Queens College, CUNY, Department of Computer Science

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7 Lecture 7

7.1 Rational option pricing

- We have come as far as the Black-Scholes-Merton equation, whose solution gives the fair value for a financial derivative (contingent claim) on a stock paying a continuous dividend yield.
- The obvious inclination now is of course to solve the Black-Scholes-Merton equation, to obtain mathematical formulas for the fair value of put and call options. It is only natural that we would wish to do so.
- However, let us not rush headlong into mathematical sophistication. The Black-Scholes-Merton equation assumes the stock price movement obeys geometric Brownian motion.
- Options have many important properties which are ***independent of any model*** for the stock price movements.
- Robert Merton derived many of these properties in his classic 1973 paper ("Theory of Rational Option Pricing").
- All of the results are proved using no-arbitrage arguments, hence they are fundamental results. Many people speak of **rational option pricing**.
- As a matter of notation, which is widely used by many experts, we shall denote the fair values of European options by lower case letters and for American options by upper case letters.
  1. European call option = $c(S, t)$.
  2. American call option = $C(S, t)$.
  3. European put option = $p(S, t)$.
  4. American put option = $P(S, t)$.
- We assume all the options are traded on an exchange, hence there is no counterparty risk. (This means the options are priced using the risk–free rate.)
- We also assume the company which issues the stock does not go bankrupt during the life of the option, hence the stock does not get delisted during the lifetime of the option.
- All of the assumptions of efficient, frictionless and infinitely liquid markets apply, also the assumption of unrestricted short selling.

7.2 Rational option pricing: fundamental results

- Some of the results below are obvious, others are more profound. All the results are independent of any model for the stock price movements.
- The stock price is S and the current time is t. The option expiration time is T (where $T \geq t$) and the strike of the option is K. Where two or more options are compared, we shall employ notations such as K_1, K_2, etc. and/or T_1, T_2, etc.
- Although interest rates will not be mentioned explicitly below, it is assumed that the (risk–free) interest rate is positive, or zero at a minimum, i.e. no negative interest rates.
- The value of an option when exercised is called the **intrinsic value** of the option.

1. Option fair values are non-negative, for all $S > 0$ and $t \leq T$. This is obvious.
$$c(S,t) \geq 0\,, \qquad C(S,t) \geq 0\,, \qquad p(S,t) \geq 0\,, \qquad P(S,t) \geq 0\,. \tag{7.2.1}$$
2. Technically, this is part of the option definition, not a deduction. The option value at expiration equals its intrinsic value.
$$c(S,T) = C(S,T) = \max\{S-K,0\}\,, \qquad p(S,T) = P(S,T) = \max\{K-S,0\}\,. \tag{7.2.2}$$
3. For all $t < T$, the fair value of an **American option** is never less than its intrinsic value.
$$C(S,t) \geq \max\{S-K,0\}\,, \qquad P(S,t) \geq \max\{K-S,0\}\,. \tag{7.2.3}$$
This is because an American option can be exercised at any time, and will be worth its intrinsic value. These inequalities may not hold for European options.
4. The fair value of a call option never exceeds the stock price.
$$c(S,t) \leq S\,, \qquad C(S,t) \leq S\,. \tag{7.2.4}$$
5. The fair value of a put option never exceeds the strike price.
$$p(S,t) \leq K\,, \qquad P(S,t) \leq K\,. \tag{7.2.5}$$
Merton proved the stronger result that the fair value of a European put option never exceeds the present value of the strike price.
$$p(S,t) \leq \mathrm{PV}(K)\,. \tag{7.2.6}$$
If the interest rate is negative the above inequalities may not be true.
6. The fair value of an American option is never less than that of the corresponding European option.
$$C(S,t) \geq c(S,t)\,, \qquad P(S,t) \geq p(S,t)\,. \tag{7.2.7}$$
This is also obvious, because an American option can do everything a European option can do, and more.

7. If the **stock pays no dividends** during the lifetime of the option, the **fair values of an American and European call option are equal.**

$$C(S,t) = c(S,t) \qquad \text{(stock dividends} = 0) \,. \tag{7.2.8}$$

This is important: if the stock pays no dividends, it is not optimal to exercise an American call option early.
This is not always true for an American put option.

