Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

Spring 2018

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due Friday, May 4, 2018, 11.59 pm

30 Homework lecture 30 & 31

You may (and should) use the FFT code in Lecture 30 to answer the questions below.

- **As experience has demonstrated, if you do not understand the above expressions/questions, THEN ASK.**
- **If you do not understand the words/sentences in the lectures, THEN ASK.**
- **Send me an email, explain what you do not understand.**
- **Do not just keep quiet and produce nonsense in exams.**

30.1 FFT simple tests

- Let us employ a simple input function

$$X_c(\theta) = \cos(m\theta)\,. \tag{30.1.1}$$

- **Set $n = 16$ (remember n should always be a power of 2.)**
- In fact it is better to set a number `num_bits` and then $n = 2^{\text{num\_bits}}$.
- Set $m = 1$.
  1. **Calculate $X_j = X_c(2\pi j/n) = \cos(2\pi mj/n)$ for $j = 0, \dots, n-1$.**
  2. **Use the FFT code to calculate F_k.**
  3. If you have done your work correctly you should obtain $\Re(F_1) = \Re(F_{15}) = 8$ and all other terms are zero.
- **Run the FFT calculation using other values of m and n (but use $m < n/2$ to avoid aliasing).**
  1. You should find that $\Re(F_m) = \Re(F_{n-m}) = n/2$ and all other terms are zero.
  2. In the special case $m = 0$, you should obtain $\Re(F_0) = n$ and all other terms are zero.
- Next employ the function

$$X_s(\theta) = \sin(m\theta)\,. \tag{30.1.2}$$

- **Run the FFT calculation using various values of m and n (but use $m < n/2$ to avoid aliasing).**
- You should find that $\Im(F_m) = -n/2$ and $\Im(F_{n-m}) = n/2$ and all other terms are zero.
- Next employ a complex input function

$$X(\theta) = \texttt{std} :: \texttt{complex} < \texttt{double} > (\cos(m\theta), \sin(m\theta))\,. \tag{30.1.3}$$

- **Run the FFT calculation using $0 \le m < n$ and various values of n.**
- You should find that $\Re(F_m) = n$ and all other terms are zero.
- **Run the FFT calculation using $m < 0$ and various values of n.**
  1. You should find that $\Re(F_{n+m}) = n$ and all other terms are zero.
  2. That is to say, if $m = -1$ then $\Re(F_{n-1}) = n$ and all other terms are zero.
  3. If $m = -2$ then $\Re(F_{n-2}) = n$ and all other terms are zero, etc.

30.2 Jacobi–Anger identity

- The **Jacobi–Anger identity** states that for real r and θ,

$$e^{ir\sin\theta} = \sum_{m=-\infty}^{\infty} e^{im\theta} J_m(r)\,. \tag{30.2.1}$$

- Here $J_m(r)$ is a **Bessel function of the first kind** (technically, *of integer order*).
- Set $r = 5$ and use $n = 64$ points. Compute X_j via $\theta_j = 2\pi j/n$ and

$$X_j = \texttt{std} :: \texttt{complex} < \texttt{double} > (\cos(m\theta_j), \sin(m\theta_j))\,. \tag{30.2.2}$$

- **Run the FFT calculation and compute the values of F_k.**
  1. You should find that all the F_k are real (all the imaginary parts should be zero).
  2. Bessel functions of integer order have the property that

$$J_{-m}(r) = (-1)^m\, J_m(r)\,. \tag{30.2.3}$$

  3. Therefore you should find that F_0 is real and for $1 \le m < n/2$, then

$$F_{n-m} = (-1)^m\, F_n\,. \tag{30.2.4}$$

  4. The above relation is valid for all values of r (and n). Try it.
- Bessel functions of integer order have the property that (just put $\theta = 0$ in eq. (30.2.1))

$$\sum_{m=-\infty}^{\infty} J_m(r) = 1\,. \tag{30.2.5}$$

- Therefore you should find that (because of the FFT normalization)

$$\sum_{k=0}^{n-1} F_k = n\,. \tag{30.2.6}$$

- The above sum is valid for all values of r (and n). Try it.
- Bessel functions of integer order also have the property that

$$\sum_{m=-\infty}^{\infty} J_m^2(r) = 1\,. \tag{30.2.7}$$

- Therefore you should find that (because of the FFT normalization)

$$\sum_{k=0}^{n-1} |F_k|^2 = n^2\,. \tag{30.2.8}$$

- The above sum is valid for all values of r (and n). Try it.

- *Let us calculate the Bessel functions.*

  1. You have been told that

  $$J_m(r) = \frac{1}{\pi} \int_0^{\pi} \cos(m\theta - r\,\sin\theta)\,d\theta\,. \tag{30.2.9}$$

  2. Use the trapezoid rule (for example) with 1000 points (for example) to compute the value of $J_m(r)$ and compare to the value of F_k.
  3. In fact, if you think about it, the FFT is evaluating eq. (30.2.9) using n points.

- **Set $r = 5$ and $n = 64$ and fill in the table below.**

m	$\Re(F_k)$	$J_m(r)$
0	4 d.p.	4 d.p.
1	4 d.p.	4 d.p.
2	4 d.p.	4 d.p.
3	4 d.p.	4 d.p.
4	4 d.p.	4 d.p.
5	4 d.p.	4 d.p.
6	4 d.p.	4 d.p.
7	4 d.p.	4 d.p.
8	4 d.p.	4 d.p.
9	4 d.p.	4 d.p.

30.3 Moving Average

- Employ the following input function

$$X(\theta) = \sin(\theta) \left[1 + 0.1 \sin(360\theta) + 0.2 \cos(90\theta) \right] . \tag{30.3.1}$$

- Hence these are high frequency oscillations, not really fluctuations, but never mind.
- Use $n = 1024$ points.
- Use the FFT to calculate a moving average $M(\theta)$ as described in the lectures.
- Use a parameter $a = 7 \times (2\pi/360)$, as described in the lectures.
- **Plot a graph of $X(\theta)$ and $M(\theta)$ for $\theta_j = 2\pi j/n$, where $j = 0, \ldots, n-1$.**