} \in U^{\perp/\langle\cdot,\cdot\rangle}$. Then

$$\begin{aligned}
\|\mathbf{u} + \mathbf{u}^{\perp/\langle\cdot,\cdot\rangle}\| &= \langle \mathbf{u} + \mathbf{u}^{\perp/\langle\cdot,\cdot\rangle}, \mathbf{u} + \mathbf{u}^{\perp/\langle\cdot,\cdot\rangle} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}^{\perp/\langle\cdot,\cdot\rangle}, \mathbf{u}^{\perp/\langle\cdot,\cdot\rangle} \rangle + \langle \mathbf{u}, \mathbf{u}^{\perp/\langle\cdot,\cdot\rangle} \rangle + \langle \mathbf{u}^{\perp/\langle\cdot,\cdot\rangle}, \mathbf{u} \rangle \\
&= \|\mathbf{u}\|^2 + \|\mathbf{u}^{\perp/\langle\cdot,\cdot\rangle}\|^2.
\end{aligned}$$

This last result is the standard Pythagorean theorem. From here, we can prove our theorem by proving that $d(\mathbf{v}, \mathbf{u'})^2 \geq d(\mathbf{v}, \mathbf{u})^2$. Then

$$\begin{aligned}
\|\mathbf{v} - \mathbf{u'}\|^2 &= \|(\mathbf{v} - \mathbf{u}) + (\mathbf{u} - \mathbf{u'})\|^2 \\
&= \|\mathbf{v} - \mathbf{u}\|^2 + \|\mathbf{u} - \mathbf{u'}\|^2 \\
&= d(\mathbf{v}, \mathbf{u}) + \|\mathbf{u} - \mathbf{u'}\|^2,
\end{aligned}$$

and since $\|\mathbf{u} - \mathbf{u'}\|^2 \geq 0$, the proof is complete. □

Orthogonal and Orthonormal Sets

We have considered to this point two types of bases; the standard basis of a vector space V, denoted $\{v_1, ..., v_n\}$, and the dual basis of the corresponding dual space V^*, with basis $\{v_1^*, ..., v_n^*\}$, where each element of the basis of V^* is a linear functional. We will now consider a special type of standard basis (of V, not V^*).

Definition 5 (Orthogonality and Orthonormality)**.** *Let, as above, V be a vector space over the reals of dimension n, equipped with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$. Consider the set of vectors $S = \{\mathbf{v}_1, ..., \mathbf{v}_l\}$ for some $l \leq n$. Then S is said to be* **orthogonal** *with respect to $\langle\cdot,\cdot\rangle$ iff*

$$\forall\, i \neq j \langle \mathbf{v_i}, \mathbf{v_j} \rangle = 0,$$

and furthermore S is **orthonormal** *iff, in addition to being orthogonal,*

$$\forall\, \mathbf{v}_i \in S \|\mathbf{v_i}\|^2 = 1.$$

A consequence of the similarity of orthogonality and orthonormality is that any orthogonal set can be *orthonormalized* by dividing every element by its norm; formally, for some orthogonal set $S = \{v_1, ..., v_n\}$, the set

$$\tilde{S} = \left\{ \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|}, ..., \frac{\mathbf{v_n}}{\|\mathbf{v_n}\|} \right\}$$

is orthonormal.

Theorem 5. *Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be an orthogonal set in an n-dimensional vector space V. Then S is a basis of V.*

Proof. Since $\#S = \dim V$, it is sufficient to prove that S is linearly independent on V to prove that it is a basis of V.

Let $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$; then, taking the inner product of both sides with respect to some $\mathbf{v}_{i_0} \neq \mathbf{0}$, we have

$$\begin{aligned} \langle \mathbf{v}, \mathbf{v_{i_0}} \rangle &= \left\langle \sum_{i=1}^{n} \alpha_i \mathbf{v}_i, \mathbf{v}_{i_0} \right\rangle \\ &= \sum_{i=1}^{n} \alpha_i \langle \mathbf{v_i}, \mathbf{v_{i_0}} \rangle \\ &= \alpha_{i_0} \langle \mathbf{v_{i_0}}, \mathbf{v_{i_0}} \rangle \\ &= \mathbf{0} \end{aligned}$$

and therefore, since $\mathbf{v}_{i_0} \neq \mathbf{0}$, $\alpha_{i_0} = 0$. Then S is linearly independent on V, and therefore S is a basis of V and the proof is complete. □

We will now briefly visit some examples of orthonormal bases.