8. If the risk–free interest rate is zero during the lifetime of the option, the fair values of an American and European put option are equal.

$$P(S,t) = p(S,t) \qquad \text{(risk–free rate} = 0) \,. \tag{7.2.9}$$

This is of course an unusual situation in practice.

9. The fair value of a call option is an non-decreasing function of the stock price. If we compare the fair values of two call options at stock prices S_1 and S_2 at the same time t, where $S_2 > S_1$, then

$$c(S_2,t) \ge c(S_1,t) \,, \qquad C(S_2,t) \ge C(S_1,t) \qquad (S_2 > S_1) \,. \tag{7.2.10}$$

If the stock price is higher, a call option is more in the money, hence its fair value is higher.

10. In addition, the difference in the call option prices is always less than $S_2 - S_1$.
Hence we have the inequalities

$$0 \le \quad c(S_2,t) - c(S_1,t) \quad \le \quad S_2 - S_1 \,, \tag{7.2.11a}$$
$$0 \le \quad C(S_2,t) - C(S_1,t) \quad \le \quad S_2 - S_1 \qquad (S_2 > S_1) \,. \tag{7.2.11b}$$

11. The fair value of a put option is a non-increasing function of the stock price. If we compare the fair values of two put options at stock prices S_1 and S_2 at the same time t, where $S_2 > S_1$, then

$$p(S_1,t) \ge p(S_2,t) \,, \qquad P(S_1,t) \ge P(S_2,t) \qquad (S_2 > S_1) \,. \tag{7.2.12}$$

If the stock price is higher, a put option is more out of the money, hence its fair value is lower.

12. In addition, the difference in the put option prices is always less than $S_2 - S_1$.
Hence we have the inequalities

$$0 \le \quad p(S_1,t) - p(S_2,t) \quad \le \quad S_2 - S_1 \,, \tag{7.2.13a}$$
$$0 \le \quad P(S_1,t) - P(S_2,t) \quad \le \quad S_2 - S_1 \qquad (S_2 > S_1) \,. \tag{7.2.13b}$$

13. The **Delta** of a call option always lies between 0 and 1. The Delta of an option, denoted by the Greek letter Δ, is defined as the partial derivative with respect to the stock price S (where the time t is held fixed)

$$\Delta_c = \frac{\partial c}{\partial S}\,, \qquad \Delta_C = \frac{\partial C}{\partial S}\,. \tag{7.2.14}$$

Then from the inequalities in eq. (7.2.11),

$$0 \le \Delta_c \le 1\,, \qquad 0 \le \Delta_C \le 1\,. \tag{7.2.15}$$

These expressions assume the call option fair value is differentiable with respect to S. The inequalities in eq. (7.2.11) do not make such an assumption.
Just because $C \ge c$, do not jump to conclusions that $\Delta_C \ge \Delta_c$.

14. The Delta of a put option is *negative* and always lies between 0 and -1. Obviously

$$\Delta_p = \frac{\partial p}{\partial S}\,, \qquad \Delta_P = \frac{\partial P}{\partial S}\,. \tag{7.2.16}$$

Then from the inequalities in eq. (7.2.13),

$$-1 \le \Delta_p \le 0\,, \qquad -1 \le \Delta_P \le 0\,. \tag{7.2.17}$$

These expressions assume the put option fair value is differentiable with respect to S. The inequalities in eq. (7.2.13) do not make such an assumption.
Just because $P \ge p$, do not jump to conclusions that $|\Delta_P| \ge |\Delta_p|$.

15. It is the other way around if we compare two options with different *strike prices* K_1 and K_2, where $K_2 > K_1$. Then we have the following inequalities (omitting the arguments S and t). Note that a present value appears for the European options.

$$0 \le \quad c(K_1) - c(K_2) \quad \le \quad \mathrm{PV}(K_2 - K_1)\,, \tag{7.2.18a}$$
$$0 \le \quad C(K_1) - C(K_2) \quad \le \quad K_2 - K_1\,, \tag{7.2.18b}$$
$$0 \le \quad p(K_2) - p(K_1) \quad \le \quad \mathrm{PV}(K_2 - K_1)\,, \tag{7.2.18c}$$
$$0 \le \quad P(K_2) - p(K_1) \quad \le \quad K_2 - K_1 \qquad (K_2 > K_1)\,. \tag{7.2.18d}$$

If the strike price is higher, a call option is less in the money, hence its fair value is lower. Conversely, a put option is more in the money, hence its fair value is higher.

