Queens College, CUNY, Department of Computer Science

Numerical Methods

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26 Lecture 26

Fourier Transforms

- We shall study some properties of the **Fourier transform.**
- We shall treat functions of one variable only.
- **This lecture is mainly theoretical, for background reading.**
- The most important topics in this lecture are:
  1. Definition of Fourier transform (and inverse).
  2. Fourier transform of **window function** (also known as rectangle function).
  3. Fourier transform of **triangle function.**
  4. **Convolution theorem,** including the application to the window and triangle functions.
- This lecture is really a precursor to the concept of **Fourier series.**
- Fourier series will be studied in later lectures.
- The algorithms and computations we shall perform in this class will be for Fourier series.
- This lecture will require knowledge of **complex numbers.**

26.1 Fourier transform and inverse (important)

- Let $f(x)$ be a function of a real–valued variable x, where $-\infty < x < \infty$.
- In general, f can be a complex–valued function, all the formulas below will be valid.
- Let k also be a real–valued variable, where $-\infty < k < \infty$.
- The **Fourier transform** of f, denoted by $F(k)$, is defined as

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\,. \tag{26.1.1}$$

- Many authors use a different notation and denote the Fourier transform by $\tilde{f}(k)$.
- There is no standard convention.
- The **inverse Fourier transform** yields $f(x)$ given $F(k)$ and is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk\,. \tag{26.1.2}$$

- Obviously the integrals in eqs. (26.1.1) and (26.1.2) may not converge.
- The integrals in eqs. (26.1.1) and (26.1.2) will converge if

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty\,, \tag{26.1.3a}$$

$$\int_{-\infty}^{\infty} |F(k)|\, dk < \infty\,. \tag{26.1.3b}$$

- We shall not worry too much about the convergence of the integrals in eqs. (26.1.1) and (26.1.2).

26.2 Alternative definition

- Notice that eqs. (26.1.1) and (26.1.2) have a very similar structure, except for the asymmetry of the factor of $1/(2\pi)$.

- For this reason, many authors employ an alternative more symmetric definition for the Fourier transform and inverse:

$$F_{\text{alt}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\,. \tag{26.2.1a}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_{\text{alt}}(k)\, e^{ikx}\, dk\,. \tag{26.2.1b}$$

- **We shall not use this alternative definition.**

- The definitions in eqs. (26.1.1) and (26.1.2) are the standard in physics and we shall use them.

26.3 Uniqueness of transform and inverse

- There is an obvious question of uniqueness.
- We can express the problem in the following way, using square roots.
  1. We begin with -2 and square it to obtain $(-2)^2 = 4$.
  2. We now invert the calculation and compute the square root $\sqrt{4}$.
  3. *Do we obtain our original value of -2?*
  4. Not necessarily.
- Hence does the inversion of a Fourier transform return the original function?
  1. We begin with an initial function $f_i(x)$.
  2. We calculate the Fourier transform $F_i(k)$ via eq. (26.1.1):

$$F_i(k) = \int_{-\infty}^{\infty} f_i(x)\, e^{-ikx}\, dx\,. \tag{26.3.1}$$

  3. We then employ eq. (26.1.2) and obtain a final function $f_f(x)$ via

$$f_f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_i(k)\, e^{ikx}\, dk\,. \tag{26.3.2}$$

  4. ***Are we guaranteed that $f_f(x) = f_i(x)$?***
- The answer is: in many cases yes, but **not necessarily.**
- It is possible that $f_f(x)$ equals $f_i(x)$ for some *but not all* values of x.
- This is especially true if $f_i(x)$ has discontinuities.
- Then $f_f(x_{\rm disc})$ may not equal $f_i(x_{\rm disc})$ if $f_i(x)$ is discontinuous at $x = x_{\rm disc}$.
- These are issues to study.
- For now, we shall assume that $f(x)$ is sufficiently well behaved so that the inverse Fourier transform yields the original function uniquely for all values of x.

26.4 Linear operator (formal mathematical proofs)

- The Fourier transform and inverse Fourier transform are linear operators.
- Given two functions $f(x)$ and $g(x)$ and $h(x) = f(x) + g(x)$, the Fourier transform of h is the sum of the Fourier transforms of f and g. The proof is as follows.

  1. By definition we have the following:

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\,, \tag{26.4.1a}$$

$$G(k) = \int_{-\infty}^{\infty} g(x)\, e^{-ikx}\, dx\,, \tag{26.4.1b}$$

$$H(k) = \int_{-\infty}^{\infty} h(x)\, e^{-ikx}\, dx\,. \tag{26.4.1c}$$

  2. Then

$$\begin{aligned} H(k) &= \int_{-\infty}^{\infty} (f(x) + g(x))\, e^{-ikx}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx + \int_{-\infty}^{\infty} g(x)\, e^{-ikx}\, dx \\ &= F(k) + G(k)\,. \end{aligned} \tag{26.4.2}$$

- Hence we may write

$$\mathrm{FT}[f+g] = \mathrm{FT}[f] + \mathrm{FT}[g]\,. \tag{26.4.3}$$

- If λ is a constant (which can be complex) and $r(x) = \lambda\, f(x)$, then the Fourier transform of r is λ times the Fourier transform of f. The proof is as follows:

$$\begin{aligned} R(k) &= \int_{-\infty}^{\infty} r(x)\, e^{-ikx}\, dx \\ &= \int_{-\infty}^{\infty} \lambda f(x)\, e^{-ikx}\, dx \;=\; \lambda \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \;=\; \lambda\, F(k)\,. \end{aligned} \tag{26.4.4}$$

- Hence we may write

$$\mathrm{FT}[\lambda f] = \lambda\, \mathrm{FT}[f]\,. \tag{26.4.5}$$

- The two formulas eqs. (26.4.3) and (26.4.5) are the two defining properties of a linear operator.
- Many authors combine them into one formula and say, using constants λ and μ,

$$\mathrm{FT}[\lambda f + \mu g] = \lambda\, \mathrm{FT}[f] + \mu\, \mathrm{FT}[g]\,. \tag{26.4.6}$$

- By a similar proof, the inverse Fourier transform is also a linear operator:

$$\mathrm{IFT}[\lambda F + \mu G] = \lambda\, \mathrm{IFT}[F] + \mu\, \mathrm{IFT}[G]\,. \tag{26.4.7}$$

