

Queens College, CUNY, Department of Computer Science
Numerical Methods
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10 Homework lecture 10

- As experience has demonstrated, if you do not understand the above expressions/questions, **THEN ASK**.
- If you do not understand the words/sentences in Lecture 6, etc. **THEN ASK**.
- Send me an email, explain what you do not understand.
- Do not just keep quiet and produce nonsense in exams.

Linear algebra: motivating example (control theory)

- There are many occasions when we have a system whose state is controlled by a set of variables. For example a spacecraft is supposed to follow a “design trajectory” but in practice we observe its motion is deviating from that trajectory, say in three coordinates (x, y, z) .
- Let us say the design trajectory is (x_d, y_d, z_d) . Let us also say the observed trajectory deviates from the design by $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$, where the notation “ ε ” suggests a small deviation.
- We can control the motion of the spacecraft with three parameters (α, β, γ) . (They might be the power to rocket thrusters.) We want to set the values of α , β and γ to cancel the unwanted deviations.

1. If we set $\alpha = 1$ (and $\beta = \gamma = 0$), we calculate that the change in the trajectory is

$$(\text{change in trajectory if } \alpha = 1) = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}. \quad (10.0.1)$$

2. If we set $\beta = 1$ (and $\alpha = \gamma = 0$), we calculate that the change in the trajectory is

$$(\text{change in trajectory if } \beta = 1) = \begin{pmatrix} b_x \\ b_y \\ c_z \end{pmatrix}. \quad (10.0.2)$$

3. If we set $\gamma = 1$ (and $\alpha = \beta = 0$), we calculate that the change in the trajectory is

$$(\text{change in trajectory if } \gamma = 1) = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}. \quad (10.0.3)$$

- Then to cancel the unwanted deviations, we want to find a set of joint values for (α, β, γ) such that the change they produce is the negative $(-\varepsilon_x, -\varepsilon_y, -\varepsilon_z)$.
- Hence the equations to solve for α , β and γ are

$$\begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -\varepsilon_x \\ -\varepsilon_y \\ -\varepsilon_z \end{pmatrix} = - \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}. \quad (10.0.4)$$

- Problems like this arise frequently in many contexts.
- The subject area is called **control theory**.
- **There is nothing for you to solve here.**
- **It is an example to motivate the subject of linear algebra.**

10.1 Linear algebra: 2×2 matrix

- The case of 2×2 matrices arises frequently in practice. There are two unknowns x_1 and x_2 .
- It is not necessary to employ fancy mathematical formalism for this simple case.
- Let the matrix equation be $A\mathbf{x} = \mathbf{r}$ which has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \quad (10.1.1)$$

- The solution is $\mathbf{x} = A^{-1}\mathbf{r}$. The inverse matrix is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad (10.1.2)$$

- You are given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}. \quad (10.1.3)$$

- **Calculate the inverse matrix A^{-1} .**
- **Solve the following matrix equation using the inverse matrix A^{-1} .**

$$\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (10.1.4)$$

- **Calculate the inverse matrix for the following set of equations.
Then solve the equations for x_1 and x_2 .**

$$\begin{pmatrix} -3 & \frac{1}{2} \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}. \quad (10.1.5)$$

10.2 Linear algebra: LU decomposition with partial pivoting

10.2.1 Eliminate a_{i1} in first column

- In this question we shall calculate the LU decomposition of the matrix A .
- You are given the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ -3 & 2 & 3 \end{pmatrix}. \quad (10.2.1)$$

- In addition, the initial array of the swap indices is $S = (1, 2, 3)$.
- **State the values of \hat{a}_i in each row $i = 1, 2, 3$.**
- **Calculate the values of $|a_{i1}|/\hat{a}_i$ for $i = 1, 2, 3$.**
- **State which pair of rows will be swapped.**
- **Write down the matrix after swapping the rows.**

$$A_1(\text{swap}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}. \quad (10.2.2)$$

- **Update the array of the swap indices and state the answer $S = (?, ?, ?)$.**
- **Subtract multiples of row 1 to eliminate the coefficient a_{i1} in rows 2 and 3.**
- **Write down the matrix after eliminating a_{i1} from rows 2 and 3.**

$$A_2(\text{eliminate } a_{i2}) = \begin{pmatrix} ? & ? & ? \\ \mathbf{0} & ? & ? \\ \mathbf{0} & ? & ? \end{pmatrix}. \quad (10.2.3)$$

- Fill in the blank spots (marked by $\mathbf{0}$ in eq. (10.2.3)) with the relevant multipliers, to obtain the “L” part of the LU decomposition.
- **Write down the matrix after filling the blank spots.**

$$A_3(\text{fill in blank spots}) = \begin{pmatrix} ? & ? & ? \\ \alpha & ? & ? \\ \beta & ? & ? \end{pmatrix}. \quad (10.2.4)$$

- **Write down the values of α and β used to fill the blank spots.**
- **The values of α and β should be negative numbers.**

10.2.2 Eliminate a_{i2} in second column

- Now we move on to the elimination of a_{i2} in the second column.
- **State the new values of \hat{a}_i in rows $i = 2, 3$ ONLY. Do NOT process row 1.**
- **Calculate the values of $|a_{i2}|/\hat{a}_i$ for $i = 2, 3$.**
- **State which pair of rows will be swapped.**
It is possible that no swap is required. Then say so.
- If a swap is required, **then write down the matrix after swapping the rows.**

$$A_4(\text{swap if necessary}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}. \quad (10.2.5)$$

- If a swap is required, **then update the array of the swap indices and state the answer $S = (?, ?, ?)$.**
- Subtract a multiple of row 2 to eliminate the coefficient a_{i2} in row 3.
- **Write down the matrix after eliminating a_{i2} from row 3.**

$$A_5(\text{eliminate } a_{i2}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & \mathbf{0} & ? \end{pmatrix}. \quad (10.2.6)$$

- Fill in the blank spot (marked by $\mathbf{0}$ in eq. (10.2.4)) with the relevant multiplier, to obtain the “L” part of the LU decomposition.
- **Write down the matrix after filling the blank spot.**

$$A_6(\text{fill in blank spot}) = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & \gamma & ? \end{pmatrix}. \quad (10.2.7)$$

- **Write down the value of γ used to fill the blank spot.**
- The value of γ should be a **positive number**.

