Queens College, CUNY, Department of Computer Science

Numerical Methods

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27 Lecture 27

Fourier Series

- We shall study some properties of **Fourier series.**
- We shall treat functions of one variable only.
- Fourier series are related to Fourier transforms.
- However, in the case of Fourier series, the function is **periodic.**
- Hence instead of continuous integrals, we obtain discrete sums.
- **This lecture contains mostly background material.**
- **The material in this lecture is a foundation for later lectures.**
- The most important topics in this lecture are:
  1. Basic concept of periodic function.
  2. Definition of Fourier series.
  3. Formula to calculate the coefficients of a Fourier series.
  4. Fourier series of **window function** (also known as rectangle function).
  5. Fourier series of **triangle function.**
  6. **Gibbs–Wilbraham phenomenon.**
- This lecture will require (a little) knowledge of **complex numbers.**

27.1 Periodic function (important)

- Let $f(x)$ be a function of a real–valued variable x, where $-\infty < x < \infty$.
- We say that $f(x)$ is a **periodic function** if there is a constant $x_p > 0$ such that

$$f(x + x_p) = f(x) \qquad \textbf{for all values of } \boldsymbol{x}. \tag{27.1.1}$$

- Actually, x_p must be the *smallest* positive constant for which eq. (27.1.1) is true.
  1. Clearly, if eq. (27.1.1) is true, then $f(x + 2x_p) = f(x)$ for all x, also $f(x + 3x_p) = f(x)$ for all x, etc.
  2. Hence, in addition to eq. (27.1.1), we must also impose the condition that there is **no other constant** x_q, where $0 < x_q < x_p$, such that $f(x + x_q) = f(x)$ for all x.
- If the above conditions are satisfied, then x_p is called the **period** of the function $f(x)$.
- We call $f(x)$ **a periodic function of x with period x_p.**
- There are many examples of periodic functions.
  1. The simplest examples are the trigonometric functions.
  2. Both $\sin(x)$ and $\cos(x)$ are periodic with period 2π. For all values of x, we have

$$\begin{aligned} \sin(x + 2\pi) &= \sin(x)\,, \\ \cos(x + 2\pi) &= \cos(x)\,. \end{aligned} \tag{27.1.2}$$

  3. The function $\tan(x)$ is periodic with period π. For all values of x, we have

$$\tan(x + \pi) = \tan(x)\,. \tag{27.1.3}$$

  4. Note that $\sin(x + \pi) = -\sin(x)$ for all x. Hence $|\sin(x)|$ is periodic with period π:

$$|\sin(x + \pi)| = |\sin(x)| \qquad \text{for all values of } x.. \tag{27.1.4}$$

  5. The function $\cos(\sqrt{3}\,x)$ is periodic with period $2\pi/\sqrt{3}$. For all values of x, we have

$$\cos\Bigl(\sqrt{3}\Bigl(x + \frac{2\pi}{\sqrt{3}}\Bigr)\Bigr) = \cos(\sqrt{3}\,x + 2\pi) = \cos(\sqrt{3}\,x)\,. \tag{27.1.5}$$

- A different and perhaps artificial example is the **sawtooth function** (see Fig. 1 top panel). The sawtooth function is periodic (with period 1) by construction, and is given by

$$f_{\text{sawtooth}}(x) = x - \lfloor x \rfloor\,. \tag{27.1.6}$$

- Here $\lfloor x \rfloor$ is the floor function, the largest integer less than or equal to x.
- A closely related example is the **triangle wave** (see Fig. 1 bottom panel).
  This consists of a set of repeating triangles, with period 2. It can be written as

$$f_{\text{tri wave}}(x) = \left| x - 2\left\lfloor \frac{x+1}{2} \right\rfloor \right|. \tag{27.1.7}$$

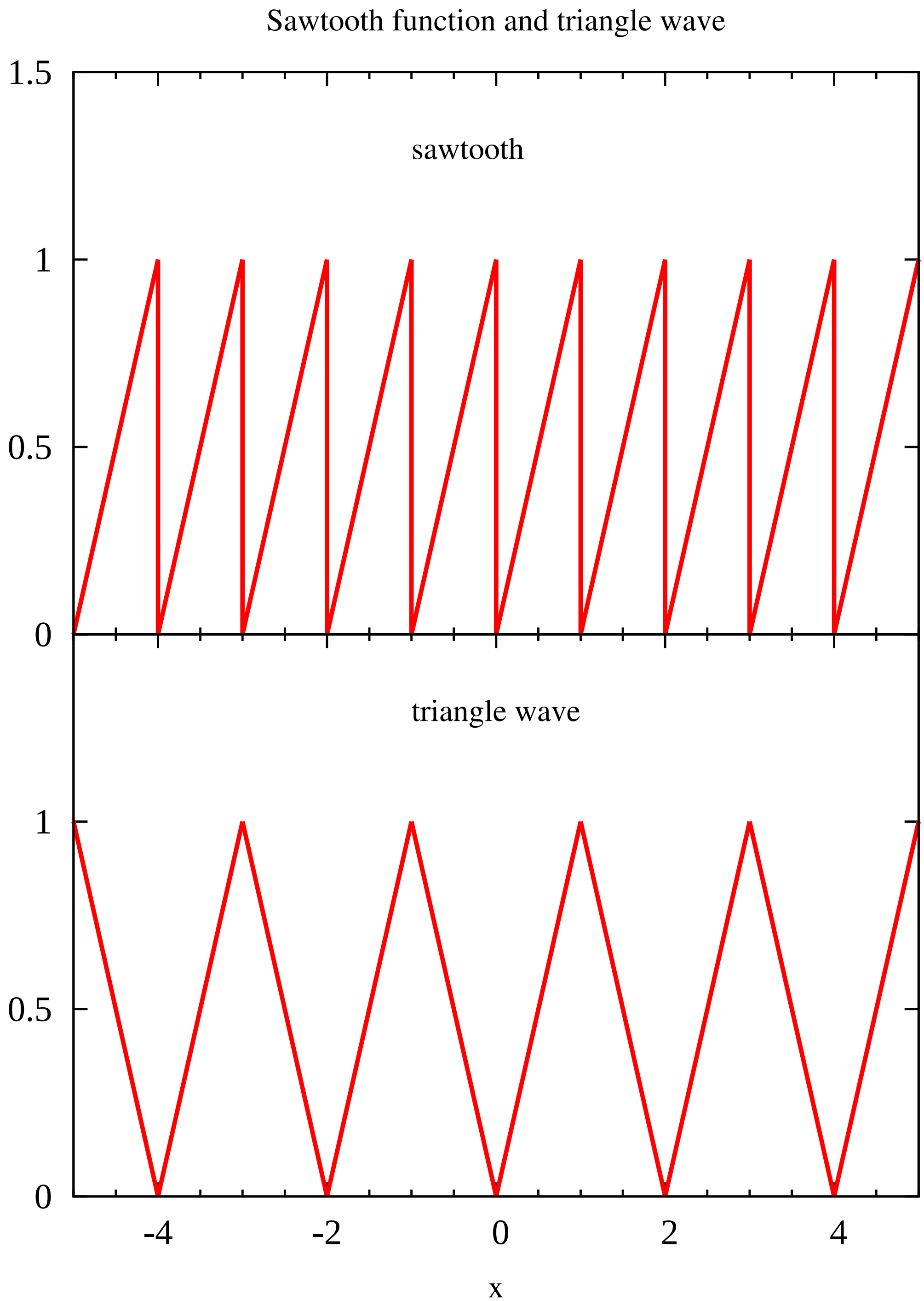

Figure 1: Plots of sawtooth function and triangle wave.

27.2 Periodic functions and a circle

- There is a close connection between periodic functions of a real variable x and functions defined around a circle using an angle θ.
- Suppose x is a real variable and $f(x)$ is a periodic function with period x_p.
- Let $\theta = 2\pi x/x_p$ and define $g(\theta) = f(x)$. Note that

$$\frac{2\pi(x + x_p)}{x_p} = \frac{2\pi x}{x_p} + 2\pi = \theta + 2\pi\,. \tag{27.2.1}$$

- Hence $g(\theta)$ is a periodic function of θ with period 2π.
- The variable θ can be viewed as the angle around the circumference of a circle.
- This is in fact a fundamental point of view, very useful, and employed by many experts.
- An increment in the value of θ by 2π is one circuit around the circumference of a circle.
- Hence, to study periodic functions, it is helpful to use functions of an angle θ around a circle.
- We shall do so in these lectures and study functions of an angle θ around a circle.

