Introduction

Radiation balance between parallel plates

This exercise is devised on the basis of the second mandatory assignment in the "Numerical Methods" course. The assignment is about radiation between two infinitely long parallel plates with a width of size w and a distance between the plates of length d, se figure 1.

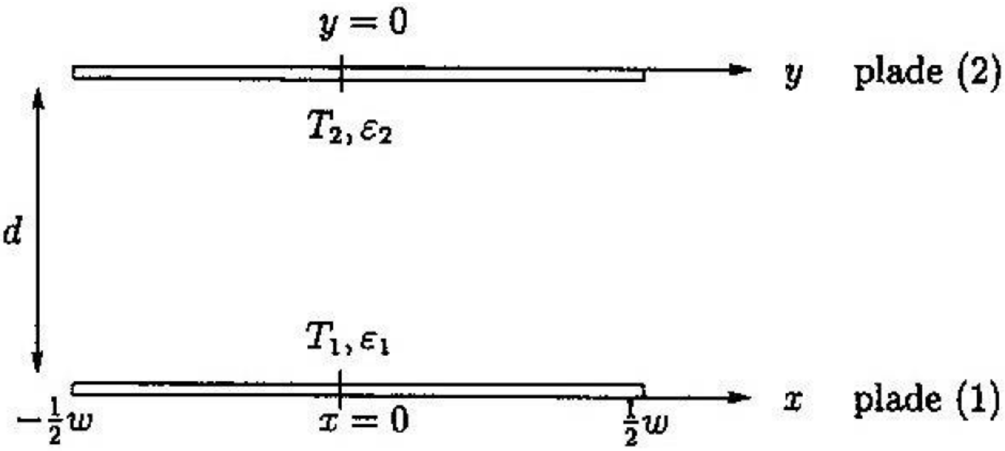

Figure 1: *Parameter relations.*

The plates radiate and reflect diffusely and have uniform temperatures T_1 and T_2 , and emmisivities (ϵ_1) and (ϵ_2) , respectively. Let $u(x)$ denote the radiation per unit time per unit area from plate 1 at the distance x from the mid axis of the plate, and let $v(y)$ be the corresponding function for plate 2. Then u and v are solutions to the coupled integral equations:

$$u(x) = \varepsilon_1 \sigma T_1^4 + (1 - \varepsilon_1) \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) v(y) dy \tag{1}$$

$$v(y) = \varepsilon_2 \sigma T_2^4 + (1 - \varepsilon_2) \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) u(x) dx \tag{2}$$

where σ is the Stefan-Bolzmann constant, and F is given by:

$$F(x, y, d) = \frac{1}{2} \frac{1}{(d^2 + (x - y)^2)^{\frac{3}{2}}} \tag{3}$$

The integrals

$$I_1(x) = \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) v(y) dy \tag{4}$$

$$I_2(y) = \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) u(x) dx \tag{5}$$

describe the incoming radiation per unit time per unit area from the other plate. The total energy (per unit time and unit length), which must be supplied to each plate in order to keep the temperature constant is

$$Q_1 = \int_{-\frac{1}{2}w}^{\frac{1}{2}w} (u(x) - I_1(x))dx \tag{6}$$

$$Q_2 = \int_{-\frac{1}{2}w}^{\frac{1}{2}w} (v(y) - I_2(y))dy \tag{7}$$

Data: $T_1 = 1000$, $T_2 = 500$, $\epsilon_1 = 0.80$, $_2 = 0.60$, $\sigma = 1.712\Delta 109$, $d = 1.00$, $w = 1.00$.

Exercise a

Use the trapezoidal method to write an approximation to (4) and (5). Hint: Introduce $(u_i)_{i=0}^N$ and $(v_i)_{i=0}^N$ *describing the values of* u & v *in the points* $(x_i)_{i=0}^N$ & $(y_i)_{i=0}^N$, *respectively. Then write the discrete problem corresponding to (1) and (2) as a system of linear equations in the variables* $(u_i)_{i=0}^N$ $(v_i)_{i=0}^N$.

