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5 Lecture 5

5.1 Derivatives are a zero sum game

- Before we begin our study of specific types of derivatives, we must learn a few basic lessons about all derivatives.
- The statements below are with respect to derivatives on a stock (or stock index), but they apply to all derivatives (on bonds, currencies, etc.).
- **Derivatives do not create wealth.**
- **Derivatives are a zero sum game.**
  1. If one side gains (makes a profit), the other side loses (makes a loss).
     It is also possible that both side break even: nobody gains and nobody loses.
  2. If derivatives do not create wealth, what do they do?
  3. Derivatives can be used to transfer risk from one party to another.
     Derivatives can be used to **manage risk.**
     Risk management is a big business.
  4. The weak point in the above statements is that we have not defined what "risk" means in this context.
  5. One example is that an investor is concerned about a risk of loss in some scenario. The investor purchases derivatives to reduce or zero out the quantity of loss, should the undesired scenario occur. Of course the investor has to pay to purchase the derivatives, which costs money. That is part of the cost of doing business. However, if the investor is now protected against loss (in the undesired scenario), it means that the risk of loss has now been taken by someone else (the person who sold the derivatives to the investor).
- Many people make an analogy with insurance, when speaking of using derivatives for protection against loss. It is an imperfect analogy. For example, if a company goes bankrupt and its stock price suddenly drops to zero, *all* the investors who own derivatives on the company's stock will be simultaneously affected.

5.2 Trading of derivatives

- Unlike the shares of stock on a company, the number of derivatives contracts is not fixed. A derivatives contract comes into existence when two parties initiate a trade (one side buys a derivative, the other sells it). Effectively, a derivative is a piece of paper (a business contract), which is created by the two parties. It is a new business contract, which did not exist before. The cost of the derivative is the cost of that piece of paper.

  1. Any pair of investors can create and trade a derivative on the stock of a company.
  2. The company which issued the stock has no involvement in the above trade.
  3. The number of derivatives contracts currently trading is called the **open volume**.
  4. **The open volume can be very large.**
  5. **There can be more derivatives on a stock that the number of shares that the company has issued.**

- Also unlike shares of stock on a company, a derivative typically has an expiration date.

  1. An investor who owns a derivative does not have to hold it until the expiration date.
  2. **Derivatives can be bought and sold in financial markets just like stocks.**
  3. Hence a trade can create a new instance of a derivative, or it can be a secondary trade, for a derivative which already exists.

- **Many derivatives are traded on exchanges.**

  1. Such derivatives are known as **exchange-listed derivatives.**
  2. **Note that the exchanges do not issue derivatives:** the exchanges provide a *financial market where derivatives can be traded* (including secondary trades). The derivatives themselves are created by pairs of investors who buy/sell the derivatives.

- It is also possible for derivatives to be traded privately between two parties.

  1. Such derivatives are known as **OTC** or **over the counter** derivatives.
  2. Typically, it is difficult to find a secondary market for OTC derivatives.

5.3 Forwards & futures

- The standard example in textbooks is a farmer who is growing a crop of wheat (or corn, etc.).
- The date is still summer, and the crop will ripen in the fall, but the farmer wishes to find customers and to lock in a price for his crops ahead of time. Doing so will guarantee the farmer an income (cashflow) at a known date in the future.
- A **forward contract** is an agreement between two parties to sell an asset at a fixed price at a specified date in the future.
- **Futures contracts** are very similar to forward contracts.
  1. Forward contracts are typically private agreements between two parties.
     They are referred to as OTC ("over the counter") contracts.
     Forward contracts can be customized to any expiration date, also any contract size.
  2. Futures contracts are listed on exchanges.
     They are referred to as **exchange listed** contracts.
     Futures are standardized contracts: their expiration dates, contract size, and details of delivery of the asset at expiration, etc. are specified by the exchange.
- Futures contracts are widely traded in the financial markets. They can be employed to guanantee the seller a known cashflow at a specified date in the future. In the above example, the farmer would sell futures contracts for his crops.
  1. There are futures contracts for many commodities, such as oil and natural gas, gold, etc.
  2. There are also futures on stock indices.
- Many futures contracts are employed for hedging. We shall study about hedging later.
- Many futures contracts are also employed for speculation in the stock market.
- **Note that a forward or futures contract does not maximize income.**
  If the farmer did not lock in a price using futures, the price of wheat (for example) might go up in the fall and the farmer could earn more income.
  Conversely, the price of wheat might go down in the fall and the farmer could lose income.
  By locking in a price (by selling futures contracts), the farmer is insulating against the fluctuations of the price of wheat in the fall. In other words, the farmer is reducing the risk to his income. However, "reducing risk" does not mean "maximizing income" and one must note the distinction.

5.3.1 Model example

- Suppose a farmer grows wheat and a bakery company bakes bread to sell in supermarkets.
- Every year the farmer produces a crop of wheat and wants to sell it and make a profit.
- The company wants to buy the wheat, as raw material to bake bread, which it sells for profit.
- Some years, the price of wheat in the fall is high, in other years it is low.
- *How can the farmer and the bakery company do business?*
- Suppose the price of wheat in the fall is high. What happens?
  1. The farmer is happy. It mean a big income, hence a big profit for the farmer.
  2. The bakery company is unhappy. It has to pay a high price to buy the wheat, hence it will make a loss.
- Suppose the price of wheat in the fall is low. What happens?
  1. The farmer is unhappy. It means very little income, hence a loss for the farmer.
  2. The bakery company is happy. It buys the wheat cheaply and makes a big profit when it sells the bread.
- Hence the uncertainty about the price of the wheat in the fall (the harvest time) causes a problem for both the farmer and the bakery company.
- Neither side wants the problems caused by the uncertainty.
- **Therefore the farmer and the bakery company make a business deal ahead of time, to agree on the price of wheat that the farmer will sell and the bakery company will buy. Both sides (farmer and bakery company) agree on a price that gives a (moderate) profit to both sides.**
- **This is a forward contract (farmer goes short and bakery company goes long).**
- Now the farmer knows that there is a customer for the crop of wheat, and the farmer knows that the sale price will be enough to pay the expenses (salaries of employees, buy new farm equipment, etc.) and make a profit at the end.
- Also, the bakery company knows that it has a source of raw material to bake the bread, and the company knows that it will not have to pay too much for the wheat, so the expenses of the company will be low enough so that the bakery company can sell the bread and there will be enough to pay the expenses (salaries of employees, buy new factory equipment, etc.) and the company will make a profit at the end.
- The forward contract **reduces the uncertainty for both sides.**
- The forward contract **does not increase the profit for either side.**
- Do not confuse those two concepts.
- Understand why forward contracts are useful for business people.

