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24 Lecture 24

Asymptotic Series

- This lecture contains some notes about **asymptotic series.**
- Specifically, *what is an asymptotic series?*
- Asymptotic series are esoteric objects, even for mathematics majors.
- This material in this lecture is an **advanced topic not for examination.**

24.1 Power series

- We begin with a quick review of power series.
- A **power series** is an infinite sum of powers of a complex variable z

$$p(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots = \sum_{n=0}^{\infty} p_n z^n \,. \qquad (24.1.1)$$

- Here the coefficients p_n are constants (i.e. not functions of z) and are complex in general.
- In general, the sum in eq. (24.1.1) converges for all z in a disk centered on the origin $z = 0$.
- The radius of the disk, say ρ, is called the **radius of convergence** and the circle $|z| = \rho$ is called the **circle of convergence.**
- Exactly on the circle of convergence, the behavior of a power series is complicated. The sum in eq. (24.1.1) may converge for all points, no points, or some but not all points on the circle of convergence. Outside the circle of convergence, a power series *diverges for all* $|z| > \rho$*.*
- *Inside* the circle of convergence $|z| < \rho$, a power series **converges absolutely.**
- Absolute convergence means the sum of the *amplitudes* of all the terms converges, i.e. even without any $\pm$ cancellations between the terms

$$|p_0| + |p_1 z| + |p_2 z^2| + \cdots + |p_n z^n| + \cdots = \sum_{n=0}^{\infty} |p_n z^n| < \infty \qquad (|z| < \rho)\,. \qquad (24.1.2)$$

- The following power series has a radius of convergence $\rho = 1$.

$$p_{\text{ex}\,1}(z) = 1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n \,. \qquad (24.1.3)$$

- For all $|z| < 1$, the series in eq. (24.1.3) describes the function $1/(1-z)$.
- The following power series describes an entire function (infinite radius of convergence).

$$p_{\text{ex}\,2}(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \,. \qquad (24.1.4)$$

- The series in eq. (24.1.4) describes the exponential function e^z, for all complex numbers z.
- Define the **partial sums** $P_m(z)$ via the sum up to z^m for $m = 0, 1, \ldots$:

$$P_m(z) = \sum_{n=0}^{m} p_n z^n \,. \qquad (24.1.5)$$

- **Inside the circle of convergence, the magnitudes of the partial sums in eq.** (24.1.5) **may initially get large, but as we sum more and more terms of the series, the values of the partial sums will eventually yield a better and better approximation for the value of the power series.**

24.2 Asymptotic series

- An **asymptotic series** is also an infinite sum of powers of a complex variable z

$$a(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = \sum_{n=0}^{\infty} a_n z^n \,. \tag{24.2.1}$$

- Here the coefficients a_n are constants (i.e. not functions of z) and are complex in general.
- However, unlike a power series, an asymptotic series **diverges for all values $z \neq 0$.**
- *What's the use of that?*
- Asymptotic series can be useful for practical computations.
- Suppose we have a function $A(z)$ and we expand it in a Taylor series in powers of z and the result is an asymptotic series as in eq. (24.2.1).
- *It is possible to employ the series in eq.* (24.2.1) *to compute an approximate value for* $A(z)$.
- Once again we form the partial sums via the sum up to z^m for $m = 0, 1, \ldots$:

$$A_m(z) = \sum_{n=0}^{m} a_n z^n \,. \tag{24.2.2}$$

- As we sum more and more terms of an asymptotic series, the values of the partial sums ***initially yield better and better approximations for the value of the function $A(z)$.***
- *However, if we keep adding more and more terms in the partial sums, the accuracy of the approximation will stop getting better and the value of the partial sums will diverge to ∞.*
- **Hence we must judge how many term to include in the partial sums, and when to stop including more terms.**
- Usually this means we include terms in the partial sums as long as the magnitudes of the terms keep getting smaller. But when the magnitudes of the terms start getting larger, then we stop. Hence we determine the value of m_*, which is the smallest value of m such that

$$|a_{m_*} z^{m_*}| < |a_{m_*+1} z^{m_*+1}| \,. \tag{24.2.3}$$

- Obviously the value of m_* depends on the value of z. It is not a fixed constant.
- The best approximation for the value of $A(z)$ is given by the partial sum $A_{m_*}(z)$.
- Note that that I wrote "best approximation" above for the value of $A(z)$.
- **It may not be a *good approximation* for the value of $A(z)$.**
- The value of the the partial sum $A_{m_*}(z)$ is simply the best approximation to $A(z)$ that the asymptotic series in eq. (24.2.1) can deliver, for the input value of z.

