Queens College, CUNY, Department of Computer Science
Numerical Methods
CSCI 361 / 761
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10 Homework lecture 10

- Please email your solution, as a file attachment, to `Sateesh.Mane@qc.cuny.edu`.
- Please submit one zip archive with all your files in it.
  1. The zip archive should have either of the names (CS361 or CS761):

     `StudentId_first_last_CS361_hw10.zip`

     `StudentId_first_last_CS761_hw10.zip`
  2. The archive should contain one "text file" named "hw10.[txt/docx/pdf]" and one cpp file per question named "Q1.cpp" and "Q2.cpp" etc.
  3. Note that not all homework assignments may require a text file.
  4. Note that not all questions may require a cpp file.

Linear algebra: motivating example (control theory)

- There are many occasions when we have a system whose state is controlled by a set of variables. For example a spacecraft is supposed to follow a "design trajectory" but in practice we observe its motion is deviating from that trajectory, say in three coordinates (x, y, z).

- Let us say the design trajectory is (x_d, y_d, z_d). Let us also say the observed trajectory deviates from the design by $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$, where the notation "$\varepsilon$" suggests a small deviation.

- We can control the motion of the spacecraft with three parameters (α, β, γ). (They might be the power to rocket thrusters.) We want to set the values of α, β and γ to cancel the unwanted deviations.

  1. If we set $\alpha = 1$ (and $\beta = \gamma = 0$), we calculate that the change in the trajectory is
  $$(\text{change in trajectory if } \alpha = 1) = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}. \tag{10.0.1}$$
  2. If we set $\beta = 1$ (and $\alpha = \gamma = 0$), we calculate that the change in the trajectory is
  $$(\text{change in trajectory if } \beta = 1) = \begin{pmatrix} b_x \\ b_y \\ c_z \end{pmatrix}. \tag{10.0.2}$$
  3. If we set $\gamma = 1$ (and $\alpha = \beta = 0$), we calculate that the change in the trajectory is
  $$(\text{change in trajectory if } \gamma = 1) = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}. \tag{10.0.3}$$

- Then to cancel the unwanted deviations, we want to find a set of joint values for (α, β, γ) such that the change they produce is the negative $(-\varepsilon_x, -\varepsilon_y, -\varepsilon_z)$.

- Hence the equations to solve for α, β and γ are
$$\begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -\varepsilon_x \\ -\varepsilon_y \\ -\varepsilon_z \end{pmatrix} = -\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}. \tag{10.0.4}$$

- Problems like this arise frequently in many contexts.

- The subject area is called **control theory.**

- **There is nothing for you to solve here.**

- **It is an example to motivate the subject of linear algebra.**

10.1 Linear algebra: 2×2 matrix

- The case of 2×2 matrices arises frequently in practice. There are two unknowns x_1 and x_2.
- It is not necessary to employ fancy mathematical formalism for this simple case.
- Let the matrix equation be $A\boldsymbol{x} = \boldsymbol{r}$ which has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} . \tag{10.1.1}$$

- The solution is $\boldsymbol{x} = A^{-1}\boldsymbol{r}$. The inverse matrix is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \tag{10.1.2}$$

- You are given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} . \tag{10.1.3}$$

- **Calculate the inverse matrix A^{-1}.**
- **Solve the following matrix equation using the inverse matrix A^{-1}.**

$$\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{10.1.4}$$

- **Calculate the inverse matrix for the following set of equations.**
  Then solve the equations for x_1 and x_2.

$$\begin{pmatrix} -3 & \frac{1}{2} \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} . \tag{10.1.5}$$

10.2 Linear algebra: LU decomposition with partial pivoting

10.2.1 Eliminate a_{i1} in first column

- In this question we shall calculate the LU decomposition of the matrix A.
- You are given the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ -3 & 2 & 3 \end{pmatrix} . \tag{10.2.1}$$