The trapezoidal method is given by:

$$\int_{-\frac{1}{2}w}^{\frac{1}{2}w} f(x)dx \approx \frac{1}{2} \cdot h \cdot f(x_0) + \sum_{i=0}^{N-1} f(x_i) \cdot h + \frac{1}{2} \cdot g \cdot f(x_n) \tag{8}$$

Here N is the number of iterations also called steps, and h is the step-size given by $h = \frac{x_n - x_0}{N}$

The trapezoidal approximation to the integrals 4 & 5

$$I(x) \approx \frac{1}{2} \cdot h \cdot F(x, y_0, d) \cdot v_0 + h \cdot \frac{1}{2} \cdot F(x, y_n, d) \cdot v_n + \sum_{i=1}^{N-1} h \cdot F(x, y_i, d) \cdot v_i \tag{9}$$

$$I(y) \approx \frac{1}{2} \cdot h \cdot F(x_0, y, d) \cdot u_0 + h \cdot \frac{1}{2} \cdot F(x_n, y, d) \cdot u_n + \sum_{i=1}^{N-1} h \cdot F(x_i, y, d) \cdot u_i \tag{10}$$

To derive the system of linear equations from equations 1 & 2, we first define 4 constants.

$$c_1 = \varepsilon_1 \cdot \sigma \cdot T_1^4 \tag{11}$$

$$c_2 = \varepsilon_2 \cdot \sigma \cdot T_2^4 \tag{12}$$

$$k_1 = (1 - \varepsilon_1) \tag{13}$$

$$k_2 = (1 - \varepsilon_2) \tag{14}$$

From equations 9 10, and by using the constants from above, equations 1 2 can be rewritten as

$$u(x) = c_1 + k_1 \cdot (\frac{1}{2} \cdot h \cdot F(x, y_0, d) \cdot v_0 + h \cdot \frac{1}{2} \cdot F(x, y_n, d) \cdot v_n + \sum_{i=1}^{N-1} h \cdot F(x, y_i, d) \cdot v_i) \quad (15)$$

$$v(y) = c_2 + k_2 \cdot (\frac{1}{2} \cdot h \cdot F(x_0, y, d) \cdot u_0 + h \cdot \frac{1}{2} \cdot F(x_n, y, d) \cdot u_n + \sum_{i=1}^{N-1} h \cdot F(x_i, y, d) \cdot u_i) \quad (16)$$

To construct the system of linear equations, the formula $\mathbf{A} \cdot \vec{x} = \vec{b}$ is used.
We establish the components for the vectors and matrix by rewrite 17 & 18 into:

$$c_1 = u(x) - k_1 \cdot (\frac{1}{2} \cdot h \cdot F(x, y_0, d) \cdot v_0 + h \cdot \frac{1}{2} \cdot F(x, y_n, d) \cdot v_n + \sum_{i=1}^{N-1} h \cdot F(x, y_i, d) \cdot v_i) \quad (17)$$

$$c_2 = v(y) - k_2 \cdot (\frac{1}{2} \cdot h \cdot F(x_0, y, d) \cdot u_0 + h \cdot \frac{1}{2} \cdot F(x_n, y, d) \cdot u_n + \sum_{i=1}^{N-1} h \cdot F(x_i, y, d) \cdot u_i) \quad (18)$$

The $\vec{b}$ vector contains the c_1 and c_2 constants and has a size of $2 \cdot (n + 1)$. The $\vec{x}$ vector contains the approximated values to equation 1 & 2 and this is values we are solving for. The size of the $\vec{x}$ vector is equal to the $\vec{b}$ vector. The $\mathbf{A}$ matrix is the parameter matrix and contains the trapezoidal approximations to 4 & 5. It has a size of $(2 \cdot (N + 1))$ by $(2 \cdot (N + 1))$.

