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2 Lecture 2

2.1 Bond: more general pricing formula

- Recall that a bond pays cashflows of coupons and redeems its face value at maturity.
- Let us write a more general formula for the fair value of a bond.
- The face of the bond is F . We shall always set $F = 100$ in these lectures.
- Suppose the annualized coupons rates are c_1, \dots, c_n .
 1. The c_i are not necessarily equal, and some or all could be zero.
 2. Let the frequency of the coupon payments be f , so there are f coupons per year.
 3. For semiannual compounding then $f = 2$.
 4. Semiannual compounding is the typical case in the USA and many countries.
- Let the dates of the coupons be t_1, \dots, t_n . Then $t_n = T =$ maturity date of bond.
 1. We shall assume the coupons are paid at equal time intervals.
 2. Hence in these lectures, we shall set $t_i = i/f$ and the maturity is $T = n/f$.
 3. In real life there are many exceptions to this rule, but we shall keep things simple.
- Let the time today be t_0 .

- Then a naive formula for the fair value B of a bond is

$$B = \frac{c_1/f}{(1+y/f)^{f(t_1-t_0)}} + \frac{c_2/f}{(1+y/f)^{f(t_2-t_0)}} + \dots$$

$$\dots + \frac{c_{n-1}/f}{(1+y/f)^{f(t_{n-1}-t_0)}} + \frac{F + (c_n/f)}{(1+y/f)^{f(t_n-t_0)}}. \quad (2.1.1)$$

- If we set $f = 2$ and $t_0 = 0$, this agrees with the simple formula in the earlier lecture.
- **However, there is an important caveat to eq. (2.1.1).**
 1. If $t_0 > 0$, **we only include coupons where $t_i - t_0 > 0$.**
 2. We exclude coupons which are in the “past” (coupon date $t_i \leq t_0$).
 3. In this context, we assume that coupons are paid at the “start of day” so if $t_i - t_0 = 0$, we exclude the coupon.
- Hence we modify the above sum as follows (“**numerator_i**” has an obvious definition):

$$B = \left[\frac{c_1/f}{(1+y/f)^{f(t_1-t_0)}} + \frac{c_2/f}{(1+y/f)^{f(t_2-t_0)}} + \dots \right. \\ \left. \dots + \frac{c_{n-1}/f}{(1+y/f)^{f(t_{n-1}-t_0)}} + \frac{F + (c_n/f)}{(1+y/f)^{f(t_n-t_0)}} \right]_{t_i > t_0} \quad (2.1.2)$$

$$= \sum_{i=1}^n \left[\frac{(\text{numerator})_i}{(1+y/f)^{f(t_i-t_0)}} \right]_{t_i > t_0}.$$

- **This is the more general bond pricing formula.**
- As has already been pointed out in an earlier lecture, in most cases we observe the value of the bond price B in the financial markets and we invert eq. (2.1.2) to calculate the yield y . See Section 2.3.

2.2 Yield and discount factors

- The following question was tagged as an item to be answered in a future lecture.
- Why do all the discount factors have the following form (assuming $f = 2$ and $t_0 = 0$)?

$$DF_1 = \frac{1}{1 + \frac{1}{2}y}, \quad DF_2 = \frac{1}{(1 + \frac{1}{2}y)^2}, \quad \dots \quad DF_i = \frac{1}{(1 + \frac{1}{2}y)^i}. \quad (2.2.1)$$

- The answer is that this is the **definition** of how the yield is calculated.
- Basically, we pose the question: suppose all the cashflows from a bond are reinvested in a bank, and they all earn the **same annualized rate of interest**, then what would that rate of interest have to be, so that the bond price equals the target value of B (in eq. (2.1.2))?
- The yield gives an answer to that question.
- In real life, it is unrealistic to assume that all the cashflows would earn the same annualized rate of interest. We know that interest rates change over time.
- However, we do not know, *today*, what those future interest rates will be.
- However, the yield is a number which *can* be calculated, using information available *today*.

