Queens College, CUNY, Department of Computer Science

**Computational Finance**
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Instructor: Dr. Sateesh Mane



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# 19 Lecture 19

## Exotic options

- The term **exotic options** refers loosely to a wide class of options.
- We shall study a few selected cases in this lecture.
- **There is no explicit mathematical probability theory in this lecture.**

## 19.1 Binary options

### 19.1.1 Definition

- Binary options are also known as **digital options**.
- A binary option pays a fixed amount \$1 if it is in the money when exercised and zero otherwise.
- The terminal payoff formulas for European binary call and put options are as follows:

$$c_{\text{bin}}(S_T, T) = \begin{cases} 1 & S_T \geq K \,, \\ 0 & S_T < K \,. \end{cases} \tag{19.1.1a}$$

$$p_{\text{bin}}(S_T, T) = \begin{cases} 1 & S_T < K \,, \\ 0 & S_T \geq K \,. \end{cases} \tag{19.1.1b}$$

- By convention, if $S_T = K$, a binary call option pays \$1 and a binary put option pays zero.
- The Black–Scholes–Merton formula for the fair values of European binary calls and puts are:

$$c_{\text{bin}}(S,t) = e^{-r(T-t)} N(d_2) \,, \tag{19.1.2a}$$

$$p_{\text{bin}}(S,t) = e^{-r(T-t)} N(-d_2) \,, \tag{19.1.2b}$$

$$d_2 = \frac{\ln(S/K) + (r-q)(T-t)}{\sigma\sqrt{T-t}} - \tfrac{1}{2}\sigma\sqrt{T-t} \,. \tag{19.1.2c}$$

- European binary or digital options are popular for speculating on the direction of a stock price movement. **They can have expirations as short as a few minutes, for example five minutes.**
- American binary options are less common but also exist. Their payoff formulas on exercise are as follows:

$$C_{\text{bin}}(S, t) = \begin{cases} 1 & S \geq K \,, \\ 0 & S < K \,. \end{cases} \tag{19.1.3a}$$

$$P_{\text{bin}}(S, t) = \begin{cases} 1 & S < K \,, \\ 0 & S \geq K \,. \end{cases} \tag{19.1.3b}$$

  1. There exist closed form formulas (using the Black–Scholes–Merton equation) for the fair values of American binary calls and puts, but they are more complicated.
  2. Note that if the stock price is $\geq K$ at any time, an American binary call should be exercised immediately.
  3. This is because the American binary call will never pay more than \$1 and the stock price could go out of the money at a later time.
  4. Conversely, if the stock price is $< K$ at any time, an American binary put should be exercised immediately.
  5. This is because the American binary put will never pay more than \$1 and the stock price could go out of the money at a later time.

### 19.1.2 Binary options: example

- Recall the worked example of European options in Lecture 17a.
- Let us use the same inputs to value European binary options.
- The input parameters were

$$K = 100\,, \quad r = 0.1\,, \quad q_{\rm div} = 0\,, \quad \sigma = 0.5\,, \quad T = 0.3\,, \quad t_0 = 0\,. \tag{19.1.4}$$

- Graphs of the fair values of a European binary calls and puts are shown in Figs. 1 and 2, respectively. The blue curves show the valuation using a binomial model with 100 timesteps, and the red curves show the values using eqs. (19.1.2a) and (19.1.2b), respectively.
- Notice that the red curves are smooth but the valuation using a binomial model (blue curves) are choppy.
- To illustrate in more detail, let us value the European binary call using $S_0 = K - 0.01 = 99.99$ and $S_0 = K + 0.01 = 100.01$.
- To highlight the limitations of the binomial model, we employ only two timesteps $n = 2$.
- The valuation tree for the case $S_0 = K - 0.01 = 99.99$ is shown in Fig. 3.
  1. At expiration $i = 2$, **the terminal payoff at the central node is zero** because the option is **out of the money** at that node.
  2. Working backwards through the tree, the option fair value is

$$c_{\rm bin}(S_0 = K - 0.01) \simeq 0.233\,. \tag{19.1.5}$$

- The valuation tree for the case $S_0 = K + 0.01 = 100.01$ is shown in Fig. 4.
  1. At expiration $i = 2$, **the terminal payoff at the central node is 1** because the option is **in the money** at that node.
  2. Working backwards through the tree, the option fair value is

$$c_{\rm bin}(S_0 = K + 0.01) \simeq 0.719\,. \tag{19.1.6}$$

- A small change in the location of the central node at the expiration time causes an enormous change to the valuation.
- The binomial model has no flexibility to always place nodes at fixed locations, for example at and close to the strike price, independent of the value of the stock price $S_0$.
- The binomial model also has no flexibility to space the nodes closely together in regions where the terminal payoff changes rapidly (or discontinuously). To decrease the spacing of the nodes near the strike price, for example, we must decrease the spacing *everywhere*.
- This leads to the option valuation behavior exhibited in Figs. 1 and 2.