27.3 Orthogonal functions (formal mathematical proofs)

- From now on we shall treat functions of θ.
- However, before we do so, we need to know some fundamental properties of the sine and cosine functions.
- First, we know (or it is easy to prove) that

$$\begin{aligned} \int_0^{2\pi} \sin(n\theta)\,d\theta &= 0 \qquad \text{all integers } n\,, \\ \int_0^{2\pi} \cos(n\theta)\,d\theta &= 0 \qquad (n \neq 0)\,. \end{aligned} \tag{27.3.1}$$

27.3.1 Orthogonality

- Next, if m and n are integers, then $\sin(m\theta)$, $\sin(n\theta)$, $\cos(m\theta)$ and $\sin(n\theta)$ are a 'family' of **orthogonal functions.**
- The concept of 'orthogonal functions' means the following integrals are all zero:
  1. If $m \neq n$ then

$$\int_0^{2\pi} \cos(m\theta)\,\cos(n\theta)\,d\theta = \int_0^{2\pi} \sin(m\theta)\,\sin(n\theta)\,d\theta = 0\,. \tag{27.3.2}$$

  2. For *all values of m and n,* then

$$\int_0^{2\pi} \sin(m\theta)\,\cos(n\theta)\,d\theta = 0\,. \tag{27.3.3}$$

- The proof involves simply the straightforward but tedious evaluation of integrals.
- **Orthogonality of $\cos(m\theta)$ and $\cos(n\theta)$.** We employ eq. (27.3.1) (note $m \neq n$):

$$\int_0^{2\pi} \cos(m\theta)\,\cos(n\theta)\,d\theta = \int_0^{2\pi} \frac{\cos((m-n)\theta) + \cos((m+n)\theta)}{2}\,d\theta = 0\,. \tag{27.3.4}$$

- **Orthogonality of $\sin(m\theta)$ and $\sin(n\theta)$.** We employ eq. (27.3.1) (note $m \neq n$):

$$\int_0^{2\pi} \sin(m\theta)\,\sin(n\theta)\,d\theta = \int_0^{2\pi} \frac{\cos((m-n)\theta) - \cos((m+n)\theta)}{2}\,d\theta = 0\,. \tag{27.3.5}$$

- **Orthogonality of $\sin(m\theta)$ and $\cos(n\theta)$.** We employ eq. (27.3.1):

$$\int_0^{2\pi} \sin(m\theta)\,\cos(n\theta)\,d\theta = \int_0^{2\pi} \frac{\sin((m-n)\theta) + \sin((m+n)\theta)}{2}\,d\theta = 0\,. \tag{27.3.6}$$

- This establishes the orthogonality of $\sin(m\theta)$, $\sin(n\theta)$, $\cos(m\theta)$ and $\sin(n\theta)$.

27.3.2 Normalization

- Next we must calculate the **normalization** of the functions $\sin(n\theta)$ and $\cos(n\theta)$.
- For any integer $n \neq 0$,

$$\frac{1}{\pi}\int_0^{2\pi} \sin^2(n\theta)\,d\theta = \frac{1}{\pi}\int_0^{2\pi} \cos^2(n\theta)\,d\theta = 1\,. \tag{27.3.7}$$

- Also if $n = 0$, $\sin(n\theta) = \sin(0) = 0$ and $\cos(n\theta) = \cos(0) = 1$, hence obviously

$$\frac{1}{\pi}\int_0^{2\pi} \sin^2(0)\,d\theta = 0\,, \qquad \frac{1}{\pi}\int_0^{2\pi} \cos^2(0)\,d\theta = 2\,. \tag{27.3.8}$$

- The derivation of the normalization for $n \neq 0$ goes as follows.
  1. We begin with $\sin(n\theta)$ and employ eq. (27.3.1):

$$\frac{1}{\pi}\int_0^{2\pi} \sin^2(n\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \bigl[\,1-\cos(2n\theta)\,\bigr]\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1\,. \tag{27.3.9}$$

  2. Next we treat $\cos(n\theta)$ and employ eq. (27.3.1):

$$\frac{1}{\pi}\int_0^{2\pi} \cos^2(n\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \bigl[\,1+\cos(2n\theta)\,\bigr]\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1\,. \tag{27.3.10}$$

- **We can summarize the above results elegantly using the Kronecker delta:**

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta &= \delta_{mn} && (m,n)\neq(0,0)\,,\\
\frac{1}{\pi}\int_0^{2\pi} \sin(m\theta)\sin(n\theta)\,d\theta &= \delta_{mn} && (m,n)\neq(0,0)\,,\\
\frac{1}{\pi}\int_0^{2\pi} \sin(m\theta)\cos(n\theta)\,d\theta &= 0 && \text{all } (m,n)\,.
\end{aligned} \tag{27.3.11}$$

- The case $(m,n) = (0,0)$ requires special treatment.

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta &= 2 && (m,n)=(0,0)\,,\\
\frac{1}{\pi}\int_0^{2\pi} \sin(m\theta)\sin(n\theta)\,d\theta &= 0 && (m,n)=(0,0)\,.
\end{aligned} \tag{27.3.12}$$

27.4 Fourier series (important)

- Let $f(\theta)$ be a periodic function of θ with period 2π.
- Then f can be expressed (or expanded) in a **Fourier series** as follows:

$$\begin{aligned} f(\theta) &= \tfrac{1}{2}a_0 + a_1\cos(\theta) + a_2\cos(2\theta) + a_3\cos(3\theta) + \cdots \\ &\qquad + b_1\sin(\theta) + b_2\sin(2\theta) + b_3\sin(3\theta) + \cdots \\ &= \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty}\big[\, a_j\cos(j\theta) + b_j\sin(j\theta)\,\big]\,. \end{aligned} \tag{27.4.1}$$

- The a_j and b_j are called **Fourier coefficients. They can be complex.**
- There is no term b_0 because $\sin(j\theta) = 0$ for $j = 0$.
- The term a_0 is multiplied by $\frac{1}{2}$ because of the normalization results from Sec. 27.3.
- Using the orthogonality and normalization results from Sec. 27.3, the coefficients a_j and b_j can be calculated via

$$\begin{aligned} a_j &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(j\theta)\,d\theta \qquad (j \ge 0)\,, \\ b_j &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(j\theta)\,d\theta \qquad (j > 0)\,. \end{aligned} \tag{27.4.2}$$

- **We can also choose θ to lie in the interval $-\pi < \theta \le \pi$.**
- Then the coefficients a_j and b_j are calculated via

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(j\theta)\,d\theta\,, \qquad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(j\theta)\,d\theta\,. \tag{27.4.3}$$

- ***However, why should we believe eq. (27.4.1)?***
- There are some obvious questions about the validity of eq. (27.4.1).
  1. Can every periodic function $f(\theta)$ with period 2π be expressed using the series in eq. (27.4.1)? *Surely not.*
  2. Even if a periodic function $f(\theta)$ can be expanded in a Fourier series, in general it is an infinite series. Hence: **does the series in eq. (27.4.1) converge?**
  3. Even if the series in eq. (27.4.1) converges, **does it converge to the original function $f(\theta)$?** *Is the 'inverse operation' (summing the series) well defined and unique?*
- The above are all questions to be dealt with.
  After all, we want sone assurance that we are not calculating nonsense.
- **However, there is a more fundamental question.**
  1. **If we already have the function $f(\theta)$, *why do we need a Fourier series?***
  2. The answer to this is given in Sec. 27.5. It is classified as 'advanced material' to avoid interrupting the flow of the rest of this lecture.

27.5 Motivation: why do we need a Fourier series?

- As stated above, if we already know the function $f(\theta)$, why do we need a Fourier series?
- The answer is that, in general, we do *not* know the function $f(\theta)$.

