Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Spring 2018

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due Friday, April 27, 2018, 11.59 pm

27 Homework lecture 27

- **As experience has demonstrated, if you do not understand the above expressions/questions, THEN ASK.**
- **If you do not understand the words/sentences in the lectures, THEN ASK.**
- **Send me an email, explain what you do not understand.**
- **Do not just keep quiet and produce nonsense in exams.**

27.1 Periodic functions

- **For each function below, state if it is periodic or not.**
- **If the function is periodic, state the period of the function.**
- *You do not need to display calculations, just state the period.*
- In all cases, x is a real valued variable and $k > 0$ is a real positive constant.
- A periodic function can be complex.
- A periodic function need not be continuous, differentiable or bounded.

$$f_1 = |\sin(x)| + \cos(2x) \tag{27.1.1}$$

$$f_2 = \sin^2\Big(\frac{k+x}{\sqrt{2}}\Big) \tag{27.1.2}$$

$$f_3 = \tan(x) \tag{27.1.3}$$

$$f_4 = e^{-kx^2} \tag{27.1.4}$$

$$f_5 = \cos(kx^2) \tag{27.1.5}$$

$$f_6 = e^{-i\sin(kx)} \tag{27.1.6}$$

$$f_7 = \cos(\cos(kx)) \tag{27.1.7}$$

$$f_8 = \sin(\sin(kx)) \tag{27.1.8}$$

$$I_9 = \frac{1}{1+\cos(kx)} \tag{27.1.9}$$

$$I_{10} = \cos(x)\,\cos(\sqrt{2}\,x) \tag{27.1.10}$$

$$I_{11} = e^{-kx}\,\sin(x) \tag{27.1.11}$$

27.2 Example: $|\cos(\frac{1}{2}\theta)|$

- Consider the following periodic function:

$$f_c(\theta) = \left|\cos(\tfrac{1}{2}\theta)\right| . \tag{27.2.1}$$

- It is an even function hence all the sine terms in the Fourier series are zero.
- In the interval $-\pi \le \theta \le \pi$, the function is given by $f_c(\theta) = \cos(\frac{1}{2}\theta)$.
- The cosine Fourier coefficients are derived as follows.

$$\begin{aligned}
a_j &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(\tfrac{1}{2}\theta)\, \cos(j\theta)\, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big[\cos((j+\tfrac{1}{2})\theta) + \cos((j-\tfrac{1}{2})\theta) \Big]\, d\theta \\
&= \frac{1}{2\pi} \left[\frac{\sin((j+\frac{1}{2})\theta)}{j+\frac{1}{2}} + \frac{\sin((j-\frac{1}{2})\theta)}{j-\frac{1}{2}} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[\frac{\sin((j+\frac{1}{2})\pi)}{j+\frac{1}{2}} + \frac{\sin((j-\frac{1}{2})\pi)}{j-\frac{1}{2}} \right] \\
&= \frac{1}{\pi} \left[\frac{\cos(j\pi)}{j+\frac{1}{2}} - \frac{\cos(j\pi)}{j-\frac{1}{2}} \right] \\
&= \frac{1}{\pi}\, \frac{(-1)^j}{\frac{1}{4} - j^2} \,.
\end{aligned} \tag{27.2.2}$$

- Hence the Fourier series is

$$f_{\mathrm{series}}(\theta) = \frac{2}{\pi} + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{\frac{1}{4} - j^2} \cos(j\theta) \,. \tag{27.2.3}$$

- The Dirichlet partial sums are $S_0 = 2/\pi$ and

$$S_n(\theta) = \frac{2}{\pi} + \frac{1}{\pi} \sum_{j=1}^{n} \frac{(-1)^j}{\frac{1}{4} - j^2} \cos(j\theta) \qquad\qquad (n \ge 1) \,. \tag{27.2.4}$$

- The Fejér partial sums are

$$D_n(\theta) = \frac{1}{n+1} \sum_{j=0}^{n} S_n(\theta) \,. \tag{27.2.5}$$

- **Compute the function value $f_c(\theta)$, the Dirichlet partial sum $S_3(\theta)$ and the Fejér partial sum $D_6(\theta)$ and fill in the table below.**