5.4 Forward contracts

- Suppose there is a stock S, whose price today is S_0. (It could be any underlying asset, but for convenience we begin with a stock.) However, we do not wish to buy the stock today. Instead we wish to enter into a contract to buy the stock at a time T in the future. We agree to pay a price F when we buy the stock at the time T.

- This is known as a **forward contract**.
  The value of F is called the **forward price**.
  The time T is the **expiration date** of the forward contract.

- Note that a forward contract requires no cash payment up front (at time t_0). We take delivery of the stock, and pay the price F, only at the future time T. A forward agreement is signed today, but the payment of money and delivery of the asset occurs on the expiration date T.

- The obvious question is: *what should the value of F be?* The answer is

$$F = e^{r(T-t_0)}\, S_0\,. \tag{5.4.1}$$

  Note: the above formula only applies for a stock which pays no dividends. After we learn about dividends, eq. (5.4.1) *must be modified.*

- We prove eq. (5.4.1) is the correct answer by formulating an arbitrage argument. Our arbitrage argument will require trades at two different times (t_0 and T). It will also require borrowing or lending of money (cash) and interest payments.

- There are obviously only two possibilities: either (i) $F > S_0\, e^{r(T-t_0)}$ or (ii) $F < S_0\, e^{r(T-t_0)}$.

- We formulate our arbitrage argument as follows. Recall that in all cases we begin with a position of zero. First suppose $F = F_+$ where $F_+ > e^{r(T-t_0)}\, S_0$.

  1. We enter into a forward contract to *sell* the stock at the forward price F_+. *(We* **go short** *the forward contract.)* This requires no cash payment or stock delivery up front.
  2. We buy one share of stock today. To buy the stock, we borrow cash worth S_0, say by selling a zero coupon bond with maturity date T. Our portfolio today is one share of stock worth S_0 and a loan worth B_0. The overall value is $S_0 - B_0 = 0$.
  3. At the future time T, the forward contract expires and we sell the stock at the price F_+. Hence we receive cash worth F_+. The money to be repaid on the bond has compounded to $B(T) = B_0 e^{r(T-t_0)} = S_0 e^{r(T-t_0)}$ because the (continuously compounded) interest rate is r. We repay our loan. Now we have no stock and no loan, but we have excess money. By hypothesis $B(T)$ is less than F_+, so our net profit is

$$\text{profit} = F_+ - B(T) = F_+ - e^{r(T-t_0)} S_0 > 0\,. \tag{5.4.2}$$

     This is positive by hypothesis and is our arbitrage profit.
  4. *Note that it does not matter what the actual stock price is at the future time T.*
     We agreed (in the forward contract) to sell the stock at the price F_+ at the time T and that is all that matters.

- Next suppose $F = F_-$ where $F_- < e^{r(T-t_0)} S_0$. We again begin with a position of zero. The arbitrage strategy is clearly the reverse of the above, but there is an important detail to understand.

  1. We enter into a forward contract to *buy* the stock at the forward price F_-. *(We* **go long** *the forward contract.)* This requires no cash payment or stock delivery up front.
  2. We now *sell* one share of stock today.
  3. *How is this possible?* Remember we begin with a position of zero, hence we do not own any stock.
  4. This is the important concept of **short selling**: in the stock market, it is possible to sell a stock first and buy it back later. We can sell stock we do not own.
  5. It was stated above that we shall assume that the financial markets allow **unrestricted short selling**. This means we can short sell a stock any time we wish. In reality, there are laws governing the short selling of stocks. Our proof of eq. (5.4.1) depends on unrestricted short selling.
  6. Because we *sold* stock, we have cash in hand, worth S_0. We save the money in a bank, where it will earn interest. Our portfolio today is short one share of stock worth S_0 and money in the bank worth B_0, where $B_0 = S_0$. The overall value of our portfolio is $-S_0 + B_0 = 0$.
  7. At the future time T, the forward contract expires and we buy the stock at the price F_-. To pay the forward price F_-, recall that our money saved in the bank has compounded to $B(T) = B_0 e^{r(T-t_0)} = S_0 e^{r(T-t_0)}$. By hypothesis $B(T)$ is greater than F_-, hence we have enough money to pay the price F_- to buy the stock. This cancels our short position in the stock. Now we have no stock, but we have excess money in the bank. Our net profit is
  $$\text{profit} = B(T) - F_- = e^{r(T-t_0)} S_0 - F_- > 0\,. \tag{5.4.3}$$
  This is positive by hypothesis and is our arbitrage profit.
  8. *Note again that it does not matter what the actual stock price is at the future time T.*

- In practice, eq. (5.4.1) is still valid even if there are restrictions on short selling. This is because there are investors who own the stock already. If $F < S_0 e^{r(T-t_0)}$, they will simply sell their stock at the price S_0 (which is *not* a short sale) and purchase a forward contract and make a profit on the expiration date of the forward contract. Effectively, they will buy back their stock (at the time T) and make a profit on the trade. Hence the existence of such investors will cause eq. (5.4.1) to still be valid.

- We say that the forward price given in eq. (5.4.1) is the **fair value** (or "fair market value") of the forward contract. It is the price consistent with "no arbitrage" in the market.

- The concept of determining the fair value of a derivative by using a "no arbitrage" argument is very important in finance. We shall see many examples in this course.

- If the market is not efficient or not liquid, etc., then the fair value will not be unique. There may be a range of fair values for which an arbitrage strategy cannot be formulated.

- **The "no arbitrage: fair value is aslo called the arbitrage free fair value.**

5.5 Spreads & arbitrage: non-unique forward price

- The arbitrage-free fair value for the forward price is not unique if the stock price has a nonzero bid-ask spread and/or the borrowing and lending interest rates are unequal.

- Let the bid price be S_b and the ask price be S_a, where $S_a > S_b$.
  Let the borrow rate be r_b and the lending rate be r_l, where $r_b > r_l$.

