1 Richardson extrapolation

Assume that A (i.e. an integral) is approximated by $A(h)$, that depends on a parameter h. h can be the size of an interval. We will assume that

$$A(h) \to A \text{ for } h \to 0,$$

when we neglect numerical rounding errors.

$A(h)$ is said to be a k'th order approximation of A, if it applies that

$$|A(h) - A| \leq c\, h^k \quad \text{for } h \leq h_0 \tag{1}$$

for a $h_0 > 0$ (the error is $O(h^k)$). $A(h)$ is also said to converge to the order k.

We now assume that the approximation $A(h)$ as a function af h has sufficiently many derivatives in $h = 0$. The Taylor's formula gives

$$A(h) = A + a_1 h + a_2 h^2 + \cdots + a_p h^p + O(h^{p+1}) \tag{2}$$

If k is the order of the method, then $a_1 = \cdots = a_{k-1} = 0$, and (2) looks like

$$A(h) = A + a_k h^k + a_m h^m + O(h^{m+1}), \tag{3}$$

where we have enabled that $a_{k+1} = \cdots = a_{m-1} = 0$. In the following it is assumed that (3) applies. It is important that this assumption is verified.

Let's now calculate $A(h)$ for two values of h, h_1 and h_2, where $h_1 > h_2$.

$$\begin{aligned} A(h_1) &= A + a_k h_1^k + a_m h_1^m + O(h_1^{m+1}) \\ A(h_2) &= A + a_k h_2^k + a_m h_2^m + O(h_2^{m+1}) \end{aligned}$$

a_k is eliminated by multiplying the second equation with $(h_1/h_2)^k$ and subtracting. With the notation $\alpha = h_1/h_2$ we get

$$A = \frac{\alpha^k A(h_2) - A(h_1)}{\alpha^k - 1} + a_m \frac{\alpha^m - \alpha^k}{\alpha^k - 1} h_2^m + O(h_2^{m+1}) \tag{4}$$

Thus

$$\begin{aligned} A_R(h_2, h_1) &\equiv \frac{\alpha^k A(h_2) - A(h_1)}{\alpha^k - 1} \\ &= A(h_2) + \frac{A(h_2) - A(h_1)}{\alpha^k - 1}, \quad \alpha = \frac{h_1}{h_2} \end{aligned} \tag{5}$$

is a order m approximation of A. (5) is called the *Richardson extrapolation* to the order k. In most applications $h_1/h_2 = 2$ is chosen.

Often the order of the method is known, and sometimes it is possible to show that (2) or (3) applies. In typical applications, the extrapolation is justified from the results, as we shall see. Actually, extrapolation must always be justified by the results. Even if it is shown theoretically that (3) applies, it can happen that $|a_m| >> |a_k|$ such that the h^m term dominates the error. In that case extrapolation is useless.

To see if extrapolation is justified, we need three values of $A(h)$. Let $h_1 > h_2 > h_3$ and assume that the ratio $h_1/h_2 = h_2/h_3 = \alpha$ is constant. If (3) applies, we have

$$A(h_1) - A(h_2) \approx (\alpha^k - 1)a_k h_2^k$$

and

$$A(h_2) - A(h_3) \approx (\alpha^k - 1)a_k h_3^k,$$

and then

$$\frac{A(h_1) - A(h_2)}{A(h_2) - A(h_3)} \approx \alpha^k \quad \text{for} \quad h_1/h_2 = h_2/h_3 = \alpha. \tag{6}$$

The changes thus decrease with a factor α^k, from which the order of convergence appear. If k is unknown, it takes more such calculations to show that (3) applies, and to determine k. If it is possible $\alpha = 2$ is chosen.

A specific method with application of Richardson extrapolation can be seen as a new method of higher order. I.e. the trapezoidal method together with Richardson extrapolation is equivalent to the method of Simpson.

It is also seen from (5), that Richardson extrapolation can be used as error estimation:

$$A(h_2) - A \approx \frac{A(h_1) - A(h_2)}{\alpha^k - 1}, \quad \alpha = \frac{h_1}{h_2}. \tag{7}$$

The integral

$$\int_0^4 e^x dx = 53.59815003\ldots\ .$$

has been calculated by the trapezoidal method with $h = 4/n$.

n	S_n	k_1	R_n	k_2
1	111.1963000660			
2	70.3762622310		56.7695829526	
4	57.9919498671	1.72	53.8638547459	
8	54.7101530638	1.92	53.6162207960	3.55
16	53.8770167080	1.98	53.5993045895	3.87
32	53.6679211235	1.99	53.5982225953	3.97

Third column shows the experimental order of convergence determined by (6). The values confirm the expected order and the extrapolation can be done to order 2 (exact). As we have $\alpha = 2$, (6) can be used again to calculate the order of R_n in column 4. The expected order (4) is confirmed, and error estimation can be made using (7). The error on R_{32} is estimated to 7×10^{-5}. □

Use the midpoint method, the trapezoidal method and Simpson's method to approximate each of the integrals:

$$\int_0^1 \cos(x^2)\exp(-x)dx$$

$$\int_0^1 \sqrt{x}\cos(x^2)\exp(-x)\,dx$$

$$\int_0^1 1000 \exp(\frac{-1}{x}) \exp(\frac{-1}{1-x})\, dx$$

$$\int_0^1 \frac{1}{\sqrt{x}} \cos(x^2) \exp(-x) dx$$

Use Richardson extrapolation.