Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Spring 2018

Instructor: Dr. Sateesh Mane

May 3, 2018

16 Lecture 16a

Numerical solution of systems of ordinary differential equations

- We display worked examples of **initial value problems using auxiliary variables.**

16.12 Worked example 1

16.12.1 Equation

- Consider the following ordinary differential equation

$$\frac{d^2y}{dx^2} + 2x\,\frac{dy}{dx} + (1-x^2)y = e^x\,. \tag{16.12.1}$$

- It is a second order ordinary differential equation. We introduce an auxiliary variable v via

$$v = \frac{dy}{dx}\,. \tag{16.12.2}$$

- We reexpress eq. (16.12.1) as a pair of coupled first order equations as follows:

$$\frac{dy}{dx} = v\,, \tag{16.12.3a}$$

$$\frac{dv}{dx} = -2xv - (1-x^2)y + e^x\,. \tag{16.12.3b}$$

- We express this in the formal notation as follows. To avoid confusion between '$\boldsymbol{y}$' and the original variable y, define $\boldsymbol{y} = (y_1, y_2) = (u, v)$.

- Then u is the original scalar variable y and the equations are

$$\begin{aligned} \frac{d\boldsymbol{y}}{dx} &= \boldsymbol{f}(x,\boldsymbol{y})\,, \\ \frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} f_1(x,u,v) \\ f_2(x,u,v) \end{pmatrix} = \begin{pmatrix} v \\ -2xv - (1-x^2)y + e^x \end{pmatrix}\,. \end{aligned} \tag{16.12.4}$$

- Therefore the right hand side functions are

$$f_1(x,u,v) = v\,, \tag{16.12.5a}$$

$$f_2(x,u,v) = -2xv - (1-x^2)u + e^x\,. \tag{16.12.5b}$$

16.12.2 C++ code

- In terms of C++ function calls, we have $m = 2$ and

```
int f(int m, double x, const std::vector<double> & y, std::vector<double> & g)
{
  // first component "f1(x,u,v) = v"
  g[0] = y[1];

  // second component "f2(x,u,v) = -2xv -(1-x^2)y + exp(x)"
  g[1] = -2.0*x*y[1] - (1.0 - x*x)*y[0] + exp(x);

  return 0;
}
```

All the integration schemes can be called with the following inputs:

```
// m = 2
// x = x_i
// h = step size
// vector (array) y_in  = (u, v)_i
// vector (array) y_out = (u, v)_{i+1}

int Euler_explicit(int m, double x, double h,
                   std::vector<double> & y_in,
                   std::vector<double> & y_out);

int midpoint(int m, double x, double h,
             std::vector<double> & y_in,
             std::vector<double> & y_out);

int trapezoid(int m, double x, double h,
              std::vector<double> & y_in,
              std::vector<double> & y_out);

int RK4(int m, double x, double h,
        std::vector<double> & y_in,
        std::vector<double> & y_out);
```

16.12.3 Solution for $i = 0$, etc

- Let the initial conditions at $x_0 = 0$ be $y_0 = 1$ and $v_0 = y'_0 = -1$.
- We shall integrate eq. (16.12.4) using explicit Euler integration with a stepsize h.
- The formal equations are

$$\boldsymbol{y}_{i+1} = \boldsymbol{y}_i + h\boldsymbol{f}(x, \boldsymbol{y}) \,. \tag{16.12.6}$$

- In terms of u and v, the equations are

$$\begin{pmatrix} u \\ v \end{pmatrix}_{i+1} = \begin{pmatrix} u \\ v \end{pmatrix}_i + h \begin{pmatrix} v_i \\ -2x_i v_i - (1 - x_i^2)u_i + \exp(x_i) \end{pmatrix} \,. \tag{16.12.7}$$

- Step $i = 0$:
  1. Initially $u_0 = 1$ and $v_0 = -1$. Then

$$u_1 = u_0 + hv_0 = 1 - h \,. \tag{16.12.8}$$

  2. Next we have

$$\begin{aligned} v_1 &= v_0 + h\bigl(-2x_0 v_0 - (1 - x_0^2)u_0 + \exp(x_0)\bigr) \\ &= -1 + h\bigl(0 - (1 - 0) + \exp(0)\bigr) \\ &= -1 \,. \end{aligned} \tag{16.12.9}$$

- Step $i = 1$:
  1. We have

$$u_2 = u_1 + hv_1 = 1 - h - h = 1 - 2h \,. \tag{16.12.10}$$

  2. Next we have

$$\begin{aligned} v_2 &= v_1 + h\bigl(-2x_1 v_1 - (1 - x_1^2)u_1 + \exp(x_1)\bigr) \\ &= -1 + h\bigl(2h - (1 - h^2)(1 - h) + \exp(h)\bigr) \\ &= -1 + h\bigl(-1 + 3h + h^2 - h^3 + \exp(h)\bigr) \\ &= -1 - h + 3h^2 + h^3 - h^4 + h\exp(h) \,. \end{aligned} \tag{16.12.11}$$

