Queens College, CUNY, Department of Computer Science

Numerical Methods

CSCI 361 / 761

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6 Lecture 6a

This lecture contains worked examples on the numerical evaluation of integrals

- Midpoint rule.
- Trapezoid rule.
- Simpson's rule.
- Extended trapezoid rule.
- Romberg integration.

6.17 Worked example 1

- We begin with simple polynomials.
- We set $a = 0$ and $b = 2$.
- We employ $n = 2$ because we wish to use Simpson's rule.
- Hence the size of the subintervals is $h = (b - a)/n = (2 - 0)/2 = 1$.
- Also $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.

6.17.1 Exact results

- We consider five cases (they are known as 'monomials'):
- $f(x) = 1$.

$$\int_0^2 f(x)\,dx = \int_0^2 dx = 2\,. \tag{6.17.1.1}$$

- $f(x) = x$.

$$\int_0^2 f(x)\,dx = \int_0^2 x\,dx = 2\,. \tag{6.17.1.2}$$

- $f(x) = x^2$.

$$\int_0^2 f(x)\,dx = \int_0^2 x^2\,dx = \frac{8}{3} = 2.6666\dots\,. \tag{6.17.1.3}$$

- $f(x) = x^3$.

$$\int_0^2 f(x)\,dx = \int_0^2 x^3\,dx = 4\,. \tag{6.17.1.4}$$

- $f(x) = x^4$.

$$\int_0^2 f(x)\,dx = \int_0^2 x^4\,dx = \frac{32}{5} = 6.4\,. \tag{6.17.1.5}$$

6.17.2 Midpoint

- $f(x) = 1$.

$$M = h\left[f\Big(\frac{x_0+x_1}{2}\Big) + f\Big(\frac{x_1+x_2}{2}\Big)\right] = f(0.5) + f(1.5) = 1 + 1 = 2\,. \tag{6.17.2.1}$$

- $f(x) = x$.

$$M = h\left[f\Big(\frac{x_0+x_1}{2}\Big) + f\Big(\frac{x_1+x_2}{2}\Big)\right] = f(0.5) + f(1.5) = 0.5 + 1.5 = 2\,. \tag{6.17.2.2}$$

- $f(x) = x^2$.

$$M = h\left[f\Big(\frac{x_0+x_1}{2}\Big) + f\Big(\frac{x_1+x_2}{2}\Big)\right] = f(0.5) + f(1.5) = 0.25 + 2.25 = 2.75 \neq 2.6666\ldots\,. \tag{6.17.2.3}$$

- The midpoint rule yields the **exact answer if $f(x)$ is a linear polynomial $f(x) = a + bx$.**
- The midpoint rule is **not exact if $f(x)$ is a quadratic $f(x) = a + bx + cx^2$ or higher degree.**

6.17.3 Trapezoid

- $f(x) = 1$.

$$T = h\left[\frac{f(x_0)+f(x_2)}{2} + f(x_1)\right] = \frac{f(0)+f(2)}{2} + f(1) = \frac{1+1}{2} + 1 = 2\,. \tag{6.17.3.1}$$

- $f(x) = x$.

$$T = h\left[\frac{f(x_0)+f(x_2)}{2} + f(x_1)\right] = \frac{f(0)+f(2)}{2} + f(1) = \frac{0+2}{2} + 1 = 2\,. \tag{6.17.3.2}$$

- $f(x) = x^2$.

$$T = h\left[\frac{f(x_0)+f(x_2)}{2} + f(x_1)\right] = \frac{f(0)+f(2)}{2} + f(1) = \frac{0+4}{2} + 1 = 3\,. \tag{6.17.3.3}$$

- The trapezoid rule yields the **exact answer if $f(x)$ is a linear polynomial $f(x) = a + bx$.**
- The trapezoid rule is **not exact if $f(x)$ is a quadratic $f(x) = a + bx + cx^2$ or higher degree.**

6.17.4 Simpson

- Constant $f(x) = 1$.

