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28 Lecture 28

Fourier Series

- In this lecture, we continue our study of **Fourier series.**
- We shall treat functions of one variable only.
- We shall study how much information is obtained from a finite sample of function evaluations.
- We shall also study the important concept of **aliasing.**
- We shall also introduce the concept of the **Nyquist frequency.**

28.1 Fourier series

- This is a review from a previous lecture, reproduced here for ease of reference.
- Let $f(\theta)$ be a periodic function of and angle θ with period 2π.
- We assume f is sufficiently well behaved to justify the relevant calculations or algorithms.
  1. For example, the function is absolutely integrable around a circle:
  $$\int_0^{2\pi} |f(\theta)|\,d\theta < \infty\,. \tag{28.1.1}$$
  2. The function also has at most a finite number of discontinuities in the interval $0 \le \theta < 2\pi$.
- Then f can be expressed (or expanded) in a **Fourier series** as follows:
$$\begin{aligned} f(\theta) &= \tfrac{1}{2}a_0 + a_1\cos(\theta) + a_2\cos(2\theta) + a_3\cos(3\theta) + \cdots \\ &\qquad + b_1\sin(\theta) + b_2\sin(2\theta) + b_3\sin(3\theta) + \cdots \\ &= \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty}\big[\,a_j\cos(j\theta) + b_j\sin(j\theta)\,\big]\,. \end{aligned} \tag{28.1.2}$$
- The coefficients a_j and b_j are obtained from the function $f(\theta)$ via
$$\begin{aligned} a_j &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(j\theta)\,d\theta \qquad && (j \ge 0)\,, \\ b_j &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(j\theta)\,d\theta \qquad && (j > 0)\,. \end{aligned} \tag{28.1.3}$$
- We have seen that there are different ways of summing the series in eq. (28.1.2).
- We have seen that if the function is discontinuous (example of window function), the partial sums of the series may exhibit the Gibbs–Wilbraham phenomenon, and may not converge to the function $f(\theta)$ at (or near) the location of a discontinuity.
- The integrals in eq. (28.1.3) must be computed numerically, in general.
- For example, the input function $f(\theta)$ may consist of a set of experimentally measured data.
- **How many points (how many values of $f(\theta)$) are required to compute the integrals in eq. (28.1.3) to sufficient accuracy?**
- This is an obvious question for any numerical algorithm.
- We shall actually turn the question around and investigate: how much information do we obtain if we sample the function at n points?

28.2 Simple example

- Suppose we are told the function f contains Fourier harmonics up to $j = 2$ only.
- That is to say, the function is

$$f_{\rm ex1}(\theta) = \tfrac{1}{2}A_0 + A_1\cos(\theta) + B_1\sin(\theta) + A_2\cos(2\theta) + B_2\sin(2\theta)\,. \tag{28.2.1}$$

- There are five unknown parameters A_0, A_1, A_2, B_1 and B_2.
- **How many function evaluations are required to determine the values $(A_0, \dots, B_2)$?**
- Let us use **four points, spaced uniformly around the circle at $\theta = 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$.**
- Let us work through the calculation step by step to see what happens.
- We fit the function $f_{\rm ex1}$ using the Fourier seriess in eq. (28.1.2).
- We compute the integrals in eq. (28.1.3) using 4 subintervals of length $h = 2\pi/4 = \pi/2$:
- Then $\theta_k = 2\pi k/4 = k\pi/2$ and

$$\begin{aligned} a_j &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(j\theta)\,d\theta \to \frac{h}{\pi}\sum_{k=0}^{3} f(\theta_k)\cos(j\theta_k) = \frac{1}{2}\sum_{k=0}^{3} f\Bigl(\frac{k\pi}{2}\Bigr)\,\cos\Bigl(\frac{jk\pi}{2}\Bigr)\,, \\ b_j &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(j\theta)\,d\theta \to \frac{h}{\pi}\sum_{k=0}^{3} f(\theta_k)\sin(j\theta_k) = \frac{1}{2}\sum_{k=0}^{3} f\Bigl(\frac{k\pi}{2}\Bigr)\,\sin\Bigl(\frac{jk\pi}{2}\Bigr)\,. \end{aligned} \tag{28.2.2}$$

- What values does the above approximation yield for $(a_0, a_1, a_2, b_1, b_2)$?
- Let us tabulate the values of 1, $\cos(\theta)$ and $\sin(\theta)$ at the four points.

