Queens College, CUNY, Department of Computer Science

Numerical Methods

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21 Lecture 21

Ordinary differential equations (ODEs)

- In this lecture we shall study **ordinary differential equations.**
- The major focus in this course will be on **linear ordinary differential equations.**

21.1 Introduction

- Let the independent variable be x and let $y(x)$ be a function of x.
- We can form equations such as
$$y(x) = 1 + 2x\,. \tag{21.1.1}$$
- We can also form more complicated equations such as
$$y + e^y = 1 + \ln x\,. \tag{21.1.2}$$
- Note that eq. (21.1.1) yields the value of y explicitly as a function of x.
- However, eq. (21.1.2) must be solved to obtain the value of y explicitly as a function of x.
- A **differential equation** is an equation which involves derivatives of y, such as dy/dx.
- Instead of eq. (21.1.1), consider the equation
$$\frac{dy}{dx} = 1 + 2x\,. \tag{21.1.3}$$
- Then eq. (21.1.3) is an example of a differential equation.
- Another example of a differential equation is
$$\frac{d^2y}{dx^2} + 3x\,\frac{dy}{dx} + x^4 y = \frac{1}{1+x}\,. \tag{21.1.4}$$
- We can form many examples. A more complicated example is
$$\left(\frac{dy}{dx}\right)^2 + y^2 = 1\,. \tag{21.1.5}$$
- Instead of eq. (21.1.2), consider the equation
$$\frac{dy}{dx} + e^y = 1 + \ln x\,. \tag{21.1.6}$$
- Fundamentally, any equation which contains derivatives of y is a differential equation.

21.2 Solving differential equations Part 1

- Consider eq. (21.1.3). How shall we solve it to obtain $y(x)$?
- This is a simple example. It can be solved easily, by integrating with respect to x

$$\begin{aligned} y(x) &= \int (1+2x)\,dx \\ &= x + x^2 + c\,. \end{aligned} \tag{21.2.1}$$

- The term "c" in the solution in eq. (21.2.1) is a constant.
- **The value of c in eq. (21.2.1) is arbitrary.**
- We can see this because if we differentiate the solution in eq. (21.2.1), then $dc/dx = 0$ because c is a constant. Therefore, if we start from eq. (21.2.1), we obtain

$$\frac{d}{dx}(x + x^2 + c) = \frac{dx}{dx} + \frac{d(x^2)}{dx} + \frac{dc}{dx} = 1 + 2x + 0 = 1 + 2x\,. \tag{21.2.2}$$

- The above example illustrates an important feature of differential equations.
- **In general, the solution of a differential equation is not unique.**
- To obtain a unique solution for $y(x)$, it is also necessary to specify additional conditions, for example the value of y at $x = 0$.
- Suppose we say $y = 1$ at $x = 0$. Then substituting $y = 1$ and $x = 0$ in eq. (21.2.1) yields

$$1 = 0 + 0 + c\,, \qquad c = 1\,. \tag{21.2.3}$$

- Hence if $y = 1$ at $x = 0$, the unique solution of eq. (21.1.3) is

$$y = 1 + x + x^2\,. \tag{21.2.4}$$

21.3 Solving differential equations Part 2

- Consider instead the following differential equation

$$\frac{d^2y}{dx^2} = 1 + 2x\,. \tag{21.3.1}$$

- How shall we solve it?

  1. Obviously, from Sec. 21.2, integrating once with respect to x yields

  $$\frac{dy}{dx} = \int (1+2x)\,dx = x + x^2 + c_1\,. \tag{21.3.2}$$

  2. Here c_1 is an arbitrary constant.
  3. Integrating again with respect to x yields

  $$y(x) = \int (x + x^2 + c_1)\,dx = \frac{x^2}{2} + \frac{x^3}{3} + c_1 x + c_2\,. \tag{21.3.3}$$

  4. Here c_2 is also an arbitrary constant, different from c_1.

- Hence to obtain a unique solution of eq. (21.3.1), we require **two** additional conditions.
- The way the two conditions are specified is not unique.

  1. One possibility is to specify the values of y and dy/dx at some value x_0.
  2. For example we could say $y = 1$ and $dy/dx = 0$ at $x = 0$. Then $c_1 = 0$ and $c_2 = 1$.
  3. Alternatively, we can specify the value of y and two different values of x, say x_0 and x_1.
  4. For example we could say $y = 1$ at $x = 0$ and $y = 2$ at $x = 2$.
     Setting $y = 1$ at $x = 0$ yields

  $$1 = 0 + 0 + 0 + c_2\,, \qquad c_2 = 1\,. \tag{21.3.4}$$

     Setting $y = 2$ at $x = 2$ yields

  $$2 = 2 + \frac{8}{3} + 2c_1 + 1\,, \qquad c_2 = -\frac{11}{6}\,. \tag{21.3.5}$$

- Hence in general, to specify a unique solution for a differential equation, multiple additional conditions are required, and the way to specify those conditions is not unique.