θ	$f_c(\theta)$	$S_3(\theta)$	$D_6(\theta)$
$-\pi$	3 d.p.	3 d.p.	3 d.p.
$-\frac{3}{4}\pi$	3 d.p.	3 d.p.	3 d.p.
$-\frac{1}{2}\pi$	3 d.p.	3 d.p.	3 d.p.
$-\frac{1}{4}\pi$	3 d.p.	3 d.p.	3 d.p.
0	3 d.p.	3 d.p.	3 d.p.
$\frac{1}{4}\pi$	3 d.p.	3 d.p.	3 d.p.
$\frac{1}{2}\pi$	3 d.p.	3 d.p.	3 d.p.
$\frac{3}{4}\pi$	3 d.p.	3 d.p.	3 d.p.
π	3 d.p.	3 d.p.	3 d.p.

- **Plot a graph for $\underline{-3\pi \leq \theta \leq 3\pi}$ of**
  (i) function $f_c(\theta)$,
  (ii) Dirichlet partial sum $S_3(\theta)$,
  (iii) the Fejér partial sum $D_6(\theta)$.
- On the vertical axis, go from 0 to 1.2.
- On the horizontal axis, plot the value of θ/π, so the values go from **-3 to 3.**
- *Note the range: we display* <u>*three full cycles*</u> *to see the repeating pattern more clearly.*

27.3 Program

- We can calculate the Fourier coefficients a_j and b_j analytically only for simple cases.
- We require a program for more complicated (or more general) functions $f(\theta)$.
- The algorithm is to sum the function at n equally spaced points around the circle.

$$
\begin{aligned}
a_j &= \frac{2}{n} \sum_{i=0}^{n-1} f(\theta_i) \cos(j\theta_i)\,, \\
b_j &= \frac{2}{n} \sum_{i=0}^{n-1} f(\theta_i) \sin(j\theta_i) \qquad\qquad \left(\theta_i = \frac{2\pi i}{n}\right).
\end{aligned}
\tag{27.3.1}
$$

- Note that the above 'equalities' are actually numerical approximations for a_j and b_j.
- **For simplicity we treat real functions only, hence a_j and b_j are of type `double`.**
- **We impose a tolerance of 10^{-12} in this homework assignment.**

```
double tol = 1.0e-12;
if (std::abs(aj) < tol) aj = 0;
if (std::abs(bj) < tol) bj = 0;
```

- The cutoff using `tol` is just to avoid messy outputs for numbers which should be exactly zero.
- **Write loops to compute the sums in eq. (27.3.1).**
  1. There are two nested loops, an outer loop for j and an inner loop over i for the numerical integral over θ.
  2. `(for j=0; j < n/2; ++j);`
  3. `(for i=0; i < n; ++i);`
  4. Hence this is an $O(n^2)$ algorithm.
  5. Later we shall learn an $O(n \log_2 n)$ algorithm.
  6. Recall that using n points, we can obtain accurate results for at most n outputs.
  7. Hence we stop the outer loop using $n/2$ not n, because we compute two numbers a_j and b_j for each value of j.
- Note that I have written `aj` and `bj` as variables of type `double`.
- It is your responsibility to save the values of `aj` and `bj` in a suitable form, for example arrays or `std::vector<double>`, to save the information in a suitable form to answer the questions in this homework assignment.

27.4 Window and triangle functions

- Let us analyze the window and triangle functions.
- **Set** $\theta_0 = \sqrt{0.005}\,\pi$ **for use below.**
- **Set** $n = 1024$ **for the calculations in this question.**

27.4.1 Window function

- We begin with the window function.
- The lectures employed the interval $-\pi \le \theta \le \pi$, but our program sums over values $0 \le \theta \le 2\pi$.
- Hence we define the window function as follows:

$$f_{\rm win}(\theta) = \begin{cases} 1/(2\theta_0) & (0 \le \theta < \theta_0) \\ 1/(2\theta_0) & (2\pi - \theta_0 < \theta \le 2\pi) \\ 0 & (\theta_0 \le \theta \le 2\pi - \theta_0)\,. \end{cases} \tag{27.4.1}$$