  1. Frequently, we have to solve a problem, of physics or engineering, etc.
  2. There are general mathematical principles which tell us that the solution we seek is a periodic function.
  3. Hence we solve the (physics/engineering/etc.) problem by *expressing the unknown function* $f(\theta)$ as a series using eq. (27.4.1), and solve for the coefficients a_j and b_j.
  4. **The function $f(\theta)$ is then obtained by summing the Fourier series in eq. (27.4.1).**
  5. *Hence the Fourier series comes first,* and the function $f(\theta)$ is obtained from it.
  6. This naturally raises the issue: is the Fourier series in eq. (27.4.1) is well defined and does it really converge to the true function $f(\theta)$?

Example

- Find the periodic solution for $f(\theta)$ of the following differential equation:

$$\frac{d^2 f}{d\theta^2} + 2f = 1 + \sum_{j=1}^{\infty} \frac{\cos(j\theta)}{2^j} - \sum_{j=1}^{\infty} \frac{\sin(j\theta)}{j!} . \tag{27.5.1}$$

- Let us try a solution by expressing $f(\theta)$ using eq. (27.4.1), then

$$\frac{d^2 f}{d\theta^2} = -\sum_{j=1}^{\infty} j^2 a_j \cos(j\theta) - \sum_{j=1}^{\infty} j^2 b_j \sin(j\theta) . \tag{27.5.2}$$

- Substituting in eq. (27.5.1) and equating coefficients term by term yields

$$\begin{aligned} a_0 &= 1 , \\ (-j^2+2)a_j &= \frac{1}{2^j} \qquad (j \ge 1) , \\ (-j^2+2)b_j &= -\frac{1}{j!} \qquad (j \ge 1) . \end{aligned} \tag{27.5.3}$$

- The solution for $f(\theta)$ is therefore

$$f(\theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{\cos(j\theta)}{2^j(2-j^2)} - \sum_{j=1}^{\infty} \frac{\sin(j\theta)}{j!(2-j^2)} . \tag{27.5.4}$$

- ***This is our answer.*** It may or may not be possible to sum the series in eq. (27.5.4) in closed form. We need to justify that the series in eq. (27.5.4) converges and yields a well defined (periodic) function.
- **The Fourier series comes first. The function must be obtained from it.**

27.6 Existence of Fourier series (mathematical technicality)

- At a minimum, the function $f(\theta)$ must be absolutely integrable for $0 \le \theta < 2\pi$:

$$\int_0^{2\pi} |f(\theta)|\, d\theta < \infty\,. \tag{27.6.1}$$

- This is obviously necessary else the integrals eq. (27.4.2) to calculate the coefficients a_j and b_j would not be well defined.

- The function $f(\theta)$ must have at most a finite (discrete) set of discontinuities for $0 \le \theta < 2\pi$:
  1. This is an important condition, which I know from my own personal experience.
  2. There exist so–called **chaotic** functions, which have an infinite number of discontinuities.
  3. A chaotic function can be periodic *(oh yes!)*.
  4. Back in 2000–2001, I derived the solution to a physics problem, which had been a major open problem in my field of expertise since at least 1985.
  5. However, my solution had peculiar properties, and were not accepted by everyone.
  6. My solution flew in the face of the accepted wisdom in the field, and contradicted the widely accepted theory on the subject.
  7. Nevertheless, being a leading expert in the field myself, I could not be dismissed as a crank. Plus, there was experimental evidence to support my solution. (Actually the experimental data came first, my theoretical work came later.)
  8. This led to some furious *(heated? acrimonious?)* debates in the period 2000–2005 (aproximately) and strained some long–standing friendships.
  9. I eventually realized that for some parameter values, I had encountered *(stumbled on?)* a periodic chaotic function.
  10. The conventional theory in my field made use of Fourier series in the formalism, but for a periodic chaotic function the required Fourier series did not exist.
  11. Ultimately, the resolution was that the standard theory had overlooked the possibility of periodic chaotic functions, and my solution had (inadvertently) found some.
  12. Not all the experts accept my explanation, but most do.
  13. *People get animated about these things.*

- The above example from my career also illustrates the fact that we do not start with a function $f(\theta)$ with known properties.

- Instead, most theoretical research problems begin with a complicated set of equations. In fact, a major part of the research effort is to find the relevant equations in the first place. The equations must be solved (usually approximately). It may not be at all obvious what are the properties of the solution (sum of the series).

27.7 Linear operator (formal mathematical proof)

- The process of expressing a periodic function as a Fourier series is a linear operator.
- This is reasonably obvious, but must be stated formally.
- Given two functions $f(\theta)$ and $g(\theta)$ and define $h(\theta) = f(\theta) + g(\theta)$, then the Fourier series of h is the sum of the individual Fourier series of f and g. With a reasonably obvious notation:

$$\begin{aligned} a_j(h) &= a_j(f) + a_j(g)\,, \\ b_j(h) &= b_j(f) + b_j(g)\,. \end{aligned} \tag{27.7.1}$$

- If λ is a constant (which can be complex) and $r(\theta) = \lambda\, f(\theta)$, then the Fourier series of r is λ times the Fourier series of f. Again with a reasonably obvious notation:

$$\begin{aligned} a_j(r) &= \lambda\, a_j(f)\,, \\ b_j(r) &= \lambda\, b_j(f)\,. \end{aligned} \tag{27.7.2}$$

- The two formulas eqs. (27.7.1) and (27.7.2) are the two defining properties of a linear operator.
- Many authors combine them into one formula and say, using constants λ and μ,

$$\begin{aligned} a_j(\lambda f + \mu g) &= \lambda\, a_j(f) + \mu\, a_j(g)\,, \\ b_j(\lambda f + \mu g) &= \lambda\, b_j(f) + \mu\, b_j(g)\,. \end{aligned} \tag{27.7.3}$$

27.8 Fourier transforms and Fourier series (formal mathematics)

- In a previous lecture, we defined the **Fourier transform** of a function f of a real variable x.
- The definiton of the Fourier transform $F(k)$ of the function $f(x)$ was via an integral

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\,. \tag{27.8.1}$$

- The inversion procedure to obtain $f(x)$ from $F(k)$ (the inverse Fourier transform) was given by another integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk\,. \tag{27.8.2}$$

- If $f(x)$ is a periodic function of x, its Fourier transform (it it exists) is a Fourier series.
  1. In later lectures, we shall formalize this idea via the **discrete Fourier transform.**
  2. Instead of a continuous integral, we obtain a discrete sum, given by eq. (27.4.1).
  3. The calculation of the coefficients a_j and b_j (effectively the calculation of the Fourier transform) is given by eq. (27.4.2).
  4. To obtain the function from the series (effectively the inverse Fourier transform), we sum the series in eq. (27.4.1).
- The important concept of **convolution** applies also to Fourier series.

27.9 Convergence and uniqueness of Fourier series (formal mathematics)

- Rigorous statements of the material in this section are given in pure mathematics textbooks.
- As already noted, a Fourier series is in general an infinite series.
- It is not obvious or guaranteed that the sum converges to a well defined function.
- Return to eq. (27.4.1) and define the **partial sum** $S_n(f,\theta)$ for $n = 0, 1, 2 \dots$

$$S_n(f,\theta) = \frac{a_0}{2} + \sum_{j=1}^{n} \big[\, a_j \cos(j\theta) + b_j \sin(j\theta) \,\big] \,. \tag{27.9.1}$$

Theorem 1 (Dirichlet) *If $f(\theta)$ is continuous and has a bounded continuous derivative, except possibly at a finite number of points, then $S_n(f,\theta) \to f(\theta)$ as $n \to \infty$ at all values of θ where f is continuous.*

- If $f(\theta)$ is a continuous function and we are given all the Fourier coefficients a_j and b_j, can we calculate $f(\theta)$ for all values $0 \le \theta < 2\pi$? (Does the sum of the series converge?)
- The answer is yes, even if, for a given fixed value of θ, the partial sums $S_n(f,\theta)$ do not converge to a limit as $n \to \infty$.
- Define the following *weighted partial sum.* It is called a **Cesàro sum:**

$$\sigma_n(f,\theta) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f,\theta) \,. \tag{27.9.2}$$

- There exist examples of sequences S_n where S_n does not converge to a limit as $n \to \infty$ but σ_n does. The Cesàro sums converge in a wider class of situations than the partial sums.

