Queens College, CUNY, Department of Computer Science
Computational Finance
CSCI 365 / 765
Fall 2017
Instructor: Dr. Sateesh Mane

November 30, 2017

20 Lecture 20

Implied volatility

- In this lecture we study the concept of **implied volatility**.
- Suppose a derivative is trading at a price $V_{\rm mkt}$ in the financial markets. What value of the volatility must we employ in the Black–Scholes–Merton formula, so that the calculated theoretical fair value $V(\sigma)$ equals $V_{\rm mkt}$? The answer is the implied volatility.
- Fundamentally, we "invert" the Black–Scholes–Merton equation. We find the value of the volatility such that the theoretical fair value equals the market price of the derivative.
- Note that the implied volatility is not guaranteed to exist, or to be unique. There may be multiple solutions, or no solution.
- We shall also study the related concept of a **volatility smile**.
- A concept closely related to the volatility smile is an **implied volatility surface**.
- **There is no explicit mathematical probability theory in this lecture.**

20.1 Implied volatility

- Suppose a European call option is trading in the market at a price c_{mkt} today.
- ***What value of the volatility σ will yield a fair value equal to the market price?***
- Note that to answer the above question, we calculate the fair value using the Black–Scholes–Merton equation.
- The answer is the value of the **implied volatility.**
- We "invert" the Black–Scholes–Merton formula to deduce a value of the volatility from the market price of the option.
- We solve the following equation for the volatility σ:

$$Se^{-q(T-t_0)}N(d_1) - Ke^{-r(T-t_0)}N(d_2) = c_{\mathrm{mkt}}\,. \tag{20.1.1}$$

- Recall that

$$d_1 = \frac{\ln(S/K) + (r-q)(T-t_0)}{\sigma\sqrt{T-t_0}} + \tfrac{1}{2}\sigma\sqrt{T-t_0}\,, \tag{20.1.2a}$$

$$d_2 = d_1 - \sigma\sqrt{T-t_0}\,. \tag{20.1.2b}$$

- The answer is the **implied volatility.**
- The implied volatility must usually be obtained by an iterative calculation.
- Note that the concept of the implied volatility applies not only for options but also other derivatives on a stock.
- In general, e.g. American options, the fair value must be calculated using a numerical model such as the binomial model.

20.2 Example 1

- Consider a European call option with strike $K = 100$ and expiration time $T = 1$.
- The underlying stock does not pay dividends. Say that $S_0 = 100$ today, where $t_0 = 0$.
- Also for simplicity let us say the interest rate is zero $r = 0$.
- Hence the relevant parameter values are:

$$S_0 = 100\,, \qquad K = 100\,, \qquad r = 0\,, \qquad q = 0\,, \qquad T = 1\,, \qquad t_0 = 0\,. \tag{20.2.1}$$

- The option trades at a market price of $c_{\text{mkt}} = 12$ today.
- The implied volatility must usually be obtained by an iterative calculation.
  1. Let us try an initial guess of $\sigma_0 = 0.3$.
  2. The resulting fair value, using the Black–Scholes–Merton formula, is $c = 11.9235$.
  3. This is too low. Hence we guess a higher value of the volatility $\sigma_1 = 0.31$.
  4. The resulting fair value, using the Black–Scholes–Merton formula, is $c = 12.3179$.
  5. This is too high. However, we have bracketed the market price of $c_{\text{mkt}} = 12$.
  6. We now know the answer lies in the interval $0.3 < \sigma < 0.31$.
  7. Hence we try a new value of the volatility $\sigma_2 = 0.305$.
  8. The resulting fair value, using the Black–Scholes–Merton formula, is $c = 12.1207$.
  9. This is still too high. Hence we try a new value of the volatility $\sigma_3 = 0.302$.
  10. The resulting fair value, using the Black–Scholes–Merton formula, is $c = 12.0024$.
  11. We declare this to be sufficiently close to the target value of 12.
  12. In other words, there is a numerical tolerance in our iterative calculation.
- The implied volatility of this option is $\sigma_{\text{implied}} = 0.302$.

