

LeanCat Benchmark (Part I: 1-Category Theory)

LeanCat Consortium

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Problem 1 (Basic - Easy). — *Theorem: Let \mathcal{C} be a category and $\text{Id}_{\mathcal{C}}$ the identity functor. Then monoid of natural transformations $\text{End}(\text{Id}_{\mathcal{C}})$ is commutative.*

```
import Mathlib

open CategoryTheory

variable {C : Type*} [Category.{v} C]

theorem id_comm (α β : (1 C) → (1 C)) : α » β = β » α := by
  sorry
```

Problem 2 (Basic - Easy). — *Theorem: Let \mathcal{C} be a category and let f, g be morphisms in \mathcal{C} such that $f \circ g$ is monic. Then g is monic.*

```
import Mathlib

open CategoryTheory

variable {C : Type*} [Category C]

theorem monic_of_comp_monic {X Y Z : C} (g : X → Y) (f : Y → Z)
  [Mono (g » f)] : Mono g := by
  sorry
```

Problem 3 (Basic - Easy). — *Theorem: The forgetful functor $\mathcal{T}\text{op} \rightarrow \mathcal{S}\text{et}$, $\mathcal{G}\text{rp} \rightarrow \mathcal{S}\text{et}$, $\mathcal{R}\text{ing} \rightarrow \mathcal{A}\text{b}$, $\mathcal{T}\text{op}_* \rightarrow \mathcal{T}\text{op}$ are faithful but not full.*

```
import Mathlib

open CategoryTheory Limits

theorem forget_Top_faithful_not_full :
  (forget TopCat).Faithful ∧ ¬ (forget TopCat).Full := by
  sorry

theorem forget_Grp_faithful_not_full :
  (forget Grp).Faithful ∧ ¬ (forget Grp).Full := by
```

sorry

```
theorem forget_Ring_Ab_faithful_not_full :  
  (forget₂ RingCat Ab).Faithful ∧ ¬ (forget₂ RingCat Ab).Full := by  
sorry
```

```
theorem forget_TopPointed_faithful_not_full :  
  (Under.forget (terminal TopCat)).Faithful ∧ ¬ (Under.forget (terminal  
    ↪ TopCat)).Full := by  
sorry
```

Problem 4 (Basic - Easy). — *Theorem: Let $\{*\} \in \mathcal{S}et$ be the terminal object in $\mathcal{S}et$. Then $\text{hom}_{\mathcal{S}et}(\{*\}, -) : \mathcal{S}et \rightarrow \mathcal{S}et$ is an equivalence of categories.*

```
import Mathlib  
  
open CategoryTheory  
  
universe u  
  
def fromTerminalFunctor : Type u ⇒ Type u where  
  obj α := PUnit.{u} → α  
  map {α β} (f : α → β) := fun g ⇒ f ∘ g  
  map_id := by  
    intro α  
    funext g x  
    rfl  
  map_comp := by  
    intro α β γ f g  
    funext h x  
    rfl  
  
theorem fromTerminalEquivalence : fromTerminalFunctor.IsEquivalence := sorry
```

Problem 5 (Basic - Medium). — *Theorem: Let \mathcal{C} be a category, if every idempotent in \mathcal{C} can be factored into an epimorphism followed by a monomorphism, then all idempotents split in \mathcal{C} .*

```
import Mathlib  
  
open CategoryTheory Idempotents  
  
variable {C : Type*} [Category.{v} C]  
  
theorem idempotent_splitting_from_epi_mono_factorization  
  (h : ∀ (X : C) (p : X → X) (hpp : p » p = p),  
    ∃ (Y : C) (e : X → Y) (he : Epi e) (m : Y → X) (hm : Mono m), p = e »  
    ↪ m) :  
  IsIdempotentComplete C := by  
sorry
```

Problem 6 (Basic - Medium). — *Theorem: Let \mathcal{C} and \mathcal{D} be two categories. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F has a quasi-inverse if and only if*

1. F is fully faithful;
2. F is essentially surjective.

```
import Mathlib

open CategoryTheory

theorem functor_has_quasi_inverse_iff {C D : Type*} [Category C] [Category D]
  ↪ (F : C ⇒ D) :
  (∃ G : D ⇒ C, (Nonempty (Functor.id C ≅ F.comp G)) ∧ (Nonempty (G.comp F
    ↪ ≅ Functor.id D)))
  ↔ F.IsEquivalence := by
  sorry
```

Problem 7 (Basic - Medium). — *Theorem: Let \mathcal{C} be a category and $\text{Kar}(\mathcal{C})$ be its idempotent completion. Let $I : \mathcal{C} \rightarrow \text{Kar}(\mathcal{C})$ be the inclusion. Then for any category \mathcal{D} in which idempotent splits and a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, there is a unique (up to isomorphism) functor $F' : \text{Kar}(\mathcal{C}) \rightarrow \mathcal{D}$ such that $F' \circ I = F$.*

```
import Mathlib

open CategoryTheory

variable {C D : Type*} [Category C] [Category D]

theorem karoubi_universal_property [IsIdempotentComplete D] (F : C ⇒ D) :
  ∃! (F' : (Idempotents.Karoubi C) ⇒ D), (Idempotents.toKaroubi C) » F' = F
  ↪ := by
  sorry
```

Problem 8 (Basic - Medium). — *Theorem: Let G_1 and G_2 be two objects in the category $\mathcal{G}rp$ of groups. The coproduct of G_1 and G_2 in $\mathcal{G}rp$ is equivalent to the free product of G_1 and G_2 .*

```
import Mathlib

open CategoryTheory Limits

universe u
variable {G H : Grp.{u}}

def M : Bool → Type u
| true  ⇒ G
| false ⇒ H

instance : (b : Bool) → Group (M (G := G) (H := H) b) := by
  intro b
  cases b <|> dsimp [M] <|> infer_instance

abbrev FreeProdGrp : Grp.{u} :=
  Grp.of (Monoid.Coproduct (ι := Bool) (M := M (G := G) (H := H)))

theorem freeProdGrp_iso_coproduct [HasBinaryCoproduct G H] :
```

```
Nonempty (FreeProdGrp (G := G) (H := H) ≅ coprod G H) := by
sorry
```

Problem 9 (Basic - Medium). — *Theorem: There exists a morphism in $\mathcal{R}ing$ such that it is epic but not surjective.*

```
import Mathlib

open CategoryTheory

theorem exists_epic_not_surjective_in_Ring :
  ∃ (A B : RingCat) (f : A → B), Epi f ∧ ¬ Function.Surjective f := by
sorry
```

Problem 10 (Basic - High). — *Theorem: Let $F : \mathcal{G}rp \rightarrow \mathcal{S}et$ be the functor that $G \mapsto \{g \in G \mid g^2 = 1\}$. Then F is representable.*

```
import Mathlib

open CategoryTheory

def functor_involution : Grp.{u} ⇒ Type u where
  obj := fun G ⇒ { g : G.carrier | g * g = 1 }
  map := fun {G H} f x ⇒ ⟨f.hom x.val, by
    refine Set.mem_setOf.mpr ?_
    rcases x with ⟨g, hg⟩
    simp only [Set.mem_setOf_eq] at hg
    rw [← f.hom.map_mul, hg]
    simp only [map_one]
  ⟩

theorem involution_functor_representable :
  CategoryTheory.Functor.IsCorepresentable functor_involution := by
sorry
```

Problem 11 (Basic - High). — *Definition: Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A morphism $f : x \rightarrow y$ in \mathcal{C} is called **initial** if for any object $c \in \mathcal{C}$, a morphism $g : U(c) \rightarrow U(x)$ is a morphism in \mathcal{C} whenever $f \circ g : U(c) \rightarrow U(y)$ is a morphism in \mathcal{C} .*

*Definition: An initial morphism $f : x \rightarrow y$ such that the underlying morphism $U(f) : U(x) \rightarrow U(y)$ is monic is called an **embedding**.*

*Definition: If $f : x \rightarrow y$ is an embedding, then (x, f) is called an **initial subobject** of y .*

*Definition: In a concrete category an object I is called **injective** provided that for any embedding $m : A \rightarrow B$ and any morphism $f : A \rightarrow C$ there exists a morphism $g : B \rightarrow C$ extending f , i.e., $g \circ m = f$*

*Definition: A concrete category **has enough injectives** provided that each of its objects is an initial subobject of an injective object.*

Theorem: The category $\mathcal{T}op^{CH}$ of compact Hausdorff space has enough injectives.

```
import Mathlib

open CategoryTheory
```

```

namespace CAT_statement_S_0011

universe u uX

variable {X : Type uX} [Category.{vX} X]

namespace AHS

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C  $\Rightarrow$  X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsInitialHom {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
   $\forall$   $\langle$  Z : C.C  $\rangle$  (g : C.U.obj Z  $\rightarrow$  C.U.obj A),
    ( $\exists$  h : Z  $\rightarrow$  B, C.U.map h = g  $\gg$  C.U.map f)  $\rightarrow$ 
    ( $\exists$  k : Z  $\rightarrow$  A, C.U.map k = g)

def IsEmbedding {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
  IsInitialHom f  $\wedge$  Mono (C.U.map f)

def IsInjectiveObj {C : ConcreteCat (X := X)} (I : C.C) : Prop :=
   $\forall$   $\langle$  A B : C.C  $\rangle$  (m : A  $\rightarrow$  B),
    IsEmbedding m  $\rightarrow$ 
     $\forall$  (f : A  $\rightarrow$  I),  $\exists$  g : B  $\rightarrow$  I, m  $\gg$  g = f

def HasEnoughInj {C : ConcreteCat (X := X)} : Prop :=
   $\forall$  x : C.C,  $\exists$  (I : C.C) (f : x  $\rightarrow$  I),
    IsInjectiveObj I  $\wedge$  IsEmbedding f

end AHS

def CompHausConcrete : AHS.ConcreteCat (X := Type u) :=
  { C := CompHaus.{u}
    U := forget CompHaus }

theorem CompHaus_Has_EnoughInj : AHS.HasEnoughInj (C := CompHausConcrete) := by
  sorry

end CAT_statement_S_0011

```

Problem 12 (Basic - High). — *Theorem: Let \mathcal{C} be a category and let $f : x \rightarrow y$ be a morphism in \mathcal{C} . Then f is a monomorphism in \mathcal{C} if and only if there exists a category \mathcal{D} and a faithful functor $I : \mathcal{C} \rightarrow \mathcal{D}$ such that f is a section in \mathcal{D} .*

```
import Mathlib
```

```

open CategoryTheory

theorem mono_iff_exists_embedding_section
  {C : Type u} [Category C]
  {X Y : C} (f : X → Y) :
  Mono f ↔
    ∃ (D : Type v),
      ∃ (ι : Category D)
        (I : C → D)
        (g : I.obj Y → I.obj X),
        (I.map f) >= g = 1 (I.obj X)
    ^
    (∀ {A B : C}, Function.Injective (fun h : A → B → I.map h)) :=
by
  sorry

```

Problem 13 (Basic - High). — *Theorem: The category $\mathcal{T}op^{CH}$ of compact Hausdorff space is dually equivalent to the category of commutative unital C^* -algebras and algebra homomorphisms.*

```

import Mathlib

open CategoryTheory

universe u

structure CommCStarAlgCat : Type (u + 1) where
  of ::

  carrier : Type u
  [commCStarAlgebra : CommCStarAlgebra carrier]

attribute [instance] CommCStarAlgCat.commCStarAlgebra

namespace CommCStarAlgCat

instance : CoeSort CommCStarAlgCat (Type u) :=
  ⟨CommCStarAlgCat.carrier⟩

instance : Category CommCStarAlgCat where
  Hom A B := A →*a[ℂ] B
  id A := StarAlgHom.id ℂ A
  comp f g := g.comp f

end CommCStarAlgCat

theorem gelfandDuality : Nonempty (CompHaus.{u} ≅ (CommCStarAlgCat.{u})op) :=
  ↪ by
  sorry

```

Problem 14 (Basic - High). — *Definition: Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A morphism $f : x \rightarrow y$ in \mathcal{C} is called **initial** if for any object $c \in \mathcal{C}$, a morphism $g : U(c) \rightarrow U(x)$ is a morphism in \mathcal{C} whenever $f \circ g : U(c) \rightarrow U(y)$ is a morphism in \mathcal{C} .*

Definition: An initial morphism $f : x \rightarrow y$ such that the underlying morphism $U(f) : U(x) \rightarrow U(y)$ is monic is called an **embedding**.

Definition: In a concrete category an object I is called **injective** provided that for any embedding $m : A \rightarrow B$ and any morphism $f : A \rightarrow C$ there exists a morphism $g : B \rightarrow C$ extending f , i.e., $g \circ m = f$

Theorem: In \mathcal{Poset} , injective objects are precisely the suplattice.

```
import Mathlib

open CategoryTheory

namespace CAT_statement_S_0014

universe u uX

variable {X : Type uX} [Category.{vX} X]

namespace AHS2

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C  $\Rightarrow$  X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsInitialHom {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
   $\forall$  {Z : C.C} (g : C.U.obj Z  $\rightarrow$  C.U.obj A),
    ( $\exists$  h : Z  $\rightarrow$  B, C.U.map h = g  $\gg$  C.U.map f)  $\rightarrow$ 
    ( $\exists$  k : Z  $\rightarrow$  A, C.U.map k = g)

def IsEmbedding {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
  IsInitialHom f  $\wedge$  Mono (C.U.map f)

def IsInjectiveObj {C : ConcreteCat (X := X)} (I : C.C) : Prop :=
   $\forall$  {A B : C.C} (m : A  $\rightarrow$  B),
    IsEmbedding m  $\rightarrow$ 
     $\forall$  (f : A  $\rightarrow$  I),  $\exists$  g : B  $\rightarrow$  I, m  $\gg$  g = f

end AHS2

namespace Poset

def PosetConcrete : AHS2.ConcreteCat (Type u) where
  C := PartOrd.{u}
  cat := inferInstance
  U := forget PartOrd
  U_Faithful := inferInstance

theorem injective_iff_suplattice (P : PartOrd.{u}) :
  AHS2.IsInjectiveObj (C := PosetConcrete) P  $\leftrightarrow$   $\forall$  (s : Set P),  $\exists$  x, IsLUB s x
   $\rightarrow$  := by
```

```

    sorry

end Poset

end CAT_statement_S_0014

```

Problem 15 (Basic - High). — *Definition:* Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A morphism $f : x \rightarrow y$ in \mathcal{C} is called **initial** if for any object $c \in \mathcal{C}$, a morphism $g : U(c) \rightarrow U(x)$ is a morphism in \mathcal{C} whenever $f \circ g : U(c) \rightarrow U(y)$ is a morphism in \mathcal{C} .

Definition: An initial morphism $f : x \rightarrow y$ such that the underlying morphism $U(f) : U(x) \rightarrow U(y)$ is monic is called an **embedding**.

