Queens College, CUNY, Department of Computer Science

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15 Lecture 15

Risk neutral valuation, hedging and replicating portfolios

- In this lecture we study the important concepts of **risk neutral valuation** and **hedging**.
- An important concept is that of a **perfect hedge**.
- We shall also briefly mention the related concept of a **replicating portfolio**.
- They are important concepts in finance (not just for options).
- Although there will be some mention of random variables, there will not be any complicated probability theory.
- The concepts in this lecture are very general: the "random variable" does not have to be only a stock price.
- In fact there can be multiple random variables.

15.1 Risk & Hedging

- In general, the value of a derivative depends on one or more random variables.
- Since the future values of those random variables are not known today, this means that the future value of the derivative is also not known today.
    1. The uncertainty in the future value of a derivative makes the derivative **risky.**
    2. We cannot be sure how the value of the derivative is going to behave.
- **Hedging** is the procedure of combining the derivative with different financial securities so as to reduce the riskiness (the uncertainty in future value) of the overall portfolio.
    1. We buy or sell other financial securities to offset the riskiness of the derivative.
    2. The additional securities we buy or sell are collectively called the **hedge.**
    3. Note that the hedge must consist of ***different securities.*** If we bought an option (for example) and formed a hedge by selling the same option, our final portfolio would have no risk, but the portfolio would also be empty. That is *not* a hedge.
    4. The portfolio can be long on the derivative and short on the hedge, or short on the derivative and long on the hedge.
    5. The essential feature is that the overall riskiness of the portfolio is reduced.
- **A hedge is not free.**
- It costs money to buy the hedge, if we are long on the hedge and short on the derivative.
- Conversely, if we are long on the derivative and short on the hedge, the money we receive by shorting the hedge may not be enough to pay to buy the derivative.
- **Hedging makes the future outcome of the portfolio more certain (hedging reduces uncertainty).**
- **Hedging reduces the risk but it also reduces the profit of the portfolio.**

15.2 Perfect hedge

- A very important special type of hedge is that of a **perfect hedge.**
- A **perfect hedge completely eliminates all risk** (all uncertainty) from the portfolio.
- In fact, a perfect hedge not only eliminates all the risk from the portfolio, *it also eliminates all the profit (to zero).*
- If the value of a perfectly hedged portfolio Π is Π_0 at $t = t_0$, then at later times $t > t_0$ its value is completely predictable and must compound at the risk-free rate r:
$$\Pi(t) = \Pi_0\, e^{r(t-t_0)}\,. \tag{15.2.1}$$
- We can subtract the value of Π_0 (essentially a constant, i.e. cash) and say:
The value of a perfectly hedged portfolio is zero.
- This is necessary to avoid arbitrage.
  1. A perfectly hedged portfolio is one which is long the derivative and short a perfect hedge, or vice versa.
  2. If a perfectly hedged portfolio (with initial value zero) yielded a profit, then because the portfolio has no risk, the profit would be riskless, i.e. it would be an arbitrage profit.
  3. Similarly, a perfectly hedged portfolio has no loss. Otherwise we would short the portfolio and make a riskless profit (= arbitrage).
- This is not as stupid as it sounds. It is a valuable tool in derivatives pricing theory.
- Many experts **define the fair value of a derivative as the cost of a perfect hedge.**
- Symbolically, we can write, if we are long the derivative and short the perfect hedge,
$$\text{perfectly hedged portfolio} = (\text{derivative}) \text{ - } (\text{perfect hedge})\,. \tag{15.2.2}$$
- Hence if the value of the perfectly hedged portfolio is zero, we have symbolically
$$\text{fair value of derivative} = \text{fair value of perfect hedge}\,. \tag{15.2.3}$$
- Typically, the derivative is a more complicated financial instrument, while the hedge consists of simpler securities whose fair values we already know how to calculate.
- Therefore, to derive a formula for the fair value of a derivative, we construct a perfect hedge. We calculate the fair value of the perfect hedge, which by hypothesis we know how to do. The fair value of the perfect hedge then gives us a formula for the fair value of the derivative.
- In the case of options, Black and Scholes found a perfect hedge using the stock and zero coupon bonds (cash in a bank).
- **A hedge (including a perfect hedge) is not unique.**
- An option can also be perfectly hedged using futures and zero coupon bonds.
- **However, all perfect hedges must have equal fair values.** If they did not, we could form an arbitrage strategy by trading one perfect hedge against another.

