

Queens College, CUNY, Department of Computer Science
Numerical Methods
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due Friday, April 27, 2018, 11.59 pm

28 Homework lecture 28

29 Homework lecture 29

- As experience has demonstrated, if you do not understand the above expressions/questions, **THEN ASK.**
- If you do not understand the words/sentences in the lectures, **THEN ASK.**
- Send me an email, explain what you do not understand.
- Do not just keep quiet and produce nonsense in exams.

28.1 Sampling

- Let us consider the simple example

$$f_4(\theta) = \cos(4\theta). \quad (28.1.1)$$

- We wish to compute the Fourier series of $f_4(\theta)$ numerically.
- We know the correct answer is $a_4 = 1$ and all the other a_j and b_j are zero.
- However, the function $f_4(\theta)$ is only available to us as the output of a ‘black box’ routine.
- We compute the Fourier coefficients using the following sums, with n points equally spaced around a circle:

$$\begin{aligned} a_j &= \frac{2}{n} \sum_{i=0}^{n-1} f(\theta_i) \cos(j\theta_i), \\ b_j &= \frac{2}{n} \sum_{i=0}^{n-1} f(\theta_i) \sin(j\theta_i) \quad \left(\theta_i = \frac{2\pi i}{n} \right). \end{aligned} \quad (28.1.2)$$

- *How many values of n do we need in eq. (28.1.2) to compute the values of a_j and b_j accurately?*
- **Compute the values of a_j and b_j for $0 \leq j \leq 10$ using $n = 1$, then $n = 2$, up to $n = 20$.**
- **You may employ/modify the program from the previous homework assignment.**
- If you have done your work correctly, you should find that $b_j = 0$ in all cases.
- **Fill in the table below, for a_j , $j = 0, \dots, 10$, for $n = 1, \dots, 20$.**

If you have done your work correctly, the tabulated values will all be zero or positive integers.

| n | a_0 | a_1 | a_2 | a_3 | a_4 | a_5 | a_6 | a_7 | a_8 | a_9 | a_{10} |
|----------|---------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| 1 | (integer ≥ 0) | etc. | | | | | | | | | |
| 2 | (integer ≥ 0) | | | | | | | | | | |
| \vdots | \vdots | | | | | | | | | | |
| 20 | (integer ≥ 0) | | | | | | | | | | |

- If you have done your work correctly, by the time you reach $n = 20$ (last line in the table), you should obtain $a_4 = 1$ and all the others are zero.
- We shall discuss the results obtained in the above table in this homework assignment.

28.2 Aliasing

- Let us consider the following ‘original’ function:

$$\begin{aligned} f_{\text{orig}}(\theta) = & \frac{1}{2} + 2 \cos(\theta) + 3 \sin(\theta) \\ & + 4 \cos(2\theta) + 5 \sin(2\theta) \\ & + 6 \cos(3\theta) + 7 \sin(3\theta) \\ & + 8 \cos(4\theta) + 9 \sin(4\theta) . \end{aligned} \tag{28.2.1}$$

- We shall use $n = 16$ in this question.

Compute the values of a_j and b_j for $f_{\text{orig}}(\theta)$.

- **Fill in the table below, for a_j and b_j $j = 0, \dots, 7$ (where $b_0 = 0$).**

| j | a_j | b_j |
|----------|----------|----------|
| 0 | | 0 |
| 1 | | |
| \vdots | \vdots | \vdots |
| 7 | | |

- Next use the following function:

$$\begin{aligned}
 f_1(\theta) = f_{\text{orig}}(\theta) &+ 0.01 \cos(9\theta) + 0.02 \sin(10\theta) \\
 &+ 0.03 \cos(11\theta) + 0.04 \sin(12\theta) \\
 &+ 0.05 \cos(13\theta) + 0.06 \sin(14\theta) \\
 &+ 0.07 \cos(15\theta) + 0.08 \sin(16\theta).
 \end{aligned}
 \tag{28.2.2}$$

- Continue to use $n = 16$ in this question.

Compute the values of a_j and b_j for $f_1(\theta)$.