16. The fair value of an American option is an non-decreasing function of the time to expiration. If we compare the fair values of two American options with expirations T_1 and T_2, where $T_2 > T_1$, then (omitting the arguments S and t)

$$C(T_2) \ge C(T_1)\,, \qquad P(T_2) \ge P(T_1) \qquad (T_2 > T_1)\,. \tag{7.2.19}$$

The longer dated option can do everything the shorter dated option can do, and more. Hence it is worth more.

7.3 Put–call parity

- **Put–call parity** is a famous and important relation connecting the fair values of European put and call options.
- The relation is independent of any model of the stock price movements.
- In modern financial theory, it was derived by Hans Stoll, but he thought it applied to American options also. Robert Merton corrected this detail and pointed out that put–call parity only holds for European options.
- There are many ways of writing the put–call parity formula. If the stock pays no dividends during the lifetime of the option, then

$$c(S,t) - p(S,t) = S - \mathrm{PV}(K)\,. \tag{7.3.1}$$

If the stock pays discrete dividends D_1, D_2, etc. during the lifetime of the option (from the current time t to the expiration T), then

$$c(S,t) - p(S,t) = S - \bigl[\,\mathrm{PV}(D_1) + \mathrm{PV}(D_2) + \cdots\bigr] - \mathrm{PV}(K)\,. \tag{7.3.2}$$

In the theoretical model where the stock pays continuous dividends at a rate q, then the put–call parity relation is

$$c(S,t) - p(S,t) = Se^{-q(T-t)} - \mathrm{PV}(K)\,. \tag{7.3.3}$$

If the risk–free interest rate is a constant r, then

$$c(S,t) - p(S,t) = Se^{-q(T-t)} - Ke^{-r(T-t)}\,. \tag{7.3.4}$$

- Put–call parity is derived using a no-arbitrage argument. However, to formulate the argument, it is essential that the options must only be exercised at expiration. For this reason, put–call parity does not apply for American options, in general.
- Put–call parity shows that the fair value of a put and a call are not independent. Given one, we can find the other. This fact is independent of any model for the stock price movements.
- If the options are **at the money forward** (recall the definition from a previous lecture), then the right hand side is zero, hence the European put and call have equal fair values

$$c(S,t) - p(S,t) = 0 \qquad\qquad \text{(at the money forward)}\,. \tag{7.3.5}$$

If the stock price is higher than the at the money forward value, then $c > p$ and if the stock price is lower then $c < p$.

- There is one special case where put–call parity *does* apply for American options. From the previous rational option pricing results, if the stock pays no dividends during the lifetime of the options then $C = c$ for call options. Also if the risk–free interest rate is zero, then $P = p$ for put options. In that case, $C - P = c - p$ and put–call parity also holds for American options. However, we effectively changed the rules to obtain what we want: we imposed restrictions so that $C = c$ and $P = p$. It is a very special case, not really a general result.

7.4 Put–call inequalities for American options

- The best put–call relations we can achieve for American options are inequalities.
- We treat the case where the stock pays discrete dividends. Then if the risk–free interest rate is non-negative, so $\mathrm{PV}(K) \le K$, we have the inequalities

$$C(S,t) - P(S,t) \ge S - \bigl[\,\mathrm{PV}(D_1) + \mathrm{PV}(D_2) + \cdots\bigr] - K\,, \tag{7.4.1a}$$
$$C(S,t) - P(S,t) \le S - \mathrm{PV}(K)\,. \tag{7.4.1b}$$

- In the theoretical model where the stock pays continuous dividends at a rate q,

$$C(S,t) - P(S,t) \ge Se^{-q(T-t)} - K\,, \tag{7.4.2a}$$
$$C(S,t) - P(S,t) \le S - \mathrm{PV}(K)\,. \tag{7.4.2b}$$

- The above inequalities can be established using no-arbitrage arguments.
- Obviously if there are no dividends and the interest rate is zero (so $\mathrm{PV}(K) = K$), then

$$C(S,t) - P(S,t) \ge S - K\,, \tag{7.4.3a}$$
$$C(S,t) - P(S,t) \le S - K\,. \tag{7.4.3b}$$

  Hence

$$C(S,t) - P(S,t) = S - K\,. \tag{7.4.4}$$

  This is put–call parity for American options. However, from the previous rational option pricing results, if the stock pays no dividends during the lifetime of the options then $C = c$ for call options. Also if the risk–free interest rate is zero, then $P = p$ for put options. Hence in this scenario $C = c$ and $P = p$ so these are really European options in disguise.

