Queens College, CUNY, Department of Computer Science

Computational Finance

CSCI 365 / 765

Fall 2017

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March 28, 2018

7 Lecture 7a

Rational option pricing: Worked examples

- This lecture contains worked examples of arbitrage strategies to derive some of the rational option pricing inequalities in Lecture 7.
- Arbitrage strategies involving options are more complicated than those involving forwards and futures.
- This lecture will show the details of what to do.
- Note: Continuous interest rate compounding is used in all the examples.

7.1 Arbitrage 1: European call

- Suppose the market price of a stock is S_0 at time t_0.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- The market price of a European call is $c_{\text{Eur}} = S_0 + 1.5$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **Solution for European call**
  1. At time t_0, we **sell/write/go short** the call option and receive cash $= S_0 + 1.5$.
  2. We use some of the money to buy one share of stock, cost $= S_0$.
  3. Therefore we have "extra cash" $= 1.5$. This cash is saved in a bank.
- At the time t_0, our portfolio is: short one European call option, long one share of stock, cash in bank. The total value of our portfolio is zero.
- At the expiration time T, the money in the bank compounds to $1.5e^{r(T-t_0)}$.
- **However, we do not know the final value of the stock price S_T at the expiration time T.**
- Hence we must analyze all cases, i.e. all values of S_T.
- We must prove that in all cases, the profit is positive (or zero),
  but ***never negative (= loss).***
  1. We do this because S_T is the only random variable in the problem.
  2. If there were two random variables, for example stock price and interest rate, then we must analyze all the possible values of both random variables.
  3. It gets complicated when there are multiple random variables.
- There are two cases (i) $S_T \geq K$ or (ii) $S_T < K$.

7.1.1 Case $S_T \geq K$

- In this situation the option holder exercises the call option.
- The holder pays the writer (= us) cash K (= strike price).
- The writer (= us) delivers the stock to the holder and receives cash $= K$.
- We **must obey** because the option writer has **no rights, only obligations.**
- However, remember that we already bought the stock at the time t_0.
- Hence the writer (= us) delivers the stock (which we already have).
- So after the option is exercised, we have cash $= K$ and *no more stock.*
- *The stock cancels out:* we bought it at time t_0 and delivered it at time T.
- But ... remember that we also have cash in the bank $= 1.5e^{r(T-t_0)}$.
- Hence our total money is $K + 1.5e^{r(T-t_0)}$.
- This is we profit $= K + 1.5e^{r(T-t_0)}$.
- We started with zero and we have a positive amount of money at the end.
- We have no more stock and no more option, **only cash.**

7.1.2 Case $S_T < K$

- The holder does not exercise because the call option is out of the money.
- So we throw away the option (= not exercised).
- We receive nothing and we deliver nothing (= not exercised).
- So we have stock $= S_T$ plus cash $= 1.5e^{r(T-t_0)}$.
- Hence we **sell the stock,** because we do not want to lose money in case the stock price goes down some more. (But see Sec. 7.1.5 below.)
- Remember that we want a ***guaranteed profit*** (not a random number).
- Hence we sell the stock and receive *cash* $= S_T$.
- Hence our total money is $S_T + 1.5e^{r(T-t_0)}$.
- This is we profit $= S_T + 1.5e^{r(T-t_0)}$.
- We started with zero and we have a positive amount of money at the end.
- We have no more stock and no more option, **only cash.**

7.1.3 Graph of profit

A graph of the arbitrage profit for a mispriced European call is plotted against the terminal stock price S_T in Fig. 1.