20.3 Example 2

- Consider an American put option with strike $K = 100$ and expiration time $T = 0.5$.
- The relevant parameter values in this case are:

$$S_0 = 105\,, \qquad K = 100\,, \qquad r = 0.05\,, \qquad q = 0.02\,, \qquad T = 0.5\,, \qquad t_0 = 0\,. \tag{20.3.1}$$

- The option trades at a market price of $P_{\rm mkt} = 9$ today.
- Because the option is American, we must calculate its fair value using a numerical model.
- We employ the binomial model with $n = 100$ time steps.
- We must again determine the implied volatility by an iterative calculation.
  1. Let us try an initial guess of $\sigma_0 = 0.4$.
  2. The resulting fair value, using the binomial model with $n = 100$, is $P = 8.51817$.
  3. This is too low. Hence we guess a higher value of the volatility $\sigma_1 = 0.42$.
  4. The resulting fair value, using the binomial model with $n = 100$, is $P = 9.07554$.
  5. This is too high. However, we have bracketed the market price of $P_{\rm mkt} = 9$.
  6. We now know the answer lies in the interval $0.4 < \sigma < 0.42$.
  7. Hence we try a new value of the volatility $\sigma_2 = 0.415$.
  8. The resulting fair value, using the binomial model with $n = 100$, is $P = 8.94621$.
  9. This is too low. Hence we try a new value of the volatility $\sigma_3 = 0.4175$.
  10. The resulting fair value, using the binomial model with $n = 100$, is $P = 9.00588$.
  11. We declare this to be sufficiently close to the target value of 9.
- The implied volatility of this option is $\sigma_{\rm implied} = 0.4175$.
- Note that the above answer depends on our use of the binomial model, and the choice of $n = 100$ steps.

20.4 Implied volatility & market price of an option

- For options on a stock (both puts and calls and also both American and European), an option's theoretical fair value increases as the volatility increases (at least within the context of Geometric Brownian Motion).

- Hence, as long as the option's market price does not violate any arbitrage pricing bounds, there is a one to one correspondence between the implied volatility and the market price of an option.

- The implied volatility is therefore an equivalent way of specifying the price of an option.

- The situation is analogous to the yield of a bond. The yield of a bond is an equivalent way of specifying the price of a bond. Bond traders prefer to think in terms of yields rather than bond prices.

- Options traders prefer to think in terms of the implied volatility rather than the market price of an option.

- Options traders buy and sell options to trade implied volatility.

- Options traders use options to perform gamma trading, as described in Lecture 16.

  1. Options traders buy options with low implied volatility. Such options have low market prices and may be underpriced. The options traders have a view that the actual volatility of the stock over the lifetime of the option (known as the **realized volatility**) will be higher than the implied volatility of the option. This will lead to a profit via gamma trading of the option.
  2. Options traders sell options with high implied volatility. Such options have high market prices and may be overpriced. The options traders have a view that the realized volatility of the stock over the lifetime of the option will be lower than the implied volatility of the option. This will lead to a profit via gamma trading of the option.

20.5 Gamma trading and time decay

- Recall the Black–Scholes–Merton equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0\,. \tag{20.5.1}$$

- For simplicity, suppose that the risk-free interest rate and dividends are both zero.
- The Black–Scholes–Merton equation simplifies to

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0\,. \tag{20.5.2}$$

- This is a sum of a Gamma term and a Theta time decay term.
- The term in Gamma is positive and the term in Theta is negative (at least if $r = q = 0$).
- If stock prices obeyed Geometric Brownian Motion exactly, the two terms would cancel exactly and gamma trading would not yield a profit.
- As explained in Lecture 16, options traders engage in gamma trading to make a profit by trading options. This is possible because real stock prices do not obey Geometric Brownian Motion exactly.

20.6 Gamma trading: worked example

20.6.1 Basic parameters

- Consider a European call option with strike $K = 100$ and expiration time $T = 0.5$.
- For simplicity suppose the underlying stock does not pay dividends.
- Also for simplicity suppose the risk-free interest rate is zero $r = 0$.
- Let us analyze a simple scenario of gamma trading with this option.

