Queens College, CUNY, Department of Computer Science

Computational Finance

CSCI 365 / 765

Fall 2017

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November 15, 2017

17 Lecture 17a

Binomial model: worked examples

- We display worked examples to calculate option fair values using the **binomial model.**
- We calculate American options and demonstrate how to incorporste early exercise.
- **There is no explicit mathematical probability theory in this lecture.**

17.6 Binomial model: summary of tree

- We make a very simple model of the stock price movements.
- We discretize the time to expiration $T - t_0$ into n equal steps of size

$$\Delta t = \frac{T - t_0}{n} \,. \tag{17.6.1}$$

- At each step, we approximate that the stock price can go to only one of two future values at the next step.
- If the stock price is S at a node at the timestep i, then the stock price either goes up by a factor u to Su or down by a factor d to Sd at the next timestep $i+1$.
- A sketch is shown in Fig. 1 for three timesteps.
- The binomial tree **recombines.** Hence it has $O(n^2)$ nodes, not $O(2^n)$ nodes.

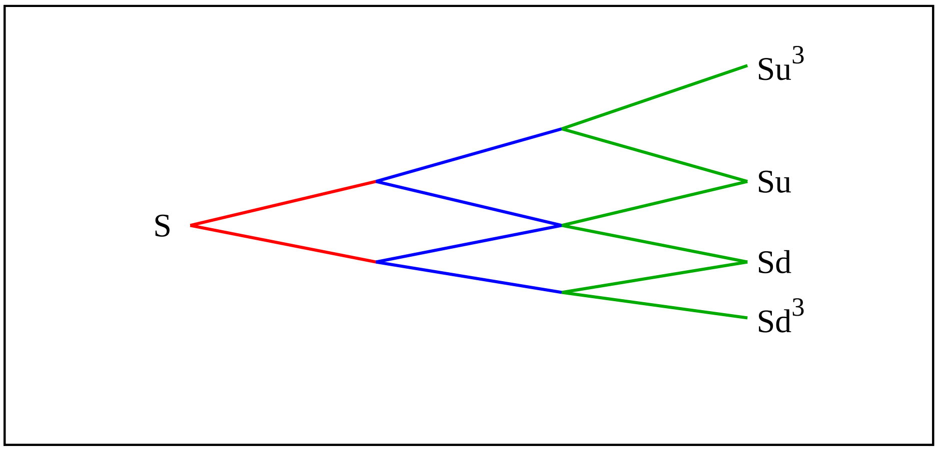

Figure 1: Sketch of binomial stock price movements for three timesteps.

17.7 Binomial model: parameters

- The probability of taking an up step is p and the probability of taking a down step is $q = 1-p$.
- Let the risk–free interest rate be r (a constant).
- Let the volatility of the stock be σ (a constant).
- Suppose the stock pays continuous dividends at a rate q.
- Then the values of u, d, p and q are given as follows:

$$u = e^{\sigma\sqrt{\Delta t}}\,, \tag{17.7.1a}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1/u\,, \tag{17.7.1b}$$

$$p = \frac{e^{(r-q)\Delta t} - d}{u-d}\,, \tag{17.7.1c}$$

$$q = \frac{u - e^{(r-q)\Delta t}}{u-d}\,. \tag{17.7.1d}$$

- Comments:
  1. There is no flexilibity in the locations of the nodes in the binomial tree.
  2. There are hidden assumptions in the above derivation. We require both p and q to be positive (or at least zero). Hence to obtain meaningful values for the probabilities we must have
  $$d \le e^{(r-q)\Delta t} \le u\,. \tag{17.7.2}$$
  3. The inequalities in eq. (17.7.2) then yield
  $$e^{-\sigma\sqrt{\Delta t}} \le e^{(r-q)\Delta t} \le e^{\sigma\sqrt{\Delta t}}\,. \tag{17.7.3}$$
  4. The inequalities in eq. (17.7.3) are usually satisfied in practice, but can fail is the value of σ is very small, or if the value of Δt is not small enough.