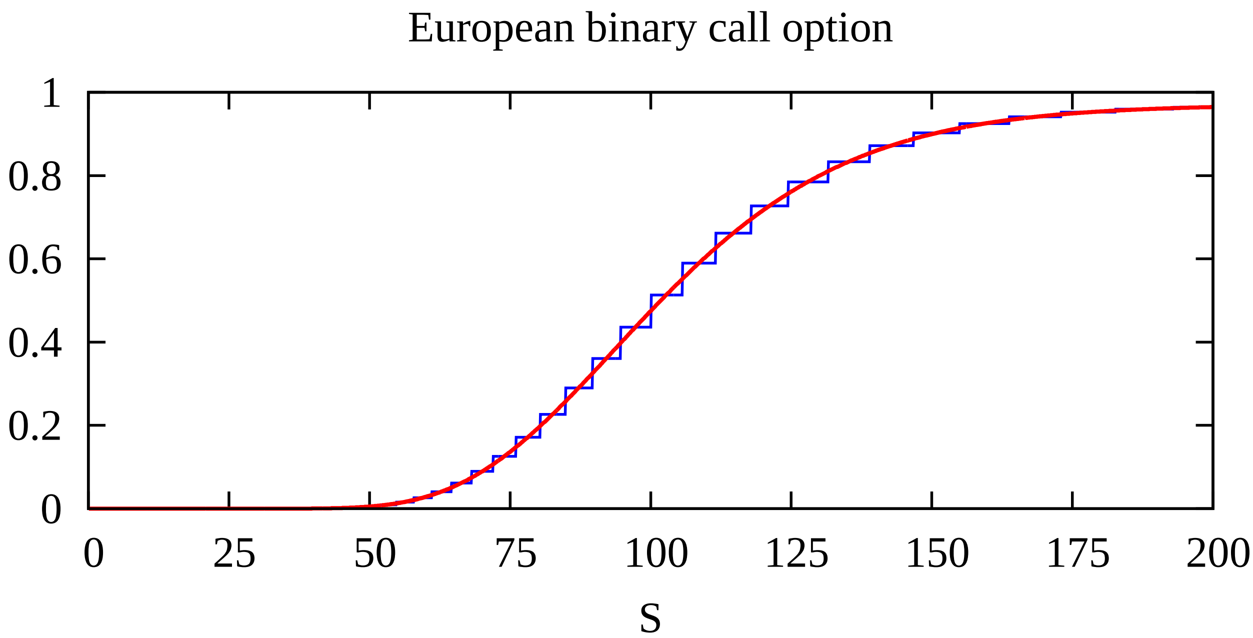


Figure 1: Fair value of European binary call option using a binomial tree with 100 timesteps (blue) and the solution of the Black–Scholes–Merton equation (smooth curve, red), using the input values in eq. (19.1.4).