Theorem 2 (Fejér) *If $f(\theta)$ is continuous then $\sigma_n(f,\theta) \to f(\theta)$ uniformly for $0 \le \theta < 2\pi$.*

- Fejér actually proved more. The above is only part of his full theorem.
  1. **Uniform convergence** is a pure mathematical concept.
  2. Uniform convergence is distinct from **pointwise convergence.**
  3. We shall skip the technical details of the different types of mathematical convergence.
- Johann Peter Gustav Lejeune Dirichlet (1805–1859) was a German mathematician.
- Lipót Fejér (1880–1959) was a Hungarian mathematician.

27.10 Notation for sum of Fourier series

- We require a notation to distinguish between the function $f(\theta)$ and the sum of the Fourier series in eq. (27.4.1), because we realize by now that the sum in eq. (27.4.1) may not always converge to the function $f(\theta)$ for all values of θ.
- We can already see that the task is ambiguous because there are at least two ways to sum the series in eq. (27.4.1) using different types of partial sums.
  1. We can employ the limit of the partial sums S_n as $n \to \infty$.
  2. We can employ the limit of the partial Cesàro sums σ_n as $n \to \infty$.
- Let us introduce the following terminology.
  1. Let us call $S_n(f, \theta)$ a **Dirichlet partial sum.**
  2. Let us call $\sigma_n(f, \theta)$ a **Fejér partial sum.**
- ***Note that these are my own terms.***
- They are *not* standard in the literature.
- The experts may scoff.

27.11 Triangle wave (important)

- Let us calculate some examples of Fourier series.
- Let us consider the triangle wave in Fig. 1.
- The period in x is 2 hence $\theta = 2\pi x/2 = \pi x$.
- It is convenient to employ the interval $-\pi < \theta \le \pi$.
- Then from Fig. 1 we see that

$$f_{\rm tri}(\theta) = \frac{|\theta|}{\pi} \qquad\qquad (-\pi < \theta \le \pi)\,. \tag{27.11.1}$$

- This is an even function of θ hence all the b_j are zero.
- We calculate a_j via eq. (27.4.3).
  1. First for $j=0$ we obtain

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|\theta|}{\pi}\,d\theta = \frac{2}{\pi^2}\int_0^{\pi} \theta\;d\theta = 1\,. \tag{27.11.2}$$

  2. Next for $j>0$ we obtain

$$\begin{aligned} a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|\theta|}{\pi}\,\cos(j\theta)\,d\theta \\ &= \frac{2}{\pi^2}\int_0^{\pi} \theta\,\cos(j\theta)\,d\theta \\ &= \frac{2}{\pi^2}\biggl\{\biggl[\theta\,\frac{\sin(j\theta)}{j}\biggr]_0^{\pi} - \int_0^{\pi}\frac{\sin(j\theta)}{j}\,d\theta\biggr\} \\ &= \frac{2}{\pi^2}\biggl[\frac{\cos(j\theta)}{j^2}\biggr]_0^{\pi} \\ &= \frac{2}{\pi^2}\,\frac{(-1)^j-1}{j^2}\,. \end{aligned} \tag{27.11.3}$$

  3. Hence $a_j = 0$ if j is even, and $a_j = -4/(j\pi)^2$ if j is odd.
- Then the Fourier series representation of the triangle function is

$$f_{\rm tri}^{\rm FS}(\theta) = \frac{1}{2} - \frac{4}{\pi^2}\biggl[\cos(\theta) + \frac{\cos(3\theta)}{9} + \frac{\cos(5\theta)}{25} + \cdots\biggr]\,. \tag{27.11.4}$$

- Let us investigate how the Dirichlet and Fejér partial sums converge to the original function.
- Note that the triangle function is continuous and has a continuous bounded derivative (except at two points $\theta = 0$ and π).
- The results are displayed in Fig. 2, for $-\pi \leq \theta \leq \pi$, with the Dirichlet partial sums in the top panel and the Fejér partial sums in the bottom panel.
- The exact triangle function $f_{\rm tri}(\theta)$ is plotted as the solid line.
- For the Dirichlet case, three partial sums $S_1(f,\theta)$ (dash), $S_5(f,\theta)$ (dotdash) and $S_9(f,\theta)$ (dots) are displayed. The partial sums converge to the exact function very well. The partial sum $S_9(f,\theta)$ is visually almost indistinguishable from the exact answer.
- For the Fejér case, three partial sums $\sigma_1(f,\theta)$ (dash), $\sigma_5(f,\theta)$ (dotdash) and $\sigma_9(f,\theta)$ (dots) are displayed. The partial sums converge to the exact function, but much more slowly than the Dirichlet partial sums. Even with ten terms in the average, the partial Fejér sum $\sigma_9(f,\theta)$ yields much poorer agreement to the exact answer than the Dirichlet partial sum $S_9(f,\theta)$.
- Note that the although the triangle wave is continuous, it is not differentiable at $\theta = 0$.
    1. However, any partial sum of the Fourier series consists of a finite sum of sines and cosines, and is therefore an infinitely differentiable function.
    2. The partial sums (both Dirichlet and Fejér) yield the poorest approximation to the function in the vicinity of the kink.
    3. This is only to be expected.
- This a common feature of the Dirichlet and Fejér partial sums. If the function $f(\theta)$ is continuous, the partial sum $S_n(f,\theta)$ yields a better approximation than the partial Fejér sum $\sigma_n(f,\theta)$, to the true value of the functiob $f(\theta)$.
- The partial Fejér sums converge to the exact function, but slowly.
- To a pure mathematician this is not a concern, but for practical computation, the slow rate of convergence is a major concern.
- *Hence why should we employ Cesàro sums?*
- To answer this question, we must examine discontinuous functions.