26.5 Example: window function (important)

- Let us calculate some examples of Fourier transforms.
- Consider the following function, where $a > 0$:

$$f_{\rm win}(x) = \begin{cases} \dfrac{1}{2a} & |x| < a\,, \\ \\ 0 & |x| \ge a\,. \end{cases} \qquad (26.5.1)$$

- This function has many names: **window function, rectangle function, top hat function.**
- It describes a rectangle with base $2a$ and height $1/(2a)$, hence has unit area for all $a > 0$.
- The Fourier transform is

$$F(k) = \int_{-a}^{a} \frac{e^{-ikx}}{2a}\,dx = \left[\frac{e^{-ikx}}{-i2ka}\right]_{-a}^{a} = \frac{e^{-ika}-e^{ika}}{-i2ka} = \frac{\sin(ka)}{ka}\,. \qquad (26.5.2)$$

- The function $\sin(\xi)/\xi$ (where $\xi = ka$ in this case), is called the **sinc function**.
- It equals 1 at $k = 0$ (we must take a limit $k \to 0$) and oscillates and decays to zero as $|k| \to \infty$.
- A graph of the window function and its Fourier transform is plotted in Fig. 1.
- Conversely, suppose the Fourier transform is a window function of width 2κ, where $\kappa > 0$.
- Let us set the area of the rectangle to be 2π, so we define

$$F_{\rm rect}(k) = \begin{cases} \dfrac{2\pi}{2\kappa} & |k| < \kappa\,, \\ \\ 0 & |k| \ge \kappa\,. \end{cases} \qquad (26.5.3)$$

- The inverse Fourier transform is obviously a sinc function:

$$f_{\rm sinc}(x) = \frac{1}{2\pi}\int_{-\kappa}^{\kappa} \frac{2\pi}{2\kappa}\,e^{ikx}\,dk = \left[\frac{e^{ikx}}{i2\kappa x}\right]_{-\kappa}^{\kappa} = \frac{e^{i\kappa x}-e^{-i\kappa x}}{i2\kappa x} = \frac{\sin(\kappa x)}{\kappa x}\,. \qquad (26.5.4)$$

- It equals 1 at $x = 0$ (we must take a limit $x \to 0$) and oscillates and decays to zero as $|x| \to \infty$.

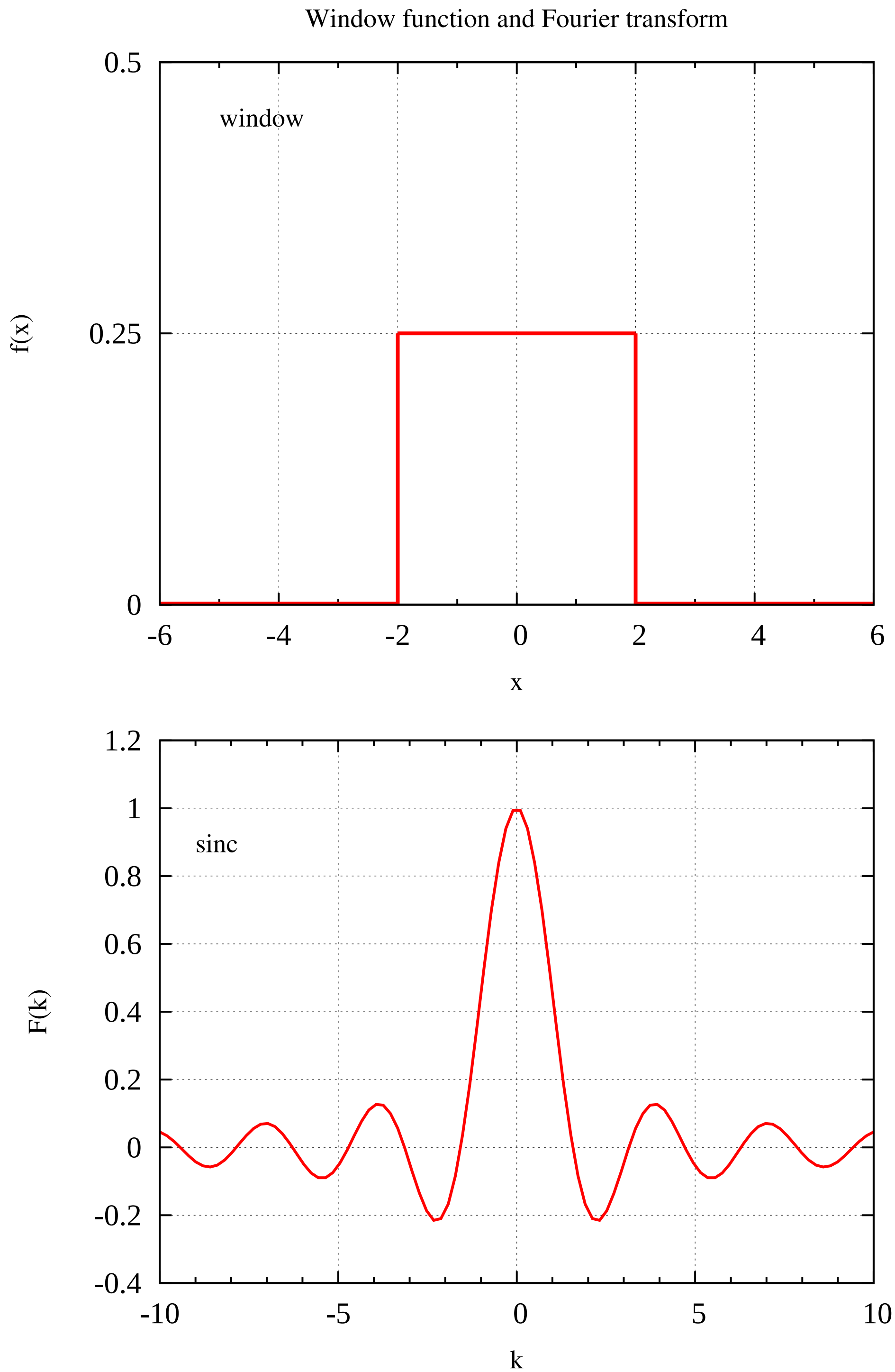

Figure 1: Plot of window function with parameter $a = 2$ and Fourier transform (sinc function).

