Queens College, CUNY, Department of Computer Science

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17 Lecture 17

Option pricing: binomial model

- In this lecture we study the **binomial model** to calculate the fair values of options.
- In fact, the binomial model can be used for any derivative on a stock, not just options.
- The binomial model can be employed to calculate the fair values of **American options.**
- To date, we have no algorithm to calculate the fair values of American options.
- The binomial model is not necessarily the most accurate option pricing model.
- However, it is simple to understand, easy to implement and computationally fast.
- For these reaons, the binomial model is important and is popular in practice.
- The binomial model assumes the stock price obeys Geometric Brownian Motion.
- **However, there is no explicit mathematical probability theory in this lecture.**

17.1 Binomial model: constructing the tree

17.1.1 One timestep

- Let us make a very simple model of the stock price movements.
- Let us discretize the time to expiration $T - t_0$ into n equal steps of size

$$\Delta t = \frac{T - t_0}{n} \,. \tag{17.1.1}$$

- At each step, we approximate that the stock price can go to only one of two future values at the next step.
- In more detail, suppose the stock price at the initial timestep $i = 1$ is S.
- At the next timestep $i = 2$, we say the stock price either goes up by a factor u to Su or down by a factor d to Sd.
- A sketch is shown in Fig. 1

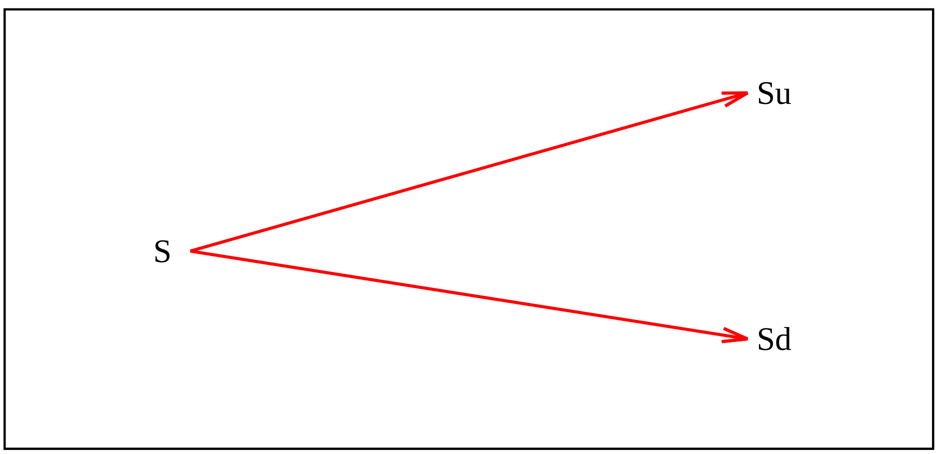

Figure 1: Sketch of binomial stock price movements for one timestep.

- The binomial model assumes the stock price obeys Geometric Brownian Motion.
- For this reason we express the stock price movements as ratios Su and Sd, as opposed to equal arithmetic steps $S \pm \delta S$.

17.1.2 Two timesteps

- After two timesteps, the final stock price levels are Su^2 (up-up), Sud (up-down), Sdu (down-up) and Sd^2 (down-down).
- Note that $Sud = Sdu$ so the up-down and down-up paths both lead to the same final node.
- We say that the paths **recombine.**
- Hence there are only three (not four) distinct "nodes" at the timestep $i + 2$.
- It is conventional to set $d = 1/u$ so $ud = 1$.
- Then $Sud = Sdu = S$ and the value equals the original stock price level S.
- A sketch is shown in Fig. 2

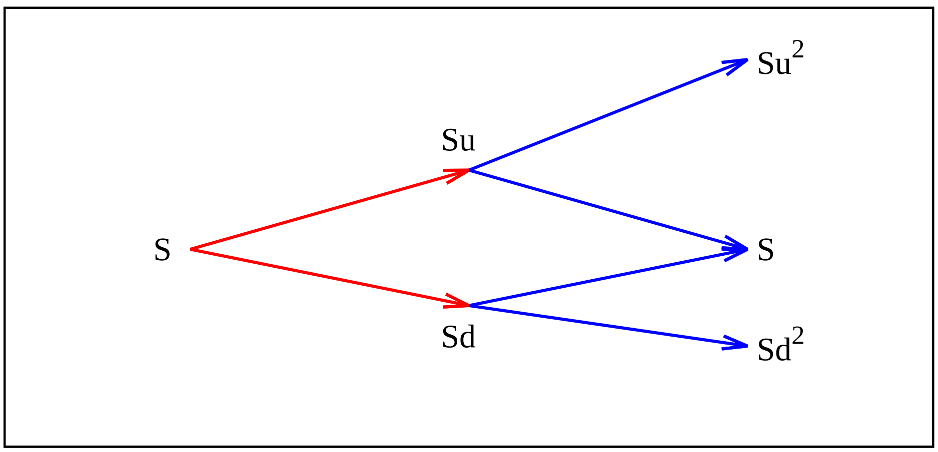

Figure 2: Sketch of binomial stock price movements for two timesteps.