7.5 Proof of put–call parity

There are many (equivalent) ways to prove put–call parity. All the proofs employ no-arbitrage arguments. The basic principle in all cases is that, if two portfolios are worth the same for all scenarios at the expiration time T, then they will have equal fair value today.

- In this case, "all scenarios" means all values of the terminal stock price S_T.
- To avoid unnecessary complications, we shall assume the stock pays no dividends during the lifetime of the option. Hence we shall derive the put–call parity formula eq. (7.3.1).
- The proof of put–call parity when the stock pays dividends also employs a no-arbitrage argument, but is simply more tedious to write down.

7.5.1 long put, short call, long stock

Form a portfolio consisting of (i) long one European put, (ii) short one European call, and (iii) long one share of stock. The value of $p(S_T,T) - c(S_T,T)$ is plotted in Fig. 1. We see that

$$p(S_T,T) - c(S_T,T) = \max(K - S_T, 0) - \max(S_T - K, 0) = K - S_T \,. \qquad (7.5.1)$$

Add one share of stock and we obtain a constant ($= K$)

$$p(S_T,T) - c(S_T,T) + S_T = K \,. \qquad (7.5.2)$$

The portfolios on both sides have equal values for all values of S_T at the expiration time T. Hence they must have equal value today. The present value of the right-hand side is $\mathrm{PV}(K)$. The fair value of the put today is $p(S,t)$ and $c(S,t)$ for the call. The value of the stock today is S. Hence we must have

$$p(S,t) - c(S,t) + S = \mathrm{PV}(K) \,. \qquad (7.5.3)$$

Rearrange terms to obtain the put–call parity formula eq. (7.3.1).

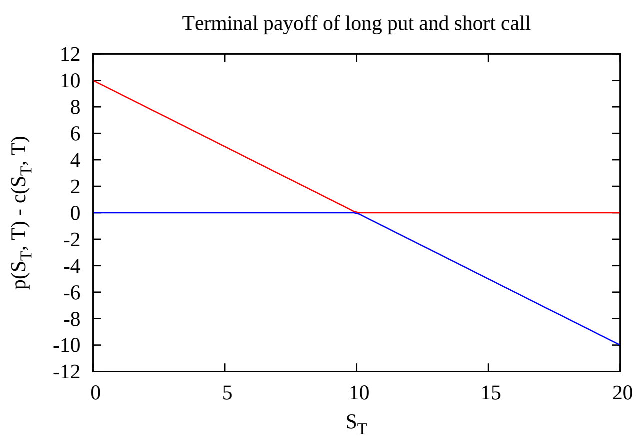

Figure 1: Graph of the terminal payoff of long one European put (red) and short one European call (blue), plotted against the terminal stock price S_T. The strike is $K = 10$.

7.5.2 long put and stock

Form a portfolio consisting of long one European put and one share of stock. The value of $p(S_T, T) + S_T$ is plotted in Fig. 2. We see that

$$p(S_T, T) + S_T = \begin{cases} K & (S_T < K)\,, \\ S_T & (S_T \geq K)\,. \end{cases} \tag{7.5.4}$$

The value of the right-hand side is simply $\max(S_T, K)$. Now note that

$$c(S_T, T) + K = \max(S_T - K, 0) + K = \max(S_T - K + K,\, 0 + K) = \max(S_T, K)\,. \tag{7.5.5}$$

Hence at expiration,

$$p(S_T, T) + S_T = c(S_T, T) + K\,. \tag{7.5.6}$$

The portfolios on both sides have equal values for all values of S_T at the expiration time T. Hence they must have equal value today. The value of the left-hand side today is $p(S, t) + S$. The value of the right-hand side today is $c(S, t) + \mathrm{PV}(K)$. Equate them to obtain

$$p(S, t) + S = c(S, t) + \mathrm{PV}(K)\,. \tag{7.5.7}$$

Rearrange terms to obtain the put–call parity formula eq. (7.3.1).