θ	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$
1	1	1	1	1
$\cos(\theta)$	1	0	-1	0
$\cos(2\theta)$	1	-1	1	-1
$\sin(\theta)$	0	1	0	-1
$\sin(2\theta)$	0	0	0	0

- Next let us tabulate the values of $f(\theta)$ at the four points.

$$\begin{aligned} f_{\rm ex1}(0) &= \tfrac{1}{2}A_0 + A_1 + A_2\,, \\ f_{\rm ex1}(\tfrac{1}{2}\pi) &= \tfrac{1}{2}A_0 + B_1 - A_2\,, \\ f_{\rm ex1}(\pi) &= \tfrac{1}{2}A_0 - A_1 + A_2\,, \\ f_{\rm ex1}(\tfrac{3}{2}\pi) &= \tfrac{1}{2}A_0 - B_1 - A_2\,. \end{aligned} \tag{28.2.3}$$

- We evaluate the sums in eq. (28.2.2).

1. We obtain the following result for a_0:

$$\begin{aligned}
a_0 &= \frac{1}{2}\sum_{k=0}^{3} f_{\rm ex1}\Big(\frac{k\pi}{2}\Big) \\
&= \frac{1}{2}\Big[(\tfrac{1}{2}A_0 + A_1 + A_2) + (\tfrac{1}{2}A_0 + B_1 - A_2) + (\tfrac{1}{2}A_0 - A_1 + A_2) + (\tfrac{1}{2}A_0 - B_1 - A_2)\Big] \\
&= A_0\,.
\end{aligned} \tag{28.2.4}$$

2. We obtain the following result for a_1:

$$\begin{aligned}
a_1 &= \frac{1}{2}\sum_{k=0}^{3} f_{\rm ex1}\Big(\frac{k\pi}{2}\Big)\,\cos\Big(\frac{k\pi}{2}\Big) \\
&= \frac{1}{2}\Big[(\tfrac{1}{2}A_0 + A_1 + A_2) + 0 - (\tfrac{1}{2}A_0 - A_1 + A_2) + 0\Big] \\
&= A_1\,.
\end{aligned} \tag{28.2.5}$$

3. We obtain the following result for b_1:

$$\begin{aligned}
b_1 &= \frac{1}{2}\sum_{k=0}^{3} f_{\rm ex1}\Big(\frac{k\pi}{2}\Big)\,\sin\Big(\frac{k\pi}{2}\Big) \\
&= \frac{1}{2}\Big[0 + (\tfrac{1}{2}A_0 + B_1 - A_2) + 0 - (\tfrac{1}{2}A_0 - B_1 - A_2) + 0\Big] \\
&= B_1\,.
\end{aligned} \tag{28.2.6}$$

4. We obtain the following result for a_2:

$$\begin{aligned}
a_2 &= \frac{1}{2}\sum_{k=0}^{3} f_{\rm ex1}\Big(\frac{k\pi}{2}\Big)\,\cos\Big(\frac{2k\pi}{2}\Big) \\
&= \frac{1}{2}\Big[(\tfrac{1}{2}A_0 + A_1 + A_2) - (\tfrac{1}{2}A_0 + B_1 - A_2) + (\tfrac{1}{2}A_0 - A_1 + A_2) - (\tfrac{1}{2}A_0 - B_1 - A_2)\Big] \\
&= \boldsymbol{2A_2}\,.
\end{aligned} \tag{28.2.7}$$

5. As for b_2, we obtain zero because $\sin(2\theta) = 0$ at all the points θ_k:

$$b_2 = \frac{1}{2}\sum_{k=0}^{3} f_{\rm ex1}\Big(\frac{k\pi}{2}\Big)\,\sin\Big(\frac{2k\pi}{2}\Big) = \frac{1}{2}\big[0+0+0+0\big] = \mathbf{0}\,. \tag{28.2.8}$$

- We obtain the correct results for a_0, a_1 and b_1.

