

April 29, 2018

22 Lecture 22

Numerical solution of systems of ordinary differential equations

- In this lecture we study some simple examples of **boundary value problems**.
- We treat only second order (inhomogeneous) linear ordinary differential equations.
- Then the equation can be solved numerically using a tridiagonal matrix algorithm.

22.1 Basic equation

- The equation we shall solve in this lecture has the general form

$$\alpha(x) \frac{dy^2}{dx^2} + \beta(x) \frac{dy}{dx} + \gamma(x)y = \zeta(x). \quad (22.1.1)$$

- Here $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $\zeta(x)$ are all functions of x only.
- We wish to solve for $y(x)$ in the interval $x_\ell \leq x \leq x_r$.
- Some of the values of $y(x)$ and/or dy/dx are given at the end point x_ℓ and others at the end point x_r .
- The above is called a **boundary value problem**.
- In this lecture, we require both x_ℓ and x_r to be finite.
- Our interest is to integrate eq. (22.1.1) numerically, using steps h_i , so $x_{i+1} = x_i + h_i$.
- We employ n subintervals, so end points are indexed as $x_0 = x_\ell$ and $x_n = x_r$.
- The steps h_i need not be of equal size, but in this lecture we shall assume equal steps ($= h$).
- We define $y_i = y(x_i)$.

22.2 Finite differences: tridiagonal matrix equations

- The differential equation is given by eq. (22.1.1).
- We discretize the derivatives using centered finite differences.
- We employ uniform steps and n subintervals, hence $h = (x_r - x_\ell)/n$.
- Then the finite differences are

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h}, \quad \frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}. \quad (22.2.1)$$

- Substituting into eq. (22.1.1) yields the approximate numerical equation

$$\alpha(x_i) \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \beta(x_i) \frac{y_{i+1} - y_{i-1}}{2h} + \gamma(x_i) y_i = \zeta(x_i). \quad (22.2.2)$$

- Multiply through by h^2 and collect terms to obtain the following:

$$\begin{aligned} \alpha(x_i) (y_{i+1} - 2y_i + y_{i-1}) + \frac{h}{2} \beta(x_i) (y_{i+1} - y_{i-1}) + h^2 \gamma(x_i) y_i &= h^2 \zeta(x_i) \\ \left[\alpha(x_i) - \frac{1}{2} h \beta(x_i) \right] y_{i-1} - \left[2\alpha(x_i) - h^2 \beta(x_i) \right] y_i + \left[\alpha(x_i) + \frac{1}{2} h \beta(x_i) \right] y_{i+1} &= h^2 \zeta(x_i). \end{aligned} \quad (22.2.3)$$

- Express eq. (22.2.3) in the form

$$b_i y_{i-1} + a_i y_i + c_i y_{i+1} = d_i, \quad (22.2.4a)$$

$$a_i = -2\alpha(x_i) + h^2 \gamma(x_i), \quad (22.2.4b)$$

$$b_i = \alpha(x_i) - \frac{1}{2} h \beta(x_i), \quad (22.2.4c)$$

$$c_i = \alpha(x_i) + \frac{1}{2} h \beta(x_i), \quad (22.2.4d)$$

$$d_i = h^2 \zeta(x_i). \quad (22.2.4e)$$

- Then eq. (22.2.4) yields a tridiagonal matrix system of equations (except at the end points). (Blanks denote zeroes in the tridiagonal matrix below.)

$$\begin{pmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & b_{i-1} & a_{i-1} & c_{i-1} & \\ & & & & b_i & a_i & c_i \\ & & & & & b_{i+1} & a_{i+1} & c_{i+1} \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ y_{i-1} \\ y_i \\ y_{i+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ d_{i-1} \\ d_i \\ d_{i+1} \\ \vdots \end{pmatrix}. \quad (22.2.5)$$

- The boundary conditions will supply the equations to use at the end points $i = 0$ and $i = n$.
- **Note that in general, the tridigonal equations are NOT diagonally dominant.**

22.3 Tridiagonal matrix equations: boundary conditions

- Since eq. (22.1.1) is a second order linear differential equation, it requires two independent conditions to specify a unique solution.
- Since we are solving a boundary value problem, one condition must be specified at x_ℓ , i.e. $i = 0$ and the other at x_r , i.e. $i = n$.
- There are four cases: (i) y_0 is given, (ii) y_n is given, (iii) y'_0 is given, (iv) y'_n is given.
- See next page(s).