17.8 Binomial model: valuation

- We formulate the valuation procedure for any derivative on a stock.
- We value the derivative by working ***backwards*** from the final timestep to the initial timestep.
- Consider a node at the timestep i and let the stock price at that node be S.
  Let the derivative value at that node be V.
- The node at the timestep i is connected to two nodes at the timestep $i+1$, with the values Su and Sd, respectively. Let the derivative fair values at those nodes be V_u and V_d, respectively.

  1. For a European style derivative, the fair value V at the timestep i is calculated as follows:
  $$V_{\mathrm{Eur}} = e^{-r\Delta t}\,(pV_u + qV_d)\,. \tag{17.8.1}$$
  2. For an American style derivative, **we compare the value from eq. (17.8.1) to the derivative intrinsic value.** If the value from eq. (17.8.1) is less than the derivative's intrinsic value, we set the fair value V at that node to the intrinsic value $V_{\mathrm{intrinsic}}$ instead. Hence for an American option
  $$V_{\mathrm{Am}} = \max\Bigl\{\, e^{-r\Delta t}\,(pV_u + qV_d)\,, V_{\mathrm{intrinsic}}\,\Bigr\}\,. \tag{17.8.2}$$

- A sketch is shown in Fig. 2.
- Notice the arrows in Fig. 2 point backwards: we work backwards through the tree.

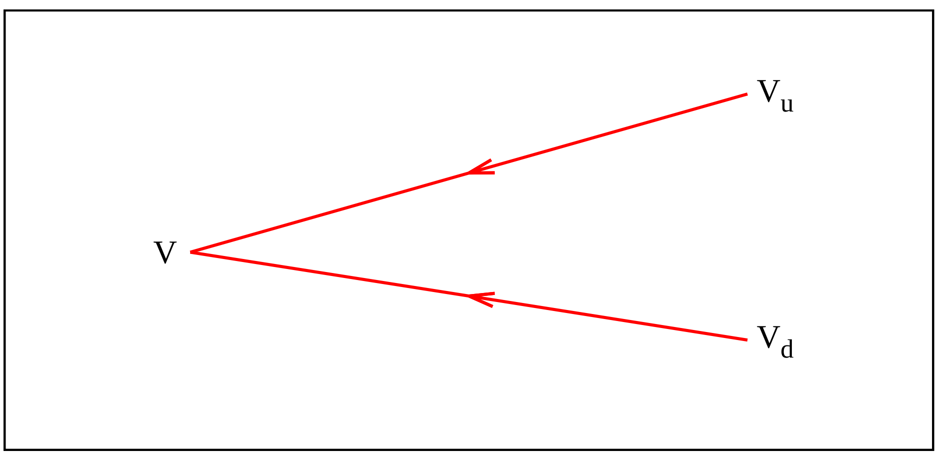

Figure 2: Sketch of binomial tree and derivative fair values at nodes at timesteps i and $i+1$.

17.9 Worked example: Put option

17.9.1 Parameter values

- We value a European put option using a binomial tree.
- The current time is $t_0 = 0$.
- The current stock price is $S_0 = 100$.
- The stock does not pay dividends.
- The stock volatility is $\sigma = 0.5$.
- The risk-free interest rate is $r = 0.1$.
- The option strike is $K = 100$ and the expiration time is $T = 0.3$.
- We make a binomial tree with three timesteps $n = 3$ so $\Delta t = 0.3/3 = 0.1$.
- Then the values of the relevant parameters are as follows:

$$e^{r\Delta t} \simeq 1.01005\,, \tag{17.9.1a}$$

$$e^{-r\Delta t} \simeq 0.99005\,, \tag{17.9.1b}$$

$$u \simeq 1.1713\,, \tag{17.9.1c}$$

$$d \simeq 0.8538\,, \tag{17.9.1d}$$

$$p \simeq 0.4922\,, \tag{17.9.1e}$$

$$q \simeq 0.5078\,. \tag{17.9.1f}$$

17.9.2 Stock price nodes

The stock price values at the nodes of the binomial tree are shown in Fig. 3.

