Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Spring 2018

Instructor: Dr. Sateesh Mane

Midterm 1 Spring 2018 (grade boost)

Students who scored F in midterm 1 are not eligible for a grade boost

due Friday March 9, 2018, 11:59 pm

- **NOTE: It is the policy of the Computer Science Department to issue a failing grade to any student who either gives or receives help on any test.**
- This is an **open-book** test.
- **Any problem to which you give two or more (different) answers receives the grade of zero automatically.**
- This is a **take home exam.**
  Please submit your solution via email, as a file attachment, to `Sateesh.Mane@qc.cuny.edu`.
  The file name should have either of the formats:

  `StudentId_first_last_CS361_midterm1_boost_Mar2018`

  `StudentId_first_last_CS761_midterm1_boost_Mar2018`

  Acceptable file types are txt, doc/docx, pdf (also cpp, with text in comment blocks).
- **Submit your programming code as part of your solution.**
- **You will be graded on your code as well as your numerical calculations.**
- **Answer ALL questions to be eligible for a grade boost.**

8 Question 8

- In this question, we compute the following sum:

$$S(x) = \frac{1}{N}\sum_{j=0}^{N-1} \cos\Big(x \sin\frac{j\pi}{N}\Big). \tag{8.1}$$

- **Set $N = 20$ in this question.**
- Define the following function

$$f(x) = 5 - \frac{1}{S(x)}. \tag{8.2}$$

- **Consider only values $x > 0$ in this question.**
- **Find brackets of size not less than 0.1 which enclose the first 6 positive roots of $f(x)$, i.e. where$f(x) = 0$.**
- **Find brackets of size not less than 0.1 which enclose the first 6 discontinuities of $f(x)$, i.e. $|f(x)| \to \infty$.**
- **Using the bisection algorithm and your brackets above, calculate the values of the first, third and fifth positive roots of $f(x)$ to an accuracy of three decimal places.**
- **Solutions which compute roots other than the first, third and fifth will receive a score of zero.**

i	x_i	$f(x_i)$
0	x_0	
1	x_1	
2	x_2 $(= (x_0 + x_1)/2)$	
$\vdots$	$\vdots$	
	converged to 3 d.p.	

- **Using the bisection algorithm and your brackets above, calculate the locations of the second, fourth and sixth discontinuities of $f(x)$ (for $x > 0$) to an accuracy of three decimal places. *Terminate the iteration if $|f(x)| > 10^6$ in the iteration.* State your reason for terminating the iteration ("converged in x" or "function value $|f(x) > 10^6$").**
- **Solutions which compute discontinuities other than the second, fourth and sixth will receive a score of zero.**

i	x_i	$f(x_i)$	reason for termination
0	x_0		
1	x_1		
2	x_2 $(= (x_0 + x_1)/2)$		
$\vdots$	$\vdots$		
	final answer		(converged in x) or ($|f(x)| > 10^6$)

Solution Question 8

- Finding an initial bracket (for bisection) or an initial iterate (for Newton–Raphson) is the hardest part of the calculation.
  1. There is no general procedure how to do it.
  2. We could, for example, calculate the value of $f(x)$ in steps of $\Delta x = 10^{-6}$.
  3. That that would locate the roots and discontinuities to an accuracy of 0.5×10^{-6}.
  4. However, that is computationally expensive, especially as the sixth positive root of $f(x)$ has a value > 37.
- Since the question asks for initial brackets of size not less than 0.1, let us plot a graph of $f(x)$ in steps of $\Delta x = 0.1$.
  1. The graph is displayed in Fig. 1.
  2. Fig. 1 reveals at a glance the approximate locations of the first six positive roots and the first six discontinuities of $f(x)$.
- Now we can employ bisection by determining some initial brackets.
- A suitable set of initial brackets for the first six positive roots of $f(x)$ is as follows.

i	x_{low}	x_{high}
1	2.0	2.1
2	6.1	6.2
3	7.8	7.9
4	12.9	13.0
5	13.7	13.8
6	37.6	37.7

- A suitable set of initial brackets for the locations of the first six positive discontinuities of $f(x)$ is as follows.

i	x_{low}	x_{high}
1	2.4	2.5
2	5.5	5.6
3	8.6	8.7
4	11.7	11.8
5	14.9	15.0
6	18.0	18.1

- **Other choices for the brackets are also acceptable.**
- ***However, each bracket should enclose only one root, or discontinuity, else bisection may not converge to the intended value. Fig. 1 will reveal what are suitable values for x_{low} and x_{high} for each case.***

- Different students submitted different choices for the initial brackets.
- The values of the iterates will depend on your initial bracket.
- Hence is is not possible to say what is a "correct" set of iterates.
- Here is a table of the first, third and fifth positive roots of $f(x)$.