10.2.3 LU factorization

- The matrix is now in LU factorized form.
- The array of swap indices is also in its final form.
- We shall test if the LU decomposition has been performed correctly.
- Denote the LU factorized matrix by

$$A_7(LU) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21} & u_{22} & u_{23} \\ \ell_{31} & \ell_{32} & u_{33} \end{pmatrix}. \quad (10.2.8)$$

- **Write down the L and U matrices as follows:**

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}. \quad (10.2.9)$$

- **Multiply the L and U matrices. Calculate and state the result.**

$$A_8 = LU = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}. \quad (10.2.10)$$

- Let the final array of swap indices be $S = (s_1, s_2, s_3)$.
 1. **Compare row 1 in the LU product with rows s_1 in the original matrix A .**
 2. **Compare row 2 in the LU product with rows s_2 in the original matrix A .**
 3. **Compare row 3 in the LU product with rows s_3 in the original matrix A .**
- If you have done everything correctly, you should obtain a match in all three cases.
- **Calculate the determinant of the matrix A .
(Remember to count the number of swaps.)**

10.2.4 Solution of equations

- You are given the following equations

$$x_1 + 2x_2 + x_3 = 1, \quad (10.2.11a)$$

$$2x_1 + 4x_2 - x_3 = 1, \quad (10.2.11b)$$

$$-3x_1 + 2x_2 + 3x_3 = 7. \quad (10.2.11c)$$

- The matrix is

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ -3 & 2 & 3 \end{pmatrix}. \quad (10.2.12)$$

- The matrix is now available in LU factorized form.
- The final array of swap indices $S = (s_1, s_2, s_3)$ is also available.
- Let the right hand side vector be

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}. \quad (10.2.13)$$

- To solve the equations using LU decomposition, we must swap the entries in the right hand side vector \mathbf{r} .
- The following steps is important. Make sure you understand and do it correctly.
- **Rearrange the entries in the right hand side vector as follows:**

$$\mathbf{r}' = \begin{pmatrix} \text{row } s_1 \text{ in } \mathbf{r} \\ \text{row } s_2 \text{ in } \mathbf{r} \\ \text{row } s_3 \text{ in } \mathbf{r} \end{pmatrix}. \quad (10.2.14)$$

- Define a temporary column vector \mathbf{y}

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \quad (10.2.15)$$

- **Calculate the values of y_1 , y_2 and y_3 by backsubstitution.**

$$L\mathbf{y} = \mathbf{r}'. \quad (10.2.16)$$

(We should really say “forward substitution” because we calculate the value of y_1 first, etc.)

- **Calculate the values of x_1 , x_2 and x_3 by backsubstitution.**

$$U\mathbf{x} = \mathbf{y}. \quad (10.2.17)$$

(This is real backsubstitution, because we calculate the value of x_3 first, etc.)

- **Substitute your solutions for x_1 , x_2 and x_3 and verify that they satisfy eq. (10.2.11a) eq. (10.2.11b) and eq. (10.2.11c).**

10.3 Linear algebra: LU decomposition, two swaps

- You are given the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ -3 & 2 & 3 \end{pmatrix}. \quad (10.3.1)$$

- In addition, the initial array of the swap indices is $S = (1, 2, 3)$.
- **Perform the LU decomposition for the matrix A in eq. (10.3.1).**
- This time you will need to do two swaps. When performing the second swap, make sure to swap the entries in the “L” part of the matrix as well.
- **Write out the steps in the LU decomposition and display the final LU matrix.**
- **Also write down the final value of the array of the swap indices S .**
- **Write down the matrices L and U and multiply them. Verify that the product LU equals the original matrix A , with permutation of rows. Verify that the permutation is given by the array of the swap indices S .**
- **Calculate the determinant of the matrix A . (Remember to count the number of swaps.)**
- **(Optional)** Solve the following equations for x_1 , x_2 and x_3 :

$$x_1 + 2x_2 + x_3 = 1, \quad (10.3.2a)$$

$$2x_1 + 4x_2 + 5x_3 = 1, \quad (10.3.2b)$$

$$-3x_1 + 2x_2 + 3x_3 = 7. \quad (10.3.2c)$$

10.4 Linear algebra: (practice for midterm)

- You are given the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (10.4.1)$$

- The initial array of the swap indices is $S = (1, 2, 3, 4)$.
- It is perfectly obvious to a human that this is a lower triangular matrix in disguise, but the computer will not know that.
- **Perform the LU decomposition for the matrix A in eq. (10.4.1). Display the steps in your calculation.**
- **Write down the final array of the swap indices.**
- **Calculate the determinant of the matrix A .**
- **Solve the equation $Ax_i = r_i$ using LU backsubstitution, $i = 1, 2, 3, 4$, for the following right hand side column vectors:**

$$r_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad r_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (10.4.2)$$

- **Calculate the matrix inverse A^{-1} .**

10.5 Linear algebra: LU decomposition, zero pivot

- Eh... this is boring. The computer will take care of it.
- *Or a midterm exam question will deal with it.*