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Output Regulation of Systems Governed by Delay Differential Equations: Approximations and Robustness

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Abstract: In this paper we consider a problem of robust geometric regulation for tracking and disturbance rejection of systems governed by delay differential equations. It is well known that geometric regulation can be highly sensitive to system parameters and hence such designs are not always robust. In particular, when employing numerical approximations to delay systems, the resulting finite dimensional models inherit natural approximation errors that can impact robustness. We demonstrate this lack of robustness and then show that robustness can be recovered by employing versions of robust regulation that have been developed for infinite dimensional systems. Numerical examples are given to illustrate the ideas.

Keywords: Distributed Parameter Control, Regulation, Tracking, Disturbance Rejection, Delay Differential Equations

1. NOTATION AND PROBLEM STATEMENT

We focus on applying geometric regulation to systems governed by delay differential equations of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-r) + B_0u(t) + w_{dist}(t) \in \mathbb{R}^n \quad (1)$$

with initial data

$$x(0) = \eta \in \mathbb{R}^n,\ x(s) = \varphi(s),\ \varphi(\cdot) \in L^2(-r,0;\mathbb{R}^n). \quad (2)$$

The controlled output to the system is defined by

$$y_c(t) = C_0x(t) \in \mathbb{R}^p. \quad (3)$$

Here, $r > 0$ is a time delay, A_0, A_1 are $n \times n$ matrices, B_0 is $n \times m$ and C_0 is $p \times n$. The goal of output regulation is to track a given reference signal $y_{ref}(t) \in \mathbb{R}^p$ while rejecting the disturbance $w_{dist}(t)$.

We assume that the reference signal $y_{ref}(t)$ and the disturbance $w_{dist}(t)$ are the outputs to a finite dimensional exogenous system. In particular, $w(t) \in \mathbb{R}^q$ satisfies the initial value problem

$$\dot{w}(t) = Sw(t),\ \ w(0) = w_0 \in \mathbb{R}^q, \quad (4)$$

where S is an $q \times q$ matrix. The disturbance and reference signals are assumed to have the form

$$w_{dist}(t) = P_0w(t) \ \text{ and }\ y_{ref}(t) = -Q_0w(t), \quad (5)$$

where P_0 is $n \times q$ and Q_0 is $p \times q$. Therefore, the error is given by

$$e(t) = y_c(t) - y_{ref}(t) = C_0x(t) + Q_0w(t). \quad (6)$$

It is well known (Banks and Burns [1978], Burns et al. [1983], Bensoussan et al. [1992]) that the initial value problem for the retarded delay equation (1) - (2) is equivalent to the infinite dimensional system on the state space $Z = \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$ of the form

$$\dot{z}(t) = \mathscr{A}z(t) + \mathscr{B}u(t) + \mathscr{P}w(t) \quad (7)$$

with initial condition

$$z(0) = [\eta \ \ \varphi(\cdot)]^\top \in \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n). \quad (8)$$

The system operator $\mathscr{A}$ is defined on the domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} : \ \varphi(\cdot) \in H^1(-r,0;\mathbb{R}^n),\ \varphi(0) = \eta \right\} \quad (9)$$

by

$$\mathscr{A} \begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = \begin{bmatrix} A_0\eta + A_1\varphi(-r) \\ \varphi'(\cdot) \end{bmatrix}, \quad (10)$$

where $\varphi'(\cdot)$ is the derivative of $\varphi(\cdot)$.

The operators $\mathscr{B} : \mathbb{R}^m \to Z$ and $\mathscr{P} : \mathbb{R}^q \to Z$ are defined by

$$\mathscr{B}u = \begin{bmatrix} B_0u \\ 0 \end{bmatrix} \ \text{ and }\ \mathscr{P}w = \begin{bmatrix} P_0w \\ 0 \end{bmatrix},$$

respectively and $\mathscr{C} : Z \to \mathbb{R}^p$ is given by

$$\mathscr{C} \begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = C_0\eta. \quad (11)$$