21.4 Linear differential equations

- We shall continue the topic of solving differential equations later.
- Here we study the topic of **linear differential equations.**
- Let a, b and c be constants. Consider the following differential equation

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0\,. \tag{21.4.1}$$

- Let $z = ky$, where k is an arbitrary nonzero oonstant. Then

$$a\,\frac{d^2z}{dx^2} + b\,\frac{dz}{dx} + cz = ak\,\frac{d^2y}{dx^2} + bk\,\frac{dy}{dx} + cky = k\,\underbrace{\left(a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy\right)}_{=0} = 0\,. \tag{21.4.2}$$

- Hence $z = ky$ is also a solution of eq. (21.4.1), for any value of k.
- Next suppose y_1 and y_2 are two linearly independent solutions of eq. (21.4.1).
- *Note: we have not proved that eq.* (21.4.1) *has two or more linearly independent solutions. For now we are assuming that at least two linearly independent solutions exist.*
- Set $\zeta = y_1 + y_2$. Then

$$\begin{aligned}
a\,\frac{d^2\zeta}{dx^2} + b\,\frac{d\zeta}{dx} + c\zeta &= a\,\frac{d^2(y_1+y_2)}{dx^2} + b\,\frac{d(y_1+y_2)}{dx} + c(y_1+y_2) \\
&= a\left(\frac{d^2y_1}{dx^2} + \frac{d^2y_2}{dx^2}\right) + b\left(\frac{dy_1}{dx} + \frac{dy_2}{dx}\right) + c\,(y_1+y_2) \\
&= \underbrace{a\,\frac{d^2y_1}{dx^2} + b\,\frac{dy_1}{dx} + cy_1}_{=0} \quad + \quad \underbrace{a\,\frac{d^2y_2}{dx^2} + b\,\frac{dy_2}{dx} + cy_2}_{=0} \\
&= 0\,.
\end{aligned} \tag{21.4.3}$$

- Hence the sum $y_1 + y_2$ is also a solution of eq. (21.4.1).
- These are the two defining properties of a **linear differential equation.**
  1. If y is a solution, then ky is also a solution, for any nonzero constant k.
  2. If y_1 and y_2 are two linearly independent solutions, then the sum y_1+y_2 is also a solution.
- Many authors combine the above two conditions into one by stating that if y_1 and y_2 are two linearly independent solutions, then if c_1 and c_2 are arbitrary constants, $c_1y_1 + c_2y_2$ is also a solution.
- Note that we did not require a, b and c to be constants in the above derivations. We only required that a, b and c did not depend on y.
- Hence if $a_1(x), \dots, a_n(x)$, are functions of x but not y, the following is a linear differential equation

$$a_n(x)\,\frac{d^ny}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy}{dx} + a_0(x)y = 0\,. \tag{21.4.4}$$

21.5 Homogeneous and inhomogeneous linear ordinary differential equations

- Let us return to eq. (21.4.4), reproduced below for convenience:

$$a_n(x)\,\frac{d^n y}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy}{dx} + a_0(x)y = 0\,. \tag{21.5.1}$$

- Technically, eq. (21.5.1) is a **homogeneous** linear ordinary differential equation.

- By "**homogeneous**" we mean the right hand side is zero.

- Technically, the statements that if y is a solution then ky is also a solution and that if y_1 and y_2 are solutions then $y_1 + y_2$ are solutions applies only to homogeneous linear ordinary differential equations.

- An **inhomogeneous** linear ordinary differential equation is similar to eq. (21.5.1) but the right hand side is a nonzero function of x, say $f(x)$:

$$a_n(x)\,\frac{d^n y}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy}{dx} + a_0(x)y = f(x)\,. \tag{21.5.2}$$

- A linear ordinary differential equation of the n^{th} order has n linearly independent solutions.

- I suppose the mathematicians have a rigorous proof of the above statement, but I do not know it. I can only demonstrate a proof in special simple cases.

21.6 Examples of linear differential equations

- The following are examples of homogeneous linear differential equations:

$$\frac{dy}{dx} + y = 0\,. \tag{21.6.1}$$

$$x^2\,\frac{d^2y}{dx^2} - 3x\,\frac{dy}{dx} + 2y = 0\,. \tag{21.6.2}$$

$$\frac{d^3y}{dx^3} + 2\,\frac{d^2y}{dx^2} - 3\,\frac{dy}{dx} - 4y = 0\,. \tag{21.6.3}$$

$$(1-x^2)\frac{d^3y}{dx^3} + \frac{2}{x}\,\frac{d^2y}{dx^2} - 3e^x\,\frac{dy}{dx} - \frac{4y}{1+x^2} = 0\,. \tag{21.6.4}$$

- The following are examples of inhomogeneous linear differential equations:

$$\frac{dy}{dx} + y = 1\,. \tag{21.6.5}$$

$$x^2\,\frac{d^2y}{dx^2} - 3x\,\frac{dy}{dx} + 2y = x\,. \tag{21.6.6}$$

$$\frac{d^3y}{dx^3} + 2\,\frac{d^2y}{dx^2} - 3\,\frac{dy}{dx} - 4y = x^2 + e^x\,. \tag{21.6.7}$$

$$(1-x^2)\frac{d^3y}{dx^3} + \frac{2}{x}\,\frac{d^2y}{dx^2} - 3e^x\,\frac{dy}{dx} - \frac{4y}{1+x^2} = \frac{\sin x}{e^x+1}\,. \tag{21.6.8}$$

