Queens College, CUNY, Department of Computer Science

Numerical Methods
CSCI 361 / 761
Spring 2018

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due Friday, Aug. 3, 2018, 11.59 pm

11 Homework lecture 11

- **As experience has demonstrated, if you do not understand the above expressions/questions, THEN ASK.**
- **If you do not understand the words/sentences in the lectures, THEN ASK.**
- **Send me an email, explain what you do not understand.**
- **Do not just keep quiet and produce nonsense in exams.**

11.1 Linear algebra: Tridiagonal 1

- The tridiagonal algorithm is straightforward.
- It is a simple "forward elimination" and "backward substitution" procedure.
- Consider the following tridiagonal set of equations in three unknowns x_1, x_2, x_3:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} . \tag{11.1.1}$$

- **Prove that the tridiagonal matrix is weakly diagonally dominant.**
- **Use the first equation to express x_1 in terms of x_2.**
  Write $x_1 = \beta_1 - \alpha_1 x_2$.
- **Substitute for x_1 in the second equation.**
- **Use the second equation to express x_2 in terms of x_3.**
  Write $x_2 = \beta_2 - \alpha_2 x_3$.
- **Substitute for x_2 in the third equation.**
- **The (processed) third equation now contains only x_3.**
- **Solve the (processed) third equation for x_3.**
- **Backsubstitute to solve for x_2 using $x_2 = \beta_2 - \alpha_2 x_3$.**
- **Backsubstitute to solve for x_1 using $x_1 = \beta_1 - \alpha_1 x_3$.**
- **Write down the solution for x_1, x_2 and x_1,**
- *That's all there is to it.*

11.2 Linear algebra: Tridiagonal 2

- Consider the following tridiagonal set of equations in three unknowns x_1, x_2, x_3:

$$\begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -14 \end{pmatrix} . \tag{11.2.1}$$

- **Prove that the tridiagonal matrix is strongly diagonally dominant.**
- **Employ the same procedure as in Sec. 11.1 to solve for x_1, x_2, x_3.**
- *That's really all there is to it.*

11.3 Diagonal dominance

- Let μ be a real number and let T be the following tridiagonal matrix:

$$T = \begin{pmatrix} 2+\mu^2 & \mu & 0 & 0 & 0 \\ \mu & 2+\mu^2 & \mu & 0 & 0 \\ 0 & \mu & 2+\mu^2 & \mu & 0 \\ 0 & 0 & \mu & 2+\mu^2 & \mu \\ 0 & 0 & 0 & \mu & 2+\mu^2 \end{pmatrix} \tag{11.3.1}$$

- Here $a_i = 2+\mu^2$, $b_i = c_i = \mu$.

- **Prove that the tridiagonal matrix T in eq. (11.3.1) is strongly diagonally dominant.**

  1. We see that $|a_i| = 2+\mu^2$ for all $-\infty < \mu < \infty$.
  2. We must analyze separate cases $\mu \geq 0$ and $\mu < 0$ to obtain the amplitudes of b_i and c_i.
  3. First suppose $\mu \geq 0$. Then $|b_i| = |c_i| = \mu$.
     (a) Then
     $$|a_i| - |b_i| - |c_i| = 2+\mu^2 - 2\mu\,. \tag{11.3.2}$$
     (b) **Prove that $2+\mu^2-2\mu > 0$ for all $\mu \geq 0$.**
     (c) **The first and last equations are special cases and require a separate analysis. Do it.**
  4. Next suppose $\mu < 0$. Then $|b_i| = |c_i| = -\mu$.
     (a) Then
     $$|a_i| - |b_i| - |c_i| = 2+\mu^2 + 2\mu\,. \tag{11.3.3}$$
     (b) **Prove that $2+\mu^2+2\mu > 0$ for all $\mu < 0$.**
     (c) **The first and last equations are special cases and require a separate analysis. Do it.**

- This proves the strong diagonal dominance.

11.4 Linear algebra: Discretized ordinary differential equations

- **This is for your information only. *** Nothing to solve. *****
- **Tridiagonal systems of equations frequently arise when we discretize linear second order ordinary differential equations.**
- Consider the linear second order ordinary differential equation

$$\alpha(x)\,\frac{d^2f}{dx^2} + \beta(x)\,\frac{df}{dx} + \gamma(x)f = \zeta(x)\,. \tag{11.4.1}$$

- Here $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $\zeta(x)$ are all known functions of x.
- The goal is to solve for $f(x)$.
- We express the derivatives using finite differences.
- We use a stepsize h and write $x_i = x(ih)$ and $f_i = f(x_i)$.
- Then eq. (11.4.1) is discretized as follows

$$\alpha(x_i)\,\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + \beta(x_i)\,\frac{f_{i+1} - f_{i-1}}{2h} + \gamma(x_i)f_i = \zeta(x_i)\,. \tag{11.4.2}$$

- Multiply through by h^2 to obtain

$$\alpha(x_i)\,(f_{i+1} - 2f_i + f_{i-1}) + \frac{h\beta(x_i)}{2}\,(f_{i+1} - f_{i-1}) + h^2\gamma(x_i)f_i = h^2\zeta(x_i)\,. \tag{11.4.3}$$

- Collect terms to obtain

$$(\alpha(x_i) - (h/2)\beta(x_i))f_{i-1} + (-2\alpha(x_i) + h^2\gamma(x_i))f_i + (\alpha(x_i) + (h/2)\beta(x_i))f_{i+1} = h^2\zeta(x_i)\,. \tag{11.4.4}$$

- This is a tridiagonal set of equations, with

$$\begin{aligned} a_i &= -2\alpha(x_i) + h^2\gamma(x_i)\,, \\ b_i &= \alpha(x_i) - (h/2)\beta(x_i)\,, \\ c_i &= \alpha(x_i) + (h/2)\beta(x_i)\,, \\ r_i &= h^2\zeta(x_i)\,. \end{aligned} \tag{11.4.5}$$

- The tridiagonal matrix equations has the form

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & b_{i-1} & a_{i-1} & c_{i-1} & & \\ & & b_i & a_i & c_i & \\ & & & b_{i+1} & a_{i+1} & c_{i+1} \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ x_{x-1} \\ x_i \\ x_{i+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ r_{x-1} \\ r_i \\ r_{i+1} \\ \vdots \end{pmatrix}\,. \tag{11.4.6}$$

- **Some decision has to be made, to truncate this to a finite set of equations.**
- **That will depend on the details of the problem.**