Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Spring 2018

Instructor: Dr. Sateesh Mane

April 29, 2018

16 Lecture 16a

Numerical solution of systems of ordinary differential equations

- We display worked examples of **initial value problems using auxiliary variables.**

16.12 Worked example 1

16.12.1 Equation

- Consider the following ordinary differential equation

$$\frac{d^2y}{dx^2} + 2x\,\frac{dy}{dx} + (1-x^2)y = e^x\,. \tag{16.12.1}$$

- It is a second order ordinary differential equation. We introduce an auxiliary variable v via

$$v = \frac{dy}{dx}\,. \tag{16.12.2}$$

- We reexpress eq. (16.12.1) as a pair of coupled first order equations as follows:

$$\frac{dy}{dx} = v\,, \tag{16.12.3a}$$

$$\frac{dv}{dx} = -2xv - (1-x^2)y + e^x\,. \tag{16.12.3b}$$

- We express this in the formal notation as follows. We define $\boldsymbol{y} = (y_1, y_2) = (y, v)$. Then

$$\begin{aligned}
\frac{d\boldsymbol{y}}{dx} &= \boldsymbol{f}(x,\boldsymbol{y})\,, \\
\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} f_1(x,y_1,y_2) \\ f_2(x,y_1,y_2) \end{pmatrix}\,. \\
\frac{d}{dx}\begin{pmatrix} y \\ v \end{pmatrix} &= \begin{pmatrix} v \\ -2xv-(1-x^2)y+e^x \end{pmatrix}\,.
\end{aligned} \tag{16.12.4}$$

- Therefore the right hand side functions are

$$f_1(x,y_1,y_2) = f_1(x,y,v) \qquad = v \quad = y_2\,, \tag{16.12.5a}$$

$$\begin{aligned}
f_2(x,y_1,y_2) = f_2(x,y,v) \qquad &= -\,2xv - (1-x^2)y + e^x \\
&= -\,2xy_2 - (1-x^2)y_1 + e^x\,.
\end{aligned} \tag{16.12.5b}$$

16.12.2 C++ code

- In terms of C++ function calls, we have $m = 2$ and

```
int f(int m, double x, const std::vector<double> & y, std::vector<double> & g)
{
  // first component "f1(x,y,v) = v"
  g[0] = y[1];

  // second component "f2(x,y,v) = -2xy -(1-x^2)y + exp(x)"
  g[1] = -2.0*x*y[1] - (1.0 - x*x)*y[0] + exp(x);

  return 0;
}
```

All the integration schemes can be called with the following inputs:

```
// m = 2
// x = x_i
// h = step size
// vector (array) y_in  = (y, v)_i
// vector (array) y_out = (y, v)_{i+1}

int Euler_explicit(int m, double x, double h,
                   std::vector<double> & y_in,
                   std::vector<double> & y_out);

int midpoint(int m, double x, double h,
             std::vector<double> & y_in,
             std::vector<double> & y_out);

int trapezoid(int m, double x, double h,
              std::vector<double> & y_in,
              std::vector<double> & y_out);

int RK4(int m, double x, double h,
        std::vector<double> & y_in,
        std::vector<double> & y_out);
```

16.12.3 Solution for $i = 0$, etc

- Let the initial conditions at $x_0 = 0$ be $y_0 = 1$ and $v_0 = y'_0 = -1$.
- We shall integrate eq. (16.12.4) using explicit Euler integration with a stepsize h.
- The formal equations are

$$\boldsymbol{y}_{i+1} = \boldsymbol{y}_i + h\boldsymbol{f}(x, \boldsymbol{y})\,. \tag{16.12.6}$$

- In terms of y and v, the equations are

$$\begin{pmatrix} y \\ v \end{pmatrix}_{i+1} = \begin{pmatrix} y \\ v \end{pmatrix}_i + h \begin{pmatrix} v_i \\ -2x_i v_i - (1-x_i^2)y_i + \exp(x_i) \end{pmatrix}\,. \tag{16.12.7}$$