24.3 Exponential integral

24.3.1 Asymptotic series

- Do not confuse the **exponential integral** with the exponential function.
- There are various definitions of the exponential integral, but one definition is

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt \qquad\qquad (|\arg(z)| < \pi)\,. \tag{24.3.1.1}$$

- Let us process this a bit.
  1. First let us treat only real values and write x instead of z.
  2. Let us also consider only $x > 0$.
  3. Then the integral in eq. (24.3.1.1) is along the positive real axis and does not cross zero.
  4. Next, change variables to $u = t - x$, so $0 \le u < \infty$. Then we obtain

$$E_1(x) = e^{-x} \int_0^\infty \frac{e^{-u}}{x+u}\,du\,. \tag{24.3.1.2}$$

- Let us drop the prefactor of e^{-x} and define the following function $E_2(x) = e^x E_1(x)$:

$$E_2(x) = \int_0^\infty \frac{e^{-u}}{x+u}\,du\,. \tag{24.3.1.3}$$

- Then $E_2(x)$ (also $E_1(x)$) has a logarithmic singularity at $x = 0$.
- Hence $E_2(x)$ cannot be expanded in a Maclaurin series in powers of x.
- However $E_2(\infty) = 0$ so we could consider a Taylor series in powers of $1/x$.
- If a series in powers of $1/x$ makes you uncomfortable, let $v = x^{-1}$ and expand in powers of v.

$$E_2(x) = \frac{1}{x}\int_0^\infty \frac{e^{-u}}{1+u/x}\,du = v\int_0^\infty \frac{e^{-u}}{1+uv}\,du = v\int_0^\infty e^{-u}\Big[1 - uv + u^2v^2 + \cdots\Big]\,du\,. \tag{24.3.1.4}$$

- Next note that for $n > 0$,

$$I_n = \int_0^\infty u^n e^{-u}\,du = \Big[-u^n e^{-u}\Big]_0^\infty + n\int_0^\infty u^{n-1}e^{-u}\,du = nI_{n-1}\,. \tag{24.3.1.5}$$

- Also $I_0 = 1$ for $n = 0$, hence $I_n = n!$.
- Hence the Taylor series in powers of v or $1/x$ is

$$E_2(x) = v - 1!v^2 + 2!v^3 - 3!v^4 + \cdots = \sum_{n=0}^\infty (-1)^n\, n!v^{n+1}\,. \tag{24.3.1.6}$$

- However the radius of convergence of the series in eq. (24.3.1.6) is zero.
- For all finite $0 < v < \infty$ (equivalently $0 < x < \infty$) the ratio of successive terms is

$$\lim_{n\to\infty} \frac{|a_n v^n|}{|a_{n-1} v^{n-1}|} = \lim_{n\to\infty} \frac{n! v^{n+1}}{(n-1)! v^n} = \lim_{n\to\infty} nv = \infty\,. \tag{24.3.1.7}$$

- The Taylor series in powers of v diverges for all finite $0 < v < \infty$.
- Hence eq. (24.3.1.6) yields an asymptotic series

$$E_2(x) = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}}\,. \tag{24.3.1.8}$$

- Note the following:
  1. The terms in the series in eq. (24.3.1.8) alternate in sign and decrease in magnitude as long as $n/x < 1$, i.e. for $n \le \lfloor x \rfloor$.
  2. The notation $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.
  3. After that, for $n > \lfloor x \rfloor$, the terms increase in magnitude and diverge as $n \to \infty$.
  4. Hence the partial sums of the asymptotic series in eq. (24.3.1.8) yield improved approximations up to $m_* = \lfloor x \rfloor$.
- It can be shown that if we compute $E_2(x)$ using the partial sum $A_m(x)$, the remainder term $R_m(x)$ is bounded by the magnitude of the next term in the series, i.e.

$$|R_m(x)| \le \frac{m!}{|x|^{m+1}}\,. \tag{24.3.1.9}$$

- For $x = 100$, the value of $R_m(x)$ is bounded by

$$|R_1(100)| \le \frac{1}{100^2}\,, \quad |R_2(100)| \le \frac{2}{100^3}\,, \quad |R_3(100)| \le \frac{6}{100^4}\,, \quad \dots \tag{24.3.1.10}$$

- Hence we can obtain very accurate estimates for $E_2(100)$ by summing only a few terms.
- However, for $x = 1$, then $m_* = 1$. The value of $R_m(x)$ is bounded by

$$|R_1(1)| \le 1\,, \quad |R_2(1)| \le 2!\,, \quad |R_3(1)| \le 3!\,, \quad |R_4(1)| \le 4!\,, \quad \dots \tag{24.3.1.11}$$

- The "best approximation" for the value of $E_2(1)$ is given by summing only one term, i.e. the partial sum $A_1(1) = 0!/1 = 1$, and the estimate for the remainder is $O(1)$. We cannot do better, because the terms for $n > 1$ (i.e. $n > x$) increase in magnitude and the asymptotic series in eq. (24.3.1.8) is then not useful.

