- Given $f(x)$ and $f(x+h_1)$, $f(x+h_2)$ $f(x+h_3)$.
- Find a numerical difference for $f'(x)$ such that the leading error term is $O(f''''(x))$.
- Taylor series:

$$
\begin{aligned}
f(x+h_1) &= f(x) + h_1 f'(x) + \frac{h_1^2}{2!} f''(x) + \frac{h_1^3}{3!} f'''(x) + \frac{h_1^4}{4!} f''''(x) + \cdots && (1)\\
f(x+h_2) &= f(x) + h_2 f'(x) + \frac{h_2^2}{2!} f''(x) + \frac{h_2^3}{3!} f'''(x) + \frac{h_2^4}{4!} f''''(x) + \cdots && (2)\\
f(x+h_3) &= f(x) + h_3 f'(x) + \frac{h_3^2}{2!} f''(x) + \frac{h_3^3}{3!} f'''(x) + \frac{h_3^4}{4!} f''''(x) + \cdots && (3)
\end{aligned}
$$

- Linear combination L:

$$
L = \frac{a}{h_1} f(x+h_1) + \frac{b}{h_2} f(x+h_2) + \frac{c}{h_3} f(x+h_3)\,. \tag{4}
$$

- Constraint equations:

$$
\begin{aligned}
a + b + c &= 1\,, && (5)\\
h_1 a + h_2 b + h_3 c &= 0\,, && (6)\\
h_1^2 a + h_2^2 b + h_3^2 c &= 0\,. && (7)
\end{aligned}
$$

- Eliminate c:

$$
h_1(h_3 - h_1) a + h_2 (h_3 - h_2) b = 0\,. \tag{8}
$$

- Eliminate b:

$$
b = -a\, \frac{h_1}{h_2} \frac{h_3 - h_1}{h_3 - h_2}\,. \tag{9}
$$

- Express c in terms of a:

$$
\begin{aligned}
h_3 c &= -h_1 a - h_2 b \\
&= -a h_1 + a\, h_1 \frac{h_3 - h_1}{h_3 - h_2} \\
&= a\, h_1 \frac{h_2 - h_1}{h_3 - h_2} \\
c &= a\, \frac{h_1}{h_3} \frac{h_2 - h_1}{h_3 - h_2}\,.
\end{aligned} \tag{10}
$$

- Solve for a:

$$
\begin{aligned}
1 &= a + b + c \\
&= a - a\, \frac{h_1}{h_2} \frac{h_3 - h_1}{h_3 - h_2} + a\, \frac{h_1}{h_3} \frac{h_2 - h_1}{h_3 - h_2} \\
&= a\, \frac{h_2 h_3 (h_3 - h_2) + h_3 h_1 (h_1 - h_3) + h_1 h_2 (h_2 - h_1)}{h_2 h_3 (h_3 - h_2)}\,.
\end{aligned} \tag{11}
$$

- Solutions via cyclic permutation of (h_1, h_2, h_3), denominator is the same:

$$a = h_2 h_3 \, \frac{h_2 - h_3}{h_1 h_2 (h_1 - h_2) + h_2 h_3 (h_2 - h_3) + h_3 h_1 (h_3 - h_1)} \,, \tag{12}$$

$$b = h_3 h_1 \, \frac{h_3 - h_1}{h_1 h_2 (h_1 - h_2) + h_2 h_3 (h_2 - h_3) + h_3 h_1 (h_3 - h_1)} \,, \tag{13}$$

$$c = h_1 h_2 \, \frac{h_1 - h_2}{h_1 h_2 (h_1 - h_2) + h_2 h_3 (h_2 - h_3) + h_3 h_1 (h_3 - h_1)} \,. \tag{14}$$

- Factorize the denominator:

$$\begin{aligned} D &= h_1 h_2 (h_1 - h_2) + h_2 h_3 (h_2 - h_3) + h_3 h_1 (h_3 - h_1) \\ &= h_1 h_2 (h_1 - h_2) + (h_1 - h_2)(h_3^2 - h_3 h_1 - h_2 h_3) \\ &= (h_1 - h_2)(h_1 h_2 - h_3 h_1 - h_2 h_3 + h_3^2) \\ &= (h_1 - h_2)(h_2 - h_3)(h_1 - h_3) \,. \end{aligned} \tag{15}$$

- Simplified solutions:

$$a = \frac{h_2 h_3}{(h_1 - h_2)(h_1 - h_3)} \,, \tag{16}$$

$$b = \frac{h_3 h_1}{(h_2 - h_1)(h_2 - h_3)} \,, \tag{17}$$

$$c = \frac{h_1 h_2}{(h_3 - h_1)(h_3 - h_2)} \,. \tag{18}$$

Vandermonde matrices

- The equations can be written in matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 1 \\ h_1 & h_2 & h_3 \\ h_1^2 & h_2^2 & h_3^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} . \tag{19}$$

- An $n \times n$ matrix of the following form is called a Vandermonde matrix:

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^n & \lambda_2^n & \dots & \lambda_n^n \end{pmatrix} \tag{20}$$

- Many texts, including Wikipedia, define a Vandermonde matrix as the transpose of the above.
- Vandermonde matrices arise in many important problems and have nice special properties.
- I have not taught Vandermonde matrices in CS361/761 up to now.
- The plan is to do linear algebra before Spring Break and numerical solution of ordinary differential equations after Spring Break.
- If time permits to squeeze in Vandermonde matrices, I shall do so.
- Remind me.