- Step $i = 0$:
  1. We have
  $$y_1 = y_0 + hv_0 = 1 - h\,. \tag{16.12.8}$$
  2. Next we have
  $$\begin{aligned} v_1 &= v_0 + h\big(-2x_0v_0 - (1-x_0^2)y_0 + \exp(x_0)\big) \\ &= -1 + h\big(0 - (1-0) + \exp(0)\big) \\ &= -1\,. \end{aligned} \tag{16.12.9}$$
- Step $i = 1$:
  1. We have
  $$y_2 = y_1 + hv_1 = 1 - h - h = 1 - 2h\,. \tag{16.12.10}$$
  2. Next we have
  $$\begin{aligned} v_2 &= v_1 + h\big(-2x_1v_1 - (1-x_1^2)y_1 + \exp(x_1)\big) \\ &= -1 + h\big(2h - (1-h^2)(1-h) + \exp(h)\big) \\ &= -1 + h\big(-1 + 3h + h^2 - h^3 + \exp(h)\big) \\ &= -1 - h + 3h^2 + h^3 - h^4 + h\exp(h)\,. \end{aligned} \tag{16.12.11}$$
- Step $i = 2$:
  1. We have
  $$\begin{aligned} y_3 = y_2 + hv_2 &= 1 - 2h + h\big(-1 - h + 3h^2 + h^3 - h^4 + h\exp(h)\big) \\ &= 1 - 3h - h^2 + 3h^3 + h^4 - h^5 + h^2\exp(h)\,. \end{aligned} \tag{16.12.12}$$
  2. Next we have
  $$\begin{aligned} v_3 &= v_2 + h\big(-2x_2v_2 - (1-x_2^2)y_2 + \exp(x_2)\big) \\ &= v_2 + h\big(-4hv_2 - (1-4h^2)y_2 + \exp(2h)\big) \\ &= \text{ugh}\,. \end{aligned} \tag{16.12.13}$$

16.13 Worked example 2

16.13.1 Equation

- Consider the following ordinary differential equation

$$\frac{d^2y}{dx^2} + y = 0\,. \tag{16.13.1}$$

- This is a simple equation. We know the general solution is

$$y(x) = c_1\cos(x) + c_2\sin(x)\,. \tag{16.13.2}$$

- It is a second order ordinary differential equation. We introduce an auxiliary variable v via

$$v = \frac{dy}{dx}\,. \tag{16.13.3}$$

- We reexpress eq. (16.13.1) as a pair of coupled first order equations as follows:

$$\frac{dy}{dx} = v\,, \tag{16.13.4a}$$

$$\frac{dv}{dx} = -y\,. \tag{16.13.4b}$$

- We express this in the formal notation as follows. We define $\boldsymbol{y} = (y_1, y_2) = (y, v)$. Then

$$\begin{aligned}\frac{d\boldsymbol{y}}{dx} &= \boldsymbol{f}(x,\boldsymbol{y})\,,\\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} f_1(x,y_1,y_2) \\ f_2(x,y_1,y_2) \end{pmatrix}\,.\\ \frac{d}{dx}\begin{pmatrix} y \\ v \end{pmatrix} &= \begin{pmatrix} v \\ -y \end{pmatrix}\,.\end{aligned} \tag{16.13.5}$$

- Therefore the right hand side functions are

$$f_1(x,y_1,y_2) = f_1(x,y,v) \qquad = v \qquad = y_2\,, \tag{16.13.6a}$$

$$f_2(x,y_1,y_2) = f_2(x,y,v) \qquad = -\,y \qquad = -\,y_1\,. \tag{16.13.6b}$$

16.13.2 C++ code

- In terms of C++ function calls, we have $m = 2$ and

```
int f(int m, double x, const std::vector<double> & y, std::vector<double> & g)
{
  // first component "f1(x,y,v) = v"
  g[0] = y[1];

  // second component "f2(x,y,v) = -y"
  g[1] = -y[1];

  return 0;
}
```