17.1.3 Multiple timesteps

- We obviously generate more "nodes" at each timestep, creating a **tree** of nodes.
- A sketch is shown in Fig. 3 for three timesteps.

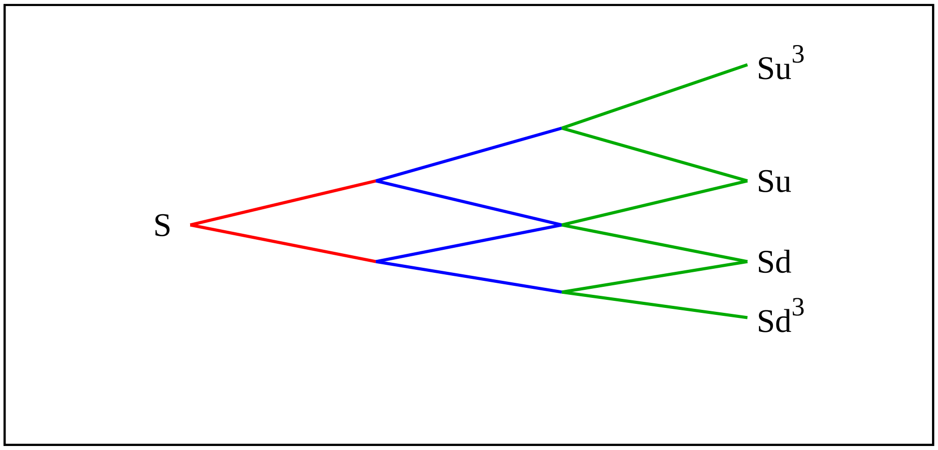

Figure 3: Sketch of binomial stock price movements for three timesteps.

- We call it a **binomial tree** because the stock price can go to only one of two values at each timestep.
- Also, because the paths recombine, there are totally i nodes at the timestep i.
- The recombination property of the binomial tree is very important.
  1. The recombination property means that a tree with n timesteps has only $\frac{1}{2}n(n+1)$ nodes.
  2. Hence the computation time for a recombining tree is of **polynomial complexity** $O(n^2)$**.**
  3. If the tree did not recombine, then a tree with n timesteps would have $2^n - 1$ nodes.
  4. **The computation time of a non-recombining tree would have exponential complexity, which is very bad.**

17.2 Binomial model: parameter values

- Let us say the probability of taking an up step is p and the probability of taking a down step is $q = 1 - p$.
- Let the risk-free interest rate be r (a constant).
- Let the volatility of the stock be σ (a constant).
- Suppose the stock pays continuous dividends at a rate q.
- Then the values of u, d, p and q are given as follows:

$$u = e^{\sigma\sqrt{\Delta t}}\,, \tag{17.2.1a}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1/u\,, \tag{17.2.1b}$$

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}\,, \tag{17.2.1c}$$

$$q = \frac{u - e^{(r-q)\Delta t}}{u - d}\,. \tag{17.2.1d}$$

- The above expressions will be derived later, in Sec. 17.5.

17.3 Binomial model: valuation

- We value an option by working ***backwards*** from the final timestep to the initial timestep.

- This is because we know the value of the option at the expiration time, at all the stock price levels (nodes) in the binomial tree. We use that information to systematically work backwards to determine the option's fair value at the nodes at earlier times.

- Let us begin with a European call option.

- Consider a node at the timestep i and let the stock price at that node be S. Let the option value be C. We wish to calculate the value of C.

- The node at the timestep i is connected to two nodes at the timestep $i+1$, with the values Su and Sd, respectively.

- Let the option fair values at those nodes be C_u and C_d, respectively.

- A sketch is shown in Fig. 4.