- The following differential equations are not linear:

$$\frac{dy}{dx} + y^2 = 0\,. \tag{21.6.9}$$

$$\left(\frac{dy}{dx}\right)^2 + 2y^2 = 3\,. \tag{21.6.10}$$

$$x^2\left(\frac{d^2y}{dx^2}\right)^4 - 3x\left(\frac{dy}{dx}\right)^3 + 2y^2 = \cos x\,. \tag{21.6.11}$$

21.7 Euler substitution Part 1

- **Euler substitution** is a technique to solve homogeneous linear ordinary differential equations where the coefficients of all the terms are constants.
- Let us also keep things simple and begin with second order linear differential equations.
- Then the differential equation has the following form:

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0\,. \tag{21.7.1}$$

- Because eq. (21.7.1) is a second order linear ordinary differential equation, it has two linearly independent solutions.
- We solve eq. (21.7.1) by the method of **Euler substitution**.
  1. We *guess* that the solution is of the form $y = e^{\alpha x}$, where α is a constant.
  2. Substituting in eq. (21.7.1) yields

$$(a\alpha^2 + b\alpha + c)\,e^{\alpha x} = 0\,. \tag{21.7.2}$$

  3. Then because $e^{\alpha x} \neq 0$ for finite values of x, we must have

$$a\alpha^2 + b\alpha + c = 0\,. \tag{21.7.3}$$

  4. This is a quadratic equation which must be solved for α.
- Let the two roots for α be α_1 and α_2, respectively.
- Note that the values of α_1 and α_2 can be complex numbers.
- If $\alpha_1 \neq \alpha_2$, the general solution of eq. (21.8.1) is

$$y(x) = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}\,. \tag{21.7.4}$$

- Here c_1 and c_2 are arbitrary constants.
- What happens if $\alpha_1 = \alpha_2$? Then we obtain a second independent solution by ***differentiating $y_1 = e^{\alpha_1 x}$ with respect to α_1***. Hence the second solution is

$$y_2(x) = \frac{d(e^{\alpha_1 x})}{d\alpha_1} = xe^{\alpha_1 x}\,. \tag{21.7.5}$$

- Then the general solution of eq. (21.7.1) is

$$y(x) = c_1 e^{\alpha_1 x} + c_2 x e^{\alpha_1 x}\,. \tag{21.7.6}$$

21.8 Euler substitution Part 2

- Next, consider more general linear ordinary differential equations with constant coefficients.
- Suppose $a_1, \dots, a_n$, are constant numbers. Consider the following linear differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0\,. \tag{21.8.1}$$

- Because eq. (21.8.1) is an n^{th} order linear ordinary differential equation, it has n linearly independent solutions.
- Use the Euler substitution $y = e^{\alpha x}$. We obtain the polynomial equation

$$a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 = 0\,. \tag{21.8.2}$$

- A polynomial equation of degree n has n roots (although some roots may be repeated).
- If the roots are all distinct, say $\alpha_1, \dots, \alpha - n$, the general solution of eq. (21.8.2) is

$$y(x) = c_1 e^{\alpha_1 x} + \cdots + c_n e^{\alpha_n x} = \sum_{j=1}^{n} c_j e^{\alpha_j x}\,. \tag{21.8.3}$$

- Here $c_1, \dots, c_n$ are arbitrary constants.
- Suppose that the α_j are not all distinct. Suppose the root α_1 occurs twice. Then the following is a solution of eq. (21.8.2)

$$y(x) = \frac{d(e^{\alpha_1 x})}{d\alpha_1} = x e^{\alpha_1 x}\,. \tag{21.8.4}$$

- More generally, suppose the root α_j occurs m_j times. Then we obtain $m_j - 1$ solutions by differentiating with respect to α_j up to $m_j - 1$ times:

$$y(x) = x e^{\alpha_j x}\,, \qquad x^2 e^{\alpha_j x}\,, \qquad \dots x^{m_j - 1} e^{\alpha_j x}\,. \tag{21.8.5}$$

- For example, for a fifth order linear ordinary differential equation where α_1 occurs twice and α_2 occurs three times, the general solution is

$$y(x) = c_1 e^{\alpha_1 x} + c_2 x e^{\alpha_1 x} + c_3 e^{\alpha_2 x} + c_4 x e^{\alpha_2 x} + c_5 x^2 e^{\alpha_2 x}\,. \tag{21.8.6}$$

