Introduction

Saturn has many moons with greatly varying sizes. Two of these moons (Janus and Epimetheus) have slightly different speed and have orbits that are similar to the point where collision would be expected. However whenever the moons approach each other, an exchange of energy occurs. This exchange effectively means that the moons swap orbits, such that the faster moon, which was on the inner path, would slow down and move to the outer path, and vice versa.
This exercise will focus on simulating this phenomenon and illustrate what happens when the moons approach each other.

Simulation means plotting each moon's distance to Saturn as a function of time, as well as plotting the angular difference as a function of time.

Some simplifying assumptions

- The moons only experience the gravity of Saturn and each Other.
- Saturn's position in space is fixed.
- We choose the center of Saturn to be the center of an ordinary coordinate system.

The following notation is introduced.

(x_i, y_i) : Position coordinates for moon i,i=1,2
m_i : Mass of moon i
M : Mass of Saturn
g : Gravitational constant
r_i $\sqrt{(x_i^2 + y_i^2)}$: Moon i's distance from Saturn
r_{12} $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$: Distance between the moons

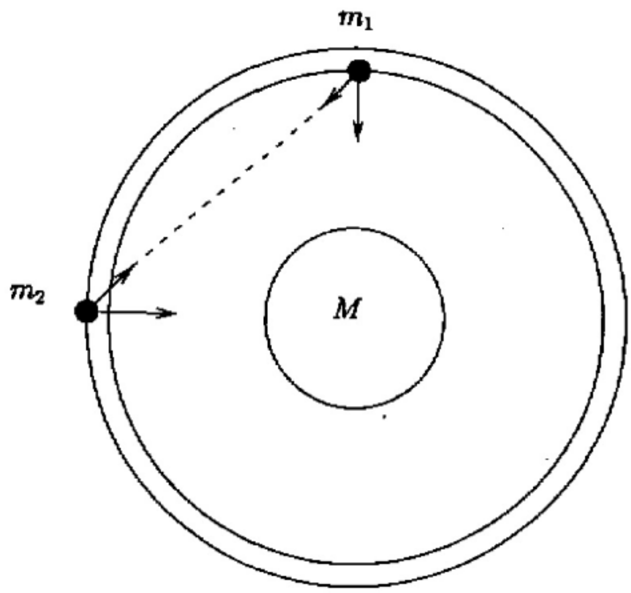

Figure 1: *Illustration of Saturn and the two moons*

The movements of the moons are determined by Newtons second law:

$$\frac{m_1 d^2}{dt^2}\begin{bmatrix}x_1\\y_1\end{bmatrix} = -\frac{m_1 \cdot Mg}{r_1^3}\begin{bmatrix}x_1\\y_1\end{bmatrix} + \frac{m_1 \cdot m_2 g}{r_{12}^3}\begin{bmatrix}x_2 - x_1\\y_2 - y_1\end{bmatrix} \tag{1}$$

$$\frac{m_2 \cdot d^2}{dt^2}\begin{bmatrix}x_2\\y_2\end{bmatrix} = -\frac{m_2 \cdot Mg}{r_1^3}\begin{bmatrix}x_2\\y_2\end{bmatrix} - \frac{m_1 \cdot m_2 g}{r_{12}^3}\begin{bmatrix}x_2 - x_1\\y_2 - y_1\end{bmatrix} \tag{2}$$

Data: Note that the units are kg, km and "$Days\ on\ earth$"

- $m_1 = m_2 = 9.20 \cdot 10^{18} kg$
- $M = 5.68 \cdot 10^{26} kg$
- $g = 4.98 \cdot 10^{-10} \frac{km^3}{kg \cdot days}$

For completeness it should be mentioned that Saturn's radius is approximately 60100 km and the radii of the moons are approximately 130 km.
Initial conditions are fitted to a circular motion $v_i^2 = \frac{Mg}{r_i}$. This yields the following initial conditions.https://www.overleaf.com/project/5cd3fe509230aa0b5c03f6c9

$$\begin{bmatrix}x_1\\y_1\end{bmatrix} = \begin{bmatrix}0\\152870\end{bmatrix}, \frac{d}{dt}\begin{bmatrix}x_1\\y_1\end{bmatrix} = \begin{bmatrix}-1360278.1\\0\end{bmatrix} \tag{3}$$

$$\begin{bmatrix}x_2\\y_2\end{bmatrix} = \begin{bmatrix}0\\-153130\end{bmatrix}, \frac{d}{dt}\begin{bmatrix}x_2\\y_2\end{bmatrix} = \begin{bmatrix}1359122.8\\0\end{bmatrix} \tag{4}$$

