Queens College, CUNY, Department of Computer Science

Computational Finance

CSCI 365 / 765

Fall 2017

Instructor: Dr. Sateesh Mane

November 15, 2017

13 Lecture 13

Derivation of the Black-Scholes-Merton formula

- In this lecture we derive the Black-Scholes-Merton formula for European call and put options.
- We do it by solving the Black-Scholes-Merton partial differential equation.
- We also derive the probability density function for geometric Brownian motion.

13.1 Black-Scholes-Merton equation

- First recall the **Black-Scholes-Merton equation** (which includes continuous dividends)

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0\,. \tag{13.1.1}$$

- For a European call, the terminal condition on V at time T is

$$V_{\rm call}(S_T, T) = \max(S_T - K, 0)\,. \tag{13.1.2}$$

- For a European put, the terminal condition on V at time T is

$$V_{\rm put}(S_T, T) = \max(K - S_T, 0)\,. \tag{13.1.3}$$

- We solve eq. (13.1.1) by integrating **backwards in time** from expiration $t = T$ to today $t = t_0$.

- Actually we can just solve eq. (13.1.1) by integrating backwards in time from expiration to an arbitrary value of the time t.

13.2 Change of variables

- We can solve eq. (13.1.1) "as is" but it is simpler to change variable to $x = \ln(S/K)$.
- Note that $0 < S < \infty$ so $-\infty < x < \infty$.
- It is also convenient to define τ as the time to expiration $\tau = T - t$.
- Then $\tau = 0$ at expiration and we solve to obtain the solution for $\tau > 0$.
- Define $U(x, \tau) = V(S, t)$ and solve for U.
- With this change of variables, eq. (13.1.1) is transformed to

$$-\frac{\partial U}{\partial \tau} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + (r - q - \tfrac{1}{2}\sigma^2)\frac{\partial U}{\partial x} - rU = 0\,. \tag{13.2.1}$$

- Then eq. (13.2.1) is a linear partial differential equation with **constant coefficients.**
- This makes eq. (13.2.1) simpler to solve than eq. (13.1.1).
- For a European call, the terminal condition on U (i.e. $\tau = 0$) is (put $y = x_T$)

$$U_{\text{call}}(y, 0) = \max(Ke^y - K, 0) = K\,\max(e^y - 1, 0)\,. \tag{13.2.2}$$

  Hence the domain of integration at expiration is $0 \le y < \infty$.
- For a European put, the terminal condition on U is

$$U_{\text{put}}(y, 0) = \max(K - Ke^y, 0) = K\,\max(1 - e^y, 0)\,. \tag{13.2.3}$$

  Hence the domain of integration at expiration is $-\infty < y \le 0$.

13.3 Discount factor

- There are many ways to solve eq. (13.2.1).
- First let us factor out the discount factor $e^{-r(T-t)} = e^{-r\tau}$.
- Let us express $U(x,t)$ in the form

$$U(x,\tau) = G(x,\tau)\, e^{-r\tau}\,. \tag{13.3.1}$$

- Substitution into eq. (13.2.1) yields the following

$$-\frac{\partial G}{\partial \tau}\, e^{-r\tau} + rG\, e^{-r\tau} + \tfrac{1}{2}\sigma^2\, \frac{\partial^2 G}{\partial x^2}\, e^{-r\tau} + (r - q - \tfrac{1}{2}\sigma^2)\, \frac{\partial G}{\partial x}\, e^{-r\tau} - rG\, e^{-r\tau} = 0\,. \tag{13.3.2}$$

- Simplifying and factoring out $e^{-r\tau}$ (which never equals zero) yields the following

$$\frac{\partial G}{\partial \tau} = \tfrac{1}{2}\sigma^2\, \frac{\partial^2 G}{\partial x^2} + (r - q - \tfrac{1}{2}\sigma^2)\, \frac{\partial G}{\partial x}\,. \tag{13.3.3}$$

- Then eq. (13.3.3) has the form of a **diffusion equation with a drift term.**
- Technically, eq. (13.3.3) is a **diffusion-convection equation.**
- Mathematically, eq. (13.3.3) is a **parabolic-hyperbolic partial differential equation.**
- We may or may not have time to discuss what that means, in these lectures.

