We are now studying a concept known as multilinear algebra.

Bilinear form: $\mathrm{Bil}(V) = \{B : V \times V \to F\}$ is a vector space

$S^2(V)$: symmetric bilinear form (subspace of $\mathrm{Bil}(V)$) and $A^2(V)$: antisymmetric bilinear form

Bases: $v_1, ..., v_n \in V$, then $v_1^*, ..., v_n^* \in V^*$; then

$$v_i^* \bigoplus v_j^*(u, v) = v_i^*(u) \cdot v_j^*(v)$$

Consider the set

$$B = \{v_i^* \bigoplus v_j^* : 1 \leq i, j \leq n\}$$

which is a basis for $\mathrm{Bil}(V)$; then the dimension of $\mathrm{Bil}(V)$ is $n^2$.

$S^2(V)$: for $i \leq j$, $v_i^* \bigodot v_j^* = v_i^* \bigoplus v_j^* + v_j^* \bigoplus v_i^* \in S^2(V)$, then

$$B_{S^2} = \{v_i^* \bigodot v_j^* \,|\, 1 \leq i \leq j \leq n\}$$

$A^2(V)$: for $i < j$, $v_i^* \wedge v_j^* = v_i^* \bigoplus v_j^* - v_j^* \bigoplus v_i^* \in A^2(V)$, then

$$B_{A^2} = \{v_i^* \wedge v_j^* \,|\, 1 \leq i < j \leq n\}$$

Theorem: $B_{A^2}$ is a basis of $A^2(V)$.

Proof: Let $B \in A^2(V)$. Then $B(u, v) = -B(v, u)$ and $B(u, u) = 0$. Assume $F \neq \mathbb{F}_2$. Since $B$ is a bilinear form,

$$B = \sum_{i,j=1}^{n} B(v_i, v_j) v_i^* \bigoplus v_j^* = \sum_{i=j} B(v_i, v_i) v_i^* \bigoplus v_i^* + \sum_{i<j} B(v_i, v_j) v_i^* \bigoplus v_j^* + \sum_{j>i} B(v_j, v_i) v_j^* \bigoplus v_i^*$$

Since $B$ is antisymmetric, $B(u, u) = 0$ and $B(v_j, v_i) = -B(v_i, v_j)$; then

$$B = \sum_{i<j} B(v_i, v_j)(v_i^* \bigoplus v_j^* - v_j^* \bigoplus v_i^*$$

and $B_{A^2}$ spans $A^2$.

Assume there exists some $a_{ij}$ such that

$$\sum_{i<j} a_{ij} v_i^* \wedge v_j^* = 0 \in A^2(V).$$

The proof is left as an exercise. (Choose an arbitrary $i_0, j_0$ and check that both sides are 0.) Then the proof is complete.

Note that $\dim V = n$, $\dim \mathrm{Bil}(V) = n^2$, $\dim S^2(V) = \frac{n(n+1)}{2}$, and $\dim A^2(V) = \frac{n(n-1)}{2}$.

# Factoriality

Given a vector space $V/F$, we can create $V \to \mathrm{Bil}(V)$, $V \to S^2(V)$, and $V \to A^2(V)$ that associates $V$ with each vector space.

Let $T : V \to W$ be a linear transformation; then there exists a linear transformation $\mathrm{Bil}(T) : \mathrm{Bil}(W) \to \mathrm{Bil}(V)$ such that

$$\mathrm{Bil}(T) : b \in \mathrm{Bil}(W) \mapsto \mathrm{Bil}(T)(b) \in \mathrm{Bil}(V).$$

Note that

$$\mathrm{Bil}(T)(b)(u, v) = b(T(u), T(v))$$

and therefore

$$\mathrm{Bil}(T)(v) = b \circ (T \times T).$$

Verify: $\mathrm{Bil}(T)(v) \in \mathrm{Bil}(V)$ is a bilinear map (since $T$ is linear).

Show $\mathrm{Bil}(T)$ is a linear transformation.

Verify: if $b \in S^2(W)$, then $\mathrm{Bil}(T)(b) \in S^2(V)$. In particular,

$$\mathrm{Bil}(T) : S^2(W) \to S^2(V)$$

where in this case if $\mathrm{Bil}(T)$ is restricted to $S^2(T)$ it is written as

$$S^2(T) : S^2(W) \to S^2(V).$$

Similarly,

$$A^2(T) : A^2(W) \to A^2(V)$$

.

$$\begin{array}{ccc}
S^2(W) \xrightarrow{S^2(T)} & & S^2(V) \\
\cap & & \cap \\
\mathrm{Bil}(W) \xrightarrow{\mathrm{Bil}(T)} & & \mathrm{Bil}(V) \\
\cup & & \cup \\
A^2(W) \xrightarrow{A^2(T)} & & A^2(V)
\end{array}$$

Theorem:

$$\mathrm{Bil}(S \circ T) = \mathrm{Bil}(T) \circ \mathrm{Bil}(S)$$

for some $T : U \to V$ and $S : V \to W$. Proof: $\mathrm{Bil}(S \circ T)(b) = b \circ (S \circ T \times S \circ T) = b \circ (S \times S) \circ (T \times T) = \mathrm{Bil}(T)(\mathrm{Bil}(S)(b))$.

Note also that

$$S^2(S \circ T) = S^2(T) \circ S^2(S)$$

and

$$A^2(S \circ T) = A^2(T) \circ A^2(S).$$

Proof of the multiplicativity of the determinant for $\dim V = 2$:

Invariant definition of the determinant (does not depend on basis):

Let $T : V \to V$. We wish to associate some scalar $\lambda_T \in F$.

Consider $A^2(V)$. Then $\dim A^2(V) = 1$ since $\dim V = 2$. By factoriality, we have $A^2(T) : A^2(V) \to A^2(V)$. Since $A^2(T)$ is one-dimensional,

$$A^2(T) = \lambda_T \cdot \mathrm{Id}\,.$$

Claim: for some $T : V \to V$ and $S : V \to V$, consider $S \circ T$. Then

$$\lambda_{S \circ T} = \lambda_S \cdot \lambda_T.$$

Proof: let $A^2(S) : A^2(V) \to A^2(V)$ and $A^2(T) : A^2(V) \to A^2(V)$. Then $A^2(S \circ T) = A^2(T) \circ A^2(S) = (\lambda_T \,\mathrm{Id}) \circ (\lambda_S \,\mathrm{Id}) = \lambda_T \cdot \lambda S \,\mathrm{Id}$, and since $A^2(S \circ T) = \lambda_{S \circ T} \,\mathrm{Id}$, $\lambda_{S \circ T} = \lambda_T \cdot \lambda_S$.