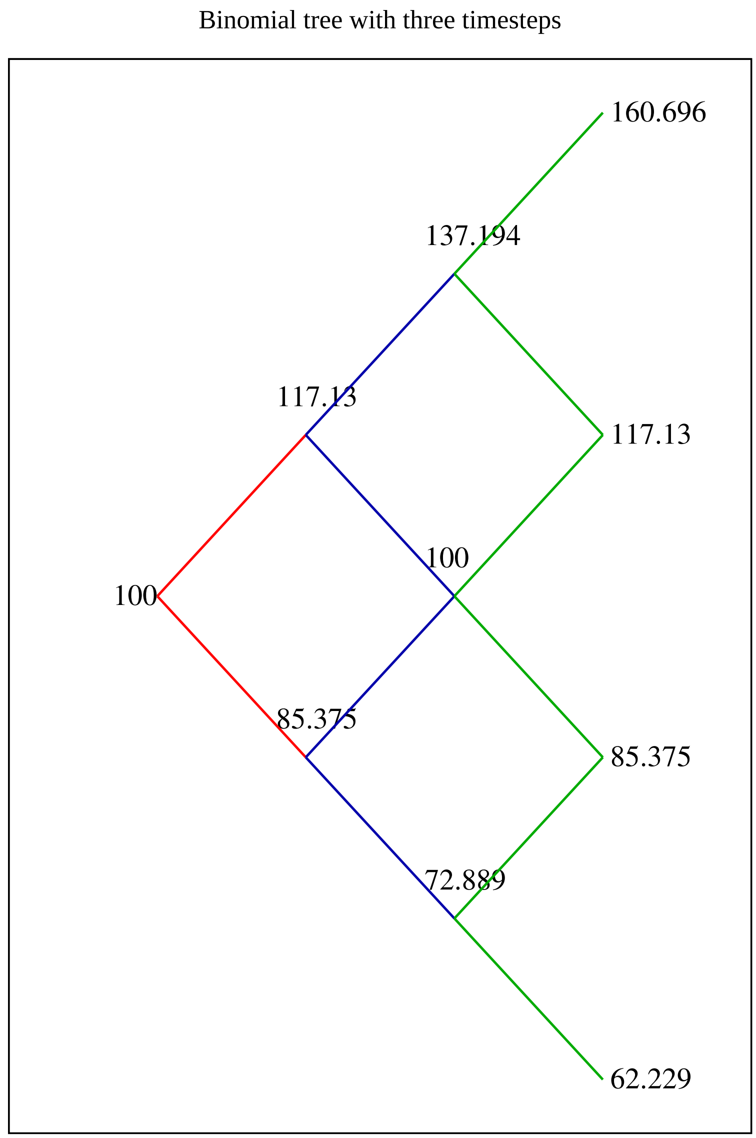

Figure 3: Example binomial tree with three timesteps. The stock prices at each node are listed.

17.9.3 Valuation of European put

- We calculate the fair value of a **European put.**
- The option valuation tree is shown in Fig. 4.
- The option fair values at expiration ($i = 3$) are filled in first.

$$V(Su^3) = \max(100 - 160.696, 0) \qquad = 0\,, \tag{17.9.2a}$$
$$V(Su) = \max(100 - 117.130, 0) \qquad = 0\,, \tag{17.9.2b}$$
$$V(Sd) = \max(100 - 85.375, 0) \qquad \simeq 14.625\,, \tag{17.9.2c}$$
$$V(Sd^3) = \max(100 - 62.229, 0) \qquad \simeq 37.771\,. \tag{17.9.2d}$$

- The option fair values at the remaining nodes are calculated using eq. (17.8.1).
- The fair values at the nodes for the step $i = 2$ are calculated as follows:

$$V(Su^2) = e^{-r\Delta t}(p \times 0 + q \times 0) \qquad = 0\,, \tag{17.9.3a}$$
$$V(S) = e^{-r\Delta t}(p \times 0 + q \times 14.625) \qquad \simeq 7.353\,, \tag{17.9.3b}$$
$$V(Sd^2) = e^{-r\Delta t}(p \times 14.625 + q \times 37.771) \qquad \simeq 26.116\,. \tag{17.9.3c}$$

