Queens College, CUNY, Department of Computer Science

**Computational Finance**

**CSCI 365 / 765**

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# 6 Homework Lecture 6, 7 & 8: Options #2

- In all questions below, time is measured in years and the time today is $t_0 = 0$.
- Continuous interest rate compounding is used in all questions.
- In all questions, the following notation is employed for American/European put/call options:
  1. **European call:** **c**
  2. **European put:** **p**
  3. **American call:** **C**
  4. **American put:** **P**
- Some of the questions below require the cumulative normal function, given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du \,. \qquad (6.0.1)$$

Fortunately C++ provides a function `erf(x)` which can be used to compute $N(x)$:

$$N(x) = \frac{1 + \mathrm{erf}(x/\sqrt{2})}{2} \,. \qquad (6.0.2)$$

**You may use the C++ function below to compute the value of $N(x)$.**

```
double cum_norm(double x)
{
  const double root = sqrt(0.5);
  return 0.5*(1.0 + erf(x*root));
}
```

### 6.1 Terminal payoff diagrams of combinations of options

- **Note: In this question, the underlying stock does not pay dividends.**
- Let us make some combinations of options and examine the payoffs at expiration.
- We shall use European options to avoid the complications of early exercise.
- The 'terminal payoff diagram' is the value of the overall combination on the expiration date.
- **Combinations can contain fractional numbers of options and/or negative values.**
- In all cases below, plot the graphs for $0 \leq S_T \leq 100$.
- **The term 'struck at $K$' means the option strike is $K$.**
- **Plot a graph of the terminal payoff diagrams of the following combinations.**
  Long 4/3 zero strike calls, short 4/3 calls struck at 40, long 1/2 calls struck at 60.
  $$V = \frac{4}{3}\,c(K=0) - \frac{4}{3}\,c(K=40) + \frac{1}{2}\,c(K=60)\,. \tag{6.1.1}$$
- **Plot a graph of the terminal payoff diagrams of the following combinations.**
  Long 2 zero strike calls, long 2 puts struck at 20, short one call struck at 20, short one call struck at 40, short one call struck at 60, long one call struck at 80.
  $$V = 2c(K=0) + 2p(K=20) - c(K=20) - c(K=40) - c(K=60) + c(K=80)\,. \tag{6.1.2}$$
- **Plot a graph of the terminal payoff diagrams of the following combinations.**
  Long one zero strike call, long one put struck at 20, short one call struck at 40, short one put struck at 60, long one put struck at 80.
  $$V = c(K=0) + p(K=20) - c(K=40) - p(K=60) + p(K=80)\,. \tag{6.1.3}$$
- **Observe that different combinations can have the same terminal payoff.**
- This demonstrates that, if we are given a terminal payoff diagram, *there may be more than one way to construct a combination of options to match it.*
- *What would happen if we used American options?*
  The terminal payoff diagram is the same, but because of the possibility of early exercise, the combination might not survive till expiration.

### 6.2 Put-call parity

**Question: For each case below, either calculate the value of the missing parameter, or else prove that there is not enough information to solve the problem.**

#### 6.2.1

1. The market price of a stock is 100 today. The stock does not pay dividends.
2. The interest rate is 10%.
3. A European call option has a strike of 101 and expiration time of 0.5 years.
4. The price of the above European call option is 8.
5. **Calculate the price of a European put option with the same strike and expiration as the call.**

#### 6.2.2

1. The market price of a stock is 100 today.
2. The stock pays continuous dividends at a rate of 3%.
3. The interest rate is 10%.
4. A European put option has a strike of 101 and expiration time of 0.75 years.
5. The price of the above European put option is 4.
6. **Calculate the price of a European call option with the same strike and expiration as the put.**

#### 6.2.3

1. The market price of a stock is 100 today. The stock does not pay dividends.
2. The interest rate is 5%.
3. The price of a European call option (with expiration time of 1 year) is 6.
4. The price of a European put option (with expiration time of 1 year) is 7.
5. Both options have the same strike price.
6. **Calculate the value of the strike price of the options.**