Note that the operator $\mathscr{Q} : \mathbb{R}^q \to \mathbb{R}^m$ and is defined by $\mathscr{Q} = Q_0$ so that the error for the distributed parameter system (7) - (8) is given by

$$e(t) = y_c(t) - y_{ref}(t) = \mathscr{C}z(t) + \mathscr{Q}w(t). \quad (12)$$

In Banks and Burns [1978] it is shown that $\mathscr{A}$ generates the C_0-semigroup $S(t) : Z \to Z$ such that for all $[\eta \ \ \varphi(\cdot)]^\top \in Z$

$$S(t) \begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = \begin{bmatrix} x(t) \\ x_t(\cdot) \end{bmatrix}, \quad (13)$$

where $x(t)$ is the solution to (1) - (2) with $u(t) = w_{dist}(t) = 0$ and $x_t(\cdot) \in H^1(-r,0;\mathbb{R}^n)$ is the past history function defined by

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$x_t(s) = x(s+t)$ for all $s \in [-r, 0]$. Moreover, it is straightforward to show that the operators $\mathscr{B}$, $\mathscr{P}$, $\mathscr{C}$ and Q_0 are all bounded.

The Problem of Output Regulation: Full Information Feedback

The goal is to find bounded linear operators $\mathscr{K}: Z \to \mathbb{R}^m$ and $\mathscr{L}: \mathbb{R}^q \to \mathbb{R}^m$ such that if the feedback controller is given by

$$u(t) = -\mathscr{K} z(t) + \mathscr{L} w(t), \tag{14}$$

then the system

$$\dot{z}_{cl}(t) = [\mathscr{A} - \mathscr{B}\mathscr{K}] z_{cl}(t) \tag{15}$$

is stable and

$$\lim_{x\to\infty} ||e(t)|| = 0, \tag{16}$$

for all $z_0 \in Z$ and $w_0 \in \mathbb{R}^q$. Observe that if the system is stable or there is a known feedback operator $\mathscr{K}$ that stabilizes the system, then the problem reduces to finding the gain $\mathscr{L}$.

The Problem of Output Regulation: Error Feedback

In this case one only assumes that error is available for measurement and consequently the feedback controller is dynamic. Now the goal is to find a (well-posed) dynamical system on a Hilbert space V of the form

$$\dot{\xi}(t) = \mathscr{G}_1 \xi(t) + \mathscr{G}_2 e(t) \tag{17}$$

where $\mathscr{G}_2: \mathbb{R}^p \to V$ is bounded and a bounded operator $\mathscr{F}: V \to \mathbb{R}^m$ defines a feedback controller by

$$u(t) = -\mathscr{F}\xi(t). \tag{18}$$

Observe that the well-posedness conditions implies that the operator $\mathscr{G}_1$ generates a C_0-semigroup on V.

The resulting closed-loop system is a dynamical system on $Z_e = Z \times V \times \mathbb{R}^q$ defined by

$$\dot{z}(t) = \mathscr{A} z_e(t) - \mathscr{B}\mathscr{F}\xi(t) + \mathscr{P} w(t), \tag{19}$$

$$\dot{\xi}(t) = \mathscr{G}_2\mathscr{C} z(t) + \mathscr{G}_1\xi(t) + \mathscr{G}_2 Q_0 w(t), \tag{20}$$

$$\dot{w}(t) = S w(t). \tag{21}$$

Lemma 1. The closed-loop system (19)- (21) is well-posed on Z_e.

Proof: Let $\mathscr{A}_{dia}$ and $\mathscr{A}_{pert}$ denote the operators

$$\mathscr{A}_{dia} = \begin{bmatrix} \mathscr{A} & 0 & 0 \\ 0 & \mathscr{G}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \mathscr{A}_{pert} = \begin{bmatrix} 0 & -\mathscr{B}\mathscr{F} & \mathscr{P} \\ \mathscr{G}_2\mathscr{C} & 0 & \mathscr{G}_2 Q_0 \\ 0 & 0 & S \end{bmatrix},$$

respectively. The diagonal operator $\mathscr{A}_{dia}$ generates a C_0-semigroup on Z_e and since $\mathscr{A}_{pert}$ is a bounded operator, the operator

$$\mathscr{A}_e = \mathscr{A}_{dia} + \mathscr{A}_{pert} = \begin{bmatrix} \mathscr{A} & -\mathscr{B}\mathscr{F} & \mathscr{P} \\ \mathscr{G}_2\mathscr{C} & \mathscr{G}_1 & \mathscr{G}_2 Q_0 \\ 0 & 0 & S \end{bmatrix}$$

also generates a C_0-semigroup on Z_e. This completes the proof.

