Queens College, CUNY, Department of Computer Science

Computational Finance

CSCI 365 / 765

Fall 2017

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November 15, 2017

8 Lecture 8a

Spreads & combinations: Worked examples

- This lecture contains worked examples of arbitrage strategies to derive some of the rational option pricing inequalities for the option spreads and combinations in Lecture 8.
- Note: Continuous interest rate compounding is used in all the examples.

8.1 Arbitrage 1: European call spread

- Suppose the market price of a stock is S at time t.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- All the options below have the same expiration time T (where $T > t$).
- We are given two European call options on the stock:
  c_1 with strike K_1 and c_2 with strike K_2 (where $K_2 > K_1 > 0$).
- The European bull call spread consists of long c_1 and short c_2, hence $c_1 - c_2$.
- **Show that at expiration, the option prices satisfy the following inequality:**

$$c_1(T) - c_2(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{8.1.1}$$

- **Solution**

1. We calculate the value of $c_1(S_T, T) - c_2(S_T, T)$ as a function of the stock price S_T at expiration.
2. Recall that

$$c_1(S_T, T) = \max(S_T - K_1, 0)\,, \qquad c_2(S_T, T) = \max(S_T - K_2, 0)\,. \tag{8.1.2}$$

3. If $S_T \le K_1$, then both $c_1(S_T, T) = c_2(S_T, T) = 0$.
4. If $K_1 \le S_T \le K_2$, then $c_1(S_T, T) = S_T - K_1$ and $c_2(S_T, T) = 0$.
   Because $S_T \le K_2$, $c_1(S_T, T) - c_2(S_T, T) \le K_2 - K_1$ in this interval.
5. If $S_T \ge K_2$, then $c_1(S_T, T) = S_T - K_1$ and $c_2(S_T, T) = S_T - K_2$.
   Hence $c_1(S_T, T) - c_2(S_T, T) = K_2 - K_1$.
6. Hence for all values of the terminal stock price S_T,

$$c_1(S_T, T) - c_2(S_T, T) \le K_2 - K_1\,. \tag{8.1.3}$$

7. A graph of the terminal payoff of a European bull call spread is displayed in Lecture 8.

- **Formulate an arbitrage strategy to show that before expiration (time $t < T$), the option prices must satisfy the following inequality:**

$$c_1(t) - c_2(t) \le \mathrm{PV}(K_2 - K_1) = e^{-r(T-t)}(K_2 - K_1) \qquad (t < T)\,. \tag{8.1.4}$$

- **Solution**

  1. Suppose that the inequality is violated and there exists $\zeta > 0$ such that

  $$c_1(t) - c_2(t) = \mathrm{PV}(K_2 - K_1) + \zeta\,. \tag{8.1.5}$$

  2. **Sell the European bull call spread** at time t_0.
  3. Put the money in a bank.
  4. The cash amount in the bank is $\mathrm{PV}(K_2 - K_1) + \zeta$.

- Hence our portfolio at time t_0 consists of short one European bull call spread, cash in bank. The total value of our portfolio is zero.

- At expiration, the money in the bank has compounded to $K_2 - K_1 + \zeta e^{r(T-t_0)}$.

- We examine what happens to the European bull call spread at expiration.

  1. If $S_T \ge K_2$, both options are in the money. The spread will be exercised against us. We pay cash $K_2 - K_1$ to the holder of the spread and are left with a profit of $\zeta e^{r(T-t_0)}$.
  2. If $K_1 \le S_T \le K_2$, the option c_1 is in the money and c_2 is not. The spread will be exercised against us. We receive a payment of K_1 from the holder of the spread, so we have cash of $K_2 + \zeta e^{r(T-t_0)}$. Since $S_T \le K_2$, we spend S_T to buy one share of stock and deliver it to the holder of the spread. We are left with a profit of $K_2 - S_T + \zeta e^{r(T-t_0)}$.
  3. If $S_T \le K_1$, both options expire worthless and the spread is not exercised. We are left with a profit of $K_2 - K_1 + \zeta e^{r(T-t_0)}$.

