Assignment for Lectures 15-16

Due 2016/1/12 23:59:59

1. In a second-order Markov chain, each state depends on the two previous states, i.e.,

$$P\left[X^{(t+1)} = S_k | X^{(t)} = S_j, X^{(t-1)} = S_i, \cdots\right] = P\left[X^{(t+1)} = S_k | X^{(t)} = S_j, X^{(t-1)} = S_i\right].$$

   Show that the second-order Markov chain can always be converted to a first-order Markov chain. (Hint: by redesign the states)

2. Prove that a Bayesian network must be a Directed Acyclic Graph (DAG).

3. Given random variables A, B, C, and D, answer true or false and justify your answer:
   (a) $\{A\} \perp\!\!\!\perp \{B\} | \{C\}$ implies $\{A\} \perp\!\!\!\perp \{B\}$;
   (b) $\{A\} \perp\!\!\!\perp \{B\}$ implies $\{A\} \perp\!\!\!\perp \{B\} | \{C\}$;
   (c) $\{A\} \perp\!\!\!\perp \{B, C\} | \{D\}$ implies $\{A\} \perp\!\!\!\perp \{B\} | \{D\}$.

4. Given a Hidden Markov model with time homogeneous Gaussian emission probability $P[\boldsymbol{x}^{(t)} | z_i^{(t)}, \theta_i] = \frac{1}{(2\pi)^{d/2} det(\boldsymbol{\Sigma}_i)^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}^{(t)} - \boldsymbol{\mu}_i)^\top \boldsymbol{\Sigma}_i^{-1} (\boldsymbol{x}^{(t)} - \boldsymbol{\mu}_i)} dx$, where $\theta_i = (\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$. Consider the problem finding $\Theta = (\boldsymbol{\pi}^{(1)}, \boldsymbol{A}, \{\theta_k\}_{k=1}^K)$ using the EM algorithm. Show that maximizing $\mathcal{Q}(\Theta; \Theta^{old})$ in the M-step gives $\boldsymbol{\mu}_i = \frac{\sum_{t=1}^T \gamma_i^{(t)} \boldsymbol{x}^{(t)}}{\sum_{t=1}^T \gamma_i^{(t)}}$ and $\boldsymbol{\Sigma}_i = \frac{\sum_{t=1}^T \gamma_i^{(t)} (\boldsymbol{x}^{(t)} - \boldsymbol{\mu}_i)(\boldsymbol{x}^{(t)} - \boldsymbol{\mu}_i)^\top}{\sum_{t=1}^T \gamma_i^{(t)}}$.