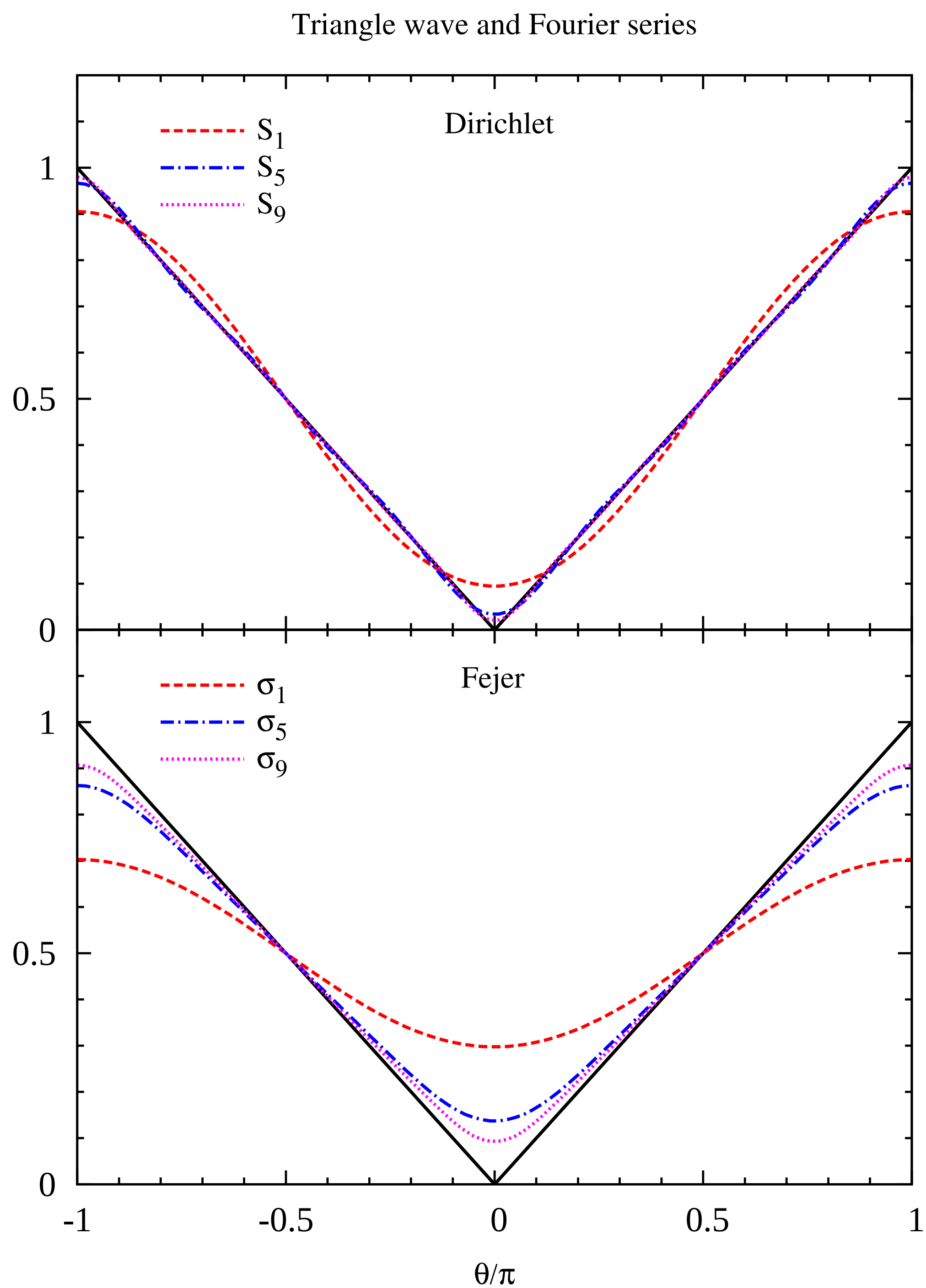

Figure 2: Plot of triangle wave and Fourier series using Dirichlet and Fejér partial sums.

27.12 Rectangle/window function (important)

- Let us calculate the Fourier series for another important function.
- It is the **window** or **rectangle** function.
- It describes a rectangle, hence it is a discontinuous function.
- Let θ_0 be a constant, where $0 < \theta \le \pi$.
- The window function of unit area in the interval $-\pi < \theta \le \pi$ is given by

$$f_{\rm win}(\theta) = \begin{cases} 1/(2\theta_0) & (|\theta| < \theta_0) \\ 0 & (\theta_0 \le |\theta| \le \pi)\,. \end{cases} \tag{27.12.1}$$

- Technically, this is a *periodic window function*. It repeats forever along the real x axis.
- Note that $f_{\rm win}(\theta)$ is discontinuous at $\theta = \pm\theta_0$.
- By construction, $f_{\rm win}(\theta)$ is an even function of θ hence all the b_j are zero.
- We calculate a_j via eq. (27.4.3):
  1. First for $j = 0$ we obtain

$$a_0 = \frac{1}{\pi}\int_{-\theta_0}^{\theta_0} \frac{1}{2\theta_0}\,d\theta = \frac{1}{\pi}\,. \tag{27.12.2}$$

  2. Next for $j > 0$ we obtain

$$a_j = \frac{1}{\pi}\int_{-\theta_0}^{\theta_0} \frac{\cos(j\theta)}{2\theta_0}\,d\theta \;=\; \frac{1}{2\theta_0\pi}\left[\frac{\sin(j\theta)}{j}\right]_{-\theta_0}^{\theta_0} \;=\; \frac{1}{\pi}\,\frac{\sin(j\theta_0)}{j\theta_0}\,. \tag{27.12.3}$$

  3. Hence a_j is given by a sinc function, but where j is an integer.
- Then the Fourier series representation of the periodic window function is

$$f^{\rm FS}_{\rm win}(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{j=1}^{\infty} \frac{\sin(j\theta_0)}{j\theta_0}\,\cos(j\theta)\,. \tag{27.12.4}$$

- The magnitudes of the Fourier coefficients decrease only as $1/j$, so we can guess this Fourier series will converge slowly.
- However, there are some surprises in store for us.

27.12.1 Graphs of partial sums

- Let us investigate how the Dirichlet and Fejér partial sums converge to the original function.
- Let us set $\theta_0 = \frac{1}{2}\pi$ and also scale its height to 1.
- Then the rectangle function is discontinuous at $\theta = \pm\frac{1}{2}\pi$.
- However, a finite sum of cosines is a continuous infinitely differentiable function.
- Hence how will the partial sums converge to a discontinuous function?
- The results are displayed in Fig. 3, for $-\pi \le \theta \le \pi$.
- The Dirichlet partial sums are shown in the top panel and the Fejér partial sums are shown in the bottom panel.
- The exact window function $f_{\rm win}(\theta)$ is plotted as the solid line.

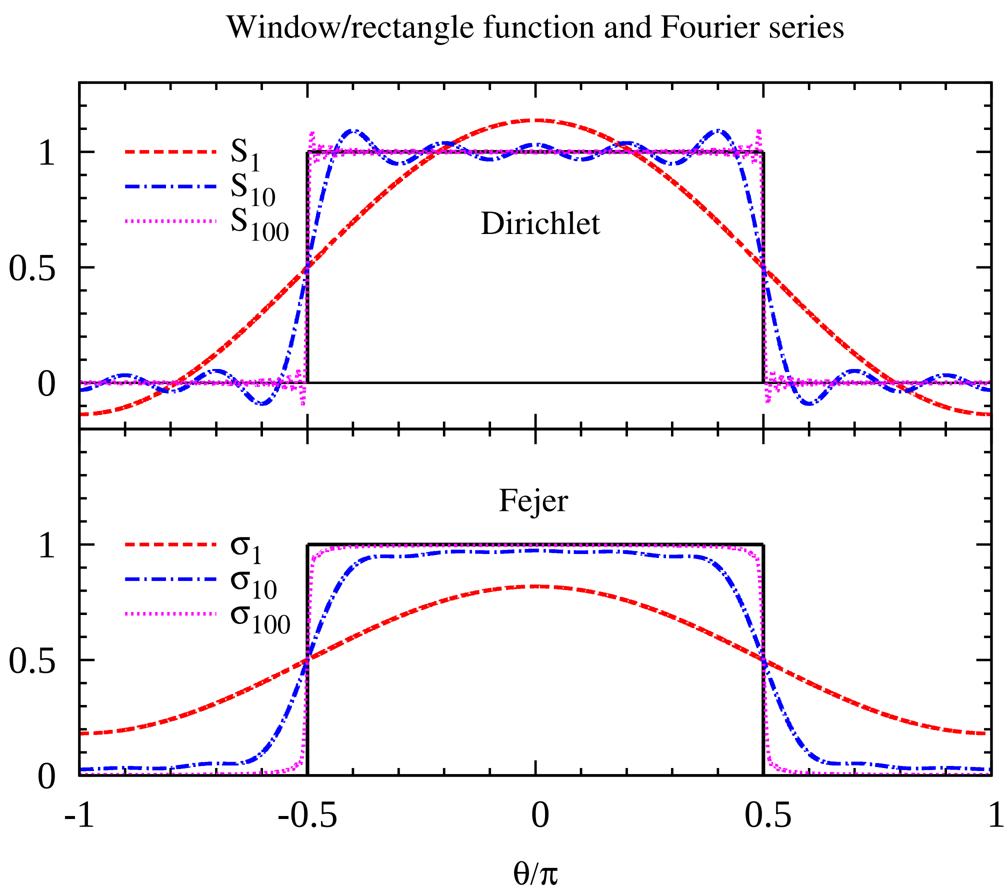

Figure 3: Plot of window function and Fourier series using Dirichlet and Fejér partial sums.