26.6 Lessons to learn

- The example of the window function teaches some important lessons about Fourier transforms.
- The Fourier transform of a *discontinuous* function (in x) can be a *continuous, infinitely differentiable* function (in k).
- ***Conversely, the Fourier transform of an infinitely differentiable function can be a discontinuous function.***
- The sinc function first crosses zero at $ka = \pm\pi$, or $k = \pm\pi/a$. The base of the central peak of the sinc function is $2\pi/a$.
- Hence for large a (broad window), the Fourier transform $F(k) = \sin(ka)/(ka)$ is sharply peaked in an interval of width $2\pi/a$. Conversely, for small a (narrow window), the Fourier transform $F(k) = \sin(ka)/(ka)$ is broad.
- This is a general feature of Fourier transforms: **there is an inverse relation between the width of the function $f(x)$ and the Fourier transform $F(k)$.**
- If the function $f(x)$ is sharply peaked in x, then the Fourier transform $F(k)$ is broad in k.
- If the function $f(x)$ is broad in x, then the Fourier transform $F(k)$ is sharply peaked in k.

26.7 Example: Triangle function (important)

- Let us consider a **triangle function**

$$f_{\rm tri}(x) = \begin{cases} \dfrac{1}{2a}\left(1-\dfrac{|x|}{2a}\right) & |x| \le 2a\,, \\ 0 & |x| > 2a\,. \end{cases} \tag{26.7.1}$$

- The base is $4a$ and the peak height is $1/(2a)$ so the triangle has unit area.
  As with the window function, the triangle function vanishes outside a finite interval.
- The Fourier transform of the triangle function is (using integration by parts)

$$\begin{aligned} F(k) &= \int_{-2a}^{2a} \frac{1-|x|/(2a)}{2a}\, e^{-ikx}\, dx \\ &= \int_{-2a}^{0} \frac{1+x/(2a)}{2a}\, e^{-ikx}\, dx + \int_{0}^{2a} \frac{1-x/(2a)}{2a}\, e^{-ikx}\, dx \\ &= \left[\frac{1+x/(2a)}{2a}\,\frac{e^{-ikx}}{-ik}\right]_{-2a}^{0} + \int_{-2a}^{0} \frac{1}{4a^2}\,\frac{e^{-ikx}}{ik}\, dx \\ &\qquad + \left[\frac{1-x/(2a)}{2a}\,\frac{e^{-ikx}}{-ik}\right]_{0}^{2a} - \int_{0}^{2a} \frac{1}{4a^2}\,\frac{e^{-ikx}}{ik}\, dx \\ &= -\frac{1}{i2ka} + \frac{1}{4a^2}\left[\frac{e^{-ikx}}{k^2}\right]_{-2a}^{0} \qquad + \frac{1}{i2ka} - \frac{1}{4a^2}\left[\frac{e^{-ikx}}{k^2}\right]_{0}^{2a} \\ &= \frac{2-e^{i2ka}-e^{-i2ka}}{4k^2a^2} \\ &= \frac{\sin^2(ka)}{(ka)^2}\,. \end{aligned} \tag{26.7.2}$$

- **The Fourier transform of the triangle function is the square of the Fourier transform of the window function.**
- The triangle function has the important property that it vanishes outside a bounded interval (in pure mathematical language, the window function has **bounded support**) *and* its Fourier transform is always nonnegative.
- This is an important property. The window function does not have such a property.
- A graph of the triangle function and its Fourier transform is plotted in Fig. 2.
- Clearly if the Fourier transform is a triangle then $f(x)$ is the square of a sinc function.

$$F_{\rm tri}(k) = \begin{cases} \dfrac{2\pi}{2\kappa}\left(1-\dfrac{|k|}{2\kappa}\right) & |k| \le 2\kappa\,, \\ 0 & |k| > 2\kappa\,. \end{cases} \tag{26.7.3}$$