All the integration schemes can be called with the following inputs:

```
// m = 2
// x = x_i
// h = step size
// vector (array) y_in  = (y, v)_i
// vector (array) y_out = (y, v)_{i+1}

int Euler_explicit(int m, double x, double h,
                   std::vector<double> & y_in,
                   std::vector<double> & y_out);

int midpoint(int m, double x, double h,
             std::vector<double> & y_in,
             std::vector<double> & y_out);

int trapezoid(int m, double x, double h,
              std::vector<double> & y_in,
              std::vector<double> & y_out);

int RK4(int m, double x, double h,
        std::vector<double> & y_in,
        std::vector<double> & y_out);
```

16.13.3 Solution for $i = 0$, etc

- Let the initial conditions at $x_0 = 0$ be $y_0 = 1$ and $v_0 = y'_0 = 0$.
- Then we know the exact solution is

$$y_{\text{exact}}(x) = \cos(x)\,. \tag{16.13.7}$$

- We shall integrate eq. (16.13.5) using explicit Euler integration with a stepsize h.
- The formal equations are

$$\boldsymbol{y}_{i+1} = \boldsymbol{y}_i + h\boldsymbol{f}(x, \boldsymbol{y})\,. \tag{16.13.8}$$

- In terms of y and v, the equations are

$$\begin{pmatrix} y \\ v \end{pmatrix}_{i+1} = \begin{pmatrix} y \\ v \end{pmatrix}_i + h\begin{pmatrix} v_i \\ -y_i \end{pmatrix}\,. \tag{16.13.9}$$

- Step $i = 0$:
  1. We have
  $$y_1 = y_0 + hv_0 = 1 - 0 = 1\,. \tag{16.13.10}$$
  2. Next we have
  $$v_1 = v_0 - hy_0 = 0 - h = -h\,. \tag{16.13.11}$$
- Step $i = 1$:
  1. We have
  $$y_2 = y_1 + hv_1 = 1 + h(-h) = 1 - h^2\,. \tag{16.13.12}$$
  2. Next we have
  $$v_2 = v_1 - hy_1 = -h - h = -2h\,. \tag{16.13.13}$$
- Step $i = 2$:
  1. We have
  $$y_3 = y_2 + hv_2 = 1 - h^2 + h(-2h) = 1 - 3h^2\,. \tag{16.13.14}$$
  2. Next we have
  $$v_3 = v_2 - hy_2 = -2h - h(1 - h^2) = -3h + h^3\,. \tag{16.13.15}$$
- Step $i = 3$:
  1. We have
  $$y_4 = y_3 + hv_3 = 1 - 3h^2 + h(-3h + h^3) = 1 - 6h^2 + h^4\,. \tag{16.13.16}$$
  2. Next we have
  $$v_4 = v_3 - hy_3 = -3h + h^3 - h(1 - 3h^2) = -4h + 4h^3\,. \tag{16.13.17}$$
- We see that the explicit Euler method is not very accurate.

16.13.4 Solution using midpoint method

- Let us integrate eq. (16.13.5) using the midpoint method with a stepsize h.
- In terms of y and v, the equations are

$$\begin{pmatrix} y \\ v \end{pmatrix}_{i+1} = \begin{pmatrix} y \\ v \end{pmatrix}_i + h \begin{pmatrix} v_i \\ -y_i \end{pmatrix} . \tag{16.13.18}$$

- However, now we require the values intermediate points.
  1. First we have

$$\boldsymbol{g}_0 \sim \begin{pmatrix} g_{y,0} \\ g_{v,0} \end{pmatrix}_i = \begin{pmatrix} v_i \\ -y_i \end{pmatrix} . \tag{16.13.19}$$

