Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Spring 2018

Instructor: Dr. Sateesh Mane

due Friday, April 27, 2018, 11.59 pm

28 Homework lecture 28

29 Homework lecture 29

- **As experience has demonstrated, if you do not understand the above expressions/questions, THEN ASK.**
- **If you do not understand the words/sentences in the lectures, THEN ASK.**
- **Send me an email, explain what you do not understand.**
- **Do not just keep quiet and produce nonsense in exams.**

28.1 Sampling

- Let us consider the simple example

$$f_4(\theta) = \cos(4\theta)\,. \tag{28.1.1}$$

- We wish to compute the Fourier series of $f_4(\theta)$ numerically.
- We know the correct answer is $a_4 = 1$ and all the other a_j and b_j are zero.
- However, the function $f_4(\theta)$ is only available to us as the output of a 'black box' routine.
- We compute the Fourier coefficients using the following sums, with n points equally spaced around a circle:

$$\begin{aligned} a_j &= \frac{2}{n}\sum_{i=0}^{n-1} f(\theta_i)\cos(j\theta_i)\,, \\ b_j &= \frac{2}{n}\sum_{i=0}^{n-1} f(\theta_i)\sin(j\theta_i) \qquad \left(\theta_i = \frac{2\pi i}{n}\right). \end{aligned} \tag{28.1.2}$$

- *How many values of n do we need in eq.* (28.1.2) *to compute the values of a_j and b_j accurately?*
- **Compute the values of a_j and b_j for $0 \le j \le 10$ using $n = 1$, then $n = 2$, up to $n = 20$.**
- **You may employ/modify the program from the previous homework assignment.**
- If you have done your work correctly, you should find that $b_j = 0$ in all cases.
- **Fill in the table below, for a_j, $j = 0, \dots, 10$, for $n = 1, \dots, 20$.**
  If you have done your work correctly, the tabulated values will all be zero or positive integers.

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	(integer ≥ 0)	etc.									
2	(integer ≥ 0)										
$\vdots$	$\vdots$										
20	(integer ≥ 0)										

- If you have done your work correctly, by the time you reach $n = 20$ (last line in the table), you should obtain $a_4 = 1$ and all the others are zero.
- We shall discuss the results obtained in the above table in this homework assignment.

28.2 Aliasing

- Let us consider the following 'orignal' function:

$$
\begin{aligned}
f_{\text{orig}}(\theta) = \tfrac{1}{2} &+ 2\cos(\theta) + 3\sin(\theta) \\
&+ 4\cos(2\theta) + 5\sin(2\theta) \\
&+ 6\cos(3\theta) + 7\sin(3\theta) \\
&+ 8\cos(4\theta) + 9\sin(4\theta)\,.
\end{aligned} \tag{28.2.1}
$$

- **We shall use $n = 16$ in this question.**

  Compute the values of a_j and b_j for $f_{\text{orig}}(\theta)$.

- **Fill in the table below, for a_j and b_j $j = 0, \ldots, 7$ (where $b_0 = 0$).**

j	a_j	b_j
0		0
1		
$\vdots$	$\vdots$	$\vdots$
7		

- Next use the following function:

$$\begin{aligned} f_1(\theta) = f_{\text{orig}}(\theta) &+ 0.01\cos(9\theta) \ + 0.02\sin(10\theta) \\ &+ 0.03\cos(11\theta) \ + 0.04\sin(12\theta) \\ &+ 0.05\cos(13\theta) \ + 0.06\sin(14\theta) \\ &+ 0.07\cos(15\theta) \ + 0.08\sin(16\theta)\,. \end{aligned} \tag{28.2.2}$$

- **Continue to use $n = 16$ in this question.**
  Compute the values of a_j and b_j for $f_1(\theta)$.
- **Fill in the table below, for a_j and b_j $j = 0, \dots, 7$ (where $b_0 = 0$).**

j	a_j	b_j
0		0
1		
⋮	⋮	⋮
7		

- Notice that the values of a_j and b_j have been changed by the aliasing.
- *Notice also the pattern of the changes caused by aliasing.*
  The values 0.01*,* 0.02*, etc. will help you to track the effects.*
- Next use the following function:

$$\begin{aligned} f_2(\theta) = f_{\text{orig}}(\theta) &+ 0.01\cos((16+9)\theta) \ + 0.02\sin((16+10)\theta) \\ &+ 0.03\cos((16+11)\theta) \ + 0.04\sin((16+12)\theta) \\ &+ 0.05\cos((16+13)\theta) \ + 0.06\sin((16+14)\theta) \\ &+ 0.07\cos((16+15)\theta) \ + 0.08\sin((16+16)\theta)\,. \end{aligned} \tag{28.2.3}$$