13.4 Green's function

- Let us try the following solution for eq. (13.3.3), including $y = x_T$ as a parameter:

$$G(x,\tau;y) = \frac{e^{-(x-y+(r-q-\sigma^2/2)\tau)^2/(2\sigma^2\tau)}}{\sqrt{2\sigma^2\tau}}\,. \tag{13.4.1}$$

- The partial derivative $\partial G/\partial x$ is given by

$$\frac{\partial G}{\partial x} = -\frac{x-y+(r-q-\frac{1}{2}\sigma^2)\tau}{\sigma^2\tau}\,G\,. \tag{13.4.2}$$

- The partial derivative $\partial^2 G/\partial x^2$ is given by

$$\begin{aligned}\frac{\partial^2 G}{\partial x^2} &= -\frac{G}{\sigma^2\tau} - \frac{x-y+(r-q-\frac{1}{2}\sigma^2)\tau}{\sigma^2\tau}\,\frac{\partial G}{\partial x} \\ &= -\frac{G}{\sigma^2\tau} + \frac{(x-y+(r-q-\frac{1}{2}\sigma^2)\tau)^2}{\sigma^4\tau^2}\,G\,.\end{aligned} \tag{13.4.3}$$

- The partial derivative $\partial G/\partial\tau$ is given by

$$\begin{aligned}\frac{\partial G}{\partial \tau} &= -\frac{G}{2\tau} + \frac{(x-y+(r-q-\frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau^2}\,G \\ &\qquad - (r-q-\tfrac{1}{2}\sigma^2)\,\frac{x-y+(r-q-\frac{1}{2}\sigma^2)\tau}{\sigma^2\tau}\,G\,.\end{aligned} \tag{13.4.4}$$

- Substituting into eq. (13.3.3) yields

$$\begin{aligned}\tfrac{1}{2}\sigma^2\,\frac{\partial^2 G}{\partial x^2} + (r-q-\tfrac{1}{2}\sigma^2)\,\frac{\partial G}{\partial x} &= -\frac{G}{2\tau} + \frac{(x-y+(r-q-\frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau^2}\,G \\ &\qquad - (r-q-\tfrac{1}{2}\sigma^2)\frac{x-y+(r-q-\frac{1}{2}\sigma^2)\tau}{\sigma^2\tau}\,G \\ &= \frac{\partial G}{\partial\tau}\,.\end{aligned} \tag{13.4.5}$$

- Hence $G(x,\tau;y)$ in eq. (13.4.1) is a solution of eq. (13.3.3).
- The function $G(x,\tau;y)$ is called a **Green's function.**
- Grammatically, it should really be called a "Green function" (named for George Green).
- However everyone writes "Green's function" and ignores grammar.
- The function $G(x,\tau;y)$ is the solution of eq. (13.3.3) with the terminal boundary condition

$$\lim_{\tau\to 0} G(x,\tau;y) = \delta(x-y)\,. \tag{13.4.6}$$

- Here $\delta(x-y)$ is the Dirac delta function.

