Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Fall 2018

Instructor: Dr. Sateesh Mane

15 Homework lecture 15

- Please email your solution, as a file attachment, to `Sateesh.Mane@qc.cuny.edu`.
- Please submit one zip archive with all your files in it.
  1. The zip archive should have either of the names (CS361 or CS761):

     `StudentId_first_last_CS361_hw15.zip`

     `StudentId_first_last_CS761_hw15.zip`
  2. The archive should contain one "text file" named "hw15.[txt/docx/pdf]" and one cpp file per question named "Q1.cpp" and "Q2.cpp" etc.
  3. Note that not all homework assignments may require a text file.
  4. Note that not all questions may require a cpp file.

15.1 Euler integration 1: explicit

15.1.1 Forward integration

- You are given the following differential equation (c is a positive constant)

$$\frac{dy}{dx} = x - cy\,. \tag{15.1.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 1$.
- You are given that the exact solution $y^{\rm ex}(x)$ is as follows.

$$y^{\rm ex}(x) = \left(1 + \frac{1}{c^2}\right) e^{-cx} + \frac{x}{c} - \frac{1}{c^2}\,. \tag{15.1.2}$$

- **You do NOT have to derive the above solution (but you can if you want to).**
- The integration stepsize is h (a constant), hence $x_i = ih$.
- **Use the explicit Euler method to derive the following.**

$$y_{i+1} = y_i + h(x_i - cy_i) = y_i(1 - hc) + hx_i\,. \tag{15.1.3}$$

- Set $h = 1/n$.
- **Show that $x_n = 1$ after n steps.**
- Set $c = 2$.
- **Compute the value of the exact solution $y^{\rm ex}(x)$ for $x = 1$ and $c = 2$.**
- **Compute the value of the numerical solution y_n at $x_n = 1$ and fill the following table of values.**
  1. **Compute the values of $y^{\rm ex}(1)$ and y_n to 5 decimal places.**
  2. **In the last column, calculate the value of $n(y^{\rm ex}(1) - y_n)$ to 2 decimal places.**

n	$y^{\rm ex}(1)$	y_n	$y^{\rm ex}(1) - y_n$	$n(y^{\rm ex}(1) - y_n)$
10	5 d.p.	5 d.p.	5 d.p.	2 d.p.
100	5 d.p.	5 d.p.	5 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.1.2 Reverse integration

- **We now reverse the direction of integration and integrate backwards to $x_0 = 0$.**
- The differential equation is therefore

$$\frac{dy}{dx} = -(x - cy)\,. \tag{15.1.4}$$

- The integration stepsize is still $h = 1/n$.
- Let the steps be indexed by $i = n+1, \dots, 2n$.
- You are given that now

$$x_i = x_n - (i-n)h = 2 - ih \qquad (i = n+1, \dots, 2n)\,. \tag{15.1.5}$$

- The initial condition is $y = y_n$ **(the computed numerical solution)** at $i = n$.
- **Use the explicit Euler method to derive the following.**

$$\begin{aligned} y_{i+1} &= y_i - h(x_i - cy_i) \\ &= y_i(1 + hc) - hx_i\,. \end{aligned} \tag{15.1.6}$$

- **Show that $x_{2n} = 0$ after $2n$ steps.**
- The exact solution will return to the initial value $y_0 = 1$ but the numerical solution will not.
- **Compute the value of the numerical solution y_{2n} at $x_{2n} = 0$ and fill the following table of values.**
  1. **Compute the value of y_{2n} to 5 decimal places.**
  2. **In the last column, calculate the value of $n(1 - y_{2n})$ to 2 decimal places.**

$2n$	$y^{\text{ex}}(0)$	y_{2n}	$1 - y_{2n}$	$n(1 - y_{2n})$
20	1	5 d.p.	5 d.p.	2 d.p.
200	1	5 d.p.	5 d.p.	2 d.p.
2000	1	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.1.3 Indexing for forward and reverse integration

- **There seems to be confusion about the notation and indexing, especially for the reverse integration.**
- **Let me use $n = 10$ as an example. Then $h = 1/n = 0.1$.**
- **Then the value of x_i is, for $i = 0, \dots, n, \dots 2n$:**

$$\begin{aligned}
x_0 &= 0\,, \\
x_1 &= 0.1\,, \\
x_2 &= 0.2\,, \\
&\vdots \\
x_9 &= 0.9\,, \\
x_{10} &= 1.0\,, \\
\boldsymbol{x_{11}} &= \mathbf{0.9}\,, \\
\boldsymbol{x_{12}} &= \mathbf{0.8}\,, \\
&\vdots \\
\boldsymbol{x_{19}} &= \mathbf{0.1}\,, \\
\boldsymbol{x_{20}} &= \mathbf{0}\,.
\end{aligned} \tag{15.1.7}$$

