Queens College, CUNY, Department of Computer Science

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1 Lecture 1

1.1 Finance & 'Money'

- The fundamental assumption is that everything (all goods, services) have a cash value.
- Hence the question: ***What is money?***

1.1.1 'Money' is a functional definition

- ***Money is anything which can perform the functions that money does.***
- It must be possible to express the value of all goods and services in terms of money.
- It must be acceptable as a means of payment for goods and services.
- It must be durable (keeps its value over time).
- It must be difficult to forge or counterfeit (security).

1.1.2 Money $\neq$ cash

- The vast majority of financial transactions are electronic.
- Financial transactions involve electronic debiting and crediting of accounts.
- The volume of 'electronic money' far exceeds the value of paper money in circulation.

1.1.3 Legal tender

- US currency is 'legal tender' *in the United States*.
- That means vendors **must** accept US currency in payment for goods and services.
- Checks and credit cards can also be used for payment, but they are not legal tender.
- A vendor can legally refuse to accept checks and/or credit cards.
- Legal tender is enforced by law, to give validity to the currency.
- Example: US dollars are legal tender in USA, but not in UK.

1.2 Stocks & Bonds & FX

1.2.1 Derivatives and basic building blocks

- The word ‘derivative’ means an object which is derived from more basic building blocks.
- The value of a derivative instrument originates from the value of those building blocks.
- In this course, the basic building blocks of all financial securities are stocks and bonds.
- For international transactions, there is also FX (exchange rates).

1.2.2 Commodities

Question: ***What about commodities? (wheat, corn, oil and gas, etc.)***

- These also exist (and are important).
- In this course we shall focus on stocks and bonds.
- ***Problem:*** crops are eaten as food, oil and gas are burned as fuel → complication.

1.2.3 Bonds & cash

- Why ‘bonds’ and not ‘cash’ (or money)?
- Key concept: ***interest rates.***
- Bonds are intimately connected with interest rates.
- Money stored in a bank (for example), accrues interest and increases in value.
- The use of bonds captures the feature of interest rate compounding.

1.3 Time value of money

1.3.1 Cashflows at different times

- ****** Very important concept ******
- Because of interest rates, \$1 today is ***not*** worth the same as \$1 one year from now.
- Suppose the (continuously compounded) interest rate is r. Then

$$\$1_{\text{future}} = e^{r(t-t_0)}\,\$1_{\text{today}}\,. \tag{1.3.1.1}$$

- To compare cashflows, we must express them all at a common point in time.

1.3.2 Discount factor

- Let today $= t_0$.
- Suppose we have a cashflow CF at a time $t > t_0$ in the future.
- The value of the cashflow today is called the **present value (PV)** of CF.
- The present value is given by a ***discount factor***, say d:

$$\text{PV}(CF) = CF \times d\,. \tag{1.3.2.1}$$

- **In response to much confusion by students in homework, let me state clearly that**

$$\text{discount factor} = \frac{\text{value of cashflow today}}{\text{value of cashflow at future date}} = \frac{CF_{\text{today}}}{CF_{\text{future}}}\,. \tag{1.3.2.2}$$

- The relation between the discount factor and the (continuously compounded) interest rate is

$$d = e^{-r(t-t_0)}\,. \tag{1.3.2.3}$$

- Note: the discount factor depends on both the future time t and the present time t_0.
- We can define a discount factor $d(t_1, t_2)$ between any two times t_1 and t_2.
- The most common example is discounting to today.

1.3.3 Present value of multiple cashflows

- If we have multiple cashflows $CF_1, CF_2, \ldots, CF_n$ etc. at times $t_1, t_2, \ldots, t_n$, respectively, the present value of the total set of cashflows is the sum of the present value of each cashflow.
- Let the respective discount factors be $d_1, d_2, \ldots, d_n$.
- Then the present value is

$$\begin{aligned} \text{PV}_{\text{total}} &= CF_1\,d_1 + CF_2\,d_2 + \cdots + CF_n\,d_n \\ &= \sum_{i=1}^{n} CF_i\,d_i\,. \end{aligned} \tag{1.3.3.1}$$