$$S = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] = \frac{1}{3}\Big[1 + 4 + 1\Big] = \frac{6}{3} = 2\,. \tag{6.17.4.1}$$

- $f(x) = x$.

$$S = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] = \frac{1}{3}\Big[0 + 4 + 2\Big] = \frac{6}{3} = 2\,. \tag{6.17.4.2}$$

- $f(x) = x^2$.

$$S = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] = \frac{1}{3}\Big[0 + 4 + 4\Big] = \frac{8}{3} = 2.6666\ldots\,. \tag{6.17.4.3}$$

- $f(x) = x^3$.

$$S = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] = \frac{1}{3}\Big[0 + 4 + 8\Big] = \frac{12}{3} = 4\,. \tag{6.17.4.4}$$

- $f(x) = x^4$.

$$S = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] = \frac{1}{3}\Big[0 + 4 + 16\Big] = \frac{20}{3} = 6.6666\ldots\,. \tag{6.17.4.5}$$

- Simpson's rule yields the **exact answer up to cubic polynomials (not just quadratic)** $f(x)$ $f(x) = a + bx + cx^2 + dx^3$**.**
- Simpson's rule is **not exact if** $f(x)$ **is a quartic** $f(x) = a + bx + cx^2 + dx^3 + ex^4$ **or higher degree.**

6.18 Worked example 2

6.18.1 Function and exact result

- The function is

$$f(x) = \frac{2}{(1+x)^2} \,. \tag{6.18.1.1}$$

- We set $a = 0$ and $b = 1$.
- The exact value of the integral is

$$I = \int_0^1 \frac{2}{(1+x)^2}\,dx = \left[-\frac{2}{1+x} \right]_0^1 = -\frac{2}{2} + \frac{2}{1} = 1\,. \tag{6.18.1.2}$$

6.18.2 Midpoint

- $n = 1$: hence $h = 1$, $x_0 = 0$, $x_1 = 1$.

$$M_1 = h\,f\Big(\frac{x_0+x_1}{2}\Big) = f(0.5) = \frac{2}{(1.5)^2} = \frac{2}{1.25} = 0.8888\ldots\,. \tag{6.18.2.1}$$

- $n = 2$: hence $h = 0.5$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

$$\begin{aligned} M_2 &= h\left[\,f\Big(\frac{x_0+x_1}{2}\Big) + f\Big(\frac{x_1+x_2}{2}\Big)\,\right] \\ &= \frac{1}{2}\left[\,f(0.25) + f(0.75)\,\right] = \frac{1}{2}\left[\,\frac{2}{(1.25)^2} + \frac{2}{(1.75)^2}\,\right] \simeq 0.966531\,. \end{aligned} \tag{6.18.2.2}$$

6.18.3 Trapezoid

- $n = 1$: hence $h = 1$, $x_0 = 0$, $x_1 = 1$.

$$T_1 = h\,\frac{f(x_0)+f(x_1)}{2} = \frac{f(0)+f(1)}{2} = \frac{1}{2}\left[\,\frac{2}{1^2} + \frac{2}{2^2}\,\right] = 1.25\,. \tag{6.18.3.1}$$

- $n = 2$: hence $h = 0.5$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

$$\begin{aligned} T_2 &= h\left[\,\frac{f(x_0)+f(x_2)}{2} + f(x_1)\,\right] \\ &= \frac{1}{2}\left[\,\frac{1}{2}\Big(\frac{2}{1^2} + \frac{2}{2^2}\Big) + \frac{2}{(1.5)^2}\,\right] = \frac{1}{2}\left[\,1.25 + 0.8888\ldots\,\right] \simeq 1.069444\,. \end{aligned} \tag{6.18.3.2}$$