- **Continue to use $n = 16$ in this question.**
  Compute the values of a_j and b_j for $f_1(\theta)$.
- **Fill in the table below, for a_j and b_j $j = 0, \dots, 7$ (where $b_0 = 0$).**

j	a_j	b_j
0		0
1		
⋮	⋮	⋮
7		

- *If you have done your work correctly, the values of a_j and b_j will be the same as for $f_1(\theta)$.*

- Next use the following function:

$$
\begin{aligned}
f_3(\theta) = f_{\text{orig}}(\theta) &+ 0.002\cos(16\theta) \;\; + 0.001\sin(17\theta) \\
&+ 0.004\cos(18\theta) \;\; + 0.003\sin(19\theta) \\
&+ 0.006\cos(20\theta) \;\; + 0.005\sin(21\theta) \\
&+ 0.008\cos(22\theta) \;\; + 0.007\sin(23\theta)\,.
\end{aligned}
\tag{28.2.4}
$$

- **Continue to use $n = 16$ in this question.**
  Compute the values of a_j and b_j for $f_1(\theta)$.
- **Fill in the table below, for a_j and b_j $j = 0, \ldots, 7$ (where $b_0 = 0$).**

j	a_j	b_j
0		0
1		
$\vdots$	$\vdots$	$\vdots$
7		

- Notice that the values of a_j and b_j have been changed by the aliasing.
- *Notice also the pattern of the changes caused by aliasing.*
  The pattern of the changes is different from that caused by $f_1(\theta)$.
  The values 0.001, 0.002, etc. will help you to track the effects.
- Next use the following function:

$$
\begin{aligned}
f_4(\theta) = f_{\text{orig}}(\theta) &+ 0.002\cos((16+16)\theta) \;\; + 0.001\sin((16+17)\theta) \\
&+ 0.004\cos((16+18)\theta) \;\; + 0.003\sin((16+19)\theta) \\
&+ 0.006\cos((16+20)\theta) \;\; + 0.005\sin((16+21)\theta) \\
&+ 0.008\cos((16+22)\theta) \;\; + 0.007\sin((16+23)\theta)\,.
\end{aligned}
\tag{28.2.5}
$$

- **Continue to use $n = 16$ in this question.**
  Compute the values of a_j and b_j for $f_1(\theta)$.
- **Fill in the table below, for a_j and b_j $j = 0, \ldots, 7$ (where $b_0 = 0$).**

j	a_j	b_j
0		0
1		
$\vdots$	$\vdots$	$\vdots$
7		

- *If you have done your work correctly, the values of a_j and b_j will be the same as for $f_3(\theta)$.*

28.3 Nyquist frequency & aliasing

- We employ the same 'orignal' function given in eq. (28.2.1).
- **For each case below:**
  1. State the Nyquist frequency (as an integer).
  2. State (without computer calculation), which values of a_j and/or b_j will be affected, for $j = 0, \ldots, 7$.
  3. ***Explain why they are affected, and not the other coefficients.***
  4. Also calculate the values of a_j and/or b_j which are affected.

28.3.1 $n = 22$

$$f(\theta) = f_{\rm orig}(\theta) + 0.0001\cos(40\theta) - 0.0002\sin(50\theta)\,. \tag{28.3.1}$$

28.3.2 $n = 24$

$$\begin{aligned} f(\theta) = f_{\rm orig}(\theta) &+ 0.0001\cos(44\theta) - 0.0002\sin(43\theta) \\ &- 0.0003\cos(54\theta) + 0.0002\sin(53\theta)\,. \end{aligned} \tag{28.3.2}$$

28.3.3 $n = 14$

$$\begin{aligned} f(\theta) = f_{\rm orig}(\theta) &+ 0.0001\cos(44\theta) - 0.0002\sin(43\theta) \\ &- 0.0003\cos(54\theta) + 0.0002\sin(53\theta)\,. \end{aligned} \tag{28.3.3}$$

