

Queens College, CUNY, Department of Computer Science
Numerical Methods
CSCI 361 / 761
Spring 2018
Instructor: Dr. Sateesh Mane

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due Friday, May 11, 2018, 11.59 pm

15 Homework lecture 15

- As experience has demonstrated, if you do not understand the above expressions/questions, **THEN ASK**.
- If you do not understand the words/sentences in the lectures, **THEN ASK**.
- Send me an email, explain what you do not understand.
- Do not just keep quiet and produce nonsense in exams.

15.1 Euler integration 1: explicit

15.1.1 Forward integration

- You are given the following differential equation (c is a positive constant)

$$\frac{dy}{dx} = x - cy. \quad (15.1.1)$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 1$.
- You are given that the exact solution $y^{\text{ex}}(x)$ is as follows.

$$y^{\text{ex}}(x) = \left(1 + \frac{1}{c^2}\right) e^{-cx} + \frac{x}{c} - \frac{1}{c^2}. \quad (15.1.2)$$

- You do NOT have to derive the above solution (but you can if you want to).**
- The integration stepsize is h (a constant), hence $x_i = ih$.
- Use the explicit Euler method to derive the following.**

$$y_{i+1} = y_i + h(x_i - cy_i) = y_i(1 - hc) + hx_i. \quad (15.1.3)$$

- Set $h = 1/n$.
- Show that $x_n = 1$ after n steps.**
- Set $c = 2$.
- Compute the value of the exact solution $y^{\text{ex}}(x)$ for $x = 1$ and $c = 2$.**
- Compute the value of the numerical solution y_n at $x_n = 1$ and fill the following table of values.**
 - Compute the values of $y^{\text{ex}}(1)$ and y_n to 5 decimal places.**
 - In the last column, calculate the value of $n(y^{\text{ex}}(1) - y_n)$ to 2 decimal places.**

n	$y^{\text{ex}}(1)$	y_n	$y^{\text{ex}}(1) - y_n$	$n(y^{\text{ex}}(1) - y_n)$
10	5 d.p.	5 d.p.	5 d.p.	2 d.p.
100	5 d.p.	5 d.p.	5 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.1.2 Reverse integration

- We now reverse the direction of integration and integrate backwards to $x_0 = 0$.
- The differential equation is therefore

$$\frac{dy}{dx} = -(x - cy). \quad (15.1.4)$$

- The integration stepsize is still $h = 1/n$.
- Let the steps be indexed by $i = n + 1, \dots, 2n$.
- You are given that now

$$x_i = x_n - (i - n)h = 2 - ih \quad (i = n + 1, \dots, 2n). \quad (15.1.5)$$

- The initial condition is $y = y_n$ (the computed numerical solution) at $i = n$.
- Use the explicit Euler method to derive the following.

$$\begin{aligned} y_{i+1} &= y_i - h(x_i - cy_i) \\ &= y_i(1 + hc) - hx_i. \end{aligned} \quad (15.1.6)$$

- Show that $x_{2n} = 0$ after $2n$ steps.
- The exact solution will return to the initial value $y_0 = 1$ but the numerical solution will not.
- Compute the value of the numerical solution y_{2n} at $x_{2n} = 0$ and fill the following table of values.
 1. Compute the value of y_{2n} to 5 decimal places.
 2. In the last column, calculate the value of $n(1 - y_{2n})$ to 2 decimal places.

$2n$	$y^{\text{ex}}(0)$	y_{2n}	$1 - y_{2n}$	$n(1 - y_{2n})$
20	1	5 d.p.	5 d.p.	2 d.p.
200	1	5 d.p.	5 d.p.	2 d.p.
2000	1	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.1.3 Indexing for forward and reverse integration

- There seems to be confusion about the notation and indexing, especially for the reverse integration.
- Let me use $n = 10$ as an example. Then $h = 1/n = 0.1$.
- Then the value of x_i is, for $i = 0, \dots, n, \dots, 2n$:

$$\begin{aligned}
 x_0 &= 0, \\
 x_1 &= 0.1, \\
 x_2 &= 0.2, \\
 &\vdots \\
 x_9 &= 0.9, \\
 x_{10} &= 1.0, \\
 \mathbf{x_{11}} &= \mathbf{0.9}, \\
 \mathbf{x_{12}} &= \mathbf{0.8}, \\
 &\vdots \\
 \mathbf{x_{19}} &= \mathbf{0.1}, \\
 \mathbf{x_{20}} &= \mathbf{0}.
 \end{aligned} \tag{15.1.7}$$

