Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Spring 2018

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due Saturday, July 14, 2018, 11.59 pm

5 Homework lecture 5

5.1 Numerical derivatives: finite difference

- Recall the Taylor series for $f(x+h)$:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}\,f''(x) + \frac{h^3}{3!}\,f'''(x) + \frac{h^4}{4!}\,f''''(x) + \cdots \tag{5.1.1}$$

- Replace h by $-h$ to obtain the partner series for $f(x-h)$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}\,f''(x) - \frac{h^3}{3!}\,f'''(x) + \frac{h^4}{4!}\,f''''(x) + \cdots \tag{5.1.2}$$

- Recall from lectures that from eqs. (5.1.1) and (5.1.2) we can obtain the forward, backward and centered finite difference approximations for the first derivative $f'(x)$.

$$\begin{aligned} f'_{\rm fwd}(x) &\simeq \frac{f(x+h)-f(x)}{h}\,, \\ f'_{\rm back}(x) &\simeq \frac{f(x)-f(x-h)}{h}\,, \\ f'_{\rm cent}(x) &\simeq \frac{f(x+h)-f(x-h)}{2h}\,. \end{aligned} \tag{5.1.3}$$

- **As experience has demonstrated, if you do not understand how the above expressions are derived, THEN ASK.**
- **If you do not understand the words/sentences in Lecture 5, etc. THEN ASK.**
- **Send me an email, explain what you do not understand.**
- **Do not just keep quiet and produce nonsense in exams.**
- **Question: derive the expressions for $f'_{\rm fwd}(x)$, $f'_{\rm back}(x)$ and $f'_{\rm cent}(x)$ in eq. (5.1.3).**

  Yes this is copying bookwork. *Just to get you to read that bookwork.*

  Look up Lecture 5 and/or textbooks and/or friends, internet, etc.

5.2 Finite difference: unequal steps

- Suppose the forward and backward steps are not equal.
- Suppose the forward step is h_1 and the backward step is h_2 and $h_2 \neq h_1$.
  1. ***Don't laugh. I have had to do this for a large part of my career.***
  2. One does not always have a choice to use equal steps.
  3. This is part of "what the textbooks don't say" that I am sharing from my career.
- **Write the Taylor series for $f(x+h_1)$ and $f(x-h_2)$ up to $O(f''''(x))$.**
  1. See eqs. (5.1.1) and (5.1.2) for guidance.
  2. **Derive a numerical expression for the first derivative as follows:**

$$\frac{f(x+h_1) - f(x-h_2)}{h_1+h_2} = f'(x) + \text{first two error terms}\,. \tag{5.2.1}$$

  3. **Hence show that if $h_2 \neq h_1$ the leading error term is $O((h_1 - h_2)f''(x))$.**
- **Using only $f(x)$, $f(x+h_1)$ and $f(x-h_2)$, derive a finite difference approximation for the first derivative $f'(x)$, where the leading error term is $O(f'''(x))$.**
- **Using only $f(x)$, $f(x+h_1)$ and $f(x-h_2)$, derive a finite difference approximation for the second derivative $f''(x)$. The leading error term is $O(f'''(x))$ if $h1 \neq h_2$.** **Fix typo 3/11/2018**
- *If you have done your work correctly, your answers should reduce to the expressions in the lectures if $h_1 = h_2 = h$.*

See next page.

5.2.1 Students are having difficulty with Question 5.2.

- Here are some hints.
- Write a Taylor series for the following linear combination using weights a and b:

$$af(x+h_1) - bf(x-h_2) = \text{weighted sum of Taylor series} \tag{5.2.2}$$

- Symbolically, the Taylor series will look like this ($U_{0,1,2,3}$ depend on a, b, h_1 amd h_2):

$$af(x+h_1) - bf(x-h_2) = U_0 f(x) + U_1 f'(x) + U_2 f''(x) + U_3 f'''(x) + \dots \tag{5.2.3}$$

- **Find the expression for the coefficient U_2 of $f''(x)$ in the above Taylor series.**
- **Choose values for a and b to make the coefficient U_2 equal to zero.**
- Then you will obtain a Taylor series of the following form (because $U_2 = 0$).

$$af(x+h_1) - bf(x-h_2) = U_0 f(x) + U_1 f'(x) + U_3 f'''(x) + \dots \tag{5.2.4}$$

- Rearrange terms to obtain a finite difference expression for $f'(x)$ as follows.

$$f'(x) = \frac{af(x+h_1) - bf(x-h_2) - U_0 f(x)}{U_1} + O(f'''(x)) \,. \tag{5.2.5}$$