- The fair values at the step $i = 1$ are calculated using the values at the step $i = 2$:

$$V(Su) = e^{-r\Delta t}(p \times 0 + q \times 7.353) \qquad \simeq 3.696\,, \tag{17.9.4a}$$
$$V(Sd) = e^{-r\Delta t}(p \times 7.353 + q \times 26.116) \qquad \simeq 16.712\,. \tag{17.9.4b}$$

- Finally, the European put option fair value using a three step binomial tree is

$$p_{\text{binom}} = e^{-r\Delta t}(p \times 3.696 + q \times 16.712) \simeq 10.203\,. \tag{17.9.5}$$

- The European put option fair value using the Black–Scholes formula is

$$p_{\text{BS}} = Ke^{-r(T-t_0)}N(-d_2) - SN(-d_1) \simeq 9.317\,. \tag{17.9.6}$$

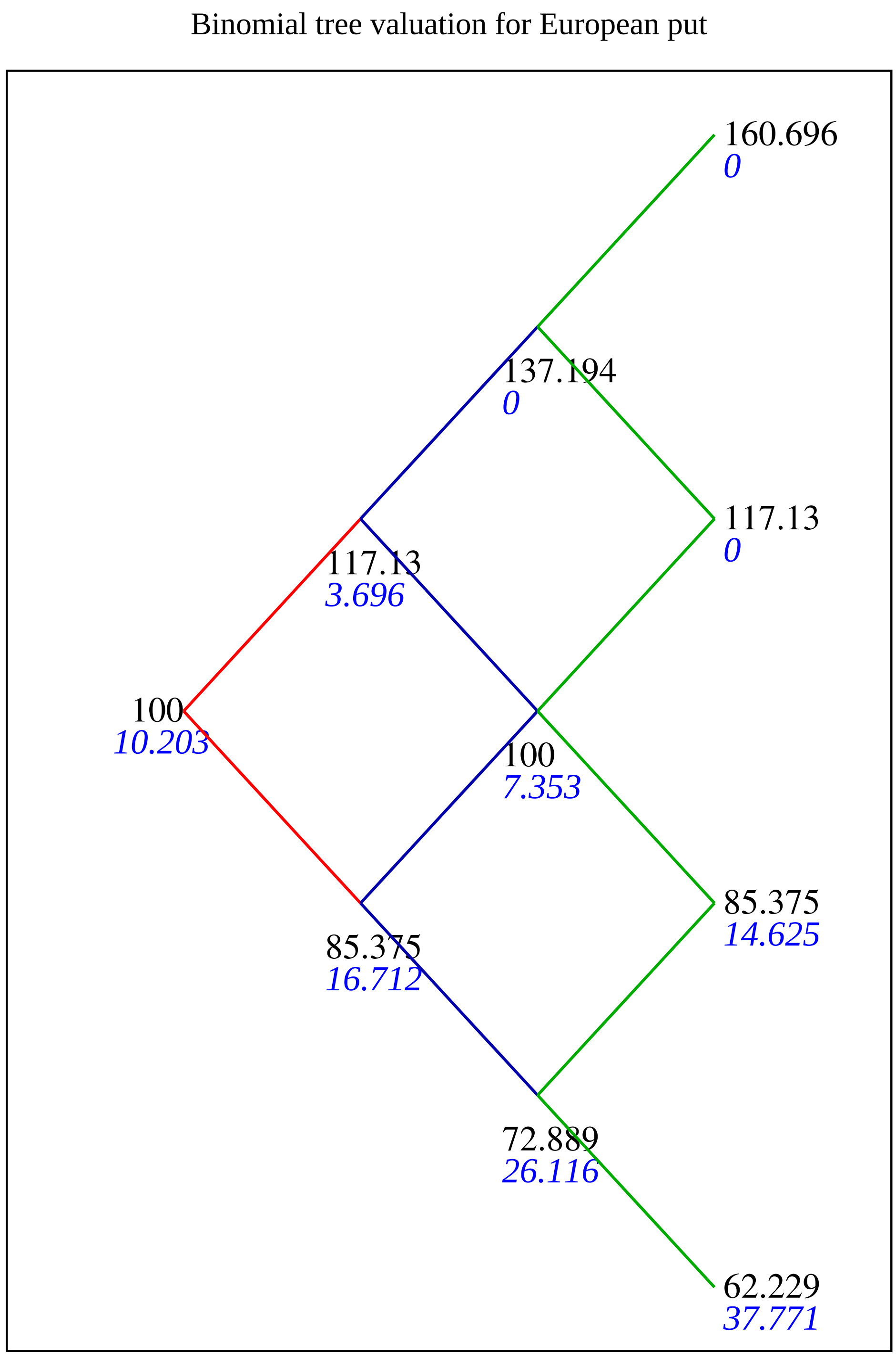

Figure 4: Valuation of European put option using the binomial tree in Fig. 3.

17.9.4 Valuation of American put

- We calculate the fair value of an **American put.**
- The option valuation tree is shown in Fig. 5.
- The option fair values at expiration ($i = 3$) are filled in first.
- The option fair values at the remaining nodes are calculated using eq. (17.8.1).
- The fair values at the nodes for the step $i = 2$ are calculated as follows:
  1. We calculate the discounted expectations:
  $$V(Su^2) = e^{-r\Delta t}(p \times 0 + q \times 0) \qquad =0\,, \tag{17.9.7a}$$
  $$V(S) = e^{-r\Delta t}(p \times 0 + q \times 14.625) \qquad \simeq 7.353\,, \tag{17.9.7b}$$
  $$\boldsymbol{X(Sd^2)} = e^{-r\Delta t}(p \times 14.625 + q \times 37.771) \qquad \simeq 26.116\,. \tag{17.9.7c}$$
  2. At the node Sd^2, the value is called "$\boldsymbol{X}$" because it is **less than the intrinsic value.**