13.5 General solution for arbitrary terminal payoff

- Let the terminal payoff function be $\mathscr{P}(y)$.
- The solution of eq. (13.2.1) with terminal payoff $\mathscr{P}(y)$ is

$$V(S,t) = U(x,\tau) = e^{-r\tau} \int_{-\infty}^{\infty} G(x,\tau;y)\, \mathscr{P}(y)\, dy\,. \tag{13.5.1}$$

- Note that eq. (13.5.1) is the **general solution** of the Black-Scholes-Merton equation eq. (13.2.1), for the terminal payoff function $\mathscr{P}(y)$, where $y = \ln(S_T/K)$.
- The structure of eq. (13.5.1) is a discount factor $e^{-r\tau} = e^{-r(T-t)}$ multiplying the **expectation value of the terminal payoff.**
- For a European call option, the terminal payoff function is

$$\mathscr{P}_{\text{call}}(y) = \max(S_T - K, 0) = \max(Ke^y - K, 0) = K\max(e^y - 1, 0)\,. \tag{13.5.2}$$

  This is nonzero in the interval $0 \le y < \infty$.
- For a European put option, the terminal payoff function is

$$\mathscr{P}_{\text{put}}(y) = \max(K - S_T, 0) = \max(K - Ke^y, 0) = K\max(1 - e^y, 0)\,. \tag{13.5.3}$$

  This is nonzero in the interval $-\infty < y \le 0$.

13.6 Geometric Brownian Motion: probability density function

- The Green's function $G(x, \tau; y)$ in eq. (13.4.1) is actually a probability density function, but for standard Brownian motion (with a drift term).
- Let us derive the probability density function for geometric Brownian motion.
- Hence we want a function in terms of S and t (with parameters S_T and T).
- Define the function $p_{\rm GBM}(S, t; S_T, T)$ via

$$p_{\rm GBM}(S, t; S_T, T) = G(x, \tau; y) \, \frac{dx}{dS} \,. \tag{13.6.1}$$

- Then $dx/dS = 1/S$, hence we obtain

$$p_{\rm GBM} = \frac{1}{S} \, \frac{e^{-(\ln(S/K) - \ln(S_T/K) + (r-q-\sigma^2/2)(T-t))^2/(2\sigma^2(T-t))}}{\sqrt{2\sigma^2(T-t)}} \,. \tag{13.6.2}$$

- The **probability density function for geometric Brownian motion** is given by $p_{\rm GBM}$.
  - The function $p_{\rm GBM}$ in eq. (13.6.2) is the probability density that, if the stock price is S at the time t, the random walk of the stock price will have the value S_T at the future time T, where $T > t$.

13.7 Solution for call option using terminal boundary condition

- The solution of eq. (13.2.1) which satisfies the terminal boundary condition is given by

$$\begin{aligned}U(x,\tau) &= e^{-r\tau}\int_{-\infty}^{\infty} G(x,\tau;y)\mathscr{P}_{\text{call}}(y)\,dy \\ &= Ke^{-r\tau}\int_{0}^{\infty} G(x,\tau;y)(e^y-1)\,dy \\ &= Ke^{-r\tau}\int_{0}^{\infty} G(x,\tau;y)e^y\,dy \quad - \quad Ke^{-r\tau}\int_{0}^{\infty} G(x,\tau;y)\,dy\,.\end{aligned} \tag{13.7.1}$$

- Hence we must evaluate the following integrals

$$I_1 = \int_0^\infty G(x,\tau;y)e^y\,dy\,, \qquad I_2 = \int_0^\infty G(x,\tau;y)\,dy\,. \tag{13.7.2}$$

- We recognize the second integral I_2 immediately as a cumulative Normal distribution:

$$\begin{aligned}\int_0^\infty G(x,\tau;y)\,dy &= \int_0^\infty \frac{e^{-(x-y+(r-q-\sigma^2/2)\tau)^2/(2\sigma^2\tau)}}{\sqrt{2\sigma^2\tau}}\,dy \\ &= N\Bigl(\frac{x+(r-q-\frac12\sigma^2)\tau}{\sigma\sqrt{\tau}}\Bigr) \\ &= N(d_2)\,.\end{aligned} \tag{13.7.3}$$