20.6.2 Day 0

- The current time is $t_0 = 0$.
- The stock price is $S_0 = 95$.
- The market price of the option is $M_0 = 8.64371$.
- The implied volatility is $\sigma_0 = 0.4$.
- The Delta of the option, using $\sigma_0 = 0.4$, is $\Delta_0 = 0.484075$.
- We begin our trading with zero options, stock and cash.
- We form a portfolio Π of long one European call option and short selling Δ_0 shares of stock.
- Observe that we delta hedged the option. This is an essential feature of gamma trading.
- We pay money to buy the option and receive money by short selling the stock.
- The resulting cash is positive and we save it in a bank. The amount of cash in the bank is

$$\text{Money}_0 = \Delta_0 S_0 - M_0 = 37.3435\,. \tag{20.6.1}$$

20.6.3 Day 1

- The current time is $t_1 = 0.01$.
- The stock price is $S_1 = 95.3$.
- The market price of the option is $M_1 = 8.81457$.
- The implied volatility is $\sigma_1 = 0.405$.
- The Delta of the option, using $\sigma_1 = 0.405$, is $\Delta_1 = 0.488808$.
- We must rebalance our hedge (short stock position) to delta hedge the option.
- We short $\Delta_1 - \Delta_0$ shares of stock. We do this at the stock price S_1.
- The amount of cash in the bank is now

$$\text{Money}_1 = \text{Money}_0 + (\Delta_1 - \Delta_0) S_1 = 37.7945\,. \tag{20.6.2}$$

20.6.4 Day 2

- The current time is $t_2 = 0.02$.
- The stock price is $S_2 = 95.1$.
- The market price of the option is $M_2 = 8.7382$.
- The implied volatility is $\sigma_2 = 0.41$.
- The Delta of the option, using $\sigma_2 = 0.41$, is $\Delta_2 = 0.486103$.
- We must rebalance our hedge (short stock position) to delta hedge the option.
- We short $\Delta_2 - \Delta_1$ shares of stock. We do this at the stock price S_2.
- Note that $\Delta_2 - \Delta_1$ is a *negative* number. Hence we really *bought shares* to *reduce* our short stock position and *paid money* to do so. We let the mathematics handle the minus signs.
- The amount of cash in the bank is now

$$\text{Money}_2 = \text{Money}_1 + (\Delta_2 - \Delta_1)S_2 = 37.6372\,. \tag{20.6.3}$$

20.6.5 Day 3

- The current time is $t_3 = 0.03$.
- The stock price is $S_3 = 94.9$.
- The market price of the option is $M_3 = 8.65842$.
- The implied volatility is $\sigma_3 = 0.415$.
- The Delta of the option, using $\sigma_3 = 0.415$, is $\Delta_3 = 0.483356$.
- We must rebalance our hedge (short stock position) to delta hedge the option.
- We short $\Delta_3 - \Delta_2$ shares of stock. We do this at the stock price S_3.
- Once again $\Delta_3 - \Delta_2$ is a negative number, so we are buying shares to reduce our short stock position and paying money to do so.
- The amount of cash in the bank is now

$$\text{Money}_3 = \text{Money}_2 + (\Delta_3 - \Delta_2)S_3 = 37.2765\,. \tag{20.6.4}$$

20.6.6 Day 4

- The current time is $t_4 = 0.04$.
- The stock price is $S_4 = 95.3$.
- The market price of the option is $M_4 = 8.86616$.
- The implied volatility is $\sigma_4 = 0.42$.
- The Delta of the option, using $\sigma_4 = 0.42$, is $\Delta_4 = 0.489402$.
- We must rebalance our hedge (short stock position) to delta hedge the option.
- We short $\Delta_3 - \Delta_2$ shares of stock. We do this at the stock price S_3.
- The amount of cash in the bank is now

$$\text{Money}_4 = \text{Money}_3 + (\Delta_4 - \Delta_3)S_4 = 37.8527\,. \tag{20.6.5}$$

20.6.7 Close out

- We close out (or liquidate) our portfolio at the end of day 4.
- Hence we sell our option at the market price M_4.
- We close out our short stock position by buying Δ_4 shares of stock at the price S_4.
- Hence we are left with only cash in the bank.
- The amount of cash in the bank is our profit.
- Our profit from gamma trading is

$$\text{Profit} = \text{Money}_4 + M_4 - \Delta_4 S_4 = 0.0788919\,. \tag{20.6.6}$$