- **Calculate the values of a_j and b_j for the Fourier series of $f_{\rm win}(\theta)$.**
- If you have done your work correctly, all the values of b_j will be zero.
- If you have done your work correctly, you should obtain

$$a_j = \frac{1}{\pi}\,\frac{\sin(j\theta_0)}{(j\theta_0)} \qquad (= 1/\pi \text{ for } j = 0)\,. \tag{27.4.2}$$

- **Those of you who have been paying attention to the class material will realize the above statement is not completely correct.**
  1. We are computing the integral by evaluating the function at a finite number of points.
  2. Hence the computed value of a_j will be only approximately equal to the theoretical value.
  3. The quality of the approximation depends on the number of sampling points n.
- **Multiply by π, calculate πa_j and fill the table of values below for $j = 0, 10, 20, 30, 40, 50, 60$.**

j	πa_j	$\sin(j\theta_0)/(j\theta_0)$	$\lvert \pi a_j - \sin(j\theta_0)/(j\theta_0) \rvert$
0	4 d.p.	4 d.p.	4 d.p.
10	4 d.p.	4 d.p.	4 d.p.
20	4 d.p.	4 d.p.	4 d.p.
30	4 d.p.	4 d.p.	4 d.p.
40	4 d.p.	4 d.p.	4 d.p.
50	4 d.p.	4 d.p.	4 d.p.
60	4 d.p.	4 d.p.	4 d.p.

- **Plot a graph of the values of πa_j for $0 \le j \le 100$. Also plot a graph of $\sin(j\theta_0)/(j\theta_0)$.**
  1. On the vertical axis, go from -0.25 to 1.25.
  2. You should observe a good but not exact match with the sinc function.

27.4.2 Triangle function

- Next let us study the triangle function.
- The lectures employed the interval $-\pi \le \theta \le \pi$, but our program sums over values $0 \le \theta \le 2\pi$.
- Hence we define the triangle function as follows:

$$f_{\rm tri}(\theta) = \begin{cases} \dfrac{1}{2\theta_0}\Bigl(1 - \dfrac{\theta}{2\theta_0}\Bigr) & (0 \le \theta < 2\theta_0) \\ \dfrac{1}{2\theta_0}\Bigl(1 - \dfrac{2\pi-\theta}{2\theta_0}\Bigr) & (2\pi - 2\theta_0 < \theta \le 2\pi) \\ 0 & (2\theta_0 \le \theta \le 2\pi - 2\theta_0)\,. \end{cases} \tag{27.4.3}$$

- **Calculate the values of a_j and b_j for the Fourier series of $f_{\rm tri}(\theta)$.**
- If you have done your work correctly, all the values of b_j will be zero.
- If you have done your work correctly, you should obtain

$$a_j = \frac{1}{\pi}\,\frac{\sin^2(j\theta_0)}{(j\theta_0)^2} \qquad (= 1/\pi \text{ for } j = 0)\,. \tag{27.4.4}$$

- As with the window function, this is only an approximate result.
- The quality of the approximation depends on the number of sampling points n.
- **Multiply by π, calculate πa_j and fill the table of values below for $j = 0, 5, 10, 15, 20$.**

j	πa_j	$\sin^2(j\theta_0)/(j\theta_0)^2$	$\lvert\pi a_j - \sin^2(j\theta_0)/(j\theta_0)^2\rvert$
0	4 d.p.	4 d.p.	4 d.p.
5	4 d.p.	4 d.p.	4 d.p.
10	4 d.p.	4 d.p.	4 d.p.
15	4 d.p.	4 d.p.	4 d.p.
20	4 d.p.	4 d.p.	4 d.p.

- **Plot a graph of the values of πa_j for $0 \le j \le 100$. Also plot a graph of $\sin^2(j\theta_0)/(j\theta_0)^2$.**
  1. On the vertical axis, go from 0 to 1.25.
  2. You should observe that the values rapidly become small.