- The inverse Fourier transform function is

$$f(x) = \frac{\sin^2(\kappa x)}{(\kappa x)^2}\,. \tag{26.7.4}$$

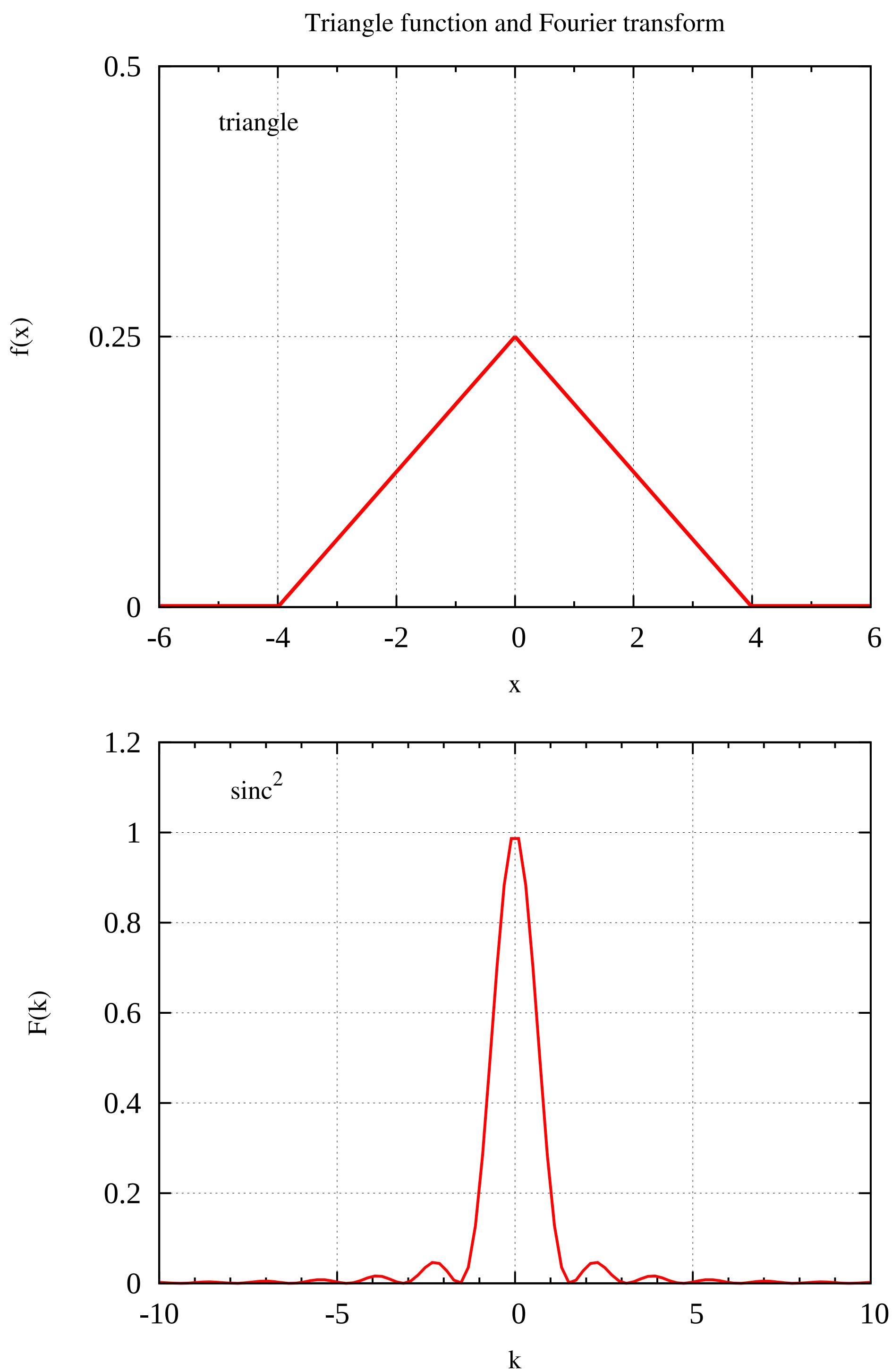

Figure 2: Plot of triangle function with parameter $a = 2$ and Fourier transform (sinc^2 function).

26.8 Example: Lorentzian

- Consider the following function, where $a > 0$:

$$f(x) = \frac{e^{-|x|/a}}{2a} \,. \tag{26.8.1}$$

- The Fourier transform is

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} \frac{e^{-|x|/a}}{2a}\, e^{-ikx}\, dx \\ &= \frac{1}{2a} \int_{-\infty}^{0} e^{x(1/a-ik)}\, dx + \frac{1}{2a} \int_{0}^{\infty} e^{-x(1/a+ik)}\, dx \\ &= \frac{1}{2} \left[\frac{e^{x(1/a-ik)}}{1-ika} \right]_{-\infty}^{0} - \frac{1}{2} \left[\frac{e^{-x(1/a+ik)}}{1+ika} \right]_{0}^{\infty} \\ &= \frac{1}{2} \left(\frac{1}{1-ika} + \frac{1}{1+ika} \right) \\ &= \frac{1}{1+(ka)^2} \,. \end{aligned} \tag{26.8.2}$$

- The function $1/(1+\xi^2)$ (where $\xi = ka$ in this example) is called a **Lorentzian.**
- The inverse Fourier transform of a Lorentzian is not so easy to calculate!
- The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+(ka)^2}\, dk \,. \tag{26.8.3}$$

- To evaluate the integral in eq. (26.8.3), we must perform a **contour integration.**
- Contour integration requires advanced complex variable theory.
- We shall not perform contour integration in these lectures.

26.9 Example: Gaussian

- Consider the following function, where $\sigma > 0$. It is called a **Gaussian:**

$$f(x) = \frac{e^{-x^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}\,. \tag{26.9.1}$$

- The Gaussian function is well known in probability theory.
- The parameter σ is the standard deviation of the Gaussian probability distribution.
- The Fourier transform is

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} \frac{e^{-x^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}\, e^{-ikx}\,dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\biggl\{-\frac{x^2+i2k\sigma^2x}{2\sigma^2}\biggr\}\,dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\biggl\{-\frac{(x+ik\sigma^2)^2+(k\sigma^2)^2}{2\sigma^2}\biggr\}\,dx \\ &= \frac{e^{-k^2\sigma^2/2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x+ik\sigma^2)^2/(2\sigma^2)}\,dx \\ &= e^{-k^2\sigma^2/2}\,. \end{aligned} \tag{26.9.2}$$

- **This is also a Gaussian, but in k.**
- The Fourier transform of a Gaussian in x, with a standard deviation σ, is a Gaussian in k, with a standard deviation $1/\sigma$.
- The more sharply peaked the Gaussian is in x (small σ), the broader is the Gaussian in k, and vice–versa.
- Technically, in the last integral in eq. (26.9.2), I used the following result without proof:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x+ik\sigma^2)^2/(2\sigma^2)}\,dx = 1\,. \tag{26.9.3}$$

- While the above result is true, a rigorous proof requires contour integration and is beyond the scope of these lectures.
- A rigorous proof of eq. (26.9.3) is given in pure mathematics textbooks.

26.10 Convolution (important)

- The **convolution** of two functions $f(x)$ and $g(x)$ is denoted by $f * g$ and is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)\,du = \int_{-\infty}^{\infty} f(x-u)g(u)\,du\,. \tag{26.10.1}$$

- Convolution is an important operation and has many practical applications.
- Although the functions f and g have equal roles in eq. (26.10.1), in practical applications we think of f as a 'signal' and g as the **kernel** of the convolution.
- The importance of convolution stems from the property of its Fourier transform.
- Let $h_{\text{conv}} = f * g$ be the convolution of f and g. The Fourier transform of h_{conv} is

$$\begin{aligned}
H_{\text{conv}}(k) &= \int_{-\infty}^{\infty} h_{\text{conv}}(x)\,e^{-ikx}\,dx \\
&= \int_{-\infty}^{\infty} \biggl(\int_{-\infty}^{\infty} f(u)g(x-u)\,du \biggr)\,e^{-ikx}\,dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bigl[\,f(u)\,e^{-iku}\,\bigr]\,\bigl[\,g(x-u)\,e^{-ik(x-u)}\,\bigr]\,dx\,du \\
&= \biggl(\int_{-\infty}^{\infty} f(u)\,e^{-iku}\,du \biggr) \biggl(\int_{-\infty}^{\infty} g(x-u)\,e^{-ik(x-u)}\,dx \biggr) \\
&= F(k)\,G(k)\,.
\end{aligned} \tag{26.10.2}$$