20.6.8 Alternative calculation of profit

- Every day, starting from day 1, we calculate the change in the option price (from the previous day) minus the change in value of the short stock position

$$\text{Change in portfolio value}_i = (M_i - M_{i-1}) - (\Delta_i - \Delta_{i-1})S_i\,. \tag{20.6.7}$$

- This yields a set of daily values

$$\begin{aligned}
\text{Change}_1 &= (8.81457 - 8.63471) - (0.488808 - 0.484075) \times 95.3 = 0.0256464\,,\\
\text{Change}_2 &= (8.7382 - 8.81457) - (0.486103 - 0.488808) \times 95.1 = 0.0213893\,,\\
\text{Change}_3 &= (8.65842 - 8.7382) - (0.483356 - 0.486103) \times 94.9 = 0.0174433\,,\\
\text{Change}_4 &= (8.86618 - 8.65842) - (0.489402 - 0.483356) \times 95.3 = 0.0144129\,.
\end{aligned} \tag{20.6.8}$$

- The sum of the daily changes in the portfolio value equals the overall profit

$$\text{Profit} = \sum_{i=1}^{4} \text{Change}_i = 0.0256464+0.0213893+0.0174433+0.0144129 = 0.0788919\,. \tag{20.6.9}$$

20.7 Historical volatility

- One can estimate the volatility of a stock from the history of the stock price over tht previous N trading days (where the value of N is chosen by the user, e.g. for the previous quarter or the previous year).

- From the above time series data, one can compute the variance of the stock price movements to obtain the statistical estimate

$$\sigma_{\rm hist}^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{S_i - S_{i-1}}{S_i}\right)^2 . \tag{20.7.1}$$

- The value of $\sigma_{\rm hist}$ so obtained is called the **historical volatility.**

- Technically, the (square of the) volatility is a variance, hence one should subtract a term in eq. (20.7.1) for the mean of the stock price movements. In practice, market practitioners do not do this and employ eq. (20.7.1) "as is" and call the answer the (square of the) historical volatility.

- As a practical matter, the historical volatility has not proved to be useful for calculating (or estimating) the theoretical fair value of an option.

  1. Technically, the volatility is a property of the underlying stock.
  2. Hence the volatility should be the same for all options on a given stock.
  3. In practice, the implied volatility is ***not*** the same for all options on a given stock.
  4. Moreover, the estimate obtained from eq. (20.7.1) depends on the value of N, which would not be the case if stock prices really obeyed Geometric Brownian Motion exactly.

- All of this is an indication that real stock do not obey Geometric Brownian Motion exactly.

20.8 Volatility smile and frown and implied volatility surface

- If stock prices really obeyed Geometric Brownian Motion exactly, then all options of a given stock would all have the same implied volatility (which would be equal to the historical volatility).

- In practice, the implied volatility varies with the strike and expiration of listed options.

- Suppose we collect all the listed options on a stock for a given expiration (e.g. three months) and plot a graph of their implied volatilities against their strike prices.

- Experience in the financial markets has shown that options which are out of the money yield more reliab;e va;ues for the implied vo;atility. The prices of options which are in the money (especially deep in the money) receive a contribution from the option intrinsic value, as opposed to value from the volatility.

- Hence if the current stock price is S_0, we use put options for strikes $K < S_0$ and call options for strikes $K > S_0$. If $K = S_0$ then we use the implied volatility from both the put and the call.

- (I am quoting from memory here. The following information may not be completely accurate.) Prior to the "Black Monday" stock market crash of 1987, the implied volatilities of listed options used to exhibit a pattern known as a **volatility smile** which meant that the implied volatility displayed a minimum for strikes close to the current stock price. See Fig. 1 for a schematic sketch of a volatility smile.

- (I am quoting from memory here. The following information may not be completely accurate.) After the stock market crash of 1987, the implied volatilities of listed options nowadays display a **volatility frown** which means that the implied volatility decreases monotonically as the option strike increases. See Fig. 2 for a schematic sketch of a volatility frown.

- What *is* a fact is that the typical pattern of the implied volatilities of listed options nowadays is a volatility frown.