- In addition, the initial array of the swap indices is $S = (1, 2, 3)$.
- **State the values of $\hat{a}_i$ in each row $i = 1, 2, 3$.**
- **Calculate the values of $|a_{i1}|/\hat{a}_i$ for $i = 1, 2, 3$.**
- **State which pair of rows will be swapped.**
- **Write down the matrix after swapping the rows.**

$$A_1(\text{swap}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} . \tag{10.2.2}$$

- **Update the array of the swap indices and state the answer $S = (?, ?, ?)$.**
- **Subtract multiples of row 1 to eliminate the coefficient a_{i1} in rows 2 and 3.**
- **Write down the matrix after eliminating a_{i1} from rows 2 and 3.**

$$A_2(\text{eliminate } a_{i2}) = \begin{pmatrix} ? & ? & ? \\ \mathbf{0} & ? & ? \\ \mathbf{0} & ? & ? \end{pmatrix} . \tag{10.2.3}$$

- Fill in the blank spots (marked by $\mathbf{0}$ in eq. (10.2.3)) with the relevant multipliers, to obtain the "L" part of the LU decomposition.
- **Write down the matrix after filling the blank spots.**

$$A_3(\text{fill in blank spots}) = \begin{pmatrix} ? & ? & ? \\ \boldsymbol{\alpha} & ? & ? \\ \boldsymbol{\beta} & ? & ? \end{pmatrix} . \tag{10.2.4}$$

- **Write down the values of α and β used to fill the blank spots.**
- **The values of α and β should be negative numbers.**

10.2.2 Eliminate a_{i2} in second column

- Now we move on to the elimination of a_{i2} in the second column.
- **State the new values of $\hat{a}_i$ in rows $i = 2, 3$ ONLY. Do NOT process row 1.**
- **Calculate the values of $|a_{i2}|/\hat{a}_i$ for $i = 2, 3$.**
- **State which pair of rows will be swapped.**
  It is possible that no swap is required. Then say so.
- **If a swap is required, then write down the matrix after swapping the rows.**

$$A_4(\text{swap if necessary}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} . \tag{10.2.5}$$

- **If a swap is required, then update the array of the swap indices and state the answer $S = (?, ?, ?)$.**
- **Subtract a multiple of row 2 to eliminate the coefficient a_{i2} in row 3.**
- **Write down the matrix after eliminating a_{i2} from row 3.**

$$A_5(\text{eliminate } a_{i2}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & \mathbf{0} & ? \end{pmatrix} . \tag{10.2.6}$$

- Fill in the blank spot (marked by **0** in eq. (10.2.4)) with the relevant multiplier, to obtain the "L" part of the LU decomposition.
- **Write down the matrix after filling the blank spot.**

$$A_6(\text{fill in blank spot}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & \boldsymbol{\gamma} & ? \end{pmatrix} . \tag{10.2.7}$$

- **Write down the value of γ used to fill the blank spot.**
- **The value of γ should be a positive number.**

10.2.3 LU factorization

- The matrix is now in LU factorized form.
- The array of swap indices is also in its final form.
- We shall test if the LU decomposition has been performed correctly.
- Denote the LU factorized matrix by

$$A_7(LU) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21} & u_{22} & u_{23} \\ \ell_{31} & \ell_{32} & u_{33} \end{pmatrix} . \tag{10.2.8}$$

- **Write down the L and U matrices as follows:**

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix} , \qquad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} . \tag{10.2.9}$$

- **Multiply the L and U matrices. Calculate and state the result.**

$$A_8 = LU = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} . \tag{10.2.10}$$

- Let the final array of swap indices be $S = (s_1, s_2, s_3)$.
  1. **Compare row 1 in the LU product with rows s_1 in the original matrix A.**
  2. **Compare row 2 in the LU product with rows s_2 in the original matrix A.**
  3. **Compare row 3 in the LU product with rows s_3 in the original matrix A.**
- If you have done everything correctly, you should obtain a match in all three cases.
- **Calculate the determinant of the matrix A.**
  (Remember to count the number of swaps.)