- *If you have done your work correctly, you should find that $a - b - U_0 = 0$.*
- *You should also find that your answer reduces to the expression in the lectures if $h_1 = h_2 = h$.*
- **Next do $f''(x)$. Write a linear combination using weights c and d.**

$$\begin{aligned} cf(x+h_1) + df(x-h_2) &= \text{weighted sum of Taylor series} \\ &= V_0 f(x) + V_1 f'(x) + V_2 f''(x) + V_3 f'''(x) + \dots \end{aligned} \tag{5.2.6}$$

- **Find the expression for the coefficient V_1 of $f'(x)$ in the above Taylor series.**
- **Choose values for c and d to make the coefficient V_1 equal to zero.**
- Then you will obtain a Taylor series of the following form (because $V_1 = 0$).

$$cf(x+h_1) + df(x-h_2) = V_0 f(x) + V_2 f''(x) + V_3 f'''(x) + \dots \tag{5.2.7}$$

- Rearrange terms to obtain a finite difference expression for $f''(x)$ as follows.

$$f''(x) = \frac{cf(x+h_1) + df(x-h_2) - V_0 f(x)}{V_2} + O(f'''(x)) \,. \tag{5.2.8}$$

- **If $h_1 \neq h_2$, the coefficient of $f'''(x)$ will not cancel to zero.**
- *If you have done your work correctly, you should find that $c + d - V_0 = 0$.*
- *You should also find that your answer reduces to the expression in the lectures if $h_1 = h_2 = h$.*

5.3 Finite differences: some observations

- Using equal forward and backward steps, the centered finite difference for $f'(x)$ is

$$f'_{\rm cent}(x) = \frac{f(x+h) - f(x-h)}{2h} \,. \tag{5.3.1}$$

- Now suppose that $f(x)$ is a perfect square $f(x) = (g(x))^2$. From calculus we know that

$$\frac{df(x)}{dx} = 2g(x)\,\frac{dg(x)}{dx} \,. \tag{5.3.2}$$

- The centered finite difference numerical derivative for $g'(x)$ is

$$g'_{\rm cent}(x) = \frac{g(x+h) - g(x-h)}{2h} \,. \tag{5.3.3}$$

- However, the two finite difference expressions are not equal in general:

$$\begin{aligned} f'_{\rm cent}(x) &\neq 2g(x)\,g'_{\rm cent}(x) \,, \\ \frac{f(x+h) - f(x-h)}{2h} &\neq 2g(x)\,\frac{g(x+h) - g(x-h)}{2h} \,. \end{aligned} \tag{5.3.4}$$

- For example if $g(x) = e^x$, then $f(x) = e^{2x}$ and

$$\begin{aligned} f'_{\rm cent}(x) &= \frac{e^{2(x+h)} - e^{2(x-h)}}{2h} = e^{2x}\,\frac{e^{2h} - e^{-2h}}{2h} = 2e^{2x}\left(1 + \frac{2h^2}{3} + \cdots\right) , \\ 2g(x)\,g'_{\rm cent}(x) &= 2e^x\,\frac{e^{x+h} - e^{x-h}}{2h} = e^{2x}\,\frac{e^h - e^{-h}}{h} = 2e^{2x}\left(1 + \frac{h^2}{6} + \cdots\right) . \end{aligned} \tag{5.3.5}$$

- Perhaps a better example (in two dimensions) is the use of Cartesian and polar coordinates. Numerical derivatives computed using Cartesian coordinates will not in general be equal to those computed using polar coordinates.

  1. According to calculus,

  $$\frac{\partial f}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{y}} = \frac{\partial f}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} \,. \tag{5.3.6}$$

  2. However, using numerical finite differences,

  $$\left(\frac{\partial f}{\partial x}\right)_{\rm cent}\hat{\boldsymbol{x}} + \left(\frac{\partial f}{\partial y}\right)_{\rm cent}\hat{\boldsymbol{y}} \neq \left(\frac{\partial f}{\partial r}\right)_{\rm cent}\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)_{\rm cent}\hat{\boldsymbol{\theta}} \,. \tag{5.3.7}$$

- **There is nothing for you to calculate here.**
- **It is information for you to know.**