- Then the arbitrage-free forward price can have any value in the interval

$$S_b e^{r_l(T-t_0)} \quad \leq \quad F \quad \leq \quad S_a e^{r_b(T-t_0)}\,. \tag{5.5.1}$$

- Let us revisit our arbitrage strategies and derive eq. (5.5.1).

  1. Suppose we buy a forward contract and short one share of stock. The forward contract costs nothing at time t_0. We sell the stock at the bid price S_b. (Because *we are the ones who want to sell,* hence buyers will not pay us more than S_b.) We invest the money ($B_0 = S_b$) in a bank, and it earns interest at the lending rate r_l.
  2. At the expiration time T, we pay the forward price F and take delivery of one share of stock. This cancels our short position in the stock. Our money in the bank has compounded to $B(T) = B_0 e^{r_l(T-t_0)} = S_b e^{r_l(T-t_0)}$. Our final amount of money is

$$\text{money at expiration} = S_b e^{r_l(T-t_0)} - F\,. \tag{5.5.2}$$

  If $F < S_b e^{r_l(T-t_0)}$, this will be a positive number and we make a riskless profit. No one will agree to trade a forward contract at such a low price, because arbitrage will be possible against them. Hence we must have

$$F \geq S_b e^{r_l(T-t_0)}\,. \tag{5.5.3}$$

  3. Next suppose we sell a forward contract and buy one share of stock. The forward contract costs nothing at time t_0 and we buy the stock at the ask price S_a. (Because *we are the ones who want to buy,* hence sellers will insist on the ask price S_a.) We borrow money $B_0 = S_a$ from a bank, and the bank charges us the borrowing rate r_b.
  4. At the expiration time T, we receive the forward price F and deliver the stock which we already own. Hence we have no more stock. We must pay the bank an amount $B(T) = B_0 e^{r_b(T-t_0)} = S_a e^{r_b(T-t_0)}$. Our final amount of money is

$$\text{money at expiration} = F - S_a e^{r_b(T-t_0)}\,. \tag{5.5.4}$$

  If $F > S_a e^{r_b(T-t_0)}$, this will be a positive number and we make a riskless profit. No one will agree to trade a forward contract at such a high price, because arbitrage will be possible against them. Hence we must have

$$F \leq S_a e^{r_b(T-t_0)}\,. \tag{5.5.5}$$

- Combining eqs. (5.5.3) and (5.5.5) yields the double inequality eq. (5.5.1).
  No arbitrage strategy is possible within this interval of forward prices.

- We must have $S_b = S_a$ and $r_l = r_b$ to obtain a unique arbitrage-free fair value.

5.6 Futures contracts

- In addition to forward contracts, there are also **futures contracts**. They are very similar to forward contracts, but there are some differences in the technical details.
- As with a forward contract, in a futures contract we agree to buy a stock at a future time T, at a future price F. The fair value of F is also given by eq. (5.4.1), and is

$$F = e^{r(T-t_0)} S_0 \,. \tag{5.6.1}$$

- Nevertheless, there is an important difference in the cashflows paid for forwards and futures.
  1. In a forward contract, no money is exchanged until the expiration date.
  2. In a futures contract, no money is exchanged today (when the contract is signed). However, a futures contract is **marked to market** every day, on the subsequent days, until the expiration date.
- **Marking to market** is an important concept. It is simplest to explain with an example.
  1. Suppose we purchase a futures contract with an expiration date three days from now (to keep the example simple). Today is $t_0 = 0$ and the expiration date is $T = 3$.
  2. The stock price today is S_0. The futures price is listed on the exchange and is F_0. No money is exchanged on the date a futures contract is initiated.
  3. A **margin account** is opened (by our stockbroker). We are required to deposit some money (a small percentage of the contract value) into this account.
  4. **In our idealized mathematical model, the amount of the deposit is zero.**
  5. After one day, the date is $t_0 = 1$, the stock price is S_1 and the futures price is F_1.
     If $F_1 > F_0$, we *receive* money into our margin account, equal to $F_1 - F_0$.
     Conversely, if $F_1 < F_0$, we *pay* money into our margin account, equal to $F_0 - F_1$.
     To simplify the mathematics, we can say that we always receive $F_1 - F_0$ (which will be a negative number if we actually pay money).
     Note that the values of S_0 and S_1 do not enter into the calculation.
  6. After two days, the date is $t_0 = 2$, the stock price is S_2 and the futures price is F_2. Again to simplify the mathematics, we receive $F_2 - F_1$ in the account (which is negative if $F_2 < F_1$). Note that the values of S_1 and S_2 do not enter into the calculation.
  7. The total accumulated money in the account is $(F_2 - F_1) + (F_1 - F_0) = F_2 - F_0$.
  8. After three days, the date is $t_0 = 3$, the stock price is S_3 and the futures price is F_3. Because this is the expiration date of the futures contract, we must have $F_3 = S_3$ (use eq. (5.6.1) and put $T - t_0 = 3 - 3 = 0$). Following our convention, we receive money into our margin account, equal to $F_3 - F_2$ (which is negative if $F_3 < F_2$).
  9. Because the futures contract has expired, we take delivery of the stock.
     However, **we pay the current stock price S_3 to take delivery of the stock, (*not the original futures price* F_0) on the expiration date.**

10. The margin account is also closed. The total accumulated money in the account is $(F_3 - F_2) + (F_2 - F_1) + (F_1 - F_0) = F_3 - F_0$. If this is positive, it is refunded to us. If negative, it is a loss. (In practice, we have to pay a deposit, as mentoned above. Hence our deposit will be refunded minus the amount of the loss.)
11. Compare the cashflows of the forward and the futures:
    (a) For a forward contract, we pay an amount F_0 on the expiration date T (= day 2).
    (b) For a futures contract, we received an amount $F_1 - F_0$ on day 1 and an amount $F_2 - F_1$ on day 2 and an amount $F_3 - F_2$ on day 3.
    (c) Our total expenditure is therefore F_3 **less the refund of the money in the account:**

$$\begin{aligned} \text{overall payment for futures contract} &= F_3 - \Big[(F_3 - F_2) - (F_2 - F_1) - (F_1 - F_0)\Big] \\ &= F_3 - (F_3 - F_0) \\ &= F_0\,. \end{aligned} \tag{5.6.2}$$

    (d) Hence overall we pay F_0, the futures price on the day we bought the futures cpntract.
    (e) **However, the overall sum of money F_0 is composed of two components:**
        i. We pay F_3 $(= S_3)$ to the **party which delivers the stock to us.**
        ii. We receive a refund of $F_3 - F_0$ from the closing of the margin account.
        This account is maintained by the clearing house.
        iii. They are two separate sets of cash flows, paid/received to/from different parties.
        iv. The total adds up to F_0.
    (f) However in a futures contract, money is paid (or received) daily in small amounts.
    (g) We say that the futures contract is **marked to market** to the current futures price every day.
    (h) We can think of the above sequence of events as follows.
    Every day, we open a new futures contract with an expiration of only one day (and close out our previous futures contract).
    We say that we "roll" the futures contract from one day to the next.
    On the expiration day, we pay the ***current market price*** of the stock.