- Therefore this is an arbitrage trade.

- In all cases we start with zero and end with a positive profit.

- Hence the inequality in eq. (8.1.4) must be satisfied, to avoid arbitrage.

8.2 Arbitrage 2: American call spread

- Suppose the market price of a stock is S at time t.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- All the options below have the same expiration time T (where $T > t$).
- We are given two American call options on the stock:
  C_1 with strike K_1 and C_2 with strike K_2 (where $K_2 > K_1 > 0$).
- The American bull call spread consists of long C_1 and short C_2, hence $C_1 - C_2$.
- **Show that at expiration, the option prices satisfy the following inequality:**

$$C_1(T) - C_2(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{8.2.1}$$

- **Solution**
- The terminal payoffs of an American and European call are the same, so this question was answered in Sec. 8.1.
- **Formulate an arbitrage strategy to show that before expiration (time $t < T$), the option prices must satisfy the following inequality:**

$$C_1(t) - C_2(t) \le K_2 - K_1 \qquad (t < T)\,. \tag{8.2.2}$$

- **Solution**
  1. Suppose that the inequality is violated and there exists $\zeta > 0$ such that
  $$C_1(t) - C_2(t) = K_2 - K_1 + \zeta\,. \tag{8.2.3}$$
  2. **Sell the American bull call spread** at time t_0.
  3. Put the money in a bank.
  4. The cash amount in the bank is $K_2 - K_1 + \zeta$.
- Hence our portfolio at time t_0 consists of short one American bull call spread, cash in bank. The total value of our portfolio is zero.
- American options can be exercised at any time, hence we must analyze what happens for arbitrary $t \le T$.
- Suppose the spread is exercised at a time t such that $t_0 \le t \le T$.
  1. The cash in the bank compounds to $(K_2 - K_1 + \zeta)e^{r(t-t_0)}$.
  2. If the spread is exercised, the value of each American option is equal to its intrinsic value.

3. Because the inrinsic value of an American option is the same as at expiration, the value of the spread, when exercised, will be equal to its value at expiration.
4. Hence using eq. (8.2.1), if the spread is exercised at the time t, its value satisfies the inequality

$$\text{value of spread (exercised)} \le K_2 - K_1 \qquad\qquad (t_0 < t < T)\,. \tag{8.2.4}$$

5. Hence we apply the same logic which was used for the European bull call spread. This time our cash in hand is even more $= (K_2 - K_1 + \zeta)e^{r(t-t_0)}$, so we have enough money to cover the spread, with a profit left over.
6. If $S_t \ge K_2$, both options are exercised and our profit is $(K_2 - K_1 + \zeta)e^{r(t-t_0)} - (K_2 - K_1)$.
7. If $K_1 \le S_t < K_2$, only the low strike option is exercised and our profit is $(K_2 - K_1 + \zeta)e^{r(t-t_0)} - (S_t - K_1)$.
8. If $S_t < K_1$, the spread is worthless (and would not be exercised if $t < T$) and our profit is $(K_2 - K_1 + \zeta)e^{r(t-t_0)}$.

- Therefore this is an arbitrage trade.
- In all cases we start with zero and end with a positive profit.
- Hence the inequality in eq. (8.2.2) must be satisfied, to avoid arbitrage.

8.2.1 Comments

- Note also that eq. (8.2.2) is the best possible no-arbitrage inequality, for American calls. If $S_t > K_2$ at any time $t_0 \le t \le T$, then the American bull call spread can be exercised to obtain a payoff of $K_2 - K_1$. Hence the right hand side in eq. (8.2.2) cannot be less than $K_2 - K_1$.
- In fact, if the stock price reaches a level $S_t > K_2$ at any time $t_0 \le t \le T$, then the American bull call spread *should* be exercised, because it will never yield a larger payoff, and the stock price could go down, resulting in a lower payoff.