- **For the *forward integration*, the loop goes from $i = 0$ through $i = 9$ (ten equations)**

$$i = 0: \qquad y_1 = y_0(1 - hc) + hx_0 \qquad = 1(1 - hc) + h \times 0\,, \tag{15.1.8a}$$

$$i = 1: \qquad y_2 = y_1(1 - hc) + hx_1 \qquad = y_1(1 - hc) + h \times 0.1\,, \tag{15.1.8b}$$

$$i = 2: \qquad y_3 = y_2(1 - hc) + hx_2 \qquad = y_2(1 - hc) + h \times 0.2\,, \tag{15.1.8c}$$

$$\vdots \tag{15.1.8d}$$

$$\boldsymbol{i = 9}: \qquad \boldsymbol{y_n} = y_{10} = y_9(1 - hc) + hx_9 \qquad = y_9(1 - hc) + h \times 0.9\,. \tag{15.1.8e}$$

- **For the *reverse integration*, the loop goes from $i = 10$ through $i = 19$ (ten equations) *We simply change sign $h \to -h$, so $h = -0.1$.***

$$\boldsymbol{i = 10}: \qquad y_{11} = y_{10}(1 + hc) - hx_{10} \qquad = \boldsymbol{y_{10}}(1 + hc) - h \times \mathbf{1.0}\,, \tag{15.1.9a}$$

$$\boldsymbol{i = 11}: \qquad y_{12} = y_{11}(1 + hc) - hx_{11} \qquad = \boldsymbol{y_{11}}(1 + hc) - h \times \mathbf{0.9}\,, \tag{15.1.9b}$$

$$\boldsymbol{i = 12}: \qquad y_{13} = y_{12}(1 + hc) - hx_{12} \qquad = \boldsymbol{y_{12}}(1 + hc) - h \times \mathbf{0.8}\,, \tag{15.1.9c}$$

$$\vdots \tag{15.1.9d}$$

$$\boldsymbol{i = 18}: \qquad y_{19} = y_{18}(1 + hc) - hx_{18} \qquad = \boldsymbol{y_{18}}(1 + hc) - h \times \mathbf{0.2}\,, \tag{15.1.9e}$$

$$\boldsymbol{i = 19}: \qquad \boldsymbol{y_{2n}} = y_{20} = y_{19}(1 + hc) - hx_{19} \qquad = \boldsymbol{y_{19}}(1 + hc) - h \times \mathbf{0.1}\,. \tag{15.1.9f}$$

- **Note the indexing and the value of x_i for the reverse integration.**

15.2 Euler integration 2: implicit

15.2.1 Forward integration

- The differential equation is given by eq. (15.1.1)

$$\frac{dy}{dx} = x - cy \,. \tag{15.2.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 1$.
- The exact solution $y^{\rm ex}(x)$ is given by eq. (15.1.2).
- The integration stepsize is h (a constant), hence $x_i = ih$.
- **Use the implicit Euler method to derive the following.**

$$y_{i+1} = y_i + h(x_{i+1} - cy_{i+1}) \,. \tag{15.2.2}$$

- **Process the above equation to derive the following.**

$$y_{i+1} = \frac{y_i + hx_{i+1}}{1 + hc} \,. \tag{15.2.3}$$

  1. **This is an important lesson to learn.**
  2. **The denominator $1 + hc$ divides the term in x_{i+1} not just y_i.**

- Set $h = 1/n$.
- **Show that $x_n = 1$ after n steps.**
- Set $c = 2$.
- Recall the value of the exact solution $y^{\rm ex}(x)$ for $x = 1$ and $c = 2$.
- **Compute the value of the numerical solution y_n and fill the following table of values.**
  1. **Compute the values of $y^{\rm ex}(1)$ and y_n to 5 decimal places.**
  2. **In the last column, calculate the value of $n(y^{\rm ex}(1) - y_n)$ to 2 decimal places.**

n	$y^{\rm ex}(1)$	y_n	$y^{\rm ex}(1) - y_n$	$n(y^{\rm ex}(1) - y_n)$
10	5 d.p.	5 d.p.	5 d.p.	2 d.p.
100	5 d.p.	5 d.p.	5 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.2.2 Reverse integration

