Let V be a vector space with dimension 2 with a linear transformation $T : V \to V$. Furthermore, we will associate $V \to A^2(V)$, the space of antisymmetric bilinear forms on V and $T \to A^2(T) : A^2(V) \to A^2(V)$.

By theorem, since $\dim V = 2$, $\dim A^2(V) = 1$, which implies that $A^2(T)$ is a scalar multiplication by the identity: $A^2(Tv) = \lambda_T \operatorname{Id}$. Furthermore, if $A^2(S) : A^2(V) \to A^2(V)$ and $A^2(T) : A^2(V) \to A^2(V)$, $A^2(S \circ T) = A^2(T) \circ A^2(S)$; then for $\dim V = 2$, $\lambda_{S \circ T} = \lambda_S \cdot \lambda_T$.

We have therefore just created a "recipe"

$$\lambda_{(-)} : \operatorname{Hom}(V, V) \to F$$

such that

$$T \mapsto \lambda_T.$$

This map λ is the **determinant** of T. Note that as above, it satisfies multiplicativity; moreover, this definition is *basis-free* or *invariant.*

Choose a basis of V $\{v_1, v_2\}$. Then

$$v_1^* \wedge v_2^* = v_1^* \bigoplus v_2^* - v_2^* \bigoplus v_1^*$$

is a basis of $A^2(V)$ by theorem. Applying $A^2(T)$ to the basis of $A^2(V)$,

$$A^2(T)(v_1^* \wedge v_2^*) = \lambda_T v_1^* \wedge v_2^*.$$

Then applying the basis of V to both sides of this result,

$$\begin{aligned}
A^2(T)(v_1^* \wedge v_2^*)(v_1, v_2) &= \lambda_T v_1^* \wedge v_2^*(v_1, v_2) \\
&= \lambda_T \\
A^2(T)(v_1^* \wedge v_2^*)(v_1, v_2) &= (v_1^* \wedge v_2^*)(Tv_1, Tv_2) \\
Tv_1 &= av_1 + cv_2 \\
Tv_2 &= bv_1 + dv_2 \\
(v_1^* \wedge v_2^*)(av_1 + cv_2, bv_1 + dv_2) &= abv_1^* \wedge v_2^*(v_1, v_1) + adv_1^* \wedge v_2^*(v_1, v_2) \\
&\quad + cbv_1^* \wedge v_2^*(v_2, v_1) + cdv_1^* \wedge v_2^*(v_2, v_2) \\
&= ad - bc \\
\therefore \lambda_T &= ad - bc
\end{aligned}$$

The Kronecker Product

The **Kronecker product** of a matrix $A \in \operatorname{Mat}_{n \times n}(F)$ is

$$\operatorname{kron}(A, A) = A \bigoplus A = \begin{bmatrix} a_{11}A & a_{12}A & \dots & a_{1n}A \\ a_{21}A & a_{22}A & \dots & a_{2n}A \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}A & a_{n2}A & \dots & a_{nn}A \end{bmatrix}$$

with dimension $(n \times n) \cdot (n \times n)$.

Let V be a vector space with dimension n, let $T : V \to V$ be a linear transformation, and let $v_1, ..., v_n$ be a basis of V. Then $v_1^*, ..., v_n^*$ is the dual basis of V. Denote T by a matrix A, such that $T^* \mapsto A^T$.

Consider the space $\mathrm{Bil}(V)$, the space of bilinear forms on V, and construct a map $\tilde{T} = \mathrm{Bil}(T) : \mathrm{Bil}(V) \to \mathrm{Bil}(V)$. We are going to introduce the **lexicographic order** on $v_1, ..., v_n$, such that $B = (v_1^* \bigoplus v_1^*, ..., v_1^* \bigoplus v_n^*, v_2^* \bigoplus v_1^*, ..., v_2^* \bigoplus v_n^*, ..., v_n^* \bigoplus v_1^*, ..., v_n^* \bigoplus v_n^*)$.

In the original matrix A, note that $a_{ij} = (Tv_j)_i = v_i^*(Tv_j)$. Consider the matrix $\tilde{A}$ corresponding to $\tilde{T}$; then the rows and columsn are indexed by the lexicographic order $(1, 1), ..., (1, n), ..., (n, 1), ..., (n, n)$. Then every element in $\tilde{A}$ is indexed by two pairs (i, j) and (k, l); then

$$\begin{aligned}
\tilde{a}_{(i,j),(k,l)} &= (\tilde{T}(v_k^* \bigoplus v_l^*))_{(i,j)} \\
&= \tilde{T}(v_k^* \bigoplus v_l^*)(v_i, v_j) \\
&= v_k^* \bigoplus v_l^*(Tv_i, Tv_j) \\
&= v_k^*(Tv_i) v_l^*(Tv_j) \\
&= a_{ki} a_{lj}
\end{aligned}$$

This matrix is exactly $A^T \bigoplus A^T$.

For $S^2(V)$, consider the basis $v_i^* \bigodot v_j^*$, $1 \le i \le j \le n$. However, since this space is symmetric, $(1, 2) = (2, 1)$, so there are only $\frac{1}{2}n(n+1)$ elements. Again, every element of $\tilde{A}$ is indexed by two pairs $(i, j), (k, l)$, where $i \le j$ and $k \le l$. Then

$$a_{(i,j),(k,l)} = a_{ki} a_{lj} + a_{li} a_{kj},$$

and if we are considering the antisymmetric space,

$$a_{(i,j),(k,l)} = a_{ki} a_{lj} - a_{li} a_{kj}.$$