$$\text{A=}\begin{bmatrix} 1 & 0 & \dots & 0 & \frac{1}{2} \cdot h \cdot F(x_0,y_0,d) \cdot V_0 \cdot -K_1 & h \cdot F(x_0,y_1,d) \cdot V_1 \cdot -K_1 & \dots & \frac{1}{2} \cdot h \cdot F(x_0,y_n,d) \cdot V_n \cdot -K_1 \\ 0 & 1 & \dots & 0 & \frac{1}{2} \cdot h \cdot F(x_1,y_0,d) \cdot V_0 \cdot -K_1 & h \cdot F(x_1,y_1,d) \cdot V_1 \cdot -K_1 & \dots & \frac{1}{2} \cdot h \cdot F(x_1,y_n,d) \cdot V_n \cdot -K_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{1}{2} \cdot h \cdot F(x_n,y_0,d) \cdot V_0 \cdot -K_1 & h \cdot F(x_n,y_1,d) \cdot V_1 \cdot -K_1 & \dots & \frac{1}{2} \cdot h \cdot F(x_n,y_n,d) \cdot V_n \cdot -K_1 \\ \frac{1}{2} \cdot h \cdot F(x_0,y_0,d) \cdot U_0 \cdot -K_2 & h \cdot F(x_0,y_1,d) \cdot U_1 \cdot -K_2 & \dots & \frac{1}{2} \cdot h \cdot F(x_0,y_n,d) \cdot U_n \cdot -K_2 & 1 & 0 & \dots & 0 \\ \frac{1}{2} \cdot h \cdot F(x_1,y_0,d) \cdot U_0 \cdot -K_2 & h \cdot F(x_1,y_1,d) \cdot U_1 \cdot -K_2 & \dots & \frac{1}{2} \cdot h \cdot F(x_1,y_n,d) \cdot U_n \cdot -K_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} \cdot h \cdot F(x_n,y_0,d) \cdot U_0 \cdot -K_2 & h \cdot F(x_n,y_1,d) \cdot U_1 \cdot -K_2 & \dots & \frac{1}{2} \cdot h \cdot F(x_n,y_n,d) \cdot U_n \cdot -K_2 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\text{X=}\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_n \\ V_0 \\ V_1 \\ \vdots \\ V_n \end{bmatrix} \quad \text{B=}\begin{bmatrix} c_1 \\ c_1 \\ \vdots \\ c_1 \\ c_2 \\ c_2 \\ \vdots \\ c_2 \end{bmatrix}$$

Exercise b

Write a program that sets up and solves the system of linear equations from Exercise a) for a given value of N . Use the results to find Q_1 Q_2. For each value of N , state the found values of $u(x), x = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$ and $v(y), y = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$ as well as Q_1 and Q_2 . Use $N = 4, 8, 16, 32, ...$

The program that solves the system of linear equations can be found in the appended file "man2.cpp".
The results extracted form the program can be seen in the table below.
The first 5 rows shows the solution to $u(x)$ evaluated at $x = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$, respectively.
The following 5 rows shows the solution to $v(y)$ evaluated at $y = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$, respectively.
The last 2 rows shows the solutions to Q_1 and Q_2.

N	4	8	16	32	64	128	256
$u(-\frac{1}{2})$	1390.15	1390.41	1390.47	1390.49	1390.49	1390.49	1390.49
$u(-\frac{1}{4})$	1394.08	1394.45	1394.54	1394.56	1394.57	1394.57	1394.57
$u(0)$	1395.6	1396.01	1396.11	1396.14	1396.14	1396.14	1396.14
$u(\frac{1}{4})$	1394.08	1394.45	1394.54	1394.56	1394.57	1394.57	1394.57
$u(\frac{1}{2})$	1390.15	1390.41	1390.47	1390.49	1390.49	1390.49	1390.49
$v(-\frac{1}{2})$	260.505	261.15	261.312	261.352	261.362	261.362	261.365
$v(-\frac{1}{4})$	297.021	298.616	299.013	299.112	299.136	299.143	299.144
$v(0)$	311.005	312.971	313.46	313.582	313.613	313.62	313.622
$v(\frac{1}{4})$	297.021	298.616	299.013	299.112	299.136	299.143	299.144
$v(\frac{1}{2})$	260.505	261.15	261.312	261.352	261.362	261.365	261.365
Q_1	1274.09	1272.09	1271.58	1271.46	1271.42	1271.42	1271.41
Q_2	-276.582	-280.87	-281.952	-282.224	-282.291	-282.308	-282.313

Table 1

From the results it can be seen that $u(x)$ and $v(y)$ converges and the results doesn't change much when increasing N from 4 to 256. Convergence is a sign of weakness!
The results for Q_1 and Q_2 also converges towards a fixed value. The positive value for Q_1 indicates that plate 1 has to be supplied with energy in order to maintain its temperature and the negative value for Q_2 indicates that plate 2 has enough energy as is.