- We can compensate for the factor of 2 in a_2 to obtain the value of A_2, but the value we obtain for $b_2(=0)$ is not useful to determine B_2.

- Hence using only points we obtain four outputs, which makes sense for a linear operation.

28.3 General case

- Suppose the function f contains a finite number of Fourier harmonics, with Fourier coefficients $(A_0, \dots, A_m)$ and $(B_1, \dots, B_{m-1})$, i.e. a set of $2m$ Fourier harmonics.

$$\begin{aligned} f(\theta) = \tfrac{1}{2}A_0 + A_1\cos(\theta) + A_2\cos(2\theta) + \cdots + A_{m+1}\cos((m+1)\theta) \\ + B_1\sin(\theta) + B_2\sin(2\theta) + \cdots + B_m\sin(m\theta)\,. \end{aligned} \tag{28.3.1}$$

- Note that we exclude B_m because the above procedure could not determine it.
- **Then we can determine all the values of $(A_0, A_1, \dots, A_m)$ and $(B_1, \dots, B_{m-1})$, using $n = 2m$ points spaced uniformly around the circle at $\theta_k = 2\pi k/n$, where $k = 0, 1, \dots, n-1$.**
- The proof follows from the orthogonality and normalization properties of the sines and cosines.
- The Fourier coefficients a_j and b_j are calculated using finite sums with n terms as follows:

$$\begin{aligned} a_j &= \frac{2}{n}\sum_{k=0}^{n-1} f(\theta_k)\cos(j\theta_k) = \frac{2}{n}\sum_{k=0}^{n-1} f\Big(\frac{2k\pi}{n}\Big)\,\cos\Big(\frac{2jk\pi}{n}\Big)\,, \\ b_j &= \frac{2}{n}\sum_{k=0}^{n-1} f(\theta_k)\sin(j\theta_k) = \frac{2}{n}\sum_{k=0}^{n-1} f\Big(\frac{2k\pi}{n}\Big)\,\sin\Big(\frac{2jk\pi}{n}\Big)\,. \end{aligned} \tag{28.3.2}$$

- For the highest harmonic $j = m$, we must divide by 2, so $a_m/2 = A_m$.
- **Don't rush to judgement. There are computationally better algorithms.**
- The purpose of this analysis is to show what information we can compute using function evaluations at only n points.

28.4 Too many harmonics/too few sampling points

- In general, we do not know how many Fourier harmonics a periodic function contains.
- Suppose we guess a value n and compute the Fourier coefficients a_j and b_j using n points.
- **What happens if the function contains Fourier harmonics beyond $m = n/2$?**
- *What will go wrong?*
- Let us employ an example with $n = 8$ points.
- Then $\theta_k = 2\pi k/8 = k\pi/4$, for $k = 0, \dots, 7$.
- The table of values of $\cos(\theta)$ and $\sin(\theta)$ at the eight points θ_k is as follows

θ	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{8}\pi$
1	1	1	1	1	1	1	1	1
$\cos(\theta)$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
$\cos(2\theta)$	1	0	-1	0	1	0	-1	0
$\cos(3\theta)$	1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	-1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos(4\theta)$	1	-1	1	-1	1	-1	1	-1
$\sin(\theta)$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$
$\sin(2\theta)$	0	1	0	-1	0	1	0	-1
$\sin(3\theta)$	0	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$

- This will work if the function contains only the following Fourier harmonics:

$$\begin{aligned} f_{\rm ex2}(\theta) = \tfrac{1}{2}A_0 &+ A_1\cos(\theta) + A_2\cos(2\theta) + A_3\cos(3\theta) + A_4\cos(4\theta) \\ &+ B_1\sin(\theta) + B_2\sin(2\theta) + B_3\sin(3\theta)\,. \end{aligned} \tag{28.4.1}$$

- Using $n = 8$ points, we compute $(a_0, a_1, a_2, a_3, a_4)$ and (b_1, b_2, b_3) using eq. (28.3.2).
- We will obtain the correct results for a_j for $j = 0, 1, 2, 3, 4$ and b_j for $j = 1, 2, 3$.
- *But what happens if $f(\theta)$ contains Fourier harmonics beyond $m = 4$?*
- Suppose instead that the function has Fourier harmonics at $j = 8$:

$$f_{\rm ex3}(\theta) = f_{\rm ex2}(\theta) + A_8\cos(8\theta) + B_8\sin(8\theta)\,. \tag{28.4.2}$$

