Queens College, CUNY, Department of Computer Science

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9 Lecture 9

9.1 The Greeks

- Let us not be in a rush to solve fancy mathematical partial differential equations. There is more to learn which does not depend on a probability model for the stock price movements.
- We introduce the **Greeks**.
- The **Greeks** are partial derivatives of the fair value of a derivative.
  1. Notice that I say **derivative** and not option.
  2. The statements below apply to *all* derivatives, not just options.
  3. In particular, they apply to forwards and futures contracts, too.
- They are called "Greeks" because they are conventionally denoted by Greek letters in mathematical formulas.
  1. To calculate partial derivatives, we must assume that the fair value of a derivative which depends on S and t is a differentiable function of both S and t.
  2. Although the above assumption does not depend on any probability model for the stock price movements, nevertheless it *is* an assumption. We have no proof that the fair value of a derivative is a differentiable function of S and t.
  3. History has shown that the prices of stocks can drop suddenly ("jump") by large amounts, such as on Black Monday on October 19, 1987.
- The fair value of a derivative also depends on other parameters, such as the interest rate r. We assume the derivative's fair value is also a differentiable function of all such parameters.
- The Greeks are also called the **sensitivities** of a derivative.

9.2 The Greeks: mathematical definitions

- Let V denote the theoretical fair value of a derivative below.
- **Delta** Δ is the partial derivative with respect to the stock price S:

$$\Delta = \frac{\partial V}{\partial S} \,. \tag{9.2.1}$$

- **Gamma** Γ is the second partial derivative with respect to the stock price S:

$$\Gamma = \frac{\partial^2 V}{\partial S^2} \,. \tag{9.2.2}$$

- **Rho** ρ is the partial derivative with respect to the (risk-free) interest rate r:

$$\rho = \frac{\partial V}{\partial r} \,. \tag{9.2.3}$$

- **Theta** Θ is the partial derivative with respect to the current time t:

$$\Theta = \frac{\partial V}{\partial t} \,. \tag{9.2.4}$$

- **Vega** is the partial derivative with respect to the volatility σ.
  Vega is not a Greek letter, which causes some problems of notation.
  Many people employ the Greek letter **nu** ν to denote vega and we shall do so:

$$\nu = \frac{\partial V}{\partial \sigma} \,. \tag{9.2.5}$$

  My understanding is the concept of a partial derivative to denote "sensitivity to volatility" was invented by a trader while he was on a visit to Las Vegas.
  He did not realize that vega is not a Greek letter.
  Some academics prefer the name **kappa** κ but this is not widely used.

- The above are the five most important (or widely used) Greeks.
- One can invent many other partial derivatives. They will not be discused in these lectures.
- One such example is the **elasticity**, denoted by Ω. The elasticity measures the ratio of the relative change in the derivative value to the relative change in the stock price:

$$\Omega = \frac{S}{V} \frac{\partial V}{\partial S} \,. \tag{9.2.6}$$

9.3 Non-equity derivatives

- There are many derivatives which do not depend on stocks.
- For example, there are derivatives on exchange rates (FX), and they are important.
- There are Greeks for such derivatives also.
- There are Greeks for all derivatives.

9.4 Stock

- Consider a stock with price S.
- The Greeks of the stock are:

$$\Delta = 1\,, \tag{9.4.1a}$$

$$\Gamma = 0\,, \tag{9.4.1b}$$

$$\rho = 0\,, \tag{9.4.1c}$$

$$\Theta = 0\,, \tag{9.4.1d}$$

$$\nu = 0\,. \tag{9.4.1e}$$

9.5 Forwards & futures

- Consider a forward or futures contract F on a stock S.
- The current time is t and the expiration time is T.
- Suppose the stock pays discrete dividends D_i at times t_i, $i = 1, 2, \dots, n$ during the lifetime of the contract, where $t < t_1 < \cdots < t_n < T$.
- The fair value formula is

$$F = \left[S - \sum_{i=1}^{n} e^{-r(t_i - t)} D_i \right] e^{r(T-t)} \,. \tag{9.5.1}$$