Definition: In a concrete category an object I is called **injective** provided that for any embedding $m : A \rightarrow B$ and any morphism $f : A \rightarrow C$ there exists a morphism $g : B \rightarrow C$ extending f , i.e., $g \circ m = f$

Theorem: In $\mathcal{L}at_{\wedge}$, the category of meet semilattice and meet preserving maps, injective objects are frames.

```

import Mathlib

open CategoryTheory

namespace CAT_statement_S_0015

universe u uX

variable {X : Type uX} [Category.{vX} X]

namespace AHS

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C  $\Rightarrow$  X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsInitialHom {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
   $\forall$  {Z : C.C} (g : C.U.obj Z  $\rightarrow$  C.U.obj A),
    ( $\exists$  h : Z  $\rightarrow$  B, C.U.map h = g  $\gg$  C.U.map f)  $\rightarrow$ 
      ( $\exists$  k : Z  $\rightarrow$  A, C.U.map k = g)

def IsEmbedding {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
  IsInitialHom f  $\wedge$  Mono (C.U.map f)

def IsInjectiveObj {C : ConcreteCat (X := X)} (I : C.C) : Prop :=
   $\forall$  {A B : C.C} (m : A  $\rightarrow$  B),
    IsEmbedding m  $\rightarrow$ 
       $\forall$  (f : A  $\rightarrow$  I),  $\exists$  g : B  $\rightarrow$  I, m  $\gg$  g = f

end AHS

namespace SemilatInfCat

```



```

def forget : SemilatInfCat.{u} ⇒ Type u where
  obj A := A
  map {A B} f := f

instance : forget.Faithful where
  map_injective {A B} f g h := by
    ext x
    simp using congrArg (fun k ⇒ k x) h

def SemilatInfCatConcrete : AHS.ConcreteCat (X := Type u) :=
{ C := SemilatInfCat.{u}
  U := forget }

class IsFrameObj (P : SemilatInfCat.{u}) (sSup : Set P.X → P.X) (sInf : Set
  ↪ P.X → P.X) : Prop where
  exists_sSup :
    (∀ (s : Set P.X), IsLUB s (sSup s))
  exists_sInf :
    (∀ (s : Set P.X), IsGLB s (sInf s))
  distributive :
    (∀ (a : P.X), ∀ (s : Set P.X),
      a ⊔ sSup s = sSup (Set.image (fun (b : P.X) ⇒ a ⊔ b) s))

theorem AHS_injective_iff_frameObj (P : SemilatInfCat) :
  AHS.IsInjectiveObj (C := SemilatInfCatConcrete) P ⇔ ∃ (sSup : Set P.X →
    ↪ P.X) (sInf : Set P.X → P.X), IsFrameObj P sSup sInf := by
  sorry

end SemilatInfCat

end CAT_statement_S_0015

```

Problem 16 (Basic - High). — *Definition:* Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A morphism $f : x \rightarrow y$ in \mathcal{C} is called **initial** if for any object $c \in \mathcal{C}$, a morphism $g : U(c) \rightarrow U(x)$ is a morphism in \mathcal{C} whenever $f \circ g : U(c) \rightarrow U(y)$ is a morphism in \mathcal{C} .

Definition: An initial morphism $f : x \rightarrow y$ such that the underlying morphism $U(f) : U(x) \rightarrow U(y)$ is monic is called an **embedding**.

Definition: In a concrete category an object I is called **injective** provided that for any embedding $m : A \rightarrow B$ and any morphism $f : A \rightarrow C$ there exists a morphism $g : B \rightarrow C$ extending f , i.e., $g \circ m = f$

Theorem: In $\mathcal{A}b$ the injective objects are precisely the divisible abelian groups.

```

import Mathlib

open CategoryTheory

namespace CAT_statement_S_0016

universe u uX

```

```

variable {X : Type uX} [Category.{vX} X]

namespace AHS

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C → X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsInitialHom {C : ConcreteCat (X := X)} {A B : C.C} (f : A → B) : Prop :=
  ∀ {Z : C.C} (g : C.U.obj Z → C.U.obj A),
    (∃ h : Z → B, C.U.map h = g >> C.U.map f) →
      (∃ k : Z → A, C.U.map k = g)

def IsEmbedding {C : ConcreteCat (X := X)} {A B : C.C} (f : A → B) : Prop :=
  IsInitialHom f ∧ Mono (C.U.map f)

def IsInjectiveObj {C : ConcreteCat (X := X)} (I : C.C) : Prop :=
  ∀ {A B : C.C} (m : A → B),
    IsEmbedding m →
      ∀ (f : A → I), ∃ g : B → I, m >> g = f

end AHS

def AddCommGrpConcrete : AHS.ConcreteCat (X := Type u) :=
  { C := AddCommGrp.{u}
    U := forget AddCommGrp }

theorem AddCommGrp.injective_iff_divisible (A : AddCommGrp.{u}) :
  AHS.IsInjectiveObj (C := AddCommGrpConcrete) A ↔ ∀ (n : ℕ) (hn : n ≠ 0) (a
    : A), ∃ b : A, n • b = a := by
  sorry

end CAT_statement_S_0016

```

Problem 17 (Basic - High). — *Definition:* Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A morphism $f : x \rightarrow y$ in \mathcal{C} is called **initial** if for any object $c \in \mathcal{C}$, a morphism $g : U(c) \rightarrow U(x)$ is a morphism in \mathcal{C} whenever $f \circ g : U(c) \rightarrow U(y)$ is a morphism in \mathcal{C} .

Definition: An initial morphism $f : x \rightarrow y$ such that the underlying morphism $U(f) : U(x) \rightarrow U(y)$ is monic is called an **embedding**.

Definition: In a concrete category an object I is called **injective** provided that for any embedding $m : A \rightarrow B$ and any morphism $f : A \rightarrow C$ there exists a morphism $g : B \rightarrow C$ extending f , i.e., $g \circ m = f$

Theorem: In $\mathcal{T}op$, the injective objects are precisely the retracts of powers C^I of the space $C := (\{0, 1, 2\}, \{\emptyset, \{0, 1\}, \{0, 1, 2\}\})$.

```

import Mathlib

open CategoryTheory Limits TopologicalSpace

```

```

namespace CAT_statement_S_0017

universe u uX

variable {X : Type uX} [Category.{vX} X]

namespace AHS

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C  $\Rightarrow$  X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsInitialHom {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
   $\forall$  {Z : C.C} (g : C.U.obj Z  $\rightarrow$  C.U.obj A),
    ( $\exists$  h : Z  $\rightarrow$  B, C.U.map h = g  $\gg$  C.U.map f)  $\rightarrow$ 
    ( $\exists$  k : Z  $\rightarrow$  A, C.U.map k = g)

def IsEmbedding {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
  IsInitialHom f  $\wedge$  Mono (C.U.map f)

def IsInjectiveObj {C : ConcreteCat (X := X)} (I : C.C) : Prop :=
   $\forall$  {A B : C.C} (m : A  $\rightarrow$  B),
    IsEmbedding m  $\rightarrow$ 
     $\forall$  (f : A  $\rightarrow$  I),  $\exists$  g : B  $\rightarrow$  I, m  $\gg$  g = f

end AHS

def S : TopCat.{u} :=
  let I : TopologicalSpace (Fin 3) := generateFrom {({0, 1} : Set (Fin 3))}
  TopCat.of (ULift.{u} (Fin 3))

def TopCatConcrete : AHS.ConcreteCat (X := Type u) :=
  { C := TopCat.{u}
    U := forget TopCat }

theorem Inj_in_TopCat {Y : TopCat.{u}} :
  AHS.IsInjectiveObj (C := TopCatConcrete) Y  $\leftrightarrow \exists$  (I : Type u), Nonempty
   $\hookrightarrow$  (Retract Y (piObj (fun _ : I  $\Rightarrow$  S))) := by
  sorry

end CAT_statement_S_0017

```

Problem 18 (Basic - High). — *Definition:* Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A morphism $f : x \rightarrow y$ in \mathcal{C} is called **initial** if for any object $c \in \mathcal{C}$, a morphism $g : U(c) \rightarrow U(x)$ is a morphism in \mathcal{C} whenever $f \circ g : U(c) \rightarrow U(y)$ is a morphism in \mathcal{C} .

Definition: An initial morphism $f : x \rightarrow y$ such that the underlying morphism $U(f) : U(x) \rightarrow U(y)$ is monic is called an **embedding**.

Definition: In a concrete category an object I is called **injective** provided that for any embedding $m : A \rightarrow B$ and any morphism $f : A \rightarrow C$ there exists a morphism $g : B \rightarrow C$ extending f , i.e., $g \circ m = f$

Theorem: In the category $\mathcal{T}op^{CH}$ of compact Hausdorff space, injective objects are precisely the retracts of powers $[0,1]^I$ of the unit interval.

```

import Mathlib

open CategoryTheory

namespace CAT_statement_S_0018

universe u uX

variable {X : Type uX} [Category.{vX} X]

namespace AHS

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C  $\Rightarrow$  X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsInitialHom {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
   $\forall$  {Z : C.C} (g : C.U.obj Z  $\rightarrow$  C.U.obj A),
    ( $\exists$  h : Z  $\rightarrow$  B, C.U.map h = g  $\gg$  C.U.map f)  $\rightarrow$ 
    ( $\exists$  k : Z  $\rightarrow$  A, C.U.map k = g)

def IsEmbedding {C : ConcreteCat (X := X)} {A B : C.C} (f : A  $\rightarrow$  B) : Prop :=
  IsInitialHom f  $\wedge$  Mono (C.U.map f)

def IsInjectiveObj {C : ConcreteCat (X := X)} (I : C.C) : Prop :=
   $\forall$  {A B : C.C} (m : A  $\rightarrow$  B),
    IsEmbedding m  $\rightarrow$ 
     $\forall$  (f : A  $\rightarrow$  I),  $\exists$  g : B  $\rightarrow$  I, m  $\gg$  g = f

end AHS

def CompHausConcrete : AHS.ConcreteCat (X := Type u) :=
{ C := CompHaus.{u}
  U := forget CompHaus}

theorem tieze_urysohn {Y : CompHaus.{u}} :
  AHS.IsInjectiveObj (C := CompHausConcrete) Y  $\leftrightarrow$   $\exists$   $\iota$  : Type u, Nonempty
     $\hookrightarrow$  (Retract Y (.of ( $\forall$  i :  $\iota$ , unitInterval))) := by
  sorry

end CAT_statement_S_0018

```

Problem 19 (Adjunction - Easy). — *Theorem: A functor $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if for each $c \in \mathcal{C}$, the comma category $(c \downarrow G)$ has an initial object.*

```

import Mathlib

open CategoryTheory Limits

variable {C : Type*} {D : Type*} [Category.{v₁} C] [Category.{v₂} D]

theorem functor_hasLeftAdjoint_iff_structuredArrow_hasInitial
  (G : D ⇒ C) :
    G.IsRightAdjoint ↔ ∀ c : C, HasInitial (StructuredArrow c G) := by
  sorry

```

Problem 20 (Adjunction - Medium). — *Theorem: Let \mathcal{C} and \mathcal{D} be categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor that admits a right adjoint G . Then F is fully faithful if and only if $u : \text{Id}_{\mathcal{C}} \rightarrow G \circ F$ is isomorphism.*

```

import Mathlib

open CategoryTheory

variable {C D : Type*} [Category C] [Category D] (F : C ⇒ D) (G : D ⇒ C)

theorem fully_faithful_iff_unit_isIso (adj : F ⊣ G) :
  (F.Full ∧ F.Faithful) ↔ IsIso adj.unit := by
  sorry

```

Problem 21 (Adjunction - Medium). — *Theorem: Let \mathcal{C} and \mathcal{D} be categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor that admits a right adjoint G . Then G is an equivalence of categories if and only if F is fully faithful and G is conservative.*

```

import Mathlib

open CategoryTheory

variable {C : Type u₁} [Category.{v₁} C] {D : Type u₂} [Category.{v₂} D]

theorem
  ⊢ right_adjoint_isEquivalence_iff_left_full_faithful_and_right_conservative
    (F : C ⇒ D) (G : D ⇒ C) (adj : F ⊣ G) :
      G.IsEquivalence ↔ (F.Full ∧ F.Faithful) ∧ G.ReflectsIsomorphisms := by
  sorry

```

Problem 22 (Adjunction - Medium). — *Theorem: Let \mathcal{C} and \mathcal{D} be locally small categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F admits a right adjoint if and only if for each $d \in \mathcal{D}$, $\text{hom}_{\mathcal{D}}(F(-), d) : \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}\text{et}$ is representable.*

```

import Mathlib

open CategoryTheory

variable {C : Type u₁} [Category.{v} C] {D : Type u₂} [Category.{v} D]

theorem isLeftAdjoint_iff_yoneda_comp_op_isRepresentable (F : C ⇒ D) :
  F.IsLeftAdjoint ↔ ∀ (d : D), (F.op » yoneda.obj d).IsRepresentable := by

```

sorry

Problem 23 (Adjunction - Medium). — *Theorem: Let $A, B \in \mathcal{R}\text{ing}$ and let $\phi : A \rightarrow B$ be a morphism in $\mathcal{R}\text{ing}$. It induces a functor $\phi_* : {}_B\mathcal{A}\mathcal{B} \rightarrow {}_A\mathcal{A}\mathcal{B}$, $(N, l_N) \mapsto (N, l_N \circ (\phi \otimes \text{id}))$. Then the functor ϕ_* admits a left adjoint $\phi^* := B \otimes_A - : {}_A\mathcal{A}\mathcal{B} \rightarrow {}_B\mathcal{A}\mathcal{B}$ and a right adjoint $\phi^! := \text{hom}_A(B, -) : {}_A\mathcal{A}\mathcal{B} \rightarrow {}_B\mathcal{A}\mathcal{B}$.*

```
import Mathlib
open CategoryTheory

theorem ring_hom_induced_functor_has_adjoints
  {A B : RingCat} (ϕ : A → B) :
  ∃ (ϕ_pull : ModuleCat B ⇒ ModuleCat A)
    (ϕ_push : ModuleCat A ⇒ ModuleCat B)
    (ϕ_coind : ModuleCat A ⇒ ModuleCat B),
  Nonempty (Adjunction ϕ_push ϕ_pull) ∧ Nonempty (Adjunction ϕ_pull
    ↪ ϕ_coind) := by
  sorry
```

Problem 24 (Adjunction - Medium). — *Theorem: Let $U : \mathcal{A}\mathcal{B} \rightarrow \mathcal{G}\text{rp}$ be the forgetful functor. Then it admits a left adjoint.*

```
import Mathlib
open CategoryTheory
universe u

theorem forget_CommGrp_to_Grp_admits_left_adjoint :
  (forget₂ CommGrp.{u} Grp.{u}).IsRightAdjoint := by
  sorry
```

Problem 25 (Monad - Medium). — *Theorem: Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $U : \mathcal{D} \rightarrow \mathcal{C}$, $V : \mathcal{E} \rightarrow \mathcal{C}$, $F : \mathcal{D} \rightarrow \mathcal{E}$ be functors such that $V \circ F = U$. Suppose U, V have left adjoints and \mathcal{D} have coequalizers. If V reflects split epimorphisms to regular epimorphisms (or equivalently, the counit of the adjunction of U is a regular epimorphism), then F has a left adjoint.*

```
import Mathlib
open CategoryTheory

variable {C D E : Type*} [Category C] [Category D] [Category E]

namespace CategoryTheory

class Functor.ReflectsSplitEpimorphismsToRegularEpimorphisms (F : Functor C
  ↪ D) : Prop where
  reflects : ∀ {X Y} {f : X → Y} [IsSplitEpi (F.map f)], Nonempty (RegularEpi
    ↪ f)

end CategoryTheory

variable (U : D ⇒ C) (V : E ⇒ C) (F : D ⇒ E)
```

```

theorem exists_left_adjoint_of_comp_eq (h : F » V = U) (hU :
  ↳ U.IsRightAdjoint) (hV : V.IsRightAdjoint)
  (hV_refl : V.ReflectsSplitEpimorphismsToRegularEpimorphisms) :
  ↳ F.IsRightAdjoint := by
sorry

```

Problem 26 (Adjunction - Medium). — *Theorem: Let F, G, H be functors such that $F \dashv G \dashv H$. Then F is fully faithful if and only if H is fully faithful.*

```

import Mathlib

open CategoryTheory Functor

variable {C : Type u1} [Category.{v1} C] {D : Type u2} [Category.{v2} D]
variable {F : C ⇒ D} {G : D ⇒ C} {H : C ⇒ D}

theorem fully_faithful_iff_of_adjoint (hFG : F ⊣ G) (hGH : G ⊣ H) :
  (F.Full ∧ F.Faithful) ↔ (H.Full ∧ H.Faithful) := by
sorry

```

Problem 27 (Adjunction - Medium). — *Theorem: Let (\mathbb{Z}, \leq) be a poset, regarded as a category, then $f \in \text{End}(\mathbb{Z})$ has left adjoint if and only if it has a right adjoint.*

```

import Mathlib

open CategoryTheory

theorem int_endofunctor_hasLeftAdjoint_iff_hasRightAdjoint (f : ℤ ⇒ ℤ) :
  f.IsRightAdjoint ↔ f.IsLeftAdjoint := by
sorry

