Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

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13 Lecture 13

Circulant matrices

- This is a "special topics" lecture on a particularly simple class of matrices.
- They are called **circulant matrices**.

13.1 Circulant matrices

- In a **circulant matrix**, all the rows are cyclic permutations of the first row.
- Equivalently, all the columns are cyclic permutations of the first column.
- It is simplest to illustrate with an example before writing a formal definition.
- The following are circulant matrices for $n = 2, 3, 4, \ldots$:

$$C_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \qquad C_3 = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \qquad C_4 = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \ldots \tag{13.1.1}$$

- Each row is a cyclic permutation of the previous row, shifted by one spot.
- Each column is a cyclic permutation of the previous row, shifted by one spot.
- Formally, for an $n \times n$ circulant matrix, the matrix elements c_{ij} satisfy

$$c_{i+k,j+k} = c_{ij} \qquad (k = 1, \ldots, n-1)\,. \tag{13.1.2}$$

- The values of "$i + k$" and "$j + k$" must be reduced modulo n, i.e. $i + n \equiv i$ and $j + n \equiv j$.
- Hence the element c_{ij} is a function of $i - j$ only.
- An $n \times n$ circulant matrix thus has the following form

$$C = \begin{pmatrix} c_0 & c_{n-1} & \ldots & & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & & c_2 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \ldots & c_2 & c_1 & c_0 \end{pmatrix}. \tag{13.1.3}$$

- Circulant matrices have some very nice remarkable properties.
- All $n \times n$ circulant matrices commute.
- All $n \times n$ circulant matrices have the **same set of eigenvectors.**

13.2 Eigenvalues and eigenvectors

- All $n \times n$ circulant matrices have the same set of eigenvectors.
- Let ζ be a primitive n^{th} root of unity, i.e. $\zeta^n = 1$ and $\zeta^m \neq 1$ for $m = 1, \dots, n-1$.
- It is simplest to choose

$$\zeta = e^{i2\pi/n}\,. \tag{13.2.1}$$

- Let C be the $n \times n$ circulant matrix in eq. (13.1.3).
- Then the following column vectors $\boldsymbol{\xi}_j$, $j = 0, 1, \dots, n-1$ are the eigenvectors of C

$$\boldsymbol{\xi}_j = \begin{pmatrix} 1 \\ \zeta^j \\ \zeta^{2j} \\ \vdots \\ \zeta^{(n-1)j} \end{pmatrix} \qquad (j = 0, 1, \dots, n-1)\,. \tag{13.2.2}$$

- The corresponding eigenvalues are

$$\lambda_j = c_0 + c_1\zeta^{n-1} + c_2\zeta^{n-2} + \cdots + c_{n-1}\zeta = \sum_{j=0}^{n-1} c_j\zeta^{n-j} = \sum_{j=0}^{n-1} c_j\zeta^{-j}\,. \tag{13.2.3}$$

13.3 Examples

- Hence for the matrix C_2 in eq. (13.1.1), the eigenvectors and eigenvalues are

$$\lambda_0 = a + b\,, \qquad \boldsymbol{\xi}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,, \tag{13.3.1a}$$

$$\lambda_1 = a - b\,, \qquad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,. \tag{13.3.1b}$$

- For the matrix C_3 in eq. (13.1.1), the eigenvectors and eigenvalues are

$$\lambda_0 = a + b + c\,, \qquad \boldsymbol{\xi}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\,, \tag{13.3.2a}$$

$$\lambda_1 = a + be^{i2\pi/3} + ce^{i4\pi/3}\,, \qquad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ e^{i2\pi/3} \\ e^{i4\pi/3} \end{pmatrix}\,, \tag{13.3.2b}$$

$$\lambda_2 = a + be^{i4\pi/3} + ce^{i2\pi/3}\,, \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ e^{i4\pi/3} \\ e^{i2\pi/3} \end{pmatrix}\,. \tag{13.3.2c}$$

- For the matrix C_4 in eq. (13.1.1), the eigenvectors and eigenvalues are

$$\lambda_0 = a + b + c + d\,, \qquad \boldsymbol{\xi}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\,, \tag{13.3.3a}$$

$$\lambda_1 = a + ib - c - id\,, \qquad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}\,, \tag{13.3.3b}$$

$$\lambda_2 = a - b + c - d\,, \qquad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\,, \tag{13.3.3c}$$

$$\lambda_3 = a - ib - c + id\,, \qquad \boldsymbol{\xi}_3 = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}\,. \tag{13.3.3d}$$