  3. The intrinsic value of the American put at this node is higher:
  $$\max(K - S, 0) \simeq 100 - 72.889 = 27.111\,. \tag{17.9.8}$$
  4. Hence the fair value at this node is set to the intrinsic value
  $$V_{\rm node} = 27.111\,. \tag{17.9.9}$$
- The fair values at the step $i = 1$ are calculated using the values at the step $i = 2$:
  1. We calculate the discounted expectations:
  $$V(Su) = e^{-r\Delta t}(p \times 0 + q \times 7.353) \qquad \simeq 3.696\,, \tag{17.9.10a}$$
  $$V(Sd) = e^{-r\Delta t}(p \times 7.353 + q \times 27.111) \qquad \simeq 17.213\,. \tag{17.9.10b}$$
  2. In each case, the result is higher than the intrinsic value at that node.
  3. Hence early exercise is not optimal at either node.
- Finally, the American put option fair value using a three step binomial tree is
$$P_{\rm binom} = e^{-r\Delta t}(p \times 3.696 + q \times 17.213) \simeq 10.455\,. \tag{17.9.11}$$
- This is higher than the intrinsic value at that node, hence early exercise is not optimal.
- The fair value of the American put $P_{\rm binom}$ in eq. (17.9.11) is higher than the fair value of a European put $p_{\rm binom}$ with the same parameters (see eq. (17.9.5)).
- The Black–Scholes formula cannot calculate the fair value of an American put option.
- The *Black–Scholes equation* (also the Black–Scholes–Merton equation) can treat American options, but the equations must be solved numerically.
- The binomial model is one of the simplest numerical algorithms for valuing derivatives.