```

Problem 28 (Adjunction - Medium). — *Theorem: Let (\mathbb{N}, \leq) be a poset, regarded as a category. There is a sequence of distinct functors $G_n : \mathbb{N} \rightarrow \mathbb{N}$ such that $G_0(x) = x + 1$ and $G_{n+1} \dashv G_n$ for each $n \in \mathbb{N}$.*

```

import Mathlib

open CategoryTheory

theorem exists_sequence_of_distinct_adjoint_nats :
  ∃ G : ℕ → (ℕ ⇒ ℕ),
    Function.Injective G ∧
    (∀ x, (G 0).obj x = x + 1) ∧
    (∀ n, Nonempty (G (n + 1) ⊣ G n)) := by
sorry

```

Problem 29 (Adjunction - High). — *Theorem: Let $(-)^{\times} : \mathcal{R}\text{ing} \rightarrow \mathcal{G}\text{rp}$ mapping a ring to its group of units. Then it admits a left adjoint.*

```

import Mathlib

open CategoryTheory

def RingCat.units : RingCat.{u}  $\Rightarrow$  Grp.{u} where
  obj R := .of  $R^\times$ 
  map f := Grp.ofHom (Units.map f.hom)

theorem exists_leftAdjoint_unitFunctor :
   $\exists$  (left : Grp.{u}  $\Rightarrow$  RingCat.{u}), Nonempty (left  $\dashv$  RingCat.units.{u}) :=
     $\hookrightarrow$  by
      sorry

```

Problem 30 (Reflective - High). — *Theorem: There are categories \mathcal{C} , \mathcal{D} and \mathcal{E} such that \mathcal{C} is a subcategory of \mathcal{D} , \mathcal{D} be a subcategory of \mathcal{E} and \mathcal{C} is reflective in \mathcal{E} , but \mathcal{C} is not reflective in \mathcal{D} .*

```

import Mathlib

open CategoryTheory Functor

universe u v

namespace CategoryTheory

open Category Adjunction

variable {C : Type u1} {D : Type u2} {E : Type u3}
variable [Category.{v1} C] [Category.{v2} D] [Category.{v3} E]

class Reflective2 (R : D  $\Rightarrow$  C) extends R.Faithful where
  L : C  $\Rightarrow$  D
  adj : L  $\dashv$  R

end CategoryTheory

theorem exists_not_reflective :
   $\exists$  (E C D : Type u)
  (_ : Category.{v} E) (_ : Category.{v} C) (_ : Category.{v} D) (i : C  $\Rightarrow$ 
     $\hookrightarrow$  D)
  (_ : Faithful i) (j : D  $\Rightarrow$  E) (_ : Faithful j),
  IsEmpty (Reflective2 i)  $\wedge$  Nonempty (Reflective2 (i  $\gg$  j)) := by
  sorry

```

Problem 31 (Reflective - High). — *Theorem: Neither $\mathcal{S}et$ nor $\mathcal{T}op$ has a proper isomorphism-closed full subcategory that is both reflective and coreflective.*

```

import Mathlib

open CategoryTheory

theorem not_reflective_and_coreflective (P : ObjectProperty (Type u))
  (h : P.IsClosedUnderIsomorphisms) (hproper :  $\exists$  X : Type u,  $\dashv$  P X) :
  IsEmpty (Reflective P.1)  $\vee$  IsEmpty (Coreflective P.1) := by
  sorry

```


Problem 32 (Reflective - High). — *Theorem: $\mathcal{S}et$ has precisely three full, isomorphism-closed, reflective subcategories.*

```
import Mathlib

open CategoryTheory Functor Limits

namespace CAT_statement_S_0032

def IsIsoClosed (P : Type u → Prop) : Prop :=
  ∀ {X Y : Type u}, Nonempty (X ≅ Y) → P X → P Y

def SubcategoryEquiv (P Q : Type u → Prop) : Prop :=
  ∀ X, P X ↔ Q X

def IsReflectiveSubcategory (P : Type u → Prop) : Prop :=
  Nonempty (Reflective (ObjectProperty.1 P))

theorem Set_has_precisely_three_reflective_subcategories :
  ∃ (P₁ P₂ P₃ : Type u → Prop),
    IsIsoClosed P₁ ∧ IsReflectiveSubcategory P₁ ∧
    IsIsoClosed P₂ ∧ IsReflectiveSubcategory P₂ ∧
    IsIsoClosed P₃ ∧ IsReflectiveSubcategory P₃ ∧
    ¬ SubcategoryEquiv P₁ P₂ ∧ ¬ SubcategoryEquiv P₂ P₃ ∧ ¬
      SubcategoryEquiv P₁ P₃ ∧
    ∀ (Q : Type u → Prop), IsIsoClosed Q → IsReflectiveSubcategory Q →
      (SubcategoryEquiv Q P₁ ∨ SubcategoryEquiv Q P₂ ∨ SubcategoryEquiv Q
        P₃) := by
  sorry

end CAT_statement_S_0032
```

Problem 33 (Reflective - High). — *Theorem: $\mathcal{T}op^{CH}$ has precisely two full, isomorphism-closed, coreflective subcategories.*

```
import Mathlib

open CategoryTheory Topology

namespace CAT_statement_S_0033

structure FullCoreflectiveSubcategory (C : Type u) [Category.{v} C] where
  obj : ObjectProperty C
  iso_closed : obj.IsClosedUnderIsomorphisms
  coreflective : Coreflective obj.1

theorem CompHaus_has_precisely_two_coreflective_subcategories :
  Nat.card (FullCoreflectiveSubcategory CompHaus) = 2 := by
  sorry

end CAT_statement_S_0033
```

Problem 34 (Concrete - Medium). — *Definition: Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A **universal arrow** over $x \in \mathcal{B}$ is a structured arrow $u : x \rightarrow U(c)$ with domain x that has the following universal property: for each structured arrow $f : x \rightarrow U(b)$ with domain x there exists a unique morphism $\underline{f} : c \rightarrow b$ such that $\underline{f} \circ u = f$.*

Definition: Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A **free object** over $x \in \mathcal{B}$ is an object $c \in \mathcal{C}$ such that there exists a universal arrow (u, c) over x .

Theorem: Let (\mathcal{C}, U) be a construct such that U is representable by an object x . Then for any set I and any object $d \in \mathcal{C}$ the following conditions are equivalent:

1. d is a free object over I .
2. d is an I -th copower of x .

```
import Mathlib

open CategoryTheory Limits Functor Opposite

namespace CAT_statement_S_0034

variable {C : Type u} [Category.{v} C]

def IsFreeObject (U : C  $\Rightarrow$  Type v) (d : C) (I : Type v) : Prop :=
   $\exists$  ( $\eta$  : I  $\rightarrow$  U.obj d),  $\forall$  {y : C} (f : I  $\rightarrow$  U.obj y),  $\exists!$  (g : d  $\rightarrow$  y), U.map g  $\circ$ 
     $\hookrightarrow$   $\eta$  = f

def IsCopower (x d : C) (I : Type v) : Prop :=
   $\exists$  ( $\iota$  : I  $\rightarrow$  (x  $\rightarrow$  d)), Nonempty (IsColimit (Cofan.mk d  $\iota$ ))

theorem free_iff_copower_of_representable
  (U : C  $\Rightarrow$  Type v) [Faithful U]
  (x : C) (hU : U  $\cong$  coyoneda.obj (op x))
  (I : Type v) (d : C) :
  IsFreeObject U d I  $\leftrightarrow$  IsCopower x d I := by
  sorry

end CAT_statement_S_0034
```

Problem 35 (Concrete - High). — *Definition:* A full concrete embedding is called a realization.

Theorem: There is a construct (\mathcal{C}, U) such that every construct has a realization to (\mathcal{C}, U) .

```
import Mathlib

open CategoryTheory

namespace CAT_statement_S_0035

structure Construct where
  C : Type u
  [str : Category.{v} C]
  U : C  $\Rightarrow$  Type u
  [faithful : Functor.Faithful U]

attribute [instance] Construct.str Construct.faithful

def IsRealization (S T : Construct.{u, v}) (F : S.C  $\Rightarrow$  T.C) : Prop :=
  F  $\gg$  T.U = S.U  $\wedge$  Functor.Full F  $\wedge$  Function.Injective F.obj

theorem exists_universal_construct :
```

```

    ∃ (T : Construct.{u, v}), ∀ (S : Construct.{u, v}), ∃ (F : S.C ⇒ T.C),
      ↪ IsRealization S T F := by
sorry

end CAT_statement_S_0035

```

Problem 36 (Concrete - High). — *Definition: A category \mathcal{C} is called **concretizable** over a category \mathcal{B} if there exists a faithful functor $U : \mathcal{C} \rightarrow \mathcal{B}$.*

Theorem: There exist categories that are not concretizable over \mathcal{Set} .

```

import Mathlib

open CategoryTheory

theorem exists_category_not_concretizable_over_Type :
  ∃ (C : Type u) ( _ : Category.{v} C), ¬ ∃ (F : C ⇒ Type v), F.Faithful :=
  ↪ by
sorry

```

Problem 37 (Concrete - High). — *Theorem: There are precisely two concrete functors from \mathcal{Set} to \mathcal{Top} , but a proper class of concrete functors from \mathcal{Top} into itself.*

```

import Mathlib

open CategoryTheory

namespace CAT_statement_S_0037

universe u v w

variable {X : Type uX} [Category.{vX} X]

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C ⇒ X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsConcreteFunc {A B : ConcreteCat (X := X)} (F : A.C ⇒ B.C) : Prop :=
  Nonempty ((F » B.U) ≅ A.U)

def SetConcrete : ConcreteCat (X := Type u) :=
{ C := Type u
  U := 1 (Type u) }

def TopConcrete : ConcreteCat (X := Type u) :=
{ C := TopCat.{u}
  U := (forget TopCat) }

```

```

def ConcreteFuncsIso (A B : ConcreteCat (X := Type u)) : Type _ :=
  { F : A.C ⇒ B.C // IsConcreteFunc (A := A) (B := B) F }

theorem only_two_concrete_functors_from_Set_to_Top_iso :
  Nat.card (ConcreteFuncsIso SetConcrete TopConcrete) = 2 := by
  sorry

end CAT_statement_S_0037

```

Problem 38 (Concrete - High). — *Definition: Let \mathcal{C} be a category and let $c \in \mathcal{C}$ be an object. A **regular subobject** of c is a pair (x, i) where i is a regular monomorphism.*

*Definition: Let \mathcal{C} be a category. \mathcal{C} is called **regular wellpowered** if no object in \mathcal{C} has a proper class of pairwise non-isomorphic regular subobjects.*

*Definition: A category \mathcal{C} is called **concretizable** over a category \mathcal{B} if there exists a faithful functor $U : \mathcal{C} \rightarrow \mathcal{B}$.*

Theorem: Let \mathcal{C} be a category that admits finite limits. Then \mathcal{C} is concretizable over $\mathcal{S}et$ if and only if \mathcal{C} is regular wellpowered.

```

import Mathlib

namespace CAT_statement_S_0038

open CategoryTheory Limits

universe u v w

variable {C : Type u} [Category C] [HasFiniteLimits C]

def IsConcretizable (X : Type v) [Category X] (D : Type u) [Category D] : Prop
  ↪ :=
  ∃ (U : D ⇒ X), U.Faithful

variable (C)

class RegularWellPowered : Prop where
  regularSubobject_small : ∀ (X : C), Small.{v} { P : Subobject X //
    ↪ Nonempty (RegularMono P.arrow) }

theorem concretizable_iff_regular_wellpowered :
  IsConcretizable (Type u) C ⇔ RegularWellPowered C := by
  sorry

end CAT_statement_S_0038

```

Problem 39 (Concrete - High). — *Theorem: Let $\mathcal{F}rm$ be the construct whose objects are frames, i.e. distributive suplattices, and whose morphisms are frame homomorphisms. Then there is a unique concrete functor $T : \mathcal{T}op_0^{op} \rightarrow \mathcal{F}rm$ over $\mathcal{S}et$, where $\mathcal{T}op_0$ is the category of T_0 topological spaces.*

```

import Mathlib

open CategoryTheory Topology

universe u v w

```

```

variable {X : Type uX} [Category.{vX} X]

namespace CAT_statement_S_0039

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C  $\Rightarrow$  X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

def IsConcreteFunc {A B : ConcreteCat (X := X)} (F : A.C  $\Rightarrow$  B.C) : Prop :=
  Nonempty ((F  $\gg$  B.U)  $\cong$  A.U)

def forgetFrm : Frm.{u}  $\Rightarrow$  Type u where
  obj X := X
  map {X Y} f := f

instance : forgetFrm.Faithful where
  map_injective {X Y} f g h := by
    ext x
    simp using congrArg (fun k  $\Rightarrow$  k x) h

structure T0TopCat where
  toTop : TopCat.{u}
  is_t0 : T0Space ( $\uparrow$ toTop)

namespace T0TopCat

instance : CoeSort T0TopCat (Type u) := (fun X  $\Rightarrow$  X.toTop)
instance (X : T0TopCat) : TopologicalSpace X := X.toTop.str
attribute [instance] T0TopCat.is_t0

instance : Category T0TopCat :=
  InducedCategory.category (fun X : T0TopCat  $\Rightarrow$  X.toTop)

def forget_0 : T0TopCat  $\Rightarrow$  TopCat :=
  inducedFunctor (fun X : T0TopCat  $\Rightarrow$  X.toTop)

instance : forget_0.Faithful :=
{ map_injective := by
  intro X Y f g h
  simp [forget_0] using h }

@[simp] def of (X : Type u) [TopologicalSpace X] [T0Space X] : T0TopCat :=
  (TopCat.of X, inferInstance)

```

```

def L : T0TopCatop ⇒ Type u :=
{ obj := fun X ⇒ TopologicalSpace.Opens ((X.unop).toTop)
  map := by
    intro ⟨X⟩ ⟨Y⟩ ⟨f⟩
    obtain ⟨g⟩ : (Y.toTop → X.toTop) := forget_0.map f
    intro U
    exact TopologicalSpace.Opens.comap g U
  map_id := by
    intro X
    ext U x
    rfl
  map_comp := by
    intro X Y Z f g
    ext U x
    rfl }

instance : L.Faithful where
  map_injective {X Y} f g h := by
    apply Quiver.Hom.unop_inj
    apply Functor.map_injective T0TopCat.forget_0
    ext x
    haveI : T0Space (T0TopCat.forget_0.obj (Opposite.unop X)) :=
      ↪ (Opposite.unop X).is_t0
    apply Inseparable.eq
    rw [inseparable_iff_forall_isOpen]
    intro U hU
    let U_op : TopologicalSpace.Opens (T0TopCat.forget_0.obj (Opposite.unop
      ↪ X)) := ⟨U, hU⟩
    have h_eq := congr_fun h U_op
    dsimp [L] at h_eq
    have h_set := congr_arg (SetLike.coe) h_eq
    rw [Set.ext_iff] at h_set
    exact h_set x

end T0TopCat

def FrmConcrete : ConcreteCat (X := Type u) :=
{ C := Frm.{u}
  U := (forgetFrm) }

def T0TopCatopConcrete : ConcreteCat (X := Type u) :=
{ C := T0TopCatop
  U := (T0TopCat.L) }

def ConcreteFuncsIso (A B : ConcreteCat (X := Type u)) : Type _ :=
{ F : A.C ⇒ B.C // IsConcreteFunc (A := A) (B := B) F }

theorem unique_concrete_functors_from_T0TopCatop_to_Frm_iso :
  Nat.card (ConcreteFuncsIso T0TopCatConcrete FrmConcrete) = 1 := by
  sorry

end CAT_statement_S_0039

```

Problem 40 (Concrete - High). — *Definition:* Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A **universal arrow** over $x \in \mathcal{B}$ is a structured arrow $u : x \rightarrow U(c)$ with domain x that has the following universal property: for each structured arrow $f : x \rightarrow U(b)$ with domain x there exists a unique morphism $\underline{f} : c \rightarrow b$ such that $\underline{f} \circ u = f$.