28.3.4 $n = 20$

$$\begin{aligned} f(\theta) = f_{\rm orig}(\theta) &+ 0.0001\cos(44\theta) - 0.0002\sin(43\theta) \\ &- 0.0003\cos(54\theta) + 0.0002\sin(53\theta)\,. \end{aligned} \tag{28.3.4}$$

28.3.5 $n = 1024$

$$\begin{aligned} f(\theta) = f_{\rm orig}(\theta) &+ 0.0001\cos(44\theta) - 0.0002\sin(43\theta) \\ &- 0.0003\cos(54\theta) + 0.0002\sin(53\theta)\,. \end{aligned} \tag{28.3.5}$$

28.4 Nested summation of Fourier series

- **Submit your program code as part of your answer to this question.**
- Suppose we are given a set of Fourier harmonics a_j and b_j for $j = 0, \dots, m$ (where $b_0 = 0$).
- It was derived in the lectures than an efficient way to sum the Fourier series is as follows.

```
const double c = cos(theta);
const double s = sin(theta);
double U = 0;
double V = 0;
for (int j = m; j > 0; --j) {
  double Utmp = a[j] + c*U + s*V;
  double Vtmp = b[j] - s*U + c*V;
  U = Utmp;
  V = Vtmp;
}
double fsum = 0.5*a[0] + c*U + s*V;
```

- We shall employ the above code to sum a Fourier series.
- **Write a loop to calculate the sum of the series for tt 2*npts+1 values of θ.**

```
const double pi = 4.0*atan2(1.0,1.0);
const int npts = (set by user);
double dtheta = pi/double(npts);
for (int i = -npts; i <= npts; ++i) {
  double theta = i*dtheta;
  // code to sum series
  // print the function and plot output
}
```

- The value of `npts` is set by the user or calling application.
- Note that the above code computes the value of `fsum` in the interval $-\pi \le \theta \le \pi$.
- Recall the Fourier harmonics of the window function of width $2\theta_0$ are given by

$$a_0 = \frac{1}{\pi}, \qquad a_j = \frac{1}{\pi}\frac{\sin(j\theta_0)}{j\theta_0} \qquad (j \ge 1). \tag{28.4.1}$$

- *But this is boring.* It will sum to the window function, which we have seen many times.

- *Let us do something different.*
- Let us make an **antiwindow function.**
- This is just my own name. I have *absolutely no idea* what the function really is.
- **Set $a_j = 0$ and b_j as follows:**

$$b_j = \frac{\sin(j\theta_0)}{j\theta_0} \qquad\qquad (j \ge 1)\,. \tag{28.4.2}$$

- **Set $\theta_0 = \pi/4$ in this question.**
- **Set $m = 256$ to sum the Fourier series.**
- **Set `npts = 1000` to compute the values of θ.**
- **Compute the value of `fsum` using the above code, with `npts = 1000` and $m = 256$.**
- **Plot a graph of `fsum` for `2*npts+1` values of θ as indicated in the above code.**
  1. On the horizontal axis, plot the value of θ/π, so the values go from -1 to 1.
  2. On the vertical axis, go from -6 to 6.
- A graph of the function is plotted in Fig. 1.
- If you have done your work correctly, you should obtain the same answer.