Example. *Consider the vector space $V = \mathbb{R}$ equipped with the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = xy$, and consider the basis $\{1\}$ (although this is technically a vector, vector notation for elements of $\mathbb{R}$ can be confusing). This basis, consisting of one element of unit length, is trivially orthonormal; however, with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = axy$ for some $a \in \mathbb{R}$, 1 has length a^2, and therefore the standard orthonormal basis is $\left\{ \frac{1}{\sqrt{a}} \right\}$.*

Example. *Consider the real plane $\mathbb{R}^2$, equipped with the standard inner product. Then the basis $B_\theta = \{(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)\}$ is orthonormal with respect to this inner product for all $\theta \in \mathbb{R}$. This basis is known as the* Fourier basis *of $\mathbb{R}^2$, since the Fourier series form of a polynomial uses this basis.*

Example. *Let $V = \mathbb{R}[X]$ for some set X, equipped with the inner product*

$$\langle f, g \rangle = \sum_{x \in X} f(x)g(x).$$

Then the basis $\{\delta_x : x \in X\}$ is orthonormal with respect to this inner product. Interestingly, for continuous functions, the set X has to be uncountably infinite, and therefore $\mathbb{R}[X]$ becomes infinite-dimensional; the study of these spaces, known as Hilbert spaces*, is a popular topic in real analysis.*

We are now prepared to state the main theorem of orthonormal sets.

Theorem 6. *Let V be a vector space over the reals, with $\dim V = n$. Then V admits an orthonormal basis with respect to some inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.*

Proof. We will begin by defining a formula for the orthogonal projection $P_U(\mathbf{v})$.

Let $U \subseteq V$, equipped with the projections $P_U(\mathbf{v}) : V \to V$ and $P_{U^{\perp/\langle \cdot,\cdot \rangle}} : V \to V$, with $\Im P_U = U$ and $\Im P_{U^{\perp/\langle \cdot,\cdot \rangle}} = U^{\perp/\langle \cdot,\cdot \rangle}$. Let $\mathbf{v}_1, ..., \mathbf{v}_r$ be an orthonormal basis of U.

Let $A = \sum_{i=1}^{r} \langle \mathbf{v}, \mathbf{v_i} \rangle \mathbf{v}_i$ for some $v \in V$. We will show that $\mathbf{v} - A \in U^{\perp/\langle \cdot,\cdot \rangle}$, or that $\langle \mathbf{v} - \mathbf{A}, \mathbf{v_i} \rangle = 0$ for all $i \in [1, r]$. Then, for some i_0, we have

$$\begin{aligned}
\langle \mathbf{v} - \mathbf{A}, \mathbf{v_{i_0}} \rangle &= \langle \mathbf{v}, \mathbf{v_{i_0}} \rangle - \langle A, \mathbf{v}_i \rangle \\
&= \langle \mathbf{v}, \mathbf{v_{i_0}} \rangle - \left\langle \sum_{i=1}^{r} \langle \mathbf{v}, \mathbf{v_i} \rangle, \mathbf{v}_{i_0} \right\rangle \\
&= \langle \mathbf{v}, \mathbf{v_{i_0}} \rangle - \sum_{i=1}^{r} \langle \mathbf{v}, \mathbf{v_i} \rangle \langle \mathbf{v_i}, \mathbf{v_{i_0}} \rangle \\
&= \langle \mathbf{v}, \mathbf{v_{i_0}} \rangle - \langle \mathbf{v}, \mathbf{v_{i_0}} \rangle \\
&= 0
\end{aligned}$$

and therefore $\mathbf{v} - A \in U^{\perp/\langle \cdot,\cdot \rangle}$. Then $A = P_U(\mathbf{v})$.