27.4.3 Graphs

- There is a relation between the Fourier coefficients of the window and triangle functions.
- The triangle function is the convolution of the window function with itself.
- Hence the Fourier coefficients for the triangle function are the square of the Fourier coefficients for the window function (up to factors of π).
- Define the following variables

$$
\begin{aligned}
w_j &= \pi, |a_j(\mathrm{win})| && (\text{window function})\,, \\
t_j &= \sqrt{\pi a_j(\mathrm{tri})} && (\text{triangle function})\,.
\end{aligned}
\tag{27.4.5}
$$

- **Plot a graph of the values of w_j and t_j for $0 \le j \le 60$.**
- You should obtain the graph displayed in Fig. 1.
- The circles plot the value of $w_j = \pi|a_j|$ for the window function.
- The triangles plot the value of $t_j = \sqrt{\pi a_j}$ for the triangle function.

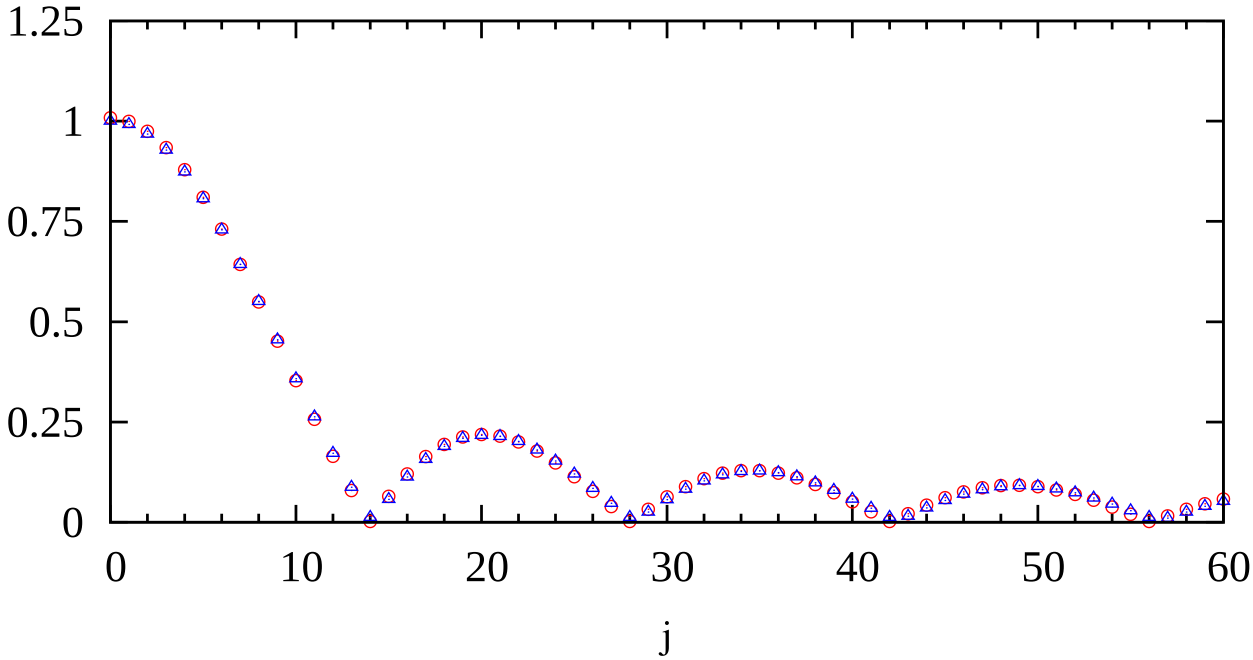

Figure 1: Graph of the scaled Fourier coefficients for the window and triangle functions. The circles plot the value of $\pi|a_j|$ for the window function and the triangles plot the value of $\sqrt{\pi a_j}$ for the triangle function, for Question 27.4.