Definition: Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A **free object** over $x \in \mathcal{B}$ is an object $c \in \mathcal{C}$ such that there exists a universal arrow (u, c) over x .

Definition: Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . \mathcal{C} is said to **have free objects**, if for each $x \in \mathcal{B}$ there is a free object over x .

Theorem: Let $\mathcal{L}at_V^\infty$ be the category of suplattices. The consturct $\mathcal{L}at_V^\infty$ has free objects.

```
import Mathlib

open CategoryTheory

namespace CAT_statement_S_0040

universe u v w

variable {X : Type uX} [Category.{vX} X]

structure ConcreteCat (X : Type v) [Category X] where
  C : Type u
  [cat : Category C]
  U : C  $\Rightarrow$  X
  [U_Faithful : U.Faithful]

attribute [instance] ConcreteCat.cat ConcreteCat.U_Faithful

abbrev StructuredArrowOver (x : X) (C : ConcreteCat (X := X)) : Type _ :=
  StructuredArrow x C.U

def IsUniversalArrowOver (x : X) {C : ConcreteCat (X := X)} (u :
   $\hookrightarrow$  StructuredArrowOver x C) : Prop :=
   $\forall$  (v : StructuredArrowOver x C),
     $\exists!$  (g : u.right  $\rightarrow$  v.right), u.hom  $\gg$  C.U.map g = v.hom

def IsFreeObjectOver (x : X) {C : ConcreteCat (X := X)} (z : C.C) : Prop :=
   $\exists$  (f : StructuredArrowOver x C), f.right = z  $\wedge$  IsUniversalArrowOver (x :=
     $\hookrightarrow$  x) (C := C) f

def HasFreeObject (C : ConcreteCat (X := X)) : Prop :=
   $\forall$  (x : X),  $\exists$  (z : C.C), IsFreeObjectOver (x := x) (z := z)

structure SupLatCat where
  carrier : Type u
  [inst : CompleteSemilatticeSup carrier]

attribute [instance] SupLatCat.inst

instance : CoeSort SupLatCat (Type u) :=  $\langle$ SupLatCat.carrier $\rangle$ 
```

```

def of (α : Type u) [CompleteSemilatticeSup α] : SupLatCat := ⟨α⟩

structure Hom (A B : SupLatCat.{u}) where
  toFun : A → B
  map_sSup' : ∀ s : Set A, toFun (sSup s) = sSup (toFun '' s)

instance (A B : SupLatCat) : CoeFun (Hom A B) (fun _ => A → B) := ⟨Hom.toFun⟩

@[simp] lemma Hom.map_sSup {A B : SupLatCat} (f : Hom A B) (s : Set A) :
  f (sSup s) = sSup (f '' s) :=
  f.map_sSup' s

@[ext] lemma Hom.ext {A B : SupLatCat} {f g : Hom A B}
  (h : ∀ a, f a = g a) : f = g := by
  cases f with
  | mk fto fmap =>
    cases g with
    | mk gto gmap =>
      have hto : fto = gto := funext (by intro a; exact h a)
      cases hto
      have : fmap = gmap := by
        apply Subsingleton.elim
      cases this
      rfl

def id (A : SupLatCat) : Hom A A :=
{ toFun := (_root_.id : A → A)
  map_sSup' := by
    intro s
    simp }

def comp {A B C : SupLatCat} (f : Hom A B) (g : Hom B C) : Hom A C :=
{ toFun := fun a => g (f a)
  map_sSup' := by
    intro s
    calc
      g (f (sSup s)) = g (sSup (f '' s)) := by
        simp
      _ = sSup (g '' (f '' s)) := by
        simp
      _ = sSup ((fun x => g (f x)) '' s) := by
        simp [Set.image_image] }

instance : Category SupLatCat where
  Hom A B := Hom A B
  id A := id A
  comp f g := comp f g
  id_comp := by intro A B f; ext a; rfl
  comp_id := by intro A B f; ext a; rfl
  assoc := by intro A B C D f g h; ext a; rfl

```



```

def forget : SupLatCat ⇒ Type u :=
{ obj := fun A ⇒ A.carrier
  map := fun {X Y} (f : X → Y) ⇒ f.toFun
  map_id := by intro A; rfl
  map_comp := by intro A B C f g; rfl }

instance : forget.Faithful where
  map_injective := by
    intro X Y f g h
    apply Hom.ext
    intro x
    simp using congrArg (fun k ⇒ k x) h

def SupLatCatConcrete : ConcreteCat (X := Type u) :=
{ C := SupLatCat.{u}
  U := (forget) }

theorem SupLat_Has_Free_Object :
  HasFreeObject SupLatCatConcrete:= by
  sorry

end CAT_statement_S_0040

```

Problem 41 (Concrete - High). — *Definition:* Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A **universal arrow** over $x \in \mathcal{B}$ is a structured arrow $u : x \rightarrow U(c)$ with domain x that has the following universal property: for each structured arrow $f : x \rightarrow U(b)$ with domain x there exists a unique morphism $\underline{f} : c \rightarrow b$ such that $\underline{f} \circ u = f$.

Definition: Let (\mathcal{C}, U) be a concrete category over \mathcal{B} . A **free object** over $x \in \mathcal{B}$ is an object $c \in \mathcal{C}$ such that there exists a universal arrow (u, c) over x .

Theorem: Let \mathcal{C} be the non-full subcategory of $\mathcal{L}at_V^\infty$ whose objects are suplattice and morphisms are meet- and join-preserving maps. In the consturct \mathcal{C} , there exists a free object over x if and only if the cardinality of x is not greater than 2, i.e. $|x| \leq 2$.

```

import Mathlib

open CategoryTheory

universe u v w

namespace CAT_statement_S_0041

structure FreeObject {C : Type u} [Category.{v} C] [HasForget.{w} C] (x :
  → Type w) where
  (obj : C)
  (emb : x → (forget C).obj obj)
  (uniq : ∀ (Y : C) (f : x → (forget C).obj Y), ∃! (g : obj → Y), emb »
    → (forget C).map g = f)

```

```

lemma complete_lattice_category (X : Type u) :
  Nonempty (FreeObject (C := CompleteLat) X) ↔ Cardinal.mk X ≤ 2 := by
  sorry

end CAT_statement_S_0041

```

Problem 42 (Limit - Easy). — *Theorem: Let \mathcal{C} and \mathcal{D} be categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful functor. Then F reflects any limits and colimits admitted in the codomain category.*

```

import Mathlib

open CategoryTheory Limits Functor

variable {C : Type u1} [Category.{v1} C] {D : Type u2} [Category.{v2} D]

lemma fully_faithful_reflects_limits_and_colimits (F : C ⇒ D) [Full F]
  ↪ [Faithful F] :
  ReflectsLimits F ∧ ReflectsColimits F := by
  sorry

```

Problem 43 (Limit - Easy). — *Theorem: The one point set $\{*\}$ form a separator in \mathbf{Set} , and the two point set $\{a, b\}$ form a coseparating set in \mathbf{Set} .*

```

import Mathlib

open CategoryTheory Function Classical

theorem PUnit_isSeparator : IsSeparator (PUnit : Type u) := by
  sorry

theorem ULiftBool_isCoseparator : IsCoseparator (ULift.{u} Bool) := by
  sorry

```

Problem 44 (Limit - Easy). — *Theorem: Filtered colimits commute with finite limits in $\mathcal{S}\mathbf{et}$.*

```

import Mathlib

open CategoryTheory Limits

variable {J : Type u} [SmallCategory J] [FinCategory J]
variable {K : Type u} [SmallCategory K] [IsFiltered K]
variable (F : J ⇒ K ⇒ Type u)

theorem filteredColimitsCommuteWithFiniteLimits :
  Nonempty (colimit (limit F) ≅ limit (colimit F.flip)) := by
  sorry

```

Problem 45 (Limit - Easy). — *Theorem: Let ω be the ordinal of natural numbers. Consider $F : \omega^{op} \rightarrow \mathcal{R}\mathbf{ing}$ with $F_n := \mathbb{Z}/p^n\mathbb{Z}$ and $f_n : F_{n+1} \rightarrow F_n$. Then the limit exists.*

```

import Mathlib

open CategoryTheory Limits Opposite

variable (p : ℕ)

noncomputable def pAdicFunctor : ℕop ⇒ RingCat where
  obj n := RingCat.of (ZMod (p ^ (unop n)))
  map {m n} f := RingCat.ofHom <|
    ZMod.castHom (pow_dvd_pow p (leOfHom f.unop)) (ZMod (p ^ (unop n)))
  map_id := by
    intro n
    ext x
    simp
  map_comp := by
    intro x y z f g
    ext x
    simp

theorem pAdic_limit_exists : HasLimit (pAdicFunctor p) := by
  sorry

```

Problem 46 (Limit - Easy). — *Theorem: Let \mathcal{C} and \mathcal{D} be a small category and let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. Then we have $\text{Nat}(F, G) \cong \int_{c \in \mathcal{C}} \text{hom}_{\mathcal{D}}(F(c), G(c))$.*

```

import Mathlib

open CategoryTheory Limits

variable {C : Type u} [SmallCategory C]
variable {D : Type u} [SmallCategory D]
variable (F G : C ⇒ D)

def homIntegrandBifunctor : Cop × C ⇒ Type u :=
  (Functor.prod F.op G) » (Functor.hom D)

theorem natTransIsoEnd :
  Nonempty (NatTrans F G ≅ end_ (curryObj (homIntegrandBifunctor F G))) :=
  by
  sorry

```

Problem 47 (Limit - Easy). — *Theorem: There is no equivalence of categories between $\mathcal{S}\text{et}$ and $\mathcal{S}\text{et}^{op}$.*

```

import Mathlib

open CategoryTheory

theorem no_equiv_between_Set_and_op : ¬ Nonempty (Equivalence (Type u) (Type
  u)op) := by
  sorry

```

Problem 48 (Limit - Easy). — *Theorem: A reflective subcategory \mathcal{C} of a cocomplete category \mathcal{D} is also cocomplete.*

```

import Mathlib

open CategoryTheory Limits

variable {C : Type u} [Category.{v} C] {D : Type u} [Category.{v} D]

theorem hasColimits_of_reflective (i : C  $\Rightarrow$  D) [Reflective i] [HasColimits D]
  ↪ :
    HasColimits C := by
  sorry

```

Problem 49 (Limit - Medium). — *Theorem: Let \mathcal{C} and \mathcal{E} be two categories and let $F : \mathcal{C} \rightarrow \mathcal{E}$ be a functor. Let \bullet be the terminal category consisting of a unique object \bullet and a unique morphism. Then a colimit of F is a left Kan extension of F along $K : \mathcal{C} \rightarrow \bullet$, i.e. $\text{Lan}_K F(\bullet) = \text{colim} F$.*

```

import Mathlib

open CategoryTheory Limits

universe u1 v1 u2 v2

variable {C : Type u1} [Category.{v1} C]
variable {E : Type u2} [Category.{v2} E]

theorem colimit_is_leftKanExtension_along_to_terminal
  (F : C  $\Rightarrow$  E) (K : C  $\Rightarrow$  PUnit) [HasColimit F] [K.HasLeftKanExtension F] :
    Nonempty ((K.leftKanExtension F).obj PUnit.unit  $\cong$  colimit F) := by
  sorry

```

Problem 50 (Limit - Medium). — *Definition: Let \mathcal{C} be a locally small category. An object $c \in \mathcal{C}$ is called compact if $\text{hom}_{\mathcal{C}}(c, -)$ preserves filtered colimits.*

Theorem: For \mathcal{S}^{et} , an object is compact if and only if it is a finite set.

```

import Mathlib

open CategoryTheory Limits

lemma isCompactObject_iff_finite_type (X : Type u) :
  PreservesFilteredColimits (coyoneda.obj (Opposite.op X))  $\leftrightarrow$  Finite X := by
  sorry

```

Problem 51 (Limit - Medium). — *Theorem: Let \mathcal{C} be a category. Then \mathcal{C} admits all small limits if and only if \mathcal{C} admits all small products and pullbacks.*

```

import Mathlib

open CategoryTheory Limits

variable {C : Type u} [Category.{v} C]

```

```

theorem has_limits_iff_has_products_and_pullbacks :
  HasLimitsOfSize.{v, v} C ↔ (∀ (J : Type v), HasLimitsOfShape (Discrete J)
    ↪ C) ∧ HasLimitsOfShape WalkingCospan C := by
  sorry

```

Problem 52 (Limit - Medium). — *Theorem: Let X, Y, Z be objects in $\mathcal{S}et$ with morphisms $f : X \rightarrow Z$ and $g : Y \rightarrow Z$. Then $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$ is the pullback $X \times_Z Y$ of X and Y over Z .*

```

import Mathlib

open CategoryTheory Limits Functor Types Function Pullback

theorem Function.isPullback_pullback {X Y Z : Type u} (f : X → Z) (g : Y →
  ↪ Z) :
  IsPullback (C := Type u) (fst (f := f) (g := g)) snd f g := by
  sorry

```

Problem 53 (Limit - Medium). — *Theorem: Let \mathcal{D} be a small-complete locally small category, a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if G is continuous and for each $c \in \mathcal{C}$, the comma category $(c \downarrow G)$ admits an initial object.*

```

import Mathlib

open CategoryTheory Limits

variable {D : Type u} [Category.{v} D] [HasLimits D] [LocallySmall.{v} D]
variable {C : Type u} [Category.{v} C]
variable (G : D ⇒ C)

theorem has_left_adjoint_iff_continuous_and_initials :
  G.IsRightAdjoint ↔ PreservesLimits G ∧ ∀ (c : C), HasInitial
    ↪ (StructuredArrow c G) := by
  sorry

```

Problem 54 (Limit - Medium). — *Theorem: Let \mathcal{B} be a complete category. Then \mathcal{B} has an initial object if and only if there exists a small set I and an I -indexed family of objects x_i such that, for every $s \in \mathcal{B}$, there is an $i \in I$ and an arrow $x_i \rightarrow s$.*

```

import Mathlib

open CategoryTheory Limits

variable {B : Type u} [Category.{v} B]

theorem hasInitial_iff_exists_weakly_initial [HasLimits B] :
  HasInitial B ↔ ∃ (I : Type v) (x : I → B), ∀ (s : B), ∃ (i : I), Nonempty
    ↪ (x i → s) := by
  sorry

```

Problem 55 (Limit - Medium). — *Theorem: The forgetful functor $U : \mathcal{G}rp, \mathcal{A}b, \mathcal{R}ing \rightarrow \mathcal{S}et$ creates limits, but they do not preserve coproducts.*

```

import Mathlib

open CategoryTheory Limits

theorem forget_Grp_createsLimits_but_not_coproducts :
  Nonempty (CreatesLimits (forget Grp.{u})) ^
  → Nonempty (PreservesColimitsOfShape (Discrete Bool) (forget Grp.{u})) :=
  ↪ by
sorry

theorem forget_Ab_createsLimits_but_not_coproducts :
  Nonempty (CreatesLimits (forget Ab.{u})) ^
  → Nonempty (PreservesColimitsOfShape (Discrete Bool) (forget Ab.{u})) :=
  ↪ by
sorry

theorem forget_RingCat_createsLimits_but_not_coproducts :
  Nonempty (CreatesLimits (forget RingCat.{u})) ^
  → Nonempty (PreservesColimitsOfShape (Discrete Bool) (forget
    ↪ RingCat.{u})) := by
sorry

```

Problem 56 (Limit - Medium). — *Theorem: Let \mathcal{C} and \mathcal{D} be categories and let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor. Then $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if the right Kan extension $\text{Ran}_G \text{Id}_{\mathcal{D}} : \mathcal{C} \rightarrow \mathcal{D}$ exists and is preserved by G (i.e. $G \circ \text{Ran}_G \text{Id}_{\mathcal{D}} \simeq \text{Ran}_K (G \circ \text{Id}_{\mathcal{D}})$).*

```

import Mathlib

open CategoryTheory Functor

variable {C : Type u1} [Category.{v1} C] {D : Type u2} [Category.{v2} D]

theorem hasLeftAdjoint_iff_ran_id_preserved (G : D ⇒ C) :
  G.IsRightAdjoint ↔
  ∃ (R : C ⇒ D) (α : G » R → 1 D),
  R.IsRightKanExtension α ^
  (R » G).IsRightKanExtension ((associator G R G).inv » whiskerRight α G
    ↪ » (leftUnitor G).hom) := by
sorry

```

Problem 57 (Limit - Medium). — *Theorem: A functor that reflects equalizers (or finite products) reflects isomorphisms.*

```

import Mathlib

open CategoryTheory Limits

variable {C : Type u} [Category.{v} C] {D : Type u'} [Category.{v'} D]

theorem reflectsIsomorphisms_of_reflects_equalizers (F : C ⇒ D)
  [ReflectsLimitsOfShape WalkingParallelPair F] : F.ReflectsIsomorphisms :=
  ↪ by
sorry

```

```

theorem reflectsIsomorphisms_of_reflects_finite_products (F : C  $\Rightarrow$  D)
  [ReflectsLimitsOfShape (Discrete PEmpty) F] [ReflectsLimitsOfShape
     $\rightarrow$  (Discrete WalkingPair) F] :
  F.ReflectsIsomorphisms := by
sorry

```

Problem 58 (Limit - Medium). — *Definition:* Let \mathcal{C} be a locally small category. An object $c \in \mathcal{C}$ is called **compact** if $\text{hom}_{\mathcal{C}}(c, -)$ preserves filtered colimits.