- Let us tabulate the values of $\cos(8\theta)$ and $\sin(8\theta)$ at the θ_k. We obtain the table

θ	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{8}\pi$	
$\cos(8\theta)$	1	1	1	1	1	1	1	1	**= cos (0)**
$\sin(8\theta)$	0	0	0	0	0	0	0	0	**= sin (0)**

- The values of $\cos(8\theta)$ are the **same as $\cos(0)(=1)$ at the θ_k.**
- The values of $\sin(8\theta)$ are the **same as $\sin(0)(=0)$ at the θ_k.**

- This means that the computed value of a_0 will be

$$\begin{aligned} a_0 = \frac{1}{4}\Big[& f_{\rm ex3}(0) + f_{\rm ex3}(\frac{\pi}{4}) + f_{\rm ex3}(\frac{\pi}{2}) + f_{\rm ex3}(\frac{3\pi}{4}) \\ & + f_{\rm ex3}(\pi) + f_{\rm ex3}(\frac{5\pi}{4}) + f_{\rm ex3}(\frac{3\pi}{2}) + f_{\rm ex3}(\frac{7\pi}{4})\Big] \\ = {} & A_0 \boldsymbol{+2A_8}\,. \end{aligned} \tag{28.4.3}$$

- **The presence of the harmonic $A_8\cos(8\theta)$ has messed up our solution for a_0.**
- The term in $B_8\sin(8\theta)$ has no effect because $\sin(8\theta) = 0$ at all the points θ_k.
- Next suppose the function has Fourier harmonics at $j = 7$:

$$f_{\rm ex4}(\theta) = f_{\rm ex2}(\theta) + A_7\cos(7\theta) + B_7\sin(7\theta)\,. \tag{28.4.4}$$

- Let us tabulate the values of $\cos(7\theta)$ and $\sin(7\theta)$ at the θ_k. We obtain the table

θ	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{8}\pi$	
$\cos(7\theta)$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\boldsymbol{=}$ $\boldsymbol{\cos(\theta)}$
$\sin(7\theta)$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	$\boldsymbol{=}$ $\boldsymbol{-\sin(\theta)}$

- The values of $\cos(7\theta)$ are the **same as $\cos(\theta)$ at the θ_k.**
- The values of $\sin(7\theta)$ are the **negative of $\sin(\theta)$ at the θ_k.**
- The computed value of a_1 is

$$\begin{aligned} a_1 &= \frac{1}{4}\Big[f_{\rm ex4}(0) + \frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{\pi}{4}) - \frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{3\pi}{4}) - f_{\rm ex4}(\pi) - \frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{5\pi}{4}) + \frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{7\pi}{4})\Big] \\ &= A_1 \boldsymbol{+A_7}\,. \end{aligned} \tag{28.4.5}$$

- The computed value of b_1 is

$$\begin{aligned} b_1 &= \frac{1}{4}\Big[\frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{\pi}{4}) + f_{\rm ex4}(\frac{\pi}{2}) + \frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{3\pi}{4}) - \frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{5\pi}{4}) - f_{\rm ex4}(\frac{3\pi}{2}) - \frac{1}{\sqrt{2}}\,f_{\rm ex4}(\frac{7\pi}{4})\Big] \\ &= B_1 \boldsymbol{-B_7}\,. \end{aligned} \tag{28.4.6}$$

- **The presence of the harmonic $A_7\cos(7\theta)$ has messed up our solution for a_1.**
- **The presence of the harmonic $B_7\sin(7\theta)$ has messed up our solution for b_1.**

- Next suppose the function has Fourier harmonics at $j = 9$:

$$f_{\rm ex5}(\theta) = f_{\rm ex2}(\theta) + A_9 \cos(9\theta) + B_9 \sin(9\theta)\,. \tag{28.4.7}$$

- Let us tabulate the values of $\cos(9\theta)$ and $\sin(9\theta)$ at the θ_k. We obtain the table