Exercise c

Determine Q_1 and Q_2 as precise as possible based on the results from Exercise b) together with an error estimate.
To determine Q_1 and Q_2 as precise as possible, Richardson extrapolation is used. In order to use Richardson extrapolation, the order of the integration method has to be passed to the method calculating the Richardson result. The trapezoidal method is a 2'th order method.
From the results of Q_1 and Q_2 an estimation of the integration methods order can be calculated by the following formula, equation 19

$$\frac{A(h_1) - A(h_2)}{A(h_2) - A(h_3)} \approx \alpha^k, \frac{h_1}{h_2} = \frac{h_2}{h_3} = \alpha, h_1 > h_2 > h_3 \tag{19}$$

From the results of Q_1 and Q_2 the approximated order is calculated and shown in the second colom of both table 2 and table 3. The approximation to the order converges towards 2 indicating that the

implemented method works as intended and reassembles the theoretical order of the trapezoidal method, i.e it is correct.
The Richardson extrapolation is a method to improve results and is the used method to improve upon Q_1 and Q_2. The formula is shown in equation 20

$$R_i = \frac{\alpha^k \cdot A(h_2) - A(h_1)}{\alpha^k - 1} \tag{20}$$

To calculate an error estimate based on the Q_1, Q_2 and on the R_i values, the equation 21 is used.

$$\frac{A(h_1) - A(h_2)}{\alpha^k - 1} \tag{21}$$

First of all the Q-values is calculated followed by an order estimation and an error estimation. We expect the order estimation to converge towards 2 and the error estimate to constantly decreases. Afterwards the improved results from the Richardson extrapolation is calculated followed by an order estimation to the extrapolation and an error estimation. We expect the order estimation to the extrapolation to converge towards 4 and the error estimation to be a smaller number than the error estimate without Richardson extrapolation.

N	Q_1	k_1	Error. EST	Rich. Extr.	k_2	Error. EST
4	1274.09	0	0	0	0	0
8	1272.09	0	0.668276	1271.42	0	0
16	1271.58	1.98463	0.168858	1271.41	0	0.000477064
32	1271.46	1.99627	0.0423237	1271.41	4.03368	$2.91286e^{-5}$
64	1271.42	1.99907	0.0105877	1271.41	4.00783	$1.81068e^{-6}$
128	1271.42	1.99977	0.00264735	1271.41	4.00193	$1.13016e^{-7}$
256	1271.41	1.99994	0.000661865	1271.41	4.00048	$7.06115e^{-9}$

Table 2: *Results of Q_1 and the Richardson improved values along with error and order estimates*

N	Q_2	k_1	Error. EST	Rich. Extr.	k_2	Error. EST
4	-276.582	0	0	0	0	0
8	-280.87	0	1.42945	-282.3	0	0
16	-281.952	1.98656	0.360706	-282.313	0	0.168858
32	-282.224	1.99676	0.0903791	-282.314	4.04564	0.0423237
64	-282.291	1.9992	0.0226073	-282.314	4.01076	0.0105877
128	-282.308	1.9998	0.00565262	-282.314	4.00266	0.00264735
256	-282.313	1.99995	0.0014132	-282.314	4.00066	0.000661865

Table 3: *Results of Q_2 and the Richardson improved values along with error and order estimates*

From the table it can be seen that the order $k1$ behaves as expected for both Q_1 and Q_2. Likewise does the order $k2$ of the Richardson extraponation and the error after Richardson extrapolation converges fast for both Q_1 and Q_2. After 32 iterations The error seems small enough for Q_1, but for Q_2 we need 256 interations. Therefore both Q_1 and Q_2 has been calculated with 256 iterations.