The goal of output regulation with error feedback is to find operators $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{F}$ such that the system

$$\dot{z}(t) = \mathscr{A} z_e(t) - \mathscr{B}\mathscr{F}\xi(t), \tag{22}$$

$$\dot{\xi}(t) = \mathscr{G}_2\mathscr{C} z(t) + \mathscr{G}_1\xi(t), \tag{23}$$

is asymptotically (exponentially) stable and the error defined by (12) satisfies

$$\lim_{x\to\infty} ||e(t)|| = 0, \tag{24}$$

for all $z_0 \in Z$, $\xi_0 \in V$ and $w_0 \in \mathbb{R}^q$.

In this setting we can apply the approaches in Hämäläinen and Pohjolainen [2010], Paunonen and Pohjolainen [2012], Paunonen [2015, 2017] and Paunonen and Phan [2019] to formulate regulator and robust regulator problems that extend the finite dimensional problems as defined in the classic book Knobloch et al. [2012]. In fact, in Paunonen and Pohjolainen [2012] the authors used a simple delay equation as an example to illustrate their theory. Also, in certain cases the dynamic regulator approach developed in the book Aulisa and Gilliam [2015] can be applied to this system without the need to use the distributed parameter formulation (see Aulisa et al. [2019]).

One of the important practical issues is the problem of computing the geometric regulator. As always, there are two basic approaches to dealing with computation. The "design-then-approximate" (DTA) approach solves the infinite dimensional regulator problem to produce an infinite dimensional controller and then approximates the controller. In the "approximate-then-design (ATD) approach one employs numerical methods to approximate the distributed parameter system, producing finite dimensional system and then solves the regulator problem for these approximate systems. Roughly speaking, the approach in the recent paper Aulisa et al. [2019] is a DTA method and the approach in Paunonen and Phan [2019] is a ATD method.

There is a very nice analysis of the convergence of the approximate controllers (and corresponding reduced order controllers) in Paunonen and Phan [2019]. As noted in that paper, dual convergence is important and, for the parabolic and hyperbolic PDE problems in that paper, standard Galerkin methods yield convergence to the dual system. Dual convergence is obtained since these types of systems are defined by normal dynamic operators $\mathscr{A}$. However, the system operator $\mathscr{A}$ defined by (9)-(10) for the delay equation is highly non-normal and dual convergence can fail for certain Galerkin methods (see Burns et al. [1988]).

2. APPROXIMATING SYSTEMS

We consider the (ATD) approach applied to problems governed by the delay system (1)-(3) by introducing approximations. Results are presented for three approximating schemes:

- (AVE) a finite volume (averaging) method Banks and Burns [1978],
- (BK) a finite element method Banks and Kappel [1979] and
- (IK) a spline based scheme Ito and Kappel [1991].

The resulting approximating models are ODE systems of the form

$$\dot{z}_N(t) = A_N z_N(t) + B_N u(t) + P_N w(t), \tag{25}$$

$$y_N(t) = C_N z_N(t), \tag{26}$$

with error

$$e_N(t) = C_N z_N(t) + Q_N w(t) \tag{27}$$

and where $Q_N = Q_0$ for all $N \geq 1$. Since we are only considering finite dimensional exogenous systems, the exogenous system is still defined by

$$\dot{w}(t) = S w(t). \tag{28}$$

All the approximations above are obtained by first selecting a finite dimensional subspace Z_N of the state space $Z = \mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n)$ and then projecting the system onto Z_N. A general framework for convergence of controllers is given in Ito [1987]. Following Ito [1987] we consider a sequence of (finite dimensional) approximating problems defined by $(Z_N, A_N, B_N, P_N, C_N, Q_N)$, where $Z_N \subset Z$ and π_N is the orthogonal projection onto Z_N. Let $S(t)$, $S^*(t)$, $S_N(t)$ and $S_N^*(t)$ denote the C_0-semigroups generated by the adjoint operators $\mathscr{A}$, $\mathscr{A}^*$, A_N and A_N^*, respectively.