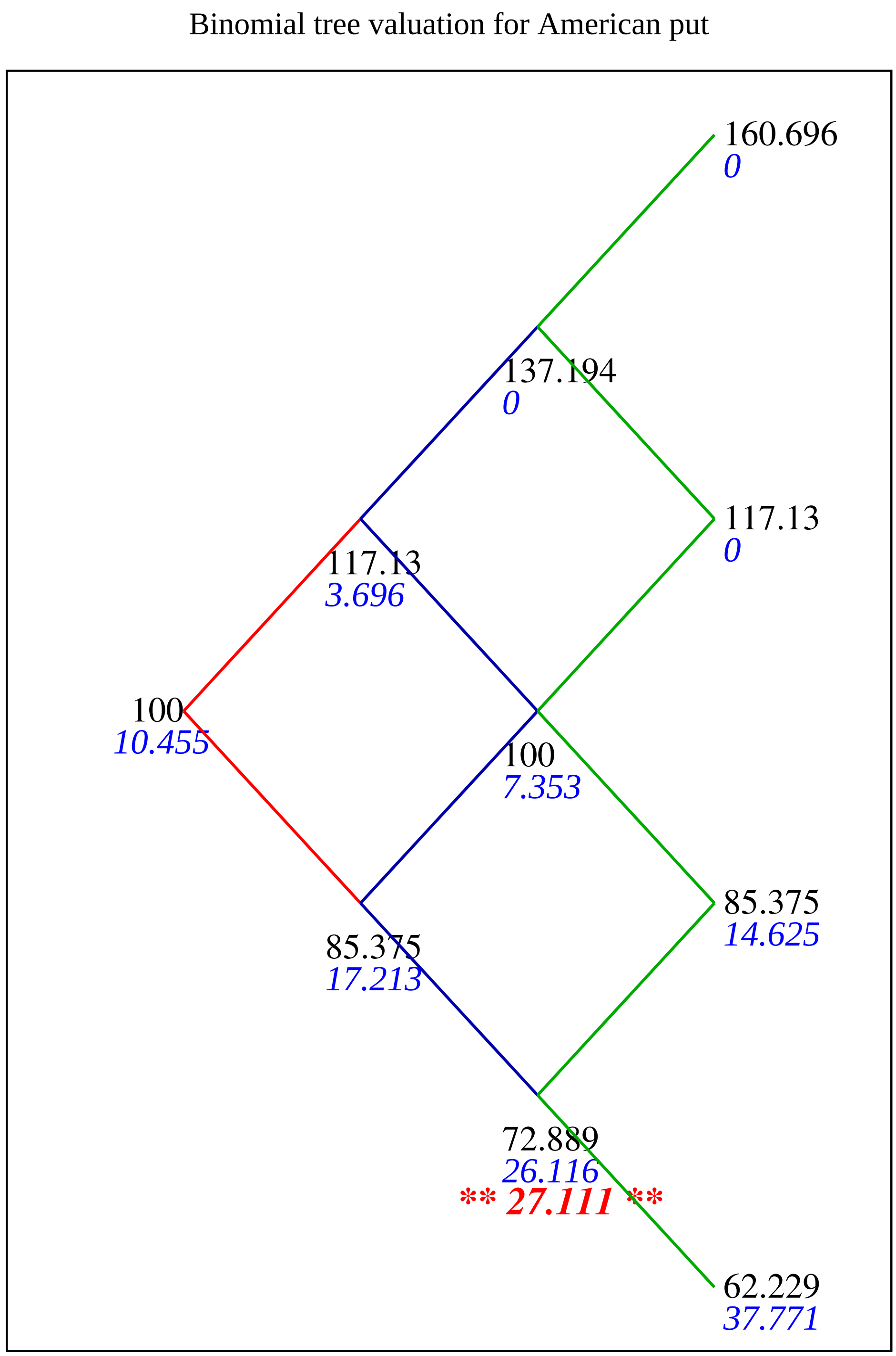

Figure 5: Valuation of American put option using the binomial tree in Fig. 3.

17.9.5 Valuation of European call

- We calculate the fair value of a **European call.**
- The option valuation tree is shown in Fig. 6.
- The option fair values at expiration ($i = 3$) are filled in first.

$$
\begin{aligned}
V(Su^3) &= \max(160.696 - 100, 0) && \simeq 60.696\,, && (17.9.12a)\\
V(Su) &= \max(117.130 - 100, 0) && \simeq 17.130\,, && (17.9.12b)\\
V(Sd) &= \max(85.375 - 100, 0) && = 0\,, && (17.9.12c)\\
V(Sd^3) &= \max(62.229 - 100, 0) && = 0\,. && (17.9.12d)
\end{aligned}
$$

