Queens College, CUNY, Department of Computer Science

Computational Finance

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12 Lecture 12

12.1 Black-Scholes formula

- In this lecture we present the **Black-Scholes formula** for the fair value of a European option.
- The formula will also be extended to include dividends.

12.2 Black-Scholes equation

- We first recall the Black-Scholes partial differential equation.
- This is the financial pricing partial differential equation for a derivative where the only random (or stochastic) variable is the price of the underlying stock S.
- Let the current time be t and denote fair value of the derivative by $V(S,t)$.
- Also let the interest rate be r and the volatility be σ. Both the values of r and σ are constants.
- The Black-Scholes equation assumes that the random walk of the stock price obeys ***geometric Brownian motion.***
- The original equation derived by Black and Scholes assumed there no stock dividends. We shall include dividends later.
- The **Black-Scholes equation** is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\,\frac{\partial V}{\partial S} - rV = 0\,. \tag{12.2.1}$$

12.3 Black-Scholes formula

- We now present the **Black-Scholes formula**.
- The Black-Scholes formula is the solution of the Black-Scholes equation eq. (12.2.1), for the fair value of a European option.
- Let the strike price of the option be K and its expiration time be T, where $T > t$.
- We require the **cumulative Normal distribution** $N(x)$, which is given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du\,. \tag{12.3.1}$$

1. The function $N(x)$ is monotonically increasing and

$$0 < N(x) < 1 \qquad\qquad (-\infty < x < \infty)\,. \tag{12.3.2}$$

2. The limiting values of $N(x)$ are 0 as $x \to -\infty$ and 1 as $x \to \infty$.
3. Note also that for all $-\infty < x < \infty$,

$$N(x) + N(-x) = 1\,. \tag{12.3.3}$$

- It is helpful to define the following functions

$$d_1 = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} + \tfrac{1}{2}\sigma\sqrt{T-t}\ , \tag{12.3.4a}$$

$$d_2 = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} - \tfrac{1}{2}\sigma\sqrt{T-t} = d_1 - \sigma\sqrt{T-t}\,. \tag{12.3.4b}$$

- Denote the fair value of a European call and put by c and p, respectively.
- Then the Black-Scholes formulas for a European call and put option are respectively

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)\ , \tag{12.3.5a}$$

$$p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)\,. \tag{12.3.5b}$$

12.4 Continuous dividends

- Merton extended the Black-Scholes equation to treat a stock which pays continuous dividends at a rate q.
- The **Black-Scholes-Merton equation** generalizes eq. (12.2.1) to include continuous dividends, and is
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0\,. \tag{12.4.1}$$
- We shall mainly use eq. (12.4.1) in these lectures.
- The definitions of d_1 and d_2 are modified as follows:
$$d_1 = \frac{\ln(S/K) + (r-q)(T-t)}{\sigma\sqrt{T-t}} + \tfrac{1}{2}\sigma\sqrt{T-t}\ , \tag{12.4.2a}$$
$$d_2 = \frac{\ln(S/K) + (r-q)(T-t)}{\sigma\sqrt{T-t}} - \tfrac{1}{2}\sigma\sqrt{T-t} = d_1 - \sigma\sqrt{T-t}\,. \tag{12.4.2b}$$
- Then the Black-Scholes-Merton formulas for a European call and put option are respectively
$$c = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)\ , \tag{12.4.3a}$$
$$p = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1)\,. \tag{12.4.3b}$$

12.5 Put-call parity

- We verify that the expressions in eq. (12.4.3) satisfy put-call parity.
- Subtracting eq. (12.4.3b) from eq. (12.4.3a) yields

$$
\begin{aligned}
c - p &= Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \\
&\qquad - Ke^{-r(T-t)}N(-d_2) + Se^{-q(T-t)}N(-d_1) \\
&= Se^{-q(T-t)}\big[\,N(d_1) + N(-d_1)\,\big] - Ke^{-r(T-t)}\big[\,N(d_2) + N(-d_2)\,\big] \\
&= Se^{-q(T-t)} - Ke^{-r(T-t)}\,.
\end{aligned}
\tag{12.5.1}
$$

- In the above derivation we used the identity eq. (12.3.3) $N(x) + N(-x) = 1$.
- Hence the expressions in eq. (12.4.3) satisfy put-call parity.

