Queens College, CUNY, Department of Computer Science

Computational Finance

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3 Lecture 3

3.1 Bond: more general pricing formula

- Recall that a bond pays cashflows of coupons and redeems its face value at maturity. Symbolically, we can write the price of a bond as a sum of the present values (PV) of n cashflows

$$B = \sum_{i=1}^{n} \mathrm{PV}(\mathrm{CF})_i \,. \tag{3.1.1}$$

- Let us write a pricing formula for a bond which is *not* newly issued. Let the time toay be t_0. Let there be n cashflows remaining to maturity, with amounts $(\mathrm{CF})_i$, paid at times t_i, $i = 1, 2, \dots, n$. (Cashflows which are in the past do not matter anymore.) We no longer assume all the coupons are equal. Note that the final cashflow includes the face of the bond. Let there be k coupon payments per year.

- The formula for the price of such a bond is, in terms of the yield y,

$$B = \sum_{i=1}^{n} \frac{(\mathrm{CF})_i}{(1+y/k)^{k(t_i-t_0)}} \,. \tag{3.1.2}$$

- If all the coupons are equal then the individual coupon payments are c/k.

- If the bond is newly issued then $t_i - t_0 = i/k$, so $k(t_i - t_0) = i$, as we have seen.

3.2 Bond duration

- The **duration** is defined as follows. It is also called the **Macaulay duration**, to avoid confusion with other terminology (see below). It was introduced by a person named Frederick Macaulay (approximately in the 1930s). It is a time weighted average of the cashflows

$$D_{\text{Macaulay}} = \frac{\sum_{i=1}^{n}(t_i - t_0)\,\mathrm{PV}(\mathrm{CF})_i}{\sum_{i=1}^{n}\mathrm{PV}(\mathrm{CF})_i} = \frac{1}{B}\sum_{i=1}^{n}(t_i - t_0)\,\mathrm{PV}(\mathrm{CF})_i\,. \tag{3.2.1}$$

- By construction, the Macaulay duration has units of time (or is measured in years).

- By analogy with physics, the Macaulay duration is analogous to a "center of mass" of a bond. Let us draw time on the horizontal axis and visualize the present values of the cashflows as "masses" located at points t_i. We can also think of the time t_i as the "maturity" (i.e. payment date) of the cashflow $(\mathrm{CF})_i$. The Macaulay duration is the "center of maturity" of the present valued cashflows of the bond. It is effectively the "average time to maturity" of the bond.

- There is also a term called the **modified duration** which is more useful in many respects.

- For this reason, the term "duration" without qualification can be confusing. It is better to say "Macaulay duration" or "modified duration" to avoid ambiguity.

- The modified duration is defined via a partial derivative with respect to the bond's yield

$$D_{\text{mod}} = -\frac{1}{B}\frac{\partial B}{\partial y}\,. \tag{3.2.2}$$

- The modified duration also has units of time (or is measured in years).

- Because the modified duration is a partial derivative, it is a measure of the sensivity of the bond price to a small change in the yield. Using a Taylor series, the change in bond price for a small change in yield is

$$B(y + \Delta y) \simeq B(y)\left[\,1 - D_{\text{mod}}\,\Delta y\,\right]. \tag{3.2.3}$$

3.3 Bond DV01

- In the financial markets, interest rates and yields are usually measured in **basis points**.
- One basis point is 1/100 of one percent, i.e. in decimal a change in yield of one basis point is $\Delta y = 0.0001$.
- The meaning of "**DV01**" (dollar value zero one) is the change in the dollar value (price) of a bond for a one basis point change in yield.
- In terms of the modified duration, the DV01 is given by (note the minus sign)

$$\mathrm{DV01} = -\frac{\partial B}{\partial y}\,\Delta y = B\,D_{\mathrm{mod}}\,\Delta y = 0.0001 \times B\,D_{\mathrm{mod}}\,. \tag{3.3.1}$$

  In the last step a one basis point value $\Delta y = 0.0001$ was substituted for Δy.

- Many times, people want the change for a 100 basis point move in yield (i.e. one percent). This is simply given by multiplying the DV01 value by 100.