- **Fill in the table below, for a_j and b_j $j = 0, \dots, 7$ (where $b_0 = 0$).**

| j | a_j | b_j |
|----------|----------|----------|
| 0 | | 0 |
| 1 | | |
| \vdots | \vdots | \vdots |
| 7 | | |

- Notice that the values of a_j and b_j have been changed by the aliasing.
- Notice also the pattern of the changes caused by aliasing.
The values 0.01, 0.02, etc. will help you to track the effects.
- Next use the following function:

$$\begin{aligned}
 f_2(\theta) = f_{\text{orig}}(\theta) &+ 0.01 \cos((16 + 9)\theta) + 0.02 \sin((16 + 10)\theta) \\
 &+ 0.03 \cos((16 + 11)\theta) + 0.04 \sin((16 + 12)\theta) \\
 &+ 0.05 \cos((16 + 13)\theta) + 0.06 \sin((16 + 14)\theta) \\
 &+ 0.07 \cos((16 + 15)\theta) + 0.08 \sin((16 + 16)\theta).
 \end{aligned}
 \tag{28.2.3}$$

- Continue to use $n = 16$ in this question.

Compute the values of a_j and b_j for $f_1(\theta)$.

- **Fill in the table below, for a_j and b_j $j = 0, \dots, 7$ (where $b_0 = 0$).**

| j | a_j | b_j |
|----------|----------|----------|
| 0 | | 0 |
| 1 | | |
| \vdots | \vdots | \vdots |
| 7 | | |

- If you have done your work correctly, the values of a_j and b_j will be the same as for $f_1(\theta)$.

- Next use the following function:

$$\begin{aligned}
 f_3(\theta) = f_{\text{orig}}(\theta) &+ 0.002 \cos(16\theta) + 0.001 \sin(17\theta) \\
 &+ 0.004 \cos(18\theta) + 0.003 \sin(19\theta) \\
 &+ 0.006 \cos(20\theta) + 0.005 \sin(21\theta) \\
 &+ 0.008 \cos(22\theta) + 0.007 \sin(23\theta).
 \end{aligned}
 \tag{28.2.4}$$

- Continue to use $n = 16$ in this question.

Compute the values of a_j and b_j for $f_1(\theta)$.

- Fill in the table below, for a_j and b_j $j = 0, \dots, 7$ (where $b_0 = 0$).

| j | a_j | b_j |
|----------|----------|----------|
| 0 | | 0 |
| 1 | | |
| \vdots | \vdots | \vdots |
| 7 | | |

- Notice that the values of a_j and b_j have been changed by the aliasing.
- Notice also the pattern of the changes caused by aliasing.
The pattern of the changes is different from that caused by $f_1(\theta)$.
The values 0.001, 0.002, etc. will help you to track the effects.
- Next use the following function:

$$\begin{aligned}
 f_4(\theta) = f_{\text{orig}}(\theta) &+ 0.002 \cos((16 + 16)\theta) + 0.001 \sin((16 + 17)\theta) \\
 &+ 0.004 \cos((16 + 18)\theta) + 0.003 \sin((16 + 19)\theta) \\
 &+ 0.006 \cos((16 + 20)\theta) + 0.005 \sin((16 + 21)\theta) \\
 &+ 0.008 \cos((16 + 22)\theta) + 0.007 \sin((16 + 23)\theta).
 \end{aligned}
 \tag{28.2.5}$$

- Continue to use $n = 16$ in this question.

Compute the values of a_j and b_j for $f_1(\theta)$.

- Fill in the table below, for a_j and b_j $j = 0, \dots, 7$ (where $b_0 = 0$).

| j | a_j | b_j |
|----------|----------|----------|
| 0 | | 0 |
| 1 | | |
| \vdots | \vdots | \vdots |
| 7 | | |

- If you have done your work correctly, the values of a_j and b_j will be the same as for $f_3(\theta)$.

28.3 Nyquist frequency & aliasing

- We employ the same ‘original’ function given in eq. (28.2.1).
- **For each case below:**
 1. State the Nyquist frequency (as an integer).
 2. State (without computer calculation), which values of a_j and/or b_j will be affected, for $j = 0, \dots, 7$.
 3. ***Explain why they are affected, and not the other coefficients.***
 4. Also calculate the values of a_j and/or b_j which are affected.