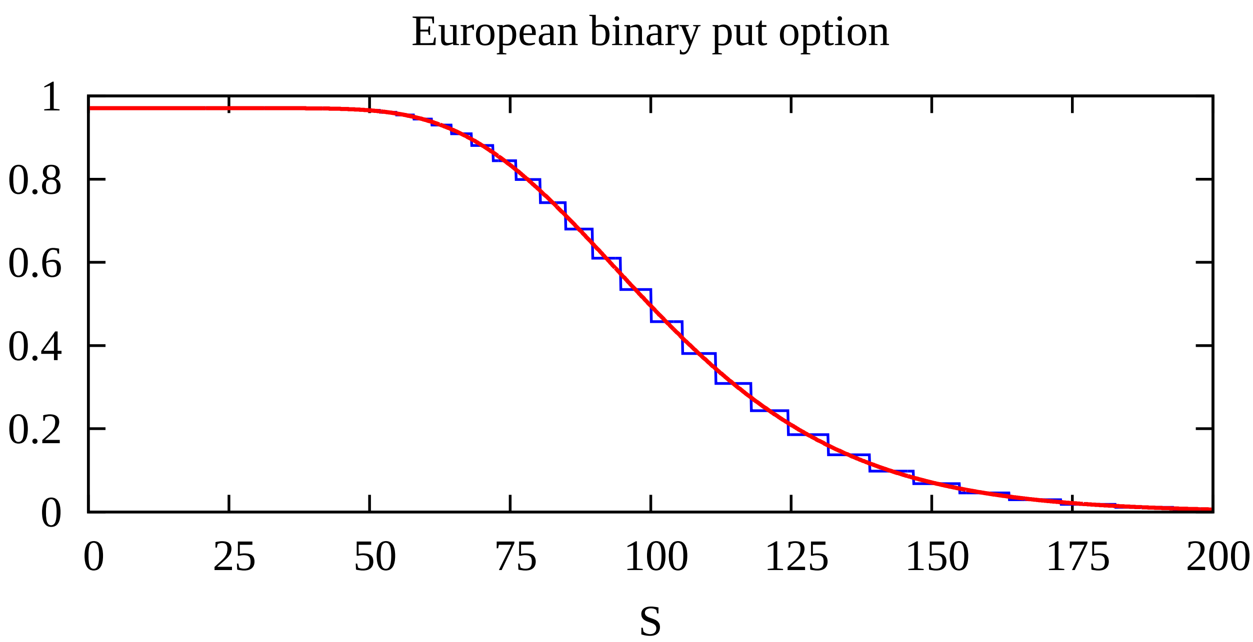


Figure 2: Fair value of European binary put option using a binomial tree with 100 timesteps (blue) and the solution of the Black–Scholes–Merton equation (smooth curve, red), using the input values in eq. (19.1.4).

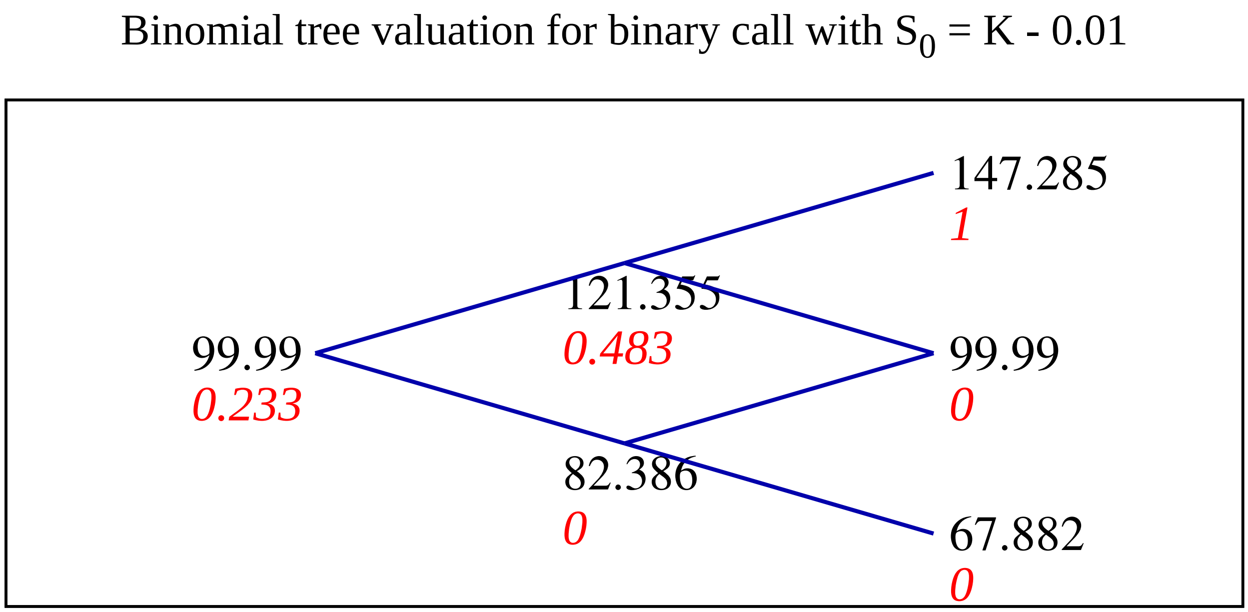


Figure 3: Valuation of European binary call option using a binomial tree with 2 timesteps. The stock price is slighly lower than the strike price. The input values are given in eq. (19.1.4).

