Queens College, CUNY, Department of Computer Science

Computational Finance

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10 Lecture 10

10.1 Delta one securities

- The Black-Scholes equation and the Black-Scholes-Merton equation have been displayed in class, and of course we are impatient to solve them!
- However, those equations are both based on geometric Brownian motion, and there are still many things to learn, which are *independent* of any probability model for the stock price movements.
- In this lecture we introduce **Delta one securities.**

10.2 Zero strike call

- Let us calculate the fair value of a **zero strike call.**
- This is simply a call option whose strike equals zero.
- A call option with a strike of zero is not a bogus thing, as we shall see.
- Let the current time be t and the option expiration time be T, where $T > t$.
- Let the stock price at time t be S.
- Let S_T denote the unknown terminal stock price at the expiration time T.

10.2.1 No dividends

- For simplicity we begin with a stock which does not pay dividends.
- Recall that the terminal payoff of a long position in a call option with strike K is given by

$$c(S_T, K) = \max(S_T - K, 0)\,. \tag{10.2.1.1}$$

- For a zero strike call, set $K = 0$ so the terminal payoff is simply

$$c(S_T, T, K = 0) = \max(S_T, 0)\,. \tag{10.2.1.2}$$

- Since the value of the stock price is always ≥ 0, then $S_T \geq 0$ so eq. (10.2.1.2) simplifies to

$$c(S_T, T, K = 0) = S_T\,. \tag{10.2.1.3}$$

- Also because the strike is $K = 0$, a zero strike call will always be in the money at expiration.
- Hence the holder of a zero strike call will always exercise the call at expiration, and receive one share of stock for free. (Recall the option holder pays the strike price, which is zero.)
- Hence to avoid arbitrage, the fair value of a zero strike call option, at time t, must be

$$c(S, t, K = 0) = S\,. \tag{10.2.1.4}$$

- For a stock which pays no dividends (during the lifetime of the option),

  the fair value of a zero strike call option equals the price of the stock.

10.2.2 Discrete dividends

- Suppose now the stock pays discrete dividends D_i at times t_i, $i = 1, 2, \dots, n$ during the lifetime of the option, where $t < t_1 < \cdots < t_n < T$.
- The terminal payoff of a zero strike call is still given by eq. (10.2.1.3).
- However, the holder of an option is **not a shareholder of record of the stock.**
- Hence the option holder receives no dividends.
- However, an investor who owns the stock will receive the dividends.
- Hence to avoid arbitrage, the fair value of a zero strike call option at time t equals the

  stock price less the sum of the present values of the dividends paid during the lifetime of the option

$$\begin{aligned} c(S,t,K=0) &= S - \mathrm{PV}(D_1) - \mathrm{PV}(D_2) - \dots \\ &= S - \sum_{i=1}^{n} \mathrm{PV}(D_i)\,. \end{aligned} \tag{10.2.2.1}$$

- This is an important fact, as we shall see.
- Note that the above analysis was for a European option.

10.3 Delta one securities

- For a stock S, the value of Delta equals one:

$$\Delta_{\text{stock}} = \frac{\partial S}{\partial S} = 1\,. \tag{10.3.1}$$

- The Delta of a zero strike call also equals one. Using eq. (10.2.2.1),

$$\frac{\partial c(S,t,K=0)}{\partial S} = \frac{\partial S}{\partial S} = 1\,. \tag{10.3.2}$$

- A **Delta one security** is a financial instrument which **behaves like a stock (equity) but *does not pay dividends.***
- It has a Delta of one, hence its name.
  1. In practice, the value of Delta does not have to be exactly 1.
  2. **The value of Delta simply has to be a number which does not depend on the stock price.**
- The fundamental concept characterizing a Delta one security is that it **trades essentially like a stock (equity) and there is negligible volatility in the fair value of a Delta one security.**
- A zero strike European call option is an example of a Delta one security.
- There are various examples of Delta one securities, but as the above analysis shows, a European call with a strike of zero is one of them and is not a bogus thing.