1.3.4 Discount factors

This was written in response to much student confusion. The concept of discount factors is puzzling. We normally think that we have a sum of money today, deposited in a bank saving account, and it earns interest and grows in value as time moves forward. It is a bit different with discount factors. It helps to visualize things in the following way. Suppose I have two loans which I must repay. I owe \$100 to creditor A and I owe \$150 to creditor B. Both loans are due immediately. How much must I pay? Clearly the answer is the sum $\$100 + \$150 = \$250$. Now suppose I have two loans, but not due immediately. I owe \$100 to creditor C, payable in one year, and I owe \$150 to creditor D, payable in two years. I do *not* need to have \$250 in my bank account today, to repay those loans. Let us say, for convenience, that the interest rate is constant and $r = 5\%$. Hence \$1 in the bank today will compound to $\$e^{0.05\times1} \simeq \$1.051$ in one year and it will compound to $\$e^{0.05\times2} \simeq \$1.105$ in two years. Then to repay a loan of \$100 one year from now, the amount X of money I need in my bank account today is

$$X = \frac{100}{e^{0.05\times1}} = \frac{100}{e^{0.05}} \simeq 95.123\,. \tag{1.3.4.1}$$

The amount X will compound at 5% (continuous compounding) and will equal \$100 in one year. For creditor D, to repay a loan of \$150 two years from now, the amount Y of money I need in my bank account today is

$$Y = \frac{150}{e^{0.05\times2}} = \frac{150}{e^{0.1}} \simeq 135.726\,. \tag{1.3.4.2}$$

The amount Y will compound at 5% (continuous compounding) and will equal \$150 in two years. The total amount of money I need in my bank account today, to repay both loans, is the sum $X + Y$, which is

$$X + Y \simeq 230.849\,. \tag{1.3.4.3}$$

This is less than \$250. We express these ideas mathematically by speaking of present value and discount factors. We say the discount factor for one year is the inverse of the compounding factor $e^{0.05}$:

$$d_1 = e^{-0.05\times1} = e^{-0.05} \simeq 0.95123\,. \tag{1.3.4.4}$$

We say the discount factor for two years is the inverse of the compounding factor $e^{0.1}$:

$$d_2 = e^{-0.05\times2} = e^{-0.10} \simeq 0.90484\,. \tag{1.3.4.5}$$

We say that X is the "present value of \$100 one year in the future"

$$X = 100\,d_1 = 100\,e^{-0.05} \simeq 95.123\,. \tag{1.3.4.6}$$

We say that Y is the "present value of \$150 two year in the future"

$$Y = 150\,d_2 = 150\,e^{-0.1} \simeq 135.726\,. \tag{1.3.4.7}$$

Notice that discount factors go backwards in time. The discount factor d_1 goes *backwards* from 1 year to today. The discount factor d_2 goes *backwards* from 2 years to today. We are accustomed to visualizing money compounding forwards in time and increasing in value. But discount factors take the opposite view: A discount factor says: if I have to pay a known fixed amount of money at a given time in the future, how much money do I need in my bank account today to meet that expense?

1.3.5 Cashflows

This was written in response to more student confusion. Do not be frightened or confused by the fancy technical terms and mathematical formulas. In the end the concept of cashflows boils down to buying and selling stuff. We do it in our daily lives all the time. If we go to a supermarket and buy stuff, the total money we have to pay is the price of all the items + tax. They are all cashflows (including the tax). They are all money going out. They all happen at the same time, and we just add them up to get the total cashflow.

$$\text{Total payment} = \text{sum of cashflows.}$$

If we have to pay rent on an apartment (every month), and if we have a car and have to pay insurance premiums (every six months), those are all cashflows, but not at the same time. You do not need money in the bank right now to cover all of your expenses for the coming year. If you have a job, you will earn a salary. Your paychecks are cashflows coming in. Expenses are cashflows going out. So we have a series of cashflows, some positive, some negative. Typically, for us, the time horizons are small enough (a few months) and the amounts of money are small enough that we do not think in terms of "discounting" and "present value" of income amd expenses. But if we have money in a savings account, it does compound and earn interest and that is extra money for us. We do this budgeting without using fancy words and mathematical formulas. But ultimately, it is "money coming in" and "money going out" and that is what cashflows are. With large sums of money and longer time intervals, especially for businesses which have to borrow (loans) for capital expansion projects, etc., then they do calculate the "present value" etc. They have millions (billions?) coming in and going out, and they need to know how much things are worth *today*, to make business decisions. Especially if they need to borrow millions of dollars in loans (so they have to pay back a lot of interest on the loan), they need to calculate how much to borrow. If they borrow too little they have not enough money for their needs and if they borrow too much they have to pay back a lot of interest. So they calculate the present values of their cashflows (positive = income or revenue, negative = expenses). But it is still money coming in and going out. The overall present value of the total set of cashflows is the sum of the present values of the individual cashflows. For that we need some mathematics. But do not be frightened or confused by it.

