Queens College, CUNY, Department of Computer Science

Numerical Methods

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19 Lecture 19

Numerical solution of systems of ordinary differential equations

- In this lecture we continue the study of initial value problems.
- This lecture lists formal mathematical theorems for the existence and uniqueness of a solution for the initial value problem, for systems of coupled ordinary differential equations.
- *This lecture does not present proofs of the stated theorems.*
- The proofs of the theorems are beyond the scope of this course.
- **The contents of this lecture are not for examination.**
- The statements of the main theorems in Sec. 19.2 are taken from the text by Alex J. Dragt, *"Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics"* and quoted with kind permission by Professor Dragt.
- Dragt's text is available for free download, see Ref. [1].

19.1 Basic notation

- Although we have denoted the independent variable by x in the lectures up to now, the independent variable in the initial value problem is usually called the time t.
- Hence the independent variable below will be called "time" and denoted by t.
- The system of coupled differential equations is

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{f}(t, \boldsymbol{y})\,. \tag{19.1.1}$$

- There are m unknown variables y_j, $j = 1, \dots, m$.
- The starting time is $t = t_0$, and the initial value $\boldsymbol{y}_0$ is given.
- The above is called the **Cauchy problem** or **initial value problem.**

19.2 Main Theorems

The theorems in this section are quoted with permission from Ref. [1].

Theorem 19.1. *Consider a set of m first order coupled ordinary differential equations as follows.*

$$\frac{dy_j}{dt} = f_j(t, y_1, \dots, y_m) \qquad j = 1, \dots, m\,. \tag{19.2.1}$$

Suppose that all the functions on the right hand side of eq. (19.2.1) *are sufficiently well behaved. In particular, suppose that all the functions f_j and all the partial derivatives $\partial f_j/\partial k$, $k = 1, \dots, n$ exist (i.e. are finite) and are continuous in the y_k and in t within some region R of the m-dimensional space $(y_1, \dots, y_m)$ and also in a region T which encloses a fixed value t_0. Let $(y_{1*}, \dots, y_{m*})$ be a point in R. Then there exists a unique solution*

$$y_j(t) = Y_j(t|y_{1*}, \dots, y_{m*}; t_0) \qquad j = 1, \dots, m\,. \tag{19.2.2}$$

of eq. (19.2.1) *with the property*

$$y_j(t_0) = Y_j(t_0|y_{1*}, \dots, y_{m*}; t_0) \qquad j = 1, \dots, m\,. \tag{19.2.3}$$

This solution is guaranteed to exist for a finite interval of time about the point t_0, and can be extended forward or backward in time as long as the f_j are continuous in the y_k and t, and the $y_j(t)$ remain within a region R' where the partial derivatives $\partial f_j/\partial y_k$ exist and are continuous in the y_k and t. Furthermore, the solution eq. (19.2.2) *is continuous (and bounded) in all the variables y_{j*}, t_0 and t. The quantities y_{j*} are called* **initial conditions** *and t_0 is called the* **initial time.**

Theorem 19.2. *Suppose the f_j also depend on a set of parameters $(\lambda_1, \dots, \lambda_\ell)$. Assume that all the partial derivatives $\partial f_j/\partial \lambda_k$ are continuous. Then the solution eq.* (19.2.2) *will also be continuous in the parameters λ_k.*

Theorem 19.3. *Suppose the f_j are* **analytic** *in the variables y_j, λ_k and t. (A function is analytic in some variable if it has a convergent Taylor series expansion in that variable when all other variables are held fixed.) Then the solution eq.* (19.2.2) *will also be analytic in the variables y_{j*}, λ_k, t_0 and t.*

- There are also theorems which employ weaker conditions than those in Theorem 19.1.
- The mathematician Giuseppe Peano proved the existence of a solution for the Cauchy problem eq. (19.2.1) under the assumption of simple continuity of the f_j in t and the y_j.
- However, under these weaker conditions the solution is not necessarily unique.