- The Greeks of the forward or futures contract are:

$$\Delta = e^{r(T-t)} \,, \tag{9.5.2a}$$

$$\Gamma = 0 \,, \tag{9.5.2b}$$

$$\rho = (T-t)F + e^{r(T-t)} \sum_{i=1}^{n} (t_i - t) e^{-r(t_i - t)} D_i \,, \tag{9.5.2c}$$

$$\Theta = -rSe^{r(T-t)} + e^{r(T-t)} \sum_{i=1}^{n} e^{-r(t_i - t)} D_i \, \delta(t - t_i) \,, \tag{9.5.2d}$$

$$\nu = 0 \,. \tag{9.5.2e}$$

- Then $\Delta > 1$ and $\rho > 0$ (positive).
- The formula for Theta is complicated because of the step change in the sum over the dividends when a dividend payment date is crossed.
- Suppose instead the stock pays continuous dividends at a rate q.
- The fair value formula is

$$F = Se^{(r-q)(T-t)} \,. \tag{9.5.3}$$

- The Greeks of the forward or futures contract are:

$$\Delta = e^{(r-q)(T-t)} \,, \tag{9.5.4a}$$

$$\Gamma = 0 \,, \tag{9.5.4b}$$

$$\rho = (T-t)F \,, \tag{9.5.4c}$$

$$\Theta = -(r-q)F \,, \tag{9.5.4d}$$

$$\nu = 0 \,. \tag{9.5.4e}$$

- The formula for Theta is much simpler.
- If $r > q$ then $\Delta > 1$ and $\Theta < 0$.
  If $r < q$ then $\Delta < 1$ and $\Theta > 0$ (and $\Delta = 1$ and $\Theta = 0$ if $r = q$).
  In all cases $\rho > 0$ (positive).

9.6 The Greeks: some properties for options

- Let now consider some properties of the Greeks for put and call options.
- The properties below apply to both American and European options.

9.6.1 Delta

- Delta is probably the most widely used of the Greeks.
- The value of Delta for a call **is positive and lies between 0 and 1.**

$$0 \le \Delta_c \le 1\,. \tag{9.6.1.1}$$

- The value of Delta for a put **is negative and lies between -1 and 0.**

$$-1 \le \Delta_p \le 0\,. \tag{9.6.1.2}$$

- The value of Δ_c increases from 0 to 1 as the value of S increases from 0 to ∞.
  The value of Δ_p increases from -1 to 0 as the value of S increases from 0 to ∞.
- Many traders interpret Delta in the following way.
  1. They think of Delta as the **probability that an option will expire in the money.**
  2. This is **not rigorous and is technically incorrect**, but is a useful approximation.
- A graph of Delta for a call and a put option is shown in Fig. 1, plotted as a function of the stock price S.

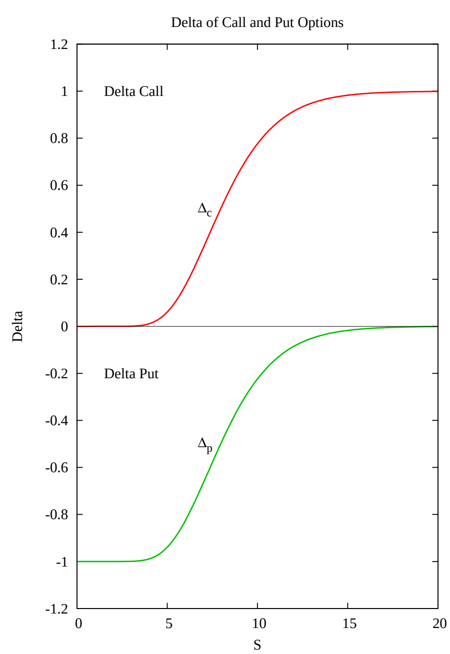

Figure 1: Graph of Delta for a call and a put option, plotted as a function of the stock price S.

9.6.2 Gamma

- Gamma is a second order partial derivative (some call it a "second order Greek").
- Many traders visualize Gamma as the rate of change of Δ.
- The value of Gamma is **always positive** for both a put and a call.
- A plot of the graph of Gamma shows a peak as the value of S increases from 0 to ∞.
- There is no simple formula for the location of the peak.
- The peak in Gamma becomes narrower and taller as the time to expiration decreases.
- A graph of Gamma for either a call or a put option is shown in Fig. 2, plotted as a function of the stock price S.

