Queens College, CUNY, Department of Computer Science

Computational Finance

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14 Lecture 14

Derivation of the Black-Scholes equation

- In this lecture we derive the Black-Scholes partial differential equation.
- **The content of this lecture involves probability theory.**
- **The content of this lecture is not for examination.**

14.1 Random variables and Geometric Brownian Motion

- Up to now we have denoted the stock price by S. It is a real valued variable which takes positive values $S > 0$ (and $S = 0$ means the company is bankrupt and out of business.) We have written formulas involving S, including partial differential equations.

  1. Obviously the "S" which appears in the Black-Scholes-Merton equation, or in the associated formula for the fair value of a European call or put, is *not* a random variable.
  2. For example, the Delta of a financial derivative with fair value $V(S,t)$ is defined as the partial derivative
  $$\Delta = \frac{\partial V}{\partial S}\,. \tag{14.1.1}$$
  3. There is no "random variable" in the above formula for Delta.

- Hence to avoid confusion, let us employ the notation S^r to denote a stock price which is a random variable.

- According to stochastic calculus, suppose the value of S^r changes by a small "infinitesimal" amount dS^r in an "infinitesimal" time interval dt.

- Then, if the random variable S^r ***obeys geometric Brownian motion,*** we have the following expression
$$dS^r = \mu S^r\,dt + \sigma S^r\,dW_t\,. \tag{14.1.2}$$

- The equation eq. (14.1.2) is called a **stochastic differential equation.**

- Here μ is the growth rate of the stock, and is a constant (not random) and is not important.

- More important is dW_t. What is dW_t or what is W_t? That we must know.

- In probability theory, W_t is a so-called **Wiener process** (named after Norbert Wiener).

- A Wiener process W_t is a random walk (technically, a **stochastic process**) with the following properties.

  1. A Wiener process W_t is a **continuous function of t.**
  2. In an infinitesimal time interval dt, the value of W_t changes by a random value dW_t.
  3. The value of dW_t is **a random variable with a normal (Gaussian) distribution, with mean zero and variance dt.**
  4. Hence in an infinitesimal time interval dt,
  $$\mathbb{E}[dW_t] = 0\,, \qquad \mathbb{E}[(dW_t)^2] = dt\,. \tag{14.1.3}$$
  5. The increments dW_{t_1} and dW_{t_2} in two non-overlapping time intervals dt_1 and dt_2 are **independent.** Hence
  $$\mathbb{E}[dW_{t_1} dW_{t_2}] = \mathbb{E}[dW_{t_1}]\,\mathbb{E}[dW_{t_2}] = 0\,. \tag{14.1.4}$$
  6. Since the value of dt is infinitesimal, we in fact write the following (because the corrections are negligible)
  $$(dW_t)^2 = dt\,. \tag{14.1.5}$$

14.2 Random change in value of derivative $V(S,t)$

- Suppose we have a derivative $V(S,t)$ on a stock S (and t is the time).
- To begin with, we assume the stock does not pay dividends.
- In a small time interval δt, we can write the following Taylor series

$$V(S+\delta S, t+\delta t) = V(S,t) + \frac{\partial V}{\partial t}\,\delta t + \frac{\partial V}{\partial S}\,\delta S + \frac{1}{2}\,\frac{\partial^2 V}{\partial S^2}\,(\delta S)^2 + \cdots \tag{14.2.1}$$

- We now say the following: ***let the change δS be a random variable.***
- Furthermore, let the random walk for the stock price obey geometric Brownian motion.
- Replace the "δt" by an infinitesimal dt and replace δS by dS^r in eq. (14.2.1).
- Then the change in the value of V, say dV, is given by

$$\begin{aligned} dV(S,t) &= \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,dS^r + \frac{1}{2}\,\frac{\partial^2 V}{\partial S^2}\,(dS^r)^2 + \cdots \\ &= \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,(\mu S^r\,dt + \sigma S^r\,dW_t) + \frac{1}{2}\,\frac{\partial^2 V}{\partial S^2}\,\sigma^2 (S^r)^2 (dW_t)^2 + \cdots \\ &= \left(\frac{\partial V}{\partial t} + \mu S^r\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 (S^r)^2\,\frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S^r\,\frac{\partial V}{\partial S}\,dW_t + \cdots \end{aligned} \tag{14.2.2}$$

- Since the time interval dt is very short (infinitesimal), we neglect all the higher order terms in eq. (14.2.2).
- We also replace the value of S^r by S in eq. (14.2.2). We know that $S^r = S$ at the time t.
- ***What we do not know is the value of dW_t,*** which is a random variable.
- Hence we can write, neglecting the higher order terms,

$$dV(S,t) = \left(\frac{\partial V}{\partial t} + \mu S\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S\,\frac{\partial V}{\partial S}\,dW_t\,. \tag{14.2.3}$$

- Then eq. (14.2.3) is also a stochastic differential equation, for V.
- **The only random term in eq. (14.2.3) is the last term, in dW_t.**
  1. The random term in dW_t in eq. (14.2.3) represents **risk.**
  2. We do not know its value, hence we do not know how the value of V will change.
  3. That poses a difficulty, how to calculate a fair value for the derivative (a formula for V).