6.18.4 Simpson

- $n = 2$: hence $h = 0.5$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

$$\begin{aligned} S_2 &= \frac{h}{3}\left[\,f(x_0) + 4f(x_1) + f(x_2)\,\right] \\ &= \frac{1}{6}\left[\,\frac{2}{1^2} + \frac{4}{(1.5)^2} + \frac{2}{2^2}\,\right] = \frac{1}{6}\left[\,2 + 3.5555\cdots + 0.5\,\right] \simeq 1.009259\,. \end{aligned} \tag{6.18.4.1}$$

- $n = 4$: hence $h = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$$\begin{aligned} S_4 &= \frac{h}{3}\left[\,f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\,\right] \\ &= \frac{1}{6}\left[\,\frac{2}{1^2} + \frac{4}{(1.25)^2} + \frac{2}{(1.5)^2} + \frac{4}{(1.75)^2} + \frac{2}{2^2}\,\right] \\ &\simeq \frac{1}{12}\left[\,2 + 5.12 + 1.777778 + 2.612245 + 0.5\,\right] \simeq 1.000835\,. \end{aligned} \tag{6.18.4.2}$$

6.18.5 Extended trapezoid

- $n = 1$: hence $h_0 = 1$, $x_0 = 0$, $x_1 = 1$.

$$E_0 = T_1 = h_0 \frac{f(x_0) + f(x_1)}{2} = \frac{f(0) + f(1)}{2} = \frac{1}{2}\left[\frac{2}{1^2} + \frac{2}{2^2}\right] = 1.25\,. \tag{6.18.5.1}$$

- $n = 2$: hence $h_1 = 0.5$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

$$E_1 = \frac{1}{2}E_0 + h_1 f(x_1) = \frac{1.25}{2} + \frac{0.8888\ldots}{2} \simeq 1.069444\,. \tag{6.18.5.2}$$

- $n = 4$: hence $h_2 = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$$\begin{aligned} E_2 &= \frac{1}{2}E_1 + h_2\Big[f(x_1) + f(x_3)\Big] \\ &\simeq \frac{1.069444}{2} + \frac{1}{4}\left[\frac{2}{(1.25)^2} + \frac{2}{(1.75)^2}\right] \simeq 1.017988\,. \end{aligned} \tag{6.18.5.3}$$

6.18.6 Romberg

- $k = 0$: $R(0,0) = E_0$, $R(1,0) = E_1$, $R(2,0) = E_2$.
- $k = 1$ (equivalent to Simpson's rule):

  1. $j = 1$:

$$\begin{aligned} R(1,1) &= \frac{4R(1,0) - R(0,0)}{3} \\ &= \frac{4E_1 - E_0}{3} \simeq \frac{4 \times 1.069444 - 1.25}{3} \simeq 1.009259 = S_2\,. \end{aligned} \tag{6.18.6.1}$$

  2. $j = 2$:

$$\begin{aligned} R(2,1) &= \frac{4R(2,0) - R(1,0)}{3} \\ &= \frac{4E_2 - E_1}{3} \simeq \frac{4 \times 1.017988 - 1.069444}{3} \simeq 1.000835 = S_4\,. \end{aligned} \tag{6.18.6.2}$$

- $j = 2$, $k = 2$:

$$R(2,2) = \frac{4^2 R(2,1) - R(1,1)}{4^2 - 1} \simeq \frac{16 \times 1.000835 - 1.009259}{15} \simeq 1.000274\,. \tag{6.18.6.3}$$

6.19 Worked example 3

- The function is

$$f(x) = \frac{1}{2\sqrt{x}}\,. \tag{6.19.1}$$

- We set $a = 0$ and $b = 1$. The exact value of the integral is

$$I = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = \Big[\sqrt{x}\Big]_0^1 = 1 - 0 = 1\,. \tag{6.19.2}$$