27.5 Superperiodicity

- Let us study the phenomenon of **superperiodicity.**
- Superperiodicity means the function (θ) has a period which is *submultiple of* 2π.
- If the period is $2\pi/P$ we say the superperiodicity is P.
- There are also concepts of strong and weak superperiodicities.
- The **Alternating Gradient Synchrotron (AGS)** *(yes, it's a particle accelerator)* at Brookhaven National Laboratory is a circular machine with a strong superperiodicity of 12 and a weak superperiodicity of 60.
- We shall study some examples.
- **Set $n = 4800$ for all the examples in this question.**
  1. This is to avoid inaccurate solution submissions with too few points $n = 10$, etc.
  2. It is also because our examples below will have symmetries of 12 and 60, etc.
  3. Hence we want n to be a multiple of 60 and a large number.

Example 1

- The following function has a superperiodicity of 12:

$$f_{12}(\theta) = \begin{cases} 1 & (|\cos(6\theta)| > 0.99) \\ 0 & \text{otherwise}\,. \end{cases} \tag{27.5.1}$$

- **Calculate the values of a_j and b_j for the Fourier series of $f_{12}(\theta)$.**
- If you have done your work correctly, all the values of b_j will be zero.
- **Explain why we must have $b_j = 0$ for this example.**
- If you have done your work correctly, you will find $a_j = 0$ *unless j is a multiple of 12.*
- This is the superperiocicity.
- **Those of you who have been paying attention to the class material will realize my claim above is not completely correct.**
  1. We are computing the integral by evaluating the function at a finite number of points.
  2. Because the function $f_{12}(\theta)$ has been constructed to have a superperiodicity of 12, we need n to be a multiple of 12 to match the symmetry of $f_{12}(\theta)$.
  3. It would be more natural to think of choosing n to be a power of 2, such as maybe 4096.
  4. However, 12 does not divide 4096, and then we would obtain nonzero (but small amplitude) values for a_j when j is not a multiple of 12.
  5. The reason to set n to a large multiple of 12 (such as 4800), is to compute the integral to good numerical accuracy.
- **Tabulate the nonzero values of a_j for $0 \le j \le 60$.**

j	a_j
0	(nonzero, 4 d.p.)
$\vdots$	$\vdots$
60	(nonzero, 4 d.p.)

Example 2

- This is a similar exercise to the previous example.
- The following function has a superperiodicity of 60:

$$f_{60}(\theta) = \begin{cases} 100 & (|\cos(30\,\theta)| > 0.99) \\ 0 & \text{otherwise}\,. \end{cases} \tag{27.5.2}$$

- Note the constant of 100, to make the magnitudes of the Fourier coefficients larger.
- **Calculate the values of a_j and b_j for the Fourier series of $f_{60}(\theta)$.**
- If you have done your work correctly, all the values of b_j will be zero.
- **Explain why we must have $b_j = 0$ for this example.**
- If you have done your work correctly, you will find $a_j = 0$ *unless j is a multiple of 60.*
- This is the superperiocicity.
- **As with $f_{12}(\theta)$, the above statement is true only because n is a multiple of 60.** The number of points at which the function is evaluated matches the symmetry of the function.
- **Tabulate the nonzero values of a_j for $0 \le j \le 300$.**

j	a_j
0	(nonzero, 4 d.p.)
$\vdots$	$\vdots$
300	(nonzero, 4 d.p.)

Example 3

- This example displays strong and weak superperiodicity.

$$f_{\mathrm{sw}}(\theta) = f_{60}(\theta)\,\bigl(1 + f_{12}(\theta)\bigr)\,. \tag{27.5.3}$$

- The exact period is $2\pi/12$. This is the strong superperiodicity:

$$f_{\mathrm{sw}}(\theta + 2\pi/12) = f_{\mathrm{sw}}(\theta)\,. \tag{27.5.4}$$

- However the function is *approximately* periodic with a period $2\pi/60$:

$$f_{\mathrm{sw}}(\theta + 2\pi/60) \simeq f_{\mathrm{sw}}(\theta)\,. \tag{27.5.5}$$

- This is the weak superperiodicity: it is not exact, but approximate.
- **Calculate the values of a_j and b_j for the Fourier series of $f_{\mathrm{sw}}(\theta)$.**
- If you have done your work correctly, all the values of b_j will be zero.
- **Explain why we must have $b_j = 0$ for this example.**
- If you have done your work correctly, you will notice a pattern for the values of the a_j.
  1. You should find that $a_j = 0$ unless j is a multiple of 12.
  2. You should *also* find that the amplitudes of the harmonics where j is a multiple of 60 are (much?) larger that those for which j is a multiple of 12 but not 60.
  3. This is the strong and weak superperiocicity.
- **Tabulate the nonzero values of a_j for $0 \le j \le 120$.**

j	a_j
0	(nonzero, 4 d.p.)
$\vdots$	$\vdots$
120	(nonzero, 4 d.p.)

