FUNCTIONS

ROHAN RAMCHAND, MICHAEL MIYAGI

In the previous section, we defined the notion of operations on sets, such as union and intersection. In this section, we formally define the notion of an operation on the elements of a set.

Definition 1. *Let A and B be sets. Then a* **function** *is a triple (F, A, B) such that*

$$F \subseteq A \times B \,|\, \forall a \in A \, \exists! \, b \in B : (a, b) \in F.$$

(Here, the operator $\exists!$ means uniquely exists*.) This triple is also denoted*

$$f : A \to B$$

where f is a map from the **domain***, A, to the* **range***, B, such that this map obeys the property of* **determinism***, or that every element in A maps to a unique element of B.*

In particular, if for some $a \in A$ and $b \in B$, $(a, b) \in F$, b **corresponds** to a via F; this is also denoted $b = f(a)$.

Example. *Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$ and let $f : A \to B$ be a map corresponding to the triple (F, A, B).*

> *The set $F = \{(1, a), (2, b), (3, d)\}$ is a function; every element of A has an element that corresponds to it in B via F.*
>
> *The set $F = \{(1, a), (2, b), (3, b)\}$ is a function, even though b has two elements in A that correspond to it.*
>
> *The set $F = \{(1, a), (2, b), (3, b), (1, d)\}$ is not a function; 1 has two elements that it corresponds to and therefore the map is not unique. (F is still a map, but this is a weaker property.)*

We now examine two unique functions, the identity and constant functions.

Definition 2. *Let f be a map from a set A to itself: $f : A \to A$. Then the* **identity function** *is the function that sends every element in a to itself:*

$$\forall a \in A \;\; f(a) = a.$$

This corresponds to the **diagonal set**

$$F = \{(a, a) \,|\, a \in A\}.$$

Definition 3. *Let f be a map from a set A to a set B. Fix a $b \in B$; then the* **constant function** *is the map $f : A \to B$ such that*

$$f(a) = b \quad \forall a \in A.$$

Note that a function is defined not just by its map, but also by its domain and range. This is illustrated by the following example.

Example. *Let $f(x) = x^2$ for all $x \in \mathbb{R}$. This corresponds to the set*

$$F = \{(x, x^2) \,|\, x \in \mathbb{R}\}.$$

This is a map $f : \mathbb{R} \to \mathbb{R}$; this differs from the map from the positive half-plane $F : \mathbb{R}^{\geq 0} \to \mathbb{R}$; this corresponds to the set

$$F = \{(x, x^2) \geq 0 | x \in \mathbb{R}\},$$

which differs from the set above. Therefore, two functions with domains, even though they might have the same map, are completely different.

The notion a function can be extended not just from one argument, as above, but to multiple arguments; for example, the set of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ takes an (ordered) n-tuple and returns another n-tuple. This type of map is usually represented as an ordered n-tuple; for example, maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ are usually represented as an n-tuple of functions

$$F : \mathbb{R}^n \to \mathbb{R}^n : \{(f_1(x_1), ..., f_n(x_n)\}, f_i : \mathbb{R} \to \mathbb{R}.$$

Terminology

We will now define several concepts associated with functions.

Definition 4. *The* **image** *of a set $U \subseteq A$ via function $F : A \to B$ is the set*

$$B \subseteq F(U) = \{b \in B \,|\, \exists\, u \in U : f(u) = b\}.$$

The **preimage** *of a set $V \subseteq B$ via F is the set*

$$A \subseteq F^{-1}(V) = \{a \in A \,|\, F(a) \in V\}.$$

Definition 5. *A function $F : A \to B$ is* **surjective** *if*

$$F(A) = B.$$

In other words, for every value in A there exists a value to which that value is mapped by F in B.

Definition 6. *A function $F : A \to B$ is* **injective** *if*

$$\forall\, a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2.$$

Definition 7. *A function $F : A \to B$ is* **bijective** *iff F is both injective and surjective. Bijective functions are the only functions that admit inverses.*

The three definitions above are worth memorizing; they are to functions what positive and negative are for real numbers. Later in the course, we will use these definitions in proofs of more advanced concepts.

Surjectivity, injectivity, and bijectivity are demonstrated in the following examples.

Example. *Let $f : \mathbb{R} \to \mathbb{R} : f(x) = x^2$. f is neither surjective nor injective; surjectivity fails on the negative numbers (there is no $x \in \mathbb{R}$ such that x^2 is negative, and $f(-2) = f(2)$ even though $-2 \neq 2$.*

Now consider $g : \mathbb{R} \to \mathbb{R}^{\geq 0}$, the mapping from the real plane to the positive half-plane. Now, we can make the statement that

$$\forall\, b \in \mathbb{R}^{\geq 0} \exists\ a \in \mathbb{R} : g(a) = b.$$

This is equivalent to the definition of surjectivity above; therefore, g is surjective.

To make g injective, we must choose a specific domain and range such that no value in the range is mapped to twice. To do this, we divide the domain of g into $\mathbb{R}^{\geq 0}$ and $\mathbb{R}^{<0}$ and construct two functions

$$\begin{cases} g^+(x) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0} : g^+(x) = x^2 \\ g^-(x) : \mathbb{R}^{<0} \to \mathbb{R}^{\geq 0} : g^-(x) = x^2 \end{cases}.$$

Both functions preserve the original mapping, but now no two values in the domain of either function map to the same value. In other words, if the domain is D,

$$\forall a_1, a_2 \in D f(a) = f(b) \implies a = b,$$

which is the definition of injectivity. Therefore, g^+ and g^- are bijective.