5.4 Finite difference: logarithmic derivative

- Don't be scared by the title. We shall work through things.
- Define $y = e^x$. Then $x = \ln(y)$, the logarithm.
- **Question: do you know that if $-\infty < x < \infty$, then $0 < y < \infty$?**
- **That is to say, do you know the value of y is always positive?**
- Let $f(x)$ be a function of x. Define $g(y) = f(e^x)$. Then from calculus we obtain the relation

$$\frac{df}{dx} = y\,\frac{dg}{dy}\,. \tag{5.4.1}$$

- Let us use the centered finite difference (more accurate).
  Let the stepsizes in x and y be h and k, respectively.
  Then define

$$\begin{aligned} F(x,h) &= \frac{f(x+h) - f(x-h)}{2h}\,, \\ G(y,k) &= \frac{g(y+k) - g(y-k)}{2k}\,. \end{aligned} \tag{5.4.2}$$

- Then according to eq. (5.4.1), in the limit $h \to 0$ and $k \to 0$, we should obtain

$$F(x, h \to 0) = y\,G(y, k \to 0)\,. \tag{5.4.3}$$

- However, eq. (5.4.3) is only correct in a pure mathematical limit.
- If we use finite differences, the values of $F(x,h)$ and $G(y,k)$ will only be approximately equal.
- Let us try some examples.

5.4.1 Example 1

1. First let us set $f(x) = x^2$. Then from calculus, $f'(x) = 2x$. If we set $x = 0.5$, then $f'(0.5) = 1$. We computed the value of $F(0.5, h)$ in class and saw that it was only approximately equal to 1. We also found out that if the magnitude of h is too small, the value of $F(x, h)$ was not accurate.
2. **Because of bad weather, the worked example might have been displayed via email.**
3. Then you are given **(don't derive this, I am giving it to you)**
$$g(y) = [\ln(y)]^2 . \tag{5.4.4}$$
4. **Question: calculate the value of $y = e^x$ for $x = 0.5$.**
Call the answer $y_{0.5}$. Use a computer, etc. and save the value of $y_{0.5}$.
You should obtain a value $y_{0.5} \simeq 1.6\ldots$ (you should save it to more decimal places).
5. From the definition,
$$F(0.5, h) = \frac{(0.5 + h)^2 - (0.5 - h)^2}{2h} . \tag{5.4.5}$$
Next, you are given **(don't derive this, I am giving it to you)**
$$G(y_{0.5}, k) = \frac{[\ln(y_{0.5} + k)]^2 - [\ln(y_{0.5} - k)]^2}{2k} . \tag{5.4.6}$$
6. **Question: (Analysis, very important) what would happen if we used an absurdly large value like $k = 2.0$ to compute the value of $G(y_{0.5}, k)$ in eq. (5.4.6)?**
7. Notice that in eq. (5.4.5), we can use any value of h that we like (except 0).
Question: Put $h = 2.0$ in eq. (5.4.5), and compute the value of $F(0.5, 2.0)$.
8. **Question: fill in the following table of values, for $F(0.5, h)$ and $y_{0.5}G(y_{0.5}, k)$.**

h	$F(0.5, h)$	k	$y_{0.5}G(y_{0.5}, k)$
0.1		0.1	
10^{-2}		10^{-2}	
10^{-4}		10^{-4}	
10^{-6}		10^{-6}	
10^{-8}		10^{-8}	
10^{-10}		10^{-10}	
10^{-12}		10^{-12}	
10^{-14}		10^{-14}	
10^{-16}		10^{-16}	

9. **Don't forget the factor of $y_{0.5}$ to multiply G in the last column.**
10. **Question: as we decrease the value of h, when does the approximation using $F(0.5, h)$ start to become worse?**
11. **Question: as we decrease the value of k, when does the approximation using $G(y_{0.5}, k)$ start to become worse?**

5.4.2 Example 2

1. Next let us set $f(x) = \ln(x)$. Then from calculus, $f'(x) = 1/x$. If we set $x = 1.0$, then $f'(1.0) = 1$.
2. Then you are given **(don't derive this, I am giving it to you)**
$$g(y) = \ln(\ln(y))\,. \tag{5.4.7}$$
3. **Question: calculate the value of $y = e^x$ for $x = 1.0$.**
Call the answer $y_{1.0}$. Use a computer, etc. and save the value of $y_{1.0}$.
You should obtain $y_{1.0} \simeq 2.7\ldots$ (you should save it to more decimal places).
4. From the definition,
$$F(1.0, h) = \frac{\ln(1.0 + h) - \ln(1.0 - h)}{2h}\,. \tag{5.4.8}$$
Next, you are given **(don't derive this, I am giving it to you)**
$$G(y_{1.0}, k) = \frac{\ln(\ln(y_{1.0} + k)) - \ln(\ln(y_{1.0} - k))}{2k}\,. \tag{5.4.9}$$
5. **Question: fill in the following table of values, for $F(1.0, h)$ and $y_{1.0}G(y_{1.0}, k)$.**

h	$F(1.0, h)$	k	$y_{1.0}G(y_{1.0}, k)$
0.1		0.1	
10^{-2}		10^{-2}	
10^{-4}		10^{-4}	
10^{-6}		10^{-6}	
10^{-8}		10^{-8}	
10^{-10}		10^{-10}	
10^{-12}		10^{-12}	
10^{-14}		10^{-14}	
10^{-16}		10^{-16}	