22.3.1 Value of y_0 is given. Boundary condition $y_0 = Y_\ell$.

- This case is easy.
- The first row (for $i = 0$) has $a_0 = 1$, $c_0 = 0$ and $d_0 = Y_\ell$
- The matrix looks like this:

$$\begin{pmatrix} 1 & 0 & 0 & \\ b_1 & a_1 & c_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} Y_\ell \\ d_1 \\ \vdots \end{pmatrix} . \quad (22.3.1)$$

22.3.2 Value of y_n is given. Boundary condition $y_n = Y_r$.

- This case is easy.
- The last row (for $i = n$) has $a_n = 1$, $b_n = 0$ and $d_n = Y_r$
- The matrix looks like this:

$$\begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} \vdots \\ d_{n-1} \\ Y_r \end{pmatrix} . \quad (22.3.2)$$

22.3.3 Value of y'_0 is given. Boundary condition $y'_0 = Y'_\ell$.

- This case requires more work.
- For the first equation, we employ a forward finite difference and write

$$\begin{aligned}\frac{y_1 - y_0}{h} &= Y'_\ell, \\ y_1 - y_0 &= hY'_\ell.\end{aligned}\tag{22.3.3}$$

- The first row (for $i = 0$) has $a_0 = -1$, $b_0 = 1$ and $d_0 = hY'_\ell$
- The matrix looks like this:

$$\begin{pmatrix} -1 & 1 & 0 & & \\ b_1 & a_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} hY'_\ell \\ d_1 \\ \vdots \end{pmatrix}.\tag{22.3.4}$$

- The forward derivative is a first order approximation.
- This reduces the overall accuracy of the numerical solution.

22.3.4 Value of y'_n is given. Boundary condition $y'_n = Y'_r$.

- This case requires more work.
- For the last equation, we employ a backward finite difference and write

$$\begin{aligned}\frac{y_n - y_{n-1}}{h} &= Y'_r, \\ y_n - y_{n-1} &= hY'_r.\end{aligned}\tag{22.3.5}$$

- The last row (for $i = n$) has $a_n = 1$, $b_n = -1$ and $d_0 = hY'_r$
- The matrix looks like this:

$$\begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & b_{n-1} & a_{n-1} & c_{n-1} \\ & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} \vdots \\ d_{n-1} \\ hY'_r \end{pmatrix}.\tag{22.3.6}$$

- The backward derivative is a first order approximation.
- This reduces the overall accuracy of the numerical solution.

22.4 Example matrices

- We set $n = 4$ to keep the matrix size small.
- Example: values of y_0 and y_n are given.

$$\begin{pmatrix} 1 & 0 & & & \\ b_1 & a_1 & c_1 & & \\ & b_2 & a_2 & c_2 & \\ & & b_3 & a_3 & c_3 \\ & & & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} Y_\ell \\ d_1 \\ d_2 \\ d_3 \\ Y_r \end{pmatrix}. \quad (22.4.1)$$

- Example: values of y'_0 and y_n are given.

$$\begin{pmatrix} -1 & 1 & & & \\ b_1 & a_1 & c_1 & & \\ & b_2 & a_2 & c_2 & \\ & & b_3 & a_3 & c_3 \\ & & & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} hY'_\ell \\ d_1 \\ d_2 \\ d_3 \\ Y_r \end{pmatrix}. \quad (22.4.2)$$

- Example: values of y_0 and y'_n are given.

$$\begin{pmatrix} 1 & 0 & & & \\ b_1 & a_1 & c_1 & & \\ & b_2 & a_2 & c_2 & \\ & & b_3 & a_3 & c_3 \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} Y_\ell \\ d_1 \\ d_2 \\ d_3 \\ hY'_r \end{pmatrix}. \quad (22.4.3)$$