We say this approximating system is *convergent* if

HC: For each $z \in Z$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^q$,

(HCa) : $S_N(t)\pi_N z \to S(t)z$,
(HCb) : $B_N u \to \mathscr{B}u$,
(HCc) : $C_N \pi_N z \to \mathscr{C}z$,
(HCp) : $P_N w \to \mathscr{P}w$,

where the convergence in (HCa) is uniform on compact time intervals.

We say this approximating system is *dual convergent* if

HD: For each $z \in Z$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$,

(HDa) : $S_N^*(t)\pi_N z \to S^*(t)z$,
(HDb) : $B_N^* \pi_N z \to \mathscr{B}^* z$,
(HDc) : $C_N^* y \to \mathscr{C}y$,
(HDp) : $P_N^* \pi_N z \to \mathscr{P}^* z$,

where the convergence in (HDa) is uniform on compact time intervals.

Remark 1. The (BK) scheme is not "dual convergent" which causes problems for optimization based controller designs (see Burns et al. [1988] and Kappel [1991] for details). This is not an issue for the regulator problem, but can be a problem for model reduction and optimization based methods that require dual convergence.

The family of pairs (A_N, B_N) is said to be *uniformly stabilizable* if there is a sequence of bounded operators $K_N : Z_N \to \mathbb{R}^m$ such that $\sup_N ||K_N|| < +\infty$ and the closed-loop system satisfies

$$||e^{(A_N - B_N K_N)t}|| \le M_1 e^{-\omega_1 t} \tag{29}$$

for fixed $M_1 \ge 1$ and $\omega_1 > 0$. Likewise, the family of pairs (A_N, C_N) is said to be *uniformly detectable* if there is a sequence of bounded operators $F_N : Z_N \to \mathbb{R}^p$ such that $\sup_N ||F_N|| < +\infty$ and the closed-loop system satisfies

$$||e^{(A_N - F_N C_N)t}|| \le M_2 e^{-\omega_2 t} \tag{30}$$

for fixed $M_2 \ge 1$ and $\omega_2 > 0$.

Remark 2. Observe that if the semigroups $S(t)$ and $S_N(t)$ are uniformly exponentially stable (as defined below), then the family of pairs are both uniformly stabilizable and uniformly detectable since we can take $K_N = F_N = 0$.

We close this section by stating two conditions that we will assume to hold going forward.

(A1): The semigroup $S(t)$ is stable and family $S_N(t)$ is *uniformly stable* in the sense that there exist $M \ge 1$ and $\omega > 0$ such that

$$||S(t)|| \le Me^{-\omega t} \text{ and } ||e^{(A_N)t}|| \le Me^{-\omega t}. \tag{31}$$

for all $N \ge 1$.

(A2) : The exogenous system (28) is neutrally stable.

In this case we take the feedback gain operators $\mathscr{K}$ and K_N to be zero. This is not essential, but it simplifies the analysis and keeps this paper within page limits. Also, we note that if condition (HCa) holds and $S(t)$ is exponentially stable, then the Trotter-Kato Theorem implies that for N sufficiently large (31) holds.

3. EXISTENCE AND CONVERGENCE

In this short paper we focus on the problem with full state information. The problem with error feedback is similar (see Byrnes et al. [2000], Paunonen [2015], Paunonen and Phan [2019]). In light of Remark 2 we set $\mathscr{K} = 0$ and $K_N = 0$ so that the solution to the output regulation problem with full information reduces to the computation of the operator $\mathscr{L} : \mathbb{R}^q \to \mathbb{R}^m$ and the feedback controller becomes

$$u(t) = \mathscr{L}w(t).$$

Also, the solutions to the approximate problems defined by (25)-(27) with exogenous system (28) are given by

$$u_N(t) = L_N w(t), \tag{32}$$

where $L_N : \mathbb{R}^q \to \mathbb{R}^m$.

Since $Q_N = Q_0$ for all $N \ge 1$, the next two results follow from Theorem 1.1 in Aulisa and Gilliam [2015] and Theorem IV.2 in Byrnes et al. [2000].