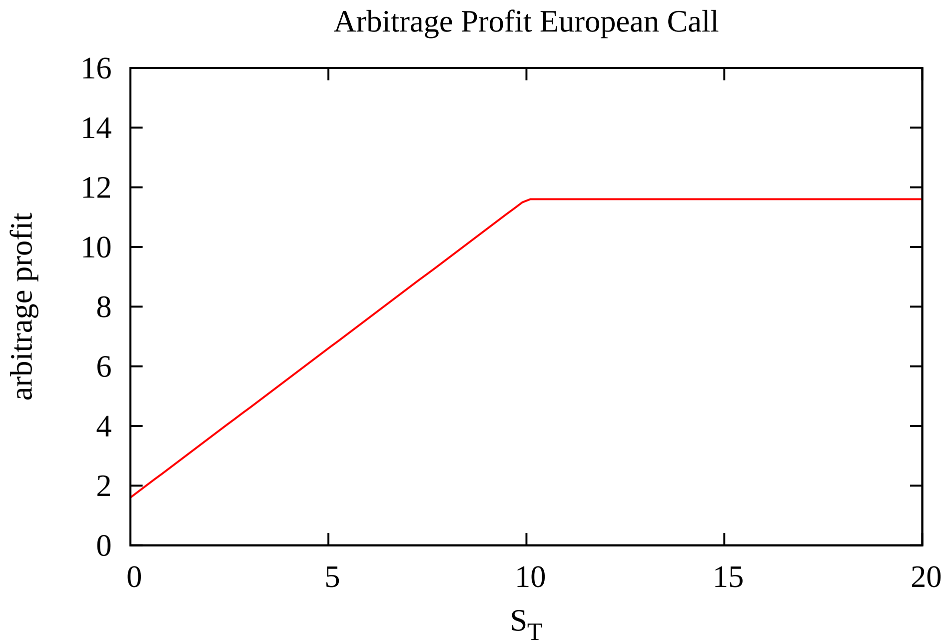

Figure 1: Graph of the arbitrage profit for a mispriced European call, plotted against the terminal stock price S_T. The strike is $K = 10$.

7.1.4 Review

- We must also understand these facts:

  1. Our total profit ***depends on the value of S_T.***
  2. Our total profit ***depends whether or not the option is exercised.***
  3. It is a different amount of profit in case (i) $S_T \geq K$ or case (ii) $S_T < K$.
  4. But the profit is ***always positive.***
  5. There is ***never a loss.***
  6. The arbitrage strategy gives a *guaranteed profit (always positive).*
  7. But ***we do not know how much profit.***
  8. It is not a constant number.
  9. And the profit is more than $1.5e^{r(T-t_0)}$.

- We must also understand that we do not need to know the strike price of the call option, to formulate we arbitrage strategy.

- It does not matter if $S_0 \geq K$ or $S_0 < K$.

- That is why the value of K was not stated in the question.

7.1.5 Comment

- A student pointed out that if the option is not exercised, we do not have to sell our stock.

- We can keep it and wait to see if its value goes up, and then sell it at a later time.

- Although the stock price is random, it is a **guaranteed positive random number.**

- Therefore it is still an arbitrage profit.

7.2 Arbitrage 2: European put

- Suppose the market price of a stock is S_0 at time t_0.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- The market price of a European put is $p_{\text{Eur}} = e^{-r(T-t_0)}K + 0.75$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **Solution for European put**
  1. At time t_0, we **sell/write/go short** the put option and receive cash $= e^{-r(T-t_0)}K + 0.75$.
  2. **We save all the money in a bank.**
  3. The arbitrage strategy for a put **does not involve buying/selling stock.**
  4. It is important to understand that an arbitrage strategy does not always have to involve the stock.
- At the time t_0, our portfolio is: short one European put option, cash in bank.
  The total value of our portfolio is zero.
- At the expiration time T, the money in the bank compounds to $K + 0.75e^{r(T-t_0)}$.
- **However, we do not know the final value of the stock price S_T at the expiration time T.**
- Hence we must analyze all cases, i.e. all values of S_T.
- We must prove that in all cases, the profit is positive (or zero),
  but ***never negative (= loss).***
- There are two cases (i) $S_T \leq K$ or (ii) $S_T > K$.

7.2.1 Case $S_T \le K$

- In this situation the option holder exercises the put option.
- The holder delivers the stock (value $= S_T$) to the writer (= us) and receives cash K (= strike price).
- The writer (= us) receives the stock (value $= S_T$) and pays cash $= K$.
- We **must obey** because the option writer has **no rights, only obligations.**
- However, remember that we already have enough cash in the bank $K + 0.75e^{r(T-t_0)}$.
- Hence the writer (= us) pays K of our cash to the holder.
- So after the option is exercised, we have cash $= 0.75e^{r(T-t_0)}$ and *one share of stock.*
- We **sell the stock** and receive cash ($= S_T$), because want a ***guaranteed profit,*** not a random number. (But see Sec. 7.2.5 below.)
- Hence our total money at the end is $S_T + 0.75e^{r(T-t_0)}$.
- This is we profit $S_T + 0.75e^{r(T-t_0)}$.
- We started with zero and we have a positive amount of money at the end.
- We have no more stock and no more option, **only cash.**

7.2.2 Case $S_T > K$

- The holder does not exercise because the put option is out of the money.
- So we throw away the option (= not exercised).
- We receive nothing and we deliver nothing (= not exercised).
- So we have only cash $= K + 0.75e^{r(T-t_0)}$.
- This is we profit $= K + 0.75e^{r(T-t_0)}$.
- We started with zero and we have a positive amount of money at the end.
- We have no stock and no option, **only cash.**

7.2.3 Graph of profit

A graph of the arbitrage profit for a mispriced European put is plotted against the terminal stock price S_T in Fig. 2.