10.2.4 Solution of equations

- You are given the following equations

$$x_1 + 2x_2 + x_3 = 1\,, \tag{10.2.11a}$$
$$2x_1 + 4x_2 - x_3 = 1\,, \tag{10.2.11b}$$
$$-3x_1 + 2x_2 + 3x_3 = 7\,. \tag{10.2.11c}$$

- The matrix is

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ -3 & 2 & 3 \end{pmatrix}. \tag{10.2.12}$$

- The matrix is now available in LU factorized form.
- The final array of swap indices $S = (s_1, s_2, s_3)$ is also available.
- Let the right hand side vector be

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}. \tag{10.2.13}$$

- To solve the equations using LU decomposition, we must swap the entries in the right hand side vector $\boldsymbol{r}$.
- **The following steps is important. Make sure you understand and do it correctly.**
- **Rearrange the entries in the right hand side vector as follows:**

$$\boldsymbol{r}' = \begin{pmatrix} \text{row } s_1 \text{ in } \boldsymbol{r} \\ \text{row } s_2 \text{ in } \boldsymbol{r} \\ \text{row } s_3 \text{ in } \boldsymbol{r} \end{pmatrix}. \tag{10.2.14}$$

- Define a temporary column vector y

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{10.2.15}$$

- **Calculate the values of y_1, y_2 and y_3 by backsubstitution.**

$$L\boldsymbol{y} = \boldsymbol{r}'\,. \tag{10.2.16}$$

  (We should really say "forward substitution" because we calculate the value of y_1 first, etc.)

- **Calculate the values of x_1, x_2 and x_3 by backsubstitution.**

$$U\boldsymbol{x} = \boldsymbol{y}\,. \tag{10.2.17}$$

  (This is real backsubstitution, because we calculate the value of x_3 first, etc.)

- **Substitute your solutions for x_1, x_2 and x_3 and verify that they satisfy eq. (10.2.11a) eq. (10.2.11b) and eq. (10.2.11c).**

10.3 Linear algebra: LU decomposition, two swaps

- You are given the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ -3 & 2 & 3 \end{pmatrix} . \tag{10.3.1}$$

- In addition, the initial array of the swap indices is $S = (1, 2, 3)$.
- **Perform the LU decomposition for the matrix A in eq. (10.3.1).**
- This time you will need to do two swaps. When performing the second swap, make sure to swap the entries in the "L" part of the matrix as well.
- **Write out the steps in the LU decomposition and display the final LU matrix.**
- **Also write down the final value of the array of the swap indices S.**
- **Write down the matrices L and U and multiply them. Verify that the product LU equals the original matrix A, with permutation of rows. Verify that the permutation is given by the array of the swap indices S.**
- **Calculate the determinant of the matrix A.**
  (Remember to count the number of swaps.)
- **(Optional)** Solve the following equations for x_1, x_2 and x_3:

$$x_1 + 2x_2 + x_3 = 1\,, \tag{10.3.2a}$$
$$2x_1 + 4x_2 + 5x_3 = 1\,, \tag{10.3.2b}$$
$$-3x_1 + 2x_2 + 3x_3 = 7\,. \tag{10.3.2c}$$

10.4 Linear algebra: (practice for midterm)

- You are given the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{10.4.1}$$

- The initial array of the swap indices is $S = (1, 2, 3, 4)$.
- It is perfectly obvious to a human that this is a lower triangular matrix in disguise, but the computer will not know that.
- **Perform the LU decomposition for the matrix A in eq. (10.4.1).**
  Display the steps in your calculation.
- **Write down the final array of the swap indices.**
- **Calculate the determinant of the matrix A.**
- **Solve the equation $A\boldsymbol{x}_i = \boldsymbol{r}_i$ using LU backsubstitution, $i = 1, 2, 3, 4$, for the following right hand side column vectors:**

$$\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \qquad \boldsymbol{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \boldsymbol{r}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \boldsymbol{r}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{10.4.2}$$

- **Calculate the matrix inverse A^{-1}.**

10.5 Linear algebra: LU decomposition, zero pivot

- Eh... this is boring. The computer will take care of it.
- *Or a midterm exam question will deal with it.*