Theorem: A topological space X is compact if and only if it is a compact object in the category $\mathcal{O}p(X)$, the category of open subsets of X .

```

import Mathlib

open CategoryTheory

namespace CAT_statement_S_0058

universe u

variable (X : Type u) [TopologicalSpace X]

abbrev Op (X : Type u) [TopologicalSpace X] := TopologicalSpace.Opens X

theorem compactSpace_iff_finitelyPresented_top :
  CompactSpace X  $\Leftrightarrow$  IsFinitelyPresentable (C := Op X) ( $\tau$  : Op X) := by
sorry

end CAT_statement_S_0058

```

Problem 59 (Limit - Medium). — *Definition:* A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to **lift limits** if for every diagram $D : \mathcal{J} \rightarrow \mathcal{C}$ and every limit L of $F \circ D$, there exists a limit $L' \in \mathcal{D}$ such that $F(L') \cong L$.

Theorem: There is a functor that lifts limits but is not faithful.

```

import Mathlib

open CategoryTheory Limits

namespace CAT_statement_S_0059

universe w' w'₁ w w₁ v₁ v₂ v₃ u₁ u₂ u₃

variable {C : Type u₁} [Category.{v₁} C]
variable {D : Type u₂} [Category.{v₂} D]
variable {J : Type w} [Category.{w'} J] {K : J  $\Rightarrow$  C}

structure LifiableCone₂ (K : J  $\Rightarrow$  C) (F : C  $\Rightarrow$  D) (c : Cone (K  $\gg$  F)) where

  liftedCone : Cone K

  validLift : F.mapCone liftedCone  $\cong$  c
  isLimit : IsLimit liftedCone

```

```

class LiftsLimit (K : J  $\Rightarrow$  C) (F : C  $\Rightarrow$  D) where

  lifts :  $\forall$  c, IsLimit c  $\rightarrow$  LiftableCone2 K F c

theorem exists_functor_lifts_limit_and_not_faithful :
   $\exists$  (C : Type (u1+1)) (C : Category.{u1} C) (D : Type u2) (D :
     $\rightarrow$  Category.{u2} D) (F : C  $\Rightarrow$  D), ( $\forall$  (J : Type u1) (J : Category.{w'} J)
     $\rightarrow$  (K : J  $\Rightarrow$  C), Nonempty (LiftsLimit K F))  $\wedge$ 
     $\rightarrow$  F.Faithful := by
  sorry

end CAT_statement_S_0059

```

Problem 60 (Limit - Medium). — *Theorem: Suppose \mathcal{B} is locally small, complete, has a small coseparating set S , and has the property that every family of subobjects has an intersection. Then \mathcal{B} has an initial object.*

```

import Mathlib

open CategoryTheory Limits

theorem has_initial_of_locally_small_complete_coseparating {B : Type u}
   $\rightarrow$  [Category.{v} B]
  [LocallySmall.{w} B] [HasLimitsOfSize.{w, w} B] {S : Set B} [Small.{w} S]
  (hS : IsCoseparating S) (h :  $\forall$  (A : B),  $\forall$  (s : Set (Subobject A)),  $\exists$  (f :
     $\rightarrow$  Subobject A),
    IsGLB s f) : HasInitial B := by
  sorry

```

Problem 61 (Limit - Medium). — *Theorem: Let $\mathcal{C} = \mathcal{D} = \text{Vec}_{\mathbb{k}}$ the category of finite dimension \mathbb{k} -vector spaces. Then the coend is the trace of matrices.*

```

import Mathlib

open CategoryTheory Limits

theorem coend_hom_is_trace_of_matrices
  (k : Type u) [Field k] :
   $\forall$  (F : (ModuleCat k)op  $\Rightarrow$  ModuleCat k  $\Rightarrow$  ModuleCat k),
  ( $\forall$  X Y, (F.obj (Opposite.op X)).obj Y  $\cong$  ModuleCat.of k (X  $\rightarrow_1$  [k] Y))  $\rightarrow$ 
   $\exists$  (T : ModuleCat k),
  ( $\exists$  (tr :  $\forall$  X, (F.obj (Opposite.op X)).obj X  $\rightarrow$  T),
    Nonempty (IsColimit (Cofan.mk T tr))) := by
  sorry

```

Problem 62 (Limit - Medium). — *Definition: Let \mathcal{C} be a category. Let S be a family of subobjects (s_n, i_n) of an object $c \in \mathcal{C}$, indexed by a class I . A subobject $(x, i : x \rightarrow c)$ of c is called an **intersection** of S provided that the following two conditions are satisfied:*

- (1) i factors through each i_n i.e., for each n there exists an $f_n : x \rightarrow s_n$ with $i = i_n \circ f_n$,
- (2) if a morphism $f : z \rightarrow c$ factors through each i_n , then it factors through i .

*Definition: A category \mathcal{C} is said to **have intersections** if for each object $c \in \mathcal{C}$ and every family of subobjects of c , there exists an intersection.*

*Definition: A category is said to be **strongly complete** if it is complete and has intersections.*

Theorem: A strongly cocomplete category with a separating set is strongly complete.


```

import Mathlib

open CategoryTheory Limits

namespace CAT_statement_S_0062

universe u v
variable {C : Type u} [Category.{v} C]

def IsIntersectionOf {B : C} (A : Subobject B) (S : Set (Subobject B)) : Prop
↪ :=
  (∀ Ai, Ai ∈ S → A ≤ Ai) ∧
  (∀ A' : Subobject B, (∀ Ai, Ai ∈ S → A' ≤ Ai) → A' ≤ A)

def HasIntersections (C : Type u) [Category.{v} C] : Prop :=
  ∀ (B : C) (S : Set (Subobject B)),
    ∃ A : Subobject B, IsIntersectionOf (C := C) (B := B) A S

class StronglyComplete (C : Type u) [Category.{v} C] : Prop where
  complete: HasLimits C
  hasinter: HasIntersections C

class StronglyCocomplete (C : Type u) [Category.{v} C] : Prop where
  dual: StronglyComplete (C:=Cop)

theorem strongly_complete_of_strongly_cocomplete_of_separating_set
↪ [StronglyComplete Cop] {G : Set C} [Small.{v} G] (hG : IsSeparating G) :
  StronglyComplete C := by
  sorry

end CAT_statement_S_0062

```

Problem 63 (Limit - High). — *Definition:* Let \mathcal{C} be a locally small category. An object $c \in \mathcal{C}$ is called **compact** if $\text{hom}_{\mathcal{C}}(x, -)$ preserves filtered colimits.

Theorem: For $\mathcal{G}\text{rp}$, an object is compact if and only if it is finitely presented as a group. Every group can be realized as a direct limit of finitely presented groups.

```

import Mathlib

open CategoryTheory

namespace CAT_statement_S_0063

universe u

def IsFinitelyPresentedGrp (X : Type u) [Group X] : Prop :=
  ∃ (α : Type u) (rels : Set (FreeGroup α)), Finite α ∧ rels.Finite ∧
  ↪ Nonempty (X ≃* PresentedGroup rels)

theorem isCompactObject_Grp_iff_finite_presented (X : Type u) [Group X] :
  CategoryTheory.IsFinitelyPresentable (Grp.of X) ↔ IsFinitelyPresentedGrp
  ↪ X := by
  sorry

```

```

theorem group_realized_as_direct_limit_of_finitely_presented_groups (X : Type
  ↳ u) [Group X] :
  ∃ (J : Type u) (inst₁ : CategoryTheory.SmallCategory J) (inst₂ :
    ↳ CategoryTheory.IsFiltered J) (F : CategoryTheory.Functor J Grp), ∀ (j
    ↳ : J), IsFinitelyPresentedGrp (F.obj j) ∧ Nonempty (X ≈*
    ↳ Grp.FilteredColimits.colimit F) := by
  sorry

end CAT_statement_S_0063

```

Problem 64 (Limit - High). — *Definition: Let \mathcal{C} be a locally small category. An object $c \in \mathcal{C}$ is called **compact** if $\text{hom}_{\mathcal{C}}(c, -)$ preserves filtered colimits.*

Theorem: Let A be a ring. For the category of right A -modules $\mathcal{A}b_A$, an object is compact if and only if it is a finitely presentable A -module. Every A -module can be realized as a direct limit of finitely presented A -module.

```

import Mathlib

open CategoryTheory

universe u v w

theorem isCompactObject_Grp_iff_finite_presented {A : Type u} [Ring A] (X :
  ↳ Type v) [Group X] [AddCommGroup X] [Module A X] :
  ↳ CategoryTheory.IsFinitelyPresentable (ModuleCat.of A X) ↔
  ↳ Module.FinitePresentation A X := by
  sorry

theorem module_realized_as_direct_limit_of_finitely_presented_modules (A :
  ↳ Type u) [Ring A] (X : Type v) [AddCommGroup X] [Module A X] :
  ∃ (J : Type w) (inst₁ : CategoryTheory.SmallCategory J) (inst₂ :
    ↳ CategoryTheory.IsFiltered J) (F : CategoryTheory.Functor J (ModuleCat
    ↳ A)), ∀ (j : J), Module.FinitePresentation A (F.obj j) ∧ Nonempty (X
    ↳ ≈₁[A] ModuleCat.FilteredColimits.colimit F) := by
  sorry

```

Problem 65 (Limit - High). — *Theorem: Let \mathcal{C} be a complete, wellpowered, cowellpowered and have a separator s . Then \mathcal{C} is cocomplete if and only if for each set I , there exists an I -th copower of S in \mathcal{C} .*

```

import Mathlib

open CategoryTheory Limits

variable {C : Type u} [Category.{v} C]

theorem hasColimits_iff_hasCoproduct_of_separator
  [HasLimits C]
  [WellPowered C]
  [WellPowered Cop]
  (S : C) (hS : IsSeparator S) :
  HasColimits C ↔ ∀ (I : Type v), HasColimit (Discrete.functor (fun (_ : I)
    ↳ ⇒ S)) := by
  sorry

```

Problem 66 (Limit - High). — *Definition:* Let \mathcal{C} be a category. Let S be a family of subobjects (s_n, i_n) of an object $c \in \mathcal{C}$, indexed by a class I . A subobject $(x, i : x \rightarrow c)$ of c is called an **intersection** of S provided that the following two conditions are satisfied:

(1) i factors through each i_n i.e., for each n there exists an $f_n : x \rightarrow s_n$ with $i = i_n \circ f_n$,

(2) if a morphism $f : z \rightarrow c$ factors through each i_n , then it factors through i .

Definition: A category \mathcal{C} is said to **have intersections** if for each object $c \in \mathcal{C}$ and every family of subobjects of c , there exists an intersection.

Definition: A category is said to be **strongly complete** if it is complete and has intersections.

Definition: A category \mathcal{C} is strongly cocomplete if \mathcal{C}^{op} is strongly complete.

Theorem: There is a strongly cocomplete category with a separator that is neither wellpowered nor co-wellpowered.

```
import Mathlib

open CategoryTheory Limits

namespace CAT_statement_S_0066

universe u v
variable {C : Type u} [Category.{v} C]

def IsIntersectionOf {B : C} (A : Subobject B) (S : Set (Subobject B)) : Prop
  ↪ :=
  (∀ Ai, Ai ∈ S → A ≤ Ai) ∧
  (∀ A' : Subobject B, (∀ Ai, Ai ∈ S → A' ≤ Ai) → A' ≤ A)

def HasIntersections (C : Type u) [Category.{v} C] : Prop :=
  ∀ (B : C) (S : Set (Subobject B)),
    ∃ A : Subobject B, IsIntersectionOf (C := C) (B := B) A S

class StronglyComplete (C : Type u) [Category.{v} C] : Prop where
  complete: HasLimits C
  hasinter: HasIntersections C

class StronglyCocomplete (C : Type u) [Category.{v} C] : Prop where
  dual: StronglyComplete (C := Cop)

theorem exists_cocomplete_separator_not_wellPowered_not_coWellPowered :
  ∃ (C : Type u) (C : Category.{v} C),
    StronglyCocomplete C ∧ HasSeparator C ∧
    ¬ WellPowered.{v} C ∧ ¬ WellPowered.{v} Cop := by
  sorry

end CAT_statement_S_0066
```

Problem 67 (Limit - High). — *Theorem:* Let ω be the ordinal of natural numbers. Consider $F : \omega^{op} \rightarrow \mathcal{R}ing$ with $F_n := k[x]/(x^n)$ and $f_n : k[x]/(x^{n+1}) \rightarrow k[x]/(x^n)$. Then the limit exists and is isomorphic to $k[[x]]$.

```
import Mathlib
open CategoryTheory Polynomial Limits

universe u
```

```

namespace CAT_statement_S_0067

variable (k : Type u) [Field k]

noncomputable def F : NatoP ⇒ RingCat :=
{
  obj := fun ⟨n⟩ ⇒ RingCat.of ((k[X] / Ideal.span {(X ^ n : k[X])}))
  map := fun {A B} f ⇒ match A, B with
    | ⟨n⟩, ⟨m⟩ ⇒ match f with
      | <<<(f : m ≤ n)>>> ⇒
        RingCat.ofHom (Ideal.Quotient.factor
          ↪ (Ideal.span_singleton_le_span_singleton.mpr (pow_dvd_pow X f)))
}

lemma quotCommTrunc {n : ℕ} (p : k[X]) : (PowerSeries.trunc n p : k[X]) = (p
  ↪ : k[X] / Ideal.span {(X ^ n : k[X])}) := by
  rw [Ideal.Quotient.eq, Ideal.mem_span_singleton, X_pow_dvd_iff]
  intro d hd
  simp [PowerSeries.coeff_trunc, hd]

noncomputable def truncQuot (n : ℕ) : PowerSeries k →+ RingCat.of ((k[X] /
  ↪ Ideal.span {(X ^ n : k[X])})) where
  toFun := fun x ⇒ PowerSeries.trunc n x
  map_zero' := by simp
  map_one' := by
    match n with
    | 0 ⇒ rw [show X^0 = 1 by simp, Ideal.span_singleton_one]
      simp [Ideal.Quotient.zero_eq_one_iff]
    | n + 1 ⇒ simp
  map_add' := by simp
  map_mul' := fun x y ⇒ by
    rw [← PowerSeries.trunc_trunc_mul_trunc, ← coe_mul, ← (Ideal.Quotient.mk
      ↪ _).map_mul, quotCommTrunc k _]

noncomputable def cone_F : Cone (F k) :=
{
  pt := RingCat.of (PowerSeries k)
  π := {
    app := fun ⟨n⟩ ⇒ RingCat.ofHom (truncQuot k n)
    naturality := by
      rintro ⟨n⟩ ⟨m⟩ <<<(l : m ≤ n)>>>
      ext (x : PowerSeries k)
      simp [F, truncQuot, ← PowerSeries.trunc_trunc_of_le x l,
        ↪ quotCommTrunc k]
  }
}

theorem power_series_islimit : Nonempty (IsLimit (cone_F k)) := by
  sorry

end CAT_statement_S_0067