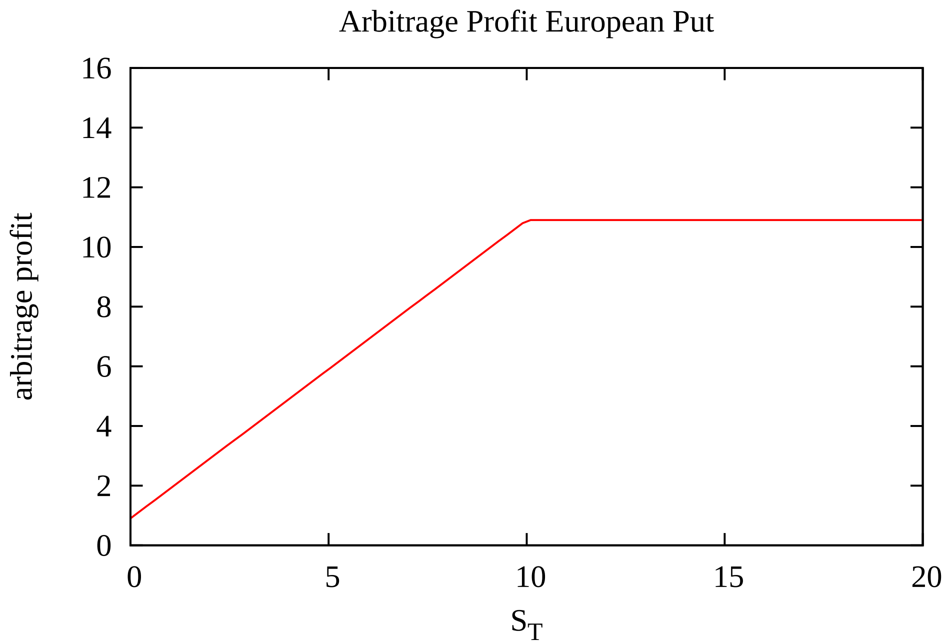

Figure 2: Graph of the arbitrage profit for a mispriced European put, plotted against the terminal stock price S_T. The strike is $K = 10$.

7.2.4 Review

- We must also understand these facts:
  1. Our total profit ***depends on the value of S_T.***
  2. Our total profit ***depends whether or not the option is exercised.***
  3. It is a different amount of profit in case (i) $S_T \leq K$ or case (ii) $S_T > K$.
  4. But the profit is ***always positive.***
  5. There is ***never a loss.***
  6. The arbitrage strategy gives a *guaranteed profit (always positive).*
  7. But ***we do not know how much profit.***
  8. It is not a constant number.
  9. And the profit is more than $0.75e^{r(T-t_0)}$.
- We must also understand that in this case, we need to know the strike price of the put option, *but we do not need to know the stock price,* to formulate we arbitrage strategy.
- It does not matter if $S_0 \leq K$ or $S_0 > K$.
- That is why the value of S_0 was not stated in the question for the put option.

7.2.5 Comment

- A student pointed out that if the option is exercised, we do not have to sell the stock.
- We can keep it and wait to see if its value goes up, and then sell it at a later time.
- Although the stock price is random, it is a **guaranteed positive random number.**
- Therefore it is still an arbitrage profit.

7.3 Arbitrage 3: American call

- Suppose the market price of a stock is S_0 at time t_0.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- The market price of an American call is $C_{\text{Am}} = S_0 + 2.5$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **Solution for American call**
  1. At time t_0, we **sell/write/go short** the call option and receive cash $= S_0 + 2.5$.
  2. We use some of the money to buy one share of stock, cost $= S_0$.
  3. Therefore we have "extra cash" $= 2.5$. This cash is saved in a bank.
- At the time t_0, our portfolio is: short one American call option, long one share of stock, cash in bank. The total value of our portfolio is zero.
- This is the same arbitrage strategy as for the European call option.
- However, there is one important additional detail in the analysis for an American option.
- Because an American option can be exercised at any time, we must prove that our arbitrage strategy yields a guaranteed profit **at any time t, where $t_0 < t \leq T$,** not only at the expiration time T.
- Suppose the option holder exercises the American call at an intermediate time t, where $t_0 < t < T$.
- The stock price at the time t is S_t, and the holder will only exercise if $S_t \geq K$.
- Hence there is only one case to analyze, which is $S_t \geq K$.
- At the time t, the money in the bank compounds to $2.5e^{r(t-t_0)}$.
- The holder exercises the option and pays the writer (= us) cash K (= strike price).
- The writer (= us) delivers the stock to the holder and receives cash $= K$.
- However, remember that we already bought the stock at the time t_0.
- Hence the writer (= us) delivers the stock (which we already have).
- So after the option is exercised, we have cash $= K$ and *no more stock.*
- *The stock cancels out:* we bought it at time t_0 and delivered it at time t (instead of T).
- But remember also that we also have cash in the bank $= 2.5e^{r(t-t_0)}$.
- Hence our total profit is $K + 2.5e^{r(t-t_0)}$ if the holder exercises at the time t.