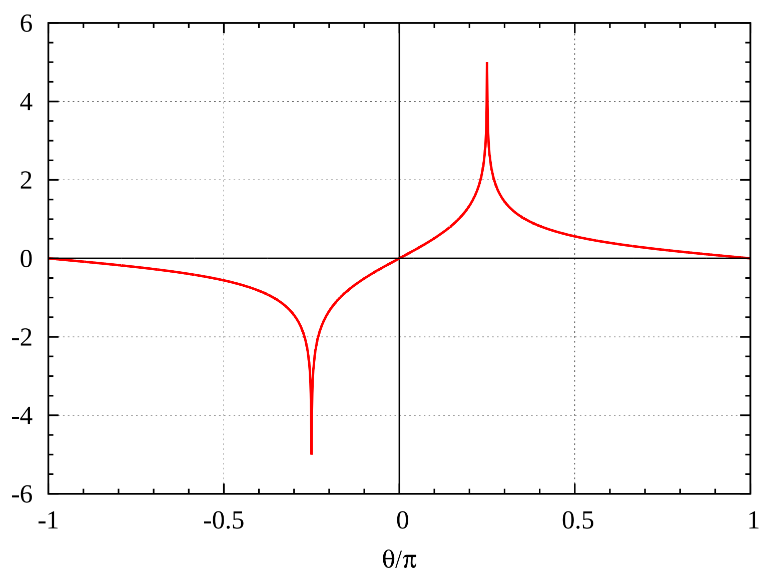

Figure 1: Plot of the 'antiwindow' function obtained by summing the Fourier series $b_j = \sin(j\theta_0)/(j\theta_0)$ with $\theta_0 = \frac{1}{4}\pi$ in Question 28.4.

- *What are we to make of the plotted function?*
- By construction, the function is

$$f(\theta) = \sum_{j=1}^{\infty} \frac{\sin(j\theta_0)}{j\theta_0} \, \sin(j\theta) \, . \tag{28.4.3}$$

- By construction, the function is odd $f(-\theta) = -f(\theta)$.
- Because $\sin(j\theta) = 0$ at $\theta = 0$ and $\theta = \pm\pi$, we must have $f(\theta) = 0$ at $\theta = 0$ and $\theta = \pm\pi$, as observed.
- The function has two spikes at $\theta = \pm\theta_0$, i.e. $\theta = \pm\frac{1}{4}\pi$ in this example.
- Set $\theta = \theta_0$ and we obtain

$$f(\theta_0) = \sum_{j=1}^{\infty} \frac{\sin^2(j\theta_0)}{j\theta_0} \, . \tag{28.4.4}$$

- Let us approximate $\sin^2(j\theta_0) \simeq \frac{1}{2}$, then

$$f(\theta_0) \simeq \frac{1}{2\theta_0} \sum_{j=1}^{\infty} \frac{1}{j} \to \infty \, . \tag{28.4.5}$$

- The sum diverges logarithmically.
- Hence $f(\theta)$ has logarithmic singularities at $\theta = \pm\theta_0$.

28.5 Solution of differential equation

- Let us solve an ordinary differential equation using a Fourier series.
- This is an important technique in physics and engineering.
- This is an example where we do not know the function $f(\theta)$ and must obtain it by summing a Fourier series.
- The system in this question is a **driven damped oscillator.**
- The differential equation is

$$\frac{d^2 f}{d\theta^2} + R\,\frac{df}{d\theta} + Q^2 f = g(\theta)\,. \tag{28.5.1}$$

- Here Q and R are constants and $g(\theta)$ is a periodic function with period 2π.
- The function $g(\theta)$ is a **driving term.**
- We seek the solution of eq. (28.5.1) which is periodic with period 2π.
- The solution for $f(\theta)$ is called the **forced (or driven) solution.**
- The forced solution would be zero if $g(\theta)$ were zero.
- We seek a procedure to calculate the forced solution.
- This is accomplished as follows.
  1. We express $g(\theta)$ as a Fourier series

$$g(\theta) = \tfrac{1}{2}c_0 + \sum_{j=1}^{\infty} \big[\, c_j \cos(j\theta) + d_j \sin(j\theta) \,\big]\,. \tag{28.5.2}$$

  2. The values of c_j and d_j are known.
  3. We express $f(\theta)$ as a Fourier series

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} \big[\, a_j \cos(j\theta) + b_j \sin(j\theta) \,\big]\,. \tag{28.5.3}$$

  4. The values of a_j and b_j are not known and our task is to solve for them.
  5. The first and second derivatives of $f(\theta)$ are given by

$$\begin{aligned} \frac{df}{d\theta} &= \sum_{j=1}^{\infty} j\,\big[\, -a_j \sin(j\theta) + b_j \sin(j\theta) \,\big]\,, \\ \frac{d^2 f}{d\theta} &= -\sum_{j=1}^{\infty} j^2 \big[\, a_j \cos(j\theta) + b_j \sin(j\theta) \,\big]\,. \end{aligned} \tag{28.5.4}$$