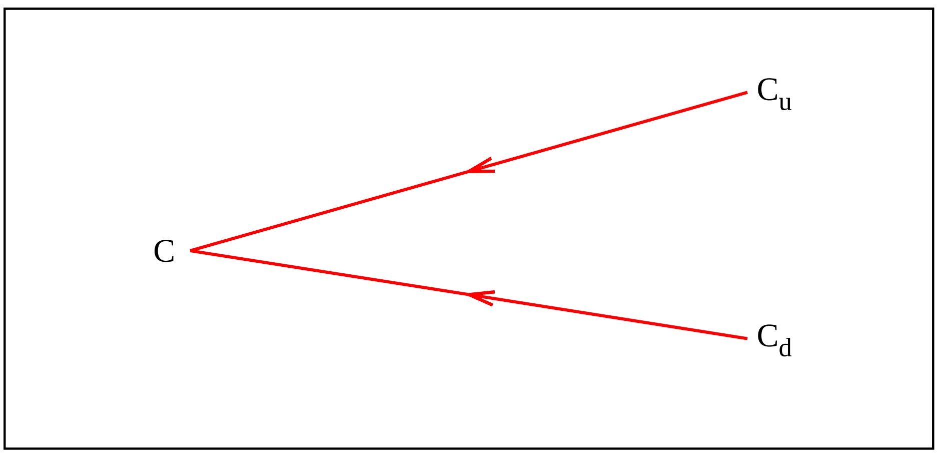

Figure 4: Sketch of binomial tree and option fair values at nodes at timesteps i and $i+1$.

- By hypothesis, we have worked backwards through the binomial tree so we know the values of C_u and C_d. We shall use that information to calculate the value of C.

- Notice the arrows in Fig. 4 point backwards: we are working backwards through the tree.

- We calculate the value of the option fair value C at the timestep i as follows:

$$C = e^{-r\Delta t} \left(pC_u + qC_d\right). \tag{17.3.1}$$

  1. Essentially, eq. (17.3.1) says that the value of C is given by calculating the expectation value (probability weighted average) of C_u and C_d, and discounting the result at the risk-free rate r.
  2. **The formal statement is that value of the option fair value at the timestep i is the discounted expected option fair value at the timestep $i+1$.**

- Hence this is the valuation procedure:

  1. We compute the option value at all the nodes at the final timestep (expiration time $t = T$), say $i = n$.
  2. We employ eq. (17.3.1) to calculate the option fair values at all the nodes at the timestep $i = n - 1$.
  3. We repeat the above procedure, looping backwards through the binomial tree to calculate the option fair values at all the nodes at the earlier timesteps $i = n-2, n-3$, etc.
  4. We loop backwards through the binomial tree until we obtain the option fair value at the initial node, say $i = 0$, where $t = t_0$ and the stock price is S_0.

- The exact same procedure also works for a European put option. The only difference is the value of the option at the terminal timestep (expiration). The formula corresponding to eq. (17.3.1) is, with an obvious notation

$$P = e^{-r\Delta t} \left(pP_u + qP_d\right). \tag{17.3.2}$$

- The same procedure also works for **American** options, with one important modification, because of the possibility of early exercise.

  1. For an American option, **we compare the value from eq. (17.3.1) to the option's intrinsic value.**
  2. If the value from eq. (17.3.1) is less than the option's intrinsic value, we set the option fair value at that node to the intrinsic value instead.
  3. In other words, it is optimal to exercise the American option early at that value of the stock price and time.
  4. This is because the value of an American option cannot be less than its intrinsic value. Else we exercise the option immediately and make an arbitrage profit.
  5. Hence the formula for the value of an American call option C is modified as follows:

$$C_{\text{American}} = \max\Big\{ e^{-r\Delta t} \left(pC_u + qC_d\right), \max(S-K, 0) \Big\}. \tag{17.3.3}$$

  6. For an American put option, the corresponding formula is as follows:

$$P_{\text{American}} = \max\Big\{ e^{-r\Delta t} \left(pP_u + qP_d\right), \max(K-S, 0) \Big\}. \tag{17.3.4}$$

17.4 Comments on the binomial model

- The binomial model is simple to understand and easy to code (implement in software).
- It runs quickly (computation time) and is popular and is widely used in the financial industry.
- Nevertheless, the binomial model does have various limitations.
- There is no flexilibity in the locations of the nodes in the binomial tree.
- For example, we might wish to have a closer spacing of nodes at values of the stock price close to the option strike. The binomial model does not allow this. If we wish to have closely spaced nodes near the strike price, we must increase the value of n, which decreases the spacing of the nodes throughout the tree.
- Let us consider an option with one year to expiration. Suppose $S_0 = K = 100$. Let the volatility be 50%, i.e. $\sigma = 0.5$. Suppose we employ $n = 1000$, which is a value used by academics in options research.

