Queens College, CUNY, Department of Computer Science

Computational Finance

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18 Lecture 18

Fugit

- In this lecture we study the concept of **fugit**.
- The **fugit** is the mean or expectation of the life of a derivative.
- The fugit is therefore measured in units of time, i.e. years.
  1. The term **fugit** is frequently called the "median life" of a derivative.
  2. This is technically incorrect, because the fugit is really an expectation, not a median.
  3. However, people in the financial industry are not big on mathematical rigor.
  4. Hence "median life" is the common description.
- The fugit of a European option is obviously the time to expiration of the option.
- The fugit of an American option can be less than the time to expiration of the option, because of the possibility of early exercise.
- The term fugit is Latin, basically "time passes" or "time flies" to indicate the passage of time.
- **There is no explicit mathematical probability theory in this lecture.**

18.1 Fugit calculation using binomial model

- The concept and definition of fugit is independent of any probability model for the stock price movements, but we shall restrict our analysis to Geometric Brownian Motion.
- We can calculate the fugit of a derivative easily using a binomial model.
- First we calculate the fair value of the derivative using a binomial model.
- In the process of calculating the fair value, we tag each node in the binomial tree:
  1. Node = "dead" if the derivative is exercised (or terminated) at that node.
  2. Node = "alive" if the derivative is not terminated at that node (therefore still alive).
- There is no standard notation for fugit. Let us denote the fugit by τ.
- We initialize the fugit calculation by setting $\tau = T - t_0$ at all the nodes at the final (terminal) timestep $i = n$.
- We then calculate the fugit by working backwards through the binomial tree.
- The fugit calculation is similar to that for the fair value.
- Recall that at a node with stock price S at the timestep i, connected to nodes Su and Sd at the timestep $i + 1$, the formula to calculate the fair value V is

$$V = e^{-r\Delta t} \left(pV_u + qV_d\right). \tag{18.1.1}$$

- To calculate the fugit, we employ the same formula but without the discount factor

$$\tau = p\tau_u + q\tau_d\,. \tag{18.1.2}$$

- If the derivative is "dead" at a node, we set $\tau = t - t_0$, where t is the time at that node.
- The value of fugit is given by the value of τ at the node at the initial timestep $i = 0$.
- By construction, the answer lies in the interval $0 \le \tau \le T - t_0$.

18.2 Fugit calculation using Black–Scholes–Merton equation

- Recall the Black–Scholes–Merton equation is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0\,. \tag{18.2.1}$$

- The partial differential equation for the fugit is the same, but without the interest rate compounding:

$$\frac{\partial \tau}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \tau}{\partial S^2} + (r-q)S\frac{\partial \tau}{\partial S} = 0\,. \tag{18.2.2}$$

- We solve eq. (18.2.2) with the terminal condition $\tau = T - t_0$ at $t = T$.
- At every point in the (S,t) plane, we set $\tau = t - t_0$ if $V(S,t)$ is dead at that value of S and t.
- The fugit of the derivative is the value of τ at the current time and stock price (S_0, t_0).
- By construction, the answer lies in the interval $0 \le \tau \le T - t_0$.

18.3 Interpretation of fugit

- The concept of fugit is similar to that of the **Macaulay duration** of a bond.
- Basically, if all the cashflows of a derivative were paid at one time t_* (and discounted to the present time t_0), what would that time be? We wish to solve for t_* such that

$$e^{-r(t_*-t_0)}(\text{All cashflows paid at time } t_*) = \text{Fair value of derivative}\,. \tag{18.3.1}$$

- The answer is the fugit $t_* = \tau$.
- There is of coourse the complication that the casjflows of a derivative are weighted by probabilities, hence the value of the fugit will change if, for example, the volatility changes.
- Some people use the fugit to estimate which of the cashflows of a derivative are the "most important" in their overall contribution to the fair value, although this is not a rigorous mathematical description.
- For example, if an American option has an expiration time T and the fugit is (say) much less than $T - t_0$, it is an indication that the option will probably be exercised early and the terminal payoff of the option is not important. One should search to check what events are occurring around the fugit time. Perhaps the underlying stock pays a big dividend on a particular date, and it may be optimal to exercise early, to own the stock and collect that dividend.

18.4 Worked example: American put

- Recall the worked example of an American put in Lecture 17a.
- The input parameters were

$$S_0 = 100\,, \tag{18.4.1a}$$
$$K = 100\,, \tag{18.4.1b}$$
$$r = 0.1\,, \tag{18.4.1c}$$
$$q_{\rm div} = 0\,, \tag{18.4.1d}$$
$$\sigma = 0.5\,, \tag{18.4.1e}$$
$$T = 0.3\,, \tag{18.4.1f}$$
$$t_0 = 0\,. \tag{18.4.1g}$$

- The binomial tree had three timesteps ($n = 3$).
- From the above data we calculated the following parameter values:

$$e^{r\Delta t} \simeq 1.01005\,, \tag{18.4.2a}$$
$$e^{-r\Delta t} \simeq 0.99005\,, \tag{18.4.2b}$$
$$u \simeq 1.1713\,, \tag{18.4.2c}$$
$$d \simeq 0.8538\,, \tag{18.4.2d}$$
$$p \simeq 0.4922\,, \tag{18.4.2e}$$
$$q \simeq 0.5078\,. \tag{18.4.2f}$$

- The option valuation tree of the American put is shown in Fig. 1.

