Queens College, CUNY, Department of Computer Science
Computational Finance
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Instructor: Dr. Sateesh Mane

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4b Homework: Stock index and continuous dividend yield

4.4 Arbitrage proof for stock index with continuous dividend yield

- It was stated in the lectures that the use of a continuous dividend yield is a good approximation for a stock index.
- We shall formulate an arbitrage argument to prove this.
- The time today is t_0.
- Let the value of the stock index be denoted by I.
- The index value is given by the weighted average of a basket of stocks.
- For simplicity we say that one index point corresponds to one dollar of the basket. Recall that forward and futures contracts on stock indices are cash settled. When a forward or futures contract on the index is settled, there is a dollar multiplier of \$1 for every index point.
- The price of the basket is denoted by S, and equals S_0 at time t_0.
- Hence by definition the index value at t_0 is $I_0 = S_0$.
- The stocks in the basket pay dividends, at all different dates.
- We approximate that the overall effect of the dividends is given by a continuous dividend yield q (a constant).
- Let the continuously compounded interest rate be r (a constant).
- Consider a forward contract on the index, with expiration T and a forward price F.
- We claim that the fair value of the forward price F is given by

$$F_{\text{fair value}} = S_0\, e^{(r-q)(T-t_0)}\,. \tag{4.4.1}$$

- We prove eq. —eqrefeq:fwdfairvalue using arbitrage arguments.
- **Formulate an arbitrage strategy to make a guaranteed profit if the forward contract is overpriced:**

$$F_{\text{index}} > S_0\, e^{(r-q)(T-t_0)}\,. \tag{4.4.2}$$

- **Formulate an arbitrage strategy to make a guaranteed profit if the forward contract is underpriced:**

$$F_{\text{index}} < S_0\, e^{(r-q)(T-t_0)}\,. \tag{4.4.3}$$

- To formulate the arbitrage trades, we need to know the following important information.
  1. The forward on the index is cash settled.
  2. At the expiration time T, the investor with a **long** forward position **receives cash in the amount $I_T - F$.**
  3. At the expiration time T, the investor with a **short** forward position **pays cash in the amount $F - I_T$.**

- To formulate the arbitrade trades, we need to go long/short the basket of stocks.
- Suppose we go long the basket at time t_0.
  1. Then we must borrow cash in the amount of S_0.
  2. This is a loan from a bank, and the loan amount will compound at the interest rate r.
  3. Because we are **long the basket,** we **receive dividends (= cash)** in the time interval $0 \le t \le T$.
  4. **We use the cash from the dividends to repay (part of) the loan. This reduces the amount of the loan.**
  5. Hence the total amount of loan to repay at the expiration time T is given by
$$(\text{loan amount at expiration}) = S_0\, e^{(r-q)(T-t_0)}\,. \tag{4.4.4}$$
  6. **You do not have to derive or prove eq. (4.4.4). It is given to you as fact.**
- Suppose we go short the basket at time t_0.
  1. Then we receive cash in the amount of S_0.
  2. We save the cash in a bank, and the saved amount will compound at the interest rate r.
  3. Because we are **short the basket,** we **must pay dividends (= cash)** to the person to whom we short sold the basket.
  4. These are cash payments in the time interval $0 \le t \le T$.
  5. **We use the cash in the bank to pay the dividends. This reduces the amount of our savings.**
  6. Hence the total amount of savings in the bank at the expiration time T is given by
$$(\text{savings amount at expiration}) = S_0\, e^{(r-q)(T-t_0)}\,. \tag{4.4.5}$$
  7. **You do not have to derive or prove eq. (4.4.5). It is given to you as fact.**
- **At the expiration time T, by definition the index value is $I_T = S_T$.**
  1. Since the forward contract is cash settled, you must close out your long/short basket position, to obtain all cash at the end.
  2. You must demonstrate that the total cash at the end is positive (= guaranteed profit).
  3. If you are **long** the basket at expiration, **sell the basket** and receive cash $= S_T = I_T$. Use the cash as part of your trades, to demonstrate an overall guaranteed profit.
  4. If you are **short** the basket, **use cash to buy the basket at the price** S_T and close out your short basket position. Therefore you must have enough cash available to close out your short basket position. You must demonstrate that overall you have positive cash at the end to obtain a guaranteed profit.