28.3.1 $n = 22$

$$f(\theta) = f_{\text{orig}}(\theta) + 0.0001 \cos(40\theta) - 0.0002 \sin(50\theta). \quad (28.3.1)$$

28.3.2 $n = 24$

$$f(\theta) = f_{\text{orig}}(\theta) + 0.0001 \cos(44\theta) - 0.0002 \sin(43\theta) \\ - 0.0003 \cos(54\theta) + 0.0002 \sin(53\theta). \quad (28.3.2)$$

28.3.3 $n = 14$

$$f(\theta) = f_{\text{orig}}(\theta) + 0.0001 \cos(44\theta) - 0.0002 \sin(43\theta) \\ - 0.0003 \cos(54\theta) + 0.0002 \sin(53\theta). \quad (28.3.3)$$

28.3.4 $n = 20$

$$f(\theta) = f_{\text{orig}}(\theta) + 0.0001 \cos(44\theta) - 0.0002 \sin(43\theta) \\ - 0.0003 \cos(54\theta) + 0.0002 \sin(53\theta). \quad (28.3.4)$$

28.3.5 $n = 1024$

$$f(\theta) = f_{\text{orig}}(\theta) + 0.0001 \cos(44\theta) - 0.0002 \sin(43\theta) \\ - 0.0003 \cos(54\theta) + 0.0002 \sin(53\theta). \quad (28.3.5)$$

28.4 Nested summation of Fourier series

- **Submit your program code as part of your answer to this question.**
- Suppose we are given a set of Fourier harmonics a_j and b_j for $j = 0, \dots, m$ (where $b_0 = 0$).
- It was derived in the lectures that an efficient way to sum the Fourier series is as follows.

```
const double c = cos(theta);
const double s = sin(theta);
double U = 0;
double V = 0;
for (int j = m; j > 0; --j) {
    double Utmp = a[j] + c*U + s*V;
    double Vtmp = b[j] - s*U + c*V;
    U = Utmp;
    V = Vtmp;
}
double fsum = 0.5*a[0] + c*U + s*V;
```

- We shall employ the above code to sum a Fourier series.
- **Write a loop to calculate the sum of the series for $2 \times \text{npts} + 1$ values of θ .**

```
const double pi = 4.0*atan2(1.0,1.0);
const int npts = (set by user);
double dtheta = pi/double(npts);
for (int i = -npts; i <= npts; ++i) {
    double theta = i*dtheta;
    // code to sum series
    // print the function and plot output
}
```

- The value of `npts` is set by the user or calling application.
- Note that the above code computes the value of `fsum` in the interval $-\pi \leq \theta \leq \pi$.
- Recall the Fourier harmonics of the window function of width $2\theta_0$ are given by

$$a_0 = \frac{1}{\pi}, \quad a_j = \frac{1}{\pi} \frac{\sin(j\theta_0)}{j\theta_0} \quad (j \geq 1). \quad (28.4.1)$$

- *But this is boring.* It will sum to the window function, which we have seen many times.

- Let us do something different.
- Let us make an **antiwindow function**.
- This is just my own name. I have *absolutely no idea* what the function really is.
- **Set $a_j = 0$ and b_j as follows:**

$$b_j = \frac{\sin(j\theta_0)}{j\theta_0} \quad (j \geq 1). \quad (28.4.2)$$

- Set $\theta_0 = \pi/4$ in this question.
- Set $m = 256$ to sum the Fourier series.
- Set $npts = 1000$ to compute the values of θ .
- **Compute the value of $fsum$ using the above code, with $npts = 1000$ and $m = 256$.**
- **Plot a graph of $fsum$ for $2*npts+1$ values of θ as indicated in the above code.**
 1. On the horizontal axis, plot the value of θ/π , so the values go from -1 to 1 .
 2. On the vertical axis, go from -6 to 6 .
- A graph of the function is plotted in Fig. 1.
- If you have done your work correctly, you should obtain the same answer.