```

Problem 68 (Limit - High). — *Theorem: There is a category \mathcal{C} such that there exists two regular epimorphisms $f : c \rightarrow d$ and $g : c' \rightarrow d'$ in which the product of f and g is not regularly epic.*

```

import Mathlib

open CategoryTheory Limits

universe u

theorem regular_epimorphism_not_product_regular_epimorphism :  $\exists$  (C : Type
   $\hookrightarrow$  (u+1)) (inst : Category C) (c d c' d' : C) (f : c  $\rightarrow$  d) (g : c'  $\rightarrow$  d')
   $\hookrightarrow$  (inst1 : RegularEpi f) (inst2 : RegularEpi g) (hasProd1 :
   $\hookrightarrow$  HasBinaryProduct c c') (hasProd2 : HasBinaryProduct d d'), IsEmpty
   $\hookrightarrow$  (RegularEpi (prod.map f g)) := by
  sorry

```

Problem 69 (Limit - High). — *Theorem: An abelian group is torsion free if and only if it is a directed colimit in $\mathcal{A}b$ of free abelian groups.*

```

import Mathlib

open CategoryTheory Limits

theorem torsionFree_iff_isFilteredColimit_free
  (A : ModuleCat  $\mathbb{Z}$ ) :
  NoZeroSMulDivisors  $\mathbb{Z}$  A  $\leftrightarrow$ 
   $\exists$  (J : Type) (_ : SmallCategory J) (_ : IsFiltered J)
    (F : J  $\Rightarrow$  ModuleCat  $\mathbb{Z}$ ),
    ( $\forall$  j : J, Module.Free  $\mathbb{Z}$  (F.obj j))  $\wedge$ 
    Nonempty (A  $\cong$  colimit F) := by
  sorry

```

Problem 70 (Limit - High). — *Definition: A concrete category (\mathcal{C}, U) over \mathcal{B} is said to **have (small) concrete limits** if \mathcal{C} has all small limits and U preserves them.*

Theorem: Let (\mathcal{C}, U) have small concrete limits. Then U reflects small limits if and only if U reflects isomorphisms.

```

import Mathlib

open CategoryTheory Limits

variable {C : Type} u { [Category.{v} C]
variable {D : Type} u' { [Category.{v'} D]
variable (U : C  $\Rightarrow$  D)

theorem reflects_limits_iff_reflects_isomorphisms_preserves_limits
  [HasLimitsOfSize.{v, v'} C]
  [PreservesLimitsOfSize.{v, v'} U]
  [CategoryTheory.Functor.Faithful U]
  :
  ReflectsLimitsOfSize.{v, v'} U  $\leftrightarrow$  U.ReflectsIsomorphisms := by
  sorry

```

Problem 71 (Limit - High). — *Definition:* A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to **lift limits** if for every diagram $D : \mathcal{J} \rightarrow \mathcal{C}$ and every limit L of $F \circ D$, there exists a limit $L' \in \mathcal{D}$ such that $F(L') \cong L$.

Theorem: A functor that lifts equalizers is faithful if and only if it reflects epimorphisms.

```
import Mathlib

open CategoryTheory Limits

namespace CAT_statement_S_0071

universe uC vC uD vD w w'

variable {C : Type uC} [Category.{vC} C]
variable {D : Type uD} [Category.{vD} D]
variable (F : C  $\Rightarrow$  D)
variable {J : Type w} [Category.{w'} J]

class LiftsLimit (K : J  $\Rightarrow$  C) (F : C  $\Rightarrow$  D) : Prop where
  lifts {c : Cone (K  $\gg$  F)} (hc : IsLimit c) :
     $\exists$  c' : Cone K, Nonempty (IsLimit c')  $\wedge$  Nonempty (F.mapCone c'  $\cong$  c)

class LiftsLimitsOfShape (J : Type w) [Category.{w'} J] (F : C  $\Rightarrow$  D) : Prop
 $\hookrightarrow$  where
  liftsLimit :  $\forall$  {K : J  $\Rightarrow$  C}, LiftsLimit K F := by infer_instance

theorem functor_faithful_iff_reflectsEpimorphisms_of_liftsEqualizers
  [LiftsLimitsOfShape Limits.WalkingParallelPair F] :
    F.Faithful  $\leftrightarrow$  F.ReflectsEpimorphisms := by
  sorry

end CAT_statement_S_0071
```

Problem 72 (Limit - High). — *Theorem:* A full subcategory of $\mathcal{T}op^{CH}$ is reflective in $\mathcal{T}op^{CH}$ if and only if it is cocomplete.

```
import Mathlib

open CategoryTheory Limits Topology

universe u

variable {D : Type (u+1)} [Category.{u} D]
variable (i : D  $\Rightarrow$  CompHaus.{u})
variable [CategoryTheory.Functor.Full i] [CategoryTheory.Functor.Faithful i]

theorem
 $\hookrightarrow$  reflective_iff_cocomplete_and_contains_nonempty_of_full_subcategory_CompHaus
 $\hookrightarrow$  :
  Nonempty (CategoryTheory.Reflective i)  $\leftrightarrow$ 
    (Nonempty (HasColimits D)  $\wedge$   $\exists$  X : D, Nonempty (i.obj X)) := by
  sorry
```

Problem 73 (Limit - High). — *Definition:* A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to **lift limits** if for every diagram $D : \mathcal{J} \rightarrow \mathcal{C}$ and every limit L of $F \circ D$, there exists a limit $L' \in \mathcal{D}$ such that $F(L') \cong L$.

Theorem: The forgetful functor $U : \mathcal{T}op \rightarrow \mathcal{S}et$ lifts limits, but not reflects limits.

```

import Mathlib

open CategoryTheory Limits

namespace CAT_statement_S_0073

universe w' w₂' w w₂ v₁ v₂ v₃ u₁ u₂ u₃

variable {C : Type u₁} [Category.{v₁} C]
variable {D : Type u₂} [Category.{v₂} D]
variable {J : Type w} [Category.{w'} J] {K : J ⇒ C}

class LiftsLimit (K : J ⇒ C) (F : C ⇒ D) : Prop where
  lifts {c : Cone (K » F)} (hc : IsLimit c) :
    ∃ c' : Cone K, Nonempty (IsLimit c') ∧ Nonempty (F.mapCone c' ≅ c)

class LiftsLimitsOfShape (J : Type w) [Category.{w'} J] (F : C ⇒ D) : Prop
  ↪ where
  liftsLimit : ∀ {K : J ⇒ C}, LiftsLimit K F := by infer_instance

@[nolint checkUnivs, pp_with_univ]
class LiftsLimitsOfSize (F : C ⇒ D) : Prop where
  liftsLimitsOfShape : ∀ {J : Type w} [Category.{w'} J], LiftsLimitsOfShape J
    ↪ F := by
    infer_instance

abbrev LiftsLimits (F : C ⇒ D) :=
  LiftsLimitsOfSize.{v₂, v₂} F

theorem TopCat_forget_lifts_and_not_reflects_limits :
  LiftsLimits (forget TopCat) ∧ IsEmpty (ReflectsLimits (forget TopCat)) :=
  ↪ by
  sorry

end CAT_statement_S_0073

```

Problem 74 (Cocompletion - Medium). — *Theorem: Let \mathcal{C} be a small category. A category \mathcal{L} containing \mathcal{C} as a full subcategory is a pro-completion of \mathcal{C} if and only if the following conditions hold:*

- (1) \mathcal{L} has cofiltered colimits,
- (2) every object of \mathcal{L} is the colimit of a cofiltered diagram in \mathcal{C} , and
- (3) every object of \mathcal{C} is finitely copresentable in \mathcal{L} .

Reference: Corollary A.5. JÍ ADÁMEK, LIANG-TING CHEN, STEFAN MILIUS and HENNINGURBAT, Reiterman's Theorem on Finite Algebras for a Monad, <https://arxiv.org/pdf/2101.00942>

```

import Mathlib

open CategoryTheory Limits

universe u v w u₁ v₁

namespace CAT_statement_S_0074

noncomputable section

```

```

abbrev Pro (C : Type u) [SmallCategory C] := (CategoryTheory.Ind (Cop))op

abbrev proYoneda (C : Type u) [SmallCategory C] : C ⇒ Pro C :=
  CategoryTheory.opOp C » (CategoryTheory.Ind.yoneda (C := Cop)).op

def HasCofilteredColimits (L : Type u1) [Category.{v1} L] : Prop :=
  ∀ (J : Type w) [SmallCategory J] [IsCofiltered J], HasColimitsOfShape J L

def IsFinitelyCopresentable {L : Type u1} [Category.{v1} L] (X : L) : Prop :=
  CategoryTheory.IsFinitelyPresentable.{w} (C := Lop) (Opposite.op X)

def IsCofilteredColimitOf
  {C : Type u} [SmallCategory C] {L : Type u1} [Category.{v1} L]
  (ι : C ⇒ L) (X : L) : Prop :=
  ∃ (J : Type w) (hJ : SmallCategory J) (hC : IsCofiltered J), by
    let _ := hJ
    let _ := hC
    exact ∃ (F : J ⇒ C) (t : Cocone (F » ι)),
      Nonempty (IsColimit t) ∧ Nonempty (t.pt ≅ X)

def IsProCompletion
  {C : Type u} [SmallCategory C] {L : Type u1} [Category.{v1} L]
  (ι : C ⇒ L) : Prop :=
  ∃ (e : L ≅ Pro C), Nonempty (ι » e.functor ≅ proYoneda C)

def ProCompletionConditions
  {C : Type u} [SmallCategory C] {L : Type u1} [Category.{v1} L]
  (ι : C ⇒ L) : Prop :=
  HasCofilteredColimits.{w} L ∧
  (∀ X : L, IsCofilteredColimitOf.{u, w} ι X) ∧
  (∀ c : C, IsFinitelyCopresentable.{w} (ι.obj c))

theorem isProCompletion_iff_intrinsic_conditions
  {C : Type u} [SmallCategory C] {L : Type u1} [Category.{v1} L]
  (ι : C ⇒ L) [CategoryTheory.Functor.Full ι]
  ↪ [CategoryTheory.Functor.Faithful ι] :
  IsProCompletion (ι := ι) ↔ ProCompletionConditions (ι := ι) := by
  sorry

end

end CAT_statement_S_0074

```

Problem 75 (Cocompletion - High). — *Definition: A category is called sifted if the category of cocones over any finite discrete family of objects in it is connected.*

Notation: $\text{Rec}(\mathcal{C})$:= free cocompletion of \mathcal{C} under reflexive coequalizers.

Theorem: For a sifted category with pullbacks \mathcal{C} , $\text{Rec}(\mathcal{C})$ is filtered.


```

import Mathlib

namespace CAT_statement_S_0075

open CategoryTheory Limits

universe u v

namespace CategoryTheory.Limits
open Limits Functor
variable {C : Type u} [Category.{v} C]

variable (C) in

abbrev Psh (C : Type u) [Category.{v} C] : Type (max u (v + 1)) :=
  CoP ⇒ Type v

inductive RecObjectPresentation : Psh C → Type (max u (v + 1))
| ofYoneda (X : C) :
  RecObjectPresentation ((yoneda : C ⇒ Psh C).obj X)
| iso {A B : Psh C} (P : RecObjectPresentation A) (i : A ≅ B) :
  RecObjectPresentation B
| reflexiveCoeq {A B : Psh C}
  (PA : RecObjectPresentation A) (PB : RecObjectPresentation B)
  (f g : A → B) [IsReflexivePair f g] [HasCoequalizer f g] :
  RecObjectPresentation (coequalizer f g)

structure IsRecObject (A : Psh C) : Prop where
  mk' :: nonempty_presentation : Nonempty (RecObjectPresentation A)

theorem IsRecObject.mk (A : Psh C) (P : RecObjectPresentation A) :
  IsRecObject A :=
  ⟨⟨P⟩⟩

theorem isRecObject_yoneda (X : C) :
  IsRecObject (C := C) ((yoneda : C ⇒ Psh C).obj X) :=
  ⟨⟨RecObjectPresentation.ofYoneda (C := C) X⟩⟩

theorem isRecObject_coequalizer
  {A B : Psh C} (hA : IsRecObject (C := C) A) (hB : IsRecObject (C := C) B)
  (f g : A → B) [IsReflexivePair f g] :
  IsRecObject (C := C) (coequalizer f g) := by
  classical
  rcases hA.nonempty_presentation with ⟨PA⟩
  rcases hB.nonempty_presentation with ⟨PB⟩
  letI : HasCoequalizer f g := by infer_instance
  exact ⟨⟨RecObjectPresentation.reflexiveCoeq (C := C) PA PB f g⟩⟩

end CategoryTheory.Limits

```

```

namespace CategoryTheory

open Limits

variable {C : Type u} [Category.{v} C]

variable (C) [LocallySmall C] in

def Rec : Type (max u (v + 1)) :=
  ShrinkHoms (ObjectProperty.FullSubcategory (IsRecObject (C := C)))

noncomputable instance : Category.{max u v} (Rec C) :=
  inferInstanceAs <| Category.{max u v}
    (ShrinkHoms (ObjectProperty.FullSubcategory (IsRecObject (C := C))))

noncomputable def Rec.equivalence :
  Rec C ≅ ObjectProperty.FullSubcategory (IsRecObject (C := C)) :=
  (ShrinkHoms.equivalence _).symm

theorem sifted_with_pullbacks_Rec_is_filtered {C : Type u} [Category.{v} C]
  [IsSifted C] [HasPullbacks C] :
  IsFiltered (Rec C) := by
  sorry

end CategoryTheory

end CAT_statement_S_0075

```

Problem 76 (Cocompletion - High). — *Theorem: Let $\mathcal{S}et^{fin}$ be the category of finite sets and functions. Its pro-completion is the category*

$$\mathrm{Pro}(\mathcal{S}et^{fin}) = \mathcal{S}tone$$

of Stone spaces, i.e. compact topological spaces in which distinct elements can be separated by clopen subsets. Morphisms are the continuous functions.

Reference: JIÍ ADÁMEK, LIANG-TING CHEN, STEFAN MILIUS and HENNINGURBAT, Reiterman's Theorem on Finite Algebras for a Monad, <https://arxiv.org/pdf/2101.00942>

```

import Mathlib

open CategoryTheory

universe v u

abbrev Pro (C : Type u) [Category.{v, u} C] : Type (max u (v + 1)) := (Ind
  ↪ (Cop)op)op

theorem pro_fintypecat_equiv_profinite : Nonempty ((Pro (FintypeCat)) ≅
  ↪ Profinite) := sorry

```

Problem 77 (Cocompletion - High). — *Notation:*

$\mathrm{Sind}(\mathcal{C}) :=$ free cocompletion of \mathcal{C} under small sifted colimits;

$\mathrm{Ind}(\mathcal{C}) :=$ free cocompletion of \mathcal{C} under small filtered colimits;

$\mathrm{Rec}(\mathcal{C}) :=$ free cocompletion of \mathcal{C} under reflexive coequalizers.