- **Plot a graph of the nonzero values of a_j for $0 \le j < 2400$.**
- If you have done your work correctly, you should obtain the graph plotted in Fig. 2.
- The graph consists essentially of two sinc functions, of different widths.
- There is a reason for this, which there may or may not be time to explain in class.
    1. The functions in this example consist of many (=60) repeating window functions, with a 'strong' (large amplitude) window repeated 12 times.
    2. Hence overall the Fourier series is a sinc function.
    3. The strong superperiodicity of 12 means only terms where j is a multiple of 12 are nonzero.
    4. The weak superperiodicity of 60 means that terms where j is a multiple of 60 have large amplitudes.
- Set $\theta_0 = \pi/690$.
- The triangle and circle data are approximately fitted by the functions

$$\begin{aligned} a_j(\mathrm{tri}) &\simeq 21\,\frac{\sin(j\theta_0)}{j\theta_0}\,, \\ a_j(\mathrm{circ}) &\simeq 3.5\,\frac{\sin(j\theta_0)}{j\theta_0}\,. \end{aligned} \tag{27.5.6}$$

- These are approximate fits.
- Logically I should obtain θ_0 by solving $\cos(30\theta_0) = 0.99$. This yields

$$\theta_0 = \frac{\arccos(0.99)}{30} \simeq \frac{\pi}{665.8763}\,. \tag{27.5.7}$$

- However, because of the product by $f_{60}(\theta)(1 + f_{12}(\theta))$ and the artifacts of sampling using $n = 4800$ points, a better fit to the data is obtained by using $\theta_0 = \pi/690$.
- ***You should check for yourself.***