### 6.3 Put-call parity & option pricing bounds

- **To answer the question below, you need to consult the inequalities in Lecture 7.**
- **Consult Lecture 7, eqs. (7.2.1), (7.2.4) and (7.2.6).**

1. The market price of a stock is 20 today. The stock does not pay dividends.
2. The interest rate is 8%.
3. A European call option has a strike of 21 and an expiration time of 1 year.
4. A European put option has a strike of 21 and an expiration time of 1 year.
5. **Calculate the present value of the strike price of the options.**
6. **Calculate the minimum fair value the call option must have, to be consistent with the above data and the inequalities in Lecture 7.**
7. **Calculate the maximum fair value of the put option, to be consistent with the above data and the inequalities in Lecture 7. Calculate the corresponding fair value of the call option, when the put option has its maximum value.**
8. **Calculate the maximum fair value of the call option, to be consistent with the above data and the inequalities in Lecture 7. Calculate the corresponding fair value of the put option, when the call option has its maximum value.**

### 6.4 Delta

1. **Skip this if not lectured in class by the due date.**
2. The market price of a stock is 10 today. The stock does not pay dividends.
3. The volatility of the stock is $\sigma = 50\%$ ($\sigma = 0.5$ in decimal).
4. The interest rate is 6%.
5. A European call option has a strike of 12 and expiration time of 0.8 years.
6. A European put option has a strike of 11 and expiration time of 0.8 years.
7. There is a futures contract on the same stock, with the same expiration time as the options.
8. The Delta of a European call option is given by
$$\Delta_c = N(d_1)\,. \tag{6.4.1}$$
9. The Delta of a European put option is given by
$$\Delta_p = -N(-d_1)\,. \tag{6.4.2}$$
10. The definition of $d_1$ is
$$d_1 = \frac{\ln(S/K) + r(T-t_0)}{\sigma\sqrt{T-t_0}} + \tfrac{1}{2}\sigma\sqrt{T-t_0}\,. \tag{6.4.3}$$
11. **Calculate the Delta of the call option.**
12. **Calculate the Delta of the put option.**
13. **Calculate the Delta of the futures contract.**
14. **Calculate the number $N_{fp}$ of futures contracts, so that the total Delta of a portfolio of long one put option and long $N_{fp}$ futures contracts equals zero.**
15. **Calculate the number $N_{fc}$ of futures contracts, so that the total Delta of a portfolio of short one call option and long $N_{fc}$ futures contracts equals zero.**

### 6.5 Rational option pricing #1

- **Work through this question carefully. It has several parts.**
- **In this question, set the interest rate to zero $r = 0$ and ignore compounding.**
- It was stated in Lecture 7 that the fair value of a call option never exceeds the stock price

$$c(S,t) \le S\,, \qquad C(S,t) \le S\,. \tag{6.5.1}$$

- The proof of eq. (6.5.1) is to construct an arbitrage argument.
  This was done in the worked examples in Lecture 7a.
- ***Conversely, it follows that, if eq.* (6.5.1) *is satisfied (as well as all the other inequalities in Lecture 7), then an arbitrage strategy does not exist.***
- In this exercise, we shall test the above claim.
- Let us analyze a European call option (easier).
- For simplicity, the stock pays no dividends. Also for simplicity the interest rate is $r = 0$.
- The strike price of the call option is $K = 100$.
- The value of the stock price at time $t_0$ is $S_0$ and suppose $S_0 > 1$.
- Since we are assuming $r = 0$, the expiration date is not important. Set $T = 1$.
- Suppose the market price of a European call option today is $c(S_0, t_0) = S_0 - 1$.
- Hence $c(S_0, t_0) > 0$, because $S_0 > 1$.
- Let us formulate a trading strategy as follows. **At time $t_0$, (a) buy one European call option, (b) short sell one share of stock, (c) save money in a bank.**
- The initial value of our portfolio is zero.
- **How much cash do we have in the bank at time $t_0$?**
- However, since it was stated above that an arbitrage strategy does not exist, *the above trading strategy will not make a guaranteed profit.*
- Conversely, it also cannot make a guaranteed loss, else we would reverse the trategy (short the call, buy the stock, borrow money) and that would yield a guaranteed profit.
- Therefore, the above trading strategy must **sometimes make a profit and sometimes make a loss.**
- Since the option is European, we cannot exercise it prior to expiration.
- Hence we hold the option to expiration, i.e. to time $T$.