27.12.2 Dirichlet partial sums & Gibbs–Wilbraham phenomenon (important)

- The three partial sums $S_1(f,\theta)$ (dash), $S_{10}(f,\theta)$ (dotdash) and $S_{100}(f,\theta)$ (dots) are displayed in the top panel of Fig. 3.
- In all cases, the partial sums oscillate around the true solution.
- They undershoot and overshoot the function value in the vicinity of a discontinuity and attain negative values even though the window function $f_{\text{win}}(\theta)$ itself is never negative.
- Notice that as the value of n increases, the oscillations get more and more localized around the locations of the discontinuities.
- ***However, the oscillations never disappear, even in the limit $n \to \infty$.***
- A close up view of the window function and Dirichlet partial sums, around the location of the discontinuity at $\theta = \frac{1}{2}\pi$, is shown in Fig. 4. The partial sums $S_{100}(f,\theta)$ (dots) and $S_{1000}(f,\theta)$ (dash) are plotted.
- The undershoot and overshoot of the Dirichlet partial sums near the location of a discontinuity is called the **Gibbs–Wilbraham phenomenon.**
- It has been proved mathematically that even in the limit $n \to \infty$, the maximum amplitude of the oscillations does not decrease to zero.
- Indeed, the maximum negative value that $S_n(f,\theta)$ attains as $n \to \infty$ (and also the maximum positive value) are known mathematically.
- The **Gibbs–Wilbraham constant** is defined as
$$\frac{1}{2}\left(\int_0^{\pi} \frac{\sin t}{t}\,dt\right) - \frac{\pi}{4} \simeq \frac{\pi}{2}\cdot(0.089489872236\dots)\,. \tag{27.12.2.1}$$
- The maximum undershoot and overshoot is about 9% of the magnitude of the jump:
$$\begin{aligned} \liminf_{n\to\infty} S_n(f,\theta) &\simeq -0.089489872236\,, \\ \limsup_{n\to\infty} S_n(f,\theta) &\simeq 1.089489872236\,. \end{aligned} \tag{27.12.2.2}$$
- **The partial sums converge to the average value of the function at a discontinuity.**
- This is in fact a general property of Fourier series, whichever set of partial sums are employed.
- If θ_d the location of a discontinuity, then the Fourier series converges to the value
$$\mathrm{FS}(\theta_d) = \lim_{\delta\to 0} \frac{f(\theta_d+\delta) + f(\theta_d-\delta)}{2}\,. \tag{27.12.2.3}$$
- In this example the average equals $\frac{1}{2}$. In fact for all n, the partial sums equal $\frac{1}{2}$ exactly:
$$S_n(f, \pm\frac{1}{2}\pi) = \frac{1}{2}\,. \tag{27.12.2.4}$$

27.12.3 Fejér partial sums

- For the Fejér case, the three partial sums $\sigma_1(f,\theta)$ (dash), $\sigma_{10}(f,\theta)$ (dotdash) and $\sigma_{100}(f,\theta)$ (dots) are displayed in the bottom panel of Fig. 3.
- A close up view of the window function and Fejér partial sums, around the location of the discontinuity at $\theta = \frac{1}{2}\pi$, is shown in Fig. 4. The values of $\sigma_{100}(f,\theta)$ (dots) and $\sigma_{1000}(f,\theta)$ (dash) are plotted.
- **The Fejér partial sums do not exhibit the Gibbs–Wilbraham phenomenon.**
- There is no undershoot and overshoot around the location of a discontinuity. The window function $f_{\rm win}(\theta)$ is never negative and the Fejér partial sums do not attain negative values.
- As the value of n increases, the Fejér partial sums approximate the true function value more and more closely.
- Mathematically, in the limit $n \to \infty$, the graph of σ_n looks like the graph of $f(\theta)$ for all values of θ. The formal mathematical term for this is **uniform convrgence.** We say that the Fejér partial sums 'converge uniformly' to the function f.
- The Dirichlet partial sums do not exhibit uniform convergence when the function f has discontinuities. Instead they display the Gibbs–Wilbraham phenomenon, and the graph of S_n does *not* look like the graph of $f(\theta)$ for all values of θ.
- The Fejér partial sums also converge to the average value of the function at a discontinuity.
- As with the Dirichlet partial sums S_n, the Fejér partial sums σ_n equal $\frac{1}{2}$ exactly at the location of the discontinuity:

$$\sigma_n(f, \pm\frac{1}{2}\pi) = \frac{1}{2}\,. \qquad (27.12.3.1)$$

27.12.4 *Which is better: Dirichlet or Fejér?*

- I suppose it is impossible to settle such a question by purely mathematical arguments.
- Even the term 'mathematics' must be balanced against 'computation/engineering' and so on.
- Let us compare the two partial sums $S_{1000}(f, \theta)$ and $\sigma_{1000}(f, \theta)$.
- A close up view of the Dirichlet and Fejér partial sums, around the location of the discontinuity of the window function at $\theta = \frac{1}{2}\pi$, is shown in Fig. 6. The values of $S_{1000}(f, \theta)$ (dotdash) and $\sigma_{1000}(f, \theta)$ (dash) are plotted.
  1. The Dirichlet partial sum yields a better approximation to the function than the Fejér partial sum, except in a small region close to the location of the discontinuity.
  2. Although the Dirichlet partial sum exhibits the Gibbs–Wilbraham phenomenon, and does not converge uniformly to the function $f(\theta)$, this defect only occirs in a small interval near the location of the discontinuity.
  3. The Fejér partial sum avoids the Gibbs–Wilbraham phenomenon and also converges uniformly to the function $f(\theta)$, but these features come at a heavy price.
- Furthermore, for a continuous function, the Gibbs–Wilbraham phenomenon does not appear.
  1. We observed this in the case of the triangle wave.
  2. For a given value of n, the accuracy of the Dirichlet partial sum was far superior to that of the Fejér partial sum.
- Hence for most practical applications, the Dirichlet partial sum is satisfactory.
- There exist other procedures to sum the Fourier series, which improve upon the Dirichlet partial sum, i.e. reduce the effect of the Gibbs–Wilbraham phenomenon, but without decreasing the accuracy of the approximation as much as the Fejér partial sums do.
- We shall not study such techniques in these lectures. Essentially, they are a compromise between the Dirichlet and Fejér partial sums.
- In the rest of these lectures, we shall employ the Dirichlet partial sums.
- We shall accept the penalty of the Gibbs–Wilbraham phenomenon.

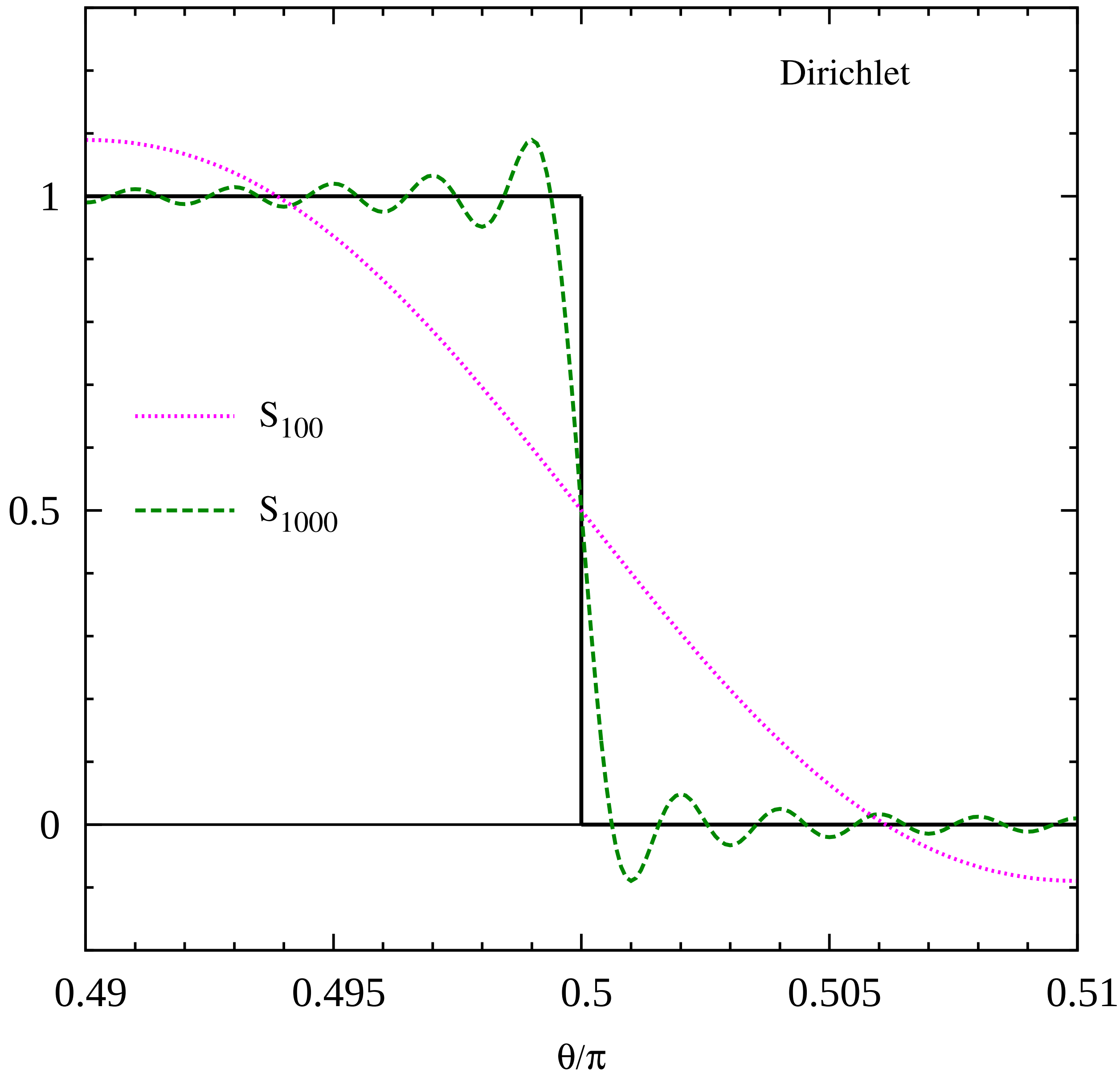

Figure 4: Close up view of window function and Dirichlet partial sums around a discontinuity.