Theorem: Let \mathcal{C} be a category with pullbacks. Then $\text{Sind}(\mathcal{C}) = \text{Ind}(\text{Rec}(\mathcal{C}))$

Reference: Theorem 5.1, Chen Ruiyuan 2021, On sifted colimits in the presence of pullbacks, arXiv:2109.12708

```

import Mathlib

namespace CAT_statement_S_0077

open CategoryTheory Limits

universe u v

namespace CategoryTheory.Limits

open Limits Functor
variable {C : Type u} [Category.{v} C]

abbrev Psh (C : Type u) [Category.{v} C] : Type (max u (v + 1)) :=
  CoP ⇒ Type v

inductive RecObjectPresentation : Psh C → Type (max u (v + 1))
| ofYoneda (X : C) :
  RecObjectPresentation ((yoneda : C ⇒ Psh C).obj X)
| iso {A B : Psh C} (P : RecObjectPresentation A) (i : A ≅ B) :
  RecObjectPresentation B
| reflexiveCoeq {A B : Psh C}
  (PA : RecObjectPresentation A) (PB : RecObjectPresentation B)
  (f g : A → B) [IsReflexivePair f g] [HasCoequalizer f g] :
  RecObjectPresentation (coequalizer f g)

structure IsRecObject (A : Psh C) : Prop where
  mk' :: nonempty_presentation : Nonempty (RecObjectPresentation A)

theorem IsRecObject.mk (A : Psh C) (P : RecObjectPresentation A) :
  IsRecObject A :=
  <<P>>

theorem isRecObject_yoneda (X : C) :
  IsRecObject (C := C) ((yoneda : C ⇒ Psh C).obj X) :=
  <<RecObjectPresentation.ofYoneda (C := C) X>>

theorem isRecObject_coequalizer
  {A B : Psh C} (hA : IsRecObject (C := C) A) (hB : IsRecObject (C := C) B)
  (f g : A → B) [IsReflexivePair f g] :
  IsRecObject (C := C) (coequalizer f g) := by
  classical
  rcases hA.nonempty_presentation with <PA>
  rcases hB.nonempty_presentation with <PB>
  letI : HasCoequalizer f g := by infer_instance
  exact <<RecObjectPresentation.reflexiveCoeq (C := C) PA PB f g>>

end CategoryTheory.Limits

```

```

namespace CategoryTheory

open Limits

variable {C : Type u} [Category.{v} C]

variable (C) in

def Rec : Type (max u (v + 1)) :=
  ShrinkHoms (ObjectProperty.FullSubcategory (IsRecObject (C := C)))

noncomputable instance : Category.{max u v} (Rec C) :=
  inferInstanceAs <| Category.{max u v}
    (ShrinkHoms (ObjectProperty.FullSubcategory (IsRecObject (C := C))))

noncomputable def Rec.equivalence :
  Rec C ≅ ObjectProperty.FullSubcategory (IsRecObject (C := C)) :=
  (ShrinkHoms.equivalence _).symm

end CategoryTheory

namespace CategoryTheory.Limits
open Limits Functor
variable {C : Type u} [Category.{v} C]

structure SindObjectPresentation (A : Cop ⇒ Type v) where

  I : Type v

  [I : SmallCategory I]
  [hI : IsSifted I]

  F : I ⇒ C

  ι : F » yoneda → (Functor.const I).obj A

  isColimit : IsColimit (Cocone.mk A ι)

namespace SindObjectPresentation

@[simps]
def yoneda (X : C) : SindObjectPresentation (yoneda.obj X) where
  I := Discrete PUnit.{v + 1}
  F := Functor.fromPUnit X
  ι := { app := fun _ ⇒ 1! _ }
  isColimit :=
    { desc := fun s ⇒ s.ι.app ⟨PUnit.unit⟩
      uniq := fun _ _ h ⇒ h ⟨PUnit.unit⟩ }

end SindObjectPresentation

```

```

structure IsSindObject (A : Cop ⇒ Type v) : Prop where
  mk' :: nonempty_presentation : Nonempty (SindObjectPresentation A)

theorem IsSindObject.mk {A : Cop ⇒ Type v} (P : SindObjectPresentation A) :
  ↪ IsSindObject A :=
  ⟨⟨P⟩⟩

theorem isSindObject_yoneda (X : C) : IsSindObject (yoneda.obj X) :=
  .mk <| SindObjectPresentation.yoneda X

end CategoryTheory.Limits

namespace CategoryTheory

open Limits

variable {C : Type u} [Category.{v} C]

variable (C) in

def Sind : Type (max u (v + 1)) :=
  ShrinkHoms (ObjectProperty.FullSubcategory (IsSindObject (C := C)))

noncomputable instance : Category.{max u v} (Sind C) :=
  inferInstanceAs <| Category.{max u v}
    (ShrinkHoms (ObjectProperty.FullSubcategory (IsSindObject (C := C))))

variable (C) in

noncomputable def Sind.equivalence :
  Sind C ≅ ObjectProperty.FullSubcategory (IsSindObject (C := C)) :=
  (ShrinkHoms.equivalence _).symm

end CategoryTheory

open CategoryTheory

theorem SindC_is_Ind_of_RecC {C : Type u} [SmallCategory C] :
  Nonempty (Sind C ≅ Ind (Rec C)) := by
  sorry

end CAT_statement_S_0077

```

Problem 78 (Cocompletion - High). — *Def.* For $F : \mathcal{C} \rightarrow \mathcal{D}$, we define the induced cocontinuous functor $\text{Lan}_{\text{Fop}} : \mathcal{P}sh(\mathcal{C}) \rightarrow \mathcal{P}sh(\mathcal{D})$, by $\phi \mapsto \phi \star yF$, where $\phi \star yF$ is the ϕ -weighted colimit of the diagram yF and y is the Yoneda embedding.

Notation: $\text{Sind}(\mathcal{C}) :=$ free cocompletion of \mathcal{C} under small sifted colimits;

Theorem: For any full and faithful $I : \mathcal{C} \rightarrow \mathcal{D}$ between small categories, $\phi \in [\mathcal{C}^{op}, \mathcal{S}et]$ is in $\text{Sind}(\mathcal{C})$ iff Lan_{Fop} is in $\text{Sind}(\mathcal{D})$.

Reference: Lemma 6.2, Chen Ruiyuan 2021, On sifted colimits in the presence of pullbacks, arXiv:2109.12708

```

import Mathlib

namespace CAT_statement_S_0078

open CategoryTheory Limits Functor

universe u v

namespace CategoryTheory

namespace Limits

variable {C : Type u} [Category.{v} C]

structure SindObjectPresentation (A : Cop ⇒ Type v) where
  I : Type v
  [I : SmallCategory I]
  [hI : IsSifted I]
  F : I ⇒ C
  ι : F » yoneda → (Functor.const I).obj A
  isColimit : IsColimit (Cocone.mk A ι)

structure IsSindObject (A : Cop ⇒ Type v) : Prop where
  mk' :: nonempty_presentation : Nonempty (SindObjectPresentation A)

theorem IsSindObject.mk {A : Cop ⇒ Type v} (P : SindObjectPresentation A) :
  ↪ IsSindObject A :=
  ⟨⟨P⟩⟩

end Limits

namespace Functor

def weightedColimitFunctor {J : Type v} [SmallCategory J] {E : Type u}
  ↪ [Category.{v} E]
  (W : Jop ⇒ Type v) (G : J ⇒ E) : E ⇒ Type v where
  obj X := W → G.op » (yoneda.obj X)
  map f h := h » (NatTrans.id G.op ▯ yoneda.map f)

abbrev WeightedColimitData {J : Type v} [SmallCategory J] {E : Type u}
  ↪ [Category.{v} E]
  (W : Jop ⇒ Type v) (G : J ⇒ E) (colim : E) :=
  (weightedColimitFunctor W G).CorepresentableBy colim

abbrev HasWeightedColimit {J : Type v} [SmallCategory J] {E : Type u}
  ↪ [Category.{v} E]
  (W : Jop ⇒ Type v) (G : J ⇒ E) :=
  (weightedColimitFunctor W G).IsCorepresentable

```

```

noncomputable def weightedColimit {J : Type v} [SmallCategory J] {E : Type u}
  ↳ [Category.{v} E]
  (W : Jop ⇒ Type v) (G : J ⇒ E) [h : HasWeightedColimit W G] : E :=
    h.has_corepresentation.choose

noncomputable def weightedColimitData {J : Type v} [SmallCategory J] {E :
  ↳ Type u} [Category.{v} E]
  (W : Jop ⇒ Type v) (G : J ⇒ E) [h : HasWeightedColimit W G] :
    WeightedColimitData W G (weightedColimit W G) :=
    h.has_corepresentation.choose_spec.some

end Functor

end CategoryTheory

open CategoryTheory Limits Functor

variable {C D : Type u} [SmallCategory C] [SmallCategory D]

def lanDiagram (F : C ⇒ D) : C ⇒ (Dop ⇒ Type u) := F » yoneda

noncomputable def lanPresheaf (F : C ⇒ D) (φ : Cop ⇒ Type u)
  [HasWeightedColimit φ (lanDiagram F)] : Dop ⇒ Type u :=
  weightedColimit φ (lanDiagram F)

theorem isSindObject_iff_isSindObject_lanPresheaf
  (I : C ⇒ D) [Full I] [Faithful I] (φ : Cop ⇒ Type u)
  [HasWeightedColimit φ (lanDiagram I)] :
    IsSindObject φ ↔ IsSindObject (lanPresheaf I φ) := by
  sorry

end CAT_statement_S_0078

```

Problem 79 (Abelian - Easy). — *Theorem: Let \mathcal{A} be an additive category. Let x, y, z be objects in \mathcal{A} . Then the composition $\text{hom}_{\mathcal{A}}(y, z) \times \text{hom}_{\mathcal{A}}(x, y) \rightarrow \text{hom}_{\mathcal{A}}(x, z)$ is bilinear map.*

```

import Mathlib

open CategoryTheory

variable {C : Type u} [Category.{v} C] [Preadditive C]

structure IsBilinear {X Y Z : C} (f : (Y → Z) → ((X → Y) → (X → Z))) : Prop
  ↳ where
    map_add_left : ∀ (a b : Y → Z) (g : X → Y),
      f (a + b) g = f a g + f b g

    map_add_right : ∀ (a : Y → Z) (g h : X → Y), f a (g + h) = f a g + f a h

    map_zero_left : ∀ (g : X → Y), f 0 g = 0

    map_zero_right : ∀ (a : Y → Z), f a 0 = 0

```

```

theorem compIsBilinear {X Y Z : C} :
  IsBilinear (fun (g : Y → Z) => (fun (f : (X → Y)) => f >> g)) := sorry

```

Problem 80 (Abelian - Easy). — *Theorem: Let \mathcal{A} be an abelian category and let f be a morphism in \mathcal{A} . Then f is an isomorphism if and only if f is monic and epic.*

```

import Mathlib

open CategoryTheory

variable {C : Type*} [Category C] [Abelian C]

theorem isIso_iff_mono_and_epi {X Y : C} (f : X → Y) :
  IsIso f ↔ (Mono f ∧ Epi f) := by
  sorry

```

Problem 81 (Abelian - Easy). — *Theorem: Let \mathcal{A} be an abelian category and let f be a morphism in \mathcal{A} . Then f is monic if and only if $\ker(f) = 0$.*

```

import Mathlib

open CategoryTheory Limits Category

variable {C : Type*} [Category C] [Abelian C]

theorem mono_iff_isZero_kernel {X Y : C} (f : X → Y) :
  Mono f ↔ IsZero (kernel f) := by
  sorry

```

Problem 82 (Abelian - Easy). — *Theorem: \mathbb{k} is the unique (up to isomorphism) simple object in $\text{Vect}_{\mathbb{k}}$.*

```

import Mathlib

open Module

variable (k : Type u) [Field k]

instance isSimpleModule_self : IsSimpleModule k k := by
  constructor
  intro N
  have : IsSimpleOrder (Submodule k k) := by infer_instance
  exact eq_bot_or_eq_top N

theorem unique_simple_object (M : Type v) [AddCommGroup M] [Module k M]
  ↪ [IsSimpleModule k M] :
  Nonempty (M ≃1 [k] k) := by
  sorry

```

Problem 83 (Abelian - Easy). — *Theorem: \mathbb{Z}_p is simple object in \mathcal{A}^b when p is prime number.*


```

import Mathlib

open CategoryTheory

variable (p : ℕ) [Fact p.Prime]

theorem ZMod_simple : CategoryTheory.Simple (ModuleCat.of ℤ (ZMod p)) := by
  sorry

```

Problem 84 (Abelian - Easy). — *Theorem: $\mathcal{G}rp$ is not an additive category.*

```

import Mathlib

open CategoryTheory Limits

def IsAdditiveCategory (C : Type u) [Category.{v} C] : Prop :=
  ∃ (_ : Preadditive C), HasZeroObject C ∧ HasFiniteBiproducts C

theorem Grp_not_is_additive : IsEmpty (IsAdditiveCategory Grp.{u}) := by
  sorry

```

Problem 85 (Abelian - Medium). — *Definition: A functor is called **left exact** if it preserves all finite limits.*

Theorem: Let \mathcal{A} and \mathcal{B} be abelian categories and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. Then F is left exact if and only if F is additive and F maps exact sequence $0 \rightarrow x \rightarrow y \rightarrow z$ to $0 \rightarrow F(x) \rightarrow F(y) \rightarrow F(z)$.

```

import Mathlib

open CategoryTheory Functor Limits ShortComplex

variable {C D : Type*} [Category C] [Category D]
variable [Abelian C] [Abelian D]

theorem preservesFiniteLimits_tfae
  (F : C ⇒ D) [F.Additive] : List.TFAE
  [
    ∀ (S : ShortComplex C), S.ShortExact → (S.map F).Exact ∧ Mono (F.map
      ↪ S.f),
    ∀ (S : ShortComplex C), S.Exact ∧ Mono S.f → (S.map F).Exact ∧ Mono
      ↪ (F.map S.f),
    ∀ {X Y : C} (f : X → Y), PreservesLimit (parallelPair f 0) F,
    PreservesFiniteLimits F
  ] := by
  sorry

```

Problem 86 (Abelian - Medium). — *Theorem: Let \mathcal{A} be an abelian category and let $P \in \mathcal{A}$. Then $\text{hom}_{\mathcal{A}}(P, -) : \mathcal{A} \rightarrow \mathcal{A}b$ is right exact if and only if $\text{hom}_{\mathcal{A}}(P, -) : \mathcal{A} \rightarrow \mathcal{A}b$ preserves epimorphism.*

```

import Mathlib

open CategoryTheory Limits Opposite

```

```

variable {A : Type u} [Category.{v} A] [Abelian A]

theorem hom_rightExact_iff_preserves_epi (P : A) :
  PreservesFiniteColimits (preadditiveCoyoneda.obj (op P)) ↔
  Functor.PreservesEpimorphisms (preadditiveCoyoneda.obj (op P)) := by
  sorry

```

Problem 87 (Abelian - Medium). — *Definition: An Abelian category \mathcal{A} is called **semisimple** if any short exact sequence in \mathcal{A} is splittable.*

Theorem: Let \mathcal{A} be an abelian category. Then the followings are equivalent:

1. \mathcal{A} is semisimple;
2. any object in \mathcal{A} is injective;
3. any object in \mathcal{A} is projective.

```

import Mathlib

open CategoryTheory Limits

variable {A : Type u} [Category.{v} A] [Abelian A]

def IsSemisimple (A : Type u) [Category.{v} A] [Abelian A] : Prop :=
  ∀ (S : ShortComplex A), S.ShortExact → Nonempty S.Splitting

theorem isSemisimple_iff_injective_iff_projective :
  (IsSemisimple A ↔ ∀ (X : A), Injective X) ∧
  (IsSemisimple A ↔ ∀ (X : A), Projective X) := by
  sorry

```

Problem 88 (Abelian - Medium). — *Theorem: Let \mathcal{A} be an abelian category. If x, y are simple objects in \mathcal{A} . Then each non-zero $f : x \rightarrow y$ are isomorphism. In particular, if x is simple, then $\text{hom}_{\mathcal{A}}(x, x)$ is a division ring; if $x \neq y$, then $\text{hom}_{\mathcal{A}}(x, y) = 0$.*

```

import Mathlib

open CategoryTheory

variable {A : Type*} [Category A] [Abelian A]

theorem simple_objects_nonzero_morphisms_iso
  {x y : A} [Simple x] [Simple y] (f : x → y) (h : f ≠ 0) :
  IsIso f := by
  sorry

theorem simple_object_end_is_division_ring
  (x : A) [Simple x] :
  Nonempty (DivisionRing (CategoryTheory.End x)) := by
  sorry