Therefore, by carefully selecting our domain and range, we have constructed a bijective function using the same mapping that, at first glance with a naively selected domain and range, is neither surjective nor injective.

Operations on Functions

Definition 8. *Let $F : A \to B$ and $G : B \to C$. Then the* **composition of F and G** *is the map*

$$G \circ F : A \to C$$

such that

$$(G \circ F)(a) = G(F(a)), a \in A.$$

Composition of functions is one of the most important concepts in this course; we will study it especially closely when discussing certain types of groups.

Theorem 1 (Associativity of Composition)**.** *Let $F : A \to B$, $G : B \to C$ and $H : C \to D$ be functions. Then*

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

Proof. Let $a \in A$. Then

$$\begin{aligned} H \circ G(F(a)) &= H(G(F(a))) \\ &= H(G \circ F(a)) \\ &= (H \circ (G \circ F))(a) \end{aligned}$$

and the proof is complete. □

For a set X, let the set $F(X)$ be the set of all functions from X to itself:

$$F(X) = \{F : X \to X\}.$$

(Later in the course this notation will be reused to describe functions over a field F; one of the more confusing aspects of matrix theory is how often notation is reused.) Then

$$\circ : F(X) \times F(X) \to F(X),$$

which is to say that the composition operator takes the Cartesian product of the set of functions over X and maps it to a single function over X. This type of operator – one that takes two elements of the same set and maps it to the same set – is known as a **binary operator**. (Other binary operators include $+$ and $\times$; these operators will be studied in more depth during our discussion of groups.)

Let $\mathrm{Id} : X \to X$ be the mapping $f(x) = x$. (This is also written $x \mapsto x$, where $x \in X$.) This is known as the **identity function**, and it satisfies the property that

$$\mathrm{Id} \circ f = f \circ \mathrm{Id} = f$$

for all $f \in F(X)$.

Finally, let $F : A \to B$ be a bijective mapping. By definition, F admits an inverse $F^{-1} : B \to A$ such that $F^{-1}(b) = a$ is the unique solution to the equation $F(a) = b$. Note that if F is surjective, *an* inverse exists; if F is injective, the solution is unique. Therefore, only bijective functions admit a unique inverse. The main property of inverse function, presented here without proof, is that

$$F^{-1} \circ F = \mathrm{Id}_A$$

and that

$$F \circ F^{-1} = \mathrm{Id}_B,$$

where Id_X is the identity mapping on $F(X)$. This last property therefore implies the following theorem.

Theorem 2. *Let $F : A \to B$ be a function. Then F is bijective iff there exists a function $G : B \to A$ such that*

$$G \circ F = \mathrm{Id}_A,$$

which implies that F is injective, and that

$$F \circ G = \mathrm{Id}_B,$$

which implies that F is surjective. Then $G = F^{-1}$.

The concept of the inverse function is explained in the following examples.

Example. *Let $f : \mathbb{R} \to \mathbb{R}$ be given as $f(x) = 2x + 5$. Then f^{-1} is the solution of $f(x) = y$:*

$$\begin{aligned} 2x + 5 &= y \\ 2x &= y - 5 \\ x &= \frac{y-5}{2} \end{aligned}$$

Then $f^{-1} = \frac{y-5}{2}$; therefore, f is bijective.

Example. *Le $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ be the mapping $f(x) = x^2$. We proved earlier that f itself is not bijective; to induce bijectivity, we define the functions*

$$\begin{cases} f^+ : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0} \\ f^- : \mathbb{R}^{<0} \to \mathbb{R}^{\geq 0} \end{cases}.$$

Both f^+ and f^- are bijective and therefore admit inverses; these are given as

$$\begin{cases} (f^+)^{-1}(y) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0} = \sqrt{y} \\ (f^-)^{-1}(y) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{<0} = -\sqrt{y} \end{cases}.$$

Functions in higher dimensions also admit inverses, and the process for computing them is nearly identical.

Example. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $f(x, y) = (x + y, x - y)$. Then the inverse is the solution to the equation $f(x, y) = (a, b)$:*

$$\begin{aligned} x + y &= a \\ x - y &= b \\ 2x &= a + b \\ \to x &= \frac{a+b}{2} \\ \to y &= \frac{a-b}{2} \end{aligned}$$

Then $f^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right))$.

Although not necessary, it can be verified that $f \circ f^{-1} = \mathrm{Id}_{\mathbb{R}^2}$ and that $f^{-1} \circ f = \mathrm{Id}_{\mathbb{R}^2}$.

It's often difficult to gain intuition about the image of a function just by looking at it. On the other hand, the domain of a function is often clear just by studying the mapping; functions with asymptotes have obvious "holes" in the domain, and so with a little careful study the domain can usually be found fairly quickly. Since inverse functions map the image of a function to its range, the image of a function becomes the domain of its inverse; therefore, it is often easier to compute the inverse and find its domain to find the image of a function. This idea is examined in the following example.

Example. *Let* $F : \mathbb{R} - \{1\} \to \mathbb{R}$ *given by the mapping* $F(x) = \frac{x+1}{x-1}$*. Then the solution to the equation* $F(x) = y$ *is*

$$\begin{aligned}
\frac{x+1}{x-1} &= y \\
x + 1 &= xy - y \\
xy - x &= y + 1 \\
x(y-1) &= y + 1 \\
x &= \frac{y+1}{y-1}
\end{aligned}$$

Then $F^{-1}(y) = \frac{y+1}{y-1}$ *for all* $y \neq 1$*; then the image of* F*, the domain of* F^{-1}*, is* $\mathbb{R} - \{1\}$*. (Here, conveniently,* $F^{-1} = F$*, but this is rarely the case.)*