Let V be a vector space over the field $\mathbb{R}$, equipped with a positive, symmetric inner product $(-,-) : V \times V \to R$.

A set of vectors $v_1, ..., v_j$ is known as an orthogonal set if

$$(v_i, v_j) = \begin{cases} 0 & i \neq j \\ > 0 & i = j \end{cases}$$

and orthonormal if $(v_i, v_i) = 1$. We have shown that any orthogonal set is a linearly independent set; any orthonormal set with n elements is a basis and is known as the orthonormal basis.

We will prove that any Euclidean vector space admits an orthonormal basis.

We will begin by defining a formula for the orthogonal projection. Let $U \subseteq V$, with $P_U : V \to V$ and $P_{U^\perp} : V \to V$, satisfying $\operatorname{Im} P_U = U$ and $\operatorname{Im} P_{U^\perp} = U^\perp$. Let $v_1, ..., v_r \in U$ be an orthonormal basis of U. Then for some $v \in V$,

$$P_U(v) = \sum_{i=1}^{r} (v, v_i) v_i.$$

This is proven below.

Let $A = \sum_{i=1}^{r} (v, v_i) v_i$; then $A \in U$. Then we want to show that $v - A = U^\perp$, which is equivalent to showing that $(v - A, v_i) = 0, i = 1, ..., r$. Then for some i_0, we have

$$\begin{aligned}
(v - A, v_{i_0}) &= (v, v_{i_0}) - (A, v_{i_0}) \\
&= (v, v_{i_0}) - \left(\sum_{i=1}^{r} (v, v_i) v_i, v_{i_0} \right) \\
&= (v, v_{i_0}) - \sum_{i=1}^{r} (v, v_i)(v_i, v_{i_0}) \\
&= (v, v_{i_0}) - (v, v_{i_0}) \\
&= 0
\end{aligned}$$

and therefore $P_U(v) = A$.

We will now prove the main theorem above.

We are, as above, given a vector space V of dimension n over the field $\mathbb{R}$, equipped with an inner product $(-,-)$. Let $v_1, ..., v_n \in V$ be a basis of V. We will denote $V_r = \mathbb{R}\langle v_1, ..., v_r \rangle$, which is the subspace spanned by the first r vectors in the basis. Then $\dim V_r = r$. We will orthonormalize $v_1, ..., v_n$.

Consider $V_1 = \mathbb{R} v_1$. v_1 is not necessarily an orthonormal basis of V_1, since it is not necessarily normalized; then define $u_1 = \frac{1}{||v_1||} v_1$ and $U_1 = \mathbb{R} u_1$. Then u_1 is an orthonormal basis of U_1; further, we claim that $V_1 = U_1$, and therefore u_1 is an orthonormal basis of V_1.

Consider V_2, and let $\tilde{v}_2 = v_2 - P_{U_1}(v_2) = P_{U^\perp}(v_2) = v_2 - (v_2, u_1)u_1$ be the orthogonal projection of v_2 onto the orthogonal projection of U_1. Then $(\tilde{v}_2, u_1) = 0$, and letting $u_2 = \frac{1}{||\tilde{v}_2||}\tilde{v}_2$, we have $U_2 = \mathbb{R}\langle u_1, u_2\rangle$. Then we claim that $V_2 = U_2$, since u_1 and u_2 are linear combinations of v_1 and v_2, and since u_1 and u_2 form an orthonormal basis of U_2, we have constructed an orthonormal basis of V_2.

Finally, consider V_3, and let $\tilde{v}_3 = v_3 - P_{U_2}(v_3) = v_3 - (v_3, u_1)u_1 - (v_3, u_2)u_2$. Then define $u_3 = \frac{1}{||\tilde{v}_3||}\tilde{v}_3$. Then as above, $U_3 = V_3$, and $\{u_1, u_2, u_3\}$ is an orthonormal basis of U_3; then we have an orthonormal basis of V_3.

This process is known as the **Gram-Schmidt Orthonormalization Procedure** for forming an orthonormal basis from a standard basis. The general recursion procedure is as follows: given $u_1, u_2, ..., u_i$ as an orthonormal basis of $U_i = V_i$, then

$$\tilde{v}_{i+1} = v_{i+1} - \sum_{j=1}^{i}(v_{i+1}, u_j)u_j$$

and

$$u_{i+1} = \frac{1}{||u_{i+1}||}u_{i+1}.$$

We will demonstrate this procedure with some examples.

Let $V = \mathbb{R}^2$; consider the vectors $x = (x_1, x_2), y = (y_1, y_2)$ and their inner product $(x, y) = x_1y_1 + x_2y_2$. Consider the basis $(1, 0), (1, 1)$. Then $u_1 = v_1$ as before; $\tilde{v}_2 = v_2 - (v_2, u_1)u_1 = (1, 1) - (1, 0) = (0, 1)$; then an orthonormal basis of $\mathbb{R}^2$ is $\{(1, 0), (0, 1)\}$.

Consider the basis $(1, 1), (1, 0)$. Then $u_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $\tilde{v}_2 = v_2 - (v_2, u_1)u_1 = (1, 0) - \frac{1}{2}(1, 1) = (\frac{1}{2}, -\frac{1}{2})$. Then another orthonormal basis of $\mathbb{R}^2$ is $\frac{1}{\sqrt{2}}\{(1, 1), (1, -1)\}$. It is clear, then, that the Gram-Schmidt Procedure gives different results for different orders of vectors.

Now consider $V = \mathbb{R}^3$, with basis $(1, 0, 0), (1, 0, 1), (1, 1, 0)$. Then $u_1 = v_1$, $\tilde{v}_2 = v_2 - (v_2, u_1)(u_1) = (1, 0, 1) - (1, 0, 0) = (0, 0, 1)$ and $u_2 = (0, 0, 1)$, and $\tilde{v}_3 = v_3 - (v_3, u_2)u_2 - (v_3, u_1)u_1 = (1, 1, 0) - (1, 0, 0) = (0, 1, 0)$ and $u_3 = (0, 1, 0)$. Then an orthonormal basis of $\mathbb{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, which is the standard basis in a different order.