Queens College, CUNY, Department of Computer Science

Computational Finance

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2 Lecture 2

2.1 Bond: more general pricing formula

- Recall that a bond pays cashflows of coupons and redeems its face value at maturity.
- Let us write a more general formula for the fair value of a bond.
- The face of the bond is F. We shall always set $F = 100$ in these lectures.
- Suppose the annualized coupons rates are $c_1, \dots, c_n$.
  1. The c_i are not necessarily equal, and some or all could be zero.
  2. Let the frequency of the coupon payments be f, so there are f coupons per year.
  3. For semiannual compounding then $f = 2$.
  4. Semiannual compounding is the typical case in the USA and many countries.
- Let the dates of the coupons be $t_1, \dots, t_n$. Then $t_n = T =$ maturity date of bond.
  1. We shall assume the coupons are paid at equal time intervals.
  2. Hence in these lectures, we shall set $t_i = fi$ and the maturity is $T = fn$.
  3. In real life there are many exceptions to this rule, but we shall keep things simple.
- Let the time today be t_0.

- Then a naive formula for the fair value B of a bond is

$$\begin{aligned} B = {} & \frac{c_1/f}{(1+y/f)^{f(t_1-t_0)}} + \frac{c_2/f}{(1+y/f)^{f(t_2-t_0)}} + \dots \\ & \quad \dots + \frac{c_{n-1}/f}{(1+y/f)^{f(t_{n-1}-t_0)}} + \frac{F + (c_n/f)}{(1+y/f)^{f(t_n-t_0)}} \,. \end{aligned} \tag{2.1.1}$$

- If we set $f = 2$ and $t_0 = 0$, this agrees with the simple formula in the earlier lecture.

- **However, there is an important caveat to eq. (2.1.1).**

    1. If $t_0 > 0$, **we only include coupons where $t_i - t_0 > 0$.**
    2. We exclude coupons which are in the "past" (coupon date $t_i \le t_0$).
    3. In this context, we assume that coupons are paid at the "start of day" so if $t_i - t_0 = 0$, we exclude the coupon.

- Hence we modify the above sum as follows ("`numerator_i`" has an obvious definition):

$$\begin{aligned} B &= \biggl[\frac{c_1/f}{(1+y/f)^{f(t_1-t_0)}} + \frac{c_2/f}{(1+y/f)^{f(t_2-t_0)}} + \dots \\ & \qquad \dots + \frac{c_{n-1}/f}{(1+y/f)^{f(t_{n-1}-t_0)}} + \frac{F + (c_n/f)}{(1+y/f)^{f(t_n-t_0)}} \biggr]_{t_i > t_0} \\ &= \sum_{i=1}^{n} \biggl[\frac{(\text{numerator})_i}{(1+y/f)^{f(t_i-t_0)}} \biggr]_{t_i > t_0} \,. \end{aligned} \tag{2.1.2}$$

- ***This* is the more general bond pricing formula.**

- As has already been pointed out in an earlier lecture, in most cases we observe the value of the bond price B in the financial markets and we invert eq. (2.1.2) to calculate the yield y. See Section 2.3.

2.2 Yield and discount factors

- The following question was tagged as an item to be answered in a future lecture.
- Why do all the discount factors have the following form (assuming $f = 2$ and $t_0 = 0$)?

$$\mathrm{DF}_1 = \frac{1}{1+\frac{1}{2}y}\,, \qquad \mathrm{DF}_2 = \frac{1}{(1+\frac{1}{2}y)^2}\,, \quad \dots \quad \mathrm{DF}_i = \frac{1}{(1+\frac{1}{2}y)^i}\,. \tag{2.2.1}$$

- The answer is that this is the **definition** of how the yield is calculated.
- Basically, we pose the question: suppose all the cashflows from a bond are reinvested in a bank, and they all earn the **same annualized rate of interest**, then what would that rate of interest have to be, so that the bond price equals the target value of B (in eq. (2.1.2))?
- The yield gives an answer to that question.
- In real life, it is unrealistic to assume that all the cashflows would earn the same annualized rate of interest. We know that interest rates change over time.
- However, we do not know, *today*, what those future interest rates will be.
- However, the yield is a number which *can* be calculated, using information available *today*.