θ	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{8}\pi$	
$\cos(9\theta)$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$=$ **cos** $(\boldsymbol{\theta})$
$\sin(9\theta)$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	$=$ **sin** $(\boldsymbol{\theta})$

- The values of $\cos(9\theta)$ are the **same as** $\cos(\theta)$ **at the** θ_k**.**

- The values of $\sin(9\theta)$ are the **same as** $\sin(\theta)$ **at the** θ_k**.**

- From the previous calculations, it is easy to see that the computed values of a_1 and b_1 are

$$\begin{aligned} a_1 &= A_1 \boldsymbol{+A_9}\,, \\ b_1 &= B_1 \boldsymbol{+B_9}\,. \end{aligned} \tag{28.4.8}$$

- It is easy to derive that if f has Fourier harmonics at $j = 8 \pm 2$, then the results as

$$\begin{aligned} a_2 &= A_2 \boldsymbol{+A_6 + A_{10}}\,, \\ b_2 &= B_2 \boldsymbol{-B_6 + B_{10}}\,. \end{aligned} \tag{28.4.9}$$

- Similarly, if f has Fourier harmonics at $j = 8 \pm 3$, then the results are

$$\begin{aligned} a_3 &= A_3 \boldsymbol{+A_5 + A_{11}}\,, \\ b_3 &= B_3 \boldsymbol{-B_5 + B_{11}}\,. \end{aligned} \tag{28.4.10}$$

28.5 Aliasing

- The phenomenon we observed above is called **aliasing.**
- Aliasing occurs when a function is sampled using too few points.
- Our computed values for a_j and b_j contain unwanted contributions from other (higher) Fourier harmonics, and therefore do not yield the correct results for A_j and B_j.
- Let us state the problem that occurs in the general case.
- We are given a periodic function f with an unknown number of Fourier harmonics.
- We choose a positive integer m and set $n = 2m$.
- We numerically estimate the Fourier harmonics of $f(\theta)$ by computing the sums in eq. (28.3.2), using n points spaced uniformly around the circle, i.e. $\theta_j = 2\pi k/n$, where $k = 0, \dots, n-1$.
- If the function f contains nonzero Fourier harmonics up to A_m (cosines) and B_{m-1} (sines), the above procedure works.
- This is the pattern of results when the function contains additional Fourier harmonics.
  1. The Fourier harmonic $j = n$ (actually only the $\cos(n\theta)$ term) messes up the value of a_0.
  2. The Fourier harmonics $j = n \pm 1$ mess up the values of a_1 and b_1.
  3. The Fourier harmonics $j = n \pm 2$ mess up the values of a_2 and b_2.
  4. In general, the Fourier harmonics $j = n \pm \ell$ mess up the values of a_ℓ and b_ℓ, for $\ell = 0, \dots, m-1$.
- **This is obviously not the complete pattern. What about higher harmonics $j > \frac{3}{2}n$?**
  1. Let $r \geq 1$ be any positive integer.
  2. The harmonic $j = rn$ (actually only the $\cos(rn\theta)$ term) messes up the value of a_0.
  3. The harmonics $j = rn \pm \ell$ mess up the values of a_ℓ and b_ℓ, for $\ell = 0, \dots, m-1$.
- In general we obtain, for $\ell = 1, \dots, m-1$ (recall $n = 2m$),

$$
\begin{aligned}
a_0 &= A_0 + 2\sum_{r=1}^{\infty} A_{rn} \,, \\
a_\ell &= A_\ell + \sum_{r=1}^{\infty} \bigl(A_{rn+\ell} + A_{rn-\ell}\bigr) \,, \\
b_\ell &= B_\ell + \sum_{r=1}^{\infty} \bigl(B_{rn+\ell} - B_{rn-\ell}\bigr) \,.
\end{aligned}
\tag{28.5.1}
$$

28.6 Anti–aliasing

- Aliasing is generally considered to be undesirable.
- Aliasing means that the sum of the Fourier series will not yield a good quality approximation (or representation) of original function.
- The Fourier reconstruction (sum of series) will contain unwanted artifacts.
- What can be done if a digitally sampled signal contains aliasing?
- If there is a known model for the aliased data (from physics or engineering, if there is additional information about the properties of the original data), one can formulate a model to subtract the aliasing.
- There exist **anti–aliasing** algorithms, to minimize the effects of aliasing.
- Anti–aliasing algorithms are important in computer graphics, for example.
- We shall not study anti–aliasing algorithms, but they are important.
- Basically, the field of signal or image processing contains many sophisticated algorithms.