i	x_{root}
1	2.042
3	7.874
5	13.739

- Here is a table of the locations of the second, fourth and sixth positive discontinuities of $f(x)$.
- For the discontinuities, it is also not possible to say if the iteration terminated because it "converged in x" or because $|f(x_i)| > 10^6$ for some iterate x_i.
- The answer depends on your choice of the initial bracket, because that determines the iterates.

i	$x_{\text{discontinuity}}$
2	5.520
4	11.792
6	18.071

- **Your numbers might differ in the last decimal place.**
  1. ***That is acceptable.***
  2. **The point of the exercise is to demonstrate you know what you are doing.**
  3. **Experience has shown that different computers and compilers (run time libraries?) yield slightly different answers.**

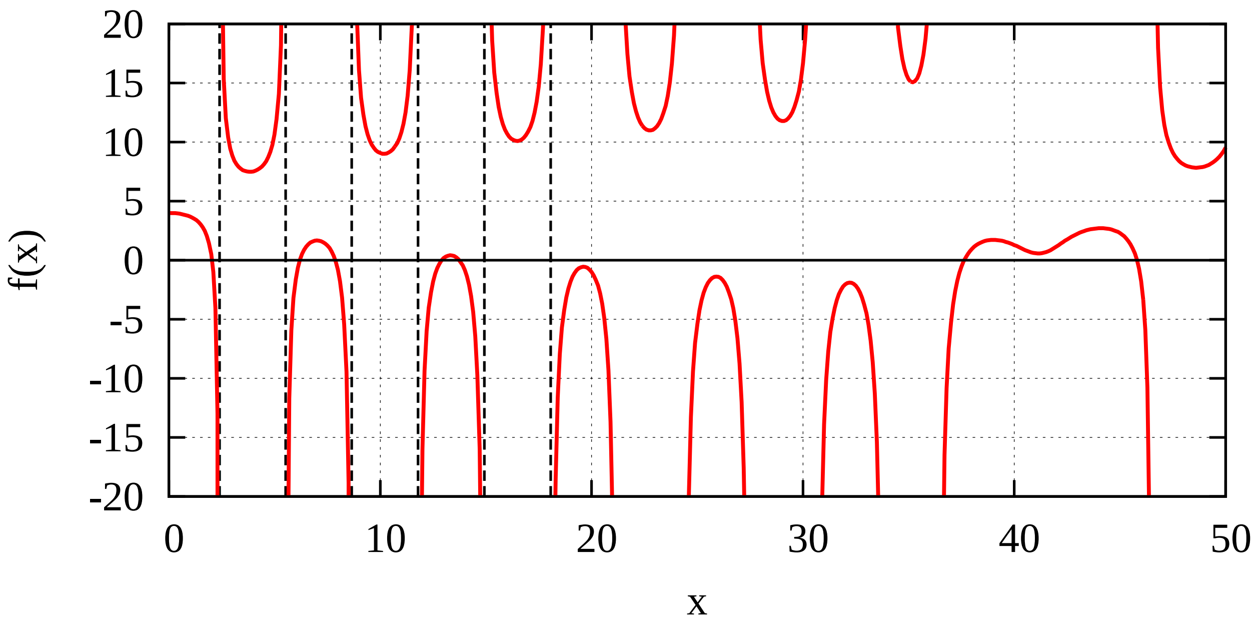

Figure 1: Graph of function $f(x)$ in Question 8.

9 Question 9

- The derivative of $f(x)$ is given by

$$f'(x) = \frac{S'(x)}{(S(x))^2} \, . \tag{9.1}$$

- **Use Newton–Raphson to compute the values of the first, third and fifth positive roots of $f(x)$ to an accuracy of three decimal places.**
- **Your starting iterate must be at a distance not less than 0.2 from the root.**
- Hence if a root is located at $x = 1.234$ then your starting iterate must be $x_0 < 1.034$ or $x_0 > 1.434$.
- **Solutions which compute roots other than the first, third and fifth will receive a score of zero.**

i	x_i	$f(x_i)$
0	x_0	
1	x_1	
$\vdots$	$\vdots$	
	converged to 3 d.p.	

- **It is not possible to say much for the solution of Question 9.**
- **The only way to know what is a suitable value for the initial iterate x_0 is to read the answers in Question 8 and back off by ≥ 0.2.**
- **Do not back off too far or Newton–Raphson may not converge to the desired root.**
- **Basically, I did not want you to set x_0 equal to the root from Question 8, else there would be nothing to iterate.**
- **Hence use a "minimum distance of 0.2" from the root and display a few iteration steps.**
- **Here is a table of suitable values for x_0 for the first, third and fifth positive roots of $f(x)$.**

i	x_0
1	**1.8**
3	**7.6**
5	**13.5**

- **Other choices are acceptable.**
- **The table of the first, third and fifth positive roots of $f(x)$ is the same as in Question 8.**