2.3 Yield from bond price

- Let us return to Section 2.1: how do we determine the yield y given the bond price B?
- We can regard the sum in eq. (2.1.2) as a function of the yield, i.e. $B = B(y)$.
- Suppose the bond trades with a market price $B_{\rm market}$.
- More generally, we can set any target value $B_{\rm target}$.
- We wish to solve the following equation for the yield y:

$$B(y) = B_{\rm target} \,. \tag{2.3.1}$$

- There are many mathematical algorithms to solve an equation such as eq. (2.3.1).
- We shall study only one method, which is in many ways the simplest.
- It is known as the method of **bisection.**
- The fundamental idea is simple. It goes as follows:
  1. We know from eq. (2.1.2) that $B(y)$ is a continuous function of y.
  2. We also know that $B(y)$ decreases as the value of y increases.
  3. Hence we find a low yield $y_{\rm low}$ such that $B(y_{\rm low}) > B_{\rm target}$ and a high yield $y_{\rm high}$ such that $B(y_{\rm high}) < B_{\rm target}$.
  4. Then the solution of eq. (2.3.1) lies somewhere between $y_{\rm low}$ and $y_{\rm high}$.
  5. It is possible by luck that either $y_{\rm low}$ and $y_{\rm high}$ is a solution of eq. (2.3.1).
  6. If so we exit the algorithm immediately.
  7. Else we iterate as follows.
  8. We use the midpoint $y_{\rm mid} = (y_{\rm low} + y_{\rm high})/2$ and calculate $B(y_{\rm mid})$.
  9. If $|B(y_{\rm mid}) - B_{\rm target}|$ is less than a prespecified tolerance, we exit the calculation and say that $y_{\rm mid}$ is the solution of eq. (2.3.1).
  10. Else if $B(y_{\rm mid}) > B_{\rm target})$ then $y_{\rm mid}$ is too low. We update $y_{\rm low} := y_{\rm mid}$.
  11. Else obviously $B(y_{\rm mid}) < B_{\rm target}$ so $y_{\rm mid}$ is too high. We update $y_{\rm high} := y_{\rm mid}$.
  12. **There are mathematically better ways to formulate the comparison tests.**
  13. We repeat the iteration using the updated values of $y_{\rm low}$ and $y_{\rm high}$.
  14. Hence the interval $|y_{\rm high} - y_{\rm low}|$ is cut by a factor of two at each iteration step.
  15. This is why it is called the "bisection" algorithm.
  16. If $|y_{\rm high} - y_{\rm low}|$ is less than a prespecified tolerance, we exit the calculation and say that $y_{\rm mid}$ is the solution of eq. (2.3.1).
  17. The bisection algorithm is not necessarily the fastest, but it is guaranteed to be stable and will converge after a finite number of iterations.
  18. The weak point is to find suitable initial values for $y_{\rm low}$ and $y_{\rm high}$.
- **It is an important part of the "computational" part of this course to write a working bisection program to compute the yield.**

2.4 Bond duration: Macaulay and modified duration

Let us now study some other formulas pertaining to bonds. The most important is the **duration.**

2.4.1 Macaulay duration

- The **Macaulay duration** is defined as follows. We write "Macaulay" to avoid confusion with other terminology (see below). It was introduced by a person named Frederick Macaulay (approximately in the 1930s). It is a time weighted average of the cashflows

$$D_{\rm Mac} = \frac{1}{B}\sum_{i=1}^{n}\biggl[(t_i - t_0)\,\frac{(\text{numerator})_i}{(1+y/f)^{f(t_i-t_0)}}\biggr]_{t_i>t_0}. \qquad (2.4.1)$$

- By construction, the Macaulay duration has units of time (or is measured in years).