- The option fair values at the remaining nodes are calculated using eq. (17.8.1).
- The fair values at the nodes for the step $i = 2$ are calculated as follows:

$$
\begin{aligned}
V(Su^2) &= e^{-r\Delta t}(p \times 60.696 + q \times 17.130) && \simeq 38.190\,, && (17.9.13a)\\
V(S) &= e^{-r\Delta t}(p \times 17.130 + q \times 0) && \simeq 8.348\,, && (17.9.13b)\\
V(Sd^2) &= e^{-r\Delta t}(p \times 0 + q \times 0) && = 0\,. && (17.9.13c)
\end{aligned}
$$

- The fair values at the step $i = 1$ are calculated using the values at the step $i = 2$:

$$
\begin{aligned}
V(Su) &= e^{-r\Delta t}(p \times 38.190 + q \times 8.348) && \simeq 22.807\,, && (17.9.14a)\\
V(Sd) &= e^{-r\Delta t}(p \times 8.348 + q \times 0) && \simeq 4.068\,. && (17.9.14b)
\end{aligned}
$$

- Finally, the European call option fair value using a three step binomial tree is

$$
c_{\text{binom}} = e^{-r\Delta t}(p \times 22.807 + q \times 4.068) \simeq 13.159\,. \qquad (17.9.15)
$$

- The European call option fair value using the Black–Scholes formula is

$$
c_{\text{BS}} = SN(d_1) - Ke^{-r(T-t_0)}N(d_2) \simeq 12.272\,. \qquad (17.9.16)
$$