Theorem 1. If conditions **(A1)** and **(A2)** hold, then the regulator problem with full information is solvable if and only if there exist bounded linear operators $\Pi : \mathbb{R}^q \to Z$ with $Range(\Pi) \subset \mathscr{D}(\mathscr{A})$ and $\mathscr{L} : \mathbb{R}^q \to \mathbb{R}^m$ satisfying the regulator equations

$$\Pi S = \mathscr{A}\Pi + \mathscr{B}\mathscr{L} + \mathscr{P}, \tag{33}$$
$$0 = \mathscr{C}\Pi + Q_0. \tag{34}$$

Here, $u(t) = \mathscr{L}w(t)$ solves the infinite dimensional problem.

Theorem 2. If conditions **(A1)** and **(A2)** hold, then the approximate regulator problems with full information are solvable if and only if there exist bounded linear operators $\Pi_N : \mathbb{R}^q \to Z_N$ and $L_N : \mathbb{R}^q \to \mathbb{R}^m$ satisfying the regulator equations

$$\Pi_N S = A_N \Pi_N + B_N L_N + P_N, \tag{35}$$
$$0 = C_N \Pi_N + Q_0. \tag{36}$$

Here, $u_N(t) = L_N w(t)$ solves the finite dimensional problem.

Theorem 3. If conditions **(A1)**, **(A2)** and **(HC)** hold, then the approximate gain operators L_N converge in norm to $\mathscr{L}$.

The proof of this result is not difficult, but is too long to be given here (see (Paruchuri [2020]). Note that dual convergence is not needed to establish convergence of the gain matrices. However, if one were to use an optimization based method to first stabilize the system (say a LQR controller), then dual convergence is also necessary.

Although the theoretical results above provide a basis to claim convergence, there remains the questions of robustness with respect to numerical approximations and to model parameters. In the next section we demonstrate that the solution to the problem with full information need not be robust with respect to errors caused by numerical computation and with respect to model parameters. However, we also show that the robust methods developed by Paunonen and co-workers can be used to recover robustness for these delay systems.

4. AN EXAMPLE AND NUMERICAL RESULTS

The three numerical schemes above are detailed in Kappel [1991]. For example, for $N \ge 1$ the finite volume (AVE) scheme in Banks and Burns [1978] produces the following approximating system of size $n(N+1)$:

$$A_N = \begin{bmatrix} A_0 & 0 & 0 & \cdots & A_1 \\ \frac{N}{r}I_n & -\frac{N}{r}I_n & 0 & \cdots & 0 \\ 0 & \frac{N}{r}I_n & -\frac{N}{r}I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{N}{r}I_n & -\frac{N}{r}I_n \end{bmatrix}, \tag{37}$$

$$B_N = [B_0 \ 0 \ \cdots \ 0]^\top, \ P_N = [P_0 \ 0 \ \cdots \ 0]^\top, \tag{38}$$

$$C_N = [C_0 \ 0 \ \cdots \ 0] \ \text{ and } \ Q_N = Q_0. \tag{39}$$

The form of the approximating systems for the (BK) finite element method and the (IK) spline based Galerkin scheme can be found in Banks and Kappel [1979] and Ito and Kappel [1991], respectively. In all cases we note that $A_N = A_N(A_0, A_1, r)$, so that perturbations of the system parameters A_0, A_1 and the delay r can impact performance (e.g., the tracking error does not go to zero). Indeed, we will show this is the case and show that this issue can be addressed by applying the robust controller given in Hämäläinen and Pohjolainen [2010], Paunonen [2015, 2017], Paunonen and Phan [2019].

Example 1. Consider the scalar system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-r) + u(t) + w_{dist}(t), \tag{40}$$

$$y(t) = cx(t), \tag{41}$$

with disturbance and reference signals

$$w_{dist}(t) = cos(t) \ \text{ and } \ y_{ref}(t) = 1 - sin(t), \tag{42}$$

respectively.

Thus, the exogenous system $\dot{w}(t) = Sw(t)$ is defined by

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ \text{with } w(0) = [1 \ 1 \ 1]^\top, \tag{43}$$

$$P_0 = [(1/2) \ \ (1/2) \ \ 0], \ Q_0 = [(1/2) \ \ (-1/2) \ \ -1]. \tag{44}$$

Note that the system (40) is stable for any a_0, a_1 satisfying $a_0 < 1/r$ and $-\pi/2r < a_1 < 0$ (see page 54 in Hale [1971]). For the numerical example we set $a_0 = a_1 = -1$ and $r = 1$.