- By analogy with physics, the Macaulay duration is analogous to a "center of mass" of a bond.

  1. Let us draw time on the horizontal axis and visualize the discounted cashflows as "masses" located at points $t_i - t_0$.
  2. The Macaulay duration is the "center of maturity" of all the discounted cashflows.
  3. Another way to say it is: if all the discounted cashflows were paid at one time, what would that time be?
  4. The Macaulay duration is effectively the "average lifetime" of the bond.

2.4.2 Modified duration

- There is also a term called the **modified duration.**

- For this reason, the term "duration" without qualification is confusing.

- It is better to say "Macaulay duration" or "modified duration" to avoid ambiguity.

- The modified duration is defined via a partial derivative with respect to the bond's yield

$$D_{\rm mod} = -\frac{1}{B}\,\frac{\partial B}{\partial y}\,. \qquad (2.4.2)$$

- The modified duration also has units of time (or is measured in years).

- Because the modified duration is a partial derivative, it is a measure of the sensivity of the bond price to a small change in the yield.

- The change in the bond price δB for a small change in the yield δy is

$$\delta B(y) \simeq -D_{\rm mod}B(y)\,\delta y\,. \qquad (2.4.3)$$

- There is a simple but important relation between the Macaulay and modified duration:

$$D_{\rm mod} = \frac{D_{\rm Mac}}{1+y/f}\,. \qquad (2.4.4)$$

2.5 Bond DV01

- In the financial markets, interest rates and yields are usually measured in **basis points**.
- One basis point is 1/100 of one percent, i.e. in decimal a change in yield of one basis point is $\delta y = 0.0001$.
- The meaning of "**DV01**" (dollar value zero one) is the change in the dollar value (price) of a bond for a one basis point change in yield.
- In terms of the modified duration, the DV01 is given by (note the minus sign)

$$\mathrm{DV01} = -\frac{\partial B}{\partial y}\,\delta y = B\,D_{\rm mod}\,\delta y = 0.0001 \times B\,D_{\rm mod}\,. \tag{2.5.1}$$

  In the last step a one basis point value $\delta y = 0.0001$ was substituted for δy.
- Many times, people want the change for a 100 basis point move in yield (i.e. one percent). This is usually computed by multiplying the DV01 value by 100.

2.6 Bond convexity

- Why stop at the first partial derivative?
- The **convexity** of a bond is given by the second partial derivative

$$C = \frac{1}{B}\,\frac{\partial^2 B}{\partial y^2}\,. \tag{2.6.1}$$

- In terms of the modified duration,

$$\begin{aligned} C &= \frac{1}{B}\,\frac{\partial}{\partial y}\Big(\frac{\partial B}{\partial y}\Big) \\ &= \frac{1}{B}\,\frac{\partial}{\partial y}(-BD_{\rm mod}) \\ &= \frac{1}{B}\left[-\frac{\partial B}{\partial y}\,D_{\rm mod} - B\,\frac{\partial D_{\rm mod}}{\partial y}\right] \\ &= D_{\rm mod}^2 - \frac{\partial D_{\rm mod}}{\partial y}\,. \end{aligned} \tag{2.6.2}$$

- For the simple examples in these lectures, the convexity of a bond is positive.
- More complicated types of bonds can exhibit negative convexity.