6. **Don't forget the factor of $y_{1.0}$ to multiply G in the last column.**
7. **Question: as we decrease the value of h, when does the approximation using $F(1.0, h)$ start to become worse?**
8. **Question: as we decrease the value of k, when does the approximation using $G(y_{1.0}, k)$ start to become worse?**

5.5 Finite difference: call option

- Let us analyze an example taken from finance.
- We cannot go into too many details about financial derivatives theory in this course, hence the following information will be given to you.
  You do not have to prove or derive it.
- We shall consider a very popular type of financial derivative known as a **call option**.
  1. The "fair value" of a call option c is a function of the stock price S.
  2. The value of c also depends on many other parameters, but we shall set most of their values to 0 or 1 and not worry about them.
  3. We shall include only one parameter, called the **volatility**, denoted by the symbol $\boldsymbol{\sigma}$.
- Hence $c = c(S, \sigma)$ is a function of S and σ (stock price and volatility).
- Both S and σ are positive real variables

$$S > 0\,, \qquad \sigma > 0\,. \tag{5.5.1}$$

- The value of $c(S, t)$ is given as follows.
  1. Recall the cumulative normal distribution, which can be obtained from the error function

$$N(x) = \frac{1 + \mathrm{erf}(x\sqrt{0.5})}{2}\,. \tag{5.5.2}$$

  2. Then define

$$d_1(S,\sigma) = \frac{\ln(S)}{\sigma} + \frac{\sigma}{2}\,, \qquad d_2(S,\sigma) = d_1(S,\sigma) - \sigma\,. \tag{5.5.3}$$

  3. Then the value of $c(S, \sigma)$ is given by

$$c(S,\sigma) = S\,N(d_1(S,\sigma)) - N(d_2(S,\sigma))\,. \tag{5.5.4}$$

     Note that this is a very simplified expression. We are omitting many important details of financial option pricing theory.
  4. We shall also require the partial derivative, which is given by the formula

$$\frac{\partial c}{\partial S} = N(d_1(S,\sigma))\,. \tag{5.5.5}$$

     This partial derivative is called **Delta**, denoted by Δ.
- **Don't panic about all this.**
- **There is nothing for you to calculate in this section. It is information.**

5.6 Finite difference: call option graphs

- **There is nothing for you to calculate in this section. It is information.**
- A graph of $c(S, \sigma)$ is plotted as a function of S, for $\sigma = 0.5$, and displayed in Fig. 1. Note that $C = 0$ at $S = 0$ and that (for fixed σ), $c(S, \sigma)$ is a monotonically increasing function of S. It is a differentiable function of S.
- A graph of $\Delta(S, \sigma)$ is plotted as a function of S, for $\sigma = 0.5$, and displayed in Fig. 2. Note that $\Delta = 0$ at $S = 0$ and that (for fixed σ), $\Delta(S, \sigma)$ is a monotonically increasing function of S. It is also a differentiable function of S. Clearly $0 \leq \Delta(S, \sigma) \leq 1$.

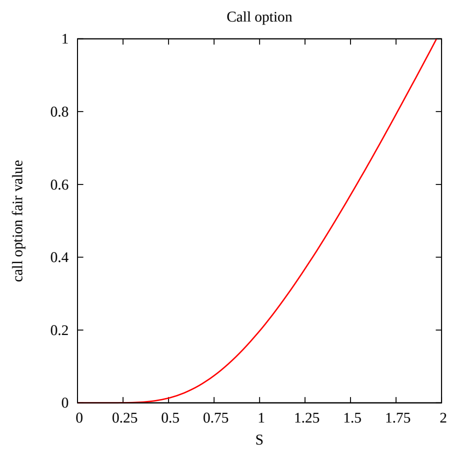

Figure 1: Graph of fair value of call option $c(S)$ as a function of the stock price S.