  2. Next we have

$$\boldsymbol{g}_1 \sim \begin{pmatrix} g_{y,1} \\ g_{v,1} \end{pmatrix}_i = \begin{pmatrix} f_1(x_i + \frac{1}{2}h, y_i + \frac{h}{2}(g_{y,0})_i, v_i + \frac{h}{2}(g_{v,0})_i) \\ f_2(x_i + \frac{1}{2}h, y_i + \frac{h}{2}(g_{y,0})_i, v_i + \frac{h}{2}(g_{v,0})_i) \end{pmatrix} = \begin{pmatrix} v_i - \frac{h}{2}y_i \\ -y_i - \frac{h}{2}v_i \end{pmatrix} . \tag{16.13.20}$$

- Because of too many subscripts, for clarity let us define, at the step i,

$$\begin{pmatrix} G_y \\ G_v \end{pmatrix}_i = \begin{pmatrix} v_i - \frac{h}{2}y_i \\ -y_i - \frac{h}{2}v_i \end{pmatrix} . \tag{16.13.21}$$

- Step $i = 0$:
  1. We have

$$(G_y)_0 = v_0 - \frac{h}{2}\, y_0 = 0 - \frac{h}{2} . \tag{16.13.22}$$

  2. Next

$$(G_v)_0 = -y_0 - \frac{h}{2}\, v_0 = -1 . \tag{16.13.23}$$

  3. Then

$$\begin{aligned} y_1 &= y_0 + h(G_y)_0 = 1 - \frac{h^2}{2} , \\ v_1 &= v_0 + h(G_v)_0 = 0 - h = -h . \end{aligned} \tag{16.13.24}$$

- Step $i = 1$:
  1. We have

$$(G_y)_1 = v_1 - \frac{h}{2}\, y_1 = -h - \frac{h}{2}\Big(1 - \frac{h^2}{2}\Big) = -\frac{3h}{2} + \frac{h^3}{4} . \tag{16.13.25}$$

  2. Next

$$(G_v)_1 = -y_1 - \frac{h}{2}\, v_1 = -1 + \frac{h^2}{2} + \frac{h^2}{2} = -1 + h^2 . \tag{16.13.26}$$

3. Then

$$\begin{aligned} y_2 = y_1 + h(G_y)_1 &= 1 - \frac{h^2}{2} + h\Big(-\frac{3h}{2} + \frac{h^3}{4}\Big) \\ &= 1 - 2h^2 + \frac{h^4}{4}\,, \\ v_2 = v_1 + h(G_v)_1 &= -h + h(-1 + h^2) \\ &= -2h + h^3\,. \end{aligned} \tag{16.13.27}$$

- Step $i = 2$:

1. We have

$$(G_y)_2 = v_2 - \frac{h}{2}\,y_2 = -2h + h^3 - \frac{h}{2}\Big(1 - 2h^2 + \frac{h^4}{4}\Big) = -\frac{5h}{2} + 2h^3 - \frac{h^5}{8}\,. \tag{16.13.28}$$

2. Next

$$(G_v)_2 = -y_2 - \frac{h}{2}\,v_2 = -1 + 2h^2 - \frac{h^4}{4} + h^2 - \frac{h^4}{2} = -1 + 3h^2 - \frac{3h^4}{4}\,. \tag{16.13.29}$$

3. Then

$$\begin{aligned} y_3 = y_2 + h(G_y)_2 &= 1 - 2h^2 + \frac{h^4}{4} + h\Big(-\frac{5h}{2} + 2h^3 - \frac{h^5}{8}\Big) \\ &= 1 - \frac{9h^2}{2} + \frac{9h^4}{4} - \frac{h^6}{8}\,, \\ v_3 = v_2 + h(G_v)_2 &= -2h + h^3 + h\Big(-1 + 3h^2 - \frac{3h^4}{4}\Big) \\ &= -3h + 4h^3 - \frac{3h^5}{4}\,. \end{aligned} \tag{16.13.30}$$

- This is more accurate than the explicit Euler method. It matches the exact solution to $O(h^2)$.