- The Simulation should run until the moons have swapped places at least twice
- The required accuracy for position should be about 10 km at all times

For simplicity the data should be presented in the format of polar coordinates,(r_i, φ) and the angular difference $\varphi_1 - \varphi_2$

The exercise revolves around approximating a solution to ordinary differential equations by the use of a numerical method. The chosen method is the fourth order Runge-Kutta method.
The method works by estimating the $y(t_{n+1})$ from the original continious time function by discretized the function into y_n steps. From any given point/step the next step is approximated as the y_{n+1} step. This is done by adding a weighted average of four increments to the y_n step. The Runge-Kutta method is given by the equations shown below.

$$y_{n+1} = y_n + \frac{1}{6} \cdot (k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4) \tag{5}$$

$$x_{n+1} = x_n + h \tag{6}$$

$$k_1 = h \cdot f(x_n, y_n) \tag{7}$$

$$k_2 = h \cdot f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \tag{8}$$

$$k_3 = h \cdot f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \tag{9}$$

$$k_4 = h \cdot f(x_n + h, y_n + k_3) \tag{10}$$

In figure 2 the Runge-Kutta is graphically explaining

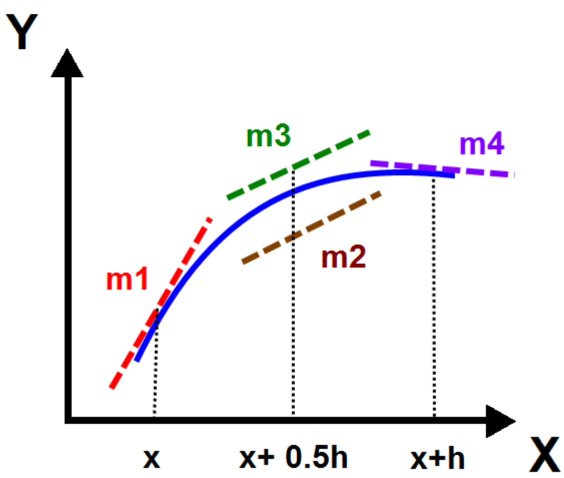

Figure 2: *Illustration of the 4. order Runge-Kutta method*

The goal of the exercise is stated below and:

- Plot the moons distance to Saturn as a function of time
- Plot the angular difference as a function of time

- Determine how close the moons are to each other during their first swap

To start this exercise the equations in 1 and 2 has to be rewritten into first order differential equations.
First the m_1 term in equation 1 is cancelled out which yields a simpler equation.

$$\frac{d^2}{dt^2}\begin{bmatrix}x_1\\y_1\end{bmatrix} = -\frac{Mg}{r_1^3}\begin{bmatrix}x_1\\y_1\end{bmatrix} + \frac{m_2 \cdot g}{r_{12}^3}\begin{bmatrix}x_2 - x_1\\y_2 - y_1\end{bmatrix} \tag{11}$$

Then the m_2 term in equation 2 is cancelled out, also yielding a simpler equation.

$$\frac{d^2}{dt^2}\begin{bmatrix}x_2\\y_2\end{bmatrix} = -\frac{Mg}{r_2^3}\begin{bmatrix}x_2\\y_2\end{bmatrix} - \frac{m_1 \cdot g}{r_{12}^3}\begin{bmatrix}x_2 - x_1\\y_2 - y_1\end{bmatrix} \tag{12}$$

After simplifying, the equations are broken up into 4 different equation as seen in 13 to 16.