19.3 Convex set

- To state another theorem on the solution of the Cauchy problem, we require the some additional definitions.
- We require the notion of a **convex set.**
- A set D in an m-dimensional space of real variables $(y_1, \dots, y_m)$ is called a **convex domain** if it has the following property. For any pair of points $\boldsymbol{y}_a$ and $\boldsymbol{y}_b$ in D, the straight line joining $\boldsymbol{y}_a$ and $\boldsymbol{y}_b$ lies entirely in D (possibly touching the boundary of D). Hence, for all $0 \le \lambda \le 1$,

$$(1-\lambda)\boldsymbol{y}_a + \lambda \boldsymbol{y}_b \in D \qquad\qquad (0 \le \lambda \le 1)\,. \tag{19.3.1}$$

- Every triangle is convex.
- An example of a convex and non-convex quadrilateral is shown in Fig. 1.

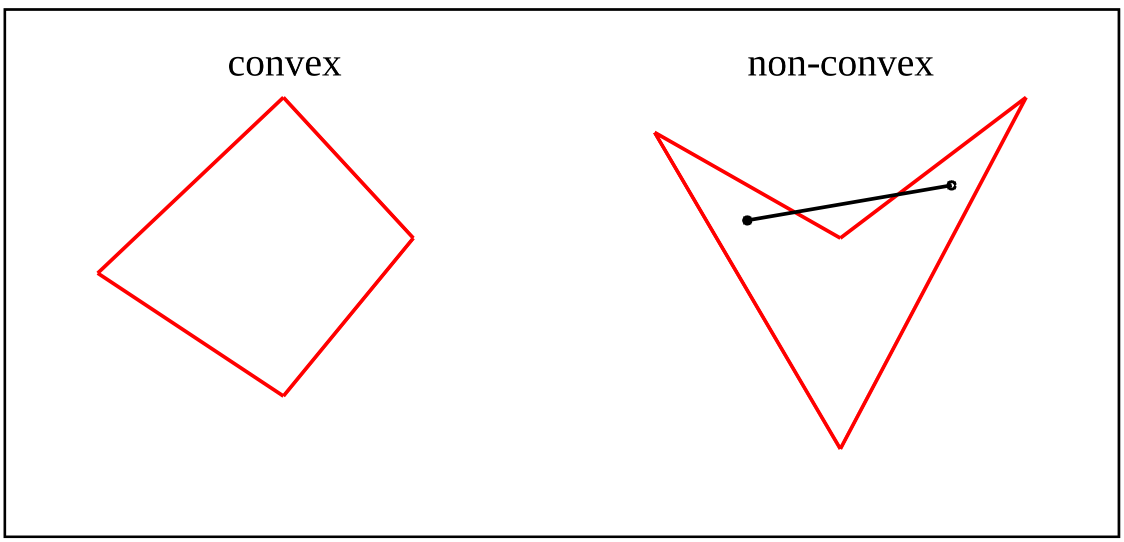

Figure 1: Examples of convex and non-convex sets (quadrilaterals).

19.4 Lipschitz continuity

- We also require the notion of **Lipschitz continuity.**
- A real valued function $f(x)$ of a real variable x is said to be **Lipschitz continuous** in an interval $a \le x \le b$ if there exists a real constant $L \ge 0$ such that for all $a \le \alpha < \beta \le b$,