21.9 Power law substitutions

- Consider a second order homogeneous linear ordinary differential equation of the form

$$ax^2\,\frac{d^2y}{dx^2} + bx\,\frac{dy}{dx} + cy = 0\,. \tag{21.9.1}$$

- There is a standard technique to solve equations such as eq. (21.9.1). I do not know if the technique has a name, so let me call it **power law substitution.** Be advised this is not a standard name.
- Try a solution of the form $y(x) = x^\beta$, i.e. a power law. Substituting into eq. (21.9.1) yields

$$\bigl[\,a\beta(\beta-1) + b\beta + c\,\bigr]\,x^\beta = 0\,. \tag{21.9.2}$$

- Because $x^\beta \neq 0$ in general, we obtain the following quadratic equation for β:

$$a\beta^2 + (b-a)\beta + c = 0\,. \tag{21.9.3}$$

- This has two roots β_1 and β_2.
- If $\beta_1 \neq \beta_2$, the general solution of eq. (21.9.1) is

$$y(x) = c_1x^{\beta_1} + c_2x^{\beta_2}\,. \tag{21.9.4}$$

- If $\beta_1 = \beta_2$, we differentiate with respect to β_1 to obtain a second solution

$$y_2(x) = \frac{d(x^{\beta_1})}{d\beta_1} = \frac{d(e^{\beta_1\ln x})}{d\beta_1} = x^{\beta_1}\ln x\,. \tag{21.9.5}$$

- The general solution of eq. (21.9.1) is then

$$y(x) = c_1x^{\beta_1} + c_2x^{\beta_1}\ln x\,. \tag{21.9.6}$$

- Here c_1 and c_2 are arbitrary constants.
- The technique generalizes in an obvious way to the n^{th} order equation

$$a_nx^n\,\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1x\,\frac{dy}{dx} + a_0y = 0\,. \tag{21.9.7}$$

21.10 Solutions of inhomogeneous linear ordinary differential equations

- Solving inhomogeneous linear ordinary differential equations is more of an art than a science.
- There are a few general techniques, but mostly one must learn a bag of tricks for special cases.
- Let us return to the simple case of eq. (21.7.1) and add an inhomogeneous term:

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = f(x)\,. \tag{21.10.1}$$

- Suppose $y = P(x)$ is a **particlar solution** of eq. (21.10.1), so

$$a\,\frac{d^2P}{dx^2} + b\,\frac{dP}{dx} + cP = f(x)\,. \tag{21.10.2}$$

- Suppose also that $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation eq. (21.7.1), so

$$a\,\frac{d^2y_j}{dx^2} + b\,\frac{dy_j}{dx} + cy_j = 0 \qquad (j = 1, 2)\,. \tag{21.10.3}$$

- Then $y = P + c_1y_1 + c_2y_2$ is also a solution of the inhomogeneous equation eq. (21.10.1).
- The proof is simple:

$$\begin{aligned} a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy &= \underbrace{a\,\frac{d^2P}{dx^2} + b\,\frac{dP}{dx} + cP}_{=f(x)} \\ &\quad + c_1\underbrace{\left(a\,\frac{d^2y_1}{dx^2} + b\,\frac{dy_1}{dx} + cy_1\right)}_{=0} + c_2\underbrace{\left(a\,\frac{d^2y_2}{dx^2} + b\,\frac{dy_2}{dx} + cy_2\right)}_{=0} \\ &= f(x)\,. \end{aligned} \tag{21.10.4}$$

- The same idea applies to the general n^{th} order linear inhomogeneous linear ordinary differential equation eq. (21.5.2), reproduced below for ease of reference:

$$a_n(x)\,\frac{d^ny}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy}{dx} + a_0(x)y = f(x)\,. \tag{21.10.5}$$

- If we can find a particular solution $P(x)$, then the general solution of eq. (21.10.5) is given by the sum of $P(x)$ and a linear combination of the solutions of the corresponding homogeneous equation

$$y(x) = P(x) + c_1y_1(x) + \cdots + c_ny_n(x)\,. \tag{21.10.6}$$

- **How do we find the particular solution $P(x)$?**
- ***That is the difficulty.*** Nobody knows a general procedure to find the particular solution $P(x)$. What we can do is to learn a bag of tricks for special cases.

21.11 Particular solution Part 1

- Consider the equation

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = d\,. \tag{21.11.1}$$

- Here d is a constant.
- Try a solution $y = k =$ constant. Then $dy/dx = 0$ and $d^2y/dx^2 = 0$, so

$$0 + 0 + ck = d\,. \tag{21.11.2}$$

- Then $k = d/c$ and the particular solution is

$$P(x) = \frac{d}{c}\,. \tag{21.11.3}$$

- ***However, what happens if $c = 0$?***
- If $c = 0$, try $y = k_1 x$, where k_1 is a constant. Then $dy/dx = k_1$ and $d^2y/dx^2 = 0$, so

$$0 + bk_1 + 0 = d\,. \tag{21.11.4}$$

- Then $k_1 = d/b$ and the particular solution is

$$P(x) = \frac{d}{b}\,x\,. \tag{21.11.5}$$

- ***However, what happens if $b = c = 0$?***
- If $b = c = 0$, try $y = k_2 x^2$, where k_2 is a constant. Then $d^2y/dx^2 = 2k_2$, so

$$2ak_2 + 0 + 0 = d\,. \tag{21.11.6}$$

- Then $k_2 = d/(2a)$ and the particular solution is

$$P(x) = \frac{d}{2a}\,x^2\,. \tag{21.11.7}$$

- ***However, what happens if $a = b = c = 0$?*** Umm . . . let's not be silly.
- The basic lesson here is that if $f(x)$ equals a constant, then try a simple guess such as $y =$ constant or $y \propto x$ or $y \propto x^2$.
- The same idea will work for the n^{th} order inhomogeneous linear ordinary equation

$$a_n\,\frac{d^ny}{dx^n} + a_{n-1}\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\,\frac{dy}{dx} + a_0 y = d\,. \tag{21.11.8}$$