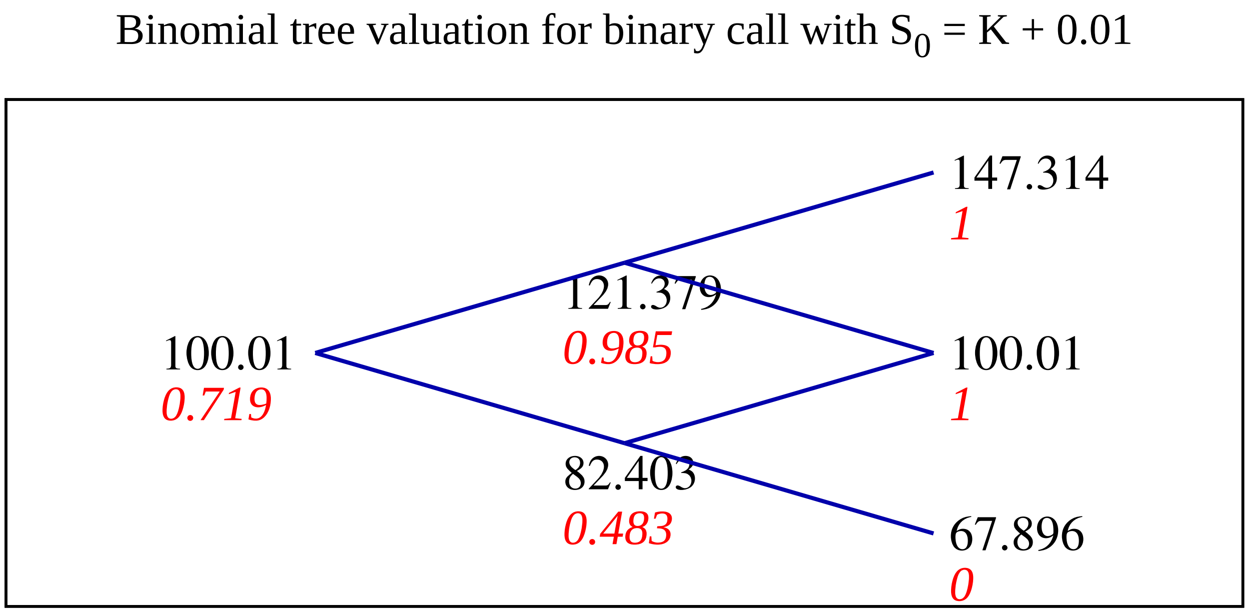


Figure 4: Valuation of European binary call option using a binomial tree with 2 timesteps. The stock price is slighly higher than the strike price. The input values are given in eq. (19.1.4).

### 19.1.3 Binary options: put–call parity relations

- Note that a European binary call plus a European binary put pays exactly \$1 at expiration, for all values of the final stocl price $S_T$.

- Hence a European binary call plus a European binary put must be worth PV(\$1) today.

- **Hence the put–call parity relation for European binary puts and calls is**

$$c_{\rm bin} + p_{\rm bin} = {\rm PV}(1) = e^{-r(T-t)}\,. \tag{19.1.7}$$

  1. Note that the right hand side of eq. (19.1.7) makes no reference to the stock price.
  2. **Hence eq. (19.1.7) is valid even if the underlying stock pays dividends.**

- As was the case with ordinary options, the put–call parity relation is an identity.

- Hence we can deduce several relations for the Greeks from eq. (19.1.7).

- Differentiating eq. (19.1.7) partially with respect to $S$ yields:

$$\Delta_{c_{\rm bin}} + \Delta_{p_{\rm bin}} = 0\,. \tag{19.1.8}$$

  **The Delta of a European binary put is negative, and the opposite of the Delta of the corresponding European binary call.**