6. We substitute the Fourier series for f and g into eq. (28.5.1).
7. For $j = 0$ we obtain
$$a_0 = \frac{c_0}{Q^2}\,. \tag{28.5.5}$$
8. For $j \geq 1$ we obtain
$$\begin{aligned}&\sum_{j=1}^{\infty}(Q^2 - j^2)\big[\,a_j\cos(j\theta) + b_j\sin(j\theta)\,\big] \\ &\quad + R\sum_{j=1}^{\infty} j\,\big[-a_j\sin(j\theta) + b_j\cos(j\theta)\,\big] = \sum_{j=1}^{\infty}\big[\,c_j\cos(j\theta) + d_j\sin(j\theta)\,\big]\,.\end{aligned} \tag{28.5.6}$$
9. Equating terms yields a pair of coupled equations for a_j and b_j:
$$\begin{aligned}(Q^2 - j^2)a_j + Rjb_j &= c_j\,, \\ -Rja_j + (Q^2 - j^2)b_j &= d_j\,.\end{aligned} \tag{28.5.7}$$
10. This can be formulated as a 2×2 matrix equation:
$$\begin{pmatrix} Q^2 - j^2 & Rj \\ -Rj & Q^2 - j^2 \end{pmatrix}\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{pmatrix} c_j \\ d_j \end{pmatrix}\,. \tag{28.5.8}$$
11. The solution for a_j and b_j for $j \geq 1$ is
$$\begin{aligned}\begin{pmatrix} a_j \\ b_j \end{pmatrix} &= \frac{1}{(Q^2 - j^2)^2 + (Rj)^2}\begin{pmatrix} Q^2 - j^2 & -Rj \\ Rj & Q^2 - j^2 \end{pmatrix}\begin{pmatrix} c_j \\ d_j \end{pmatrix} \\ &= \frac{1}{(Q^2 - j^2)^2 + (Rj)^2}\begin{pmatrix} (Q^2 - j^2)c_j - Rjd_j \\ Rjc_j + (Q^2 - j^2)d_j \end{pmatrix}\,.\end{aligned} \tag{28.5.9}$$
12. The solution for $f(\theta)$ is
$$f(\theta) = \frac{c_0}{2Q^2} + \sum_{j=1}^{\infty}\frac{(Q^2 - j^2)c_j - Rjd_j}{(Q^2 - j^2)^2 + (Rj)^2}\cos(j\theta) + \frac{Rjc_j + (Q^2 - j^2)d_j}{(Q^2 - j^2)^2 + (Rj)^2}\sin(j\theta)\,. \tag{28.5.10}$$

- For the function $g(\theta)$, set $c_0 = 0$ and for $j \geq 1$ use
$$c_j = \exp(-0.01\,j)\,, \qquad d_j = \frac{\sin(j\theta_0)}{j\theta_0}\,. \tag{28.5.11}$$
- **Set $\theta_0 = \pi/100$, $Q = 3.41$ and $R = 0.51$.**
- **Set $m = 256$ to sum the Fourier series.**
- **Set `npts = 1000` to compute the values of θ.**
- **Compute the value of `fsum` using the code in Question 28.4.**
- **Plot a graph of `fsum` for `2*npts+1` values of θ as in the code.**
  1. On the horizontal axis, plot the value of θ/π, so the values go from -1 to 1.
  2. On the vertical axis, go from -1 to 1.

28.6 Phase space plots

- However, solving a differential equation to know $f(\theta)$ as a function of θ is frequently *not* the primary object of interest in physics.
- What is of greater interest is a **phase space diagram.**
- That raises the obvious question: ***What is phase space?***
- **Phase space** is a mathematical space of (position, momentum).
  1. Because position and momentum are vectors (3 components each), phase space is really a six-dimensional space.
  2. In fact, if there are N particles, it is a $6N$-dimensional space.
  3. We shall consider only one particle.
  4. The 'position' is $f(\theta)$.
  5. The 'momentum' is $df/d\theta$ (this is an adequate approximation for us).
- Hence what we want is a graph of $(f, df/d\theta)$.
- The system in this question is a **driven undamped oscillator.**
- Hence we set $R = 0$ (the term in R is the damping) and the differential equation is

$$\frac{d^2 f}{d\theta^2} + Q^2 f = g(\theta)\,. \tag{28.6.1}$$

- This simplifies the solution a lot. It is given by $a_0 = c_0/Q^2$ and

$$a_j = \frac{c_j}{Q^2 - j^2}\,, \qquad b_j = \frac{d_j}{Q^2 - j^2} \qquad\qquad (j \ge 1)\,. \tag{28.6.2}$$