- **This is an improper integral because the integrand diverges at $x = a = 0$.**
- **However, it can be evaluated using the midpoint rule.**
- $n = 1$: hence $h = 1$, $x_0 = 0$, $x_0 = 1$

$$\begin{aligned}
n = 1: \qquad & T_1 = f\left(\frac{x_0 + x_1}{2}\right) = \frac{1}{2\sqrt{0.5}} \simeq 0.707107\,, && (6.19.3a)\\
n = 2: \qquad & T_2 = \frac{1}{2}\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right)\right] \\
& = \frac{1}{2}\left[\frac{1}{2\sqrt{1/4}} + \frac{1}{2\sqrt{3/4}}\right] \\
& \simeq 0.788675\,, && (6.19.3b)\\
n = 3: \qquad & T_3 = \frac{1}{3}\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right)\right] \\
& = \frac{1}{3}\left[\frac{1}{2\sqrt{1/6}} + \frac{1}{2\sqrt{3/6}} + \frac{1}{2\sqrt{5/6}}\right] \\
& \simeq 0.826525\,, && (6.19.3c)\\
n = 4: \qquad & T_4 = \frac{1}{4}\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right)\right] \\
& = \frac{1}{4}\left[\frac{1}{2\sqrt{1/8}} + \frac{1}{2\sqrt{3/8}} + \frac{1}{2\sqrt{5/8}} + \frac{1}{2\sqrt{7/8}}\right] \\
& \simeq 0.849422\,. && (6.19.3d)
\end{aligned}$$

- **The rate of convergence is slow, only $O(1/n)$ not $O(1/n^2)$, because the integrand is unbounded in the domain of integration. Hence derivatives such as $f'(x)$, $f''(x)$, etc. in the error analysis are not necessarily bounded.**
- We can form a table of the errors $\varepsilon_n = 1 - T_n$, tabulate the inverses $1/\varepsilon_n$.
- The table clearly exhibits the $O(1/n)$ rate of convergence.

n	$1/\varepsilon_n$
1	3.414
2	4.732
3	5.765
4	6.641

6.20 Worked example 4

- The function is $f(x) = x^9$. We set $a = 0$ and $b = 1$, so the exact value of the integral is

$$I = \int_0^1 x^9 \, dx = \left[\frac{x^{10}}{10} \right]_0^1 = 0.1\,. \tag{6.20.1}$$

- **Extended trapezoid This is the only part where we evaluate the function $f(x)$.**

  1. We use the extended trapezoid rule to compute $E[j]$ for $n = 2^j$ intervals, $j = 0, 1, \ldots$.
  2. The points at which to evaluate the function are:

n	x
1	0 $\quad$ 1
2	$\frac{1}{2}$
4	$\frac{1}{4}$ $\quad$ $\frac{3}{4}$
8	$\frac{1}{8}$ $\quad$ $\frac{3}{8}$ $\quad$ $\frac{5}{8}$ $\quad$ $\frac{7}{8}$
$\cdots$	$\cdots$

- **Romberg integration. This is simply a set of subtractions.**

$$\text{initial} \qquad R(j,0) = E[j]\,, \tag{6.20.2a}$$

$$1^{st} \text{ order} \qquad R(j,1) = \frac{4R(j,0) - R(j-1,0)}{3}\,, \tag{6.20.2b}$$

$$2^{nd} \text{ order} \qquad R(j,2) = \frac{16R(j,1) - R(j-1,1)}{15}\,, \tag{6.20.2c}$$

$$3^{rd} \text{ order} \qquad R(j,3) = \frac{64R(j,2) - R(j-1,2)}{63}\,. \tag{6.20.2d}$$

- **The results are tabulated below for $E[j]$ and Romberg first, second, third order**

j	n	$E[j]$	$R(j,1)$	$R(j,2)$	$R(j,3)$
0	1	0.5			
1	2	0.250977	0.167969		
2	4	0.14426	0.108688	0.104736	
3	8	0.11155	0.100646	0.10011	0.100037
4	16	0.102919	0.100042	0.100002	0.1
5	32	0.100732	0.100003	0.1	0.1
6	64	0.100183	0.1	0.1	0.1
7	128	0.100046	0.1	0.1	0.1
8	256	0.100011	0.1	0.1	0.1
9	512	0.100003	0.1	0.1	0.1
10	1024	0.100001	0.1	0.1	0.1