- **The Fourier transform of the <u>convolution of two functions</u> f and g is the <u>product of their Fourier transforms</u> $F(k)$ and $G(k)$.**
- The Fourier transform of the product fg is the convolution of the Fourier transforms $F * G$.
- Let $h_{\text{prod}}(x) = f(x)g(x)$ be the product of f and g. The Fourier transform of h_{prod} is

$$\begin{aligned}
H_{\text{prod}}(k) &= \int_{-\infty}^{\infty} f(x)g(x)\,e^{-ikx}\,dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \biggl(\int_{-\infty}^{\infty} F(k_1)\,e^{ik_1x}\,dk_1 \biggr)\,g(x)\,e^{-ikx}\,dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_1)\,g(x)\,e^{-i(k-k_1)x}\,dx\,dk_1 \\
&= \int_{-\infty}^{\infty} F(k_1) \biggl(\frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)\,e^{-i(k-k_1)x}\,dx \biggr)\,dk_1 \\
&= \int_{-\infty}^{\infty} F(k_1)\,G(k-k_1)\,dk_1 \\
&= (F * G)(k)\,.
\end{aligned} \tag{26.10.3}$$

- **These are both very important properties:**
  1. **FT(convolution $f * g$) = product $F(k)G(k)$.**
  2. **FT(product fg) = convolution $F * G$.**

26.11 Convolution example: window & triangle (important)

- Let us calculate an important example of a convolution.
- The example is the **convolution of the window function with itself.**
- It is perfectly possible to have $g(\theta) = f(\theta)$ in a convolution.
- Hence our functions are, with $a > 0$ (see eq. (26.5.1))

$$f(x) = g(x) = \begin{cases} \dfrac{1}{2a} & |x| < a\,, \\[2ex] 0 & |x| \geq a\,. \end{cases} \tag{26.11.1}$$

- The convolution $h = f * g$ is calculated using eq. (26.10.1).
- Note that $f(u) \neq 0$ if $|u| < a$, so the interval of integration is $-a < u < a$.
- Next note that $g(x-u) \neq 0$ if $x - a < u < x + a$.
- First consider $x \geq 0$.
  1. Then $f(u)g(x-u)$ is nonzero if $x - a < u < a$. See Fig. 3 (top panel).
  2. This means we must have $0 \leq x < 2a$ because the condition $x - a < u < a$ cannot be satisfied if $x \geq 2a$.
  3. Hence for $0 \leq x < 2a$, the value of the convolution integral is

$$h(x) = \int_{x-a}^{a} \frac{1}{(2a)^2}\,du = \frac{2a - x}{4a^2} = \frac{1}{2a}\left(1 - \frac{x}{2a}\right). \tag{26.11.2}$$

- Next consider $x \leq 0$.
  1. Then $f(u)g(x-u)$ is nonzero if $-a < u < x + a$. See Fig. 3 (bottom panel).
  2. This means we must have $-2a < x \leq 0$ because the condition $-a < u < x + a$ cannot be satisfied if $x \leq -2a$.
  3. Hence for $-2a < x \leq 0$, the value of the convolution integral is

$$h(x) = \int_{-a}^{x+a} \frac{1}{(2a)^2}\,du = \frac{2a + x}{4a^2} = \frac{1}{2a}\left(1 + \frac{x}{2a}\right). \tag{26.11.3}$$

- Hence overall the convolution function is

$$h(x) = \begin{cases} \dfrac{1}{2a}\left(1 - \dfrac{|x|}{2a}\right) & |x| \leq 2a\,, \\[2ex] 0 & |x| > 2a\,. \end{cases} \tag{26.11.4}$$

- **This is the triangle function (see eq. (26.7.1)).**

- The convolution of a window function with itself is a triangle function.
- Recall that it was proved above that the Fourier transform of a convolution $f*g$ is the product of the Fourier transforms $F(k)G(k)$.
- Hence if $f(x) = g(x)$, then $F(k) = G(k)$, **hence the Fourier transform of $f*f$ is $(F(k))^2$.**
- *That is exactly what we derived in Secs. 26.5 and 26.7.*
- The Fourier transform of the window function is (see eq. (26.5.2))

$$F_{\text{win}}(k) = \frac{\sin(ka)}{ka} \,. \tag{26.11.5}$$

- The Fourier transform of the triangle function is (see eq. (26.7.2))

$$F_{\text{tri}}(k) = \frac{\sin^2(ka)}{(ka)^2} = \big(F_{\text{win}}(k)\big)^2 \,. \tag{26.11.6}$$

- This is an important reason why the Fourier transform of the triangle function is nonnegative.

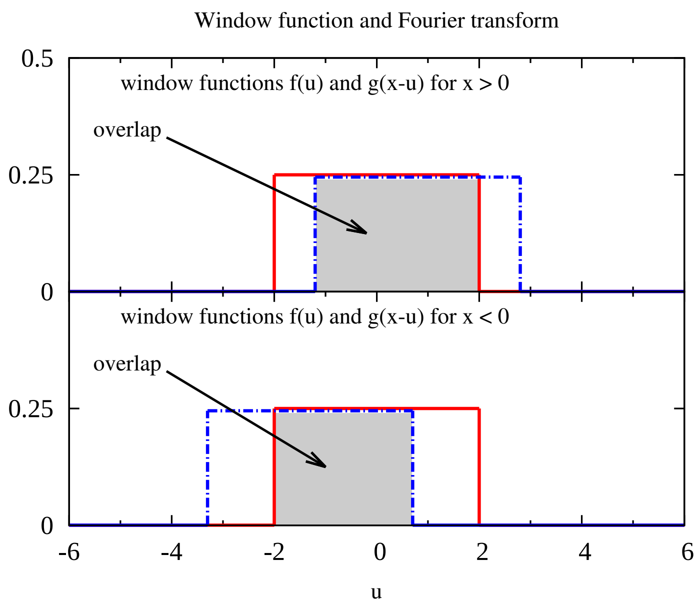

Figure 3: Convolution of window function with itself.

26.12 Cross–correlation & autocorrelation

- The concept of **cross–correlation** is closely related to convolution.
- Cross–correlation is really no more complicated than convolution.
- We shall not do much with cross–correlation and autocorrelation.
- Notice that in a convolution, as the argument of $f(u)$ increases, the argument of $g(x-u)$ decreases (see eq. (26.10.1)).
- *What if both function arguments increase together?* That is cross–correlation.
- The **cross–correlation** of two functions $f(x)$ and $g(x)$ is denoted by $f \star g$ and is defined as

$$(f \star g)(x) = \int_{-\infty}^{\infty} f^*(u)g(x+u)\,du\,. \tag{26.12.1}$$

- Note the complex conjugate on f, hence $f \star g \neq g \star f$ in general.
- The cross–correlation is a measure of how similar two functions f and g are.
- Cross–correlation is related to convolution via

$$\text{Cross–correlation}(f(x), g(x)) = \text{Convolution}(f^*(-x), g(x))\,. \tag{26.12.2}$$