- For the *forward integration*, the loop goes from $i = 0$ through $i = 9$ (ten equations)

$$i = 0 : \quad y_1 = y_0(1 - hc) + hx_0 = 1(1 - hc) + h \times 0, \tag{15.1.8a}$$

$$i = 1 : \quad y_2 = y_1(1 - hc) + hx_1 = y_1(1 - hc) + h \times 0.1, \tag{15.1.8b}$$

$$i = 2 : \quad y_3 = y_2(1 - hc) + hx_2 = y_2(1 - hc) + h \times 0.2, \tag{15.1.8c}$$

$$\vdots \tag{15.1.8d}$$

$$\mathbf{i = 9 :} \quad \mathbf{y_n = y_{10} = y_9(1 - hc) + hx_9 = y_9(1 - hc) + h \times 0.9.} \tag{15.1.8e}$$

- For the *reverse integration*, the loop goes from $i = 10$ through $i = 19$ (ten equations) We simply change sign $h \rightarrow -h$, so $h = -0.1$.

$$\mathbf{i = 10 :} \quad y_{11} = y_{10}(1 + hc) - hx_{10} = \mathbf{y_{10}}(1 + hc) - h \times \mathbf{1.0}, \tag{15.1.9a}$$

$$\mathbf{i = 11 :} \quad y_{12} = y_{11}(1 + hc) - hx_{11} = \mathbf{y_{11}}(1 + hc) - h \times \mathbf{0.9}, \tag{15.1.9b}$$

$$\mathbf{i = 12 :} \quad y_{13} = y_{12}(1 + hc) - hx_{12} = \mathbf{y_{12}}(1 + hc) - h \times \mathbf{0.8}, \tag{15.1.9c}$$

$$\vdots \tag{15.1.9d}$$

$$\mathbf{i = 18 :} \quad y_{19} = y_{18}(1 + hc) - hx_{18} = \mathbf{y_{18}}(1 + hc) - h \times \mathbf{0.2}, \tag{15.1.9e}$$

$$\mathbf{i = 19 :} \quad \mathbf{y_{2n} = y_{20} = y_{19}(1 + hc) - hx_{19} = \mathbf{y_{19}}(1 + hc) - h \times \mathbf{0.1.} \tag{15.1.9f}$$

- Note the indexing and the value of x_i for the reverse integration.

15.2 Euler integration 2: implicit

15.2.1 Forward integration

- The differential equation is given by eq. (15.1.1)

$$\frac{dy}{dx} = x - cy. \quad (15.2.1)$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 1$.
- The exact solution $y^{\text{ex}}(x)$ is given by eq. (15.1.2).
- The integration stepsize is h (a constant), hence $x_i = ih$.
- **Use the implicit Euler method to derive the following.**

$$y_{i+1} = y_i + h(x_{i+1} - cy_{i+1}). \quad (15.2.2)$$

- **Process the above equation to derive the following.**

$$y_{i+1} = \frac{y_i + hx_{i+1}}{1 + hc}. \quad (15.2.3)$$

1. **This is an important lesson to learn.**
2. **The denominator $1 + hc$ divides the term in x_{i+1} not just y_i .**

- Set $h = 1/n$.
- **Show that $x_n = 1$ after n steps.**
- Set $c = 2$.
- Recall the value of the exact solution $y^{\text{ex}}(x)$ for $x = 1$ and $c = 2$.
- **Compute the value of the numerical solution y_n and fill the following table of values.**
 1. **Compute the values of $y^{\text{ex}}(1)$ and y_n to 5 decimal places.**
 2. **In the last column, calculate the value of $n(y^{\text{ex}}(1) - y_n)$ to 2 decimal places.**

n	$y^{\text{ex}}(1)$	y_n	$y^{\text{ex}}(1) - y_n$	$n(y^{\text{ex}}(1) - y_n)$
10	5 d.p.	5 d.p.	5 d.p.	2 d.p.
100	5 d.p.	5 d.p.	5 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.2.2 Reverse integration

