Queens College, CUNY, Department of Computer Science

Computational Finance

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16 Lecture 16

Convexity, delta hedging and gamma trading with options

- In this lecture we study the important concepts of **delta hedging** and **gamma trading**.
- They are important concepts in the trading of options.
- They are related to a concept known as **convexity.**
- **There is no explicit mathematical probability theory in this lecture.**

16.1 Convex function

- We require the concept of a **convex function**.

  1. It is simpler to motivate the notion of a convex function with an illustrative example.
  2. The parabola $f(x) = x^2$ is convex for all values $-\infty < x < \infty$.
  3. A graph of the parabola $f(x) = x^2$ is plotted in Fig. 1. The points marked A and B are joined by a straight line (dotdash). The function (the curve) always lies below the straight line joining A and B, for all $x_A < x < x_B$.

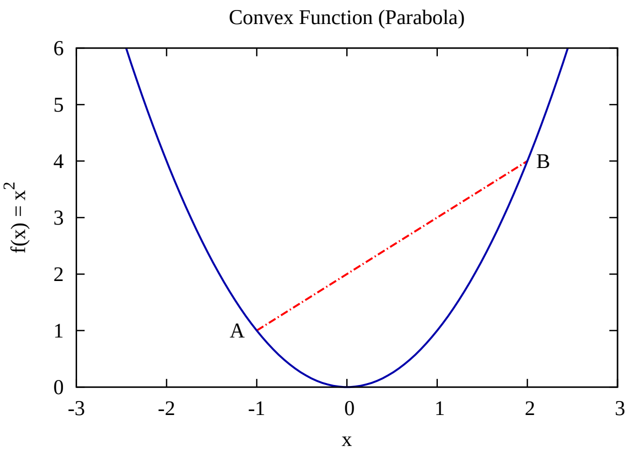

Figure 1: Graph of the parabola $f(x) = x^2$ plotted as a function of x, as an example of a convex function.

- The mathematical definition is as follows. A function $f(x)$ is called a **convex function** in an interval $x_A \le x \le x_B$ if it has the the following property. Choose any two points x_1 and x_2 such that $x_A \le x_1 < x_2 \le x_B$. Then for all $x_1 \le x \le x_2$, the value of $f(x)$ always lies on or below the straight line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

$$f(x) \le \frac{x_2 - x}{x_2 - x_1}\, f(x_1) + \frac{x - x_1}{x_2 - x_1}\, f(x_2) \qquad (x_1 \le x \le x_2)\,. \tag{16.1.1}$$

- If a convex function $f(x)$ is (at least) twice differentiable, then its second derivative is non-negative:

$$f''(x) \ge 0 \qquad (x_1 \le x \le x_2)\,. \tag{16.1.2}$$

- An informal way of expressing the notion of a convex function is that it **curves upwards.**

16.2 Examples of convex functions

- The exponential function $f(x) = e^x$ is convex for all $-\infty < x < \infty$ because $f''(x) = e^x > 0$.

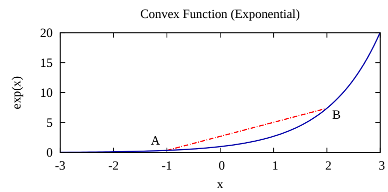

Figure 2: Convex function: graph of the exponential function $f(x) = e^x$ plotted as a function of x.

- The *negative* exponential function $f(x) = e^{-x}$ is also convex for all $-\infty < x < \infty$ because $f''(x) = e^{-x} > 0$. Hence a monotonically decreasing function can be a convex function.

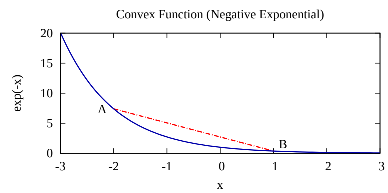

Figure 3: Convex function: graph of the negative exponential function $f(x) = e^x$ plotted as a function of x.