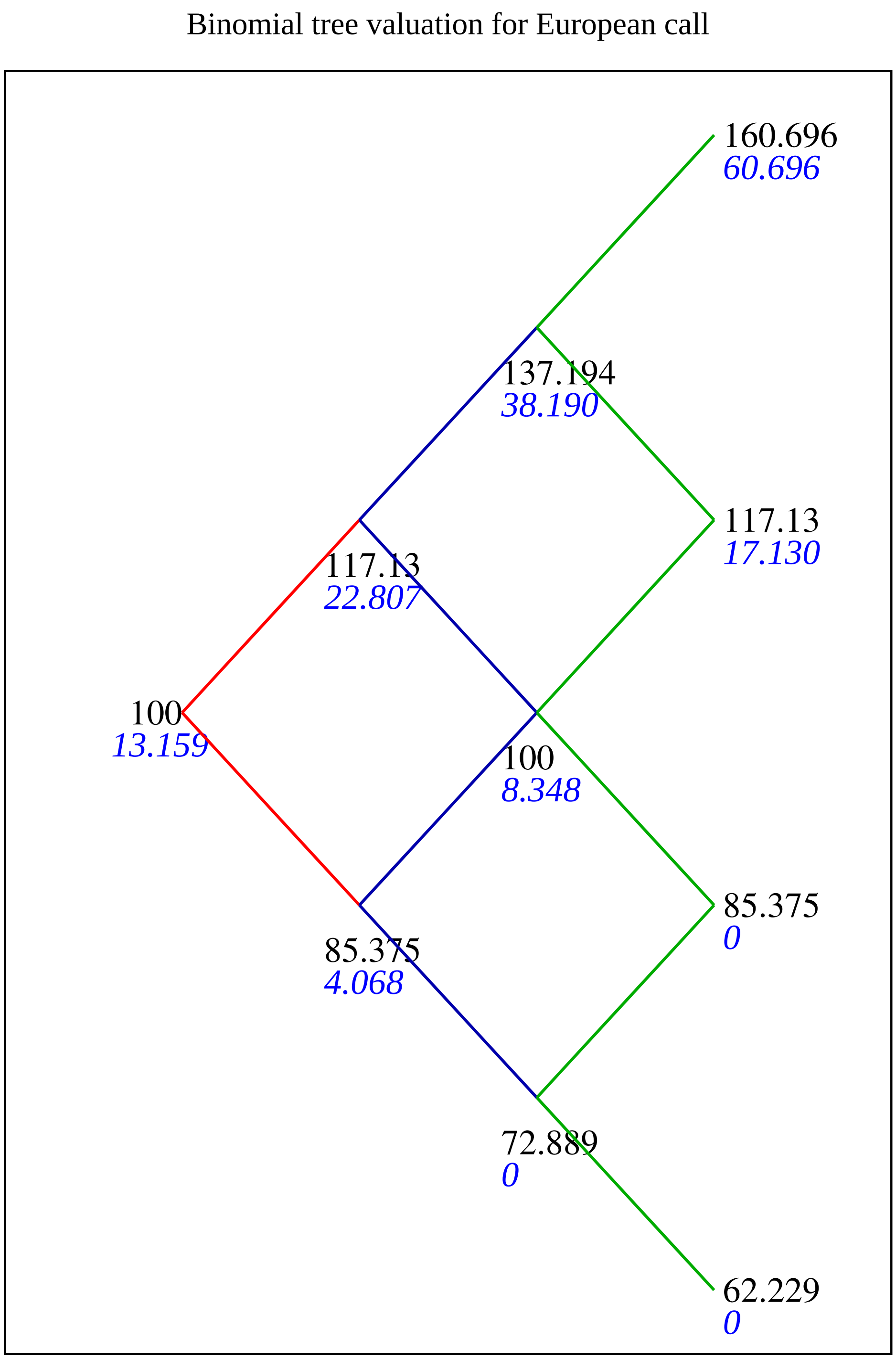

Figure 6: Valuation of European call option using the binomial tree in Fig. 3.

17.9.6 Valuation of American call

- If the stock does not pay dividends, then the value of an American call equals the value of a European call on the same stock, with the same strike and expiration.
- The binomial model confirms this behavior.
- The valuation of an American call using the binomial model yields the same results, at all nodes, as the valuation of the European call.
- Hence the American call option fair value using a three step binomial tree is

$$C_{\text{binom}} \simeq 13.159\,. \tag{17.9.17}$$

17.10 Tests: put–call parity

- If the underlying stock does not pay dividends, the put–call parity formula is

$$c - p = S - \mathrm{PV}(K) = S - Ke^{-r(T-t_0)}\,. \tag{17.10.1}$$

- For the given parameter values, we obtain

$$S - Ke^{-r(T-t_0)} = 100 - 100\,e^{-0.1\times 0.3} = 100 - 100\,e^{-0.03} \simeq 2.955\,. \tag{17.10.2}$$

- The fair values using the Black–Scholes formula agree with eq. (17.10.1):

$$c_{\mathrm{BS}} - p_{\mathrm{BS}} \simeq 12.272 - 9.317 = 2.955\,. \tag{17.10.3}$$

- Using eqs. (17.9.5) and (17.9.15), the fair values using the binomial model yield

$$c_{\mathrm{binom}} - p_{\mathrm{binom}} \simeq 13.159 - 10.203 = 2.956\,. \tag{17.10.4}$$

- **The fair values using the binomial model also agrees with eq. (17.10.1).**
- Remember that put–call parity **does not depend on a probability model** for the stock price movements.
- **Therefore all option valuation models, including in particular the binomial model, must satisfy put–call parity.**

17.11 Tests: inequalities for American options

- If the underlying stock does not pay dividends, the fair values of American calls and puts satisfy the following inequalities (derived from rational option pricing theory)

$$S - K \quad \le \quad C - P \quad \le \quad S - \mathrm{PV}(K)\,. \tag{17.11.1}$$

- From the data, $S = K = 100$, hence $S - K = 0$.
- Also from eq. (17.10.2), $S - \mathrm{PV}(K) \simeq 2.955$.
- Hence using these parameter values in eq. (17.11.1), we must have

$$0 \quad \le \quad C - P \quad \le \quad 2.955\,. \tag{17.11.2}$$

- Using eqs. (17.9.11) and (17.9.17), the fair values using the binomial model yield

$$C_{\mathrm{binom}} - P_{\mathrm{binom}} \simeq 13.159 - 10.455 = 2.704\,. \tag{17.11.3}$$

- Hence eq. (17.11.3) satisfies eq. (17.11.2), and therefore (for these parameter values), the rational option pricing inequalities in eq. (17.11.1).