21.12 Particular solution Part 2

- Consider the equation

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = d_0 + d_1 x\,. \qquad (21.12.1)$$

- Here d_0 and d_1 are constants.
- It is obvious what to do. Try a solution $y = k_0 + k_1 x$, where k_1 and k_2 are constants. Then $dy/dx = k_1$ and $d^2y/dx^2 = 0$, so

$$0 + bk_1 + c(k_0 + k_1 x) = d_0 + d_1 x\,. \qquad (21.12.2)$$

- Now we must equate some coefficients. First we equate the terms in x to obtain $ck_1 = d_1$ or $k_1 = d_1/c$. Then we obtain

$$ck_0 = d_0 - bk_1 = d_0 - \frac{bd_1}{c} = \frac{cd_0 - bd_1}{c}\,. \qquad (21.12.3)$$

- The particular solution is

$$P(x) = \frac{cd_0 - bd_1}{c^2} + \frac{d_1}{c}\,x\,. \qquad (21.12.4)$$

- ***However, what happens if $c = 0$?***
- If $c = 0$, try $y = k_1 x + k_2 x^2$, where k_1 and k_2 are constants. Then $dy/dx = k_1 + 2k_2 x$ and $d^2y/dx^2 = 2k_2$, so

$$2ak_2 + b(k_1 + 2k_2 x) + 0 = d_0 + d_1 x\,. \qquad (21.12.5)$$

- We equate the terms in x to obtain $2bk_2 = d_1$ or $k_2 = d_1/(2b)$. Then we obtain

$$bk_1 = d_0 - 2ak_2 = d_0 - \frac{ad_1}{b} = \frac{bd_0 - ad_1}{b}\,. \qquad (21.12.6)$$

- The particular solution is

$$P(x) = \frac{bd_0 - ad_1}{b^2} + \frac{d_1}{2b}\,x\,. \qquad (21.12.7)$$

- ***However, what happens if $b = c = 0$?***
- If $b = c = 0$, try $y = k_2 x^2 + k_3 x^3$, where k_2 and k_3 are constants. Then $d^2y/dx^2 = 2k_2 + 6k_3 x$, so

$$2ak_2 + 6ak_3 x + 0 + 0 = d_0 + d_1 x\,. \qquad (21.12.8)$$

- Then $k_2 = d_0/(2a)$ and $k_3 = d_1/(6a)$ and the particular solution is

$$P(x) = \frac{d_0}{2a}\,x^2 + \frac{d_1}{6a}\,x^3\,. \qquad (21.12.9)$$

- The basic lesson here is that if $f(x)$ is a polynomial in x, then try setting $y(x)$ to a (different) polynomial in x, and equate coefficients. The particular solution $P(x)$ is a polynomial in x.
- The same idea will work for the n^{th} order inhomogeneous linear ordinary equation

$$a_n\,\frac{d^ny}{dx^n} + a_{n-1}\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\,\frac{dy}{dx} + a_0 y = (\text{polynomial in } x)\,. \qquad (21.12.10)$$

21.13 Particular solution Part 3

- Consider the equation

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = de^{\gamma x}\,. \tag{21.13.1}$$

- Here d and γ are constants and moreover γ is ***not a root*** of the polynomial:

$$a\gamma^2 + b\gamma + c \neq 0\,. \tag{21.13.2}$$

- Now we try a solution $y = ke^{\gamma x}$, where k is a constant. Then we obtain

$$(a\gamma^2 + b\gamma + c)\,ke^{\gamma x} = de^{\gamma x}\,. \tag{21.13.3}$$

- Hence

$$k = \frac{d}{a\gamma^2 + b\gamma + c}\,. \tag{21.13.4}$$

- The particular solution is

$$P(x) = \frac{de^{\gamma x}}{a\gamma^2 + b\gamma + c}\,. \tag{21.13.5}$$

- Suppose the right hand side consists of a sum of exponentials

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = d_1e^{\gamma_1 x} + d_2e^{\gamma_2 x} + \cdots \tag{21.13.6}$$

- Here the d_j and γ_j are all constants, $j = 1, 2, \dots$ and

$$a\gamma_j^2 + b\gamma_j + c \neq 0 \qquad\qquad (j = 1, 2, \dots)\,. \tag{21.13.7}$$

- Then the particular solution is

$$P(x) = \frac{d_1e^{\gamma_1 x}}{a\gamma_1^2 + b\gamma_1 + c} + \frac{d_2e^{\gamma_2 x}}{a\gamma_2^2 + b\gamma_2 + c} + \cdots \tag{21.13.8}$$