- A convex function need not be differentiable at all points.
  The terminal payoff of a call option is a convex function of the terminal stock price S_T.

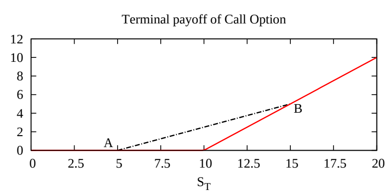

Figure 4: Convex function: terminal payoff of a call option plotted as a function of the stock price S_T.

- The terminal payoff of a put option is also a convex function of the terminal stock price S_T. This is an example of a convex function which is (a) non-increasing with S_T and (b) not differentiable at values of S_T.

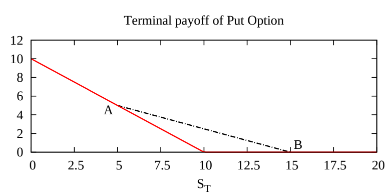

Figure 5: Convex function: terminal payoff of a put option plotted as a function of the stock price S_T.

16.3 Convexity of option fair value

- It is not only the terminal value of an option which is a convex function of the stock price.
- Under reasonable assumptions, which are usually valid not only mathematically but also in the financial markets, the fair value of an option (both put and call, also American and European) is a convex function of the stock price.
- An example fair value of a call option is plotted against the stock price in Fig. 6.
- An example fair value of a put option is plotted against the stock price in Fig. 7.
- The convexity of the option fair value, with respect to the stock price, is of crucial importance to option traders, as we shall see in this lecture.
- It was stated above that, if a convex function is at least twice differentiable, then its second derivative is non-negative.
  1. The second (partial) derivative of the option fair value with respect to the stock price S is the **Gamma** of the option.
  2. The convexity of the option fair value, with respect to the stock price means that Gamma is non-negative for an option.
  3. We shall see that the value of Gamma is important to options traders.

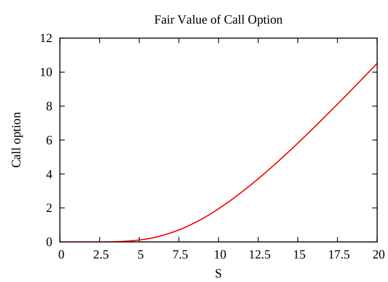

Figure 6: Graph of the fair value of a call option, plotted as a function of the stock price S.

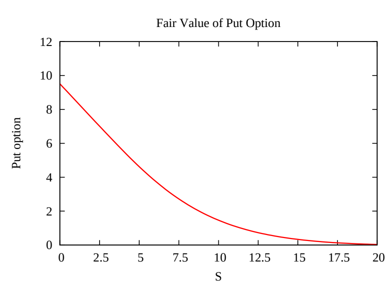

Figure 7: Graph of the fair value of a put option, plotted as a function of the stock price S.

16.4 Convexity and delta hedging

16.4.1 Delta hedging

- Begin with a call option. Suppose we are long one call. It can be either American or European.
- Let us examine the graph of a long position in a call option in more detail.
  A sample graph is shown in Fig. 8.
  The tangent line (dashed) and the points labelled α_1 and α_2 will be explained later.

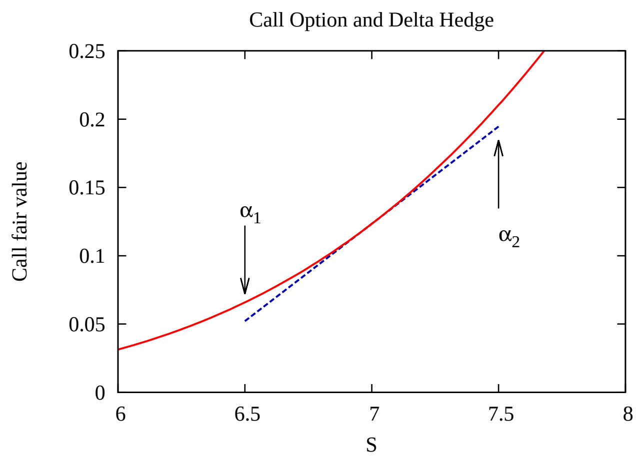

Figure 8: Graph of the fair value of a call option, with tangent line for Delta hedging.