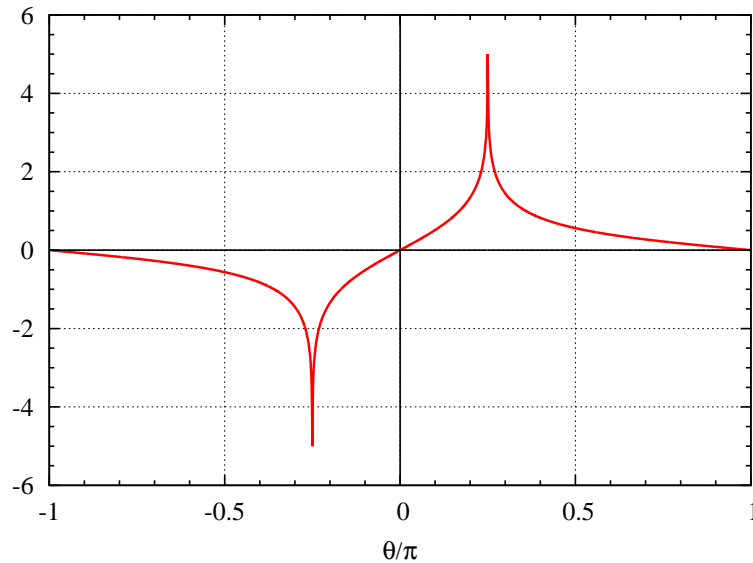


Figure 1: Plot of the ‘antiwindow’ function obtained by summing the Fourier series $b_j = \sin(j\theta_0)/(j\theta_0)$ with $\theta_0 = \frac{1}{4}\pi$ in Question 28.4.

- *What are we to make of the plotted function?*
- By construction, the function is

$$f(\theta) = \sum_{j=1}^{\infty} \frac{\sin(j\theta_0)}{j\theta_0} \sin(j\theta). \quad (28.4.3)$$

- By construction, the function is odd $f(-\theta) = -f(\theta)$.
- Because $\sin(j\theta) = 0$ at $\theta = 0$ and $\theta = \pm\pi$, we must have $f(\theta) = 0$ at $\theta = 0$ and $\theta = \pm\pi$, as observed.
- The function has two spikes at $\theta = \pm\theta_0$, i.e. $\theta = \pm\frac{1}{4}\pi$ in this example.
- Set $\theta = \theta_0$ and we obtain

$$f(\theta_0) = \sum_{j=1}^{\infty} \frac{\sin^2(j\theta_0)}{j\theta_0}. \quad (28.4.4)$$

- Let us approximate $\sin^2(j\theta_0) \simeq \frac{1}{2}$, then

$$f(\theta_0) \simeq \frac{1}{2\theta_0} \sum_{j=1}^{\infty} \frac{1}{j} \rightarrow \infty. \quad (28.4.5)$$

- The sum diverges logarithmically.
- Hence $f(\theta)$ has logarithmic singularities at $\theta = \pm\theta_0$.

28.5 Solution of differential equation

- Let us solve an ordinary differential equation using a Fourier series.
- This is an important technique in physics and engineering.
- This is an example where we do not know the function $f(\theta)$ and must obtain it by summing a Fourier series.
- The system in this question is a **driven damped oscillator**.
- The differential equation is

$$\frac{d^2 f}{d\theta^2} + R \frac{df}{d\theta} + Q^2 f = g(\theta). \quad (28.5.1)$$

- Here Q and R are constants and $g(\theta)$ is a periodic function with period 2π .
- The function $g(\theta)$ is a **driving term**.
- We seek the solution of eq. (28.5.1) which is periodic with period 2π .
- The solution for $f(\theta)$ is called the **forced (or driven) solution**.
- The forced solution would be zero if $g(\theta)$ were zero.
- We seek a procedure to calculate the forced solution.
- This is accomplished as follows.