- Consequently, the Fourier transform of the cross–correlation of $f(x)$ and $g(x)$ is given by a complex conjugate product

$$\mathrm{FT}[(f \star g)(x)] = F^*(k)G(k)\,. \tag{26.12.3}$$

- If $f = g$, we use the term **autocorrelation.**

$$(f \star f)(x) = \int_{-\infty}^{\infty} f^*(u)f(x+u)\,du\,. \tag{26.12.4}$$

- The Fourier transform of the autocorrelation of $f(x)$ is therefore the absolute square of $F(k)$:

$$\mathrm{FT}[(f \star f)(x)] = F^*(k)F(k) = |F(k)|^2\,. \tag{26.12.5}$$

- Taking the inverse Fourier transform in eq. (26.12.3) yields the relation

$$\int_{-\infty}^{\infty} f^*(u)g(x+u)\,du = (f \star g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,F^*(k)G(k)\,dk\,. \tag{26.12.6}$$

- Put $x = 0$ and we obtain

$$\int_{-\infty}^{\infty} f^*(u)g(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(k)G(k)\,dk\,. \tag{26.12.7}$$

- Put $f = g$ and we obtain **Plancherel's theorem**

$$\int_{-\infty}^{\infty} |f(u)|^2\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(k)|^2\,dk\,. \tag{26.12.8}$$

- Actually eq. (26.12.7) is sometimes also called Plancherel's theorem.

26.13 Bandwidth limited function

- A **bandwidth limited function** is a function $f(x)$ whose Fourier transform $F(k)$ equals zero for all $|k| > K$ for some constant $K \geq 0$.
- The Fourier transform in eq. (26.5.3) equals zero for all $|k| > \kappa$.
- Hence the sinc function in eq. (26.5.4) is bandwidth limited, with $K = \kappa$.
- Bandwidth limited functions will be important in later lectures, in our study of Fourier series and the discrete Fourier transform.

Advanced material

26.14 Superposition of waves

- The following is a very insightful way to visualize Fourier transforms.
- It has an actual physics interpretation.
- We can think of e^{ikx} as a wave (oscillations as a function of x)

$$e^{ikx} = \cos(kx) + i\,\sin(kx)\,. \tag{26.14.1}$$

- In the complex plane, the value of e^{ikx} moves in a circle of unit radius as x increases.
  1. If $k > 0$, the value of e^{ikx} circulates counterclockwise as x increases.
  2. If $k < 0$, the value of e^{ikx} circulates clockwise as x increases.
  3. The wavelength of the oscillations is λ, where $|k\lambda| = 2\pi$, so $\lambda = 2\pi/|k|$.
  4. The variable k is called the **wave vector.**
- Next, we also know that an integral can be viewed as a sum.
  1. Suppose we are given an integral of a function $F_1(k)$

$$I = \int_a^b F_1(k)\,dk\,. \tag{26.14.2}$$

  2. If we divide the domain of integration into little subintervals of length $h > 0$ (where the value of h is assumed small), we can approximate the above integral by a sum over n subintervals

$$I \simeq h\sum_{j=1}^{n} F_1(k_j)\,. \tag{26.14.3}$$

- Return to eq. (26.1.2) and set $F_1(k) = F(k)\,e^{ikx}/(2\pi)$.
  1. We can write, approximately (with $a = -\infty$ and $b = \infty$ and using an infinite sum)

$$f(x) \simeq h\sum_{j=-\infty}^{\infty} \frac{F(k_j)\,e^{ik_jx}}{2\pi}\,. \tag{26.14.4}$$

  2. **We can interpret eq. (26.14.4) as a sum of a large number of waves.**
  3. **Each complex exponential e^{ikx} is a wave (dropping the subscript j).**
  4. **The factor of $F(k)$ is the 'weight' of that wave in the sum.**
  5. The integral in eq. (26.1.2) is the limit as $h \to 0$ in eq. (26.14.4).
  6. The integral in eq. (26.1.2) is a weighted sum of infinitely many waves.
- **We can interpret the inverse Fourier transform as a weighted sum of waves.**

- This is not merely a mathematical artifact.
- This point of view has real physical significance.
- We can visualize that the function $f(x)$ is a weighted sum of waves.
- The value of $F(k)$ is the weight of each wave in the sum.
- The Fourier transform in eq. (26.1.1) tells us, given $f(x)$, how to calculate the weight of each wave, i.e. how to calculate $F(k)$.
- For example when we hear a sound, it is made up of waves and what we hear is a weighted sum of sound waves.
- A beam of light is made up of waves and what we see is a weighted sum of light waves.
- The Fourier transform and inverse tell us how to go back and forth between the 'coordinate picture' (function of x) and the 'wave picture' (function of k).
- *The transformation to/from the 'coordinate picture' and 'wave picture' is essential in the quantum theory.*
- We shall not study the quantum theory in this class, but the above interpretation of $f(x)$ as a weighted sum of waves, via the Fourier transform and inverse, is fundamental.

26.15 Are $f(x)$ and $F(k)$ different functions?

- **There is another fundamental point of view, which is that $f(x)$ and $F(k)$ are two different ways to visualize the same function.**
- The function $f(x)$ is the **representation in the coordinate picture.**
- The function $F(k)$ is the **representation in the wave picture.**
- In a manner of speaking, you have seen such concepts before.
  1. A vector $\boldsymbol{v}$ can be expressed in components using either Cartesian or polar coordinates.
  2. They are simply two different ways of representing the same fundamental quantity.
  3. There is a fundamental object (the vector $\boldsymbol{v}$) and it does not depend on any particular coordinate system.
- From that point of view, we are visualizing the same underlying function using either x or k, two different ways of representing the same fundamental quantity.
- For example, if we have a beam of light, we can calculate its energy in two different ways.
  1. We calculate the intensity of the light at each point in space and sum over all the points.
  2. Alternatively, we can express the light as a sum of waves and calculate the energy in each wave, and sum the energies of all the waves.
  3. The value of the energy is the same either way.
  4. The coordinate picture and the wave picture are two different ways to calculate the same fundamental quantity.