- The slope of the curve of the option fair value in Fig. 8 is $\Delta_c(S)$, by the definition of Delta.
- Suppose the current value of the stock price is $S_0 = 7$. Let the value of the slope at $S_0 = 7$ be $\Delta_0 = \Delta_c(S_0)$. Let us also define $\Gamma_0 = \Gamma_c(S_0)$.
- **Now comes a key idea: We create a second portfolio, with a number Δ_0 of shares.**
  1. Note that this is a **fractional number of shares.**
  2. We therefore have two portfolios: (i) long one call option, and (ii) long Δ_0 shares of stock (there is also some cash, but that does not matter in the analysis below).
  3. For convenience, let us label the two portfolios Π_1 and Π_2, respectively.
  4. Also for convenience, let us suppose the two portfolios have equal value at $S = S_0$.
  5. This is always possible by borrowing/lending a suitable amount of cash, but we shall see later that this step is actually not necessary.
- Let us now analyze what happens to both portfolios Π_1 and Π_2 if the stock price changes by a small random value δS.
  1. For Geometric Brownian Motion, $|\delta S| \propto \sqrt{\delta t}$, and $\sqrt{\delta t} \gg \delta t$ for small δt.
  2. (This requires some knowledge of probability theory from Lecture 14, but you can ignore that and just accept that it is a good approximation to neglect δt.)
  3. Hence if the value of δt is very small, it is a good approximation to neglect any interest rate compounding in the time interval δt and to only consider the effects of $\delta S \neq 0$.

- With the approximation of neglecting δt, the graph of the option fair value will remain approximately the same as in Fig. 8.

  1. Hence for $\delta S \neq 0$, the fair value of the call option will lie along the curve in Fig. 8.
  2. Mathematically, $\Pi_1 = C(S)$ and the change in the value of the portfolio Π_1 is approximately
  $$\delta\Pi_1 = \Delta_0\,\delta S + \tfrac{1}{2}\,\Gamma_0(\delta S)^2 + \dots \tag{16.4.1}$$

- The portfolio Π_2 contains only Δ_0 shares of stock (plus cash), hence its value of Delta is a constant $= \Delta_0$ and its value of Gamma is zero.

  1. Mathematically,
  $$\Pi_2 = S\,\Delta_0 + (\text{cash})\,. \tag{16.4.2}$$
  2. Therefore the fair value of the portfolio Π_2 lies on a straight line, given by the tangent line marked in Fig. 8.
  3. The change in the value of the portfolio Π_2 is therefore
  $$\delta\Pi_2 = \Delta_0\,\delta S\,. \tag{16.4.3}$$

- Now comes the notion of **convexity**. Because the option fair value is a convex function of the stock price, **the curve always lies above the tangent line in Fig. 8.**

- See for example the points marked α_1 and α_2 in Fig. 8.

- Why is the convexity important?

  1. Let us form a new portfolio Π_3 which is long a call option and short Δ_0 shares of stock:
  $$\Pi_3 = C - S\Delta_0\,. \tag{16.4.4}$$
  2. We can ignore any cash in Π_3, for now.
  3. Then for small changes δS, the change in the value of Π_3 is **always non-negative:**
  $$\delta\Pi_3 = \delta\Pi_1 - \delta\Pi_2 \simeq \tfrac{1}{2}\Gamma\,(\delta S)^2 \geq 0\,. \tag{16.4.5}$$
  4. In general the value of Gamma is positive. Then if Gamma > 0, for small changes δS the portfolio Π_3 **always yields a profit, even if the value of ΔS is negative.**

- This sounds magical.

- By going long one call option and shorting Δ_0 shares of stock, an options trader can make a profit *regardless of whether the stock price goes up or down.*

- How is this possible, without violating arbitrage?