28.7 Nyquist frequency

- If a function is sampled at n uniformly spaced points around the circle, aliasing will occur if the function contains nonzero Fourier harmonics for values $j > n/2$.
- It is common to speak of Fourier analysis in terms of time and frequencies.
- The Fourier harmonics correspond to waves of different frequencies.
- Then the value $n/2$ is called the **Nyquist frequency.**
- Technically, if the value $j = 1$ corresponds to a frequency unit f_0, then the Nyquist frequency is $(n/2)f_0$.
- If the signal (the function f) contains Fourier harmonics at higher than the Nyquist frequency, aliasing will occur.
- A **bandwidth limited** (periodic) function is one for which there is a constant $K > 0$ such that all of its Fourier harmonics are zero for $k > K$.
- Hence if a bandwidth limited periodic function is sampled using $n > 2K$ points, the Fourier series will reproduce the function exactly.
- Some authors employ the term **Nyquist rate** to mean **twice the maximum frequency that a signal contains.**
- For a bandwidth limited periodic function, the Nyquist rate is $2K$.
- Hence if a bandwidth limited periodic function is sampled at higher than the Nyquist rate, the Fourier series will reproduce the function exactly.

28.8 Oversampling

- From a pure mathematical perspective, the Fourier coefficients are arbitrary numbers.
- Nevertheless, from a practical standpoint, it is reasonable to expect that the magnitudes of a_j and b_j will decrease to zero as $j \to \infty$.
- Recall **Parseval's theorem.** It states that

$$\frac{1}{\pi}\int_0^{2\pi} |f(\theta)|^2 \, d\theta = \frac{|a_0|^2}{2} + \sum_{j=1}^{\infty}(|a_j|^2 + |b_j|^2) \,. \tag{28.8.1}$$

- Since we expect the left hand side of eq. (28.8.1) to be finite (that is to say, the signal has a finite power, to borrow from physics), the infinite sum on the right hand side must converge.
- Hence we expect that both $|a_j| \to 0$ and $|b_j| \to 0$ as $j \to \infty$.
- Let us suppose that a periodic function (or signal) is not bandwidth limited, but the amplitudes of its Fourier coefficients are negligible for $k > K_1$, where $K_1 > 0$ is a constant.
- The definition of 'negligible' of course depends on the application.
- Then if the signal is sampled using $n \gg 2K_1$, the aliasing will be negligible for the Fourier harmonics in the range $0 \le j \le K_1$.
- This is called **oversampling.**
- Oversampling is extensively employed in digital audio.
- Human beings can hear sounds up to about 16 kHz (to use a convenient power of 2).
- Compact discs are recorded at a frequency of 44.1 kHz (half of that is 22.05 kHz, well above the threshold for humans).
- Oversampling at 4× means a frequency of 176.4 kHz.
- Then any aliasing introduced by digital audio filters or other signal processing has little effect on the content of the signal (the Fourier harmonics) at frequencies below 22.05 kHz.

28.9 Why use Fourier series? Why use uniformly spaced points?

- Why do we measure the function $f(\theta)$ using equally spaced points around the circle?
- *Why do we use Fourier series?*
- Suppose a function was significantly nonzero in only a small localized region and was almost zero (or exactly zero) for most value of θ.
- For example, consider the function $\sin^{200}(\frac{1}{2}\theta)$, plotted in Fig. 1.
- The function is sharply peaked near $\theta = \pi$ and is almost zero for most other values of θ.
- If we expand the function as a sum of cosines, we obtain a large number of Fourier harmonics, which clearly cancel for most values of θ:

$$\begin{aligned}\sin^{200}(\tfrac{1}{2}\theta) &= \frac{1}{2^{200}}\left(e^{i\theta/2} - e^{-i\theta/2}\right)^{200} = \frac{1}{2^{200}}\sum_{j=0}^{200}(-1)^j\binom{200}{j}e^{i(100-j)\theta} \\ &= \frac{1}{2^{200}}\left[\binom{200}{100} + 2\sum_{j=0}^{99}(-1)^j\binom{200}{j}\cos((100-j)\theta)\right].\end{aligned} \tag{28.9.1}$$