1. We express $g(\theta)$ as a Fourier series

$$g(\theta) = \frac{1}{2}c_0 + \sum_{j=1}^{\infty} [c_j \cos(j\theta) + d_j \sin(j\theta)]. \quad (28.5.2)$$

2. The values of c_j and d_j are known.
3. We express $f(\theta)$ as a Fourier series

$$f(\theta) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty} [a_j \cos(j\theta) + b_j \sin(j\theta)]. \quad (28.5.3)$$

4. The values of a_j and b_j are not known and our task is to solve for them.
5. The first and second derivatives of $f(\theta)$ are given by

$$\begin{aligned} \frac{df}{d\theta} &= \sum_{j=1}^{\infty} j [-a_j \sin(j\theta) + b_j \cos(j\theta)], \\ \frac{d^2 f}{d\theta^2} &= - \sum_{j=1}^{\infty} j^2 [a_j \cos(j\theta) + b_j \sin(j\theta)]. \end{aligned} \quad (28.5.4)$$

6. We substitute the Fourier series for f and g into eq. (28.5.1).

7. For $j = 0$ we obtain

$$a_0 = \frac{c_0}{Q^2}. \quad (28.5.5)$$

8. For $j \geq 1$ we obtain

$$\begin{aligned} & \sum_{j=1}^{\infty} (Q^2 - j^2) [a_j \cos(j\theta) + b_j \sin(j\theta)] \\ & + R \sum_{j=1}^{\infty} j [-a_j \sin(j\theta) + b_j \cos(j\theta)] = \sum_{j=1}^{\infty} [c_j \cos(j\theta) + d_j \sin(j\theta)]. \end{aligned} \quad (28.5.6)$$

9. Equating terms yields a pair of coupled equations for a_j and b_j :

$$\begin{aligned} (Q^2 - j^2)a_j + Rjb_j &= c_j, \\ -Rja_j + (Q^2 - j^2)b_j &= d_j. \end{aligned} \quad (28.5.7)$$

10. This can be formulated as a 2×2 matrix equation:

$$\begin{pmatrix} Q^2 - j^2 & Rj \\ -Rj & Q^2 - j^2 \end{pmatrix} \begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{pmatrix} c_j \\ d_j \end{pmatrix}. \quad (28.5.8)$$

11. The solution for a_j and b_j for $j \geq 1$ is

$$\begin{aligned} \begin{pmatrix} a_j \\ b_j \end{pmatrix} &= \frac{1}{(Q^2 - j^2)^2 + (Rj)^2} \begin{pmatrix} Q^2 - j^2 & -Rj \\ Rj & Q^2 - j^2 \end{pmatrix} \begin{pmatrix} c_j \\ d_j \end{pmatrix} \\ &= \frac{1}{(Q^2 - j^2)^2 + (Rj)^2} \begin{pmatrix} (Q^2 - j^2)c_j - Rjd_j \\ Rjc_j + (Q^2 - j^2)d_j \end{pmatrix}. \end{aligned} \quad (28.5.9)$$

12. The solution for $f(\theta)$ is

$$f(\theta) = \frac{c_0}{2Q^2} + \sum_{j=1}^{\infty} \frac{(Q^2 - j^2)c_j - Rjd_j}{(Q^2 - j^2)^2 + (Rj)^2} \cos(j\theta) + \frac{Rjc_j + (Q^2 - j^2)d_j}{(Q^2 - j^2)^2 + (Rj)^2} \sin(j\theta). \quad (28.5.10)$$

- For the function $g(\theta)$, set $c_0 = 0$ and for $j \geq 1$ use

$$c_j = \exp(-0.01j), \quad d_j = \frac{\sin(j\theta_0)}{j\theta_0}. \quad (28.5.11)$$

- Set $\theta_0 = \pi/100$, $Q = 3.41$ and $R = 0.51$.
- Set $m = 256$ to sum the Fourier series.
- Set $npts = 1000$ to compute the values of θ .
- Compute the value of $fsum$ using the code in Question 28.4.
- Plot a graph of $fsum$ for $2*npts+1$ values of θ as in the code.

1. On the horizontal axis, plot the value of θ/π , so the values go from -1 to 1 .
2. On the vertical axis, go from -1 to 1 .