2.3 Yield from bond price

- Let us return to Section 2.1: how do we determine the yield y given the bond price B ?
- We can regard the sum in eq. (2.1.2) as a function of the yield, i.e. $B = B(y)$.
- Suppose the bond trades with a market price B_{market} .
- More generally, we can set any target value B_{target} .
- We wish to solve the following equation for the yield y :

$$B(y) = B_{\text{target}} . \quad (2.3.1)$$

- There are many mathematical algorithms to solve an equation such as eq. (2.3.1).
- We shall study only one method, which is in many ways the simplest.
- It is known as the method of **bisection**.
- The fundamental idea is simple. It goes as follows:
 1. We know from eq. (2.1.2) that $B(y)$ is a continuous function of y .
 2. We also know that $B(y)$ decreases as the value of y increases.
 3. Hence we find a low yield y_{low} such that $B(y_{\text{low}}) > B_{\text{target}}$ and a high yield y_{high} such that $B(y_{\text{high}}) < B_{\text{target}}$.
 4. Then the solution of eq. (2.3.1) lies somewhere between y_{low} and y_{high} .
 5. It is possible by luck that either y_{low} and y_{high} is a solution of eq. (2.3.1).
 6. If so we exit the algorithm immediately.
 7. Else we iterate as follows.
 8. We use the midpoint $y_{\text{mid}} = (y_{\text{low}} + y_{\text{high}})/2$ and calculate $B(y_{\text{mid}})$.
 9. If $|B(y_{\text{mid}}) - B_{\text{target}}|$ is less than a prespecified tolerance, we exit the calculation and say that y_{mid} is the solution of eq. (2.3.1).
 10. Else if $B(y_{\text{mid}}) > B_{\text{target}}$ then y_{mid} is too low. We update $y_{\text{low}} := y_{\text{mid}}$.
 11. Else obviously $B(y_{\text{mid}}) < B_{\text{target}}$ so y_{mid} is too high. We update $y_{\text{high}} := y_{\text{mid}}$.
 12. **There are mathematically better ways to formulate the comparison tests.**
 13. We repeat the iteration using the updated values of y_{low} and y_{high} .
 14. Hence the interval $|y_{\text{high}} - y_{\text{low}}|$ is cut by a factor of two at each iteration step.
 15. This is why it is called the “bisection” algorithm.
 16. If $|y_{\text{high}} - y_{\text{low}}|$ is less than a prespecified tolerance, we exit the calculation and say that y_{mid} is the solution of eq. (2.3.1).
 17. The bisection algorithm is not necessarily the fastest, but it is guaranteed to be stable and will converge after a finite number of iterations.
 18. The weak point is to find suitable initial values for y_{low} and y_{high} .
- **It is an important part of the “computational” part of this course to write a working bisection program to compute the yield.**

2.4 Bond duration: Macaulay and modified duration

Let us now study some other formulas pertaining to bonds. The most important is the **duration**.

2.4.1 Macaulay duration

- The **Macaulay duration** is defined as follows. We write “Macaulay” to avoid confusion with other terminology (see below). It was introduced by a person named Frederick Macaulay (approximately in the 1930s). It is a time weighted average of the cashflows

$$D_{\text{Mac}} = \frac{1}{B} \sum_{i=1}^n \left[(t_i - t_0) \frac{(\text{numerator})_i}{(1 + y/f)^{f(t_i - t_0)}} \right]_{t_i > t_0} . \quad (2.4.1)$$

- By construction, the Macaulay duration has units of time (or is measured in years).
- By analogy with physics, the Macaulay duration is analogous to a “center of mass” of a bond.
 1. Let us draw time on the horizontal axis and visualize the discounted cashflows as “masses” located at points $t_i - t_0$.
 2. The Macaulay duration is the “center of maturity” of all the discounted cashflows.
 3. Another way to say it is: if all the discounted cashflows were paid at one time, what would that time be?
 4. The Macaulay duration is effectively the “average lifetime” of the bond.