- Example: values of y'_0 and y'_n are given.

$$\begin{pmatrix} -1 & 1 & & & \\ b_1 & a_1 & c_1 & & \\ & b_2 & a_2 & c_2 & \\ & & b_3 & a_3 & c_3 \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} hY'_\ell \\ d_1 \\ d_2 \\ d_3 \\ hY'_r \end{pmatrix}. \quad (22.4.4)$$

22.5 Worked example

- Let us solve the following equation

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + \nu y = e^x. \quad (22.5.1)$$

- Here ν is a constant. We set $\nu = 1.5$ in the numerical work below.
- The domain of integration is $-1 \leq x \leq 1$.
- We set $n = 1000$, hence $h = 2/1000 = 0.002$.
- From eq. (22.2.4), the tridiagonal equations are given by:

$$b_i y_{i-1} + a_i y_i + c_i y_{i+1} = d_i, \quad (22.5.2a)$$

$$a_i = -2x_i^2 + h^2 \nu, \quad (22.5.2b)$$

$$b_i = x_i^2 - hx_i, \quad (22.5.2c)$$

$$c_i = x_i^2 + hx_i, \quad (22.5.2d)$$

$$d_i = h^2 \exp(x_i). \quad (22.5.2e)$$

- For $h < x_i \leq 1$, we see that $|b_i| = x_i^2 - hx_i$ and $|c_i| = x_i^2 + hx_i$.
Also $|a_i| = 2x_i^2 - h^2 \nu$ for $\nu = 1.5$. Then

$$\begin{aligned} |a_i| - (|b_i| + |c_i|) &= 2x_i^2 - h^2 \nu - (x_i^2 - hx_i + x_i^2 + hx_i) \\ &= -h^2 \nu \\ &< 0. \end{aligned} \quad (22.5.3)$$

- Hence the equations are NOT diagonally dominant.
- Notice also that for $x \simeq 0$, the coefficients of both $d^2 y/dx^2$ and dy/dx vanish, and the differential equation reduces to $\nu y \simeq 1$.
- *But the tridiagonal algorithm does not fail.*
- **See next page.**

22.5.1 Case 1: $y_0 = 0.75$, $y_n = 0.25$

- The equations are

$$\begin{pmatrix} 1 & 0 & & & \\ b_1 & a_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} 0.75 \\ h^2 \exp(x_1) \\ \vdots \\ h^2 \exp(x_{n-1}) \\ 0.25 \end{pmatrix}. \quad (22.5.4)$$

- The solution is plotted as the black curve in Fig. 1, for $\nu = 1.5$.

22.5.2 Case 2: $y'_0 = -1$, $y_n = 1$

- The equations are

$$\begin{pmatrix} -1 & 1 & & & \\ b_1 & a_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} -h \\ h^2 \exp(x_1) \\ \vdots \\ h^2 \exp(x_{n-1}) \\ 1 \end{pmatrix}. \quad (22.5.5)$$

- The solution is plotted as the red curve in Fig. 1, for $\nu = 1.5$.

22.5.3 Case 3: $y_0 = 1$, $y'_n = 1$

- The equations are

$$\begin{pmatrix} 1 & 0 & & & \\ b_1 & a_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} 1 \\ h^2 \exp(x_1) \\ \vdots \\ h^2 \exp(x_{n-1}) \\ h \end{pmatrix}. \quad (22.5.6)$$

- The solution is plotted as the blue curve in Fig. 1, for $\nu = 1.5$.

22.5.4 Case 4: $y'_0 = 0$, $y'_n = 0$

- The equations are

$$\begin{pmatrix} -1 & 1 & & & \\ b_1 & a_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ h^2 \exp(x_1) \\ \vdots \\ h^2 \exp(x_{n-1}) \\ 0 \end{pmatrix}. \quad (22.5.7)$$

- The solution is plotted as the green curve in Fig. 1, for $\nu = 1.5$.