- There must be something wrong with the above scenario.

- *Well, not completely . . . read on.*

16.4.2 Caveats: mitigating the magical profit

- First, let us dispose of one detail: it is not necessary for the portfolios Π_1 and Π_2 to have equal value at $S = S_0$.

  1. From the above analysis, all that matters is the *changes* $\delta\Pi_1$ and $\delta\Pi_2$.
  2. The actual values of Π_1 and Π_2 do not matter, and need not be equal.
  3. Then the convexity of the option price tells us that $\delta\Pi_1 \geq \delta\Pi_2$ so the value of $\delta\Pi_3 = \delta\Pi_1 - \delta\Pi_2$ is always ≥ 0.
  4. This yields a profit for the portfolio $\Pi_3 = C - S\Delta_0$, regardless of the sign of δS.

- Now we must realize that there are hidden assumptions in the above options trading strategy.

- First, the value of S will obviously move away from S_0.

  1. Hence the value of Delta (and Gamma) of the option will change.
  2. To maintain the above trading strategy, the shorting of shares in Π_3 must be updated.
  3. Depending on the sign of δS, more shares may have to be sold (if the value of Delta increases), or some shares may have to be bought back (if the value of Delta decreases).
  4. Adjusting the short stock position in the portfolio Π_3 will involve cash inflows or outflows.
  5. Hence we cannot completely ignore the cash in the portfolio Π_3.

- Next, although we ignored the change in time δt above, we cannot ignore δt forever.

- There *will* be interest rate compounding, in particular for the cash inflows or outflows in Π_3.

- The option fair value will not remain on the curve in Fig. 8 forever.

  1. The curve will change as time passes, and this cannot be neglected forever.
  2. The change in the shape of the curve will also change the value of the Delta of the option, which will in turn affect the shorting of stock in Π_3.
  3. It is not only changes in the value of S which affect the value of the Delta of the option.

- If we combine all of the above effects, we arrive at the following conclusion:

  > If the stock price really obeys Geometric Brownian Motion, and the volatility does not change over the life of the option (changes in the interest rate are less important, but can also exist) the overall effect of all the above caveats is that the total profit produced by trading the portfolio Π_3 is **zero.**

- **There is no violation of arbitrage.**

- ***But . . . what if the volatility is not constant?***

- *What if the stock price does* **not** *obey Geometric Brownian Motion?*

- This brings us to the important concept of **gamma trading.**

16.5 Gamma trading

- It was derived in Sec. 16.4 that the portfolio Π_3 yields a profit regardless of whether the stock price goes up or down, for a small change in the stock price δS and a short time interval δt.
- We are long one call option and short Δ_0 shares of the stock.
- Note that the option need not be European. It can be an American option.
- We must update the short stock position, in principle continuously.
- The magnitude of the profit yielded by the change in value $\delta\Pi_3$ is proportional to the value of the Gamma of the option.
- Hence the above trading strategy is called **gamma trading.**
- Gamma trading is widely employed by options traders.
- It was also pointed out in Sec. 16.4 that there are caveats to the gamma trading strategy.
- If indeed the stock price obeys Geometric Brownian Motion exactly and the volatility does not change over the life of the option (and other caveats in listed in Sec. 16.4) then gamma trading does not yield a net profit.
- However, stock prices in real life do *not* obey Geometric Brownian Motion exactly.
- **However, what is more important is that the volatility is *not a constant.***
- Under these circumstances, gamma trading ***can*** yield a profit.
- Gamma trading ***can*** yield a profit, but is not a *guaranteed profit.*
- There is no violation of arbitrage.
- The rational option pricing inequalities in Lecture 7 are not violated.
- The options traders are really **trading volatility.**
- One can buy or sell stocks and futures and options in the financial markets, but one cannot directly buy or sell volatility.
- Gamma trading is a mechanism for the options traders to buy and sell volatility.
- To understand gamma trading in more detail, we must learn about **implied volatility.**
- **Implied volatility** is a very important concept in options trading.
- Implied volatility will be the subject of a separate lecture.

