Queens College, CUNY, Department of Computer Science
Numerical Methods
CSCI 361 / 761
Summer 2018
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due Friday, August 3, 2018, 11.59 pm

15 Homework lecture 15

- **As experience has demonstrated, if you do not understand the above expressions/questions, THEN ASK.**
- **If you do not understand the words/sentences in the lectures, THEN ASK.**
- **Send me an email, explain what you do not understand.**
- **Do not just keep quiet and produce nonsense in exams.**

15.1 Euler integration 1: explicit

15.1.1 Forward integration

- You are given the following differential equation (c is a positive constant)

$$\frac{dy}{dx} = x - cy\,. \tag{15.1.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 1$.
- You are given that the exact solution $y^{\text{ex}}(x)$ is as follows.

$$y^{\text{ex}}(x) = \left(1 + \frac{1}{c^2}\right) e^{-cx} + \frac{x}{c} - \frac{1}{c^2}\,. \tag{15.1.2}$$

- **You do NOT have to derive the above solution (but you can if you want to).**
- The integration stepsize is h (a constant), hence $x_i = ih$.
- **Use the explicit Euler method to derive the following.**

$$y_{i+1} = y_i + h(x_i - cy_i) = y_i(1 - hc) + hx_i\,. \tag{15.1.3}$$

- Set $h = 1/n$.
- **Show that $x_n = 1$ after n steps.**
- Set $c = 2$.
- **Compute the value of the exact solution $y^{\text{ex}}(x)$ for $x = 1$ and $c = 2$.**
- **Compute the value of the numerical solution y_n at $x_n = 1$ and fill the following table of values.**
    1. **Compute the values of $y^{\text{ex}}(1)$ and y_n to 5 decimal places.**
    2. **In the last column, calculate the value of $n(y^{\text{ex}}(1) - y_n)$ to 2 decimal places.**

n	$y^{\text{ex}}(1)$	y_n	$y^{\text{ex}}(1) - y_n$	$n(y^{\text{ex}}(1) - y_n)$
10	5 d.p.	5 d.p.	5 d.p.	2 d.p.
100	5 d.p.	5 d.p.	5 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.1.2 Reverse integration

- **We now reverse the direction of integration and integrate backwards to $x_0 = 0$.**
- The differential equation is therefore

$$\frac{dy}{dx} = -(x - cy)\,. \tag{15.1.4}$$

- The integration stepsize is still $h = 1/n$.
- Let the steps be indexed by $i = n+1, \dots, 2n$.
- You are given that now

$$x_i = x_n - (i-n)h = 2 - ih \qquad (i = n+1, \dots, 2n)\,. \tag{15.1.5}$$

- The initial condition is $y = y_n$ **(the computed numerical solution)** at $i = n$.
- **Use the explicit Euler method to derive the following.**

$$\begin{aligned} y_{i+1} &= y_i - h(x_i - cy_i) \\ &= y_i(1+hc) - hx_i\,. \end{aligned} \tag{15.1.6}$$

- **Show that $x_{2n} = 0$ after $2n$ steps.**
- The exact solution will return to the initial value $y_0 = 1$ but the numerical solution will not.
- **Compute the value of the numerical solution y_{2n} at $x_{2n} = 0$ and fill the following table of values.**
  1. **Compute the value of y_{2n} to 5 decimal places.**
  2. **In the last column, calculate the value of $n(1 - y_{2n})$ to 2 decimal places.**

$2n$	$y^{\text{ex}}(0)$	y_{2n}	$1 - y_{2n}$	$n(1 - y_{2n})$
20	1	5 d.p.	5 d.p.	2 d.p.
200	1	5 d.p.	5 d.p.	2 d.p.
2000	1	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.1.3 Indexing for forward and reverse integration

- **There seems to be confusion about the notation and indexing, especially for the reverse integration.**
- **Let me use $n = 10$ as an example. Then $h = 1/n = 0.1$.**
- **Then the value of x_i is, for $i = 0, \dots, n, \dots 2n$:**

$$
\begin{aligned}
x_0 &= 0\,, \\
x_1 &= 0.1\,, \\
x_2 &= 0.2\,, \\
&\vdots \\
x_9 &= 0.9\,, \\
x_{10} &= 1.0\,, \\
\boldsymbol{x_{11}} &= \mathbf{0.9}\,, \\
\boldsymbol{x_{12}} &= \mathbf{0.8}\,, \\
&\vdots \\
\boldsymbol{x_{19}} &= \mathbf{0.1}\,, \\
\boldsymbol{x_{20}} &= \mathbf{0}\,.
\end{aligned}
\tag{15.1.7}
$$

