Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Fall 2017

Instructor: Dr. Sateesh Mane

Final Exam Fall 2018

Monday Dec. 17, 2018

Take home, grade boost
Due Sunday December 23, 2018 11:59 pm

- **You are permitted to update any program code you sent for the in-class final.**
- **Put ALL your program code into the zip along with your answers to the exam questions.**
- **Submit your answers (and code) via email, as a file attachment, to `Sateesh.Mane@qc.cuny.edu`.**

  `StudentId_first_last_CS361_take_home_final_Dec2018.zip`
  `StudentId_first_last_CS761_take_home_final_Dec2018.zip`

This is for the in class exam, mostly not applicable.

- **NOTE: It is the policy of the Computer Science Department to issue a failing grade to any student who either gives or receives help on any test.**
- This is an **open-book** test.
- Once you leave the classroom, you cannot come back to the test.
- **Any problem to which you give two or more (different) answers receives the grade of zero automatically.**
- Submit your solution in the envelope provided, with your name and student id on the cover.
  1. Write your answers in the blue book provided, with your name and student id on the cover of the blue book.
  2. If you require extra sheets of paper, write your name and student id at the top of each page and place the sheets in the envelope provided.
  3. **Answers must be written in legible handwriting: a failing grade will be awarded if the examiner is unable to decipher your handwriting.**
- Some questions require you to perform computations using a computer program.
  1. **Answers to questions which require a computer program will not be accepted if you do not submit your program code.**
  2. **Submit your program code on or before the date of the exam.**
  3. The code should implement the following:
     (a) Runge-Kutta RK4 algorithm.
     (b) Tridiagonal matrix algorithm.
  4. Programs may be written in C++ or Java.
  5. You are permitted to use the code in the online lectures (else write your own code).
  6. **You are NOT permitted to use online software (free or commercial software).**
  7. You ARE permitted to use Excel on your computer, and/or a pocket calculator.

1 Question 1

- Solve the following linear equations for x_1, x_2 and x_3 using LU decomposition:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 3\,, \\
x_1 + 2x_2 + 4x_3 &= -6\,, \\
2x_1 + 2x_2 + 3x_3 &= -1\,.
\end{aligned}
\tag{1.1}
$$

- Write the matrix A associated with eq. (1.1).
- Write out the steps in the LU decomposition of A.
- Display the final matrix in LU form.
- Also write down the final value of the array of the swap indices.

$$(\text{swap array}) = \ldots$$

- Also write down the total number of swaps performed.
- Calculate the determinant of the matrix A.
- Solve eq. (1.1) for x_1, x_2 and x_3.

2 Question 2

- You are given the following ordinary differential equation:

$$\frac{dy}{dx} = \cos(\alpha x)\, y^{1/3}\,. \tag{2.1}$$

- The following is the exact solution of eq. (2.1) with the initial condition $y(0) = 1$.
  You do not have to prove that this is the answer.

$$y_{\rm exact}(x) = \left[1 + \frac{2}{3\alpha}\,\sin(\alpha x)\right]^{3/2}. \tag{2.2}$$

- **Set $\alpha = \frac{1}{2}\pi$ below for this question.**
- **Calculate the value of $y_{\rm exact}(x)$ at $x = 2$. Call this value y_* below.**
- **If you use C++, you can obtain the value of π numerically via the following code:**

```
const double pi = 4.0*atan2(1.0,1.0);
```

- **Use Runge–Kutta fourth order `RK4` to integrate eq. (2.1) from $x = 0$ to $x = 2$ with the initial condition $y(0) = 1$.**
  1. Use n steps to calculate the value of y_n, i.e. the numerical solution for $y(x)$ at $x = 2$.
  2. Set $n = 10, 100, \ldots$ and fill the table below until you find a value of n such that $|y_n - y_*| < 10^{-4}$.
  3. **Using Java, you may attain the tolerance requirement using only $n = 10$. *That is acceptable.***

n	$\lvert y_n - y_*\rvert$
10	...
100	...
$\vdots$	...
...	stop when $\lvert y_n - y_*\rvert < 10^{-4}$