- The same idea will work for the n^{th} order inhomogeneous linear ordinary equation

$$a_n\,\frac{d^ny}{dx^n} + a_{n-1}\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\,\frac{dy}{dx} + a_0y = \sum_j d_je^{\gamma_j x}\,. \tag{21.13.9}$$

- We require that

$$a_n\gamma_j^n + a_{n-1}\gamma_j^{n-1} + \cdots + a_1\gamma_j + a_0 \neq 0 \qquad\qquad (j = 1, 2, \dots)\,. \tag{21.13.10}$$

- The particular solution is

$$P(x) = \sum_j \frac{d_je^{\gamma_j x}}{a_n\gamma_j^n + a_{n-1}\gamma_j^{n-1} + \cdots + a_1\gamma_j + a_0}\,. \tag{21.13.11}$$

21.14 Particular solution Part 4

- Consider the equation

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = d_c\cos(\omega x) + d_s\sin(\omega x)\,. \tag{21.14.1}$$

- Here d_c, d_s and ω are constants. We shall determine restriction on the value of ω later.
- Now we try a solution

$$y = k_c\cos(\omega x) + k_s\sin(\omega x)\,. \tag{21.14.2}$$

- Then we obtain

$$\begin{aligned}&(-a\omega^2 k_c + b\omega k_s + ck_c)\cos(\omega x)\\ &\quad+(-a\omega^2 k_s - b\omega k_c + ck_s)\sin(\omega x) = d_c\cos(\omega x) + d_s\sin(\omega x)\,.\end{aligned} \tag{21.14.3}$$

- Equating the coefficients yields a pair of simultaneous equations

$$(c - a\omega^2)k_c + b\omega k_s = d_c\,, \tag{21.14.4a}$$

$$(c - a\omega^2)k_s - b\omega k_c = d_s\,. \tag{21.14.4b}$$

- We can express this as a 2×2 matrix equation

$$\begin{pmatrix} c - a\omega^2 & b\omega \\ -b\omega & c - a\omega^2 \end{pmatrix}\begin{pmatrix} k_c \\ k_s \end{pmatrix} = \begin{pmatrix} d_c \\ d_s \end{pmatrix}\,. \tag{21.14.5}$$

- The determinant of the matrix is

$$\Delta = (c - a\omega^2)^2 + b^2\omega^2 = c^2 + (b^2 - 2ac)\omega^2 + a^4\omega^4\,. \tag{21.14.6}$$

- Hence the restriction on the value of ω is $\Delta(\omega) \neq 0$.
- We invert the matrix to obtain

$$\begin{pmatrix} k_c \\ k_s \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} c - a\omega^2 & -b\omega \\ b\omega & c - a\omega^2 \end{pmatrix}\begin{pmatrix} d_c \\ d_s \end{pmatrix}\,. \tag{21.14.7}$$

- The particular solution is

$$P(x) = \frac{(c - a\omega^2)d_c - b\omega d_s}{\Delta}\,\cos(\omega x) + \frac{(c - a\omega^2)d_s + b\omega d_c}{\Delta}\,\sin(\omega x)\,. \tag{21.14.8}$$

- This can be generalized to include multiple terms on the right hand side and also for an n^{th} order linear ordinary differential equation.

21.15 Integrating factor

- Consider the equation

$$\frac{dy}{dx} + g(x)y = h(x)\,. \tag{21.15.1}$$

- Here $g(x)$ and $h(x)$ are functions of x.
- Define a function

$$G(x) = \exp\biggl\{\int_{x_0}^{x} g(u)\,du\biggr\}\,. \tag{21.15.2}$$

- Here x_0 is an arbitrary lower limit. Its value is not important for now.
- The function $G(x)$ is called an **integrating factor.**
- We shall see below how it works.
- First note that

$$\frac{d(Gy)}{dx} = G\,\frac{dy}{dx} + \frac{dG}{dx}\,y = G\,\frac{dy}{dx} + gG\,y = G\left(\frac{dy}{dx} + gy\right)\,. \tag{21.15.3}$$

- Hence multiply eq. (21.15.1) through by $G(x)$ to obtain

$$\begin{aligned} G\left(\frac{dy}{dx} + gy\right) &= h \\ \frac{d(Gy)}{dx} &= Gh\,. \end{aligned} \tag{21.15.4}$$

- We integrate this from x_0 to x to obtain (note that $G(x_0) = 1$ by construction)

$$G(x)y(x) - y_0 = \int_{x_0}^{x} G(u)h(u)\,du\,. \tag{21.15.5}$$

- Here $y_0 = y(x_0)$.
- The solution for $y(x)$ is therefore

$$y(x) = y_0\,e^{-\int_{x_0}^{x} g(u)\,du} + e^{-\int_{x_0}^{x} g(u)\,du}\int_{x_0}^{x} e^{\int_{x_0}^{w} g(v)\,dv} h(w)\,dw\,. \tag{21.15.6}$$

- Note that this solution automatically incorporates the initial condition $y = y_0$ at $x = x_0$.

