Queens College, CUNY, Department of Computer Science

Computational Finance

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10 Lecture 10

10.1 Delta one securities

- The Black-Scholes equation and the Black-Scholes-Merton equation have been displayed in class, and of course we are impatient to solve them!
- However, those equations are both based on geometric Brownian motion, and there are still many things to learn, which are *independent* of any probability model for the stock price movements.
- In this lecture we introduce **Delta one securities.**

10.2 Zero strike call

- Let us calculate the fair value of a **call option whoes strike is zero.**
- A zero strike call option is not a bogus thing, as we shall see.
- Let the current time be t and the option expiration time be T, where $T > t$.
- Let the stock price at time t_0 be S_0.
- Let S_T denote the unknown terminal stock price at the expiration time T.

10.2.1 No dividends

- For simplicity we begin with a stock which does not pay dividends.
- **First consider a European call option.**
- Recall that the terminal payoff of a long position in a call option with strike K is given by

$$c(S_T, K) = \max(S_T - K, 0)\,. \tag{10.2.1.1}$$

- For a zero strike call, set $K = 0$ so the terminal payoff is simply

$$c(S_T, T, K = 0) = \max(S_T, 0)\,. \tag{10.2.1.2}$$

- Since the value of the stock price is always ≥ 0, then $S_T \geq 0$ so eq. (10.2.1.2) simplifies to

$$c(S_T, T, K = 0) = S_T\,. \tag{10.2.1.3}$$

- Also because the strike is $K = 0$, a zero strike call will always be in the money at expiration.
- Hence the holder of a zero strike call will always exercise the call at expiration, and **receive one share of stock for free.** (Recall the option holder pays the strike price, which is zero.)
- Hence to avoid arbitrage, the fair value of a zero strike call option, at time t_0, must be

$$c(S_0, t_0, K = 0) = S_0\,. \tag{10.2.1.4}$$

- Actually we can drop the "0" subscript, eq. (10.2.1.4) is obviously true at any time $t \leq T$.

$$c(S, t, K = 0) = S\,. \tag{10.2.1.5}$$

- For a stock which pays no dividends (during the lifetime of the option),

  the fair value of a zero strike call option equals the current price of the stock S.

- One way to prove this is to note that if two portfolios are worth exactly the same **in all scenarios** at a future time T, *then they must be worth the same today.* Else we buy (go long) the cheaper portfolio and sell (go short) the more expensive portfolio today. At the future time T, they cancel out to zero. The money we have in the bank at the time t_0 will compound to yield a guaranteed profit = arbitrage.
- The other arbitrage proof is as follows.
  1. Let $\varepsilon > 0$ be an arbitrarily small positive number.
  2. Suppose the option price at time t is $c_{\text{zero strike}} = S - \varepsilon$. Go long the option (and pay cash $= S - \varepsilon$) and short one share of stock (and receive cash $= S$). Our total cash in the bank is $S - S + \varepsilon = \varepsilon$. At expiration, we exercise the option (because it is always in the money) and receive one share of stock, which we use to close out our short stock position. Our final portfolio is positive cash in the bank, worth $\varepsilon e^{r(T-t)} > 0$.

3. Suppose the option price at time t is $c_{\text{zero strike}} = S+\varepsilon$. Go *short* the option (and receive cash $= S+\varepsilon$) and long one share of stock (and pay cash $= S$). Our total cash in the bank is $S+\varepsilon-S=\varepsilon$. At expiration, the option holder will exercise the option against us. *(Remember we are <u>short</u> the option, so it is the <u>holder</u> who makes the decision to exercise the option.)* We deliver our share of stock to the option holder (and receive nothing) and close out the option. Our final portfolio is positive cash in the bank, worth $\varepsilon e^{r(T-t)} > 0$, hence guaranteed profit.

4. Hence if the option price is not equal to S, we make a guaranteed profit = arbitrage.

- **Note that eq. (10.2.1.4) is also true for an American call option.**
  If we own one share of stock at time t_0, it will always exactly match an American zero strike call option which is exercised at any time $t > t_0$.

10.2.2 Discrete dividends

- Suppose now the stock pays discrete dividends D_i at times t_i, $i = 1, 2, \dots, n$ during the lifetime of the option, where $t < t_1 < \cdots < t_n < T$.
- **First consider a European call option.**
- The terminal payoff of a zero strike call option is still given by eq. (10.2.1.3).
- However, the holder of an option is **not a shareholder of record of the stock.**
- Hence the option holder receives no dividends.
- However, an investor who owns the stock will receive the dividends.
- Hence to avoid arbitrage, the fair value of a zero strike call option at time t equals the

  stock price less the sum of the present values of the dividends paid during the lifetime of the option

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\begin{aligned}
c(S_0, t_0, K = 0) &= S_0 - \mathrm{PV}(D_1) - \mathrm{PV}(D_2) - \dots \\
&= S_0 - \sum_{i=1}^{n} \mathrm{PV}(D_i)\,.
\end{aligned}
\tag{10.2.2.1}
$$

- This is an important fact, as we shall see.
- **The situation is different for an American zero strike call option.**
  1. The holder of an American option can exercise at any time, hence exercise immediately and receive the stock (for free) and then also receive all the dividends.
  2. Hence the fair value of an American zero strike call option is the current stock price:
  $$C(S_0, t_0, K = 0) = S_0\,. \tag{10.2.2.2}$$
  3. A simple way to see this is to note that an American option must always be worth at least the intrinsic value.
  4. The intrinsic value of a zero strike call is $\max\{S - K, 0\} = \max\{S - 0, 0\} = S$.
  5. Hence we must have $C \geq S$ at any time t.
  6. We also cannot have $C > S$ because we would arbitrage by going short the call and long one share of stock, save the net cash $= C - S$ in the bank. If the option is exercised at any future time, we deliver our share of stock to close out the option. Our arbitrage profit is the cash in the bank plus any dividends received by owning the stock.
  7. Hence for an American zero strike call option, we must have $C = S$ at any time t, regardless of dividends.