- Differentiating eq. (19.1.7) twice partially with respect to $S$ yields:

$$\Gamma_{c_{\rm bin}} + \Gamma_{p_{\rm bin}} = 0\,. \tag{19.1.9}$$

  **Unlike ordinary options, the Gamma of a European binary put and call cannot both be positive.**

- Differentiating eq. (19.1.7) partially with respect to $\sigma$ yields:

$$\nu_{c_{\rm bin}} + \nu_{c_{\rm bin}} = 0\,. \tag{19.1.10}$$

  **The Vega of a European binary put and call cannot both be positive.**

- Differentiating eq. (19.1.7) partially with respect to $r$ yields:

$$\rho_{c_{\rm bin}} + \rho_{p_{\rm bin}} = -(T-t)e^{-r(T-t)}\,. \tag{19.1.11}$$

- Differentiating eq. (19.1.7) partially with respect to $t$ yields:

$$\Theta_{c_{\rm bin}} + \Theta_{p_{\rm bin}} = re^{-r(T-t)}\,. \tag{19.1.12}$$

- An American binary call plus an American binary put will always pay \$1 if one or the other is exercised at any time. Hence the inequality for American binary puts and calls is

$$C_{\rm bin} + P_{\rm bin} \ge 1\,. \tag{19.1.13}$$

## 19.2 Barrier options

### 19.2.1 Definition

- For a **barrier option,** in addition to the strike price $K$, there is also an additional threshold called the **barrier** $B$. It is simplest to explain by listing the various cases.

- **Up and out** barrier call option.

  1. The barrier level $B$ is higher than the strike price $K$, i.e. $B > K$. The option terminates immediately if $S \geq B$ at any time on or before expiration.
  2. The above is an example of a **knockout** barrier option. The option terminates ("knocks out") if the stock price hits the barrier.
  3. Obviously we must have $S_0 < B$ at $t = t_0$, else the option is dead at $t_0$.

- **Up and in** barrier call option.

  1. The barrier level $B$ is also higher than the strike price $K$, i.e. $B > K$. However, the option will pay *zero* **unless the stock price hits the barrier level at any time on or before expiration.**
  2. This means that we require $S_t \geq B$ for some value $t_0 \leq t \leq T$, else the option will pay zero, even if it is in the money at the expiration time $T$.
  3. The above is an example of a **knockin** barrier option. The option only comes to life ("knocks in") if the stock price hits the barrier.
  4. Obviously we must have $S_0 < B$ at $t = t_0$, else this is just a regular call option.

- **Down and out** barrier call option.

  1. The barrier level $B$ is lower than the strike price $K$, i.e. $B > K$. The option terminates immediately if $S \leq B$ at any time on or before expiration.
  2. The above is an example of a **knockout** barrier option. The option terminates ("knocks out") if the stock price hits the barrier.
  3. Obviously we must have $S_0 > B$ at $t = t_0$, else the option is dead at $t_0$.

- **Down and in** barrier call option.

  1. The barrier level $B$ is also lower than the strike price $K$, i.e. $B < K$. However, the option will pay *zero* **unless the stock price hits the barrier level at any time on or before expiration.**
  2. This means that we require $S_t \leq B$ for some value $t_0 \leq t \leq T$, else the option will pay zero, even if it is in the money at the expiration time $T$.
  3. The above is an example of a **knockin** barrier option. The option only comes to life ("knocks in") if the stock price hits the barrier.
  4. Obviously we must have $S_0 > B$ at $t = t_0$, else this is just a regular call option.

- The sum of an (up and out call) and (up and in call) equals a European call option.

- The sum of a (down and out call) and (down and in call) equals a European call option.

- **Up and out** barrier put option.
  1. The barrier level $B$ is higher than the strike price $K$, i.e. $B > K$. The option terminates immediately if $S \geq B$ at any time on or before expiration.
  2. The above is an example of a **knockout** barrier option. The option terminates ("knocks out") if the stock price hits the barrier.
  3. Obviously we must have $S_0 < B$ at $t = t_0$, else the option is dead at $t_0$.
- **Up and in** barrier put option.
  1. The barrier level $B$ is also higher than the strike price $K$, i.e. $B > K$. However, the option will pay *zero* **unless the stock price hits the barrier level at any time on or before expiration.**
  2. This means that we require $S_t \geq B$ for some value $t_0 \leq t \leq T$, else the option will pay zero, even if it is in the money at the expiration time $T$.
  3. The above is an example of a **knockin** barrier option. The option only comes to life ("knocks in") if the stock price hits the barrier.
  4. Obviously we must have $S_0 < B$ at $t = t_0$, else this is just a regular call option.
- **Down and out** barrier put option.
  1. The barrier level $B$ is lower than the strike price $K$, i.e. $B > K$. The option terminates immediately if $S \leq B$ at any time on or before expiration.
  2. The above is an example of a **knockout** barrier option. The option terminates ("knocks out") if the stock price hits the barrier.
  3. Obviously we must have $S_0 > B$ at $t = t_0$, else the option is dead at $t_0$.
- **Down and in** barrier put option.
  1. The barrier level $B$ is also lower than the strike price $K$, i.e. $B < K$. However, the option will pay *zero* **unless the stock price hits the barrier level at any time on or before expiration.**
  2. This means that we require $S_t \leq B$ for some value $t_0 \leq t \leq T$, else the option will pay zero, even if it is in the money at the expiration time $T$.
  3. The above is an example of a **knockin** barrier option. The option only comes to life ("knocks in") if the stock price hits the barrier.
  4. Obviously we must have $S_0 > B$ at $t = t_0$, else this is just a regular call option.
- The sum of an (up and out put) and (up and in put) equals a European put option.
- The sum of a (down and out put) and (down and in put) equals a European put option.