$$|f(\alpha) - f(\beta)| \le L\,|\alpha - \beta|\,. \tag{19.4.1}$$

- It is possible to have $a = -\infty$ and/or $b = \infty$. The interval can be the entire real line.
- The constant L is called the **Lipschitz constant.** It clearly depends on the interval (the values of a and b).
- Every Lipschitz continuous function is continuous.
- The converse is not true: not every continuous function is Lipschitz continuous.
- Example: consider the function $f(x) = \sqrt{x}$ for $x \ge 0$.
  1. For any value $a > 0$, the function $\sqrt{x}$ is Lipschitz continuous on the interval $a \le x < \infty$.
  2. The value of the Lipschitz constant is $f'(a) = 1/(2\sqrt{a})$.
  3. However if $a = 0$, the function $\sqrt{x}$ is **not** Lipschitz continuous on the interval $0 \le x < \infty$.
  4. Nevertheless, the function $\sqrt{x}$ is continuous on the interval $0 \le x < \infty$.
- Example: consider the function $f(x) = x^2$ for $x \ge 0$.
  1. For any finite value $0 < b < \infty$, the function x^2 is Lipschitz continuous on the interval $0 \le x \le b$.
  2. The value of the Lipschitz constant is $f'(b) = 2b$.
  3. However the function x^2 is not Lipschitz continuous on the interval $0 \le x < \infty$.
  4. Nevertheless, the function x^2 is continuous on the interval $0 \le x < \infty$.
- In multi-dimensions with real valued variables $(y_1, \dots, y_m)$, a real valued function $f(y_1, \dots, y_m)$ is Lipschitz continuous in a convex domain D if for all pairs of points $\boldsymbol{y}_a$ and $\boldsymbol{y}_b$ in D, there exists a real constant $L \ge 0$ such that

$$|f(\boldsymbol{y}_a) - f(\boldsymbol{y}_b)| \le L\,|\boldsymbol{y}_a - \boldsymbol{y}_b|\,. \tag{19.4.2}$$

- The definition of the absolute value on the right hand side employs the **Euclidean norm**

$$|\boldsymbol{y}_a - \boldsymbol{y}_b| = \sqrt{(y_{1,a} - y_{1,b})^2 + \cdots + (y_{n,a} - y_{n,b})^2} = \left[\sum_{j=1}^{m} (y_{j,a} - y_{j,b})^2\right]^{1/2}. \tag{19.4.3}$$

19.5 Theorem with weaker assumptions

Theorem 19.4. *If the functions f_j in eq.* (19.2.1) *are continuous in t and Lipschitz continuous in the y_j, then a solution of the Cauchy problem eq.* (19.2.1) *exists and is unique. (See eqs.* (19.2.2) *and* (19.2.3)*.) The domain D for the Lipschitz continuity is a convex domain D which includes the initial point $(y_{1*}, \dots, y_{m*})$. For our purposes we can choose D to be a rectangular hypercube in m dimensions, hence it will automatically be convex.*

19.6 Well Posed Problem

- The mathematician Jacques Hadamard defined the notion of a **well posed problem.**
- An initial value problem is said to be **well posed** if it has the following properties:
  1. A solution exists.
  2. The solution is unique.
  3. The behavior of the solution changes continuously with the initial conditions (or equivalently, the solution is a continuous function of the initial conditions).
- A problem which is not well posed is said to be **ill posed.**
- The terminology "ill posed" may be misleading. An ill posed problem may have a solution, which is unique. The solution simply may not depend continuously on the initial conditions, but this may not be a bad thing.
  1. Consider a rocket which is launched from the surface of the Earth.
  2. If the rocket is launched with less than the escape velocity, then after a finite time it will return to the surface of the Earth.
  3. However, if the rocket is launched with a velocity greater than or equal to the escape velocity, it will travel to infinity and never return to the surface of the Earth.
  4. In other words, as the initial velocity of the rocket crosses the threshold of the escape velocity, the path of the rocket changes from a bounded orbit to an unbounded trajectory.
  5. Nevertheless, the motion of the rocket can be calculated from the initial conditions, for all values of the initial velocity.
  6. Rockets have been launched many times, both to carry satellites to orbits around the Earth or space probes to other planets, as well as sending astronauts to the Moon.
  7. Admittedly, there have also been many failed launches. Rocketry is difficult.
  8. Fundamentally, we need to define the concept of "the behavior of the solution" more precisely, and it is not always clear how to do so.

References

[1] Alex J. Dragt, *"Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics"* (University of Maryland Press, 2017).
A copy of Dragt's text is available for free dowload at the following website:
`http://www.physics.umd.edu/dsat/dsatliemethods.html`