- Note that as the expiration date approaches, the price of a futures contract must converge to the stock price. If not, there would be an arbitrage opportunity.

- For our purposes, i.e. computation, we do not need to worry about the above details. The futures price is given by eq. (5.6.1).

5.7 Futures on stock indices: physical and cash settlement

- In the above analysis, it was stated that on the expiration date, we take delivery of the underlying asset (the stock).
  This is called **physical settlement**.

- There are also futures contracts on **stock indices**.

  1. For a futures contract on a stock index, physical settlement is not possible.
  2. Futures contracts on stock indices employ **cash settlement**.
  3. The value of the stock index (a number) is multiplied by a dollar multipler (in the USA; obviously other currencies are employed in other countries). The marking to market and "delivery" on the expiration date is all in terms of cash, using the above multiplier.

- Cash or physical settlement makes no difference to the pricing formula. In both cases the fair value of the futures contract is given by eq. (5.6.1).

5.8 Futures on commodities

- There are also futures contracts on **commodities**.
- The pricing formula for a futures contract commodity presents some complications because a commodity can be consumed.
- As opposed to eqs. (5.4.1) or (5.6.1), the best we can say for the fair value of a futures on a commodity is an inequality

$$F \leq S_0 \, e^{r(T-t_0)} \, . \tag{5.8.1}$$

- We shall not concern ourselves with commodities in this course.

5.9 Stock dividends

- Some (but not all) companies pay a cash dividend on the stock of their company.
- A cash dividend means that if an investor owns shares of the company, the company will pay cash to the investor at regular time intervals.
- In the USA, dividends are usually paid quarterly (at intervals of three months).
- The set of cash payments is called a **dividend stream**.
- For example, suppose a company declares an annual divident of $D_{\rm ann}$ per share, paid quarterly. If an investor owns N shares of stock in the company, every three months the investor will receive the following dividend payment in cash

$$\text{cash dividend payment} = \tfrac{1}{4} N D_{\rm ann} \,. \tag{5.9.1}$$

- The actual cash amount paid will be rounded to the nearest penny.
- There are several variations on the above picture:
  1. The frequency does not have to be quarterly. In many countries, dividends are paid annually or semiannually.
  2. The dividend payments do not have to be equal. In some countries where the dividends are paid semiannually, one dividend is (much) larger than the other (there is a "large dividend" and a "small dividend").
  3. The dividends do not have to be paid at equal time intervals. In some cases, again for example where the dividends are paid twice a year, the time intervals between the dividend payments could be (3 months, 9 months).
  4. Not all companies pay dividends, and the dividend amount may not remain constant in time. If the company's stock price goes up, the company may increase its dividend payments. Conversely, if the company's stock price goes down, the company may decrease its dividend payments. (Typically, companies are reluctant to decrease their dividends unless forced to do so. Decreasing the dividend payment is an admission the company is in trouble and carries bad sociological consequences.)
  5. Sometimes, companies declare "special dividends" which are one-time payments and are not repeated. The date of a special dividend does not have to coincide with the date of a regulaar dividend.
- There are also **stock dividends**. A company may not always pay cash. If an investor owns N shares of stock in the company, the company may award more shares to the investor, so the customer will own $N + \Delta N$ ahares of stock.
  1. For example $\Delta N = 0.001\,N$, so an investor who owns 1000 shares will now own 1001 shares after the stock dividend is paid.
  2. The number of shares is rounded to the nearest integer, so if $\Delta N = 0.001\,N$, then an investor who owns only 100 shares of stock will receive no extra dividends.
- We shall only consider cash dividends in these lectures.

5.10 Forwards & Futures on stocks paying dividends

- Suppose a stock pays a dividend stream D_1, D_2, etc. during the time interval from today t_0 to the expiration date T. Then the fair value formulas in eqs. (5.4.1) and (5.6.1) must be modified as follows:

$$F = e^{r(T-t_0)} \left[S_0 - \mathrm{PV}(D_1) - \mathrm{PV}(D_2) - \dots \right] . \tag{5.10.1}$$

- The proof of eq. (5.10.1) again employs a "no arbitrage" argument and is basically the same as the derivation of eq. (5.4.1).

- We must take into account that if we buy one share of the stock, then we shall receive cash dividends during the lifetime of the forward contract. It is important to remember that the dividends are cash amounts so we **reinvest the dividends in a bank account and earn interest (at the interest rate r)**.

- However, if we buy a forward contract, we are *not* shareholders of the company, hence we receive *no* dividends. These details are important, to correctly formulate the no-arbitrage argument to derive eq. (5.10.1).

- Conversely, if we short sell the stock, then **we are responsible for paying the dividends** to the investor to whom we sold the stock. Remember that the investor is a shareholder of the company and will expect to receive the dividend payments. We have short sold shares of stock which technically were not issued by the company. Hence we have to pay the investor the relevant dividends. All of this contributes to the no-arbitrage derivation of eq. (5.10.1).

- In writing eq. (5.10.1), it is essential to baseline all the cashflows to today's date t_0, hence we must use the *present value* of the dividends on the right hand side in eq. (5.10.1).

- There is another theoretical model of cash dividend payments which is widely used (and which we shall employ extensively). This is the model of a **continuous dividend yield**.