3 Question 3

- Set $\alpha = \frac{1}{2}\pi$ below for this question.
- Multiply your student id by 10^{-8} and define β as follows (hence $0 < \beta < 1$):

$$\beta = (\text{your student id}) \times 10^{-8}\,. \tag{3.1}$$

- You are given the following ordinary differential equation:

$$\frac{dy}{dx} = \cos(\alpha x)\, y^{1/3} - \beta\,. \tag{3.2}$$

- **Use Runge–Kutta fourth order `RK4` to integrate eq. (3.2) from $x = 0$ to $x = 2$ with the initial condition $y(0) = 1$.**
  1. **Use $n = 1000$ steps** to calculate the value of y_i for $i = 1, 2, \dots, n$.
  2. **Find the value of i and x_i where y_i attains its peak (maximum) value.**
  3. Write the values of i, x_i and y_i where y_i attains its peak (maximum) value.

$$\begin{aligned} i &= \dots \\ x_i &= \dots \\ y_i &= \dots \end{aligned}$$

- Sketch a graph of $y(x)$ for $0 \le x \le 2$.
  1. *The sketch is only approximate and does not have to be "to scale" etc.*
  2. Mark the peak (values of x and y).
  3. Write the value of y_n at $x = 2$. **Optional**
  4. **It should be possible to copy and paste an Excel chart of the graph.**
  5. **If you use Excel (or another charting program), you do not need to mark the peak, etc.**

4 Question 4

- Multiply your student id by 10^{-8} and define β as follows (hence $0 < \beta < 1$):

$$\beta = (\text{your student id}) \times 10^{-8} \,. \tag{4.1}$$

- You are given the following inhomogeneous linear second order differential equation:

$$\frac{d^2y}{dx^2} + \beta \, \frac{dy}{dx} + xy = 1 \,. \tag{4.2}$$

- We shall employ the tridiagonal matrix algorithm to solve eq. (4.2) numerically.
  1. Use centered finite differences (with a stepsize h) and derive equations of the form
  $$b_i y_{i-1} + a_i y_i + c_i y_{i+1} = d_i \qquad\qquad (i = 1, \dots, n-1) \,. \tag{4.3}$$
  2. **Write the expressions for a_i, b_i, c_i and d_i below, for $1 \le i \le n-1$:**
  $$\begin{aligned} a_i &= \text{function of } (x_i, \beta, h) \,, \\ b_i &= \text{function of } (x_i, \beta, h) \,, \\ c_i &= \text{function of } (x_i, \beta, h) \,, \\ d_i &= \text{function of } (x_i, \beta, h) \,. \end{aligned}$$
  3. **For sufficiently small $|h| \ll 1$ and $x_i > 0$, show that:**
  $$|a_i| < |b_i| + |c_i| \,. \tag{4.4}$$
  4. *Hence the coefficients in eq.* (4.3) *are NOT diagonally dominant. Do not worry.*
- Set $n = 10000 = 10^4$ in this question.
  1. Define a set of $n+1$ equally spaced points x_i with $x_0 = 0$ **and** $x_n = 10$**.**
  2. Hence the interval size we shall employ in this question is $h = 10/n = 0.001$.
  3. **The boundary conditions are $y = -1$ at $x = 0$ and $y = 1$ at $x = 10$.**
- **Solve for y_i numerically using the tridiagonal matrix algorithm with eq.** (4.3) **and the given boundary conditions.**
- The solution for $y(x)$ crosses zero multiple times in the interval $0 \le x \le 10$.
  1. Find the values of i such that y_i and y_{i+1} have opposite signs.
  2. Then because $y(x)$ is a continuous function, it crosses zero between x_i and x_{i+1}.
  3. Fill the following table with the relevant values of i and x_i.

i	x_i
...	...
...	...
etc.	

- **Copy and paste an Excel chart of the graph of $y(x)$ for $0 \le x \le 10$ (or use some other charting program).**

5 Question 5

- You are given the following linear equations in the variables x_1, x_2 and x_3:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1\,, \\ \boldsymbol{a_{21}} x_1 + 2x_2 + 7x_3 &= \boldsymbol{r_2}\,, \\ 2x_1 + 2x_2 + 3x_3 &= 3\,. \end{aligned} \tag{5.1}$$

- Here a_{21} and r_2 are constants.
- **Find the value of a_{21} such that the LU decomposition encounters a zero pivot.**
- Denote that value of a_{21} by α_{21}.
- *Hint: Process the equations "as is" and do not attempt to swap rows.*
- **Set $a_{21} = \alpha_{21}$ and then find the value of r_2 such that the equations are consistent.**
- *Note: Do NOT attempt to solve the resulting equations.*
  The equations are consistent but not linearly independent, hence there is no unique solution.