### 19.2.2 Barrier options with rebate

- It is not necessarily the case that a knockout barrier option pays zero when the barrier is hit.
- In some cases the knockout barrier option pays a **rebate** when the barrier is hit.
- The rebate is a nonzero sum of money, i.e. cash (not stock).
- The amount of the rebate can be fixed, or can depend on the time at which the barrier is hit.

### 19.2.3 Barrier options with rebate: worked example

- Let us display a worked example of a barrier option with a rebate.
- Consider the European call option valued in Lecture 17a. Now impose an up and out barrrier $B = 125$ and a rebate of $R = B - K = 25$. The input values are listed in eq. (19.1.4). We set $S_0 = 100$ and employ a binomial model with $n = 3$ timesteps.
- The valuation tree for $S_0 = 100$ is shown in Fig. 5.
- The option fair values which are set to the rebate value $R = 25$ are indicated in boldface red.
- Note that there are no nodes exactly at the barrier level of $B = 125$. This makes the option pricing less accurate.
- Some authors recommend the use of "fake nodes" to interpolate to the barrier level. The details are beyond the scope of these lectures.

Binomial tree valuation for up and out barrier call

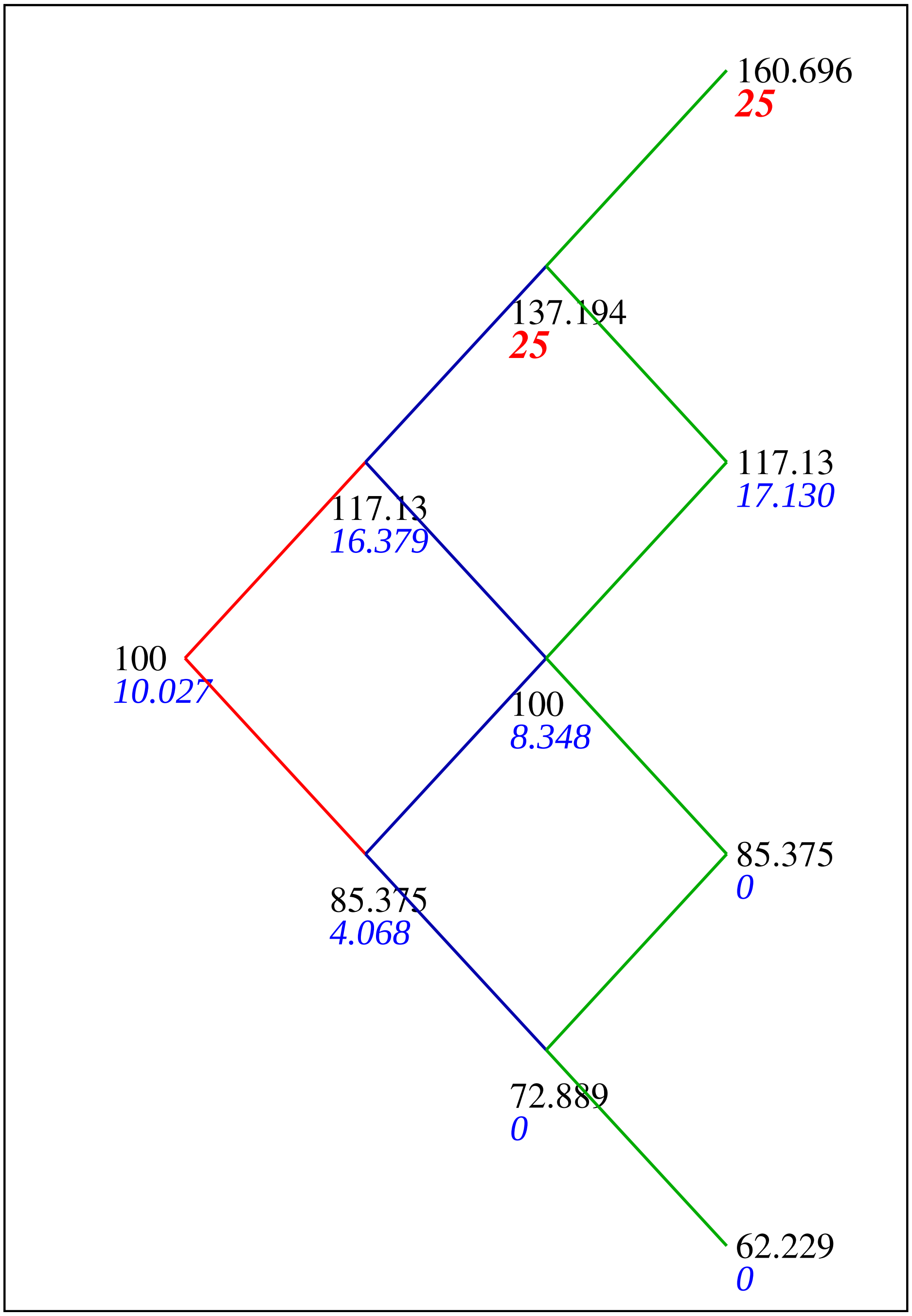


Figure 5: Valuation of knockout up and out barrier call option with strike $K = 100$, barrier $B = 125$ and rebate $R = B - K = 25$. The input values are given in eq. (19.1.4).

## 19.3 Double barrier options

- It is possible for an option to have two barriers.
- There are many possibilities:
  1. Both barriers could be knockouts.
  2. One barrier could be a knockout and the other could be a knockin.
  3. Both barriers could be knockins.
- There could be rebates at one or both barriers.
- The binomial model can handle all such cases. It is simply a matter of imposing new valuation tests at each node in the binomial tree.

## 19.4 Bermudan options

- A **Bermudan option** can be exercised at selected times prior to expiration.
- The name "Bermudan" is a play on geography, because Bermuda is located in between Europe and America.
- A binomial model can value a Bermudan option in the same way as it can value an American option.
- The only difference is that the early exercise tests are applied in specific dates, not at every timestep.