- **For the *forward integration*, the loop goes from $i = 0$ through $i = 9$ (ten equations)**

$$
\begin{aligned}
i = 0 :& \quad y_1 = y_0(1 - hc) + hx_0 &&= 1(1 - hc) + h \times 0\,, && (15.1.8a) \\
i = 1 :& \quad y_2 = y_1(1 - hc) + hx_1 &&= y_1(1 - hc) + h \times 0.1\,, && (15.1.8b) \\
i = 2 :& \quad y_3 = y_2(1 - hc) + hx_2 &&= y_2(1 - hc) + h \times 0.2\,, && (15.1.8c) \\
&\vdots && && (15.1.8d) \\
\boldsymbol{i = 9} :& \quad \boldsymbol{y_n} = y_{10} = y_9(1 - hc) + hx_9 &&= y_9(1 - hc) + h \times 0.9\,. && (15.1.8e)
\end{aligned}
$$

- **For the *reverse integration*, the loop goes from $i = 10$ through $i = 19$ (ten equations) *We simply change sign $h \to -h$, so $h = -0.1$.***

$$
\begin{aligned}
\boldsymbol{i = 10} :& \quad y_{11} = y_{10}(1 + hc) - hx_{10} &&= \boldsymbol{y_{10}}(1 + hc) - h \times \mathbf{1.0}\,, && (15.1.9a) \\
\boldsymbol{i = 11} :& \quad y_{12} = y_{11}(1 + hc) - hx_{11} &&= \boldsymbol{y_{11}}(1 + hc) - h \times \mathbf{0.9}\,, && (15.1.9b) \\
\boldsymbol{i = 12} :& \quad y_{13} = y_{12}(1 + hc) - hx_{12} &&= \boldsymbol{y_{12}}(1 + hc) - h \times \mathbf{0.8}\,, && (15.1.9c) \\
&\vdots && && (15.1.9d) \\
\boldsymbol{i = 18} :& \quad y_{19} = y_{18}(1 + hc) - hx_{18} &&= \boldsymbol{y_{18}}(1 + hc) - h \times \mathbf{0.2}\,, && (15.1.9e) \\
\boldsymbol{i = 19} :& \quad \boldsymbol{y_{2n}} = y_{20} = y_{19}(1 + hc) - hx_{19} &&= \boldsymbol{y_{19}}(1 + hc) - h \times \mathbf{0.1}\,. && (15.1.9f)
\end{aligned}
$$

- **Note the indexing and the value of x_i for the reverse integration.**

15.2 Euler integration 2: implicit

15.2.1 Forward integration

- The differential equation is given by eq. (15.1.1)

$$\frac{dy}{dx} = x - cy\,. \tag{15.2.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 1$.
- The exact solution $y^{\rm ex}(x)$ is given by eq. (15.1.2).
- The integration stepsize is h (a constant), hence $x_i = ih$.
- **Use the implicit Euler method to derive the following.**

$$y_{i+1} = y_i + h(x_{i+1} - cy_{i+1})\,. \tag{15.2.2}$$

- **Process the above equation to derive the following.**

$$y_{i+1} = \frac{y_i + hx_{i+1}}{1+hc}\,. \tag{15.2.3}$$

  1. **This is an important lesson to learn.**
  2. **The denominator $1 + hc$ divides the term in x_{i+1} not just y_i.**

- Set $h = 1/n$.
- **Show that $x_n = 1$ after n steps.**
- Set $c = 2$.
- Recall the value of the exact solution $y^{\rm ex}(x)$ for $x = 1$ and $c = 2$.
- **Compute the value of the numerical solution y_n and fill the following table of values.**
  1. **Compute the values of $y^{\rm ex}(1)$ and y_n to 5 decimal places.**
  2. **In the last column, calculate the value of $n(y^{\rm ex}(1) - y_n)$ to 2 decimal places.**

n	$y^{\rm ex}(1)$	y_n	$y^{\rm ex}(1) - y_n$	$n(y^{\rm ex}(1) - y_n)$
10	5 d.p.	5 d.p.	5 d.p.	2 d.p.
100	5 d.p.	5 d.p.	5 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.2.2 Reverse integration