  1. This is a theoretical model, because no real company pays dividends continuously.
  2. However, it is a good approximation for stock indices. Stock indices are weighted averages of many stocks, which pay dividends amounts (of varying amounts) at multiple different dates. This can be approximated as a continuous dividend stream.
  3. Denote the continuous dividend yield by q. In a *short* time interval δt, the cash dividend amount paid $D_{\rm cts}$, if the stock price is S, is

  $$D_{\rm cts} = qS\,\delta t\,. \tag{5.10.2}$$

     This leads to an exponential continuous compounding formula. The fair value formula in eq. (5.10.1) is modified to

  $$F = e^{(r-q)(T-t_0)}\,S_0\,. \tag{5.10.3}$$

  4. Many authors define a parameter b, called the **cost of carry**, given by

  $$b = r - q\,. \tag{5.10.4}$$

     Then we can rewrite eq. (5.10.3) as

  $$F = e^{b(T-t_0)}\,S_0\,. \tag{5.10.5}$$

5.11 Stock indices & continuous dividend yields

- **This section was written in response to an excellent question from a student.**
- With correction of a few minor typos and adding boldface, the question/observation is:

  I am confused by the **stock index dividend yield.**
  I remember you said that you can't actually buy a stock index.
  So what does 'index dividend yield' mean?

- A value of a stock index is a weighted average of real stocks (30 stocks for the Dow Jones).
- The individual stocks (which make up the index) pay dividends at all different dates.
  1. Every day, if some of the stocks pay dividends, the index value must track those dividends.
  2. Hence we cannot buy the index but nevertheless the index has a dividend yield.
- Consider a forward contract on the index with a forward price F (with expiration time $T > t_0$).
- Compare it with a forward contract on a 'basket of stocks' where the basket is the weighted average of all the stocks which make up the index.
- The forward price F_2 of the second contract must be the same value $F_2 = F$ else there will be arbitrage.
- It is convenient to model the weighted dividends using a continuous yield.
- The formula for the forward price on the contract with the weighted average is

$$F_2 = S_0\, e^{(r-q)(T-t0)}\,. \tag{5.11.1}$$

- Here S_0 is the weighted average value of the stocks, which equals the value of the stock index.
- Hence the forward price F of the forward on the stock index is given by $F = F_2$, or

$$F_{\text{index}} = S_0\, e^{(r-q)(T-t0)}\,. \tag{5.11.2}$$

- The stock index must be priced with the same continuous dividend yield q to avoid arbitrage.
- Hence the concept of a continuous dividend yield on a stock index is a mathematical formula, but it is backed up by an arbitrage argument using the weighted average of the stocks which make up the index.

5.12 Basket of stocks

- When the number of stocks in a weighted average is small (e.g. two stocks), we do not use the term 'stock index' instead we call it a **basket of stocks.**
- This is called an **equity basket.**
- There also exist baskets of other financial securities, e.g. bonds.
- It is possible to buy shares of a basket of stocks.
- A company A may offer for sale a basket on the stocks of two companies B and C.
- Typically, such a basket is not an exchange-listed financial security and is traded privately.
- One share of the basket equals a weighted average of the stocks which make up the basket.
- There exist derivatives on baskets of stocks.
- When the stocks in the basket pay dividends, the weighted average dividend must be calculated and used in formulas to calculate the fair value of a derivative on the basket.

5.13 Spreads & arbitrage II: non-unique forward price with dividends

- Let the bid price be S_b and the ask price be S_a, where $S_a > S_b$.
  Let the borrow rate be r_b and the lending rate be r_l, where $r_b > r_l$.

- Suppose the stock pays a dividend stream D_1, D_2, etc. at times t_1, t_2, etc. during the time interval from today t_0 to the expiration date T. Then the fair value formulas eqs. (5.5.1) and (5.10.1) are modified as follows:

$$\left[S_b - \sum_{i=1,2,\dots} e^{-r_l(t_i-t_0)} D_i \right] e^{r_l(T-t_0)} \le F \le \left[S_a - \sum_{i=1,2,\dots} e^{-r_b(t_i-t_0)} D_i \right] e^{r_b(T-t_0)} . \qquad (5.13.1)$$

- *Present value:* the dividends are discounted using the
  (i) **lending rate r_l on the left**,
  (ii) **borrowing rate r_b on the right.**

- In the theoretical model where the stock pays dividends continuously at a rate q, then

$$S_b\, e^{(r_l-q)(T-t_0)} \quad \le \quad F \quad \le \quad S_a\, e^{(r_b-q)(T-t_0)} . \qquad (5.13.2)$$

5.14 Forwards & futures: obligations

- **Both parties are obligated to fulfil the terms of a forward or futures contract at expiration.**

- The buyer realizes, as the expiration date approaches, that the price of the underlying asset is going down (so $S < F$) and it would be cheaper to buy the underlying asset instead of honoring the forward contract (and pay the price F). The buyer cannot simply refuse to pay and instead purchase the underlying asset on the stock market at a lower price.

- The seller realizes, as the expiration date approaches, that the price of the underlying asset is going up (so $S > F$) and it would be cheaper to sell the underlying asset on the stock market instead of honoring the forward contract (and receve the price F). The seller cannot simply refuse to make delivery and instead sell the underlying asset on the stock market at a higher price.

- Suppose an investor buys a forward/futures contract. The next day the buyer finds another party who is willing to accept a lower forward price. Or perhaps the investor finds another party who offers better quality merchandise. The buyer cannot simply abandon the first contract.

- Suppose a person sells a forward/futures contract (for example a farmer, to secure a buyer for the farmer's crop of wheat, corn, etc.). Then next day the seller finds another party who is willing to pay a higher forward price. The seller cannot simply abandon the first contract.

- This is part of our assumption that there is no counterparty risk: a party to a trade cannot default on the contract. In practice, such things have to be enforced by contract law. In some countries the laws are not enforced strictly, which can lead to difficulty in doing business.

5.15 Unwinding of futures contracts

- The vast majority of futures contracts are **unwound.**
- An investor who has a position in a futures contract can **unwind** the futures contract by taking an **exactly offsetting position in the same futures contract.**
  1. An investor who is long futures contracts would sell futures contracts (of the same specification) to close out the long futures position.
  2. An investor who is short futures contracts would buy futures contracts (of the same specification) to close out the short futures position.
- Why are the vast majority of futures contracts unwound?
  1. There are multiple reasons. Most investors do not in fact want to take delivery of the underlying asset.
  2. The futures contracts are purchased to speculate in the financial markets. Instead of buying and selling stocks for speculation, they buy and sell futures contracts.
  3. **Speculators provide liquidity in the financial markets.**
     By actively buying and selling (not only futures but other financial instruments also), they keep the market prices of the financial instruments up to date.
  4. The futures contracts are purchased or sold for **hedging**. This is something we shall study in later lectures.