Binomial tree valuation for American put

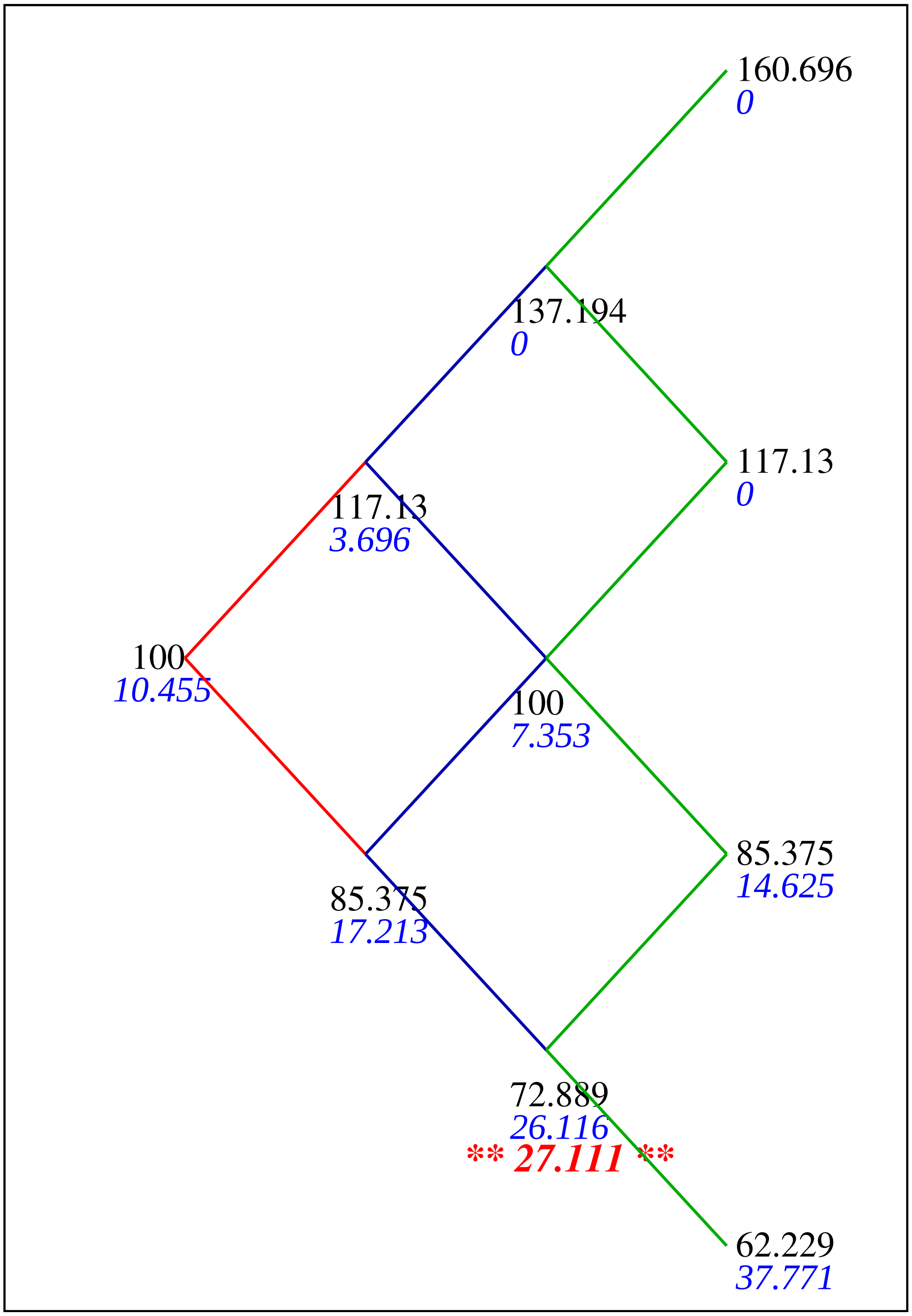

Figure 1: Valuation of American put option using a binomial tree with three timesteps, using the parameter values in eq. (18.4.2).

