1 Question 1

- You are given the following integral (β is a real number):

$$I(\beta) = \int_0^1 \frac{1}{\sqrt{1 - \beta x^2}}\, dx\,. \tag{1.1}$$

- Calculate the value of $I(\beta)$ numerically using Simpson's rule and $n = 10$ steps.

- **Find a value β_* such that:**

$$0.69 < I(\beta_*) < 0.71\,. \tag{1.2}$$

- **State your value for β_* to one decimal place.**

- **State your value for $I(\beta_*)$ to three decimal places.**

2 Proper integral

- Some students claimed the integral is not proper hence there is no solution.
- Let us begin by analyzing if the interal is proper.
- This was a question in HW6, which no one answered completely correctly.
- The domain of integration is $0 \leq x \leq 1$.
- There is no problem with the integrand at $x = 0$.
- At $x = 1$ there will be a problem if $1 - \beta x^2 \leq 0$.
- The integral is proper as long as $1 - \beta x^2 > 0$ for all $0 \leq x \leq 1$.
- Hence we want $\beta x^2 < 1$, for all $0 \leq x \leq 1$.
- This will be the case if $\beta < 1$.
- In particular if $\beta \leq 0$, then the value of $1 - \beta x^2$ is always ≥ 0.
- **The integral is proper if $\beta < 1$ (including negative values).**

3 Simpson's rule

- Let us write the integrand as $f(x) = 1/\sqrt{1-\beta x^2}$.
- We are given $n = 10$, hence $h = 0.1$ and $x_i = 0, 0.1, \ldots, 0.9, 1.0$ for $i = 0, \ldots, 10$.
- Then applying Simpson's rule yields

$$\begin{aligned}
I_{\text{Simpson}} &= \frac{h}{3}\biggl[f(0) + f(1) \\
&\qquad + 4\Bigl(f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9) \Bigr) \\
&\qquad + 2\Bigl(f(0.2) + f(0.4) + f(0.6) + f(0.8) \Bigr)\biggr] \\
&= \frac{1}{30.0}\biggl[1 + \frac{1}{\sqrt{1-\beta}} \\
&\qquad + 4\biggl(\frac{1}{\sqrt{1-\beta(0.1)^2}} + \frac{1}{\sqrt{1-\beta(0.3)^2}} + \frac{1}{\sqrt{1-\beta(0.5)^2}} + \frac{1}{\sqrt{1-\beta(0.6)^2}} + \frac{1}{\sqrt{1-\beta(0.9)^2}} \biggr) \\
&\qquad + 2\biggl(\frac{1}{\sqrt{1-\beta(0.2)^2}} + \frac{1}{\sqrt{1-\beta(0.4)^2}} + \frac{1}{\sqrt{1-\beta(0.6)^2}} + \frac{1}{\sqrt{1-\beta(0.8)^2}} \biggr)\biggr]
\end{aligned} \tag{3.1}$$

- This can be coded in Excel, for example. Set a value for β and see what you get.
- The question wants a value for β_* such that $0.69 < I(\beta_*) < 0.71$.
- **A value $-5.0 \le \beta_* \le -4.4$ will work.**
- Since the question asks for the value of β_* to one decimal place, acceptable values are