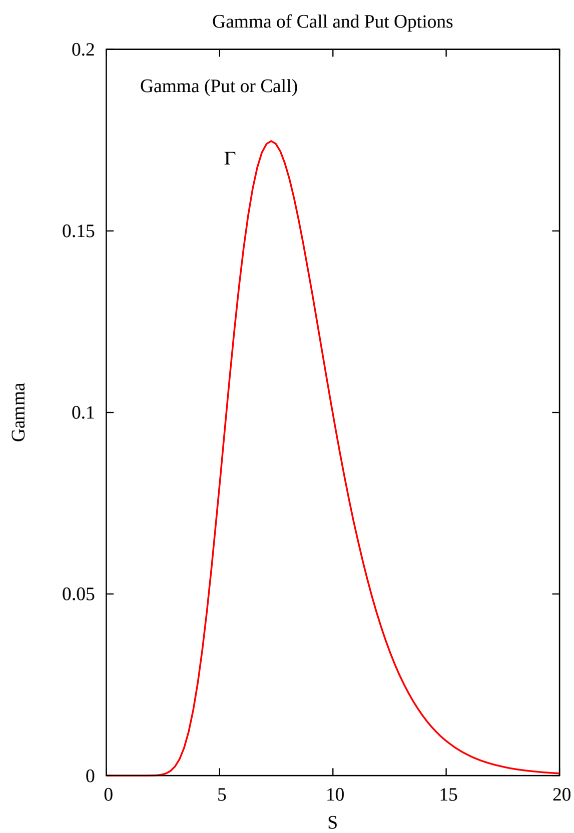

Figure 2: Graph of Gamma for an option, plotted as a function of the stock price S. Only one graph is displayed because Gamma is positive for both call and put options.

9.6.3 Vega

- The value of Vega is also always positive for both a put and a call.
- A plot of the graph of Vega shows a peak as the value of S increases from 0 to ∞.
- There is no simple formula for the location of the peak.
- A plot of the graph of Vega shows a broadly similar shape to the graph of Gamma.
- However, although the peak in Vega becomes narrower as the time to expiration decreases, the peak height decreases to zero as the time to expiration decreases.
- A graph of Vega for either a call or a put option is shown in Fig. 3, plotted as a function of the stock price S.

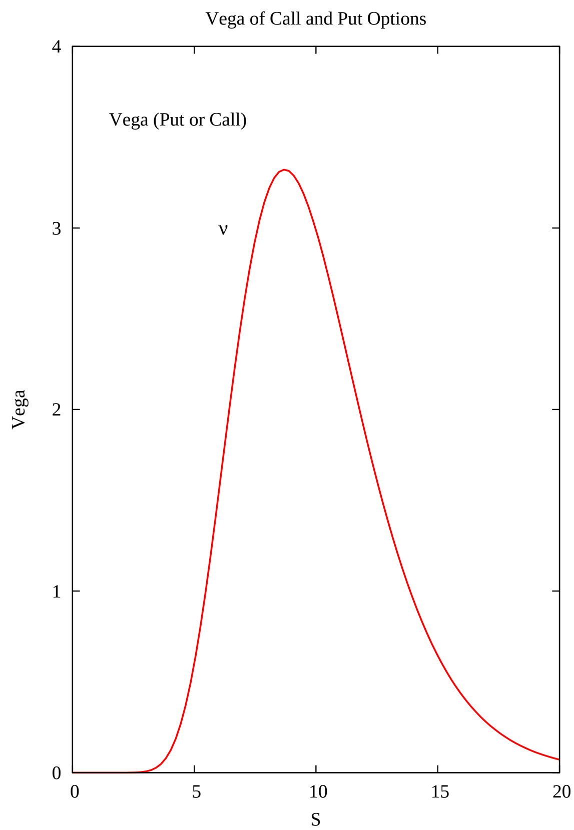

Figure 3: Graph of Vega for an option, plotted as a function of the stock price S. Only one graph is displayed because Vega is positive for both call and put options.

9.6.4 Theta

- Theta has a different status from the other Greeks because time is not a random variable.
- Nevertheless, Theta is also considered to be a Greek.
- The value of Theta is always negative for American options (both put and call).
- The value of Theta is usually negative for European options (both put and call), but can sometimes be positive (for example a very deep in the money put).

9.6.5 Rho

- The value of Rho is positive for a call and negative for a put.
- The fair value of an option is typically not very sensitive to the interest rate.
- Hence Rho is generally considered to be the least important Greek for options.