### 19.5 Forward start options

- Just as one can enter into a forward contract to buy a stock at a fixed price on a future date, one can also enter into a forward contract to buy an option at a fixed price on a future date.
- They are called **forward start options.**

## 19.6 Asian options

- An **Asian option** is a call or put option where the payoff is determined by the ***average value of the stock price over the lifetime of the option.***
- Asian options are used to avoid stock price manipulations at the expiration time.
- To value an Asian option using a binomial model, we must employ a non-recombining tree, with $2^{n+1} - 1$ nodes. Essentially, we must sum over all $2^n$ random paths of the stock price.
- An Asian option which is not newly issued but has already acquired some history (for the stock price) is called a **seasoned Asian option.**
- If the stock price history is sufficiently high, it is possible for a seasoned Asian call option to never go out of the money at future times until expiration.
- If the stock price history is sufficiently high, it is possible for a seasoned Asian put option to never go in the money at future times until expiration, in which case it is worth zero.

### 19.7 Options on options

- *Yes one can buy or sell an option on an option!*
- A **compound option** is an option with expiration time $T_1$ to buy/sell an option with a later expiration time $T_2 > T_1$.
- There are four types of compound options:
  1. Call on call: a call option to buy a call option.
  2. Call on put: a call option to buy a put option.
  3. Put on call: a put option to sell a call option.
  4. Put on put: a put option to sell a put option.
- For European options to buy/sell European options, there exist closed form formulas for all four types of compound options, using the Black–Scholes–Merton equation.
- The resulting formulas involve the **cumulative bivariate normal distribution**

$$M(X \le x, Y \le y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} du \int_{-\infty}^{y} dv \, \exp\left\{-\frac{u^2+v^2-2\rho uv}{2(1-\rho^2)}\right\}. \tag{19.7.1}$$

- Here $\rho$ is the correlation coefficient between the two random variables $X$ and $Y$, where both $X$ and $Y$ are normally distributed with zero mean and unit variance.