3.4 Bond convexity

- Why stop at the first partial derivative?
- The **convexity** of a bond is given by the second partial derivative

$$C = \frac{1}{B}\,\frac{\partial^2 B}{\partial y^2}\,. \tag{3.4.1}$$

- In terms of the modified duration,

$$\begin{aligned}
C &= \frac{1}{B}\,\frac{\partial}{\partial y}\Big(\frac{\partial B}{\partial y}\Big) \\
&= \frac{1}{B}\,\frac{\partial}{\partial y}(-BD_{\rm mod}) \\
&= \frac{1}{B}\left[-\frac{\partial B}{\partial y}\,D_{\rm mod} - B\,\frac{\partial D_{\rm mod}}{\partial y}\right] \\
&= D_{\rm mod}^2 - \frac{\partial D_{\rm mod}}{\partial y}\,.
\end{aligned} \tag{3.4.2}$$

- The convexity of a bond is usually positive.
- We can extend the Taylor series to include a second order change in the yield

$$B(y+\Delta y) \simeq B(y)\left[\,1 - D_{\rm mod}\,\Delta y + \tfrac{1}{2}\,C\,(\Delta y)^2\,\right]. \tag{3.4.3}$$

3.5 Bond: formulas using yield

- Recall the pricing formula

$$B = \sum_{i=1}^{n} \frac{(\mathrm{CF})_i}{(1+y/k)^{k(t_i-t_0)}} \,. \tag{3.5.1}$$

- The Macaulay duration is

$$D_{\mathrm{Macaulay}} = \frac{1}{B} \sum_{i=1}^{n} (t_i - t_0)\, \frac{(\mathrm{CF})_i}{(1+y/k)^{k(t_i-t_0)}} \,. \tag{3.5.2}$$

- The modified duration is

$$\begin{aligned} D_{\mathrm{mod}} &= \frac{1}{B} \sum_{i=1}^{n} (t_i - t_0)\, \frac{(\mathrm{CF})_i}{(1+y/k)^{1+k(t_i-t_0)}} \\ &= \frac{1}{1+y/k}\, \frac{1}{B} \sum_{i=1}^{n} (t_i - t_0)\, \frac{(\mathrm{CF})_i}{(1+y/k)^{k(t_i-t_0)}} \,. \end{aligned} \tag{3.5.3}$$

- We obtain the following (important) relation between the two expressions

$$D_{\mathrm{mod}} = \frac{D_{\mathrm{Macaulay}}}{1+y/k} \,. \tag{3.5.4}$$

- The convexity is given by

$$\begin{aligned} D_{\mathrm{mod}} &= \frac{1}{B} \sum_{i=1}^{n} (t_i - t_0)\, \frac{1+k(t_i-t_0)}{k}\, \frac{(\mathrm{CF})_i}{(1+y/k)^{2+k(t_i-t_0)}} \\ &= \frac{1}{(1+y/k)^2}\, \frac{1}{B} \sum_{i=1}^{n} (t_i - t_0)\, \frac{1+k(t_i-t_0)}{k}\, \frac{(\mathrm{CF})_i}{(1+y/k)^{k(t_i-t_0)}} \,. \end{aligned} \tag{3.5.5}$$

  We see from this expression that the convexity is positive.

3.6 Zero coupon bonds

- There is an important class of bonds which are called **zero coupon bonds**.
- As the name suggests, zero coupon bonds pay no coupons. They pay only one cashflow, on the maturity date, which is the face of the bond.
- Zero coupon bonds are popular financial instruments. They allow an investor to perform hedging of cashflows at a particular point in time, without the complicaton of juggling additional cashflows (the coupons), which might interfere with other activities the investor is performing.
- From the point of view of the finance academics, zero coupons bonds are theoretically simpler to analyze.
- Since there is only one cashflow ($= F$, the face value), let us say the maturity date is T (where obviously $T > t_0$). In terms of the yield, the price of a zero coupon bond is given by

$$B = \frac{F}{(1 + y/k)^{k(T-t_0)}} \,. \tag{3.6.1}$$

- As stupid as it sounds, there is still a parameter k, simply because of market conventions. It is awkward to maintain inventory of bonds in a database if some bonds have a parameter k (frequency of cashflows) and for zero coupon bonds the parameter k is absent. Hence for quoting conventions, in the USA the yield is typically quoted on a semi-annual basis ($k = 2$).
- It is easy to invert the above formula to calculate the yield of a zero coupon bond from the market price of the bond.
- To the extent that interest rates and yields are positive, the price of a zero coupon bond is less than par.
- **The Macaulay duration of a zero coupon bond is equal to its time to maturity**

$$D_{\text{Macaulay}} = \frac{(T - t_0)\,\text{PV(CF)}_{\text{maturity}}}{\text{PV(CF)}_{\text{maturity}}} = T - t_0 \,. \tag{3.6.2}$$

  If there is only one cashflow then the "weighted average" is obviously the time to that cashflow.
- The modified duration has a slightly different value

$$D_{\text{mod}} = \frac{D_{\text{Macaulay}}}{1 + y/k} = \frac{T - t_0}{1 + y/k} \,. \tag{3.6.3}$$