- We started with zero and we have a positive amount of money after the holder exercises the option.

- We have no more stock and no more option, **only cash.**

- At the expiration time T the analysis is the same as for a European call option, because the terminal payoffs of an American and European call option are the same.

- Therefore the above arbitrage strategy yields a positive guaranteed profit at any time $t_0 < t \leq T$.

- A graph of the arbitrage profit for a mispriced American call (exercised early) is plotted in Fig. 3.

- At expiration, the graph of the arbitrage profit for a mispriced American call looks the same as Fig. 1 for a European call.

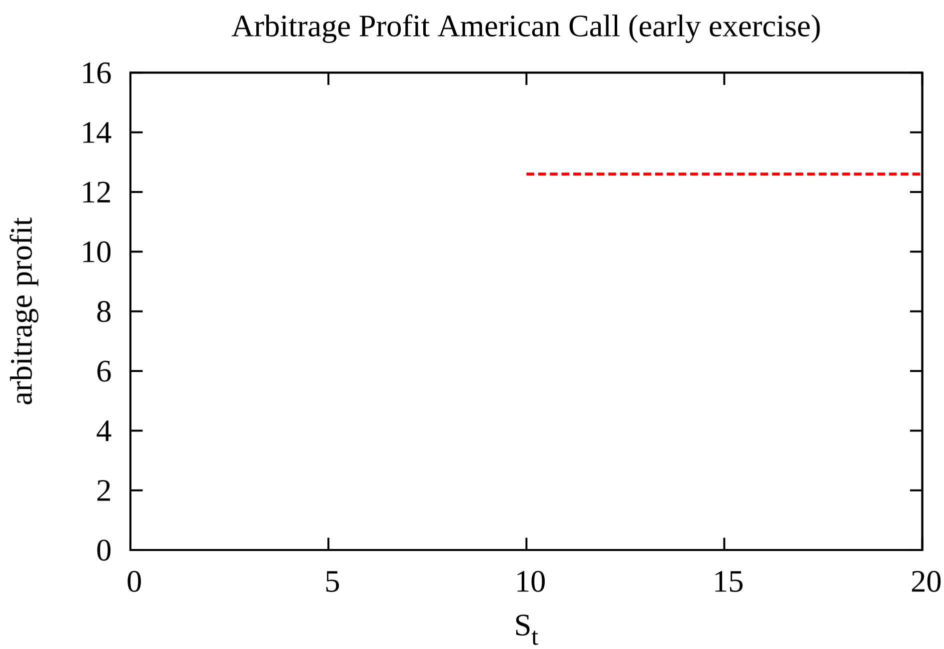

Figure 3: Graph of the arbitrage profit for a mispriced American call with early exercise, plotted against the stock price S_t at time t. The strike is $K = 10$.

7.4 Arbitrage 4: American put

- Suppose the market price of a stock is S_0 at time t_0.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- The market price of an American put is $P_{\rm Am} = K + 1.75$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **Solution for American put**

  1. At time t_0, we **sell/write/go short** the put option and receive cash $= K + 1.75$.
  2. We save all the money in a bank.