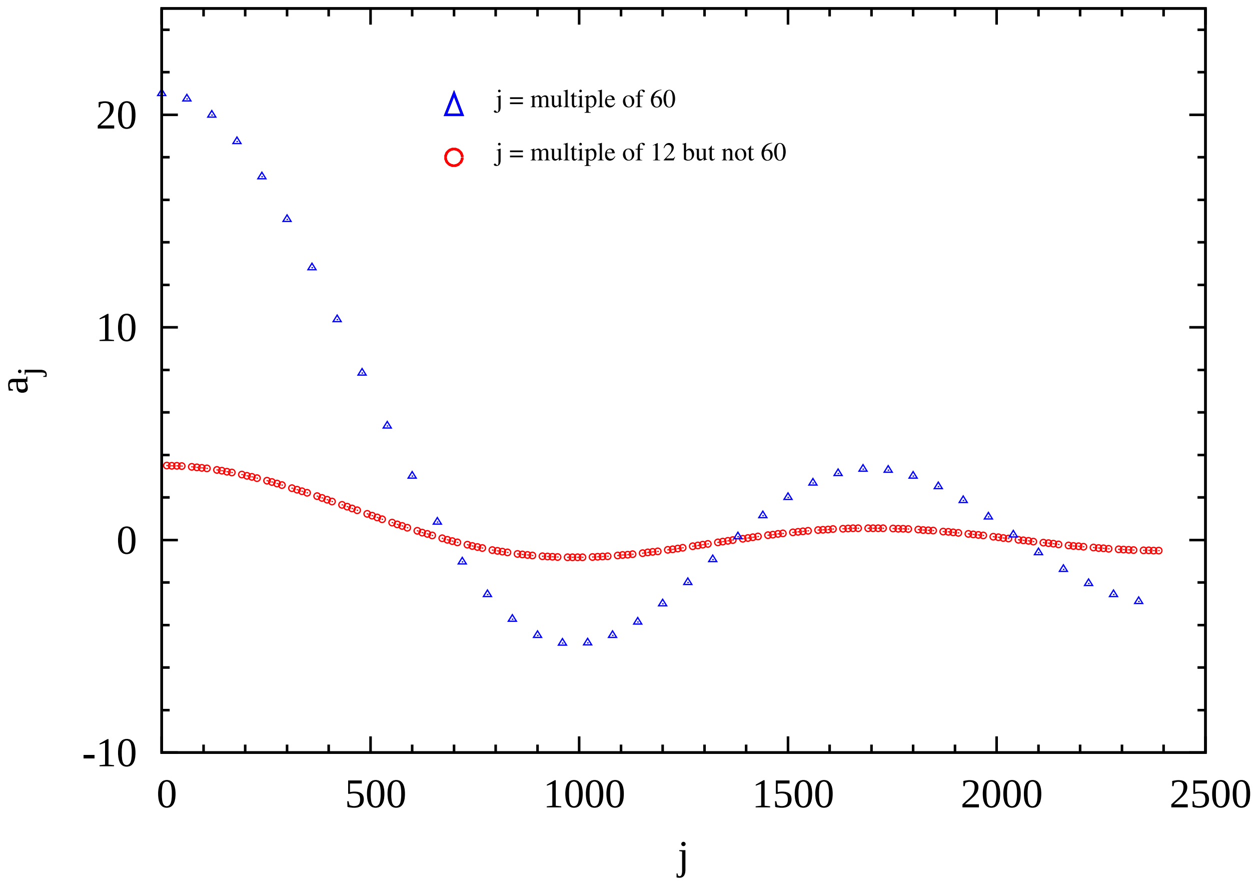

Figure 2: Plot of Fourier coefficients a_j exhibiting strong and weak superperiodicity (12 and 60, respectively), for Question 27.5, Example 3.

27.6 Example: asymmetric function

- Consider the following asymmetric function:

$$f(\theta) = \frac{1}{1 - \frac{1}{2}\sin(\theta)} . \qquad (27.6.1)$$

- A graph of the function is plotted in Fig. 3.
- **Set $n = 256$ in this question.**
- **Calculate the values of a_j and b_j for the Fourier series of $f_{\rm sw}(\theta)$.**
- Because the function is asymmetric, both the a_j and b_j coefficients will be nonzero.
- Nevertheless, there is a pattern.
- *If you have done your work correctly, you should find that:*
  (i) all the a_j for odd j are zero, (ii) all the b_j for even j are zero.
- If you have done your work correctly, you should also notice another fact about the values of a_j and b_j.
  1. The magnitudes $|a_j|$ and $|b_j|$ decrease rapidly as j increases,
  2. For $j > 10$, you should obtain $|a_j| < 10^{-6}$ and $|b_j| < 10^{-6}$.
- **Tabulate the nonzero values of a_j and b_j for $0 \le j \le 20$.**
  Use the tolerance, so if $|a_j| < 10^{-12}$ set $a_j = 0$ and if $|b_j| < 10^{-12}$ set $b_j = 0$.

j	a_j	b_j
0	(nonzero, 4 d.p.)	(nonzero, 4 d.p.)
$\vdots$	$\vdots$	$\vdots$
20	(nonzero, 4 d.p.)	(nonzero, 4 d.p.)

- What this means is, even though the values of a_j and b_j and nonzero as $j \to \infty$, in practice we can obtain a very good approximation of the function using only a few Fourier harmonics.
- **Define the following Dirichlet partial sum up to $j = 10$ only:**

$$S_{10}(\theta) = \tfrac{1}{2}a_0 + \sum_{j=1}^{10} \big[\, a_j\cos(j\theta) + b_j\sin(j\theta) \,\big] . \qquad (27.6.2)$$

- **Calculate the maximum value of $|f(\theta_i) - S_{10}(\theta_i)|$ for $\theta_i = 2\pi i/n$ for $i = 0, \dots, n-1$.**
- If you have done your work correctly, you should obtain $\max\{|f(\theta_i) - S_{10}(\theta_i)|\} < 2 \times 10^{-6}$.
- This is an important fact in the digital sampling of signals in many practical applications.
- The signal is sampled only up to a finite maximum frequency, but the additional (unsampled) data is negligible.