1.3.6 Forward discount factors and forward rates

- Discount factors do not always have to go back from a future date to today.
- It is possible to discount from a future time t_2 to an earlier time t_1.
- Let the time today be t_0.
- The discount factor from t_1 to today is d_1. Technically it is $d_1 = d(t_1, t_0)$.
- The discount factor from t_2 to today is d_2. Technically it is $d_2 = d(t_2, t_0)$.
- Instead of discounting from t_2 to t_0, we can discount in two steps from t_2 to t_1 and t_1 to t_0.
- With an obvious notation, this means

$$d_2 = d(t_2, t_1) d_1 \,. \tag{1.3.6.1}$$

- Here $d(t_2, t_1)$ is the **forward discount factor** from t_2 to t_1. Obviously

$$d(t_2, t_1) = \frac{d_2}{d_1} \,. \tag{1.3.6.2}$$

- Equally obviously, there is a forward interest rate $r_{\rm fwd}$, given by the formula

$$d(t_2, t_1) = e^{-r_{\rm fwd}(t_2 - t_1)} \,. \tag{1.3.6.3}$$

- We shall not really deal with forward rates and discount factors in this class.
- But they do exist and are important to some people.
- There is a simple formula for the forward rate in terms of the interest rates (r_1, r_2) at (t_1, t_2).

  1. Using $d_1 = e^{-r_1 t_1}$ and $d_2 = e^{-r_2 t_2}$ we obtain

$$e^{-r_{\rm fwd}(t_2 - t_1)} = d(t_2, t_1) = \frac{d_2}{d_1} = \frac{e^{-r_2 t_2}}{e^{-r_1 t_1}} \,. \tag{1.3.6.4}$$

  2. Taking logarithms (and cancelling lots of minus signs) we obtain

$$r_{\rm fwd}(t_2 - t_1) = r_2 t_2 - r_1 t_1 \,. \tag{1.3.6.5}$$

  3. Hence we obtain the simple formula

$$r_{\rm fwd} = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} \,. \tag{1.3.6.6}$$

  4. Notice that if r_1 and r_2 are equal, say $r_1 = r_2 = r$, then $r_{\rm fwd} = r$ also.

1.4 Bonds and coupons

1.4.1 Bonds

- Bonds encapsulate the concept of borrowing money (and repaying with interest).
- The US govt borrows money by issuing US Treasury bonds.
- People buy the bonds (= loan money to US govt).
- The US govt pays back the money at a later time, with interest.
- Local govts do the same (municipal bonds) also companies (corporate bonds).

1.4.2 Coupons

- Bonds are issued with a maturity date (5 years, 10 years, 30 years, etc).
- On the maturity date, a bond repays its face value (usually $10,000 in the USA).
- In addition, a bond pays cashflows called **coupons**.
- In the USA, bonds typically pay coupons semi-annually (twice a year).

1.5 What does "coupon" mean?

A student asked an important question: ***why are the payments called "coupons"?***

The word "coupon" suggests a piece of paper that one presents to get a discount in sales at a shop.

- **That is correct.**
- **Originally, bonds were paper certificates with actual paper coupons attached.**
- The owner of a bond would tear off a coupon and redeem it for money.
- *So yes! "Coupons" were pieces of paper with redemption dates on them.*
- Today we use the word "coupon" to denote a cashflow from a bond.

***** I encourage such astute questions from students *****

1.6 Bonds, yields & interest rates

1.6.1 Yield

- Suppose we have a bond, whose price is B.
- To keep the calculations simple, suppose it is a newly issued bond (today).
- Let the bond have a face value F.
- **We shall always set $F = 100$ in these lectures.**
- Let the annualized coupon rate be c.
- For now c is a constant. We shall treat variable rate coupons later.
- Suppse the coupons are paid semi-annually (two coupons per year).
- Suppose there are totally n cashflows.
- The intermediate cashflows pay $c/2$ each and the final cashflow is $F + (c/2)$.
- The (annualized) **yield** of the bond, call it y, is defined via the formula

$$B = \frac{\frac{1}{2}c}{1+\frac{1}{2}y} + \frac{\frac{1}{2}c}{(1+\frac{1}{2}y)^2} + \cdots + \frac{\frac{1}{2}c}{(1+\frac{1}{2}y)^{n-1}} + \frac{F+\frac{1}{2}c}{(1+\frac{1}{2}y)^n}\,. \tag{1.6.1.1}$$

1.6.2 Yield and discount factors

- Why do all the discount factors have the form $1/(1+\frac{1}{2}y)$, $1/(1+\frac{1}{2}y)^2$, $1/(1+\frac{1}{2}y)^3$, etc.?
- ****** This is an important question, an item for future lectures ******

1.6.3 How do we know the value of the yield of a bond?

- **In practice it is the other way around.**
- We observe the market price of the bond B (from trading in the financial markets).
- We **invert** the above formula to calculate the yield.