10.3 Delta one securities

- For a stock S, the value of Delta equals one:

$$\Delta_{\text{stock}} = \frac{\partial S}{\partial S} = 1\,. \tag{10.3.1}$$

- The Delta of a zero strike call also equals one. Using eq. (10.2.2.1),

$$\frac{\partial c(S,t,K=0)}{\partial S} = \frac{\partial S}{\partial S} = 1\,. \tag{10.3.2}$$

- A **Delta one security** is a financial instrument which **behaves like a stock (equity) but *does not pay dividends.***

- It has a Delta of one, hence its name.

  1. In practice, the value of Delta does not have to be exactly 1.
  2. **The value of Delta simply has to be a number which does not depend on the stock price.**

- The fundamental concept characterizing a Delta one security is that it **trades essentially like a stock (equity) and there is negligible volatility in the fair value of a Delta one security.**

- A zero strike European call option is an example of a Delta one security.

- There are various examples of Delta one securities, but as the above analysis shows, a European call with a strike of zero is one of them and is not a bogus thing.

10.4 Forwards & futures

- Consider a forward or futures contract with expiration time T.
- Suppose the stock pays discrete dividends as before.
- Let the interest rate be r.
- The fair value formula for the forward or futures contract is

$$F = \left[S - \sum_{i=1}^{n} e^{-r(t_i - t)} D_i \right] e^{r(T-t)} \,. \tag{10.4.1}$$

- The Delta of the forward or futures contract is

$$\Delta_F = \frac{\partial F}{\partial S} = e^{r(T-t)} \,. \tag{10.4.2}$$

- The value of Delta does not depend on the stock price.
- The holder of a forward or futures contract does not receive dividends.
- **Hence a forward or futures contract is an example of a Delta one security.**
- The value of Delta of a forward or futures contract does not equal 1 exactly, but nevertheless the fundamental concept characterizing a Delta one security is that it trades like a stock (equity) but does not pay dividends. There is little or no volatility in the fair value of a Delta one security.

10.5 Call with negative strike

- Let us calculate the fair value of a call option with a **negative strike price.**
- This is not completely stupid. It is a lesson in financial derivatives pricing theory.
- Let the strike be $-|K|$, where $K < 0$.
- **We begin with a European call option.**
- The terminal payoff of the call option, at time T, is given by

$$c(S_T, T, -|K|) = \max(S_T - K, 0) = \max(S_T + |K|, 0)\,. \tag{10.5.1}$$

- The right hand side is always a positive number, hence the payoff is

$$c(S_T, T, -|K|) = S_T + |K|\,. \tag{10.5.2}$$

- We recognize this as simply a **sum of one share of stock plus cash $|K|$.**
- Hence to avoid arbitrage, the fair value of a negative strike call option at time t is equal to **a zero strike call plus the present value of cash $|K|$**

$$\begin{aligned} c(S_0, t_0, -|K|) &= c(S_0, t_0, K = 0) + \mathrm{PV}(|K|) \\ &= S_0 - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) + \mathrm{PV}(|K|)\,. \end{aligned} \tag{10.5.3}$$

- **The formula is different for an American call option.**
  1. An American call option with a negative strike can be exercised at any time t.
  2. The payoff is worth $S_t + |K|$.
  3. Hence the fair value of an American call option with a negative strike, at any time t, is

  $$C(S, t, -|K|) = S + |K|\,. \tag{10.5.4}$$

  4. It should be easy for you to formulate an arbitrage argument to prove eq. (10.5.4).
- All of the above can be derived ***without reference to a probability model for the stock price movements.***

10.6 Put with non-positive strike

- Let us calculate the fair value of a put option with a strike price ≤ 0.
- One *can* argue that this is stupid, but it is an academic exercise in arbitrage.
- Let the strike be $-|K|$, where $K \leq 0$.
- The terminal payoff of the put option, at time T, is given by

$$p(S_T, T, -|K|) = \max(K - S_T, 0) = \max(-|K| - S_T, 0)\,. \tag{10.6.1}$$

- The value of $-|K| - S_T$ is never positive, hence the payoff is

$$p(S_T, T, -|K|) = 0\,. \tag{10.6.2}$$

- Hence a put option with a non-positive strike will never pay anything to an investor.
- Hence to avoid arbitrage a put with a strike $K \leq 0$ must be worth zero today

$$p(S, t, -|K|) = 0\,. \tag{10.6.3}$$

- The above analysis applies for **both European and American put options.**
- *Why would anyone pay money to buy something that is worth zero?*

10.7 Put call parity

- For a stock which pays discrete dividends during the lifetime of the options, the put-call parity relation is

$$c - p = S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) - \mathrm{PV}(K)\,. \tag{10.7.1}$$

- Consider what happens if K is zero or negative. Set $K = -|K|$, where $K \le 0$.
- Then from eqs. (10.5.3) and (10.6.3),

$$\begin{aligned} c(S,t,-|K|) &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) + \mathrm{PV}(|K|)\,, \\ p(S,t,-|K|) &= 0\,. \end{aligned} \tag{10.7.2}$$

- Subtraction yields

$$\begin{aligned} c(S,t,-|K|) - p(S,t,-|K|) &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) + \mathrm{PV}(|K|) \\ &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) - \mathrm{PV}(K)\,. \end{aligned} \tag{10.7.3}$$

- This is the put-call parity relation eq. (10.7.1).
- Hence put-call parity works even if the strike price is zero or negative.