- We now reverse the direction of integration and integrate backwards to $x_0 = 0$.
- The differential equation is given by eq. (15.1.4)

$$\frac{dy}{dx} = -(x - cy). \quad (15.2.4)$$

- Let the steps be indexed by $i = n + 1, \dots, 2n$.
- The integration stepsize is still $h = 1/n$.
- You are given that now $x_i = 1 - (i - n)h$.
- **Use the implicit Euler method to derive the following.**

$$y_{i+1} = y_i - h(x_{i+1} - cy_{i+1}) \quad (15.2.5)$$

- **Process the above equation to derive the following.**

$$y_{i+1} = \frac{y_i - hx_{i+1}}{1 - hc}. \quad (15.2.6)$$

- **Show that $x_{2n} = 0$ after $2n$ steps.**
- The exact solution will return to the initial value $y_0 = 1$ but the numerical solution will not.
- **Compute the value of the numerical solution y_{2n} and fill the following table of values.**
 1. **Compute the value of y_{2n} to 5 decimal places.**
 2. **In the last column, calculate the value of $n(1 - y_{2n})$ to 2 decimal places.**

$2n$	$y^{\text{ex}}(0)$	y_{2n}	$1 - y_{2n}$	$n(1 - y_{2n})$
20	1	5 d.p.	5 d.p.	2 d.p.
200	1	5 d.p.	5 d.p.	2 d.p.
2000	1	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.2.3 Indexing for forward and reverse integration

- Let me use $n = 10$ as an example. Then $h = 1/n = 0.1$.
- The value of x_i is the same as previously listed.
- For the *forward integration*, the loop goes from $i = 0$ through $i = 9$ (ten equations)

$$i = 0 : \quad y_1 = \frac{y_0 + h\mathbf{x}_1}{1 + hc} = \frac{1 + h \times \mathbf{0.1}}{1 + hc}, \quad (15.2.7a)$$

$$i = 1 : \quad y_2 = \frac{y_1 + h\mathbf{x}_2}{1 + hc} = \frac{y_1 + h \times \mathbf{0.2}}{1 + hc}, \quad (15.2.7b)$$

$$i = 2 : \quad y_3 = \frac{y_2 + h\mathbf{x}_3}{1 + hc} = \frac{y_2 + h \times \mathbf{0.3}}{1 + hc}, \quad (15.2.7c)$$

$$\vdots \quad (15.2.7d)$$

$$i = 9 : \quad \mathbf{y}_n = y_{10} = \frac{y_9 + h\mathbf{x}_{10}}{1 + hc} = \frac{y_9 + h \times \mathbf{1.0}}{1 + hc}. \quad (15.2.7e)$$

- For the *reverse integration*, the loop goes from $i = 10$ through $i = 19$ (ten equations) We simply change sign $h \rightarrow -h$, so $h = -0.1$.

$$\mathbf{i} = 10 : \quad y_{11} = \frac{y_{10} - h\mathbf{x}_{11}}{1 - hc} = \frac{y_{10} - h \times \mathbf{0.9}}{1 - hc}, \quad (15.2.8a)$$

$$\mathbf{i} = 11 : \quad y_{12} = \frac{y_{11} - h\mathbf{x}_{12}}{1 - hc} = \frac{y_{11} - h \times \mathbf{0.8}}{1 - hc}, \quad (15.2.8b)$$

$$\mathbf{i} = 12 : \quad y_{13} = \frac{y_{12} - h\mathbf{x}_{13}}{1 - hc} = \frac{y_{12} - h \times \mathbf{0.7}}{1 - hc}, \quad (15.2.8c)$$

$$\vdots \quad (15.2.8d)$$

$$\mathbf{i} = 18 : \quad y_{19} = \frac{y_{18} - h\mathbf{x}_{19}}{1 - hc} = \frac{y_{18} - h \times \mathbf{0.1}}{1 - hc}, \quad (15.2.8e)$$

$$\mathbf{i} = 19 : \quad \mathbf{y}_{2n} = y_{20} = \frac{y_{19} - h\mathbf{x}_{20}}{1 - hc} = \frac{y_{19} - h \times \mathbf{0}}{1 - hc}. \quad (15.2.8f)$$

- Note the indexing and the value of x_i for the reverse integration.

15.3 Euler integration 3

- You are given the following differential equation

$$\frac{dy}{dx} = 2x. \quad (15.3.1)$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 0$.
- The exact solution $y^{\text{ex}}(x)$ is given as follows:

$$y^{\text{ex}}(x) = x^2. \quad (15.3.2)$$

- The integration stepsize is h (a constant), hence $x_i = ih$.
- Denote the explicit numerical solution by $y_i^{\text{exp}} = y^{\text{exp}}(x_i)$.
- Denote the implicit numerical solution by $y_i^{\text{imp}} = y^{\text{imp}}(x_i)$.
- Use the explicit Euler method to derive the following.**

