The co-orbital moons of Saturn

Saturn has many moons with greatly varying sizes. Two of these moons(Janus and Epimetheus) have slightly different speed and have orbits that are similar to the point where collision would be expected. However whenever the moons approach each other, an exchange of energy occurs. This exchange effectively means that the moons swap orbits, such that the faster moon, which was on the inner path, would slow down and move to the outer path, and vice versa.

This exercise will focus on simulating this phenomenon and illustrate what happens when the moons approach each other.

We make some simplifying assumptions.

- The moons only experience the gravity of Saturn and each other.
- Saturn's position in space is fixed.
- We choose the center of Saturn to be the center of an ordinary coordinate system

We introduce the following notation.

(x_i, y_i)	:Position coordinates for moon $i, i = 1, 2$
m_i	:Mass of moon i
M	:Mass of Saturn
g	:Gravitational constant
r_i	$\sqrt{(x_i^2 + y_i^2)}$:Moon i's distance from Saturn
r_{12}	$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$:Distance between the moons

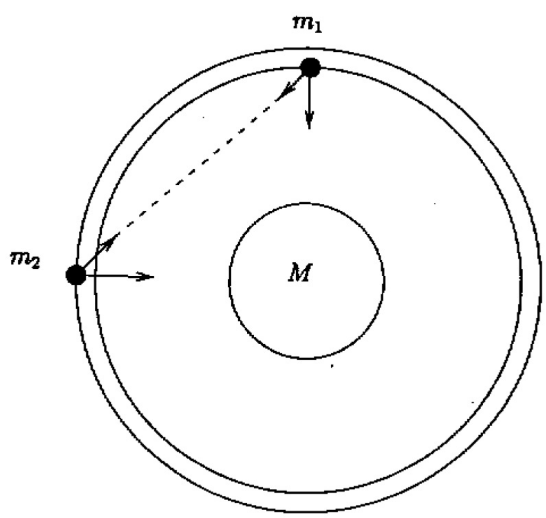

Figure 1: Illustration of Saturn and the two moons

The movements of the moons are determined by Newtons second law:

$$\left.\begin{array}{l} m_1\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = -\frac{m_1 M g}{r_1^3}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \frac{m_1 m_2 g}{r_{12}^3}\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \\ m_2\frac{d^2}{dt^2}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = -\frac{m_2 M g}{r_2^3}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \frac{m_1 m_2 g}{r_{12}^3}\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \end{array}\right\} \quad (1)$$

Data: Note that the units are kg,km and "Days on earth"

- $m_1 = m_2 = 9.20 \cdot 10^{18} kg$
- $M = 5.68 \cdot 10^{26} kg$
- $g = 4.98 \cdot 10^{-10} \frac{km^3}{kg \cdot days}$

for completeness it should be mentioned that Saturn's radius is approximately 60100 km and the radii of the moons are approximately 130 km.

Initial conditions are fitted to a circular motion $v_i^2 = \frac{Mg}{r_i}$. This yields the following initial conditions.

$$\left.\begin{array}{l} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 152870 \end{bmatrix}, \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1360278.1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -153130 \end{bmatrix}, \quad \frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1359122.8 \\ 0 \end{bmatrix} \end{array}\right\} \quad (2)$$

- The Simulation should run until the moons have swapped places at least twice
- The required accuracy for position should be about 10 km at all times

For simplicity the data should be presented in the format of polar coordinates,(r_i, φ_i) and the angular difference $\varphi_1 - \varphi_2$

- Plot the moons distance to Saturn as a function of time
- Plot the angular difference as a function of time.
- Determine how close the moons are to each other during their first swap