24.3.2 Power series

- There is in fact a power series expansion for $E_1(z)$ (with a logarithm):

$$E_1(z) = -\gamma - \ln(z) - \sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n\,n!} \qquad (|\arg(z)| < \pi)\,. \tag{24.3.2.1}$$

- Here γ is the **Euler–Mascheroni constant:**

$$\gamma = \lim_{n\to\infty}\biggl(-\ln(n) + \sum_{k=1}^{n}\frac{1}{k}\biggr) \simeq 0.5772156649\,. \tag{24.3.2.2}$$

- The series in eq. (24.3.2.1) converges for all complex values of z, but we must also evaluate the complex logarithm $\ln(z)$.
- For real $x > 0$, the value of $E_2(x)$ can therefore be obtained via $E_2(x) = e^x E_1(x)$, i.e.

$$E_2(x) = e^x\biggl(-\gamma - \ln(x) - \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n\,n!}\biggr)\,. \tag{24.3.2.3}$$

24.3.3 Sample numerical evaluations

- **$x = 100$.** Consider how we would evaluate $E_2(x)$ for $x = 100$, to an accuracy of 10^{-6}.

  1. Even though the power series in eq. (24.3.2.3) converges for all x, how many terms in that series would we need to compute, bearing in mind the prefactor of $e^x = e^{100}$ in eq. (24.3.2.3)?
  2. On the other hand, the evaluation of $E_2(x)$ for $x = 100$, to an accuracy of 10^{-6}, can be accomplished using only three terms of the asymptotic the series in eq. (24.3.1.8):

  $$E_2(100) \simeq \frac{0!}{100} - \frac{1!}{100^2} + \frac{2!}{100^3} = 10^{-2} - 10^{-4} + 2 \times 10^{-6} = 0.009902\,. \qquad (24.3.3.1)$$

  3. The accuracy of the above value is bounded by

  $$|R_3| \le \frac{3!}{100^4} = 6 \times 10^{-8}\,. \qquad (24.3.3.2)$$

- **$x = 1$.** Next consider the evaluation of $E_2(x)$ for $x = 1$, to an accuracy of 10^{-6}.

  1. Using the asymptotic series in eq. (24.3.1.8), as explained above, for $x = 1$ the best approximation is given using only one term:

  $$E_2(1) \simeq \frac{0!}{1} = 1\,. \qquad (24.3.3.3)$$

  2. The accuracy of the above value is bounded by

  $$|R_1| \le \frac{1!}{1^2} = 1\,. \qquad (24.3.3.4)$$

  3. This is no good.
  4. On the other hand, using eq. (24.3.2.3), we sum the series to nine terms and obtain

  $$\begin{aligned} E_2(1) &\simeq e\left(-\gamma - \ln(1) - \sum_{n=1}^{9} (-1)^n \,\frac{1}{n\,n!}\right) \\ &\simeq e\left(-0.5772156649 - \sum_{n=1}^{9} (-1)^n \,\frac{1}{n\,n!}\right) \\ &\simeq 0.596347\,. \end{aligned} \qquad (24.3.3.5)$$

  5. The remainder R using eq. (24.3.2.3) to nine terms is bounded by $e^x (= e)$ times the next term in the series, and is

  $$|R| \le e \times \frac{1}{10 \times 10!} \simeq 7.5 \times 10^{-8}\,. \qquad (24.3.3.6)$$

24.4 Particle accelerators

- Asymptotic series are not completely a joke or a frivolous exercise.
- I have encountered them in my own career, which in this case means particle accelerators.
- Subatomic particles circulate in an accelerator, in a closed loop. There is a reference orbit, which we can say is a circle, and the orbits of the actual particles oscillate around that circle, as the particles travel around the circumference of the accelerator.
- I had to calculate the strengths of so-called **spin resonances** and never mind what *they* are.
- The calculation required an average over the statistical distribution of the particle orbits.
- An oscillation has both an amplitude and a phase, i.e. two independent parameters.
- The phase is uniformly distributed in the interval $[0, 2\pi]$ and is not important.
- We can use a variable 'ξ' to denote the square of the orbit oscillation amplitude.
- I ended up with the need to compute expressions of the form

$$\varepsilon = \Big\langle 1 - \xi + \xi^2 - \xi^3 + \cdots \Big\rangle \,. \tag{24.4.1}$$

- The average $\langle \cdots \rangle$ was over the distribution of the particle orbits.
- The statistical average was calculated using an exponential distribution:

$$\Big\langle f(\xi) \Big\rangle = \int_0^\infty f(\xi)\, \frac{e^{-\xi/\sigma^2}}{\sigma^2}\, d\xi \,. \tag{24.4.2}$$

- Here σ was a parameter, the average of the value of ξ.
- You can easily work out that

$$\langle 1 \rangle = 1 \,, \quad \langle \xi \rangle = \sigma^2 \,, \quad \langle \xi^2 \rangle = 2!\sigma^4 \,, \quad \dots \quad \langle \xi^n \rangle = n!\sigma^{2n} \,. \tag{24.4.3}$$

- Hence the average I obtained was the following series

$$\varepsilon = 1 - \sigma^2 + 2!\sigma^4 - 3!\sigma^6 + \cdots + (-1)^n n!\sigma^{2n} + \dots \,. \tag{24.4.4}$$

- The sum of the series, i.e. the output from my program, did not converge very well.
- It took me a while to realize my program was producing the terms of an asymptotic series.
- You should recognize the value of ε is really $E_2(1/\sigma^2)/\sigma^2$.
- The true value of ε is finite, but my program had difficulty computing it.
- The basic algorithm had to be formulated in a different way, which was not easy to do.