8.3 Arbitrage 3: European put spread

- Suppose the market price of a stock is S at time t.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- All the options below have the same expiration time T (where $T > t$).
- We are given two European put options on the stock:
  p_1 with strike K_1 and p_2 with strike K_2 (where $K_2 > K_1 > 0$).
- The European bear put spread consists of long p_2 and short p_1, hence $p_2 - p_1$.
- **Show that at expiration, the option prices satisfy the following inequality:**

$$p_2(T) - p_1(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{8.3.1}$$

- **Solution**

1. We calculate the value of $p_2(S_T, T) - p_1(S_T, T)$ as a function of the stock price S_T at expiration.
2. Recall that

$$p_1(S_T, T) = \max(K_1 - S_T, 0)\,, \qquad p_2(S_T, T) = \max(K_2 - S_T, 0)\,. \tag{8.3.2}$$

3. If $S_T > K_2$, then both $p_1(S_T, T) = p_2(S_T, T) = 0$.
4. If $K_1 \le S_T \le K_2$, then $p_1(S_T, T) = 0$ and $p_2(S_T, T) = K_2 - S_T$.
   Because $S_T \ge K_1$, $p_2(S_T, T) - p_1(S_T, T) \le K_2 - K_1$ in this interval.
5. If $S_T \le K_1$, then $p_1(S_T, T) = K_1 - S_T$ and $p_2(S_T, T) = K_2 - S_T$.
   Hence $p_2(S_T, T) - p_1(S_T, T) = K_2 - K_1$.
6. Hence for all values of the terminal stock price S_T,

$$p_2(S_T, T) - p_1(S_T, T) \le K_2 - K_1\,. \tag{8.3.3}$$

7. A graph of the terminal payoff of a European bear put spread is displayed in Lecture 8.

- **Formulate an arbitrage strategy to show that before expiration (time $t < T$), the option prices must satisfy the following inequality:**

$$p_2(t) - p_1(t) \le \mathrm{PV}(K_2 - K_1) = e^{-r(T-t)}(K_2 - K_1) \qquad (t < T)\,. \tag{8.3.4}$$

- **Solution**

  1. Suppose that the inequality is violated and there exists $\zeta > 0$ such that
  $$p_2(t) - p_1(t) = \mathrm{PV}(K_2 - K_1) + \zeta\,. \tag{8.3.5}$$
  2. **Sell the European bear put spread** at time t_0.
  3. Put the money in a bank.
  4. The cash amount in the bank is $\mathrm{PV}(K_2 - K_1) + \zeta$.

- Hence our portfolio at time t_0 consists of short one European bear put spread, cash in bank. The total value of our portfolio is zero.

- At expiration, the money in the bank has compounded to $K_2 - K_1 + \zeta e^{r(T-t_0)}$.

- We examine what happens to the European bear put spread at expiration.

  1. If $S_T \le K_1$, both options are in the money. The spread will be exercised against us. We pay cash $K_2 - K_1$ to the holder of the spread and are left with a profit of $\zeta e^{r(T-t_0)}$.
  2. If $K_1 \le S_T \le K_2$, the option p_2 is in the money and p_1 is not. The spread will be exercised against us. We pay K_2 to the holder of the spread and receive one share of stock, which is worth S_T. We sell the stock immediately to obtain cash S_T. Hence we have cash of $S_T - K_1 + \zeta e^{r(T-t_0)}$. Since $S_T \ge K_1$, we have a positive amount of cash.
  3. If $S_T \ge K_2$, both options expire worthless and the spread is not exercised. We are left with a profit of $K_2 - K_1 + \zeta e^{r(T-t_0)}$.

- Therefore this is an arbitrage trade.

- In all cases we start with zero and end with a positive profit.

- Hence the inequality in eq. (8.3.4) must be satisfied, to avoid arbitrage.