  1. Then the highest and lowest stock price levels in the binomial tree are

$$S_0(e^{\sigma\sqrt{\Delta t}})^n = 100\left(e^{0.5\sqrt{0.001}}\right)^{1000} \simeq 7.4 \times 10^8\,. \tag{17.4.1a}$$

$$S_0(e^{-\sigma\sqrt{\Delta t}})^n = 100\left(e^{-0.5\sqrt{0.001}}\right)^{1000} \simeq 1.4 \times 10^{-5}\,. \tag{17.4.1b}$$

  2. These are absurdly large and small values, but there is nothing we can do about this.
  3. We cannot arbitrarily truncate the binomial tree without upsetting the option valuation.

17.5 Binomial model: derivation of parameters

- This section presents the derivation of the parameter values in Sec. 17.2.
- This section contains a small amount of mathematical probability theory.
- Let us say the probability of taking an up step is p and the probability of taking a down step is $q = 1 - p$.
- We determine the values of p and q by equating the expressions for mean and variance of S after one timestep, calculated using the binomial model, to the known expressions in terms of the risk-free interest rate and volatility.
- We assume that the stock pays continuous dividends at a rate q.
- Let the risk-free interest rate be r (a constant).
  1. Then if the stock price is S at the timestep i, after one timestep the expectation value of the stock price is
  $$\mathbb{E}[S]_{i+1} = Se^{(r-q)\Delta t}\,. \tag{17.5.1}$$
  2. The above expression does **not** require Geometric Brownian Motion.
  3. Using the binomial tree, the expectation value of S at the timestep $i+1$ is
  $$\mathbb{E}[S]_{i+1} = pSu + qSd = S(pu + qd)\,. \tag{17.5.2}$$
  4. Equating the two expressions for the expectations yields
  $$pu + qd = e^{(r-q)\Delta t}\,. \tag{17.5.3}$$
  5. This can be solved for p and q by using the additional relation $p + q = 1$.
  6. The answer is
  $$p = \frac{e^{(r-q)\Delta t} - d}{u - d}\,, \qquad q = \frac{u - e^{(r-q)\Delta t}}{u - d}\,. \tag{17.5.4}$$
- However, this is not a complete solution because we have not specified the values of u and d.
- There are of course hidden assumptions in the above derivation. We require both p and q to be positive (or at least zero). Hence to obtain meaningful values for the probabilities we must have
$$d \le e^{(r-q)\Delta t} \le u\,. \tag{17.5.5}$$
- **Technical mathematical note about eq. (17.5.1).**
  1. It was stated in Lecture 14 that the growth of the stock price random variable dS^r is
  $$dS^r = \mu S^r\,dt + \sigma S^r\,dW_t\,. \tag{17.5.6}$$
  2. However, eq. (17.5.1) makes no reference to μ and contains the risk-free rate r instead.
  3. Technically, we have performed a mathematical transformation, as in the derivation of the Black-Scholes equation.
  4. The values of p and q are the called **risk-neutral probabilities.**

- We determine the values of u and d by fitting to the variance of S in one timestep, which is
- Let the volatility of the stock be σ (a constant).

1. Then if the stock price is S at the timestep i, after one timestep the variance of the stock price is
$$\mathrm{Var}(S)_{i+1} = \sigma^2 S^2 \Delta t \,. \tag{17.5.7}$$
2. **The above expression is derived using Geometric Brownian Motion.**
3. Using the binomial tree, the variance of S at the timestep $i+1$ is
$$\mathrm{Var}(S)_{i+1} = pS^2u^2 + qS^2d^2 - S^2e^{2(r-q)\Delta t} \tag{17.5.8}$$
4. Equating the two expressions for the variance (and retaining terms to $O(\Delta t)$ only) yields
$$pu^2 + qd^2 - (1 + 2(r-q)\Delta t) = \sigma^2 \Delta t \,. \tag{17.5.9}$$
5. The solution is
$$u = e^{\sigma\sqrt{\Delta t}} \,, \qquad d = e^{-\sigma\sqrt{\Delta t}} \,. \tag{17.5.10}$$

- Hence the probabilities p and q are given by eq. (17.5.4) where the values of u and d are given by eq. (17.5.10). Writing it out explicitly, the expressions are
$$p = \frac{e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \,, \qquad q = \frac{e^{\sigma\sqrt{\Delta t}} - e^{(r-q)\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \,. \tag{17.5.11}$$
- The inequalities in eq. (17.5.5) then yield
$$e^{-\sigma\sqrt{\Delta t}} \le e^{(r-q)\Delta t} \le e^{\sigma\sqrt{\Delta t}} \,. \tag{17.5.12}$$
- The inequalities in eq. (17.5.12) are usually satisfied in practice, but can fail is the value of σ is very small, or if the value of Δt is not small enough.