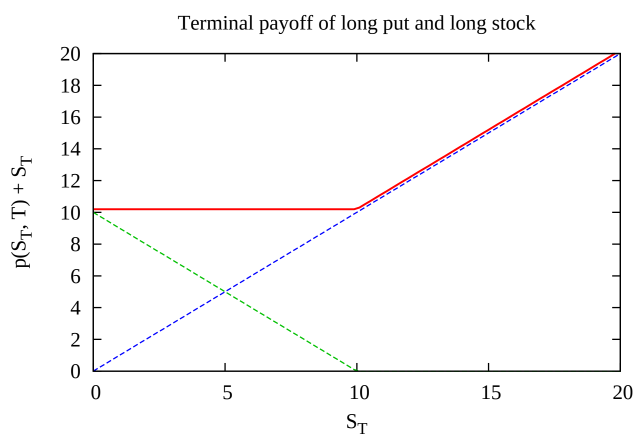

Figure 2: Graph of the terminal payoff of long one European put (green, dashed), long stock (blue, dashed) and their sum (red, solid), plotted against the terminal stock price S_T. The strike is $K = 10$. The red curve has been displaced upwards slightly to avoid overlap with the blue line.

7.6 Married put & lower bound on put fair value

- **Jianqiang Li, a student in my Spring 2018 class, found the following lower bound on the fair value of a European put option.**
- **It is not in Merton's 1973 paper.**
- Li actually expressed the relation using a **European married put.**
- Recall that a married put is a portfolio of long a put and long one share of stock.
- 'European married put' means the option is a European put.
- We employ discrete dividends and the put–call parity relation in eq. (7.3.2), which we write in the following form

$$p(S,t) + S = c(S,t) + \mathrm{PV}(K) + \bigl[\,\mathrm{PV}(D_1) + \mathrm{PV}(D_2) + \cdots\bigr]\,. \tag{7.6.1}$$

- The left hand side is the fair value of a European married put.
- Since $c(S,t) \ge 0$, a *lower bound* for the fair value of a European married put is:

$$p(S,t) + S \ge \mathrm{PV}(K) + \bigl[\,\mathrm{PV}(D_1) + \mathrm{PV}(D_2) + \cdots\bigr]\,. \tag{7.6.2}$$

- The proof of eq. (7.6.2) is as follows.
  1. Suppose that the value of $p(S,t) + S$ is *less* than the right hand side of eq. (7.6.2), say
  $$p(S,t) + S = \mathrm{PV}(K) + \bigl[\,\mathrm{PV}(D_1) + \mathrm{PV}(D_2) + \cdots\bigr] - \varepsilon \qquad (\varepsilon > 0)\,. \tag{7.6.3}$$
  2. We go long the put and long one share of stock.
  3. This costs us cash $p(S,t) + S$ at the time t, which we borrow from a bank.
  4. Then we wait until the option expires at time T (because it is a European option).
  5. From the time t until expiration T, we will receive the dividends D_1, D_2, etc. We invest the dividend amounts to earn interest.
     (a) At the expiration time, suppose $S_T \le K$. Then we exercise the put. We deliver our share of stock to close out the put, and receive cash K. Then we have enough money, from the strike price K and the income from the dividends, to repay our bank loan. Our net profit is $\varepsilon e^{r(T-t)}$.
     (b) Conversely, suppose $S_T > K$ at the option expiration. We do not exercise the put and it expires worthless. We sell our share of stock and receive cash S_T, which by definition is more than K. Then we have enough money, from the sale of the stock at the price $S_T > K$ and the income from the dividends, to repay our bank loan. Our net profit is $\varepsilon e^{r(T-t)} + S_T - K$.
  6. *Therefore we start from zero and we make a profit at the option expiration, for all values of the stock price at expiration.* This is arbitrage.
- We can also express eq. (7.6.2) as a lower bound on the fair value of a European put option (in combination with the lower bound $p \ge 0$):

$$p(S,t) \ge \max\Bigl\{\mathrm{PV}(K) - S + \bigl[\,\mathrm{PV}(D_1) + \mathrm{PV}(D_2) + \cdots\bigr]\,,\ 0\Bigr\}\,. \tag{7.6.4}$$