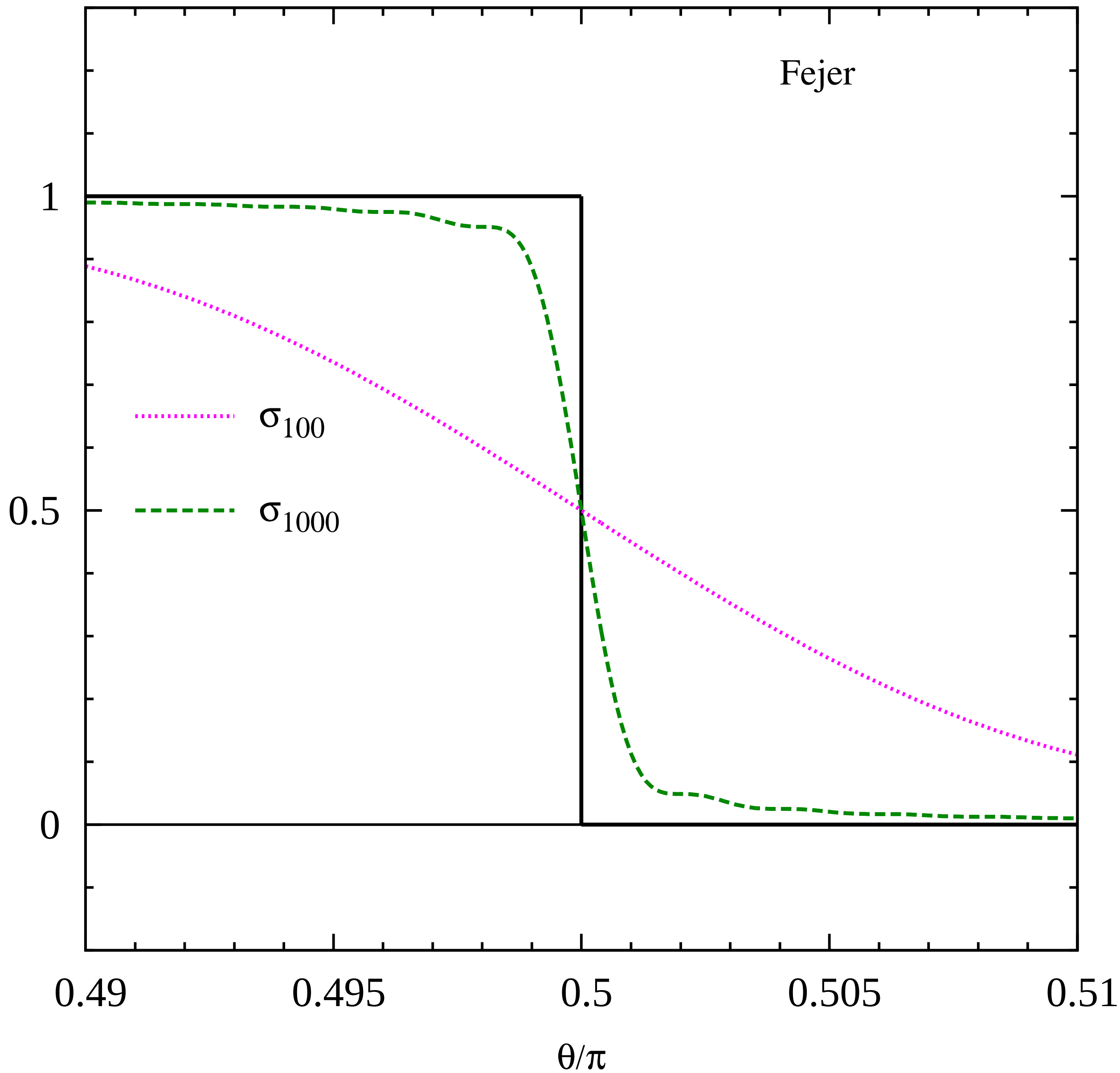

Figure 5: Close up view of window function and Fejér partial sums around a discontinuity.

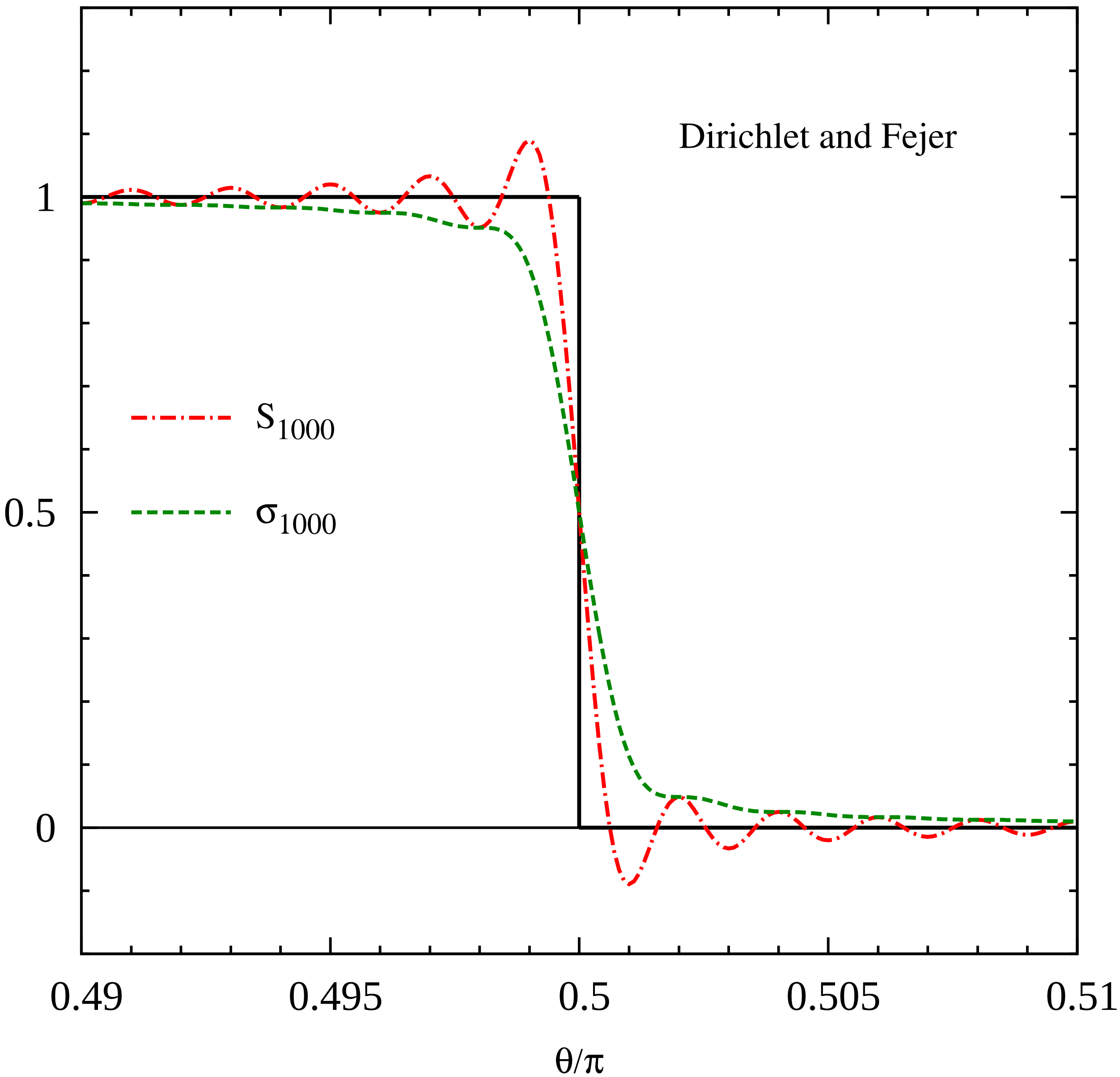

Figure 6: Comparison of Dirichlet and Fejér partial sums around a discontinuity of the window function.