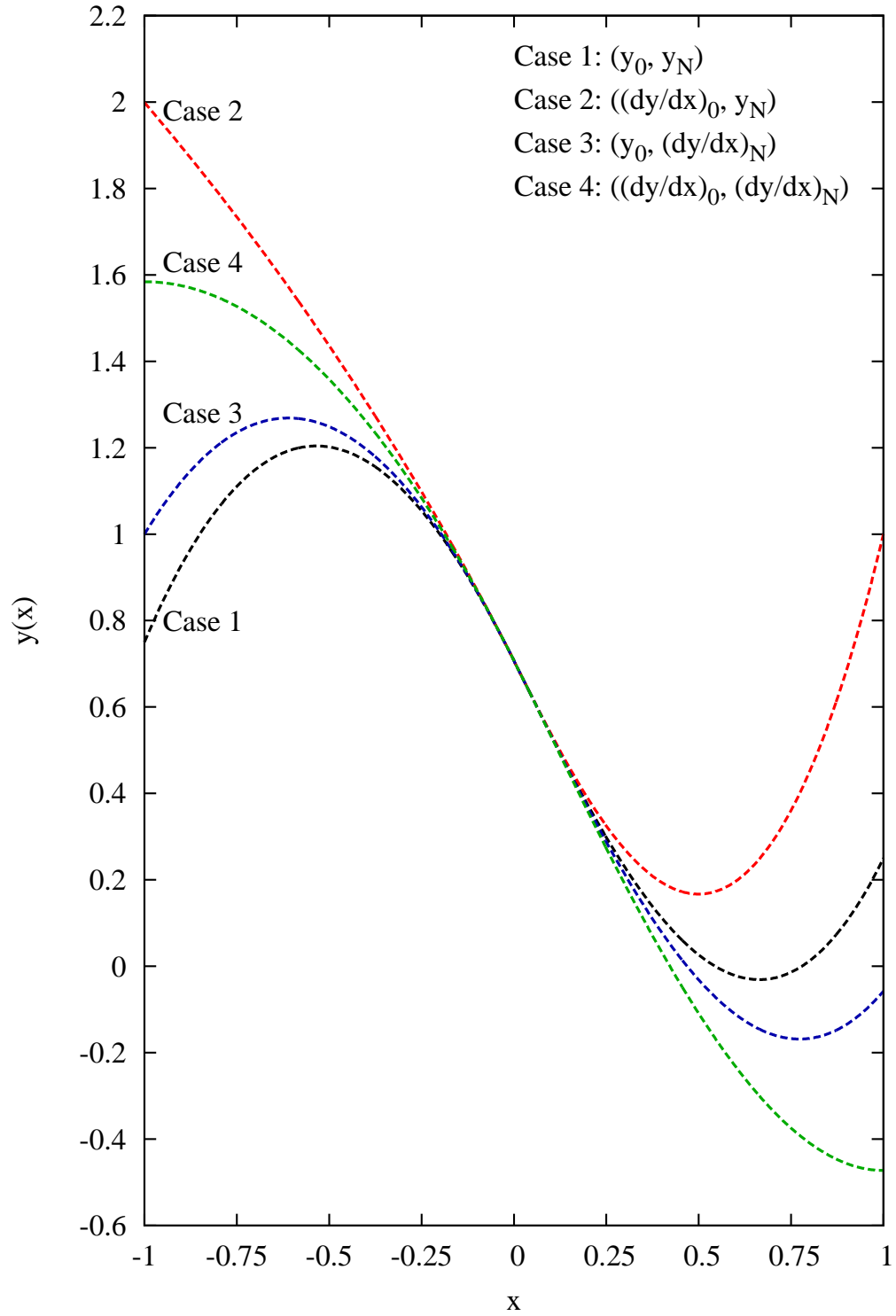


Figure 1: Solution of worked example in the text, for the four cases of boundary conditions: (i) (y_0, y_n) , (ii) (y'_0, y_n) , (iii) (y_0, y'_n) and (iv) (y'_0, y'_n) .