1.6.4 Yield curve

- There are many bonds in the financial markets, with different yields and maturities.
- From these we obtain a **"yield curve"** (for example the US Treasury yield curve).
- From the yield curve, *we back out the interest rates and discount factors.*
- ****** This is a very important procedure in finance ******
- ****** It is an item for future lectures ******

1.7 Zero coupon bonds

- There is a very important class of bonds called **zero coupon bonds.**
- **Zero coupon bonds are very popular and important financial instruments.**
- As the name suggests, a zero coupon bonds pays no coupons. There is only one cashflow, which is to pay the face F at the maturity T.
- Let us set $c = 0$ in eq. (1.6.1.1). Then the price of a newly issued zero coupon bond is

$$B = \frac{F}{(1 + \frac{1}{2}y)^n}\,. \tag{1.7.1}$$

- **Notice a peculiarity in eq. (1.7.1):**
  1. We still have the factor of $\frac{1}{2}$ in the denominator $1 + \frac{1}{2}y$ and we still have $n = 2T$.
  2. ***Why is this?***
- Logically, a zero coupon has no "coupon frequency" because there are no coupons. Nevertheless, for accountng purposes, the yields of zero coupon bonds are calculated with a semiannual frequency (in the USA).
- Look at eq. (1.6.1.1) again. If c is a very small number (but not zero), we would calculate the yield with a semiannual frequency. Then if $c = 0$, there would be a disconnect if we switched to a different convention for the yield.
- So we employ the same frequency (= semiannual in USA) to quote the yields of all bonds (zero and nonzero coupons).
- Since $y \geq 0$, obviously $B \leq F$. This makes sense. If we receive cash F at a future time T, then to obtain a positive return on investment we would pay less than F today, to buy a zero coupon bond.
- The yield of a zero coupon bond is easy to calculate from the market price. For a newly issued zero coupon bond, we invert eq. (1.7.1) to obtain

$$\begin{aligned} 1 + \tfrac{1}{2}y &= (F/B_{\text{market}})^{1/n}\,, \\ y &= 2\left[(F/B_{\text{market}})^{1/n} - 1\right]. \end{aligned} \tag{1.7.2}$$

- The discount factor of the face value F of a zero coupon bond is

$$d = \frac{B}{F} = \frac{1}{(1 + \frac{1}{2}y)^n}\,. \tag{1.7.3}$$

- The associated interest rate r is (in decimal)

$$r = -\frac{\ln(d)}{T - t_0} = -\frac{\ln[1/(1 + \frac{1}{2}y)^n]}{n/2} = \frac{n\ln(1 + \frac{1}{2}y)}{n/2} = 2\ln(1 + \tfrac{1}{2}y)\,. \tag{1.7.4}$$

1.8 Worked examples: discount factor & interest rate

- **Example: Calculate the discount factor given the interest rate and time.**

  1. Inputs $t_0 = 0$, $t = 1.5$ and $r = 8\%$. Therefore decimal $r = 0.08$.
  2. The value of the discount factor is

  $$d = e^{-r(t-t_0)} = e^{-0.08\times 1.5} = e^{-0.12} \simeq 0.88692\,. \tag{1.8.1}$$

- **Example: Calculate the interest rate given the discount factor and time.**

  1. Inputs $t_0 = 0$, $t = 0.8$ and $d = 0.95$.
  2. The value of the interest rate in decimal is

  $$r = -\frac{\ln(d)}{t - t_0} = -\frac{\ln(0.95)}{0.8} \simeq \frac{0.051293}{0.8} \simeq 0.064117\,. \tag{1.8.2}$$

  3. Therefore in percent $r \simeq 6.41\,\%$ (two decimal places).

- **Example: Given the value of cashflows at present and future times. Calculate the discount factor.**

  1. Inputs $\mathrm{CF}_{\mathrm{today}} = 93.456$ and $\mathrm{CF}_{\mathrm{future}} = 124.567$.
  2. The value of the discount factor is

  $$d = \frac{\mathrm{CF}_{\mathrm{today}}}{\mathrm{CF}_{\mathrm{future}}} = \frac{93.456}{124.567} \simeq 0.750247\,. \tag{1.8.3}$$