27.13 Bandwidth limited functions

- Consider the simple example $f_s(\theta) = \sin\theta$.
- Its Fourier series is obvious. The Fourier coefficients are $b_1 = 1$ and all the rest are zero.
- The Dirichlet partial sums are given by $S_0 = 0$, $S_1 = S_2 = S_3 = \cdots = \sin\theta$.
- **The Dirichlet partial sums yield the exact value of f_s for all $n \ge 1$.**
- What are the Fejér partial sums? They are given by

$$\begin{aligned}
\sigma_0(f_s,\theta) &= S_0 &&= 0\,, \\
\sigma_1(f_s,\theta) &= \frac{S_0+S_1}{2} &&= \frac{1}{2}\sin\theta\,, \\
\sigma_2(f_s,\theta) &= \frac{S_0+S_1+S_2}{3} &&= \frac{2}{3}\sin\theta\,, \\
&\vdots && \vdots \\
\sigma_n(f_s,\theta) &= \frac{S_0+S_1+\cdots+S_n}{n+1} &&= \frac{n}{n+1}\sin\theta\,.
\end{aligned} \tag{27.13.1}$$

- *The Fejér partial sums never equal f_s exactly.*
- A **bandwidth limited function** is a function $f(x)$ whose Fourier transform $F(k)$ equals zero for all $k > K$ for some constant $K \ge 0$.
- If a bandwidth limited function is periodic, its Fourier series is a finite sum, say up to m:

$$f_m(\theta) = \tfrac{1}{2}a_0 + \sum_{j=1}^{m} \big[\, a_j\cos(j\theta) + b_j\sin(j\theta) \,\big]\,. \tag{27.13.2}$$

- **By construction, the Dirichlet partial sums S_n equal f_m exactly for all $n \ge m$.**
- The Fejér partial sums σ_n never equal f_m exactly for any n (unless $m = 0$ and f_m is constant).
- For $m \ge n$, the Fejér partial sum σ_n is given by

$$\sigma_n(f_p,\theta) = \tfrac{1}{2}a_0 + \sum_{j=1}^{m} \left(1 - \frac{2j+1}{2m+1}\right) \big[\, a_j\cos(j\theta) + b_j\sin(j\theta) \,\big]\,. \tag{27.13.3}$$

- We shall return to bandwidth limited functions when we study the discrete Fourier transform.

27.14 Parseval's theorem

- Let us suppose $f(\theta)$ is a square integrable function.
- Let us integrate $|f(\theta)|^2$ over a full period.
- The orthogonality and normalization relations in Sec. 27.3 yield

$$\frac{1}{\pi}\int_0^{2\pi} |f(\theta)|^2\,d\theta = \frac{|a_0|^2}{2} + \sum_{j=1}^{\infty}(|a_j|^2 + |b_j|^2)\,. \tag{27.14.1}$$

- This is called **Parseval's theorem.**
- If the above seems like an absurdly oversimplified derivation, indeed it is.
- To formulate a rigorous proof, we need to show that the following limit holds true:

$$\lim_{n\to\infty}\int_0^{2\pi}\Bigl|f(\theta) - S_n(f,\theta)\Bigr|^2\,d\theta = 0\,. \tag{27.14.2}$$

- The rigorous proof is given in pure mathematics textbooks.
- If f and g are continuous for all values $0 \le \theta < 2\pi$, then, with an obvious notation,

$$\frac{1}{\pi}\int_0^{2\pi} f^*(\theta)g(\theta)\,d\theta = \frac{a_0^*(f)a_0(g)}{2} + \sum_{j=1}^{\infty}\bigl[\,a_j^*(f)a_j(g) + b_j^*(f)b_j(g)\,\bigr]\,. \tag{27.14.3}$$

- This is also proved rigorously in pure mathematics textbooks.

27.15 Computation: trapezoid rule for periodic functions

- We can calculate the Fourier coefficients a_j and b_j analytically only for simple cases.
- We require a numerical algorithm for more complicated (or more general) functions $f(\theta)$.
- Let us set aside Fourier series and consider a periodic function $F(\theta)$ with period 2π.
- We wish to numerically compute the integral of $F(\theta)$ over a full period 2π.
- A simple algorithm is to employ the trapezoid rule, for $0 \le \theta \le 2\pi$.
- We employ n subintervals of length $h = 2\pi/n$, then $\theta_i = 2\pi i/n$ for $i = 0, \dots, n$.
- Let T_n denote the computed value of the integral using the trapezoid rule with n subintervals:

$$T_n = h \left[\frac{F(0) + F(2\pi)}{2} + \sum_{i=1}^{n-1} F(\theta_i) \right]. \tag{27.15.1}$$

- **However, because $F(\theta)$ is periodic with period 2π, by definition $F(0) = F(2\pi)$.**
- **Hence we can simplify eq. (27.15.1) and write a sum over n points $i = 0, \dots, n-1$:**

$$T_n = h \sum_{i=0}^{n-1} F(\theta_i). \tag{27.15.2}$$

- Hence we can compute a_j and b_j by summing the function at n points around a circle.
- Note that $h/\pi = (2\pi/n)/\pi = 2/n$, hence

$$a_j = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(j\theta)\, d\theta \;\simeq\; \frac{2}{n} \sum_{i=0}^{n-1} f(\theta_i) \cos(j\theta_i). \tag{27.15.3}$$

- Similarly we obtain

$$b_j \simeq \frac{2}{n} \sum_{i=0}^{n-1} f(\theta_i) \sin(j\theta_i). \tag{27.15.4}$$

- Technically, we should employ a notation to denote that these are *numerical approximations* for the true values of a_j and b_j, using n points.
- If we compute $2m+1$ Fourier coefficients in this way, for a_0 and $j = 1, \dots, m$ for a_j and b_j, then we must compute totally $2m+1$ sums.
- Hence this is an $O(mn)$ algorithm.
- We shall see later that if we evaluate the function at n points, we must set $2m+1 \le n$.
- Using only n points, we cannot obtain accurate results for more than n output values.

27.16 Euler–Maclaurin series (mathematical technicality)

- There is another fact to note about the use of the trapezoid rule for a periodic function integrated over a full period.
- In this section, let us use x instead of θ, where $-\infty < x < \infty$.
- Consider the integral of a function $F(x)$:

$$I = \int_a^b F(x)\,dx\,. \tag{27.16.1}$$

- Suppose $F(x)$ is a periodic function with period $b-a$, so we integrate over a full period $a \le x \le b$.
- Denote the exact value of the integral by $I_{\rm ex}$ and the trapezoid rule approximation by T_n.
- The **Euler–Maclaurin formula** states that, if $F(x)$ is $2p$ times continuously differentiable for all $x \in [a,b]$, then

$$T_n - I_{\rm ex} = \biggl(\sum_{k=1}^{p} \frac{h^{2k}B_{2k}}{(2k)!}(F^{(2k-1)}(b) - F^{(2k-1)}(a))\biggr) + R_{2p}\,. \tag{27.16.2}$$

- Here the B_k are the **Bernoulli numbers** and R_p is a remainder term.
- **However, because $F(x)$ is periodic with period $b-a$, therefore**

$$F^{(k)}(a) = F^{(k)}(b) \qquad\qquad (\text{all } k \ge 0)\,. \tag{27.16.3}$$

- **Hence all the terms in the sum in eq. (27.16.2) vanish.**
- **The numerical error in eq. (27.16.2) is given entirely by the remainfer term R_{2p}:**

$$T_n - I_{\rm ex} = R_{2p}\,. \tag{27.16.4}$$

- We shall observe some consequences of this fact in the homework exercises.