- This is wasteful: many cosines and they almost cancel out for most values of θ.
- It would make more sense to sample the function non–uniformly, with closely spaced points near the peak, and it would be satisfactory to use sparsely spaced points far from the peak.
- The problem is: **what next?**
- *How would we express our set of numbers in a way that is suitable for mathematical analysis?*
- **Fourier series have many nice, well understood mathematical properties.**
  1. Using uniformly spacing, it is easy to go from the function to the Fourier series and back.
  2. We require additional information, to concentrate the points nonuniformly.
  3. We know how much information content a set of uniformly spaced points contains. There are well understood facts (such as the Nyquist frequency) which we can employ for practical applications.
- Nevertheless, there *are* alternative ways to represent a set of data.
- **Wavelets** are a modern mathematically powerful way to represent a set of data.
  1. Wavelets can represent some types of data using only a few 'wavelet basis functions' in situations where a Fourier series might require many Fourier harmonics.
  2. There is in fact a lot of mathematical theory about wavelets.
- **Hence alternatives to Fourier series (and uniform spacing) do exist.**
- However, they are mathematically more sophisticated and beyond the scope of these lectures.

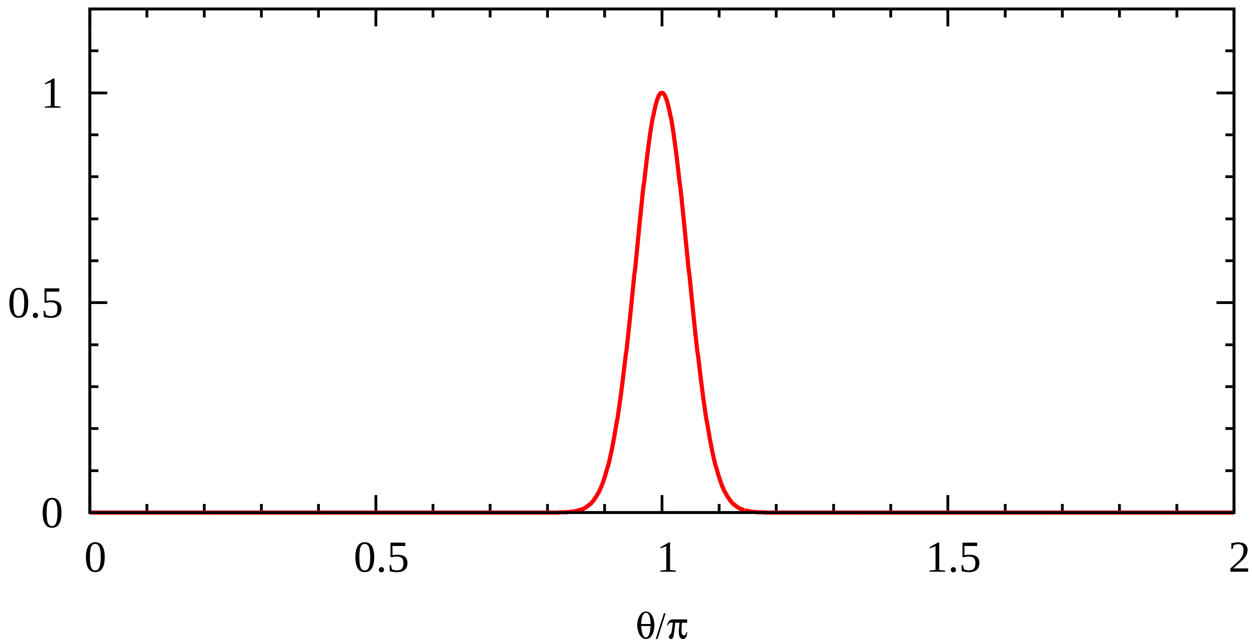

Figure 1: Plot of a highly peaked function $\sin^{200}(\frac{1}{2}\theta)$.