15.3 Risk neutral valuation

- It was pointed out in Sec. 15.1 that the future value of a derivative is risky, because (in general) the value of a derivative depends on one or more random variables, and the future values of those random variables are not known today.

- It was also pointed out that the risk could be reduced to zero by combining the derivative with a perfect hedge.

- **Risk neutral valuation** is a mathematical theory of derivatives pricing.

- It is beyond the scope of these lectures to go into the technical details of the theory, but for our purposes, we can value all financial instruments (forwards, futures, options, anything else) by **discounting all cashflows at the risk-free rate.**

- A technical mathematical detail of risk neutral valuation is that expectations are calculated using something called a **martingale measure.** This is sophisticated probability theory which we shall not discuss.

- **This is possible if and only if the financial markets are arbitrage-free.**

- We always assume that the financial markets are arbitrage-free.

- Hence we employ the risk-free rate in all our calculations for the fair values of derivatives.

- The Black-Scholes formulas for European put and call options are derived under the condition of risk neutral valuation.

- The derivation by Black and Scholes of their partial differential equation came first, and the theory of risk neutral valuation was developed later. Essentially the work of Black and Scholes pioneered a lot of mathematical developments in derivatives pricing theory.

15.4 Static & dynamic hedging

- A **static hedge** is a hedge which does not change with time.

  1. In a static perfect hedge, we buy or sell securities to create the hedge at time t_0, and the portfolio is perfectly hedged for all times $t > t_0$.
  2. A static perfect hedge is very nice if we can find one. It yields a simple formula for the fair value of the derivative.

- A **dynamic hedge** is a hedge which changes with time.

  1. A dynamic hedge requires active trading of the securities which make up the hedge (this can include borrowing and lending of cash, with interest).
  2. In a dynamic perfect hedge, the hedge is only perfect for a short (in principle infinitesimal) time interval.
  3. We must update the dynamic perfect hedge frequently (in principle continuously) to maintain the riskless property of the portfolio.

- It is rare to be able to construct a static perfect hedge.

- A dynamic perfect hedge is much more common.

- In the case of options, the perfect hedge found by Black and Scholes is a dynamic hedge.

- For a long position in a call option, the Black-Scholes perfect hedge requires shorting Delta shares of the underlying stock. Because the value of the Delta of the option changes continuously, the Black-Scholes hedge must be updated continuously.

- The Black-Scholes formula for the fair value of a European call (or put) option is the cost of the Black-Scholes trading strategy to maintain a dynamic perfect hedge for the option.

15.5 Replicating portfolio

- Given a financial security, such as a derivative, a **replicating portfolio** is a portfolio of other securities (including bonds/cash) with the same properties as the original security.
- **Hence a replicating portfolio is a perfect hedge.**
- **The cost of the replicating portfolio yields a formula for the fair value of the security.**
- A static replicating portfolio is a static perfect hedge.
- A dynamic replicating portfolio (involving active trading) is a dynamic perfect hedge.
- Black and Scholes found a combination of stock and cash which was a dynamic replicating portfolio for a European option.

15.6 Examples of static perfect hedges

- Consider a stock S which does not pay dividends. For a forward contract on the stock with expiration T and forward price F, the static hedge consists of (i) long one share of stock and (ii) borrow cash $= S_0$ to pay for the stock. The total cost of the hedge is zero.