Since $q = 3$, the gains L_N and $\mathscr{L}$ are 3×1 matrices of the form $L = [l_1 \ l_2 \ l_3]$. The following tables demonstrate convergence of the gains for all three approximation schemes. Observe that the (BK) finite element method produces convergent gains even though this scheme is not dual convergent. Also, note that the higher order scheme (IK) has converged to the "infinite dimensional gain" $\mathscr{L} = [-1.3494 \ 0.19089 \ 2.0000]$ when $N = 64$. Although all schemes are convergent (as stated in Theorem 3), the (IK) scheme converges much more rapidly.

AVE System:

N	l_1	l_2	l_3
8	-1.3618	0.15006	2.0000
16	-1.3549	0.16986	2.0000
32	-1.3520	0.18023	2.0000
64	-1.3506	0.18552	2.0000
128	-1.3500	0.18820	2.0000

BK System:

N	l_1	l_2	l_3
8	-1.3286	0.17416	2.0000
16	-1.3396	0.18254	2.0000
32	-1.3447	0.18671	2.0000
64	-1.3471	0.18888	2.0000
128	-1.3483	0.18984	2.0000

IK System:

N	l_1	l_2	l_3
8	-1.3503	0.19108	2.0000
16	-1.3496	0.19094	2.0000
32	-1.3495	0.19090	2.0000
64	-1.3494	0.19089	2.0000
128	-1.3494	0.19089	2.0000

For the simulations we assume the initial data for the delay system is the constant function $\varphi(s) = 1$ and $\eta = 1$. As one can see in Figure 1, the controller does an excellent job of tracking the reference signal. Note that the tracking error is essentially zero for time $t > 20$. In Figure 2 we show the reference signal y_{ref} and the controlled output $y(t)$.

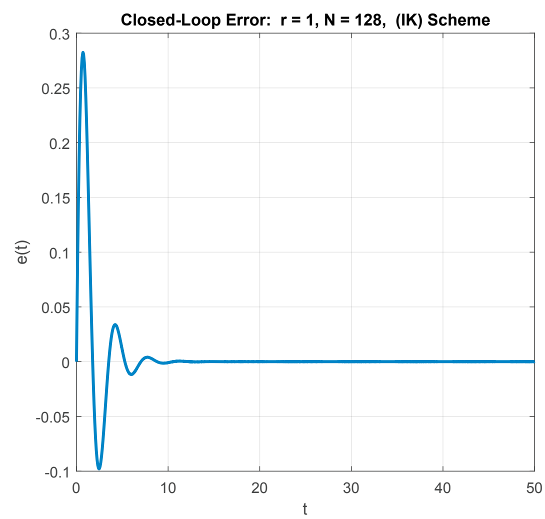

Fig. 1. $N = 128$ (IK) Model: Tracking Error

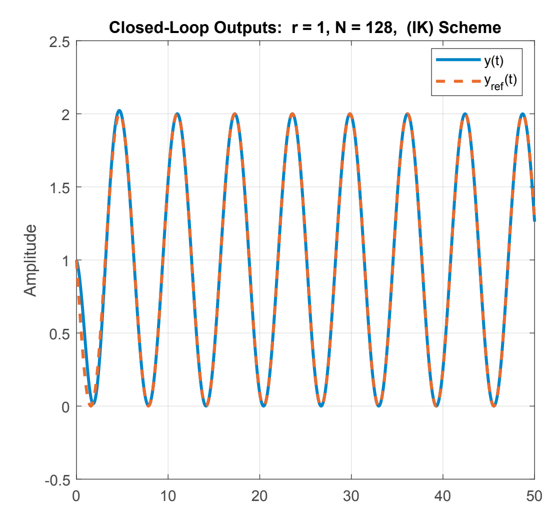

Fig. 2. $N = 128$ (IK) Model: Outputs $y_{ref}(t)$ and $y(t)$

Observe that the gain $l_3^N = l_3 = 2$ for all $N \geq 8$ and for all three schemes. This is due to the special structure of the delay system and the corresponding approximation schemes. A natural question is, What happens if the numerical gain computed for a low order model such as the (AVE) scheme is applied to delay system (or a sufficiently accurate high order model)? A more general question is, What is the impact of perturbations in gains on regulator performance?