16.6 Gamma trading with put options

- Gamma trading can also be performed using put options.
- The analogous graph to Fig. 8 is plotted in Fig. 9, for a put option.

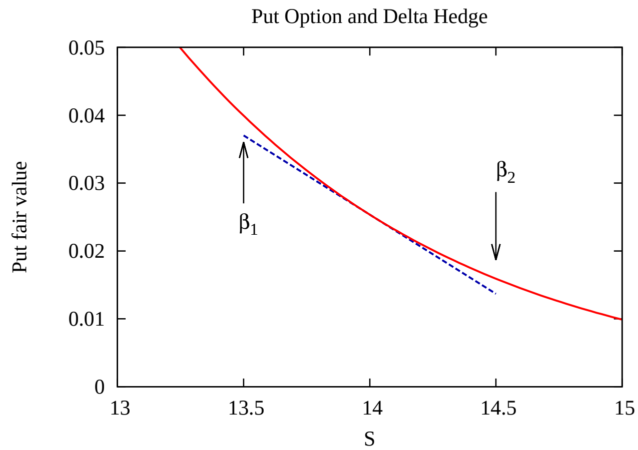

Figure 9: Graph of the fair value of a put option, with tangent line for Delta hedging.

- The graph in Fig. 9 should be self-explanatory. Because of the convexity of the option price, the curve always lies above the tangent line.
- In the case of a put option, because its Delta is negative, the gamma trading portfolio, say Π_4, is formed by long one put option and **long** $|\Delta_0|$ shares of the stock (note the absolute value):

$$\Pi_4 = P + S\,|\Delta_0|\,. \tag{16.6.1}$$

- Then for a small change δS in the stock price, the change in the value of Π_4 is

$$\begin{aligned}\delta\Pi_4 &\simeq \Delta_0\,\delta S + \tfrac{1}{2}\Gamma_0\,(\delta S)^2 + |\Delta_0|\,\delta S \\ &= \tfrac{1}{2}\Gamma_0(\delta S)^2\,.\end{aligned} \tag{16.6.2}$$

- Hence $\delta\Pi_4 \geq 0$ because Gamma is non-negative (and usually positive) for a put option.
- All of the caveats about gamma trading with call options also apply for put options.
- To repeat, if the stock price really obeys Geometric Brownian Motion, and the volatility does not change over the life of the option (changes in the interest rate are less important, but can also exist) then the net profit from gamma trading (with puts or calls) is zero.

16.7 Gamma trading using futures

- The delta hedging in gamma trading can perfectly well be carried out using futures contracts.
- Indeed, for options on stock indices, the index is not a tradeable underlying asset hence one *must* hedge the option using futures.
- Note that the Delta of a futures contract is not unity.
- However, the key fact is that the Gamma of a futures contract is zero.
- Suppose the stock index has a dividend yield q. Then the fair values of a futures contract with expiration T is (using the usual notation exployed in these lectures)

$$F = S\,e^{(r-q)(T-t)}\,. \tag{16.7.1}$$

- Hence the value of Delta and Gamma of the futures are

$$\Delta_F = e^{(r-q)(T-t)}\,, \qquad \Gamma_F = 0\,. \tag{16.7.2}$$

- For a call, the gamma trading portfolio using futures to hedge the Delta of the call is

$$\Pi_5 = C - \frac{\Delta_0}{\Delta_F}\,F\,. \tag{16.7.3}$$

- For a put, the gamma trading portfolio using futures to hedge the Delta of the put is

$$\Pi_6 = P + \frac{|\Delta_0|}{\Delta_F}\,F\,. \tag{16.7.4}$$

- As the stock index value and the time change, both the values of Δ_0 and Δ_F will change.
- *The above analysis assumed the expiration time of the futures and the option are the same.*
- **The expiration time of the futures and the option need not be the same.**
- Sometimes it is not possible to find a futures with the same expiration time as an option.