- Step $i = 2$:
  1. We have

$$\begin{aligned} u_3 = u_2 + hv_2 &= 1 - 2h + h\bigl(-1 - h + 3h^2 + h^3 - h^4 + h\exp(h)\bigr) \\ &= 1 - 3h - h^2 + 3h^3 + h^4 - h^5 + h^2\exp(h) \,. \end{aligned} \tag{16.12.12}$$

  2. Next we have

$$\begin{aligned} v_3 &= v_2 + h\bigl(-2x_2 v_2 - (1 - x_2^2)u_2 + \exp(x_2)\bigr) \\ &= v_2 + h\bigl(-4hv_2 - (1 - 4h^2)u_2 + \exp(2h)\bigr) \\ &= \text{ugh} \,. \end{aligned} \tag{16.12.13}$$

16.13 Worked example 2

16.13.1 Equation

- Consider the following ordinary differential equation

$$\frac{d^2y}{dx^2} + y = 0\,. \tag{16.13.1}$$

- This is a simple equation. We know the general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x)\,. \tag{16.13.2}$$

- It is a second order ordinary differential equation. We introduce an auxiliary variable v via

$$v = \frac{dy}{dx}\,. \tag{16.13.3}$$

- We reexpress eq. (16.13.1) as a pair of coupled first order equations as follows:

$$\frac{dy}{dx} = v\,, \tag{16.13.4a}$$

$$\frac{dv}{dx} = -y\,. \tag{16.13.4b}$$

- We express this in the formal notation as follows. Again define $\boldsymbol{y} = (y_1, y_2) = (u, v)$. Then

$$\begin{gathered} \frac{d\boldsymbol{y}}{dx} = \boldsymbol{f}(x, \boldsymbol{y})\,, \\ \frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1(x,u,v) \\ f_2(x,u,v) \end{pmatrix} = \begin{pmatrix} v \\ -u \end{pmatrix}\,. \end{gathered} \tag{16.13.5}$$

- Therefore the right hand side functions are

$$f_1(x,u,v) = \quad v\,, \tag{16.13.6a}$$

$$f_2(x,u,v) = -u\,. \tag{16.13.6b}$$

16.13.2 C++ code

- In terms of C++ function calls, we have $m = 2$ and

```
int f(int m, double x, const std::vector<double> & y, std::vector<double> & g)
{
  // first component "f1(x,u,v) = v"
  g[0] = y[1];

  // second component "f2(x,u,v) = -u"
  g[1] = -y[0];

  return 0;
}
```

All the integration schemes can be called with the following inputs:

```
// m = 2
// x = x_i
// h = step size
// vector (array) y_in  = (u, v)_i
// vector (array) y_out = (u, v)_{i+1}

int Euler_explicit(int m, double x, double h,
                   std::vector<double> & y_in,
                   std::vector<double> & y_out);

int midpoint(int m, double x, double h,
             std::vector<double> & y_in,
             std::vector<double> & y_out);

int trapezoid(int m, double x, double h,
              std::vector<double> & y_in,
              std::vector<double> & y_out);

int RK4(int m, double x, double h,
        std::vector<double> & y_in,
        std::vector<double> & y_out);
```

16.13.3 Solution for $i = 0$, etc

- Let the initial conditions at $x_0 = 0$ be $y_0 = 1$ and $v_0 = y'_0 = 0$.
- Then we know the exact solution is
$$y_{\text{exact}}(x) = \cos(x)\,. \tag{16.13.7}$$
- We shall integrate eq. (16.13.5) using explicit Euler integration with a stepsize h.
- The formal equations are
$$\boldsymbol{y}_{i+1} = \boldsymbol{y}_i + h\boldsymbol{f}(x, \boldsymbol{y})\,. \tag{16.13.8}$$
- In terms of u and v, the equations are
$$\begin{pmatrix} u \\ v \end{pmatrix}_{i+1} = \begin{pmatrix} u \\ v \end{pmatrix}_i + h \begin{pmatrix} v_i \\ -u_i \end{pmatrix}\,. \tag{16.13.9}$$
- Step $i = 0$:
  1. Initially $u_0 = 1$ and $v_0 = 0$ and so
  $$u_1 = u_0 + hv_0 = 1 - 0 = 1\,. \tag{16.13.10}$$
  2. Next we have
  $$v_1 = v_0 - hu_0 = 0 - h = -h\,. \tag{16.13.11}$$
- Step $i = 1$:
  1. We have
  $$u_2 = u_1 + hv_1 = 1 + h(-h) = 1 - h^2\,. \tag{16.13.12}$$
  2. Next we have
  $$v_2 = v_1 - hu_1 = -h - h = -2h\,. \tag{16.13.13}$$
- Step $i = 2$:
  1. We have
  $$u_3 = u_2 + hv_2 = 1 - h^2 + h(-2h) = 1 - 3h^2\,. \tag{16.13.14}$$
  2. Next we have
  $$v_3 = v_2 - hu_2 = -2h - h(1 - h^2) = -3h + h^3\,. \tag{16.13.15}$$
- Step $i = 3$:
  1. We have
  $$u_4 = u_3 + hv_3 = 1 - 3h^2 + h(-3h + h^3) = 1 - 6h^2 + h^4\,. \tag{16.13.16}$$
  2. Next we have
  $$v_4 = v_3 - hu_3 = -3h + h^3 - h(1 - 3h^2) = -4h + 4h^3\,. \tag{16.13.17}$$
- We see that the explicit Euler method is not very accurate.