- **We now reverse the direction of integration and integrate backwards to $x_0 = 0$.**
- The differential equation is given by eq. (15.1.4)

$$\frac{dy}{dx} = -(x - cy)\,. \tag{15.2.4}$$

- Let the steps be indexed by $i = n + 1, \dots, 2n$.
- The integration stepsize is still $h = 1/n$.
- You are given that now $x_i = 1 - (i - n)h$.
- **Use the implicit Euler method to derive the following.**

$$y_{i+1} = y_i - h(x_{i+1} - cy_{i+1}) \tag{15.2.5}$$

- **Process the above equation to derive the following.**

$$y_{i+1} = \frac{y_i - hx_{i+1}}{1 - hc}\,. \tag{15.2.6}$$

- **Show that $x_{2n} = 0$ after $2n$ steps.**
- The exact solution will return to the initial value $y_0 = 1$ but the numerical solution will not.
- **Compute the value of the numerical solution y_{2n} and fill the following table of values.**
  1. **Compute the value of y_{2n} to 5 decimal places.**
  2. **In the last column, calculate the value of $n(1 - y_{2n})$ to 2 decimal places.**

$2n$	$y^{\text{ex}}(0)$	y_{2n}	$1 - y_{2n}$	$n(1 - y_{2n})$
20	1	5 d.p.	5 d.p.	2 d.p.
200	1	5 d.p.	5 d.p.	2 d.p.
2000	1	5 d.p.	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately equal.
- This is because Euler integration is a first order method.

15.2.3 Indexing for forward and reverse integration

- **Let me use $n = 10$ as an example. Then $h = 1/n = 0.1$.**
- **The value of x_i is the same as previously listed.**
- **For the *forward integration*, the loop goes from $i = 0$ through $i = 9$ (ten equations)**

$$i = 0: \qquad y_1 = \frac{y_0 + h\boldsymbol{x_1}}{1 + hc} = \frac{1 + h \times \mathbf{0.1}}{1 + hc}, \tag{15.2.7a}$$

$$i = 1: \qquad y_2 = \frac{y_1 + h\boldsymbol{x_2}}{1 + hc} = \frac{y_1 + h \times \mathbf{0.2}}{1 + hc}, \tag{15.2.7b}$$

$$i = 2: \qquad y_3 = \frac{y_2 + h\boldsymbol{x_3}}{1 + hc} = \frac{y_2 + h \times \mathbf{0.3}}{1 + hc}, \tag{15.2.7c}$$

$$\vdots \tag{15.2.7d}$$

$$i = 9: \qquad \boldsymbol{y_n} = y_{10} = \frac{y_9 + h\boldsymbol{x_{10}}}{1 + hc} = \frac{y_2 + h \times \mathbf{1.0}}{1 + hc}. \tag{15.2.7e}$$

- **For the *reverse integration*, the loop goes from $i = 10$ through $i = 19$ (ten equations) *We simply change sign $h \to -h$, so $h = -0.1$.***

$$\boldsymbol{i = 10}: \qquad y_{11} = \frac{y_{10} - h\boldsymbol{x_{11}}}{1 - hc} = \frac{y_{10} - h \times \mathbf{0.9}}{1 - hc}, \tag{15.2.8a}$$

$$\boldsymbol{i = 11}: \qquad y_{12} = \frac{y_{11} - h\boldsymbol{x_{12}}}{1 - hc} = \frac{y_{11} - h \times \mathbf{0.8}}{1 - hc}, \tag{15.2.8b}$$

$$\boldsymbol{i = 12}: \qquad y_{13} = \frac{y_{12} - h\boldsymbol{x_{13}}}{1 - hc} = \frac{y_{11} - h \times \mathbf{0.7}}{1 - hc}, \tag{15.2.8c}$$

$$\vdots \tag{15.2.8d}$$

$$\boldsymbol{i = 18}: \qquad y_{19} = \frac{y_{18} - h\boldsymbol{x_{19}}}{1 - hc} = \frac{y_{18} - h \times \mathbf{0.1}}{1 - hc}, \tag{15.2.8e}$$

$$\boldsymbol{i = 19}: \qquad \boldsymbol{y_{2n}} = y_{20} = \frac{y_{19} - h\boldsymbol{x_{20}}}{1 - hc} = \frac{y_{19} - h \times \mathbf{0}}{1 - hc}. \tag{15.2.8f}$$

- **Note the indexing and the value of x_i for the reverse integration.**

15.3 Euler integration 3

- You are given the following differential equation

$$\frac{dy}{dx} = 2x\,. \tag{15.3.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 0$.
- The exact solution $y^{\rm ex}(x)$ is given as follows:

$$y^{\rm ex}(x) = x^2\,. \tag{15.3.2}$$

- The integration stepsize is h (a constant), hence $x_i = ih$.
- Denote the explicit numerical solution by $y_i^{\rm exp} = y^{\rm exp}(x_i)$.
- Denote the implicit numerical solution by $y_i^{\rm imp} = y^{\rm imp}(x_i)$.
- **Use the explicit Euler method to derive the following.**