$$\frac{d^2x_1}{dt^2} = -\frac{Mg}{r_1^3} \cdot x_1 + \frac{m_2}{r_{12}^3} \cdot (x_2 - x_1) \tag{13}$$

$$\frac{d^2y_1}{dt^2} = -\frac{Mg}{r_1^3} \cdot y_1 + \frac{m_2}{r_{12}^3} \cdot (y_2 - y_1) \tag{14}$$

$$\frac{d^2x_2}{dt^2} = -\frac{Mg}{r_2^3} \cdot x_2 + \frac{m_2}{r_{12}^3} \cdot (x_2 - x_1) \tag{15}$$

$$\frac{d^2y_2}{dt^2} = -\frac{Mg}{r_2^3} \cdot y_2 + \frac{m_2}{r_{12}^3} \cdot (y_2 - y_1) \tag{16}$$

From here on the 4 equation are rewritten into coupled first order differential equations. By doing this we end up with 8 differential equations.

$$z(x_1) = \frac{dx_1}{dt} \tag{17}$$

$$\frac{dz(x_1)}{dt} = \frac{d^2x_1}{dt^2} = -\frac{Mg}{r_1^3} \cdot x_1 + \frac{m_2}{r_{12}^3} \cdot (x_2 - x_1) \tag{18}$$

$$z(y_1) = \frac{dy_1}{dt} \tag{19}$$

$$\frac{dz(y_1)}{dt} = \frac{d^2y_1}{dt^2} = -\frac{Mg}{r_1^3} \cdot y_1 + \frac{m_2}{r_{12}^3} \cdot (y_2 - y_1) \tag{20}$$

$$z(x_2) = \frac{dx_2}{dt} \tag{21}$$

$$\frac{dz(x_2)}{dt} = \frac{d^2x_2}{dt^2} = -\frac{Mg}{r_2^3} \cdot x_2 + \frac{m_2}{r_{12}^3} \cdot (x_2 - x_1) \tag{22}$$

$$z(y_2) = \frac{dy_2}{dt} \tag{23}$$

$$\frac{dz(y_2)}{dt} = \frac{d^2y_2}{dt^2} = -\frac{Mg}{r_2^3} \cdot y_2 + \frac{m_2}{r_{12}^3} \cdot (y_2 - y_1) \tag{24}$$

These equations are feed to the Runge-Kutta method along with the initial values states in equation 3 and 4. Through Runge-Kutta the initial values are updated as the method progresses along the function.
These equations are used to determine the movement of the 2 moons.

In figure 3, it can be seen that the accuracy is below 10Km when the step-size is below 1/70. The minus sign should be ignored.

```
runge@E7440-ARCH:~/Documents/Numerical Methods/Mandatory Exercise 3$ g++ sourceNY.cpp
runge@E7440-ARCH:~/Documents/Numerical Methods/Mandatory Exercise 3$ ./a.out
results 1'th moon       Vector 10D:
    1.14436e+09    6.33196e+07         130481         151619         152640         152809         152847         152862         152866         152868
K order 1'th moon       Vector 10D:
              0              0        4.09659            nan        4.37232        2.59336        2.12999        1.43245        1.79569       0.978161
Rich. error est 1'th moon       Vector 10D:
              0    3.60345e+08     2.1063e+07       -7045.87         -340.2       -56.3711       -12.8785       -4.77151       -1.37435      -0.697659
Rich. extrap 1'th moon  Vector 10D:
              0   -2.97026e+08   -2.09326e+07         158665         152980         152865         152860         152866         152867         152869
Rich. order 1'th moon   Vector 10D:
              0              0              0        3.71044            nan        5.63104        4.56182            nan        3.09617      -0.964251


results 2'nd moon       Vector 10D:
    1.14087e+09    7.94397e+08         131369         151893         152904         153069         153108         153122         153126         153128
K order 2'nd moon       Vector 10D:
              0              0       -1.19686            nan        4.34265        2.61646        2.10066        1.44019         1.8043        0.95977
Rich. error est 2'nd moon       Vector 10D:
              0    1.15493e+08    2.64755e+08       -6841.25       -337.185       -54.9839       -12.8196       -4.72425       -1.35265      -0.695448
Rich. extrap 2'nd moon  Vector 10D:
              0    6.78905e+08   -2.64624e+08         158734         153241         153124         153120         153127         153127         153129
Rich. order 2'nd moon   Vector 10D:
              0              0              0            nan            nan        5.54982         4.9837            nan        3.14647       -1.05816
done
```