10.4 Forwards & futures

- Consider a forward or futures contract with expiration time T.
- Suppose the stock pays discrete dividends as before.
- Let the interest rate be r.
- The fair value formula for the forward or futures contract is

$$F = \left[S - \sum_{i=1}^{n} e^{-r(t_i - t)} D_i \right] e^{r(T-t)} . \tag{10.4.1}$$

- The Delta of the forward or futures contract is

$$\Delta_F = \frac{\partial F}{\partial S} = e^{r(T-t)} . \tag{10.4.2}$$

- The value of Delta does not depend on the stock price.
- The holder of a forward or futures contract does not receive dividends.
- **Hence a forward or futures contract is an example of a Delta one security.**
- The value of Delta of a forward or futures contract does not equal 1 exactly, but nevertheless the fundamental concept characterizing a Delta one security is that it trades like a stock (equity) but does not pay dividends. There is little or no volatility in the fair value of a Delta one security.

10.5 Call with negative strike

- Let us calculate the fair value of a call option with a **negative strike price.**
- This is not completely stupid. It is a lesson in financial derivatives pricing theory.
- Let the strike be $-|K|$, where $K < 0$.
- The terminal payoff of the call option, at time T, is given by

$$c(S_T, T, -|K|) = \max(S_T - K, 0) = \max(S_T + |K|, 0)\,. \tag{10.5.1}$$

- The right hand side is always a positive number, hence the payoff is

$$c(S_T, T, -|K|) = S_T + |K|\,. \tag{10.5.2}$$

- We recognize this as simply a sum of one share of stock plus cash $|K|$.
- Hence to avoid arbitrage, the fair value of a negative strike call option at time t is equal to **a zero strike call plus the present value of cash** $|K|$

$$\begin{aligned} c(S, t, -|K|) &= c(S, t, K = 0) + \mathrm{PV}(|K|) \\ &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) + \mathrm{PV}(|K|)\,. \end{aligned} \tag{10.5.3}$$

- All of the above can be derived ***without reference to a probability model for the stock price movements.***

10.6 Put with non-positive strike

- Let us calculate the fair value of a put option with a strike price ≤ 0.
- One *can* argue that this is stupid, but it is an academic exercise in arbitrage.
- Let the strike be $-|K|$, where $K \leq 0$.
- The terminal payoff of the put option, at time T, is given by

$$p(S_T, T, -|K|) = \max(K - S_T, 0) = \max(-|K| - S_T, 0)\,. \tag{10.6.1}$$

- The value of $-|K| - S_T$ is never positive, hence the payoff is

$$p(S_T, T, -|K|) = 0\,. \tag{10.6.2}$$

- Hence to avoid arbitrage a put with a strike $K \leq 0$ must be worth zero today

$$p(S, t, -|K|) = 0\,. \tag{10.6.3}$$

- For put options, the above analysis **also applies to American options.**

10.7 Put call parity

- For a stock which pays discrete dividends during the lifetime of the options, the put-call parity relation is

$$c - p = S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) - \mathrm{PV}(K)\,. \tag{10.7.1}$$

- Consider what happens if K is zero or negative. Set $K = -|K|$, where $K \le 0$.

- Then from eqs. (10.5.3) and (10.6.3),

$$\begin{aligned} c(S,t,-|K|) &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) + \mathrm{PV}(|K|)\,, \\ p(S,t,-|K|) &= 0\,. \end{aligned} \tag{10.7.2}$$

- Subtraction yields

$$\begin{aligned} c(S,t,-|K|) - p(S,t,-|K|) &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) + \mathrm{PV}(|K|) \\ &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) - \mathrm{PV}(K)\,. \end{aligned} \tag{10.7.3}$$

- This is the put-call parity relation eq. (10.7.1).

- Hence put-call parity works even if the strike price is zero or negative.