26.16 Time & frequency

- Fourier transforms have a close connection to physics.
- We have already remarked on this in Sec. 26.14, introducing the notion of the inverse Fourier transform as a sum of waves.
- It was tacitly assumed in Sec. 26.14 that x is a length ('coordinate') and k (the wave vector) is an inverse wavelength.
- There is another point of view, also closely connected to physics, which is consider $f(t)$ as a function of the time t.
- We employ ω instead of k and define the Fourier transform and inverse via

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt\,, \tag{26.16.1a}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega\,. \tag{26.16.1b}$$

- As opposed to eq. (26.1.1), the exponent is $e^{i\omega t}$ not $e^{-i\omega t}$ in the forward transform.
- As in Sec. 26.14, we can visualize the inverse Fourier transform as a sum of waves.
- In this case, the waves are oscillations in time rather than in space.
- The variable ω is called the **angular frequency** of the wave.
- The frequency of the wave is given by $f = \omega/(2\pi)$. Do not confuse this use of f with the function $f(x)$ or $f(t)$. We shall avoid using the term 'frequency' in this class for precisely this reason.
- If the period of a wave is τ, then $|\omega\tau| = 2\pi$ so $\tau = 2\pi/|\omega|$.
- For those of you who have heard of the concept of 'space–time' (from Einstein's Theory of Relativity or more likely from science fiction) there is indeed a joint Fourier transform in both space and time:

$$F(\omega, k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, x)\, e^{i(\omega t - kx)}\, dt\, dx\,, \tag{26.16.2a}$$

$$f(t, x) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega, k)\, e^{i(kx - \omega t)}\, dt\, dx\,. \tag{26.16.2b}$$

- Electromagntic waves (also sound waves) oscillate in both space and time.
- We shall not deal with space–time in these lectures.

26.17 Filtering of signals part 1

- Let $f(x)$ be a function and its Fourier transform be $F(k)$.
- However, suppose our measuring apparatus (signal analyzer?) can only measure waves up to a maximum value $|k| \le K$, where $K > 0$ is a constant.
- Hence the filtered Fourier transform F_{lpf} that our apparatus actually measures is

$$F_{lpf}(k) = \begin{cases} F(k) & |k| \le K\,, \\ 0 & |k| > K\,. \end{cases} \tag{26.17.1}$$

- Alternatively, perhaps we deliberately cut the Fourier tranform for all values $|k| > K$.
  1. This is called a **low pass filter.**
  2. Technically, this is an ideal low pass filter.
  3. A practical low pass filter passes $F(k)$ for all values $|k| \le K$ and suppresses (but not completely) $F(k)$ for $|k| > K$.
  4. We shall study only ideal filters.
- *What is the filtered function $f_{lpf}(x)$ which the filtering yields?*
- To put it another way, what is the effect of filtering on the function $f(x)$?
- This question is elegantly answered using convolutions.
  1. Let us set $K = \kappa$. Recall the window Fourier transform in eq. (26.5.3).
  2. Then the filtered Fourier transform is the product

$$F_{lpf}(k) = \frac{\kappa}{\pi}\,F(k)F_{\rm rect}(k)\,. \tag{26.17.2}$$

- **Hence $f_{lpf}(x)$ equals the convolution of $f(x)$ and $(\kappa/\pi)f_{\rm sinc}(x)$ (see eq. (26.5.4))**

$$f_{lpf}(x) = \frac{\kappa}{\pi}\,(f * f_{\rm sinc})(x) = \frac{\kappa}{\pi}\int_{-\infty}^{\infty} f(x-u)\,\frac{\sin(\kappa u)}{\kappa u}\,du\,. \tag{26.17.3}$$

  1. Hence $f_{lpf}(x)$ equals a weighted average of $f(x)$ with other values $f(x-u)$ with a weight function of $\sin(\kappa u)/(\pi u)$, wich is called the **kernel** of the convolution.
  2. The weighted average 'smears out' the original value $f(x)$ with other values of $f(x)$.
  3. If κ is large, the smearing involves a small interval of nearby values of f.
  4. If κ is small, the smearing will include values over a large interval of values of f.
  5. This makes sense: if κ is large we lose little information about the original function, but if κ is small we lose a lot of information about the original function.
- Note that $\sin(\kappa x)/(\kappa x)$ is sometimes negative. Hence if the original function $f(x)$ is always positive, the filtered function $f_{lpf}(x)$ can sometimes be negative. This may be 'unphysical' or undesirable in some contexts.

26.18 Filtering of signals part 2

- There are other possible filtering procedures.
- For example we could take a convolution with the square of the sinc function

$$f_{lpf,\,\mathrm{tri}}(x) = \frac{\kappa}{\pi} \int_{-\infty}^{\infty} f(x-u)\, \frac{\sin^2(\kappa u)}{(\kappa u)^2}\, du\,. \tag{26.18.1}$$

- If $f(x)$ is always nonnegative then $f_{lpf,\,\mathrm{tri}}(x)$ will also be always nonnegative.
- This procedure corresponds to multiplying the Fourier transform $F(k)$ by the triangle transform function $F_{\mathrm{tri}}(k)$ (see eq. (26.7.3)).
- This is effectively a different filtering procedure.
- It does not impose a sharp cutoff on $F(k)$.

26.19 Operations on Fourier transform and inverse (advanced topic)

Assuming all the functions are well behaved and all the relevant integrals converge, there are several useful operations we can perform on Fourier transforms and their inverses. The following is a lookup table. The derivations are given below.

property	function	transform
real	$f(x) = f^*(x)$	$F(-k) = F^*(k)$
pure imaginary	$f(x) = -f^*(x)$	$F(-k) = -F^*(k)$
even	$f(x) = f(-x)$	$F(-k) = F(k)$
odd	$f(x) = -f(-x)$	$F(-k) = -F(k)$
shift $x - x_0$	$f(x - x_0)$	$e^{-ikx_0}F(k)$
shift $k - k_0$	$e^{ik_0x}f(x)$	$F(k - k_0)$
scaling x/a	$f(x/a)$	$\|a\|F(ka)$
scaling k/κ	$\|\kappa\|f(\kappa x)$	$F(k/\kappa)$
derivative	df/dx	$ikF(k)$
	d^2f/dx^2	$-k^2F(k)$
	d^nf/dx^n	$(ik)^nF(k)$
multiplication by x	$xf(x)$	$i\,dF/dk$
	$x^2f(x)$	$-d^2F/dk^2$
	$x^nf(x)$	$i^n\,d^nF/dk^n$

26.19.1 $f(x)$ is real

- This is the most common situation. Then $f^*(x) = f(x)$ and

$$\begin{aligned} F^*(k) &= \left(\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \right)^* \\ &= \int_{-\infty}^{\infty} f^*(x)\, e^{ikx}\, dx = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx = F(-k)\,. \end{aligned} \tag{26.19.1}$$

- Hence $F(-k) = F^*(k)$.
- We observed this property in all of our examples above.