5.16 Open volume: how many futures contracts are there?

- The number of futures contracts on a asset is called the **open volume.**
- Unlike shares of stock on a company, the number of futures contracts is not fixed. A futures contract comes into existence when two parties initiate a futures trade (one side is long, the other is short). If the trade is for N futures contracts, then N futures contracts are created by that trade. These are new futures contracts, which did not exist before.
- There is **no upper limit** to the number of futures contracts which can be created (or traded).
- When I write that "futures contracts are listed on an exchange" or "futures contracts are exchange listed financial instruments" it means they are not provate deals. Anyone can buy of sell futures contracts.
- Just as stocks are publicly traded on a stock exchange, futures contracts are publicly traded on futures exchanges. (And we shall see later that options are publicly traded on options exchanges.)
- The open volume of a futures contract is equal to the sum of all the long positions in that futures. Because derivatives are a zero sum game, the open volume is also equal to the (negative?) sum of all the short positions in that futures.
- Obviously, there are different open volumes for futures with different expiration dates.
- The open volume of futures contracts can be (and is) huge. It is possible that there are so many futures contracts on an asset (for example crude oil) that the total quantity of the asset to be delivered is more than is available in reality.
- What will happen on the futures expiration date? The buyers unwind their positions prior to expiration and do not take delivery of the underlying asset. We shall discuss this below.
- **The same concept of "open volume" applies to all derivatives (options, etc,) not just futures.**

5.17 Futures: primary and secondary markets

- The **primary market** means buying/selling (going long/short) futures contracts which did not previously exist.
- The **secondary market** means selling (unwinding) futures contracts which the investor already owns. Or buying futures contracts from someone who already owns those contract.
- Therefore new futures contract come into existence when a primary market trade is initiated.
- In the secondary market, futures contracts which already exist are **transferred from one investor to another.**
- A trade in the primary market will change the open volume.
- A trade in the secondary market may or may not change the open volume.
  1. On day 0, suppose there are no futures contracts (the open volume is zero).
  2. On day 1, investor A goes long 1000 futures contracts (trading with investor B).
  3. This is a trade on the primary market and the open volume is now 1000 contracts.
  4. On day 2, investor A sells 700 of the 1000 futures contracts to investor C.
  5. This is a trade on the secondary market.
  6. The open volume does not change (1000 contracts).
  7. On day 3, A sells 300 futures contracts to B and C sells 700 futures contracts to B.
  8. These are trades on the secondary market and the open volume decreases to zero.
  9. All the investors A, B and C now have a position of zero futures contracts.
- **The same concepts of primary and secondary markets apply to all derivatives (options, etc,) not just futures.**
- It is helpful to make a table of the dates, positions and trades.

day	A	B	C	open volume	explanation
0	0	0 0	0	0	nothing has happened yet
1	1000	-1000	0	1000	primary market trade (A and B) contracts did not exist prior to the trade new contracts come into existence
2	300	-1000	700	1000	secondary market trade (A and C) A sold existing contracts to C no new contracts created by the trade
3	0	0	0	0	secondary market 2 trades (A,B) and (B,C) A and C both sold existing contracts to B all the contracts are unwound hence open volume = 0

5.18 Futures contracts: standardized terms and mismatches

- Unlike forwards, which are customized private agreements between two parties, the terms of a futures contract are ***specified by the exchange*** and are publicly known. The standardized terms include
  (i) specification of the asset.
  (ii) size of contract (quantity of asset to be delivered).
  (iii) terms of delivery (date & location).

- The standardization of futures contracts makes it easier for investors to trade them, but it does entail some potential difficulties. We examine the details in turn.

5.18.1 Specification of the asset

- Up to now we have visualized the underlying asset as a unique thing. In practice, the futures contract will list a variety of items which will constitute an acceptable asset. For example, for crude oil, there may be a variety of grades of crude oil (for example min/max sulfur content) which are considered to be acceptable.

- The seller of a futures contract can choose to deliver anything which meets the terms of the contract.

- The buyer of a futures contract has to accept whatever is delivered, and it may not be exactly what the buyer desires.

- Basically, the asset is whatever is the cheapest to deliver within the terms of the contract, and that may not be known at the time the contract is initiated.

5.18.2 Size of contract

- One crude oil futures is for 1000 barrels per contract (it could be 5000 bushels for corn, etc.) Hence by purchasing futures contracts, an investor will receive a multiple of 1000 barrels of crude oil. This may not be exactly the quantity the buyer wants. Conversly, it may not be exactly the quantity the seller wants. (Crude oil is probably a bad example in this context because the quantities traded are so large that 1000 barrels is negligible.)

5.18.3 Terms of delivery (date & location)

- After a futures contract has expired, the seller must deliver the asset to the buyer. However, it is too complicated and too expensive to perform door-to-door delivery for every trade. The exchange will specify the terms of delivery.

- The terms will state a begin/end date for delivery to be made. The seller has to deliver the asset in this time interval.

- The terms will state a set of locations where delivery can be made (for example a warehouse in a rail freight yard, or airport). The buyer may select the details. The seller must make delivery to that location. The buyer then has to transport the asset to its final location. It is not a door-to-door process.

5.19 "Interest rate" – counterparty risk

- Up to now, I have employed the term "interest rate" rather casually. It is time to make this concept more precise.

- Consider the following scenario. Two borrowers X and Y go to a bank to take out (separate) loans of amount L. We can say $L = \$1000$ for convenience, if we want explicit numbers. In both cases, the loan will be repaid, with interest, at a time T in the future. We can say $T = 1$ year. For convenience say $t = 0$ today.

  1. The bank checks the credit ratings of both X and Y.
  2. Person X has a good credit rating but Y has a bad credit rating. What will happen?
  3. In the case of X, the bank will approve the loan, and will charge a low interest rate, say $r_{\rm low}$. The amount to be repaid by X is
  $$\text{Repayment by X} = L\, e^{r_{\rm low}T}\,. \tag{5.19.1}$$
  In practice the loan will be repaid in installments, but the individual installmenrs will be small.
  4. In the case of Y, the bank may not approve the loan at all. Even if the bank approves the loan, it will charge a high interest rate, say $r_{\rm high}$. The amount to be repaid by Y is
  $$\text{Repayment by Y} = L\, e^{r_{\rm high}T}\,. \tag{5.19.2}$$
  This is a higher amount than X has to repay, for the same loan amount L. In practice the loan will be repaid in installments, but the individual installmenrs for Y will be higher than for X. This is because the bank is worried that Y might default and not repay, so the bank demands larger installments from Y.