  3. ***It really is as simple as that!***

- **Followup to previous example: Calculate the associated interest rate.**

  1. For this we need to know the dates of the cashflows.
  2. Inputs $t_0 = 0$ and $t_{\mathrm{future}} = 2.5$.
  3. The value of the interest rate in decimal is

  $$r = -\frac{\ln(d)}{t_{\mathrm{future}} - t_0} = -\frac{\ln(0.750247)}{2.5} \simeq \frac{0.287353}{2.5} \simeq 0.114941\,. \tag{1.8.4}$$

  4. Therefore in percent $r \simeq 11.49\,\%$ (two decimal places).

1.9 Worked examples: bond & yield

- **Example: Calculate the bond price given the yield.**

1. Inputs: newly issued bond, face $F = 100$, annualized coupon rate $c = 5$, maturity $T = 1.5$ years, semiannual coupons.
2. Hence $n = 3$ cashflows, at times $t_1 = 0.5$, $t_2 = 1.0$ and $t_3 = 1.5$.
3. **Input: yield $y = 3\%$ (decimal $y = 0.03$).** The bond price is

$$\begin{aligned} B &= \frac{\frac{1}{2}c}{1+\frac{1}{2}y} + \frac{\frac{1}{2}c}{(1+\frac{1}{2}y)^2} + \frac{F+\frac{1}{2}c}{(1+\frac{1}{2}y)^3} \\ &= \frac{2.5}{1+0.015} + \frac{2.5}{(1+0.015)^2} + \frac{100+2.5}{(1+0.015)^3} \\ &\simeq 102.9122\,. \end{aligned} \tag{1.9.1}$$

4. **Input: yield $y = 5\%$ (decimal $y = 0.05$).** The bond price is

$$\begin{aligned} B &= \frac{\frac{1}{2}c}{1+\frac{1}{2}y} + \frac{\frac{1}{2}c}{(1+\frac{1}{2}y)^2} + \frac{F+\frac{1}{2}c}{(1+\frac{1}{2}y)^3} \\ &= \frac{2.5}{1+0.025} + \frac{2.5}{(1+0.025)^2} + \frac{100+2.5}{(1+0.025)^3} \\ &\simeq 100\,. \end{aligned} \tag{1.9.2}$$

5. **Input: yield $y = 6\%$ (decimal $y = 0.06$).** The bond price is

$$\begin{aligned} B &= \frac{\frac{1}{2}c}{1+\frac{1}{2}y} + \frac{\frac{1}{2}c}{(1+\frac{1}{2}y)^2} + \frac{F+\frac{1}{2}c}{(1+\frac{1}{2}y)^3} \\ &= \frac{2.5}{1+0.03} + \frac{2.5}{(1+0.03)^2} + \frac{100+2.5}{(1+0.03)^3} \\ &\simeq 98.58569\,. \end{aligned} \tag{1.9.3}$$

6. **As the yield increases, the bond price decreases.**
7. **For newly issued bonds, $B = 100$ if the yield equals the annualized coupon rate.**

1.10 Worked examples: zero coupon bond

- **Example: Calculate the bond price given the yield.**

  1. Inputs: newly issued bond, face $F = 100$, maturity $T = 2.5$ years. Then $n = 2T = 5$.
  2. **Input: yield $y = 5\%$ (decimal $y = 0.05$).** The bond price is

$$B = \frac{F}{(1+\frac{1}{2}y)^n} = \frac{100}{(1+0.025)^5} \simeq 88.38543\,. \tag{1.10.1}$$

- **Example: Calculate the yield given the bond price.**

  1. Inputs: newly issued bond, face $F = 100$, maturity $T = 5$ years. Then $n = 2T = 10$.
  2. **Input: bnd price = 80.** The yield is (decimal)

$$y = 2\left[(F/B_{\rm market})^{1/n} - 1\right] = 2\left[(100/80)^{1/10} - 1\right] \simeq 0.04513\,. \tag{1.10.2}$$

  3. Therefore in percent the yield is $y \simeq 4.51\,\%$ (two decimal places).
  4. The **discount factor** is *(this should be easy)*

$$d = \frac{B_{\rm market}}{F} = \frac{80}{100} = 0.8\,. \tag{1.10.3}$$

  5. The associated **interest rate** is (decimal)

$$r = -\frac{\ln(d)}{T - t_0} = -\frac{\ln(0.8)}{5} \simeq 0.044629\,. \tag{1.10.4}$$

  6. Therefore in percent $r \simeq 4.46\,\%$ (two decimal places).

1.11 Worked examples: calculate yield given the price of a coupon bond

This is a more complicated calculation. Worked examples will be displayed in Lecture 2.