12.6 Greeks in the Black-Scholes-Merton model

12.6.1 Delta

$$\begin{aligned} \Delta_c &= \quad e^{-q(T-t)} N(d_1)\,, \\ \Delta_p &= -e^{-q(T-t)} N(-d_1)\,. \end{aligned} \tag{12.6.1}$$

12.6.2 Gamma

$$\Gamma_c = \Gamma_p = \frac{e^{-q(T-t)}}{S\sigma\sqrt{T-t}}\,\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\,. \tag{12.6.2}$$

12.6.3 Vega

$$\nu_c = \nu_p = Se^{-q(T-t)}\sqrt{T-t}\,\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\,. \tag{12.6.3}$$

Notice that in the Black-Scholes-Merton model, Vega is proportional to Gamma:

$$\nu_{c,p} = S^2\sigma(T-t)\,\Gamma_{c,p}\,. \tag{12.6.4}$$

12.6.4 Rho

$$\begin{aligned} \rho_c &= \quad (T-t)Ke^{-r(T-t)} N(d_2)\,, \\ \rho_p &= -(T-t)Ke^{-r(T-t)} N(-d_2)\,. \end{aligned} \tag{12.6.5}$$

12.6.5 Theta

$$\begin{aligned} \Theta_c &= qSe^{-q(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - Se^{-q(T-t)}\,\frac{\sigma}{2\sqrt{T-t}}\,\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\,, \\ \Theta_p &= rKe^{-r(T-t)}N(-d_2) - qSe^{-q(T-t)}N(-d_1) - Se^{-q(T-t)}\,\frac{\sigma}{2\sqrt{T-t}}\,\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\,. \end{aligned} \tag{12.6.6}$$

12.7 Discrete dividends

- The Black-Scholes expressions in eq. (12.3.5) can be generalized to include discrete dividends.
- Suppose the stock pays discrete dividends D_i at times t_i, $i = 1, 2, \dots, n$ during the lifetime of the option, where $t < t_1 < \cdots < t_n < T$.
- It is helpful to employ the variable

$$\tilde{S} = S - \sum_{i=1}^{n} \mathrm{PV}(D_i) = S - \sum_{i=1}^{n} e^{-r(t_i - t)} D_i \,. \tag{12.7.1}$$

- The definitions of d_1 and d_2 are modified as follows:

$$d_1 = \frac{\ln(\tilde{S}/K) + r(T-t)}{\sigma\sqrt{T-t}} + \tfrac{1}{2}\sigma\sqrt{T-t} \ , \tag{12.7.2a}$$

$$d_2 = \frac{\ln(\tilde{S}/K) + r(T-t)}{\sigma\sqrt{T-t}} - \tfrac{1}{2}\sigma\sqrt{T-t} = d_1 - \sigma\sqrt{T-t}\,. \tag{12.7.2b}$$

- Then the expressions in eq. (12.3.5) are modifid to

$$c = \tilde{S}N(d_1) - Ke^{-r(T-t)}N(d_2) \ , \tag{12.7.3a}$$

$$p = Ke^{-r(T-t)}N(-d_2) - \tilde{S}N(-d_1)\,. \tag{12.7.3b}$$

- Subtraction confirms these expressions satisfy put-call parity with discrete dividends:

$$\begin{aligned} c - p &= \tilde{S}N(d_1) - Ke^{-r(T-t)}N(d_2) \\ &\qquad - Ke^{-r(T-t)}N(-d_2) + \tilde{S}N(-d_1) \\ &= \tilde{S}\big[\,N(d_1) + N(-d_1)\,\big] - Ke^{-r(T-t)}\big[\,N(d_2) + N(-d_2)\,\big] \\ &= \tilde{S} - Ke^{-r(T-t)}\,. \\ &= S - \left(\sum_{i=1}^{n} \mathrm{PV}(D_i)\right) - \mathrm{PV}(K)\,. \end{aligned} \tag{12.7.4}$$

12.8 Greeks with discrete dividends

12.8.1 Delta

$$\begin{aligned} \Delta_c &= \quad N(d_1)\,, \\ \Delta_p &= -N(-d_1)\,. \end{aligned} \tag{12.8.1}$$

12.8.2 Gamma

$$\Gamma_c = \Gamma_p = \frac{1}{\tilde{S}\sigma\sqrt{T-t}}\,\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\,. \tag{12.8.2}$$

12.8.3 Vega

$$\nu_c = \nu_p = \tilde{S}\sqrt{T-t}\,\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\,. \tag{12.8.3}$$

Notice again that Vega is proportional to Gamma:

$$\nu_{c,p} = \tilde{S}^2\sigma(T-t)\,\Gamma_{c,p}\,. \tag{12.8.4}$$

12.8.4 Rho

The expression for Rho is messy, with discrete dividends.

12.8.5 Theta

The expression for Theta is messy, with discrete dividends.
The expression especially complicated when a dividend payment date is crossed.