16.8 Gamma trading with options that never go in the money

- It is perfectly possible to make a profit by performing gamma trading with an option that never goes in the money during its entire lifetime.
- *Read Secs. 16.4, 16.5 and 16.6 very carefully.*
- **Nowhere in Secs. 16.4, 16.5 and 16.6 was it stated that the call or the put option must be in the money.**
- Gamma trading only requires that the Gamma of the option should be positive (i.e. not zero).
- An out of the money option has a positive value of Gamma.
- An option can remain out of the money for its entire lifetime, but because the value of its Gammma is always positive, an options trader can make a profit by gamma trading the option. As a matter of fact, this is not at all an unusual situation.

16.9 Gamma trading and option expiration

- What happens to a gamma trading portfolio when the option expires?
- Let us begin with a call option.
  1. If the option expires in the money, its Delta will equal one. Hence the short stock position will have been updated to short one share of stock.
  2. We exercise the call, pay the strike price and receive one share of stock and close out of short stock position. We use money saved in the bank from our short stock sales (and some of our profit from gamma trading) to pay the strike price.
  3. In the ideal Geometric Brownian Motion model, our final cash and stock both equal zero: there is no profit.
  4. If the option expires out of the money, its Delta will equal zero. Hence the short stock position will have been updated to short zero shares of stock.
  5. The option expires worthless and we have no short stock position. Hence nothing to do.
  6. We use our profit from gamma trading plus cash saved in the bank to pay to buy back the stock, to reduce our short stock position to zero.
  7. In the ideal Geometric Brownian Motion model, our final cash and stock both equal zero: there is no profit.
  8. The option expires exactly at the money ($S_T = K$). This is complicated and will be discussed below.
- Next consider a put option.
  1. If the option expires in the money, its Delta will equal negative one. Hence the long stock position will have been updated to one share of stock.
  2. We exercise the put, deliver the stock in our long stock position to close out the put, and receive cash equal to the strike price. We use that money (and some of our profit from gamma trading) to repay the money borrowed to buy the stock to establish our long stock position.
  3. In the ideal Geometric Brownian Motion model, our final cash and stock both equal zero: there is no profit.
  4. If the option expires out of the money, its Delta will equal zero. Hence the long stock position will have been updated to zero shares of stock.
  5. The option expires worthless and we have no stock. Hence nothing to do.
  6. We use our profit from gamma trading and cash from selling off stock (as we reduce our long stock position to zero) to repay the money borrowed to buy shares purchased at earlier times.
  7. In the ideal Geometric Brownian Motion model, our final cash and stock both equal zero: there is no profit.
  8. The option expires exactly at the money ($S_T = K$). This is complicated and will be discussed below.

- Suppose the option (put or call) expires exactly at the money, or anyway the final stock price is very close to the option strike price.

  1. This is a complicated scenario for gamma trading.
  2. As the option expiration approaches, if the option is even slightly in the money, its Delta will equal almost one, whereas if the option is even slightly out of the money, its Delta will equal almost zero.
  3. Hence even small fluctuations in the stock price (close to the expiration time), will cause large changes in the stock position (long or short) in the gamma trading portfolio.
  4. An options trader can be caught with a large (unwanted) long or short stock position at the expiration time.
  5. In the ideal Geometric Brownian Motion model, the above scenario does not matter. By hypothesis, as expiration nears, the price of the stock is close to the strike price. Hence the value of any money borrowed or saved in a bank will approximately match the value of the stock price in the corresponding long or short stock position. Because the time to expiration is by definition small, inteest rate compounding is negligible. In the ideal theory, because all trades can be executed instantaneously, our final cash and stock will both equal zero: there is no profit.
  6. In real life, this is not so.
  7. Trading in the stock markets can become chaotic, by options traders performing gamma trading, as listed options approach their expiration times.