SETS AND OPERATIONS ON SETS

ROHAN RAMCHAND, MICHAEL MIYAGI

Definition 1. *A* **set** *is a collection of elements.*

The **natural numbers** $0, 1, 2...$ form a set:

$$\mathbb{N} = \{0, 1, 2, ...\}.$$

The **integers** $..., -2, -1, 0, 1, 2...$ form a set:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}.$$

The **rational numbers** are defined using **set-builder notation**:

$$Q = \{\frac{m}{n} | \, m, n \in \mathbb{Z}\}.$$

This statement is equivalent to saying *the rational numbers are the set of numbers of the form* $\frac{m}{n}$*, where* m *and* n *are both in the set of integers.* The set of **real numbers**, $\mathbb{R}$, has no formal definition; it is merely the union of the rational and irrational numbers. (At this point, it serves no purpose to further define the irrational numbers; other texts provide a more concrete explanation.) Finally, the **complex numbers** are also defined using set-builder notation:

$$\mathbb{C} = \{a + bi \, | \, a, b \in \mathbb{R}\}.$$

Finally, consider the set with zero elements. This set, denoted $\varnothing$, is the set consisting of no elements. The set $S_1 = \{\varnothing\}$, however, contains one element, which is the empty set; $S_2 = \{\varnothing, \{\varnothing\}\}$, and through this system the natural numbers can be expressed as increasingly nested empty sets.

We now introduce a few symbols as a notational convenience. Let A be a set and x be an element in A; this can be denoted $x \in A$. If A and B are sets and all elements of A are in B, then A is a **subset** of B; this is denoted $A \subseteq B$, or, in the case where B has elements other than those in A, $A \subset B$. This is equivalent to the logical statement $x \in A \wedge x \in B$. Note that the empty set is a subset of every set; furthermore, the sets introduced above are subsets of each other:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Finally, if A and B are sets, and A and B have exactly the same elements, then A **equals** B: $A = B$. This is equivalent to the statement $A \subseteq B \wedge B \subseteq A$; in fact, it is that relation that must be proved to show that two sets are equal.

We define a few operations on sets as well. Let A and B be sets. Then the **union** of A and B is

$$A \cup B = \{x | x \in A \vee x \in B\}$$

and the **intersection** of A and B is

$$A \cap B = \{x | x \in A \wedge x \in B\}.$$

Let X be a set such that $A \subseteq X$; then the **complement** of A is

$$A^c = \{x | x \in X \wedge x \notin A\}.$$

In particular, for a universe X,

$$X^c = \varnothing \quad \text{and} \quad \varnothing^c = X.$$

The following theorems on operations on sets are given without proof.

Theorem 1 (Distributivity of Sets)**.** *Let X be a universe and let $A, B, C \subseteq X$ be sets. Then*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

and

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Theorem 2 (DeMorgan's Law)**.** *Let X be a universe and let $A, B, C \subseteq X$ be sets. Then*

$$(A \cap B)^c = A^c \cup B^c$$

and

$$(A \cup B)^c = A^c \cap B^c.$$

Cartesian Product

Whereas a set is an unordered collection of elements, an **ordered pair** or **tuple** of elements imposes an order. Ordered pairs are denoted using parentheses instead of braces, such that $(1, 2) \neq (2, 1)$ (but $\{1, 2\} = \{2, 1\}$). We now define one final operation on sets, the **Cartesian product**, as

$$A \times B = \{(a, b) \,|\, a \in A, b \in B\}.$$

Note that the Cartesian product is a **non-commutative** operation; in other words, for sets A, B,

$$A \times B \neq B \times A.$$

(It is worth mentioning now that these sets, although not equal, are **isomorphic**; this is explained further in later sections.) In particular, note that for any set A,

$$\varnothing \times A = \varnothing = A \times \varnothing.$$

Let $A = \mathbb{R}$ and $B = \mathbb{R}$. Then

$$A \times B = \underbrace{\mathbb{R} \times \mathbb{R}}_{\mathbb{R}^2} = \{(x, y) \,|\, x, y \in \mathbb{R}\}.$$

This product is referred to as the **real plane** or the **Cartesian plane**. The product $\mathbb{Z} \times \mathbb{Z}$ also has a special name, the **2-D lattice**, and is studied extensively in later courses.

Cartesian products can be extended to arbitrary dimensions. To do so, we define an n-**tuple** as an extension of the tuple into n dimensions; given $x_1, ..., x_n$, $(x_1, ..., x_n)$ is an ordering of the x_i such that differing orders are unequal. Then for the sets $A_1, ..., A_n$, the Cartesian product of these sets with each other is

$$A_1 \times \ldots \times A_n = \{(a_1, ..., a_n) \,|\, a_i \in A_i\}.$$

Like the Cartesian plane above, the **Cartesian space** is defined as

$$\underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R}}_{\mathbb{R}^3} = \{(x, y, z) \,|\, x, y, z \in \mathbb{R}\}.$$