14.3 Cancellation of risk: Delta hedging

- Note that the risky term in eq. (14.2.3) is proportional to $\partial V/\partial S$, which is the value of the Delta of the derivative.
- Hence let us form a portfolio $U(S,t)$ consisting of long V and short a number Delta of shares:

$$U(S,t) = V(S,t) - S\Delta\,. \tag{14.3.1}$$

- Then using eq. (14.2.3) and eq. (14.1.2),

$$\begin{aligned} dU(S,t) &= dV(S,t) - \Delta\, dS^r \\ &= \left(\frac{\partial V}{\partial t} + \mu S\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S\,\frac{\partial V}{\partial S}\,dW_t - \Delta\,(\mu S\,dt + \sigma S\,dW_t) \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2}\right) dt\,. \end{aligned} \tag{14.3.2}$$

- ***The term in dW_t cancels out. There is no risky term in eq.* (14.3.2).**
- This procedure is called **hedging**.
- **Hedging** is the procedure of **reducing the riskiness of a portfolio.**
    1. The risk in this case is proportional to the Delta of the derivative.
    2. Hence if we are long one derivative, we hedge our risk by selling Delta shares of stock.
    3. Hence the above procedure is called **Delta hedging.**
- We now come to a key observation:
    1. Since the change in U has no risk, **the value of U grows at the risk-free rate**.
    2. Mathematically, this means $dU/dt = rU$ at the time t.
    3. Expressed in infinitesimals, we have

$$dU(S,t) = rU\,dt\,. \tag{14.3.3}$$

- Equating the two expressions for dU in eqs. (14.3.2) and (14.3.3) yields

$$r(V - S\Delta) = \frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2}\,. \tag{14.3.4}$$

- Rearrange terms and write $\Delta = \partial V/\partial S$ to obtain a partial differential equation for V:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + rS\,\frac{\partial V}{\partial S} - rV = 0\,. \tag{14.3.5}$$

- This is the Black-Scholes equation.

14.4 Dynamic hedging

- Note that in general the value of $\partial V/\partial S$ is not a constant.
- Hence the hedge must be continuously updated, to keep the portfolio U riskless.
- This procedure is called **dynamic hedging.**
- In **static hedging,** the hedge does not change with time.

14.4.1 Technical subtlety

- The fact that $\partial V/\partial S$ is not a constant in time raises a technical subtlety.
- Recall that it was stated above that to eliminate the risk in the portfolio U, it was necessary to short a number Delta of shares (see eq. (14.3.1)).
- Now by definition $\Delta = \partial V/\partial S$.
- This raises the point: when it was stated that the portfolio U grows at the risk-free rate r (see eq. (14.3.3)), then **the value of Δ should change in the time interval dt also.**
- However, the correct formulation is to say that we hedge by shorting a number Delta of shares, and this number ***does not change*** in the time interval dt. It is only the values of S and t, and therefore V, which change.
- The technical formulation is that the portfolio U is assumed to be **self-financing.**
- This is a detail of stochastic calculus which is beyond the scope of these lectures to justify.

14.5 Black-Scholes-Merton equation

- Extra care is required to derive the Black-Scholes-Merton equation, to include the contribution from the stock dividends in the time interval dt in eq. (14.3.3).

- If the stock pays continuous dividends with a yield q, then the growth of the riskless portfolio U in the infinitesimal time interval dt is

$$dU(S,t) = rU\,dt + qS\Delta\,dt\,. \tag{14.5.1}$$

- This yields the Black-Scholes-Merton equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + (r-q)S\,\frac{\partial V}{\partial S} - rV = 0\,. \tag{14.5.2}$$

14.6 Absence of unobservable parameters

- Notice that the **value of μ cancelled out** in eq. (14.3.2) and μ does not appear in the Black-Scholes equation eq. (14.3.5).
- Historically, this was the key breakthrough by Black and Scholes.
- The value of μ is the expected rate of return on the stock. Its value is essentially unobservable.
  1. For example, investor A might believe the stock is risky, and will demand a high rate of return (large value of μ) to agree to buy the stock.
  2. On the other hand, investor B might believe the stock is a safe investment, and will agree to buy it for a much lower rate of return (small value of μ).
  3. Hence the value of μ depends on the risk preferences of individual investors.
- If the value of V depended on μ, it would be impossible to derive a fair value for V that all investors could agree on.
- This was a major difficulty with option pricing theory in the 1960s and earlier.
- Black and Scholes succeeded in deriving an equation for valuing derivatives which did *not* contain parameters that depend on individual investors.
- Hence Black and Scholes were able to calculate a fair value for options, which all investors could agree on.

14.7 Delta hedging using futures

- **The material in this section will be repeated in other lectures because it is important and *will be tested in examinations.***
- **The overall procedure of Delta hedging will be repeated in other lectures because it is important and *will be tested in examinations.***
- It is perfectly possible to Delta hedge an option (or any other derivative on a stock) using futures.
- Options traders do this all the time.
- If the stock does not pay dividends, the fair value of the futures is

$$F = Se^{r(T-t)}\,. \tag{14.7.1}$$

- The Delta of the futures is

$$\Delta_{\text{futures}} = e^{r(T-t)}\,. \tag{14.7.2}$$

- Hence the number of futures contracts to short to Delta hedge the derivative is

$$N_{\text{fut}} = \frac{\Delta_{\text{derivative}}}{\Delta_{\text{futures}}} = e^{-r(T-t)}\,\frac{\partial V}{\partial S}\,. \tag{14.7.3}$$

- It is also possible to use a futures contract whose expiration is not the same as the option.
  1. This can happen if there is no suitable exchange listed futures contract.
  2. Suppose the futures expiration time is T', where $T' \neq T$.
  3. Then the number of futures contracts to short to Delta hedge the derivative is

$$N_{\text{fut}} = \frac{\Delta_{\text{derivative}}}{\Delta_{\text{futures}}} = e^{-r(T'-t)}\,\frac{\partial V}{\partial S}\,. \tag{14.7.4}$$

  4. This is also done, in practice.