2.7 Zero coupon bonds

- There is an important class of bonds which are called **zero coupon bonds**.
- As the name suggests, zero coupon bonds pay no coupons. They pay only one cashflow, on the maturity date, which is the face of the bond.
- Zero coupon bonds are popular financial instruments. They allow an investor to perform hedging of cashflows at a particular point in time, without the complicaton of juggling additional cashflows (the coupons), which might interfere with other activities of the investor.
- For finance academics, zero coupons bonds are theoretically simpler to analyze.
- Since there is only one cashflow, i.e. the face value F, let us say the maturity date is T (where obviously $T > t_0$). In terms of the yield, the price of a zero coupon bond is given by

$$B = \frac{F}{(1+y/f)^{f(T-t_0)}} \,. \tag{2.7.1}$$

- As stupid as it sounds, there is still a parameter f, simply because of market conventions. It is awkward to maintain inventory of bonds in a database if some bonds have a parameter f (frequency of cashflows) and for zero coupon bonds the parameter f is absent. Hence for quoting conventions, in the USA the yield is typically quoted on a semi-annual basis ($f = 2$).
- It is easy to invert the above formula to calculate the yield of a zero coupon bond from the market price of the bond.
- To the extent that interest rates and yields are positive, the price of a zero coupon bond is less than par.
- **The Macaulay duration of a zero coupon bond is equal to its time to maturity.**
- **This is a very important fact.**
- Proof:

$$D_{\rm Mac} = \frac{1}{B}\,\frac{(T-t_0)F}{(1+y/f)^{f(T-t_0)}} = \frac{(T-t_0)B}{B} = T - t_0 \,. \tag{2.7.2}$$

- If there is only one cashflow then the "weighted average" is obviously the time to that cashflow.

2.8 Worked example: Bond price

- Consider a bond with two years to maturity.
- For simplicity, we consider only semiannual coupons (two coupons per year).
- Hence there are four coupons, paid at times $t_1 = 0.5$, $t_2 = 1.0$, $t_3 = 1.5$ and $t_4 = 2.0$.
- Let $F = 100$ and $c_1 = 4.1$, $c_2 = 4.2$, $c_3 = 4.3$, $c_4 = 4.4$ and the yield be $y = 6.0\%$.
- Suppose $t_0 < t_1$ (date of first coupon). Then the bond price is (see eq. (2.1.2))

$$B = \frac{\frac{1}{2}c_1}{(1+\frac{1}{2}y)^{2(t_1-t_0)}} + \frac{\frac{1}{2}c_2}{(1+\frac{1}{2}y)^{2(t_2-t_0)}} + \frac{\frac{1}{2}c_3}{(1+\frac{1}{2}y)^{2(t_3-t_0)}} + \frac{F+\frac{1}{2}c_4}{(1+\frac{1}{2}y)^{2(t_4-t_0)}}\,. \tag{2.8.1}$$

- **Let $t_0 = 0.0$.** Then the bond price is

$$\begin{aligned} B &= \frac{2.05}{(1.03)^{1.0}} + \frac{2.1}{(1.03)^{2.0}} + \frac{2.15}{(1.03)^{3.0}} + \frac{102.2}{(1.03)^{4.0}} \\ &\simeq 96.74067\,. \end{aligned} \tag{2.8.2}$$

- **Let $t_0 = 0.1$.** Then the bond price is

$$\begin{aligned} B &= \frac{2.05}{(1.03)^{0.8}} + \frac{2.1}{(1.03)^{1.8}} + \frac{2.15}{(1.03)^{2.8}} + \frac{102.2}{(1.03)^{3.8}} \\ &\simeq 97.31428\,. \end{aligned} \tag{2.8.3}$$

- **Let $t_0 = 0.55$.** Then $t_1 < t_0$ so we skip the first coupon. The bond price is

$$\begin{aligned} B &= \frac{\frac{1}{2}c_2}{(1+\frac{1}{2}y)^{2(t_2-t_0)}} + \frac{\frac{1}{2}c_3}{(1+\frac{1}{2}y)^{2(t_3-t_0)}} + \frac{F+\frac{1}{2}c_4}{(1+\frac{1}{2}y)^{2(t_4-t_0)}} \\ &= \frac{2.1}{(1.03)^{0.9}} + \frac{2.15}{(1.03)^{1.9}} + \frac{102.2}{(1.03)^{2.9}} \\ &\simeq 97.88179\,. \end{aligned} \tag{2.8.4}$$