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### 6.5.1 Scenario 1: option expires out of the money

- The stock price at time $T$ is $S_T = 0.5$.
- The option has expired out of the money and is worth zero.
- **Calculate the profit/loss of the above trading strategy.**

### 6.5.2 Scenario 2: option expires in the money

- The stock price at time $T$ is $S_T = K + 2 = 102$.
- The option has expired in the money.
- **We can choose to exercise or not.**
- **Just because an option expires in the money, we are not forced to exercise it.**
- **Remember that the holder of an option has a choice.**
- **The holder decides to exercise or not.**

  1. **Calculate the profit/loss of the trading strategy if the option is exercised.**
  2. **Calculate the profit/loss of the trading strategy if the option is not exercised.**

### 6.6 Rational option pricing #2

- **We shall repeat the exercise in Question 6.5 using an American call option.**
- We again set the interest rate to zero $r = 0$ and ignore compounding.
- Recall eq. (6.5.1) that we must have $C(S,t) \le S$, as well as other inequalities derived in Lecture 7.
- As in Question 6.5, the stock pays no dividends. Also the interest rate is $r = 0$.
- The strike price of the call option is $K = 100$.
- The value of the stock price at time $t_0$ is $S_0$ and now suppose $S_0 > 1.5$.
- Since we are assuming $r = 0$, the expiration date is not important. Set $T = 1$.
- Suppose the market price of an American call option today is $C(S_0, t_0) = S_0 - 1.5$.
- Hence $C(S_0, t_0) > 0$, because $S_0 > 1.5$.
- Note also that $C(S_0, t_0) \ge S_0 - K$, because $K = 100$.
- **Explain why we must have $C(S_0, t_0) \ge S_0 - K$ for an Americal call option.**
- Let us formulate a trading strategy as follows. **At time $t_0$, (a) buy one American call option, (b) short sell one share of stock, (c) save money in a bank.**
- The initial value of our portfolio is zero.
- **How much cash do we have in the bank at time $t_0$?**
- By the same logic as in Question 6.5, the above trading strategy must sometimes make a profit and sometimes make a loss.
- However, since the option is American, we can exercise it prior to expiration.
- Hence we need not hold the option to expiration, i.e. to time $T$.
- We can exercise at any time $t_0 \le t \le T$.

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### 6.6.1 Scenario 1: option expires out of the money

- Suppose we hold the option to expiration.
- The stock price at time $T$ is $S_T = 0.8$.
- The option has expired out of the money and is worth zero.
- **Calculate the profit/loss of the above trading strategy.**

### 6.6.2 Scenario 2: option expires in the money

- Suppose we hold the option to expiration.
- The stock price at time $T$ is $S_T = K + 2 = 102$.
- The option has expired in the money.
  1. **Calculate the profit/loss of the trading strategy if the option is exercised.**
  2. **Calculate the profit/loss of the trading strategy if the option is not exercised.**

### 6.6.3 Scenario 3: option is exercised early

- Suppose we exercise the option prior to expiration.
- The option can only be exercised if it is in the money.
- The stock price at time $t$ is $S_t$.
- Hence we can exercise early only if $S_t \geq 100$, because the strike is $K = 100$.
- **Calculate the profit/loss of the trading strategy if the option is exercised early.**