26.19.2 $f(x)$ is pure imaginary

- This is rare but not impossible. Then $f^*(x) = -f(x)$ and

$$\begin{aligned} F^*(k) &= \left(\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \right)^* \\ &= \int_{-\infty}^{\infty} f^*(x)\, e^{ikx}\, dx = -\int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx = F(-k)\,. \end{aligned} \tag{26.19.2}$$

- Hence $F(-k) = -F^*(k)$.

26.19.3 $f(x)$ is even in x

- This is also a common situation. Then $f(-x) = f(x)$ and

$$
\begin{aligned}
F(k) &= \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \\
&= \int_{-\infty}^{\infty} f(-u)\, e^{iku}\, du \qquad\qquad \text{(change variables } u = -x\text{)} \\
&= \int_{-\infty}^{\infty} f(u)\, e^{iku}\, du \\
&= F(-k)\,.
\end{aligned}
\tag{26.19.3}
$$

- Hence $F(-k) = F(k)$.
- If $f(x)$ is even in x, then $F(k)$ is even in k.
- We observed this property in all of our examples above.

26.19.4 $f(x)$ is odd in x

- This is also common. Then $f(-x) = -f(x)$ and

$$
\begin{aligned}
F(k) &= \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \\
&= \int_{-\infty}^{\infty} f(-u)\, e^{iku}\, du \qquad\qquad \text{(change variables } u = -x\text{)} \\
&= -\int_{-\infty}^{\infty} f(u)\, e^{iku}\, du \\
&= -F(-k)\,.
\end{aligned}
\tag{26.19.4}
$$

- Hence $F(-k) = -F(k)$.
- If $f(x)$ is odd in x, then $F(k)$ is odd in k.

26.19.5 Shifts in x and k

- Let $g(x) = f(x - x_0)$. The Fourier transform of $g(x)$ is

$$\begin{aligned} G(k) &= \int_{-\infty}^{\infty} f(x - x_0)\, e^{-ikx}\, dx \\ &= e^{-ikx_0} \int_{-\infty}^{\infty} f(x - x_0)\, e^{-ik(x-x_0)}\, dx = e^{-ikx_0}\, F(k)\,. \end{aligned} \tag{26.19.5}$$

- Shifting x by x_0 multiplies the Fourier transform $F(k)$ by e^{-ikx_0}.
- Conversely, let $H(k) = F(k - k_0)$. The inverse Fourier transform of $H(k)$ is

$$\begin{aligned} h(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k - k_0)\, e^{ikx}\, dx \\ &= \frac{e^{ik_0 x}}{2\pi} \int_{-\infty}^{\infty} F(k - k_0)\, e^{i(k-k_0)x}\, dx = e^{ik_0 x}\, f(x)\,. \end{aligned} \tag{26.19.6}$$

- Shifting k by k_0 multiplies the function $f(x)$ by $e^{ik_0 x}$.

26.19.6 Changes of scale in x and k

- Let $a \neq 0$ be a real constant and let $g(x) = f(x/a)$.
- The Fourier transform of $g(x)$ is

$$\begin{aligned} G(k) &= \int_{-\infty}^{\infty} f(x/a)\, e^{-ikx}\, dx \\ &= |a| \int_{-\infty}^{\infty} f(u)\, e^{-ikau}\, du \qquad\qquad \text{(change variables } u = x/a\text{)} \\ &= |a|\, F(ka)\,. \end{aligned} \tag{26.19.7}$$

- Note that the integral is multiplied by $|a|$ because if $a < 0$ the limits of integration must be interchanged, which introduces a minus sign.
- Conversely, let $\kappa \neq 0$ be a real constant and let $G(k) = F(k/\kappa)$.
- The inverse Fourier transform of $G(k)$ is

$$\begin{aligned} g(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k/\kappa)\, e^{ikx}\, dk \\ &= \frac{|\kappa|}{2\pi} \int_{-\infty}^{\infty} F(K)\, e^{iK\kappa x}\, dK \qquad\qquad \text{(change variables } K = k/\kappa\text{)} \\ &= |\kappa|\, f(\kappa x)\,. \end{aligned} \tag{26.19.8}$$

- Note that the integral is multiplied by $|\kappa|$ because if $\kappa < 0$ the limits of integration must be interchanged, which introduces a minus sign.

26.19.7 Derivatives with respect to x and k

- Let us differentiate $f(x)$ with respect to x:

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{1}{2\pi}\frac{d}{dx}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, \frac{d(e^{ikx})}{dx}\, dk \\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty} kF(k)\, e^{ikx}\, dk \\ &= i\, \mathrm{IFT}[kF]\,.\end{aligned} \tag{26.19.9}$$

- Hence the inverse Fourier transform of $ikF(k)$ is df/dx.
- We can also say the Fourier transform of df/dx is $ikF(k)$.
- Note that to justify the above derivation, the following integral must converge:

$$\int_{-\infty}^{\infty} |kF(k)|\, dk < \infty\,. \tag{26.19.10}$$

- Conversely, let us differentiate $F(k)$ with respect to k:

$$\begin{aligned}\frac{dF(k)}{dk} &= \frac{d}{dk}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\, \frac{d(e^{-ikx})}{dk}\, dx \\ &= -i\int_{-\infty}^{\infty} xf(x)\, e^{-ikx}\, dx \\ &= -i\, \mathrm{FT}[xf]\,.\end{aligned} \tag{26.19.11}$$

- Hence the Fourier transform of $xf(x)$ is $i\, dF/dk$.
- Note that to justify the above derivation, the following integral must converge:

$$\int_{-\infty}^{\infty} |xf(x)|\, dx < \infty\,. \tag{26.19.12}$$

- It is easy to extend the process.
- The Fourier transform of $x^2 f(x)$ is $-d^2F/dk^2$, in general $x^n f(x)$ and $i^n\, d^nF/dk^n$,
- The Fourier transform of d^2f/dx^2 is $-k^2F(k)$, in general d^nf/dx^n is $(ik)^nF(k)$,
- Multiplying $f(x)$ by powers of x corresponds to differentiating $F(k)$ multiple times.
- Differentiating $f(x)$ multiple times corresponds to multiplying $F(k)$ by powers of k.