16.13.4 Solution using midpoint method

- Let us integrate eq. (16.13.5) using the midpoint method with a stepsize h.
- In terms of u and v, the equations are

$$\begin{pmatrix} u \\ v \end{pmatrix}_{i+1} = \begin{pmatrix} u \\ v \end{pmatrix}_{i} + h \begin{pmatrix} v_i \\ -u_i \end{pmatrix} . \tag{16.13.18}$$

- However, now we require the values intermediate points.
  1. First we have, at the step i,

$$\boldsymbol{g}_1 = \begin{pmatrix} g_i^u \\ g_i^v \end{pmatrix} = \begin{pmatrix} v_i \\ -u_i \end{pmatrix} . \tag{16.13.19}$$

  2. Next we have

$$\boldsymbol{g}_2 = \begin{pmatrix} g_1^u \\ g_1^v \end{pmatrix} = \begin{pmatrix} f_1(x_i + \frac{1}{2}h, u_i + \frac{h}{2}g_1^u, v_i + \frac{h}{2}g_1^v) \\ f_2(x_i + \frac{1}{2}h, u_i + \frac{h}{2}g_1^u, v_i + \frac{h}{2}g_1^v) \end{pmatrix} = \begin{pmatrix} v_i - \frac{h}{2}u_i \\ -u_i - \frac{h}{2}v_i \end{pmatrix} . \tag{16.13.20}$$

- Step $i = 0$:
  1. We have

$$g_2^u = v_0 - \frac{h}{2}\, u_0 = 0 - \frac{h}{2} . \tag{16.13.21}$$

  2. Next

$$g_2^v = -u_0 - \frac{h}{2}\, v_0 = -1 . \tag{16.13.22}$$

  3. Then

$$\begin{aligned} u_1 &= u_0 + hg_2^u = 1 - \frac{h^2}{2} , \\ v_1 &= v_0 + hg_2^v = 0 - h = -h . \end{aligned} \tag{16.13.23}$$

- Step $i = 1$:

1. **Now g_2^u and g_2^v refer to the step $i = 1$.**

$$(g_2^u)_{i=1} = v_1 - \frac{h}{2}\,u_1 = -h - \frac{h}{2}\Big(1 - \frac{h^2}{2}\Big) = -\frac{3h}{2} + \frac{h^3}{4}\,. \tag{16.13.24}$$

2. Next

$$(g_2^v)_{i=1} = -u_1 - \frac{h}{2}\,v_1 = -1 + \frac{h^2}{2} + \frac{h^2}{2} = -1 + h^2\,. \tag{16.13.25}$$

3. Then

$$\begin{aligned} u_2 = u_1 + h(g_2^u)_{i=1} &= 1 - \frac{h^2}{2} + h\Big(-\frac{3h}{2} + \frac{h^3}{4}\Big) \\ &= 1 - 2h^2 + \frac{h^4}{4}\,, \\ v_2 = v_1 + h(g_2^v)_{i=1} &= -h + h(-1 + h^2) \\ &= -2h + h^3\,. \end{aligned} \tag{16.13.26}$$

- Step $i = 2$:

1. **Now g_2^u and g_2^v refer to the step $i = 2$.**

$$(g_2^u)_{i=2} = v_2 - \frac{h}{2}\,u_2 = -2h + h^3 - \frac{h}{2}\Big(1 - 2h^2 + \frac{h^4}{4}\Big) = -\frac{5h}{2} + 2h^3 - \frac{h^5}{8}\,. \tag{16.13.27}$$

2. Next

$$(g_2^v)_{i=2} = -u_2 - \frac{h}{2}\,v_2 = -1 + 2h^2 - \frac{h^4}{4} + h^2 - \frac{h^4}{2} = -1 + 3h^2 - \frac{3h^4}{4}\,. \tag{16.13.28}$$

3. Then

$$\begin{aligned} u_3 = u_2 + h(g_2^u)_{i=2} &= 1 - 2h^2 + \frac{h^4}{4} + h\Big(-\frac{5h}{2} + 2h^3 - \frac{h^5}{8}\Big) \\ &= 1 - \frac{9h^2}{2} + \frac{9h^4}{4} - \frac{h^6}{8}\,, \\ v_3 = v_2 + h(g_2^v)_{i=2} &= -2h + h^3 + h\Big(-1 + 3h^2 - \frac{3h^4}{4}\Big) \\ &= -3h + 4h^3 - \frac{3h^5}{4}\,. \end{aligned} \tag{16.13.29}$$

- This is more accurate than the explicit Euler method. It matches the exact solution to $O(h^2)$.