- At the time t_0, our portfolio is: short one American put option, cash in bank.
  The total value of our portfolio is zero.
- This is the same arbitrage strategy as for the European put option.
- However, notice that the amount of money involved is K, not the present value $e^{-r(T-t_0)}K$.
- This is because an American option can be exercised at any time, as we shall see below.
- Now we must prove that our arbitrage strategy yields a guaranteed profit **at any time t, where $t_0 < t \le T$,** not only at the expiration time T.
- Suppose the option holder exercises the American put at an intermediate time t, where $t_0 < t < T$.
- The stock price at the time t is S_t, and the holder will only exercise if $S_t \le K$.
- Hence there is only one case to analyze, which is $S_t \le K$.
- At the time t, the money in the bank compounds to $(K + 1.75)e^{r(t-t_0)}$.
- The holder exercises the option and delivers the stock to the writer (= us).
- The writer (= us) receives the stock and pays cash $= K$ to the option holder.
- However, we have enough cash available to pay K to the option holder.
- Hence the writer (= us) pays K and has money $K(e^{r(t-t_0)} - 1) + 1.75e^{r(t-t_0)}$ in the bank.
- So after the option is exercised, we have cash $= K(e^{r(t-t_0)} - 1) + 1.75e^{r(t-t_0)}$ and *one share of stock.*
- We **sell the stock,** because we want a guaranteed profit, not a random number.
- Hence we profit is $S_t + K(e^{r(t-t_0)} - 1) + 1.75e^{r(t-t_0)}$ if the holder exercises at the time t.

- We started with zero and we have a positive amount of money after the holder exercises the option.
- We have no stock and no option, **only cash.**
- At the expiration time T the analysis is the same as for a European put option, because the terminal payoffs of an American and European put option are the same.
- Therefore the above arbitrage strategy yields a positive guaranteed profit at any time $t_0 < t \leq T$.

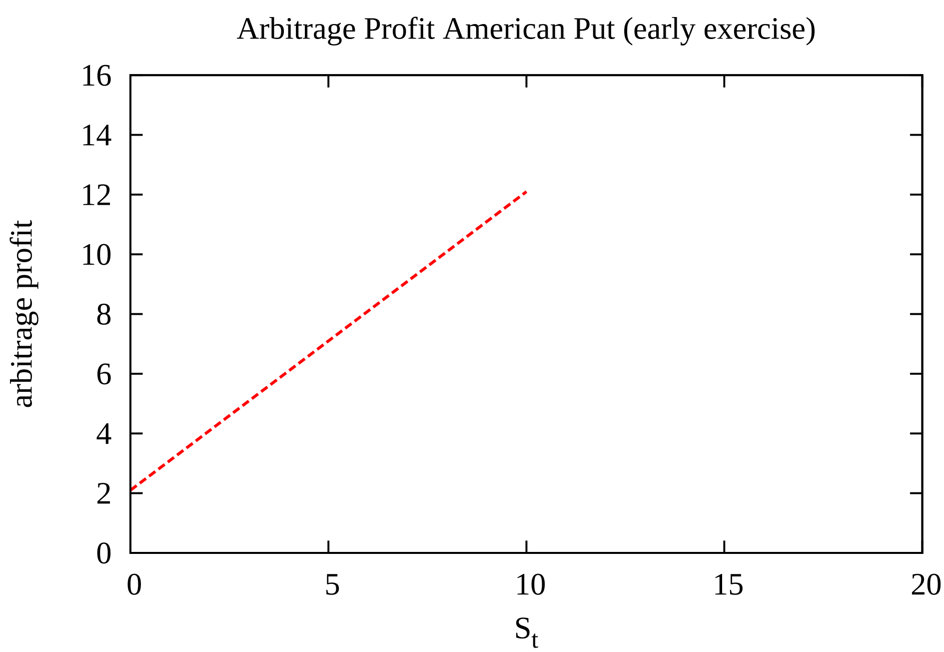

Figure 4: Graph of the arbitrage profit for a mispriced American put with early exercise, plotted against the stock price S_t at time t. The strike is $K = 10$.

7.5 Arbitrage 5: dividends

- For all the cases in Secs. 7.1 – 7.4, suppose that stock pays dividends during the lifetime of the options.
- There are n dividends, of amounts D_i, paid at times t_i, where $i = 1, \dots, n$, and $t_0 < t_1 < \cdots < t_n < T$.
- **Formulate an arbitrage strategy to take advantage of the mispriced options.**
- **Solution**
  1. All of the arbitrage strategies in Secs. 7.1 – 7.4 are **unchanged.**
  2. This is because dividends do not affect the relevant rational option pricing inequalities.
- For the European and American calls, our arbitrage strategy contains a **long position in the stock**, hence we *collect the dividends and our arbitrage profit is even higher.*
- For the European and American puts, our arbitrage strategy **does not contain the stock at all.** For the European and American puts, if the option holder exercises the option against us, the option holder delivers the stock to us and we **only have to pay the option strike price.** We **do not have to pay the dividends.** Hence for the European and American puts, the dividends are not our problem and do not affect our arbitrage profit.