Thus, we consider two robustness questions:

Q1: Is the control robust with respect to numerical errors that naturally occur in the computation of the gains?
Q2: Is the control robust with respect to changes in the system parameters?

For the problem here, the answer to both questions is no. To illustrate this lack of robustness for question Q1, we ran several cases with small perturbations in the computed gains and the resulting closed-loop systems do not track the reference signal. For example, consider the case where one perturbs the third gain $l_3 = 2$. If one sets $\delta = 0.1$ and perturb the infinite dimensional gain $\mathscr{L}$ to $\mathscr{L}_{delta} = [-1.3494\ \ 0.19089\ \ 2.00+\delta]$, then as shown in Figure 3 and Figure 4 below, the error does not go to zero so that the control fails to track $y_{ref} = 1 - sin(t)$. The case with $l_3 = 2 - \delta = 1.9$ produces the same type of results.

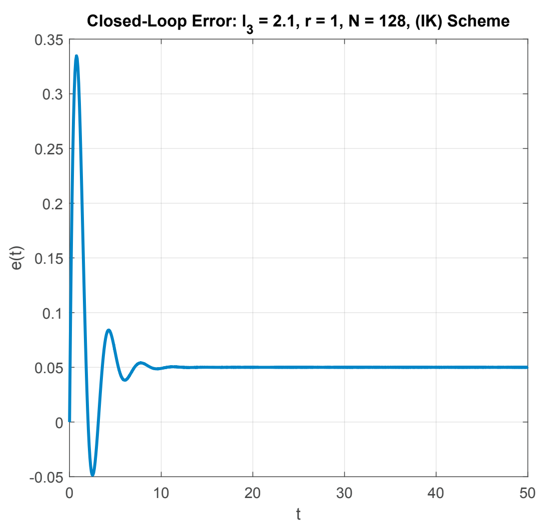

Fig. 3. Error for $l_3 = 2 + \delta = 2.1$: Controller Fails

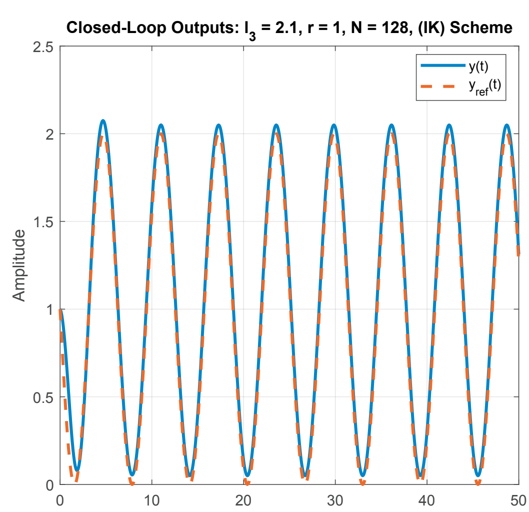

Fig. 4. $l_3 = 2.1$: Outputs $y_{ref}(t)$ and $y(t)$

Now consider question Q2. As noted above $A_N = A_N(A_0, A_1, r)$, so it is natural to consider perturbations in the delay $r = 1$. In this numerical experiment we apply the infinite dimensional gain $\mathscr{L}$ (computed for the nominal value $r = 1$) to the perturbed systems defined with $r_{pert} = 1.1$. As shown in Figure 5, the controller designed for the delay $r = 1$ applied to the perturbed system with $r = 1.1$ fails to drive the error to zero. Thus, there is no robustness with respect to the delay parameter. The case with $r = 1 - 0.1 = 0.9$ is essentially the same.

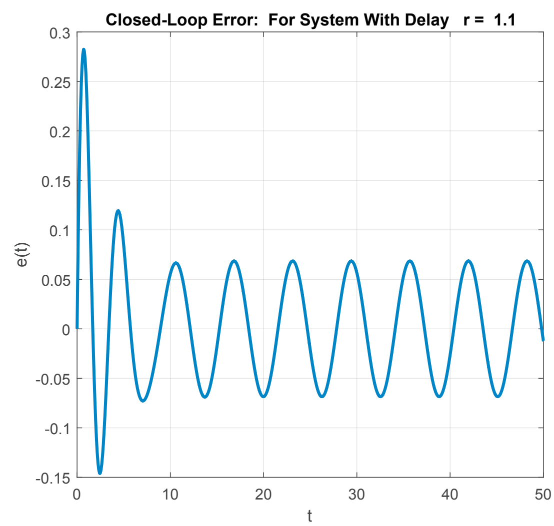

Fig. 5. Error for $r = 1.1$: Controller Fails

In view of the results in Example 1, it is clear that robustness is an issue with output regulation even with full information. In the next section we apply the method described in Hämäläinen and Pohjolainen [2010], Paunonen [2015, 2017], Paunonen and Phan [2019] to produce a robust regulator.

