PARTITIONS AND EQUIVALENCE RELATIONS

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Definition 1. *A* **partition** *of a set X is a set*

$$P = \{C_i \subseteq X \,|i \in I\}$$

such that

$$\bigcup_{i \in I} C_i = X \qquad \textit{(covering property)}$$

$$\forall i \neq s \; C_i \cap C_s = \varnothing \qquad \textit{(mutual disjointness)}$$

In essence, a set is completely divided into *mutually disjoint* partitions – no two partitions share any elements. Moreover, there is no element of X that is not contained in one of its partitions.

The concept of a partition is illustrated by the following examples.

Example. *Let $X = \{1, 2, 3\}$. The set $P_1 = \{\{1\}, \{2, 3\}\}$ is a valid partitioning of X, since $\{1\}$ and $\{2, 3\}$ share no elements; moreover, every element of X is contained in P_1.*

Let $P_2 = \{\{1\}, \{2\}\}$. P_2 is not a valid partitioning of X; even though $\{1\}$ and $\{2\}$ share no elements and P_2 is therefore mutually disjoint, P_2 does not contain the element 3 and is therefore not covering.

Let $P_3 = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. P_3 covers X, unlike P_2, but is not mutually disjoint; therefore, P_3 is not a valid partitioning of X.

There are two special partitions of any set X. The first is the **minimal partition**:

$$P_{\min} = \{X\}.$$

The second is the **maximal partition**:

$$P_{\max} = \{\{x\} \,|\, x \in X\}.$$

These are valid partitions for any set.

Example. *Let $X = \{1, 2, 3\}$. Then*

$$P_{min} = \{\{1, 2, 3\}\}$$

and

$$P_{max} = \{\{1\}, \{2\}, \{3\}\}.$$

Partitions of $\mathbb{Z}$

Let $n \in \mathbb{N}$ for some $n \geq 2$. Let $P_n = \{C_0, C_1, ..., C_{n-1}\}$, where

$$C_r = \{a \in \mathbb{Z} \mid \underbrace{n|(a-r)}_{n \text{ divides } a-r}\}.$$

Example. *Let $n = 3$. Then*

$$C_0 = \{3k \mid k \in \mathbb{Z}\} = \{0, 3, 6, ...\}$$
$$C_1 = \{3k+1 \mid k \in \mathbb{Z}\} = \{1, 4, 7, ...\}$$
$$C_2 = \{3k+2 \mid k \in \mathbb{Z}\} = \{2, 5, 8, ...\}$$

Note that $C_0 \cup C_1 \cup C_2 = \mathbb{Z}$ and $C_0 \cap C_1 = C_1 \cap C_2 = C_0 \cap C_2 = \varnothing$.

We now state the following theorems of P_n.

Theorem 1. *Let C_r be defined as above. Then*

$$\bigcup_{r=0}^{n-1} C_r = \mathbb{Z}$$
$$\forall r_1 \neq r_2 \; C_{r_1} \cap C_{r_2} = \varnothing$$

Then P_n is a valid partitioning of $\mathbb{Z}$.

The proof of this theorem is left as an exercise to the reader.

Equivalence Relations

Definition 2. *Let X be a set. Then a* **relation** *on X is a subset*

$$R \subseteq X \times X.$$

Let $a, b \in X$ and let $(a, b) \in R$. Then a is related to b via R; this is denoted

$$aRb.$$

Relations are somewhat general, and don't say very much about sets; therefore, we introduce the concept of the equivalence relation, which is a slightly more specifically-defined relation.

Definition 3. *Let X be a set and let $R \subseteq X \times X$. Then R is an* **equivalence relation** *on X if it satisfies the following properties.*

Reflexivity: *For every $x \in X$,*

$$(x, x) \in R.$$

Symmetry: *Let $a, b \in X$. Then*

$$(a, b) \in R \Leftrightarrow (b, a) \in R.$$

Transitivity: *Let $a, b, c \in X$. Then*

$$(a, b) \in R \wedge (b, c) \in R \Leftrightarrow (a, c) \in R.$$

If R is an equivalence class, $a \sim b$ is a more common way of denoting $(a, b) \in R$ and will be used from here on.

The most famous example of an equivalence relation on practically any set is equality; the proof is trivial and relies more on definition than any actual algebra. (We will revisit equality later.)

Equivalence relations are demonstrated in the following example.

Example. *Let $X = \{1, 2, 3\}$.*

Consider the set $R_1 = \varnothing$. Since there are no elements in R_1, reflexivity fails and therefore R_1 is not an equivalence relation.

Consider the set $R_2 = \{(1, 1), (1, 2)\}$. Although $1 \sim 1$, $2 \neg \sim 2$ and theref¡F5¿ore reflexivity fails. (Symmetry and transitivity also fail, but it is not necessary to show this.) Then R_2 is not an equivalence relation.

Consider the set $R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Every element is related to itself, and therefore R_3 satisfies reflexivity. Since both $1 \sim 2$ and $2 \sim 1$, R_3 satisfies symmetry. (Note that it is not *a requirement that 3 be related to anything other than itself; however, if it is, it must be symmetrically related.) There are no transitive relations in R_3, but again, this is not necessary; however, if for example, $(1, 3) \in R_3$, then $(2, 3)$ (and $(3, 1)$) should also be in R_3. Therefore, all three conditions are satisfied and R_3 is an equivalence relation.*

As with most other structures previously explored, there are two canonical equivalence relations for any set X.

Definition 4. *Let X be a set. Then the* **maximal equivalence relation** *is the set*

$$R = X \times X.$$

Definition 5. *Let X be a set. Then the* **minimal equivalence relation** *is the set*

$$R = \{(x, x) \mid x \in X\}.$$

This relation is also referred to as equality *and is denoted in set form by Δ.*

We now return to the divisibility partition above. Recall that $n|k$ is shortform for "n divides k". Then consider the equivalence relation

$$\sim_N = \mathbb{Z} \times \mathbb{Z}$$

for some $N \in \mathbb{N}^{\geq 2}$. Let $a, b \in \mathbb{Z}$; then

$$a \sim_N b \Leftrightarrow n|(b-a).$$

We will now prove that $\sim_N$ is an equivalence relation.

Proof. **Reflexivity:** Let $a \in \mathbb{Z}$:

$$a \sim_N a \Leftrightarrow n|0$$

Since n always divides 0, $\sim_N$ satisfies reflexivity.

Symmetry: Let $a, b \in \mathbb{Z}$:

$$\begin{aligned} a \sim_N b &\Leftrightarrow n|(b-a) \\ &\Leftrightarrow n|(a-b) \\ &\Leftrightarrow b \sim_N a \end{aligned}$$

Then $\sim_N$ satisfies symmetry.

Transitivity: Let $a, b, c \in \mathbb{Z}$ and let $a \sim_N b$ and $b \sim_N c$:

$$\begin{aligned} a \sim_N b \Leftrightarrow n|(b-a) &\qquad \rightarrow b-a = k_1 n \\ b \sim_N c \Leftrightarrow n|(c-b) &\qquad \rightarrow c-b = k_2 n \\ &\qquad c-a = (k_1+k_2)n \\ &\qquad \rightarrow a \sim_N c \end{aligned}$$

Then $\sim_N$ satisfies transivity.

Therefore, $\sim_N$ is an equivalence relation. $\square$