- The solution for $f(\theta)$ is

$$\begin{aligned} f(\theta) &= \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty}\big[\, a_j\cos(j\theta) + b_j\sin(j\theta)\,\big] \\ &= \frac{c_0}{2Q^2} + \sum_{j=1}^{\infty} \frac{c_j}{Q^2-j^2}\,\cos(j\theta) + \frac{d_j}{Q^2-j^2}\,\sin(j\theta)\,. \end{aligned} \tag{28.6.3}$$

- We require the expression for $f'(\theta) = df/d\theta$. It is given by

$$f'(\theta) = \frac{df}{d\theta} = \sum_{j=1}^{\infty} j\,\big[\, b_j\cos(j\theta) - a_j\sin(j\theta)\,\big]\,. \tag{28.6.4}$$

- We require an efficient nested sum to compute $df/d\theta$. The nested sum is as follows.

```
const double c = cos(theta);
const double s = sin(theta);
double U = 0;
double V = 0;
for (int j = m; j > 0; --j) {
  double Utmp = j*b[j] + c*U + s*V;
  double Vtmp = -j*a[j] - s*U + c*V;
  U = Utmp;
  V = Vtmp;
}
double fprime = c*U + s*V;
```

28.6.1 Example 1

- **Set $c_j = 0$ for all j and**

$$d_j = \frac{\sin(j\theta_0)}{j\theta_0} \qquad (j \geq 1)\,. \tag{28.6.5}$$

- **Set $\theta_0 = \pi/100$ and $Q = 3.01$.**
- **Set $m = 256$ and `npts=1000`.**
- **Compute the values of $f(\theta)$ and $f' = df/d\theta$ and plot a graph of (f, f').**
- The graph displayed in Fig. 2 was obtained using $\boldsymbol{Q = 4.01}$.
- You should obtain a different graph but it will also be a closed self-intersecting loop.
- The curve is called a **Lissajous figure.**

28.6.2 Example 2

- **Set $c_0 = 0$ and**

$$c_j = \exp(-0.01j)\,, \qquad d_j = \frac{\sin(j\theta_0)}{j\theta_0} \qquad (j \geq 1)\,. \tag{28.6.6}$$

- **Set $\theta_0 = \pi/100$ and $Q = 3.01$.**
- **Set $m = 256$ and `npts=1000`.**
- **Compute the values of $f(\theta)$ and $f' = df/d\theta$ and plot a graph of (f, f').**
- The graph displayed in Fig. 3 was obtained using $\boldsymbol{Q = 4.01}$.
- You should obtain a different graph but it will also be a closed self-intersecting loop.
- The curve is called a **Lissajous figure.**