- **We now reverse the direction of integration and integrate backwards to $x_0 = 0$.**
- The differential equation is given by eq. (15.1.4)

$$\frac{dy}{dx} = -(x - cy)\,. \tag{15.2.4}$$

- Let the steps be indexed by $i = n+1, \dots, 2n$.
- The integration stepsize is still $h = 1/n$.
- You are given that now $x_i = 1 - (i - n)h$.
- **Use the implicit Euler method to derive the following.**

$$y_{i+1} = y_i - h(x_{i+1} - cy_{i+1}) \tag{15.2.5}$$

- **Process the above equation to derive the following.**

$$y_{i+1} = \frac{y_i - hx_{i+1}}{1 - hc}\,. \tag{15.2.6}$$

- **Show that $x_{2n} = 0$ after $2n$ steps.**
- The exact solution will return to the initial value $y_0 = 1$ but the numerical solution will not.
- **Compute the value of the numerical solution y_{2n} and fill the following table of values.**
  1. **Compute the value of y_{2n} to 5 decimal places.**
  2. **In the last column, calculate the value of $n(1 - y_{2n})$ to 2 decimal places.**

$2n$	$y^{\text{ex}}(0)$	y_{2n}	$1 - y_{2n}$	$n(1 - y_{2n})$
20	1	5 d.p.	5 d.p.	2 d.p.
200	1	5 d.p.	5 d.p.	2 d.p.
2000	1	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.2.3 Indexing for forward and reverse integration

- **Let me use $n = 10$ as an example. Then $h = 1/n = 0.1$.**
- **The value of x_i is the same as previously listed.**
- **For the *forward integration*, the loop goes from $i = 0$ through $i = 9$ (ten equations)**

$$
\begin{aligned}
i = 0: &\quad y_1 = \frac{y_0 + h\boldsymbol{x_1}}{1 + hc} &&= \frac{1 + h \times \mathbf{0.1}}{1 + hc}, && \text{(15.2.7a)} \\
i = 1: &\quad y_2 = \frac{y_1 + h\boldsymbol{x_2}}{1 + hc} &&= \frac{y_1 + h \times \mathbf{0.2}}{1 + hc}, && \text{(15.2.7b)} \\
i = 2: &\quad y_3 = \frac{y_2 + h\boldsymbol{x_3}}{1 + hc} &&= \frac{y_2 + h \times \mathbf{0.3}}{1 + hc}, && \text{(15.2.7c)} \\
&\quad \vdots && && \text{(15.2.7d)} \\
i = 9: &\quad \boldsymbol{y_n} = y_{10} = \frac{y_9 + h\boldsymbol{x_{10}}}{1 + hc} &&= \frac{y_2 + h \times \mathbf{1.0}}{1 + hc}. && \text{(15.2.7e)}
\end{aligned}
$$

- **For the *reverse integration*, the loop goes from $i = 10$ through $i = 19$ (ten equations) *We simply change sign $h \to -h$, so $h = -0.1$.***

$$
\begin{aligned}
\boldsymbol{i = 10}: &\quad y_{11} = \frac{y_{10} - h\boldsymbol{x_{11}}}{1 - hc} &&= \frac{y_{10} - h \times \mathbf{0.9}}{1 - hc}, && \text{(15.2.8a)} \\
\boldsymbol{i = 11}: &\quad y_{12} = \frac{y_{11} - h\boldsymbol{x_{12}}}{1 - hc} &&= \frac{y_{11} - h \times \mathbf{0.8}}{1 - hc}, && \text{(15.2.8b)} \\
\boldsymbol{i = 12}: &\quad y_{13} = \frac{y_{12} - h\boldsymbol{x_{13}}}{1 - hc} &&= \frac{y_{11} - h \times \mathbf{0.7}}{1 - hc}, && \text{(15.2.8c)} \\
&\quad \vdots && && \text{(15.2.8d)} \\
\boldsymbol{i = 18}: &\quad y_{19} = \frac{y_{18} - h\boldsymbol{x_{19}}}{1 - hc} &&= \frac{y_{18} - h \times \mathbf{0.1}}{1 - hc}, && \text{(15.2.8e)} \\
\boldsymbol{i = 19}: &\quad \boldsymbol{y_{2n}} = y_{20} = \frac{y_{19} - h\boldsymbol{x_{20}}}{1 - hc} &&= \frac{y_{19} - h \times \mathbf{0}}{1 - hc}. && \text{(15.2.8f)}
\end{aligned}
$$