Figure 3: *Program output*

Plot the moons distance to Saturn as a function of time

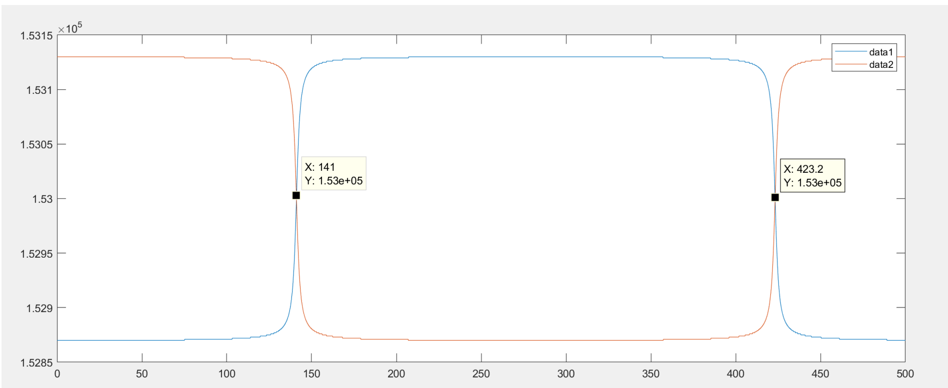

Figure 4: *Both moons distance to Saturn*

Figure 4 is a plot of the moons distance to Saturn over a period of time corresponding to 500 days on earth. The figure shows that the moons will swap 2 times during the time-period. The first swap occurs after 141 days have passed and the second swap occurs when 423 days have passed.