$$\beta_* = -4.4,\ -4.5,\ -4.6,\ -4.7,\ -4.8,\ -4.9,\ -5.0\,. \tag{3.2}$$

4 Analytical evaluation of integral & fixed-point iteration

- **The material below is NOT required by the examination question.**
- One student attempted to evaluate the integral analytically, and made a mess.
- But let us evaluate the integral analytically.
- It will serve as an educational exercise in fixed point iteration.
- First, anticipating that $\beta_* < 0$, let us write $\beta = -c^2$.
- Then the integral is
$$I_c = \int_0^1 \frac{1}{\sqrt{1+c^2x^2}}\,dx\,. \tag{4.1}$$
- To evaluate this we require the hyperbolic sine and cosine functions.
- Some of you may not be familiar with these functions.
- Just as there are sine and cosine $\sin(x)$ and $\cos(x)$, there are also the hyperbolic sine and cosine $\sinh(x)$ and $\cosh(x)$.
- They are defined as follows:
$$\sinh(x) = \frac{e^x - e^{-x}}{2}\,, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}\,. \tag{4.2}$$
- They have the following properties:
$$\begin{aligned} \cosh(-x) &= \cosh(x)\,, \\ \sinh(-x) &= -\sinh(x)\,, \\ \cosh(0) &= 1\,, \\ \sinh(0) &= 0\,, \\ \cosh^2(x) - \sinh^2(x) &= 1\,, \\ \frac{d}{dx}\sinh(x) &= \cosh(x)\,, \\ \frac{d}{dx}\cosh(x) &= \sinh(x)\,. \end{aligned} \tag{4.3}$$
- Make the substitution (change of variables)
$$x = \frac{1}{c}\sinh(u)\,. \tag{4.4}$$
- Notice the division by c, hence we must have $c \neq 0$. This will come back to bite us later.
- Then the derivative is
$$\frac{dx}{du} = \frac{1}{c}\cosh(u)\,. \tag{4.5}$$

- The limits of integration for u are $0 \le 0 \le \sinh^{-1}(c)$.
- Hence the value of the integral is

$$\begin{aligned} I_c &= \int_0^{\sinh^{-1}(c)} \frac{1}{\sqrt{1+\sinh^2(u)}}\,\frac{1}{c}\,\cosh(u)\,du \\ &= \frac{1}{c}\int_0^{\sinh^{-1}(c)} \frac{1}{\cosh(u)}\,\cosh(u)\,du \\ &= \frac{1}{c}\int_0^{\sinh^{-1}(c)} du \\ &= \frac{\sinh^{-1}(c)}{c}\,. \end{aligned} \tag{4.6}$$

- We can solve for the value of c using fixed point iteration.
- Let the target value of the integral be I_* (which is 0.7 for the exam question).
- Then we wish to find a value c_* such that

$$I_* = \frac{\sinh^{-1}(c_*)}{c_*}\,. \tag{4.7}$$

- Multiply through by c_* to obtain

$$c_* I_* = \sinh^{-1}(c_*)\,. \tag{4.8}$$

- Hence the equation to solve is

$$c_* = \sinh(c_* I_*)\,. \tag{4.9}$$

- This can be solved using fixed point iteration, but there are some difficulties.

  1. First, one obvious solution is $c_* = 0$ because $\sinh(0) = 0$. We must exclude this solution. I mentioned above that the requirement $c \neq 0$ would come back to bite us.
  2. Second, fixed point iteration using the above equation is unstable. A graph of c and $\sinh(cI_*)$ is displayed in Fig. 1, for $I_* = 0.7$. The slope of $\sinh(cI_*)$ exceeds unity in the vicinity of the fixed point. Hence the fixed point iteration is unstable. The only stable fixed point is $c_* = 0$, which is an unwanted solution.
  3. A list of iterates is shown below, starting from $c_0 = 3$.
     The iteration diverges to ∞.

i	c_i
0	3
1	4.0219
2	8.3191
3	169.0588
4	1.24×10^{51}

- Hence we must reformulate the iteration, to make it stable.

  1. Let us ‘flip’ or reflect the red curve around the 45° straight line (black). Then the slope of the reflected curve will be less than unity in the vicinity of the fixed point.
  2. How to do this? Let the above iteration be $c = g_1(c)$. We seek another function $g_2(c)$ such that the average is $\frac{1}{2}(g_1(c) + g_2(c)) = c$. Hence $g_2(c) = 2c - g_1(c) = 2c - \sinh(cI_*)$.
  3. Hence we change the iteration formula to the following:

$$c_* = 2c_* - \sinh(c_* I_*)\,. \tag{4.10}$$

  4. A graph of c and $2c - \sinh(cI_*)$ is displayed in Fig. 2, for $I_* = 0.7$. The slope of $2c - \sinh(cI_*)$ is less than unity in the vicinity of the fixed point. Hence this time the fixed point iteration is stable.
  5. A list of iterates is shown below, also starting from $c_0 = 3$.
     The value of $\beta_i = -c_i^2$ is also tabulated.
     The iteration converges to $\beta_* \simeq -4.68$.
     This is the same as the value using Simpson’s rule (with more intervals in the integration).