- **Note the indexing and the value of x_i for the reverse integration.**

15.3 Euler integration 3

- You are given the following differential equation

$$\frac{dy}{dx} = 2x\,. \tag{15.3.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 0$.
- The exact solution $y^{\text{ex}}(x)$ is given as follows:

$$y^{\text{ex}}(x) = x^2\,. \tag{15.3.2}$$

- The integration stepsize is h (a constant), hence $x_i = ih$.
- Denote the explicit numerical solution by $y_i^{\text{exp}} = y^{\text{exp}}(x_i)$.
- Denote the implicit numerical solution by $y_i^{\text{imp}} = y^{\text{imp}}(x_i)$.
- **Use the explicit Euler method to derive the following.**

$$y_{i+1}^{\text{exp}} = y_i^{\text{exp}} + 2hx_i\,. \tag{15.3.3}$$

- **Show the following, for $i = 1$:**

$$y_1^{\text{exp}} = 0\,. \tag{15.3.4}$$

- **Use the implicit Euler method to derive the following.**

$$y_{i+1}^{\text{imp}} = y_i^{\text{imp}} + 2hx_{i+1}\,. \tag{15.3.5}$$

- **Show the following, for $i = 1$:**

$$y_1^{\text{imp}} = 2h^2\,. \tag{15.3.6}$$

- Set $h = 1/n$.
- The value of the exact solution is $y^{\text{ex}}(1) = 1$ for $x = 1$.
- **Compute the value of the numerical solutions y_n^{exp} and y_n^{imp} and fill the following table of values.**

n	y_n^{exp}	y_n^{imp}	$n(1 - y_n^{\text{exp}})$	$n(1 - y_n^{\text{imp}})$
10	5 d.p.	5 d.p.	2 d.p.	2 d.p.
100	5 d.p.	5 d.p.	2 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	2 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last two columns should be approximately independent of n.
- This is because Euler integration is a first order method.

15.4 Euler integration 4

- Let us modify the differential equation in eq. (15.3.1).
- We know the exact solution is $y^{\text{ex}}(x) = x^2$ so let us integrate the following differential equation

$$\frac{dy}{dx} = 2\sqrt{y}\,. \tag{15.4.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 0$.
- The exact solution is still $y^{\text{ex}}(x) = x^2$.
- The integration stepsize is h (a constant), hence $x_i = ih$.
- Denote the explicit numerical solution by $y_i^{\text{exp}} = y^{\text{exp}}(x_i)$.
- Denote the implicit numerical solution by $y_i^{\text{imp}} = y^{\text{imp}}(x_i)$.
- **Use the explicit Euler method to derive the following.**

$$y_{i+1}^{\text{exp}} = y_i^{\text{exp}} + 2h\sqrt{y_i^{\text{exp}}}\,. \tag{15.4.2}$$

- **Compute using eq. (15.4.2) with $h = 1/n$ for $n = 10$, 100, 1000 and show that:**

$$y_n^{\text{exp}} = 0 \qquad\qquad (n = 10, 100, 1000)\,. \tag{15.4.3}$$

- *The explicit Euler solution never leaves zero!*

- **Use the implicit Euler method to derive the following.**

$$y_{i+1}^{\text{imp}} = y_i^{\text{imp}} + 2h\sqrt{y_{i+1}^{\text{imp}}}\,. \tag{15.4.4}$$

- **Process eq. (15.4.4) to derive the following.**

$$y_{i+1}^{\text{imp}} - 2h\sqrt{y_{i+1}^{\text{imp}}} = y_i^{\text{imp}}\,. \tag{15.4.5}$$

- We have our first example of a nontrivial implicit equation.
- **Show that $y_i^{\text{imp}} = 0$ (for all i) is a solution of eq. (15.4.5).**
- However, it is not the only solution of eq. (15.4.5). There is another solution.
  1. **I will derive the formula for the integration step for you.**
  2. We process eq. (15.4.5) to obtain the following:

$$\begin{aligned}\left(\sqrt{y_{i+1}^{\text{imp}}} - h\right)^2 &= h^2 + y_i^{\text{imp}} \\ \sqrt{y_{i+1}^{\text{imp}}} - h &= \sqrt{h^2 + y_i^{\text{imp}}} \\ \sqrt{y_{i+1}^{\text{imp}}} &= h + \sqrt{h^2 + y_i^{\text{imp}}}\,.\end{aligned} \tag{15.4.6}$$

- From the above we obtain the following expression:

$$y_{i+1}^{\text{imp}} = \left(h + \sqrt{h^2 + y_i^{\text{imp}}}\right)^2. \tag{15.4.7}$$

- Set $h = 1/n$.
- The value of the exact solution is $y^{\text{ex}}(1) = 1$ for $x = 1$.
- **Compute the value of the numerical solution y_n^{imp} using eq. (15.4.7) and fill the following table of values.**

n	y_n^{imp}	$n(1 - y_n^{\text{imp}})$
10	5 d.p.	2 d.p.
100	5 d.p.	2 d.p.
1000	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately independent of n.
- This is because Euler integration is a first order method.
- This example teahes us various lessons.
  1. We need to pay more attention to the conditons for the existence of a solution of a differential equation.
  2. For an implicit integration algorithm, there can be more than one solution of the implicit equation, and a root finding algorithm might converge to the wrong root.