Let $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be a basis of V, and furthermore let $V_r = \mathbb{R}(\mathbf{v}_1, ..., \mathbf{v}_r)$ be the subspace of V spanned by $\{\mathbf{v}_1, ..., \mathbf{v}_r\}$, noting that $\dim V_r = r$.

Consider $V_1 = \mathbb{R}(\mathbf{v}_1)$. Since v_1 is not necessarily of unit length, we will normalize it by considering the complement vector $\mathbf{u_1} = \frac{\mathbf{v}_1}{\|\mathbf{v_1}\|}$. Then $U_1 = \mathbb{R}(\mathbf{u}_1)$ has an orthonormal basis $\{\mathbf{u}_1\}$. Furthermore, since they are spanned by linearly dependent vectors, $V_1 = U_1$ and therefore $\{\mathbf{u}_1\}$ is an orthonormal basis of V_1.

We will now project $\mathbf{v}_2$ onto the orthogonal complement of u_1, and in so doing form a basis for V_2. Let $\tilde{\mathbf{v}}_2 = P_{U^{\perp/\langle \cdot,\cdot \rangle}}(\mathbf{v}_2) = \mathbf{v}_2 - \langle \mathbf{v_2}, \mathbf{u_1} \rangle \mathbf{u}_1$, and let $\mathbf{u}_2 = \frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|}$; then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set, and furthermore a basis of V_2.

Continuing in this manner, let $i \leq n$, and let $\{\mathbf{u}_1, ..., \mathbf{u}_{i-1}\}$ be an orthonormal basis of V_{i-1}. Then

$$\tilde{\mathbf{v}}_i = \mathbf{v}_i - \sum_{j=1}^{i-1} \langle \mathbf{v_i}, \mathbf{u_{i-1}} \rangle \mathbf{u}_{i-1}$$

and

$$u_i = \frac{\tilde{\mathbf{v}}_i}{\|\tilde{\mathbf{v}}_\mathbf{i}\|}.$$

This recursive procedure is known as the **Gram-Schmidt Orthogonalization Procedure**, and can always generate an orthonormal basis of a vector space V from any basis of V.

We have therefore constructively proven our theorem, since by an earlier theorem every vector space admits a basis. $\square$

We will briefly examine some examples of this procedure.

Example. *Let $V = \mathbb{R}^2$ with the standard inner product. Consider the basis $\{(1,0),(1,1)\}$ (the proof that this is a basis is an exercise for the reader). Since $\|(1,0)\| = 1$, $\mathbf{u}_1 = (1,0)$. Then*

$$\begin{aligned} \tilde{\mathbf{v}}_2 &= \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\mathbf{u}_1 \\ &= (1,1) - 1 \cdot (1,0) \\ &= (0,1) \end{aligned}$$

and therefore $\{(1,0),(0,1)\}$ is an orthonormal basis of $\mathbb{R}^2$.

Now consider the basis $\{(1,1),(1,0)\}$. Then $\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1,1)$ and therefore

$$\begin{aligned} \tilde{\mathbf{v}}_2 &= \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\mathbf{u}_1 \\ &= (1,0) - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(1,1) \\ &= \left(\frac{1}{2}, -\frac{1}{2}\right). \end{aligned}$$

Then $\left\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{2}(1,-1)\right\}$ is an orthonormal basis of $\mathbb{R}^2$.

Example. *Let $V = \mathbb{R}^3$ with the standard inner product. Consider the basis $\{(1,0,0),(1,0,1),(1,1,0)\}$. Then $\mathbf{u}_1 = (1,0,0)$, $\tilde{\mathbf{v}}_2 = (1,0,1) - (1,0,0) = (0,0,1) = \mathbf{u}_2$ and $\tilde{\mathbf{v}}_3 = \mathbf{v}_3 - \langle \mathbf{v_3}, \mathbf{u_2} \rangle \mathbf{u}_2 - \langle \mathbf{v_3}, \mathbf{u_1} \rangle \mathbf{u}_1 = (0,1,0)$. Then $\{(1,0,0),(0,0,1),(0,1,0)\}$ is an orthonormal basis of $\mathbb{R}^3$.*