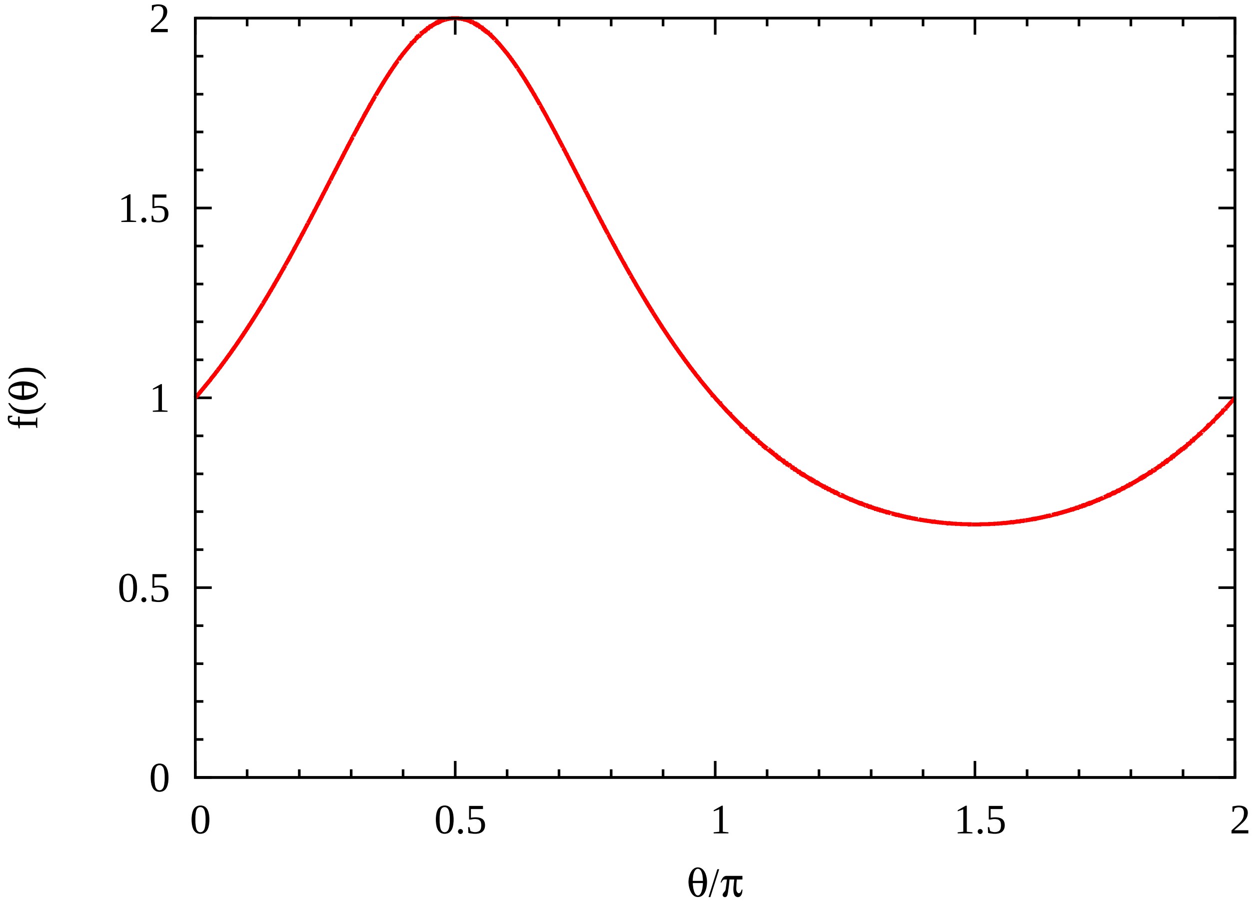

Figure 3: Plot of the function $f(\theta) = 1/(1 - \frac{1}{2}\sin(\theta))$ for $0 \le \theta/\pi \le 2$, for Question 27.6.