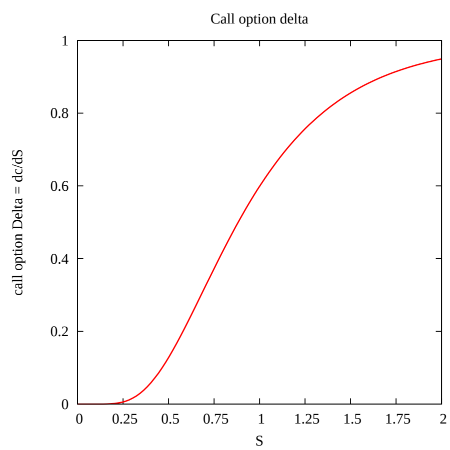

Figure 2: Graph of Delta of call option $\partial c/\partial S$ as a function of the stock price S.

5.7 Finite difference: call option C++ code

Here is a working C++ code which you can use to calculate the simplified formula for the option value and Delta for this homework. It has return type `int` because it tests for invalid inputs.
I also include the code for the cumulative normal function.

```
double cum_norm(double x)
{
  const double root = sqrt(0.5);
  return 0.5*(1.0 + erf(x*root));
}
```

Call option valuation function:

```
int call_option(double S, double sigma, double & call, double & delta)
{
  call  = 0.0;
  delta = 0.0;
  if ((S < 0.0) || (sigma < 0.0)) return 1;  // fail
  if (S == 0.0) {
    call = 0.0;
    delta = 0.0;
    return 0;  // ok
  }
  if (sigma == 0.0) {
    if (S > 1.0) {
      call = S - 1.0;
      delta = 1.0;
    }
    else {
      call = 0.0;
      delta = 0.0;
    }
    return 0;  // ok
  }

  double d1 = log(S)/sigma + 0.5*sigma;
  double d2 = d1 - sigma;
  double Nd1 = cum_norm(d1);
  double Nd2 = cum_norm(d2);
  call = S*Nd1 - Nd2;
  delta = Nd1;
  return 0;  // ok
}
```

5.8 Finite difference: call option Delta

- Let us calculate the partial derivative $\partial c/\partial S$ via a numerical finite difference.
- Let us set $S = 0.9$ and $\sigma = 0.5$.
- **Question: calculate the value of $N(d_1(S, \sigma))$ for $S = 0.9$, $\sigma = 0.5$.**

  You can use the function `call_option(S, sigma, call, delta)`.

  Save the output value of delta. Call the answer $\Delta(0.9, 0.5)$.

  You should obtain $\Delta(0.9, 0.5) \simeq 0.515666\ldots$ (you should save it to more decimal places).
- Many experts in financial option pricing theory define $S = e^x$ or $x = \ln(S)$ and use x instead of S in calculations. Hence let us define $f(x) = c(S)$. If we look at eq. (5.4.1), we see that we can also write the derivatives as

$$\frac{dc}{dS} = \frac{1}{e^x}\frac{df}{dx} = \frac{1}{S}\frac{df}{dx}. \tag{5.8.1}$$

  (Technically, we should use partial derivatives, but we shall keep the value of σ constant.)
- **You can use the code given above to compute the value of c in this homework.**

  The fair value of the call option "c" is obtained by calling `call_option(...)`.
  The output field "`call`" gives the value of c.
- Let us compute the value of dc/dS numerically. We do it in two ways, using x and S. We use a stepsize h for x and k for S. Then **(you have to make four function calls to `call_option(...)`)**

$$F(x, \sigma, h) = \frac{c(\exp(x+h), \sigma) - c(\exp(x-h), \sigma)}{2h}. \tag{5.8.2}$$

$$G(S, \sigma, k) = \frac{c(S+k, \sigma) - c(S-k, \sigma)}{2k}. \tag{5.8.3}$$

- **Question: fill in the following table of values. Note that $x = \ln(0.9) \simeq -0.105360516$.**
  For brevity write $A = (1/0.9)F(\ln(0.9), 0.5, h)$ and $B = G(0.9, 0.5, k)$.
  The last two columns tabulate the values of $A - \Delta(0.9, 0.5)$ and $B - \Delta(0.9, 0.5)$.

h	A	k	B	$A - \Delta(0.9, 0.5)$	$B - \Delta(0.9, 0.5)$
10^{-2}		10^{-2}			
10^{-3}		10^{-3}			
10^{-4}		10^{-4}			
10^{-5}		10^{-5}			
10^{-6}		10^{-6}			
10^{-7}		10^{-7}			
10^{-8}		10^{-8}			

- **Question: (Analysis) which numerical derivative yields a better approximation?**
- *Hint: have the courage to say they are about the same, and you don't know.*
  Don't panic and make up a fairy tale.
  Actually, it depends on the computer and compiler. Results are slightly different for each.