- ***The interest rate depends on the counterparty.***

- More precisely, the interest rate depends on the **"creditworthiness"** of the counterparty.

- This is an example of **counterparty risk.**

5.20 Risk free rate

- This is a fundamental difficulty when calculating the fair value of a financial instrument.

- However, there is a value of the interest rate which all market participants can agree on.
  This is the interest rate charged when there is **no counterparty risk.**
  It is the lowest interest rate, by construction.
  It is called the **risk-free rate.**

- How does the risk-free rate work in practice?
  Of course everyone would be happy to *borrow* money at the risk-free rate, but why would anyone *loan* money at the risk-free rate?

- This is an important question.
  This is where the stock and futures and options exchanges come into play.

  1. Futures contracts are listed on **futures exchanges.**
  2. Futures contracts are an example of **exchange listed** financial instruments.
  3. When an investor buys an exchange listed futures contract, **the futures exchange guarantees that the investor will receive deivery of the underlying asset.**
  4. Conversely, for the counterparty which sold the above futures contract, **the futures exchange guarantees that the counterparty will be paid.**

- In this way, the futures exchange removes the counterparty risk to both sides of the trade.

- Hence the fair value of an exchange listed futures contract can be calculated using the risk-free rate.

- Although I used the term "futures exchange" above, the organization which steps in makes sure both sides honor their obligations is actually called a **clearing house**.

- The trade takes place on an exchange, and the clearing house enforces that it will be honored by both sides. The clearing house absorbs the counterparty risk, from both sides. If the buyer of a futures contract defaults and does not pay, the clearing house will take action to obtain payment. If the seller of a futures contract defaults and does not deliver the underlying asset, the clearing house will take action to secure delivery. The clearing house is a powerful organization with large legal resources. This allows all parties who trade exchange listed financial instruments to operate without counterparty risk.

- It is the same when buying stock on a stock exchange. When an investor buys shares of stock, the clearing house guarantees that that the buyer will receive the shares and the seller will be paid. The clearing house removes the counterparty risk to both sides of the trade.

- For private trades which are ***not*** carried out on an exchange, counterparty risk can be a serious problem.

5.21 Hedge

- The arbitrage proof for the fair value of a forward contract (see eq. (5.4.1)) is actually an example of the use of a **hedge.**
- If you know the term "hedging one's bets" in gambling, it is the same concept.
  If we have made a bet, a hedge is a collection of things (or actions) to guard against loss.
  A **perfect hedge** guards against all loss.
- Let us return to the forward contract. Suppose we are short a forward contract. We have made a commitment to deliver one share of stock at a price F at a future time T.
  1. *What shall we do?* Suppose we do nothing and wait until the expiration time T.
  2. If the stock price S_T at the time T is high, we would have to buy the stock on the stock market (and pay S_T), then deliver it to close the forward contract, for which we would receive the payment F. We would suffer a loss of $S_T - F$.
  3. *How shall we guard against the possibility of loss?*
- We form a hedge. We go long one share of stock today. At the expiration time T, we have one share of stock available to deliver, independent of the stock price S_T. *However, it is not free to buy the stock today.* It costs S_0. We have to borrow cash equal to S_0, and at the time T we have to repay the loan, which will be worth $S_0 e^{r(T-t_0)}$. Our total hedge is actually the stock and the loan.
- **A hedge is not free.** Although we completely eliminated the uncertainty about the value of S_T at the time T, it cost money to set up the hedge: we have to repay the loan of $S_0 e^{r(T-t_0)}$.
- Hence if $F > S_0 e^{r(T-t_0)}$, we make an arbitage profit, because the hedge costs less than F.
- Similarly, for a long position in the forward contract, our hedge is short one share of stock and cash worth $S_0 e^{r(T-t_0)}$ at the time T. If $F < S_0 e^{r(T-t_0)}$, we make an arbitage profit, because the hedge gives up more money than the amount F that we must pay.
- The stock/cash combination is a perfect hedge of the forward contract because it eliminates all uncertainty about what will happen at the time T. However, the hedge is not free.
- Notice another thing: a hedge not only guards against loss, it also eats up some of the profits (because the hedge costs money). *In fact, a perfect hedge not only guards against all loss, it also eliminates all profit.* The net value of (short derivative, long perfect hedge) or (long derivative, short perfect hedge) is zero.
- In the above case, the cost of the perfect hedge is $S_0 e^{r(T-t_0)}$ at the time T. Hence we say the fair value of the forward is $F = S_0 e^{r(T-t_0)}$.
- **Many experts <u>define</u> the fair value of a derivative as the cost of a perfect hedge.**
- The concept of hedging is important in finance. That is how we calculate the fair value of more complicated derivatives. We employ simpler financial instruments to form a perfect hedge. By definition, we know how to calculate the fair values of the simpler financial instruments. The cost of the perfect hedge is then defined as the fair value of the derivative.

5.22 Worked Example 1: hold a futures contract till expiration

- **We neglect the interest rate and compounding in the worked example below.**
- Suppose on day 0 the stock price is $S_0 = 100$ and the futures price is $F_0 = 103.7$.
- An investor goes long a futures contract on day 0. The futures contract expires on day 5.
- Here is a list of the stock and futures prices, for $i = 1, 2, 3, 4, 5$.

i	S_i	F_i	money received	money paid	calculation
0	100.0	103.7	0	0	–
1	99.5	103.1		0.6	$F_0 - F_1 = 0.6$
2	101.3	104.8	1.7		$F_1 - F_2 = -1.7$
3	101.3	101.1		3.7	$F_2 - F_3 = 3.7$
4	100.2	100.5		0.6	$F_3 - F_4 = 0.6$
5	99.3	99.3		3.2	$F_4 - F_5 = 1.2$