Plot the angular difference as a function of time

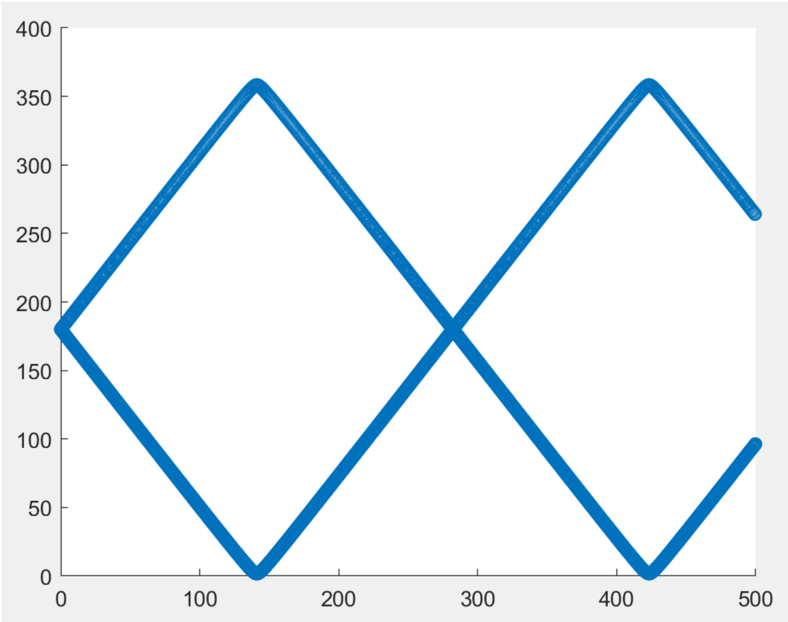

Figure 5: *Angular difference*

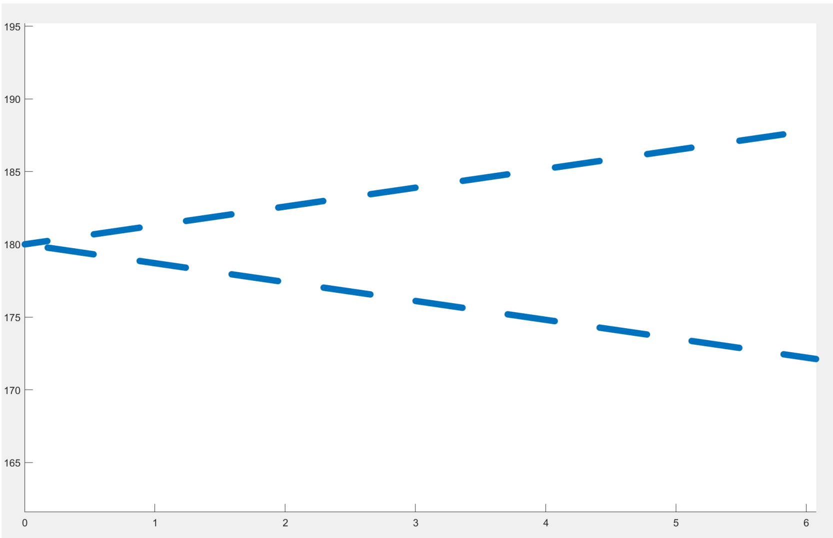

Figure 6: *Close-up angular difference*

Figure 5 show the absolute difference between the two moons angels. From figure 6, which is a close-up view of the start of figure 5, it can be seen that the angle periodically raises and decreases. This is due to the fact that sometimes moon 1 leads moon 2 and sometimes moon 1 lags moon 2.

Determine how close the moons are to each other during their first swap

To determine the distance between the 2 moons during the first swap, a graph of the $r12$ values is created. The graph can be seen in figure 7.

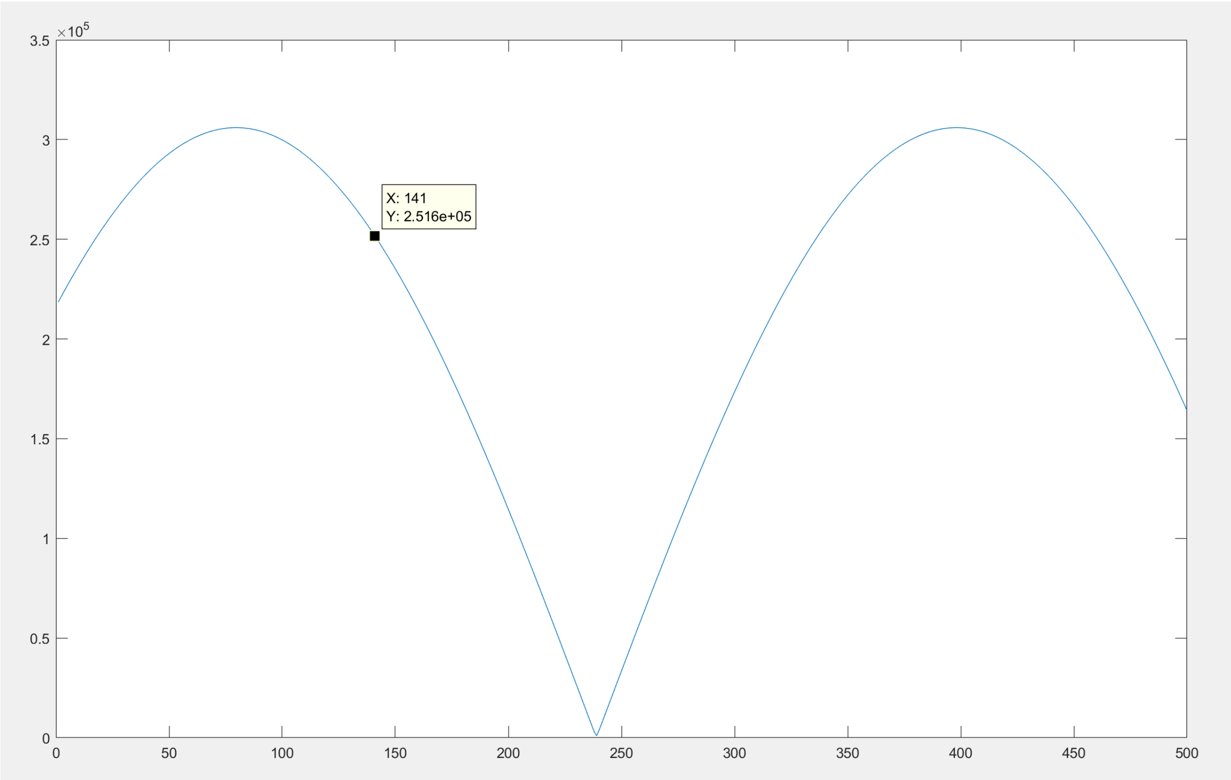

Figure 7: *Distance between moon recorded every passing day.*

The graph in figure 7 plots the r12 value recorded for every passing day. The dot is places at the 141'th day, the day of the first swap, and the distance between the moons, as can be seen, is about 251.600Km.