8.4 Arbitrage 4: American put spread

- Suppose the market price of a stock is S at time t.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- All the options below have the same expiration time T (where $T > t$).
- We are given two American put options on the stock:
  P_1 with strike K_1 and P_2 with strike K_2 (where $K_2 > K_1 > 0$).
- The American bear put spread consists of long P_2 and short P_1, hence $P_2 - P_1$.
- **Show that at expiration, the option prices satisfy the following inequality:**

$$P_2(T) - P_1(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{8.4.1}$$

- **Solution**
- The terminal payoffs of an American and European put are the same, so this question was answered in Sec. 8.1.
- **Formulate an arbitrage strategy to show that before expiration (time $t < T$), the option prices must satisfy the following inequality:**

$$P_2(t) - P_1(t) \le K_2 - K_1 \qquad (t < T)\,. \tag{8.4.2}$$

- **Solution**
  1. Suppose that the inequality is violated and there exists $\zeta > 0$ such that
  $$P_2(t) - P_1(t) = K_2 - K_1 + \zeta\,. \tag{8.4.3}$$
  2. **Sell the American bear put spread** at time t_0.
  3. Put the money in a bank.
  4. The cash amount in the bank is $K_2 - K_1 + \zeta$.
- Hence our portfolio at time t_0 consists of short one American bear put spread, cash in bank. The total value of our portfolio is zero.
- American options can be exercised at any time, hence we must analyze what happens for arbitrary $t \le T$.
- Suppose the spread is exercised at a time t such that $t_0 \le t \le T$.
  1. The cash in the bank compounds to $(K_2 - K_1 + \zeta)e^{r(t-t_0)}$.
  2. If the spread is exercised, the value of each American option is equal to its intrinsic value.

3. Because the inrinsic value of an American option is the same as at expiration, the value of the spread, when exercised, will be equal to its value at expiration.
4. Hence using eq. (8.4.1), if the spread is exercised at the time t, its value satisfies the inequality

$$\text{value of spread (exercised)} \leq K_2 - K_1 \qquad\qquad (t_0 < t < T)\,. \tag{8.4.4}$$

5. Hence we apply the same logic which was used for the European bear put spread. This time our cash in hand is even more $= (K_2 - K_1 + \zeta)e^{r(t-t_0)}$, so we have enough money to cover the spread, with a profit left over.
6. If $S_t \leq K_1$, both options are exercised and our profit is $(K_2 - K_1 + \zeta)e^{r(t-t_0)} - (K_2 - K_1)$.
7. If $K_1 \leq S_t < K_2$, only the high strike option is exercised and our profit is $(K_2 - K_1 + \zeta)e^{r(t-t_0)} - (S_t - K_1)$.
8. If $S_t > K_2$, the spread is worthless (and would not be exercised if $t < T$) and our profit is $(K_2 - K_1 + \zeta)e^{r(t-t_0)}$.

- Therefore this is an arbitrage trade.
- In all cases we start with zero and end with a positive profit.
- Hence the inequality in eq. (8.4.2) must be satisfied, to avoid arbitrage.

8.4.1 Comments

- Note also that eq. (8.4.2) is the best possible no-arbitrage inequality, for American puts. If $S_t < K_1$ at any time $t_0 \leq t \leq T$, then the American bear put spread can be exercised to obtain a payoff of $K_2 - K_1$. Hence the right hand side in eq. (8.4.2) cannot be less than $K_2 - K_1$.
- In fact, if the stock price reaches a level $S_t < K_1$ at any time $t_0 \leq t \leq T$, then the American bull call spread *should* be exercised, because it will never yield a larger payoff, and the stock price could go up, resulting in a lower payoff.

8.5 Arbitrage 5: dividends

- Suppose the stock pays n dividends D_i at times t_i, $i = 1, 2, \dots, n$ during the lifetime of the options.
- **Explain how the inequalities in eqs. (8.1.4), (8.2.2), (8.3.4) and (8.4.2) would be modified.**
- **Solution**
- The inequalities in eqs. (8.1.4), (8.2.2), (8.3.4) and (8.4.2) **would not be modified.**
- None of the inequalities in eqs. (8.1.4), (8.2.2), (8.3.4) and (8.4.2) make any reference to the stock price.
- For the call spreads, we sometimes have to deliver one share of the stock. But we buy the stock at the time t and deliver it immediately to the holder of the spread. We neither receive nor pay any dividends.
- For the put spreads, the holder of the spread sometimes delivers one share of the stock to us. But the holder of the spread delivers only the stock, not any dividends. We neither receive nor pay any dividends.