i	c_i	β
0	3	-9
1	1.9781	-3.9131
2	2.0847	-4.3459
3	2.1341	-4.5545
4	2.1533	-4.6368
5	2.1601	-4.6659
6	2.1624	-4.6758
7	2.1631	-4.6791
8	2.1634	-4.6802
9	2.1635	-4.6806
10	2.1635	-4.6807

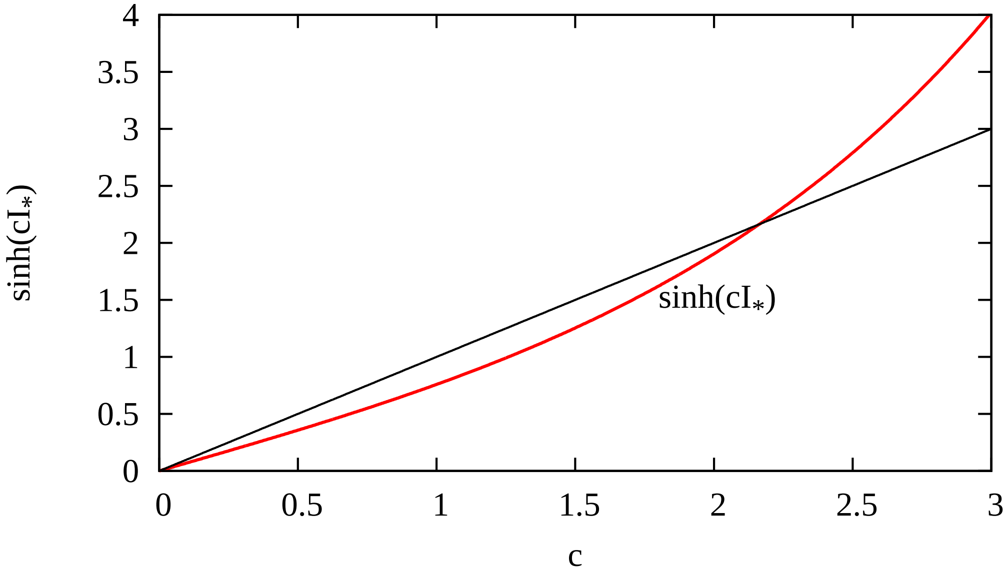

Figure 1: Graph of c and $\sinh(cI_*)$ for $I_* = 0.7$.

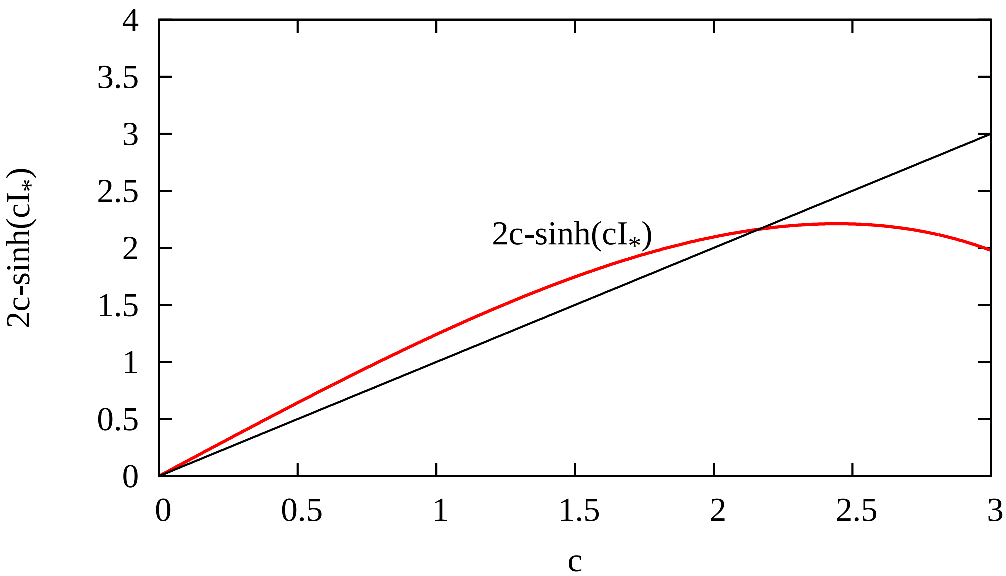

Figure 2: Graph of c and $2c - \sinh(cI_*)$ for $I_* = 0.7$.