- We now calculate the fugit for the above American put.
- The option valuation tree of the American put is shown in Fig. 2.
- We begin with the nodes at the expiration time, hence $i = 3$.
  1. We set $\tau = T - t_0 = 0.3$ at every node for $i = 3$.
  2. It does not matter if the option is in or out of the money.
- Next we calculate the fugit values at the timestep $i = 2$, i.e. $t - t_0 = 0.2$.
  1. The values at the top two nodes are calculated as follows:

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.3 = (p+q) \times 0.3 = 0.3\,. \tag{18.4.3}$$

  2. The American put was exercised early at the bottom node, hence we set $\tau = t - t_0 = 0.2$.
- Next we calculate the fugit values at the timestep $i = 1$, i.e. $t - t_0 = 0.1$.
  1. The value at the top node is calculated as follows:

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.3 = (p+q) \times 0.3 = 0.3\,. \tag{18.4.4}$$

  2. The value at the bottom node is calculated as follows:

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.2 \simeq 0.249\,. \tag{18.4.5}$$

  3. This is a weighted average of the fugit values at the nodes for $i = 2$, where there was early exercise at one of those nodes. Hence the fugit calculation now yields a more complicated result.
- Finally we calculate the fugit value at the initial timestep $i = 0$, i.e. $t = t_0 = 0$.
- The fugit of the American put, using a binomial tree with three timesteps, is given by

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.249 \simeq 0.274\,. \tag{18.4.6}$$

- The fugit value is less than the time to expiration $T - t_0 = 0.3$ because there was early exercise at one node in the binomial tree.
- A more accurate value using $n = 1000$ timesteps yields $\tau \simeq 0.259$.

Binomial tree fugit calculation for American put

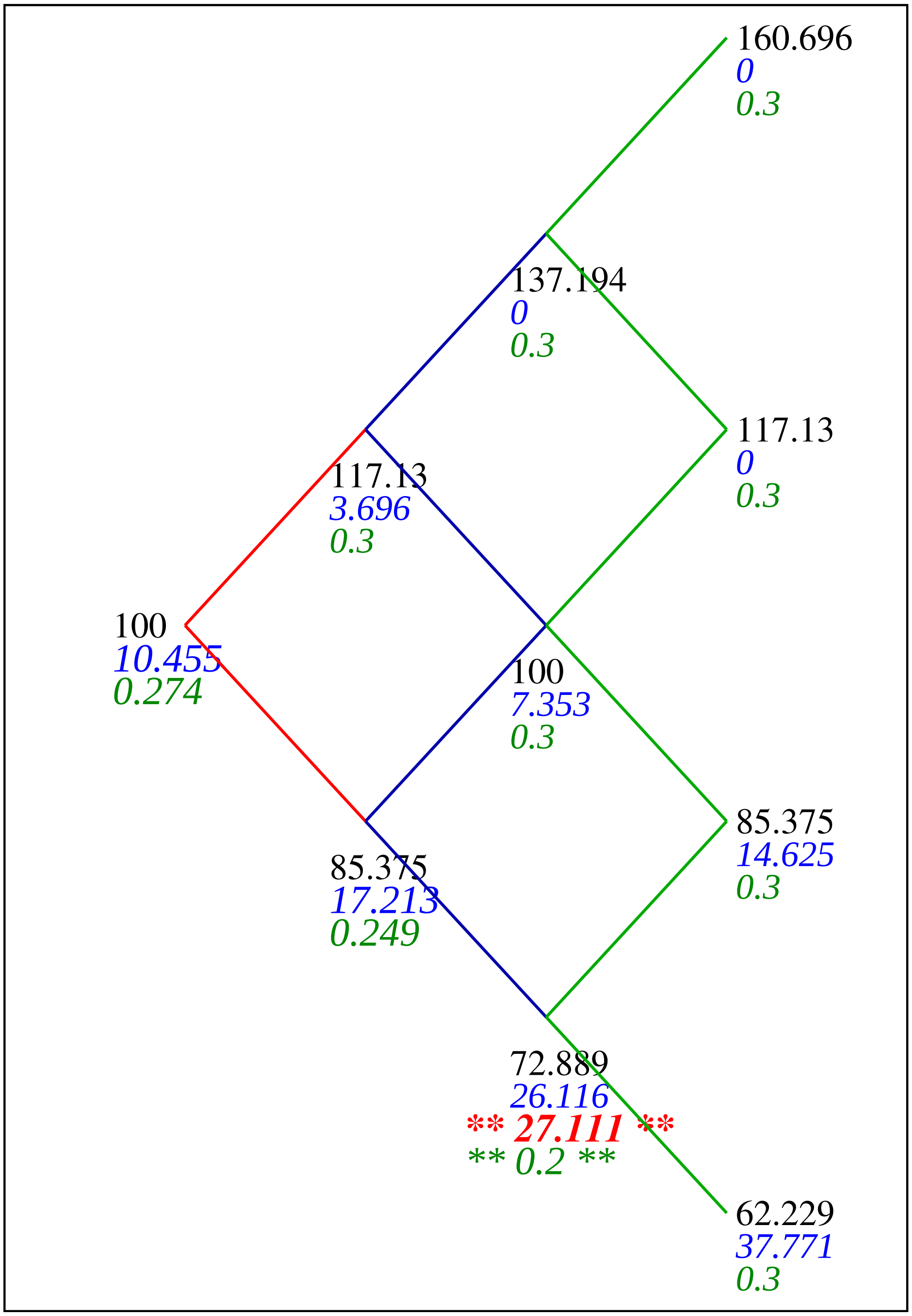

Figure 2: Calculation of fugit of American put option using the binomial tree shown in Fig. 1.