  1. After the hedge is set up at time t_0, it requires no further adjustments.
  2. The fair value of the hedge $H(t)$ at any time $t_0 \le t \le T$ is
  $$H(t) = S - S_0 e^{r(T-t)}\,. \tag{15.6.1}$$
  3. We can construct a hedged portfolio of short a forward contract and long the hedge.
  4. The value of the hedged portfolio at the time $t_0 \le t \le T$ is
  $$\text{value of hedged portfolio} = -F + S - S_0 e^{r(t-t_0)}\,. \tag{15.6.2}$$
  5. At the expiration time $t = T$, we deliver the stock to close out the forward contract. In exchange we receive cash equal to the forward price F. We use the money F to repay our loan, which has compounded to the value $S_0 e^{r(T-t_0)}$.
  6. In order to be a perfect hedge and avoid arbitrage, the forward price must have the value
  $$F = S_0 e^{(r(T-t_0)}\,. \tag{15.6.3}$$
  7. Also, it costs no money to go long a forward contract at the time t_0.

- Put-call parity can be viewed as a static hedge. Let the derivative be a European call option c, with strike K and expiration T. The option is on a stock S which does not pay dividends. The static perfect hedge consists of (i) long a European put option p with the same strike K and expiration T, (ii) long one share of stock and (iii) cash borrowing (loan) of $Ke^{-r(T-t_0)}$. Obviously both options are on the stock S.

  1. After the hedge is set up at time t_0, it requires no further adjustments.
  2. The fair value of the hedge $H(t)$ at any time $t_0 \le t \le T$ is
  $$H(t) = p(S,t) + S - Ke^{r(T-t)}\,. \tag{15.6.4}$$
  3. By put-call parity, the fair value of the hedge equals that of the call c.
  4. The hedge is also a static replicating portfolio.
  5. The hedge exactly matches the properties of the call.

15.7 Imperfect hedge

- A hedge need not be perfect.
- Consider the example of a **covered call.**
- In a covered call, we write a call option and buy the stock. That way, we are "covered" if the option is exercised against us. We have the atock available, and can deliver it upon demand.
- However, if the call option is ***not exercised,*** then we are left with a stock whose price by definition has a low value (less than the strike price). Hence we might be exposed to loss if we attempt to sell the stock.
- Hence a covered call is not a perfectly hedged portfolio.
- One share of stock is not a perfect hedge for a call option.
- In fact, we know the fair value of a call option is less than or equal to the stock price.

15.8 Martingale

- What is a martingale?
- You may see this word used in the options (or derivatives) literature.
- A **martingale** is the mathematical definition of a fair game.
- If $X(t)$ is a random variable and a function of the time t, then the expectation value of X at a time t_1 is the same as its expectation value at any other time t_2:

$$\mathbb{E}[X(t_1)] = \mathbb{E}[X(t_2)] \,. \tag{15.8.1}$$

- In other words, our expectation of gain or loss is zero in a martingale random process.
- Don't worry about it.

15.9 Arrow-Debreu securities

- An **Arrow-Debreu security** is a financial security which pays one dollar (or any other unit of currency, obviously) at a specific point in time and a specific value of the stock price.
- An Arrow-Debreu security is therefore different from a zero coupon bond.
- A zero coupon bond (with a face of one dollar) pays one dollar on a particular date, but a zero coupon bond is not linked to the value of the stock price.
- An Arrow-Debreu security is also known as a **state-price security.**
- The "state" is a point in a (time, stock price) space.
- What shall we do with Arrow-Debreu securities?
- Nothing.
- We just memorize the terminology to show the world that we are clever people who know fancy words.
- Any dynamic hedge can be constructed using a combination of Arrow-Debreu securities.
- Arrow-Debreu securities are named for Kenneth Arrow and Gérard Debreu. Both Arrow and Debreu won the Nobel Memorial Prize in Economic Sciences, but in different years.