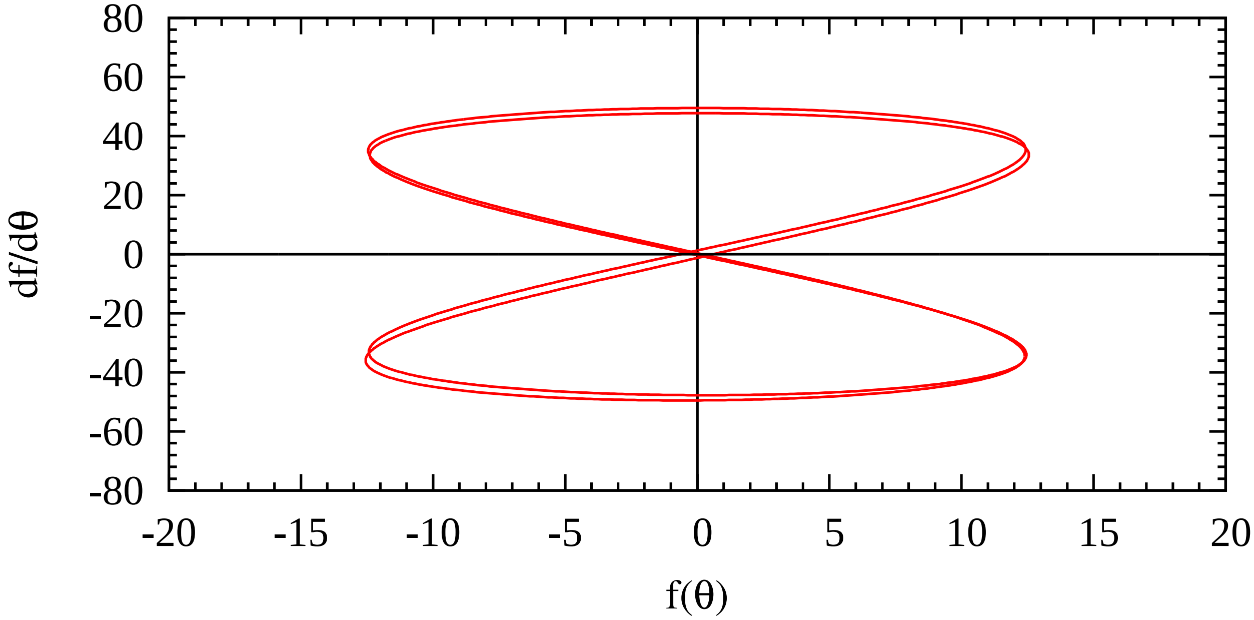

Figure 2: Phase space plot exhibiting a Lissajous figure for example 1 in Question 28.6.

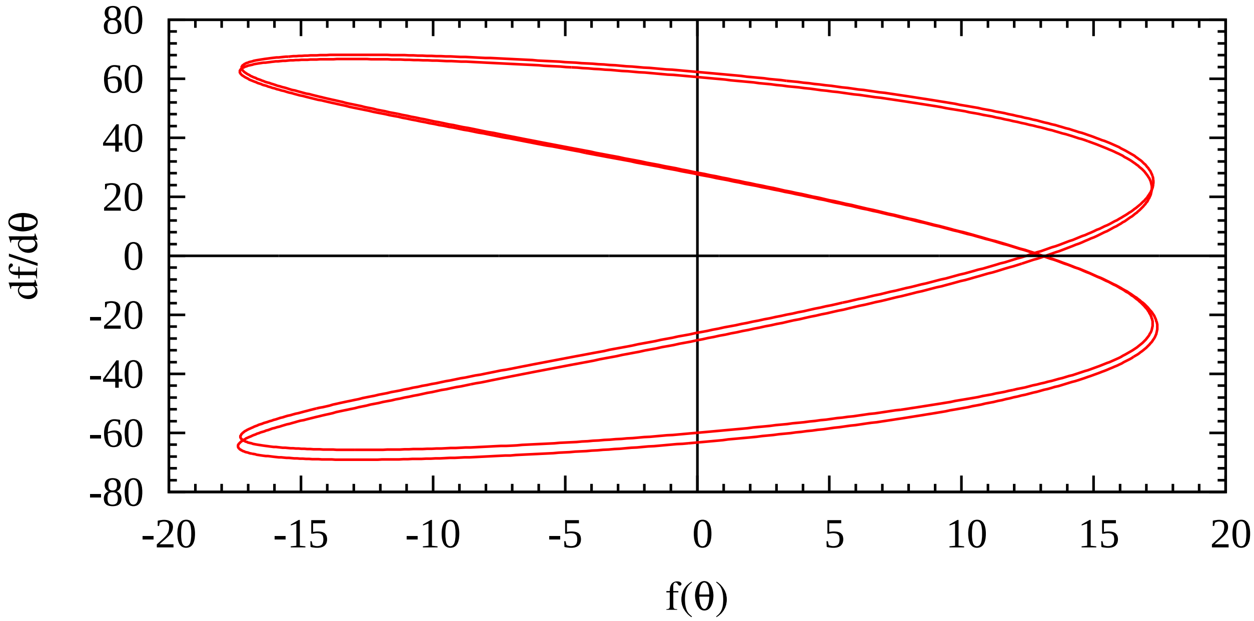

Figure 3: Phase space plot exhibiting a Lissajous figure for example 2 in Question 28.6.

Who has enough sense to realize a question in the final will be to numerically integrate the area enclosed by a Lissajous figure, obtained by solving a differential equation using a Fourier series?