9.7 Put-call parity

9.7.1 General remarks

- Put-call parity applies only for European options.
- Put-call parity can be employed to derive important relations for the Greeks of European calls and puts.
- Let the strike price of the put and the call options be K.
- The current time is t and the option expiration time is T, where $T > t$.
- The interest rate is r and the volatility is σ.

9.7.2 Discrete dividends

- Consider a stock which pays discrete dividends D_i at times t_i, $i = 1, 2, \ldots, n$ during the lifetime of the options, where $t < t_1 < \cdots < t_n < T$.
- The put-call parity relation in this case is

$$c - p = S - \left(\sum_{i=1}^{n} e^{-r(t_i - t)} D_i\right) - K e^{-r(T-t)} . \qquad (9.7.2.1)$$

- Note that eq. (9.7.2.1) is an identity, valid for all values of S, etc.
- Hence eq. (9.7.2.1) can be differentiated partially to obtain useful identities.
- Differentiating eq. (9.7.2.1) partially with respect to S yields:

$$\Delta_c - \Delta_p = 1 . \qquad (9.7.2.2)$$

This is a very important relation between the Delta of a put and call with the same strike and expiration.

- Differentiating eq. (9.7.2.1) twice partially with respect to S yields:

$$\Gamma_c - \Gamma_p = 0 . \qquad (9.7.2.3)$$

The Gamma of a put and call with the same strike and expiration are equal.

- Notice that the volatility σ does not appear in eq. (9.7.2.1).
Hence differentiating eq. (9.7.2.1) partially with respect to σ yields:

$$\nu_c - \nu_p = 0 . \qquad (9.7.2.4)$$

The Vega of a put and call with the same strike and expiration are equal.

- Differentiating eq. (9.7.2.1) partially with respect to r yields:

$$\rho_c - \rho_p = (T - t) K e^{-r(T-t)} + \left(\sum_{i=1}^{n} (t_i - t) e^{-r(t_i - t)} D_i\right) . \qquad (9.7.2.5)$$

This expression is much simpler if there are no dividends.

- Differentiating eq. (9.7.2.1) partially with respect to t yields:

$$\Theta_c - \Theta_p = -rKe^{-r(T-t)} - r\left(\sum_{i=1}^{n} e^{-r(t_i - t)} D_i\right) + \left(\sum_{i=1}^{n} e^{-r(t_i - t)} D_i \delta(t - t_i)\right) . \qquad (9.7.2.6)$$

This formula is complicated because of the step change in the sum over the dividends when a dividend payment date is crossed.

9.7.3 Continuous dividends

- Consider a stock which pays continuous dividends at a rate q.
- The put-call parity relation in this case is

$$c - p = Se^{-q(T-t)} - Ke^{-r(T-t)}\,. \tag{9.7.3.1}$$

- Note that eq. (9.7.3.1) is an identity, valid for all values of S, etc.
- Hence eq. (9.7.3.1) can be differentiated partially to obtain useful identities.
- Differentiating eq. (9.7.3.1) partially with respect to S yields:

$$\Delta_c - \Delta_p = e^{-q(T-t)}\,. \tag{9.7.3.2}$$

This is a very important relation between the Delta of a put and call with the same strike and expiration, for a continuous dividend yield.

- Differentiating eq. (9.7.3.1) twice partially with respect to S yields:

$$\Gamma_c - \Gamma_p = 0\,. \tag{9.7.3.3}$$

The Gamma of a put and call with the same strike and expiration are equal.

- Notice (as before) that the volatility σ does not appear in eq. (9.7.3.1).
  Hence differentiating eq. (9.7.3.1) partially with respect to σ yields:

$$\nu_c - \nu_p = 0\,. \tag{9.7.3.4}$$

The Vega of a put and call with the same strike and expiration are equal.

- Differentiating eq. (9.7.3.1) partially with respect to r yields:

$$\rho_c - \rho_p = (T-t)Ke^{-r(T-t)}\,. \tag{9.7.3.5}$$

This expression is much simpler than the corresponding identity for discrete dividends.

- Differentiating eq. (9.7.3.1) partially with respect to t yields:

$$\Theta_c - \Theta_p = qSe^{-q(T-t)} - rKe^{-r(T-t)}\,. \tag{9.7.3.6}$$

This formula is much simpler than the corresponding identity for discrete dividends.