21.16 Worked example 1

21.16.1 Homogeneous equation

- The equation is

$$\frac{dy}{dx} + 2y = 0\,. \tag{21.16.1}$$

- We employ an Euler substitution $y = e^{\alpha x}$ to obtain

$$\alpha + 2 = 0\,. \tag{21.16.2}$$

- The solution is $\alpha = -2$.
- This is a first order differential equation so there is only one linearly independent solution.
- The general solution is

$$y(x) = ce^{-2x}\,. \tag{21.16.3}$$

21.16.2 Inhomogeneous equation

- Next consider the equation

$$\frac{dy}{dx} + 2y = 1 - 2x + 2x^2\,. \tag{21.16.4}$$

- We try a particular solution $y = k_0 + k_1 x + k_2 x^2$, then

$$(k_1 + 2k_2 x) + 2(k_0 + k_1 x + k_2 x^2) = 1 - 2x + 2x^2\,. \tag{21.16.5}$$

- Equating the coefficient of x^2 yields $2k_2 = 2$ so $k_2 = 1$.
- Next, equating the coefficient of x yields $2k_2 + 2k_1 = -2$, hence $2k_1 = -4$, hence $k_1 = -2$.
- Next, equating the constant term yields $k_1 + 2k_0 = 1$, hence $2k_0 = 3$, hence $k_1 = \frac{3}{2}$.
- Hence the particular solution is

$$P(x) = \frac{3}{2} - 2x + x^2\,. \tag{21.16.6}$$

- The general solution is

$$y(x) = \frac{3}{2} - 2x + x^2 + ce^{-2x}\,. \tag{21.16.7}$$

21.17 Worked example 2

21.17.1 Homogeneous equation

- The equation is
$$\frac{d^2y}{dx^2} - 3\,\frac{dy}{dx} + 2y = 0\,. \tag{21.17.1}$$
- We employ an Euler substitution $y = e^{\alpha x}$ to obtain
$$\alpha^2 - 3\alpha + 2 = (\alpha - 1)(\alpha - 2) = 0\,. \tag{21.17.2}$$
- The solutions are $\alpha_1 = 1$ and $\alpha_2 = 2$.
- The general solution is
$$y(x) = c_1 e^x + c_2 e^{2x}\,. \tag{21.17.3}$$

21.17.2 Inhomogeneous equation

- Next consider the equation
$$\frac{d^2y}{dx^2} - 3\,\frac{dy}{dx} + 2y = 1 - 2x + 2x^2\,. \tag{21.17.4}$$
- We try a particular solution $y = k_0 + k_1 x + k_2 x^2$, then
$$2k_2 - 3(k_1 + 2k_2 x) + 2(k_0 + k_1 x + k_2 x^2) = 1 - 2x + 2x^2\,. \tag{21.17.5}$$
- Equating the coefficient of x^2 yields $2k_2 = 2$ so $k_2 = 1$.
- Next, equating the coefficient of x yields $-6k_2 + 2k_1 = -2$, hence $2k_1 = 4$, hence $k_1 = 2$.
- Next, equating the constant term yields $2k_2 - 3k_1 + 2k_0 = 1$, hence $2k_0 = 5$, hence $k_1 = \frac{5}{2}$.
- Hence the particular solution is
$$P(x) = \frac{5}{2} + 2x + x^2\,. \tag{21.17.6}$$
- The general solution is
$$y(x) = \frac{5}{2} + 2x + x^2 + c_1 e^x + c_2 e^{2x}\,. \tag{21.17.7}$$

21.18 Worked example 3

21.18.1 Homogeneous equation

- The equation is

$$\frac{d^2y}{dx^2} - 2\,\frac{dy}{dx} + y = 0\,. \tag{21.18.1}$$

- We employ an Euler substitution $y = e^{\alpha x}$ to obtain

$$\alpha^2 - 2\alpha + 1 = (\alpha - 1)^2 = 0\,. \tag{21.18.2}$$

- The solutions are $\alpha_1 = 1$ and $\alpha_2 = 1$, which are equal.

- Hence the solutions are

$$y_1(x) = e^x\,, \qquad y_2(x) = xe^x\,. \tag{21.18.3}$$

- Note that

$$\frac{dy_2}{dx} = (1+x)e^x\,, \qquad \frac{d^2y_2}{dx^2} = (2+x)e^x\,. \tag{21.18.4}$$

- Hence

$$\frac{d^2y}{dx^2} - 2\,\frac{dy}{dx} + y = (2+x)e^x - 2(1+x)e^x + xe^x = 0\,. \tag{21.18.5}$$

- The general solution is

$$y(x) = c_1e^x + c_2xe^x\,. \tag{21.18.6}$$

21.18.2 Inhomogeneous equation

- Next consider the equation

$$\frac{d^2y}{dx^2} - 2\,\frac{dy}{dx} + y = 1 - 2x + 2x^2\,. \tag{21.18.7}$$

- We try a particular solution $y = k_0 + k_1x + k_2x^2$, then

$$2k_2 - 2(k_1 + 2k_2x) + k_0 + k_1x + k_2x^2 = 1 - 2x + 2x^2\,. \tag{21.18.8}$$

- Equating the coefficient of x^2 yields $k_2 = 2$.

- Next, equating the coefficient of x yields $-4k_2 + k_1 = -2$, hence $k_1 = 6$.