- As explained in the lectures, on the expiration date, the futures price converges to the stock price, so $F_5 = S_5$.
- On the expiration date (= day 5), the investor takes delivery of the stock and pays the stock price $S_5 = 99.3$.
- The total money paid is (stock price on expiration date) - (money received) + (money paid)

$$\text{Total money paid} = 99.3 - 1.7 + (0.6 + 3.7 + 0.6 + 1.2) = 103.7\,. \tag{5.22.1}$$

- **The total money paid equals F_0, i.e. the futures price on the day the investor went long the futures contract (= day 0).**
- The mark to market makes things complicated/confusing, so pay attention.
- ***Note that there is no profit or loss here.***
  1. The investor bought one share of stock, and paid some amount of money to buy it.
  2. Instead of buying the stock on day 0, the investor went long a futures contract and bought the stock (took delivery) on day 5.
  3. **Profit is when an investor buys something at a lower price and sells it at a higher price. Loss is when the selling price is lower than the buying price.**
- If I buy stock today (for example), there is no profit or loss.
- It is only when I sell the stock, then there is a profit or loss.
- In the above example, *the investor bought the stock*, i.e. took delivery on day 5.
- There was no statement about selling the stock.
- ***If the investor had sold the futures contract prior to expiration, then there would be profit/loss.***
- That is a different situation. See worked example #3.

5.23 Worked Example 2: hold a futures contract till expiration

- **We neglect the interest rate and compounding in the worked example below.**
- Suppose on day 0 the stock price is $S_0 = 100$ and the futures price is $F_0 = 103.7$.
- An investor goes long a futures contract on day 0. The futures contract expires on day 5.
- Here is a different list of the stock and futures prices, for $i = 1, 2, 3, 4, 5$.

i	S_i	F_i	money received	money paid	calculation
0	100.0	103.7	0	0	–
1	102.7	103.2		0.5	$F_0 - F_1 = 0.5$
2	102.8	104.8	1.6		$F_1 - F_2 = -1.6$
3	106.3	107.1	2.3		$F_2 - F_3 = -2.3$
4	104.2	104.5		2.6	$F_3 - F_4 = 2.6$
5	105.3	105.3	0.8		$F_2 - F_3 = -0.8$

- As explained in the lectures, on the expiration date, the futures price converges to the stock price, so $F_5 = S_5$.
- On the expiration date (= day 5), the investor takes delivery of the stock and pays the stock price $S_5 = 105.3$.
- The total money paid is (stock price on expiration date) - (money received) + (money paid)

$$\text{Total money paid} = 105.3 - (1.6 + 2.3 + 0.8) + (0.5 + 2.6) = 103.7\,. \tag{5.23.1}$$

- **The total money paid equals F_0, i.e. the futures price on the day the investor went long the futures contract (= day 0).**
- ***Once again, there is no profit or loss.***
    1. The investor bought one share of stock, and paid some amount of money to buy it.
    2. Instead of buying the stock on day 0, the investor went long a futures contract and bought the stock (took delivery) on day 5.
- **In both worked examples #1 and #2, the futures price was $F_0 = 103.7$ on day 0, when the investor went long the futures contract, and F_0 is the overall amount of money the investor pays on the expiration date (= day 5), when the investor takes delivery of the underlying stock.**

5.24 Worked Example 3: unwind a futures contract before expiration

- **We neglect the interest rate and compounding in the worked example below.**
- **We use the same random walk data as in worked example #1.**
- **However, now the investor unwinds the futures contract prior to expiration.**
- Suppose on day 0 the stock price is $S_0 = 100$ and the futures price is $F_0 = 103.7$.
- An investor goes long a futures contract on day 0.
- The futures contract expires on day 5.
- Here is a list of the stock and futures prices, for $i = 1, 2, 3, 4, 5$ (same as worked example #1).

i	S_i	F_i	received	paid	unwind	money in account (on unwind date)	profit/loss
0	100.0	103.7	0	0	–	–	–
1	99.5	103.1		0.6	day 1	-0.6	loss
2	101.3	104.8	1.7		day 2	$-0.6 + 1.7 = 1.1$	profit
3	101.3	101.1		3.7	day 3	$-0.6 + 1.7 - 3.7 = -2.6$	loss
4	100.2	100.5		0.6	day 4	$-0.6 + 1.7 - 3.7 - 0.6 = -3.2$	loss
5	99.3	99.3		3.2	expiration		

- The table shows what happens if the investor unwinds the futures contract of day 1 or day 2 or day 3 or day 4.
  1. The investor receives or loses the money accumulated in the mark to market account to the unwind date.
  2. If the amount is positive, it is a profit to the investor.
  3. If the amount is negative, it is a loss to the investor.
- *On day 5 it is impossible to unwind. It is the expiration date. The investor must take delivery.*
- **A futures contract can only be unwound up to the day before expiration.**
- **Unwinding a futures contract on the day before expiration is common, in practice.**

5.25 Worked Example 4: unwind a futures contract before expiration

- **We neglect the interest rate and compounding in the worked example below.**
- **We use the same random walk data as in worked example #2.**
- **However, now the investor unwinds the futures contract prior to expiration.**
- Suppose on day 0 the stock price is $S_0 = 100$ and the futures price is $F_0 = 103.7$.
- An investor goes long a futures contract on day 0.
- The futures contract expires on day 5.
- Here is a list of the stock and futures prices, for $i = 1, 2, 3, 4, 5$ (same as worked example #2).

i	S_i	F_i	received	paid	unwind	money in account (on unwind date)	profit/loss
0	100.0	103.7	0	0	–	–	–
1	102.7	103.2		0.5	day 1	-0.5	loss
2	102.8	104.8	1.6		day 2	$-0.5 + 1.6 = 1.1$	profit
3	106.3	107.1	2.3		day 3	$-0.5 + 1.6 + 2.3 = 3.4$	profit
4	104.2	104.5		2.6	day 4	$-0.5 + 1.6 + 2.3 - 2.6 = 0.8$	profit
5	105.3	105.3	0.8		expiration		

- The table shows what happens if the investor unwinds the futures contract of day 1 or day 2 or day 3 or day 4.
  1. The investor receives or loses the money accumulated in the mark to market account to the unwind date.
  2. If the amount is positive, it is a profit to the investor.
  3. If the amount is negative, it is a loss to the investor.
- *On day 5 it is impossible to unwind. It is the expiration date. The investor must take delivery.*
- **A futures contract can only be unwound up to the day before expiration.**
- **Unwinding a futures contract on the day before expiration is common, in practice.**