2.9 Worked example: Macaulay and modified duration

- We use the same bond as in Sec. 2.8.
- **Let $t_0 = 0.0$.** The Macaulay duration is

$$\begin{aligned} D_{\text{Mac}} &= \frac{1}{B}\left[0.5 \times \frac{2.05}{(1.03)^{1.0}} + 1.0 \times \frac{2.1}{(1.03)^{2.0}} + 1.5 \times \frac{2.15}{(1.03)^{3.0}} + 2.0 \times \frac{102.2}{(1.03)^{4.0}}\right] \\ &\simeq \frac{187.5327}{96.74067} \\ &\simeq 1.938509\,. \end{aligned} \tag{2.9.1}$$

- The modified duration is

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1+\frac{1}{2}y} \simeq \frac{1.938509}{1.03} \simeq 1.882048\,. \tag{2.9.2}$$

- **Let $t_0 = 0.1$.** The Macaulay duration is

$$\begin{aligned} D_{\text{Mac}} &= \frac{1}{B}\left[0.4 \times \frac{2.05}{(1.03)^{0.8}} + 0.9 \times \frac{2.1}{(1.03)^{1.8}} + 1.4 \times \frac{2.15}{(1.03)^{2.8}} + 1.9 \times \frac{102.2}{(1.03)^{3.8}}\right] \\ &\simeq \frac{178.9132}{97.31428} \\ &\simeq 1.838509\,. \end{aligned} \tag{2.9.3}$$

- The modified duration is

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1+\frac{1}{2}y} \simeq \frac{1.838509}{1.03} \simeq 1.78496\,. \tag{2.9.4}$$

- **Let $t_0 = 0.55$.** The Macaulay duration is

$$\begin{aligned} D_{\text{Mac}} &= \frac{1}{B}\left[0.45 \times \frac{2.1}{(1.03)^{0.9}} + 0.95 \times \frac{2.15}{(1.03)^{1.9}} + 1.45 \times \frac{102.2}{(1.03)^{2.9}}\right] \\ &\simeq \frac{138.8674}{97.88179} \\ &\simeq 1.418726\,. \end{aligned} \tag{2.9.5}$$

- The modified duration is

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1+\frac{1}{2}y} \simeq \frac{1.418726}{1.03} \simeq 1.377404\,. \tag{2.9.6}$$

2.10 Worked example: Bond yield

- We use the same bond as in Sec. 2.8.
- Let the market price of the bond be $B_{\mathrm{market}} = 99.5$.
- For simplicity, we analyze only the case $t_0 = 0.0$.
- We already know that if the yield is $y = 6\%$, the bond price is $B \simeq 96.74067$.
- Next try a guess $y = 4\%$. Using the formulas in Sec. 2.8, the bond price is $B \simeq 100.4713$.
- Hence we have found a bracket because $B(4\%) > B_{\mathrm{market}} > B(6\%)$.
- **Hence we know the true yield lies somewhere between $y_{\mathrm{low}} = 4\%$ and $y_{\mathrm{high}} = 6\%$.**
- Use the midpoint $y_{\mathrm{mid}} = (y_{\mathrm{low}} + y_{\mathrm{high}})/2.0 = 5.0\%$. The bond price is $B \simeq 98.58345$.
- Hence $B(5\%)$ and $B(6\%)$ are **on the same side of B_{market}.**
- **Hence we update $y_{\mathrm{high}} := y_{\mathrm{mid}} = 5\%$** and iterate again.
- The updated midpoint is $y_{\mathrm{mid}} = (4.0 + 5.0)/2.0 = 4.5\%$. The bond price is $B \simeq 99.52164$.
- We stop here.
- This is not a very accurate "convergence" but is enough for the purposes of a worked example.
- We say that for a market price of $B_{\mathrm{market}} = 99.5$, the yield is approximately 4.5%.