- Next, equating the constant term yields $2k_2 - 2k_1 + k_0 = 1$, hence $k_0 = 9$.

- Hence the particular solution is

$$P(x) = 9 + 6x + 2x^2\,. \tag{21.18.9}$$

- The general solution is

$$y(x) = 9 + 6x + 2x^2 + c_1e^x + c_2xe^x\,. \tag{21.18.10}$$

21.19 Worked example 4

21.19.1 Homogeneous equation

- The equation is

$$\frac{d^2y}{dx^2} - 3\,\frac{dy}{dx} + 2y = 0\,. \tag{21.19.1}$$

- We saw in Sec. 21.17 that the general solution is

$$y(x) = c_1 e^x + c_2 e^{2x}\,. \tag{21.19.2}$$

21.19.2 Inhomogeneous equation

- Next consider the equation

$$\frac{d^2y}{dx^2} - 3\,\frac{dy}{dx} + 2y = e^{-x} - 2e^{3x}\,. \tag{21.19.3}$$

- First we try a particular solution $y = k_1 e^{-x}$. Then

$$(1 + 3 + 2)k_1 = 1\,. \tag{21.19.4}$$

- The solution is $k_1 = 1/6$.
- Next we try a particular solution $y = k_2 e^{3x}$. Then

$$(9 - 6 + 2)k_2 = -2\,. \tag{21.19.5}$$

- The solution is $k_2 = -2/5$.
- Hence the particular solution is

$$P(x) = \frac{e^{-x}}{6} - \frac{2e^{3x}}{5}\,. \tag{21.19.6}$$

- The general solution is

$$y(x) = \frac{e^{-x}}{6} - \frac{2e^{3x}}{5} + c_1 e^x + c_2 e^{2x}\,. \tag{21.19.7}$$

21.20 Worked example 5

- Consier again the equation

$$\frac{dy}{dx} + 2y = 1 - 2x + 2x^2 \,. \tag{21.20.1}$$

- We solve this using an integrating factor.
- Then $g(x) = 2$. Let us choose $x_0 = 0$. Then the integrating factor is

$$G(x) = \exp\biggl\{\int_0^x g(u)\,du\biggr\} = \exp\biggl\{\int_0^x 2\,du\biggr\} = e^{2x} \,. \tag{21.20.2}$$

- Then the solution is

$$y(x) = y_0 e^{-2x} + e^{-2x}\int_0^x e^{2w}(1 - 2w + 2w^2)\,dw \,. \tag{21.20.3}$$

- Let us evaluate the three integrals.
  1. First
  $$\int_0^x e^{2w}\,dw = \frac{e^{2x}-1}{2} \,. \tag{21.20.4}$$
  2. Next, integrating by parts,
  $$\begin{aligned}\int_0^x 2we^{2w}\,dw &= \Bigl[\,we^{2w}\Bigr]_0^x - \int_0^x e^{2w}\,dw \\ &= xe^{2x} - \frac{e^{2x}-1}{2} \,.\end{aligned} \tag{21.20.5}$$
  3. Next, integrating by parts,
  $$\begin{aligned}\int_0^x 2w^2e^{2w}\,dw &= \Bigl[\,w^2e^{2w}\Bigr]_0^x - \int_0^x 2we^{2w}\,dw \\ &= x^2e^{2x} - xe^{2x} + \frac{e^{2x}-1}{2} \,.\end{aligned} \tag{21.20.6}$$
- Hence overall we obtain

$$\begin{aligned}y(x) &= y_0e^{-2x} + e^{-2x}\biggl[\frac{e^{2x}-1}{2} - xe^{2x} + \frac{e^{2x}-1}{2} + x^2e^{2x} - xe^{2x} + \frac{e^{2x}-1}{2}\biggr] \\ &= y_0e^{-2x} + \frac{3}{2}(1-e^{-2x}) - 2x + x^2 \\ &= c\,e^{-2x} + \frac{3}{2} - 2x + x^2 \,.\end{aligned} \tag{21.20.7}$$

- The constant is $c = y_0 - \frac{3}{2}$.
- Hence the solution using an integrating factor agrees with that derived in Sec. 21.16.

21.21 Worked example 6

- The equation is

$$\frac{dy}{dx} + xy = x\,. \tag{21.21.1}$$

- We solve this using an integrating factor.
- Then $g(x) = x$. Let us choose $x_0 = 0$. Then the integrating factor is

$$G(x) = \exp\left\{\int_0^x g(u)\,du\right\} = \exp\left\{\int_0^x x\,du\right\} = e^{x^2/2}\,. \tag{21.21.2}$$

- The solution is

$$\begin{aligned} y(x) &= y_0 e^{-x^2/2} + e^{-x^2/2}\int_0^x e^{w^2/2} w\,dw\,. \\ &= y_0 e^{-x^2/2} + e^{-x^2/2}(e^{x^2/2} - 1) \\ &= y_0 e^{-x^2/2} + 1 - e^{-x^2/2} \\ &= ce^{-x^2/2} + 1\,. \end{aligned} \tag{21.21.3}$$

- The constant is $c = y_0 - 1$.