```

```

theorem simple_objects_hom_zero_of_ne
  {x y :  $\mathcal{A}$ } [Simple x] [Simple y] (hxy : x  $\neq$  y) :
   $\forall$  f : x  $\rightarrow$  y, f = 0 := by
  sorry

```

Problem 89 (Abelian - Medium). — *Theorem: Let \mathcal{A} be an additive category. Let x be a Schurian simple object, then it is both monosimple and episimple.*

```

import Mathlib

open CategoryTheory

class IsSplitMonoCategory (A : Type*) [Category A] where
  splitMonoOfMono {X Y : A} (f : X  $\rightarrow$  Y) [Mono f] : Nonempty (SplitMono f)

class IsSplitEpiCategory (A : Type*) [Category A] where
  splitEpiOfEpi {X Y : A} (f : X  $\rightarrow$  Y) [Epi f] : Nonempty (SplitEpi f)

variable {A : Type*} [Category A] [Preadditive A] [IsSplitMonoCategory A]
   $\hookrightarrow$  [IsSplitEpiCategory A]

theorem schur_simple_monosimple_and_episimple
  (x : A) [NoZeroDivisors (End x)] :
  ( $\forall$  (y : A) (f : y  $\rightarrow$  x) [Mono f], f = 0  $\vee$  IsIso f)  $\wedge$ 
  ( $\forall$  (y : A) (g : x  $\rightarrow$  y) [Epi g], g = 0  $\vee$  IsIso g) := by
  sorry

```

Problem 90 (Abelian - High). — *Definition: A category is called **normal** if each monomorphism is a kernel. Definition: A category is called **conormal** if each epimorphism is a cokernel. Definition: A category is called **binormal** if it is both normal and conormal. Definition: Let \mathcal{C} be a category. An object $c \in \mathcal{C}$ is called **mono-simple** if it has no proper subobjects. An object $c \in \mathcal{C}$ is called **epi-simple** if it has no proper quotient objects. Theorem: Let \mathcal{A} be a binormal category. Then an object is mono-simple if and only if it is epi-simple.*

```

import Mathlib

open CategoryTheory

variable {A : Type*} [Category A] [Limits.HasZeroMorphisms A]
  [IsNormalMonoCategory A]
  [IsNormalEpiCategory A]

  [Limits.HasKernels A]
  [Limits.HasCokernels A]

theorem binormal_mono_simple_iff_epi_simple (x : A) :
  ( $\forall$  (y : A) (f : y  $\rightarrow$  x) [Mono f], f = 0  $\vee$  IsIso f)  $\leftrightarrow$ 
  ( $\forall$  (y : A) (g : x  $\rightarrow$  y) [Epi g], g = 0  $\vee$  IsIso g) := by
  sorry

```

Problem 91 (Monad - Easy). — *Theorem: For any monad (T, μ, η) on a category \mathcal{C} and let \mathcal{C}^T be its Eilenberg-Moore category. Let $U : \mathcal{C}^T \rightarrow \mathcal{C}$ be the forgetful functor, then it admits a left adjoint.*

```

import Mathlib

open CategoryTheory

variable {C : Type u1} [Category.{v1} C]

theorem monad_forget_has_left_adjoint (T : Monad C) :
  T.forget.IsRightAdjoint := by
  sorry

```

Problem 92 (Monad - Easy). — *Theorem: The forgetful functor $U : \mathcal{A}b_R \rightarrow \mathcal{A}b$ creates all colimits that $\mathcal{A}b$ admits.*

```

import Mathlib

open CategoryTheory Limits

variable {R : Type u} [CommRing R]

theorem ModuleCat.forgetReflectsColimits :
  Nonempty (ReflectsColimits (forget2 (ModuleCat R) AddCommGrp)) :=
  sorry

```

Problem 93 (Monad - Medium). — *Theorem: Suppose \mathcal{C} is cocomplete and $G : \mathcal{D} \rightarrow \mathcal{C}$ is monadic. Then \mathcal{D} is cocomplete if and only if \mathcal{D} has coequalizers.*

```

import Mathlib

open CategoryTheory Limits

universe uC uD vC vD w w'

variable {C : Type uC} [Category.{vC} C]
variable {D : Type uD} [Category.{vD} D]
variable (G : D  $\Rightarrow$  C)

theorem cocomplete_iff_hasCoequalizers_of_monadic
  [HasColimitsOfSize.{w, w'} C] [MonadicRightAdjoint G] :
  HasColimitsOfSize.{w, w'} D  $\Leftrightarrow$  HasCoequalizers D := by
  sorry

```

Problem 94 (Monad - Medium). — *Theorem: If $U : \mathcal{C} \rightarrow \mathcal{B}$ is an isomorphism-closed full reflective embedding, then the associated monad is idempotent.*

```

import Mathlib

open CategoryTheory Functor

namespace CAT_statement_S_0094

```

```

variable {C : Type*} [Category C]
variable {B : Type*} [Category B]

noncomputable def monadOfRightAdjoint (U : Functor C B) [IsRightAdjoint U] :
  ↳ Monad B :=
  (Adjunction.ofIsRightAdjoint U).toMonad

def IsIsoClosed (U : Functor C B) := ∀ (x : C) (y : B) (f : U.obj x → y)
  ↳ [IsIso f], ∃ (z : C), y = U.obj z

variable {U : Functor C B} [Full U] [Faithful U] [IsRightAdjoint U]
{h_inj : Function.Injective U.obj}
{h_iso_closed : IsIsoClosed U}

theorem monad_idempotent_of_full_reflective_embedding :
  let T : Monad B := monadOfRightAdjoint U
  IsIso T.μ := by
  sorry

end CAT_statement_S_0094

```

Problem 95 (Monad - Medium). — *Theorem: If \mathcal{D} admits coequalizers, a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ is monadic if G has a left adjoint, conservative and preserves coequalizers.*

```

import Mathlib

open CategoryTheory Limits

universe u1 u2 v1

variable {C : Type u1} {D : Type u2} [Category.{v1} C] [Category.{v1} D]
variable {G : D ⇒ C} {F : C ⇒ D} (adjFG : F ⊣ G)
variable [HasCoequalizers D]
variable [G.ReflectsIsomorphisms]
variable [PreservesColimitsOfShape WalkingParallelPair G]

theorem monadicOfConservativePreservesCoequalizers :
  Nonempty (MonadicRightAdjoint G) := by
  sorry

```

Problem 96 (Monad - Medium). — *Theorem: Consider the adjunction $- \otimes_{\mathbb{Z}} R : \mathcal{A}^b \rightarrow \mathcal{A}^b_R$ and $U : \mathcal{A}^b_R \rightarrow \mathcal{A}^b$. We obtain a monad T . The T -modules are right R -modules.*

```

import Mathlib

open CategoryTheory

namespace CAT_statement_S_0096

universe u v

```

```

variable (R : Type u) [CommRing R]

abbrev intToR :  $\mathbb{Z} \rightarrow^{+*} R$  := Int.castRingHom R

noncomputable abbrev U : ModuleCat.{max u v} R  $\Rightarrow$  ModuleCat.{max u v}  $\mathbb{Z}$  :=
  ModuleCat.restrictScalars (intToR R)

noncomputable abbrev F : ModuleCat.{max u v}  $\mathbb{Z} \Rightarrow$  ModuleCat.{max u v} R :=
  ModuleCat.extendScalars (intToR R)

noncomputable abbrev adj : F (R := R)  $\dashv$  U (R := R) :=
  ModuleCat.extendRestrictScalarsAdj (intToR R)

noncomputable abbrev T : Monad (ModuleCat.{max u v}  $\mathbb{Z}$ ) :=
  (adj (R := R)).toMonad

theorem t_algebra_equiv_modulecat :
  Nonempty (Monad.Algebra (T (R := R))  $\cong$  ModuleCat.{max u v} R) := by
  sorry

end CAT_statement_S_0096

```

Problem 97 (Monad - Medium). — *Theorem: Let \mathcal{C}, \mathcal{D} be categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ be a left adjoint functor to $G : \mathcal{D} \rightarrow \mathcal{C}$. Denote the induced monad of the adjunction $F \dashv G$ by $T := GF$. Let $K : \mathcal{D} \rightarrow \mathcal{C}^T$ be the comparison functor. If \mathcal{D} admits coequalizers, then K has a left adjoint.*

```

import Mathlib

open CategoryTheory Monad

universe u1 u2 v1

variable {C : Type u1} [Category.{v1} C] {D : Type u2} [Category.{v1} D]
variable (F : C  $\Rightarrow$  D) (G : D  $\Rightarrow$  C) (adj : F  $\dashv$  G)

theorem comparison_adjunction
  [∀ (A : adj.toMonad.Algebra), Limits.HasCoequalizer (F.map A.a)
    ↪ (adj.counit.app (F.obj A.A))] :
  ∃ K : adj.toMonad.Algebra  $\Rightarrow$  D, Nonempty (K  $\dashv$  comparison adj) := by
  sorry

```

Problem 98 (Monad - Medium). — *Definition: For any monad T on \mathcal{C} , we define a category Adj_T whose objects are adjunctions $(F : \mathcal{C} \rightarrow \mathcal{D}, G, \eta, \epsilon)$ which induce the same monad T , and a morphism between $(F : \mathcal{C} \rightarrow \mathcal{D}, G, \eta, \epsilon)$ and $(F' : \mathcal{C} \rightarrow \mathcal{D}', G', \eta', \epsilon')$ in Adj_T is given by a functor $K : \mathcal{D} \rightarrow \mathcal{D}'$ such that $KF = F'$ and $G'K = G$.*

Theorem: Let (T, μ, η) be a monad on a category \mathcal{C} . The Kleisli category \mathcal{C}_T is initial in Adj_T and the Eilenberg-Moore category \mathcal{C}^T is terminal,

```

import Mathlib

```

```

open CategoryTheory Monad

namespace CAT_statement_S_0098

variable {C : Type*} [Category C]

structure AdjCat (T : Monad C) where
  D : Type*
  [category : Category D]
  F : Functor C D
  U : Functor D C
  adj : F  $\dashv$  U
  monad_eq : T  $\cong$  Adjunction.toMonad adj

namespace AdjCat

variable {T : Monad C}

instance (X : AdjCat T) : Category X.D := X.category

structure Hom (X Y : AdjCat T) where
  K : Functor (X.D) (Y.D)
  comm_left : X.F  $\gg$  K = Y.F
  comm_right : K  $\gg$  Y.U = X.U

instance : Category (AdjCat T) where
  Hom X Y := Hom X Y
  id X :=
    { K := Functor.id X.D
      comm_left := Functor.comp_id X.F
      comm_right := Functor.id_comp X.U }
  comp f g :=
    { K := f.K  $\gg$  g.K
      comm_left := by
        rewrite [<-Functor.assoc, f.comm_left]
        exact g.comm_left
      comm_right := by
        rewrite [Functor.assoc, g.comm_right]
        exact f.comm_right }

end AdjCat

variable (T : Monad C)

def kleisli_adj_obj : AdjCat T :=
  { D := Kleisli T
    F := Kleisli.Adjunction.toKleisli T
    U := Kleisli.Adjunction.fromKleisli T
    adj := Kleisli.Adjunction.adj T
    monad_eq :=
      { hom :=
        { app := fun X  $\Rightarrow$  1 (T.obj X)
          app_μ (X : C) := by
            simp
            rewrite [Kleisli.Adjunction.adj]

```

```

      simp
      rewrite [Equiv.refl]
      simp }
  inv :=
  { app := fun X => 1 (T.obj X)
    app_μ (X : C) := by
      simp
      rewrite [Kleisli.Adjunction.adj]
      simp
      rewrite [Equiv.refl]
      simp } } }

theorem kleisli_initial : Nonempty (Limits.IsInitial (kleisli_adj_obj T)) :=
  ↪ by
  sorry

def eilenberg_moore_adj_obj : AdjCat T :=
  { D := T.Algebra
    F := Monad.free T
    U := Monad.forget T
    adj := Monad.adj T
    monad_eq :=
      { hom := { app := fun X => 1 (T.obj X) }
        inv := { app := fun X => 1 (T.obj X) } } }

theorem eilenberg_moore_terminal : Nonempty (Limits.IsTerminal
  ↪ (eilenberg_moore_adj_obj T)) := by
  sorry

end CAT_statement_S_0098

```

Problem 99 (Monad - Medium). — *Theorem: The monad associated with the forgetful functor $\mathcal{T}op \rightarrow \mathcal{S}et$ is idempotent.*

```

import Mathlib

open CategoryTheory

theorem monad_Top_idempotent : IsIso TopCat.adj_1.toMonad.μ := by
  sorry

```

Problem 100 (Monad - High). — *Theorem: Let \mathcal{C}, \mathcal{D} be categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ be a left adjoint functor to $G : \mathcal{D} \rightarrow \mathcal{C}$. Denote the induced monad of the adjunction $F \dashv G$ by T . The following statements are equivalent:*

1. *The comparison functor $K : \mathcal{D} \rightarrow \mathcal{C}^T$ is fully faithful.*
2. *For every $d \in \mathcal{D}$, the counit $\epsilon_d : FG(d) \rightarrow d$ is a coequalizer of*

$$FGFG(d) \begin{array}{c} \xrightarrow{\epsilon_{FG(d)}} \\ \xrightarrow{FG(\epsilon_d)} \end{array} FG(d)$$

3. *The functor G reflects split epimorphisms to regular epimorphisms*


```

import Mathlib

open CategoryTheory Limits

universe v u u'

namespace CAT_statement_S0100

variable {C : Type u} [Category.{v} C]
variable {D : Type u'} [Category.{v} D]

variable (F : C  $\Rightarrow$  D) (G : D  $\Rightarrow$  C)
variable (adj : F  $\dashv$  G)

abbrev FG : D  $\Rightarrow$  D := G  $\gg$  F

abbrev K : D  $\Rightarrow$  (adj.toMonad).Algebra :=
  Monad.comparison adj

abbrev epsFG (d : D) :
  (FG (F := F) (G := G)).obj ((FG (F := F) (G := G)).obj d)
   $\rightarrow$  (FG (F := F) (G := G)).obj d :=
  adj.counit.app ((FG (F := F) (G := G)).obj d)

abbrev FGeps (d : D) :
  (FG (F := F) (G := G)).obj ((FG (F := F) (G := G)).obj d)
   $\rightarrow$  (FG (F := F) (G := G)).obj d :=
  (FG (F := F) (G := G)).map (adj.counit.app d)

def counitCofork (d : D) :
  Cofork (epsFG (F := F) (G := G) adj d) (FGeps (F := F) (G := G) adj d) :=
  Cofork.of $\pi$  (adj.counit.app d) (by
    simp [epsFG, FGeps]
  )

def cond1 : Prop :=
  (K (F := F) (G := G) adj).Full  $\wedge$  (K (F := F) (G := G) adj).Faithful

def cond2 : Prop :=
   $\forall$  d : D, Nonempty (IsColimit (counitCofork (F := F) (G := G) adj d))

def IsRegularEpi' {X Y : D} (f : X  $\rightarrow$  Y) : Prop :=
  Nonempty (RegularEpi f)

def ReflectsSplitEpiToRegularEpi' (G : D  $\Rightarrow$  C) : Prop :=
   $\forall$  {X Y : D} (f : X  $\rightarrow$  Y), IsSplitEpi (G.map f)  $\rightarrow$  IsRegularEpi' (D := D) f

def cond3 : Prop :=
  ReflectsSplitEpiToRegularEpi' (G := G)

```

```

theorem K_fullyFaithful_tfae :
  List.TFAE
    [cond1 (F := F) (G := G) adj,
     cond2 (F := F) (G := G) adj,
     cond3 (G := G)] := by
  sorry
end CAT_statement_S0100

```