- Recall the definitions of d_1 and d_2 from previous lectures (and recall $x=\ln(S/K)$):

$$d_1 = \frac{\ln(S/K)+(r-q)(T-t)}{\sigma\sqrt{T-t}} + \tfrac12\sigma\sqrt{T-t}\;, \tag{13.7.4a}$$

$$d_2 = \frac{\ln(S/K)+(r-q)(T-t)}{\sigma\sqrt{T-t}} - \tfrac12\sigma\sqrt{T-t} = d_1 - \sigma\sqrt{T-t}\,. \tag{13.7.4b}$$

- To evaluate I_1 we require the following identity

$$Ke^{-r\tau}e^{-(x-y+(r-q-\sigma^2/2)\tau)^2/(2\sigma^2\tau)}\,e^y = Se^{-q\tau}e^{-(x-y+(r-q+\sigma^2/2)\tau)^2/(2\sigma^2\tau)}\,. \tag{13.7.5}$$

- The proof of eq. (13.7.5) consists basically of completing the square in the exponent.
- Then the integral for I_1 also yields a cumulative Normal distribution:

$$\begin{aligned}Ke^{-r\tau}I_1 &= Ke^{-r\tau}\int_0^\infty G(x,\tau;y)e^y\,dy \\ &= Se^{-q\tau}\int_0^\infty \frac{e^{-(x-y+(r-q+\sigma^2/2)\tau)^2/(2\sigma^2\tau)}}{\sqrt{2\sigma^2\tau}}\,dy \\ &= Se^{-q\tau}\,N\Bigl(\frac{x+(r-q+\frac12\sigma^2)\tau}{\sigma\sqrt{\tau}}\Bigr) \\ &= Se^{-q\tau}\,N(d_1)\,.\end{aligned} \tag{13.7.6}$$

- Summing the integrals yields the Black-Scholes-Merton formula for a European call option.

13.8 Solution for put option using terminal boundary condition

- The discount factor and the Green's function is the same for a European put option.
- Only the terminal boundary condition (terminal payoff) is different.
- The solution of eq. (13.2.1) which satisfies the terminal boundary condition for a put is

$$\begin{aligned} U(x,\tau) &= e^{-r\tau}\int_{-\infty}^{\infty} G(x,\tau;y)\mathscr{P}_{\rm put}(y)\,dy \\ &= Ke^{-r\tau}\int_{0}^{\infty} G(x,\tau;y)(1-e^y)\,dy \\ &= Ke^{-r\tau}\int_{-\infty}^{0} G(x,\tau;y)\,dy \quad - \quad Ke^{-r\tau}\int_{-\infty}^{0} G(x,\tau;y)e^y\,dy\,. \end{aligned} \tag{13.8.1}$$

- Hence we must evaluate the following integrals

$$I_3 = \int_{-\infty}^{0} G(x,\tau;y)\,dy\,, \qquad I_4 = \int_{-\infty}^{0} G(x,\tau;y)e^y\,dy\,. \tag{13.8.2}$$

- The evaluation of these integrals is similar to that for the call option.
- The results are

$$I_3 = N(-d_2)\,, \qquad Ke^{-r\tau}I_4 = Se^{-q\tau}\,N(-d_1)\,. \tag{13.8.3}$$

- Summing the integrals yields the Black-Scholes-Merton formula for a European put option.

13.9 Black-Scholes-Merton formula

The Black-Scholes-Merton formulas for a European call and put option are respectively

$$c(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \ , \tag{13.9.1a}$$

$$p(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) \, . \tag{13.9.1b}$$

The definitions of d_1 and d_2 were displayed above, but for ease of reference they are

$$d_1 = \frac{\ln(S/K) + (r-q)(T-t)}{\sigma\sqrt{T-t}} + \tfrac{1}{2}\sigma\sqrt{T-t} \ , \tag{13.9.2a}$$

$$d_2 = \frac{\ln(S/K) + (r-q)(T-t)}{\sigma\sqrt{T-t}} - \tfrac{1}{2}\sigma\sqrt{T-t} = d_1 - \sigma\sqrt{T-t} \, . \tag{13.9.2b}$$