Let V be a vector space over a field F. Then the **dual** of V is defined as

$$V^* = \mathrm{Hom}(V, F).$$

For some basis $B = \{v_1, ..., v_n\}$, the **dual basis** is defined as

$$B^* = \{v_1^*, ..., v_n^*\}.$$

Let $T : V \to W$ be a linear transformation. Then the **dual map** is given as

$$T^* : W^* \to V^*$$

where

$$T^*(\alpha) = \alpha \circ T$$

for some $\alpha \in W^*$.

Definition 1. *Let V be a vector space and $U \subseteq V$. Then*

$$U^\perp = \{a \in V^* \mid \alpha(u) = 0 \,\forall\, u \in U\}.$$

Theorem 1.

$$\dim U + \dim U^\perp = \dim V = \dim U^*$$

Theorem 2. *Let $T : V \to W$ be a linear transformation. Then*

$$\ker T^\perp = \mathrm{Im}\, T^*$$

and

$$\mathrm{Im}\, T^\perp = \ker T^*.$$

Then T is injective iff T^* is surjective, and T^* is injective iff T is surjective.

Proof. To prove this theorem, we will rely on the proof that $\mathrm{Im}\, T^* \subseteq \ker T^\perp$ and that $\ker T^* \subseteq \mathrm{Im}\, T^\perp$. (The proof os the latter is left as an exercise.)

By definition, $T : V \to W$ and $T^* : W^* \to V^*$. Let $\alpha \in \mathrm{Im}\, T^*$; we want to show that $\alpha \in \ker T^\perp$. Then

$$a \in \mathrm{Im}\, T^* \to \alpha = T^*(\beta), \beta \in W^*.$$

Let $v \in \ker T$:

$$\begin{aligned} \alpha(v) &= T^*(\beta)(v) \\ &= \beta(T(v)) \\ &= \beta(0) \\ &= 0 \end{aligned}$$

Therefore $\alpha(v) = 0$ for all $v \in \ker T$ and the proof is complete.

(We will assume here that it has been proven that $\ker T^* \subseteq \operatorname{Im} T^{\perp}$.)

Therefore,

$$\dim \operatorname{Im} T^* \leq \dim \ker T^{\perp}$$

and

$$\dim \ker T^* \leq \dim \operatorname{Im} T^{\perp}.$$

Then

$$\begin{aligned}
\dim \operatorname{Im} T^* + \dim \ker T^* &= \dim W^* \\
&\leq \dim \ker T^{\perp} + \dim \operatorname{Im} T^{\perp} \\
&= (\dim V - \dim \ker T) + (\dim W - \dim \operatorname{Im} T) \\
&= \dim V + \dim W - (\dim \ker T + \dim \operatorname{Im} T) \\
&= \dim V + \dim W - \dim W \\
&= \dim W
\end{aligned}$$

Then, since $\dim W^* = \dim W$, $\dim \operatorname{Im} T^* = \dim \ker T^{\perp}$ and $\dim \ker T^* = \dim \operatorname{Im} T^{\perp}$ and the proof is complete. □

Definition 2 (Transpose)**.** *Let $T : V \to V$ be a linear transformation from a vector space V to itself. Let $B = (v_1, ..., v_n)$ be an ordered basis of V. Let $T \cong M_T \in F^{n \times n}$, where*

$$M_T = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}.$$

Let $T^* : V^* \to V^*$ be the dual map, and let $B^* = (v_1^*, ..., v_n^*)$ be the dual basis of B. Then $T^* \cong M_{T^*} \in F^{n \times n}$ as above, where

$$M_{T^*} = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}.$$

Then $a_{ij} = \left(T_{v_j}\right)_i = v_i^*(T_{v_j})$ and $b_{ij} = (T^*(v_j^*))_i = T^*(v_j^*)(v_i) = v_j^*(Tv_i) = a_{ji}$. Therefore,

$$b_{ij} = a_{ji}$$

and M_{T^*} is defined as the **transpose** *of M_T, where*

$$B = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix} = A^T.$$

Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \cong T : V \to V$$

with the columns of A denoted $A_1 ..., A_n$. Then if T is injective, this is equivalent to saying $A_1, ..., A_n$ are linearly independent. If T is surjective, this is equivalent to saying $A_1, ..., A_n$ are spanning.

As above, if $T \cong A$, then $T^* \cong A^T = B$. Then the columns of B are equal to the rows of B. Then, since T is injective iff T^* is surjective, **if the columns of A are linearly independent, the rows of A must be spanning, and if the rows of A are linearly independent, the columns of A are spanning.**