5. ROBUST REGULATOR

Here, we show that one can recover robustness by using the robust design in the papers cited above. We consider the same delay system defined in Example 1 and illustrate that robust regulators can deal with the type of perturbations discussed there.

The robust controller is a dynamic controller similar in structure to (17)-(18). Thus, one seeks operators $\mathfrak{G}_1$, $\mathfrak{G}_2$ and $\mathfrak{F}$ such that

$$\dot{z}_e(t) = \mathfrak{G}_1 z_e(t) + \mathfrak{G}_2 e(t), \tag{45}$$

where the bounded operator $\mathscr{F} : V_e \to \mathbb{R}^m$ defines a feedback controller by

$$u(t) = -\mathfrak{F} z_e(t). \tag{46}$$

Although the system (45)-(46) is similar in structure to (17)-(18), the spaces are different and the construction is more complex. Due to limited space we will not go into the details of the approach. However, this construction is well documented in Hämäläinen and Pohjolainen [2010], Paunonen [2015, 2017], Paunonen and Phan [2019] and we refer the reader to those papers for details of the method.

Applying the robust regulation theory discussed above, we generated a robust controller of the form (45)-(46) for the nominal value of the delay $r = 1$. In Figure 6 we plot the tracking error for the controlled system with $r_{pert} = 1.1$. Although the robust controller drives the tracking error to zero, good tracking does not occur until $t > 40$ as compared to $t > 20$ as shown in Figure 1. This was typical of the many numerical runs we conducted to test the method.

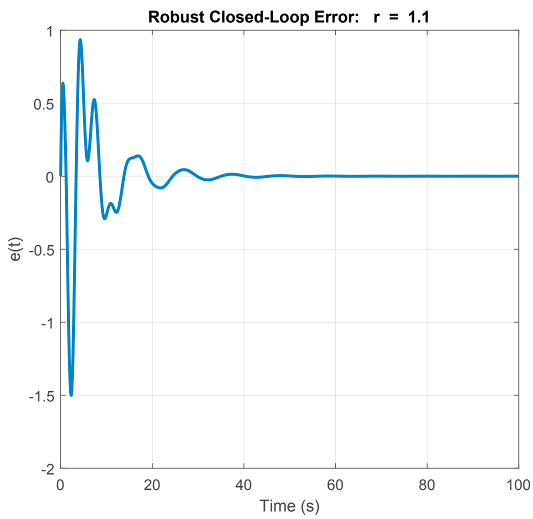

Fig. 6. Error for $r = 1 + 0.1 = 1.1$: Robust Controller

6. CONCLUSION

In this paper we use a system defined by a delay differential equation to set up and formulate the standard distributed parameter regulator problem in order to establish convergence of numerical methods in the ATD approach. We showed that forward convergence is sufficient to establish convergence of the feedback gain operators. We also investigated the robustness of the output regulator control with respect to both numerical errors and time delays. Finally, we demonstrated that by applying the recent results on robust regulation of distributed parameters systems robustness can be restored.

The delay equation is an infinite dimensional distributed parameter system and as long as the operator $\mathscr{A}$ defined by (9)-(10) generates an exponentially stable semigroup, the results in this paper hold. In Example 1, this is satisfied even for $a_0 = 0$ with $-\pi/2 < a_1 < 0$.

Under certain conditions (e.g., the lead matrix A_0 in equation (1) is stable) the methods in Aulisa and Gilliam [2015] can be applied directly to the (finite dimensional) delay system (see Aulisa et al. [2019]). Moreover, this can be done without requiring an exogenous system to generate disturbances or the reference signal. However, the approach in Aulisa et al. [2019] cannot be directly applied to Example 1 with $a_0 = 0$. We are currently working to modify the approach in Aulisa et al. [2019] to deal with such cases.

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