$$y_{i+1}^{\text{exp}} = y_i^{\text{exp}} + 2hx_i. \quad (15.3.3)$$

- Show the following, for $i = 1$:**

$$y_1^{\text{exp}} = 0. \quad (15.3.4)$$

- Use the implicit Euler method to derive the following.**

$$y_{i+1}^{\text{imp}} = y_i^{\text{imp}} + 2hx_{i+1}. \quad (15.3.5)$$

- Show the following, for $i = 1$:**

$$y_1^{\text{imp}} = 2h^2. \quad (15.3.6)$$

- Set $h = 1/n$.
- The value of the exact solution is $y^{\text{ex}}(1) = 1$ for $x = 1$.
- Compute the value of the numerical solutions y_n^{exp} and y_n^{imp} and fill the following table of values.**

n	y_n^{exp}	y_n^{imp}	$n(1 - y_n^{\text{exp}})$	$n(1 - y_n^{\text{imp}})$
10	5 d.p.	5 d.p.	2 d.p.	2 d.p.
100	5 d.p.	5 d.p.	2 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	2 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last two columns should be approximately independent of n .
- This is because Euler integration is a first order method.

15.4 Euler integration 4

- Let us modify the differential equation in eq. (15.3.1).
- We know the exact solution is $y^{\text{ex}}(x) = x^2$ so let us integrate the following differential equation

$$\frac{dy}{dx} = 2\sqrt{y}. \quad (15.4.1)$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 0$.
- The exact solution is still $y^{\text{ex}}(x) = x^2$.
- The integration stepsize is h (a constant), hence $x_i = ih$.
- Denote the explicit numerical solution by $y_i^{\text{exp}} = y^{\text{exp}}(x_i)$.
- Denote the implicit numerical solution by $y_i^{\text{imp}} = y^{\text{imp}}(x_i)$.
- **Use the explicit Euler method to derive the following.**

$$y_{i+1}^{\text{exp}} = y_i^{\text{exp}} + 2h\sqrt{y_i^{\text{exp}}}. \quad (15.4.2)$$

- **Compute using eq. (15.4.2) with $h = 1/n$ for $n = 10, 100, 1000$ and show that:**

$$y_n^{\text{exp}} = 0 \quad (n = 10, 100, 1000). \quad (15.4.3)$$

- *The explicit Euler solution never leaves zero!*

- Use the implicit Euler method to derive the following.

$$y_{i+1}^{\text{imp}} = y_i^{\text{imp}} + 2h\sqrt{y_{i+1}^{\text{imp}}}. \quad (15.4.4)$$

- Process eq. (15.4.4) to derive the following.

$$y_{i+1}^{\text{imp}} - 2h\sqrt{y_{i+1}^{\text{imp}}} = y_i^{\text{imp}}. \quad (15.4.5)$$

- We have our first example of a nontrivial implicit equation.
- Show that $y_i^{\text{imp}} = 0$ (for all i) is a solution of eq. (15.4.5).
- However, it is not the only solution of eq. (15.4.5). There is another solution.
 1. I will derive the formula for the integration step for you.
 2. We process eq. (15.4.5) to obtain the following:

$$\begin{aligned} \left(\sqrt{y_{i+1}^{\text{imp}}} - h\right)^2 &= h^2 + y_i^{\text{imp}} \\ \sqrt{y_{i+1}^{\text{imp}}} - h &= \sqrt{h^2 + y_i^{\text{imp}}} \\ \sqrt{y_{i+1}^{\text{imp}}} &= h + \sqrt{h^2 + y_i^{\text{imp}}}. \end{aligned} \quad (15.4.6)$$

- From the above we obtain the following expression:

$$y_{i+1}^{\text{imp}} = \left(h + \sqrt{h^2 + y_i^{\text{imp}}}\right)^2. \quad (15.4.7)$$

- Set $h = 1/n$.
- The value of the exact solution is $y^{\text{ex}}(1) = 1$ for $x = 1$.
- Compute the value of the numerical solution y_n^{imp} using eq. (15.4.7) and fill the following table of values.

n	y_n^{imp}	$n(1 - y_n^{\text{imp}})$
10	5 d.p.	2 d.p.
100	5 d.p.	2 d.p.
1000	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately independent of n .
- This is because Euler integration is a first order method.
- This example teaches us various lessons.
 1. We need to pay more attention to the conditions for the existence of a solution of a differential equation.
 2. For an implicit integration algorithm, there can be more than one solution of the implicit equation, and a root finding algorithm might converge to the wrong root.