$$y_{i+1}^{\rm exp} = y_i^{\rm exp} + 2hx_i\,. \tag{15.3.3}$$

- **Show the following, for $i = 1$:**

$$y_1^{\rm exp} = 0\,. \tag{15.3.4}$$

- **Use the implicit Euler method to derive the following.**

$$y_{i+1}^{\rm imp} = y_i^{\rm imp} + 2hx_{i+1}\,. \tag{15.3.5}$$

- **Show the following, for $i = 1$:**

$$y_1^{\rm imp} = 2h^2\,. \tag{15.3.6}$$

- Set $h = 1/n$.
- The value of the exact solution is $y^{\rm ex}(1) = 1$ for $x = 1$.
- **Compute the value of the numerical solutions $y_n^{\rm exp}$ and $y_n^{\rm imp}$ and fill the following table of values.**

n	$y_n^{\rm exp}$	$y_n^{\rm imp}$	$n(1 - y_n^{\rm exp})$	$n(1 - y_n^{\rm imp})$
10	5 d.p.	5 d.p.	2 d.p.	2 d.p.
100	5 d.p.	5 d.p.	2 d.p.	2 d.p.
1000	5 d.p.	5 d.p.	2 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last two columns should be approximately independent of n.
- This is because Euler integration is a first order method.

15.4 Euler integration 4

- Let us modify the differential equation in eq. (15.3.1).
- We know the exact solution is $y^{\rm ex}(x) = x^2$ so let us integrate the following differential equation

$$\frac{dy}{dx} = 2\sqrt{y}\,. \tag{15.4.1}$$

- The integration begins at $x_0 = 0$.
- The initial condition is $y_0 = 0$.
- The exact solution is still $y^{\rm ex}(x) = x^2$.
- The integration stepsize is h (a constant), hence $x_i = ih$.
- Denote the explicit numerical solution by $y_i^{\rm exp} = y^{\rm exp}(x_i)$.
- Denote the implicit numerical solution by $y_i^{\rm imp} = y^{\rm imp}(x_i)$.
- **Use the explicit Euler method to derive the following.**

$$y_{i+1}^{\rm exp} = y_i^{\rm exp} + 2h\sqrt{y_i^{\rm exp}}\,. \tag{15.4.2}$$

- **Compute using eq.** (15.4.2) **with** $h = 1/n$ **for** $n = 10$**, 100, 1000 and show that:**

$$y_n^{\rm exp} = 0 \qquad\qquad (n = 10, 100, 1000)\,. \tag{15.4.3}$$

- *The explicit Euler solution never leaves zero!*

- **Use the implicit Euler method to derive the following.**

$$y_{i+1}^{\rm imp} = y_i^{\rm imp} + 2h\sqrt{y_{i+1}^{\rm imp}}\,. \tag{15.4.4}$$

- **Process eq. (15.4.4) to derive the following.**

$$y_{i+1}^{\rm imp} - 2h\sqrt{y_{i+1}^{\rm imp}} = y_i^{\rm imp}\,. \tag{15.4.5}$$

- We have our first example of a nontrivial implicit equation.
- **Show that $y_i^{\rm imp} = 0$ (for all i) is a solution of eq. (15.4.5).**
- However, it is not the only solution of eq. (15.4.5). There is another solution.
  1. **I will derive the formula for the integration step for you.**
  2. We process eq. (15.4.5) to obtain the following:

$$\begin{aligned} \left(\sqrt{y_{i+1}^{\rm imp}} - h\right)^2 &= h^2 + y_i^{\rm imp} \\ \sqrt{y_{i+1}^{\rm imp}} - h &= \sqrt{h^2 + y_i^{\rm imp}} \\ \sqrt{y_{i+1}^{\rm imp}} &= h + \sqrt{h^2 + y_i^{\rm imp}}\,. \end{aligned} \tag{15.4.6}$$

- From the above we obtain the following expression:

$$y_{i+1}^{\rm imp} = \left(h + \sqrt{h^2 + y_i^{\rm imp}}\right)^2. \tag{15.4.7}$$

- Set $h = 1/n$.
- The value of the exact solution is $y^{\rm ex}(1) = 1$ for $x = 1$.
- **Compute the value of the numerical solution $y_n^{\rm imp}$ using eq. (15.4.7) and fill the following table of values.**

n	$y_n^{\rm imp}$	$n(1 - y_n^{\rm imp})$
10	5 d.p.	2 d.p.
100	5 d.p.	2 d.p.
1000	5 d.p.	2 d.p.

- If you have done your work correctly, the numbers in the last column should be approximately independent of n.
- This is because Euler integration is a first order method.
- This example teahes us various lessons.
  1. We need to pay more attention to the conditons for the existence of a solution of a differential equation.
  2. For an implicit integration algorithm, there can be more than one solution of the implicit equation, and a root finding algorithm might converge to the wrong root.