- If we collect graphs such as Fig. 2 for all the listed option expirations, we can form a two dimensional surface. This is called an **implied volatility surface.**

- Regrettably, lack of time does not permit us to discuss implied volatility surfaces in any detail.

- From my own personal experience, constructing an implied volatility surface from market data can be surprisingly difficult. It is frequently difficult to obtain a complete set of market data for option prices, for all the strikes and expirations, even for options on actively traded stocks. It can partly be a problem of volume of data. Especially for heavily traded stocks (and stock indices), there can be a large number of listed options. (For example, there are over a thousand listed options on the major stock indices such as the S&P 500 or the FTSE 100.) Many times, some of the data is delayed, and not all the data is available at the start of the day, to prepare a reliable implied volatility surface for use by the trading desk.

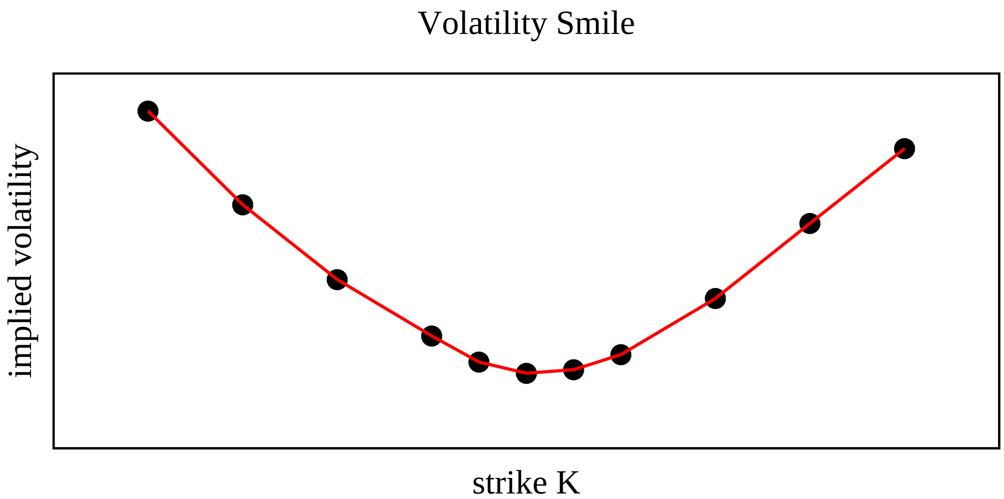

Figure 1: Schmatic graph of volatility smile of impliled volatilities plotted againt the option strike prices.

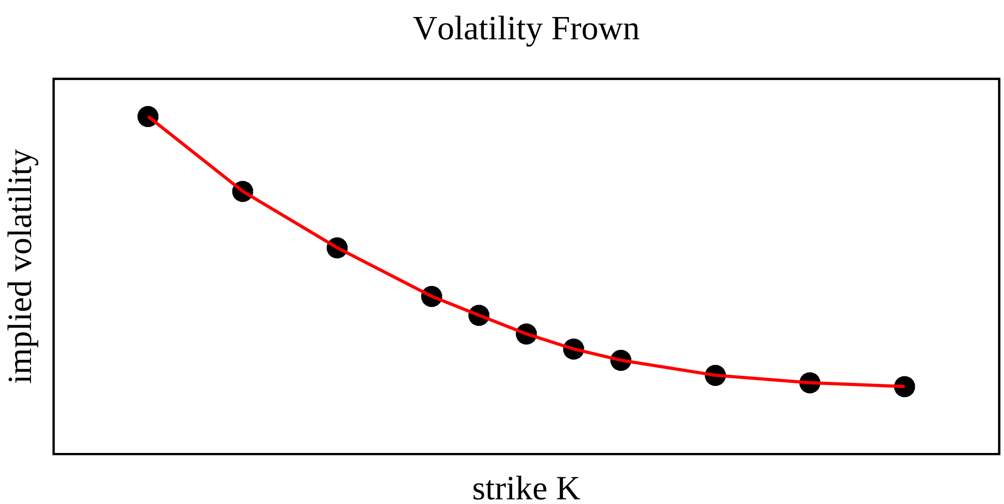

Figure 2: Schmatic graph of volatility frown of impliled volatilities plotted againt the option strike prices.