2.4.2 Modified duration

- There is also a term called the **modified duration**.
- For this reason, the term “duration” without qualification is confusing.
- It is better to say “Macaulay duration” or “modified duration” to avoid ambiguity.
- The modified duration is defined via a partial derivative with respect to the bond’s yield

$$D_{\text{mod}} = -\frac{1}{B} \frac{\partial B}{\partial y} . \quad (2.4.2)$$

- The modified duration also has units of time (or is measured in years).
- Because the modified duration is a partial derivative, it is a measure of the sensivity of the bond price to a small change in the yield.
- The change in the bond price δB for a small change in the yield δy is

$$\delta B(y) \simeq -D_{\text{mod}} B(y) \delta y . \quad (2.4.3)$$

- There is a simple but important relation between the Macaulay and modified duration:

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1 + y/f} . \quad (2.4.4)$$

2.5 Bond DV01

- In the financial markets, interest rates and yields are usually measured in **basis points**.
- One basis point is 1/100 of one percent, i.e. in decimal a change in yield of one basis point is $\delta y = 0.0001$.
- The meaning of “**DV01**” (dollar value zero one) is the change in the dollar value (price) of a bond for a one basis point change in yield.
- In terms of the modified duration, the DV01 is given by (note the minus sign)

$$\text{DV01} = -\frac{\partial B}{\partial y} \delta y = B D_{\text{mod}} \delta y = 0.0001 \times B D_{\text{mod}} . \quad (2.5.1)$$

In the last step a one basis point value $\delta y = 0.0001$ was substituted for δy .

- Many times, people want the change for a 100 basis point move in yield (i.e. one percent). This is usually computed by multiplying the DV01 value by 100.

2.6 Bond convexity

- Why stop at the first partial derivative?
- The **convexity** of a bond is given by the second partial derivative

$$C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} . \quad (2.6.1)$$

- In terms of the modified duration,

$$\begin{aligned} C &= \frac{1}{B} \frac{\partial}{\partial y} \left(\frac{\partial B}{\partial y} \right) \\ &= \frac{1}{B} \frac{\partial}{\partial y} (-BD_{\text{mod}}) \\ &= \frac{1}{B} \left[-\frac{\partial B}{\partial y} D_{\text{mod}} - B \frac{\partial D_{\text{mod}}}{\partial y} \right] \\ &= D_{\text{mod}}^2 - \frac{\partial D_{\text{mod}}}{\partial y} . \end{aligned} \quad (2.6.2)$$

- For the simple examples in these lectures, the convexity of a bond is positive.
- More complicated types of bonds can exhibit negative convexity.

2.7 Zero coupon bonds

- There is an important class of bonds which are called **zero coupon bonds**.
- As the name suggests, zero coupon bonds pay no coupons. They pay only one cashflow, on the maturity date, which is the face of the bond.
- Zero coupon bonds are popular financial instruments. They allow an investor to perform hedging of cashflows at a particular point in time, without the complication of juggling additional cashflows (the coupons), which might interfere with other activities of the investor.
- For finance academics, zero coupons bonds are theoretically simpler to analyze.
- Since there is only one cashflow, i.e. the face value F , let us say the maturity date is T (where obviously $T > t_0$). In terms of the yield, the price of a zero coupon bond is given by

$$B = \frac{F}{(1 + y/f)^{f(T-t_0)}} . \quad (2.7.1)$$

- As stupid as it sounds, there is still a parameter f , simply because of market conventions. It is awkward to maintain inventory of bonds in a database if some bonds have a parameter f (frequency of cashflows) and for zero coupon bonds the parameter f is absent. Hence for quoting conventions, in the USA the yield is typically quoted on a semi-annual basis ($f = 2$).
- It is easy to invert the above formula to calculate the yield of a zero coupon bond from the market price of the bond.
- To the extent that interest rates and yields are positive, the price of a zero coupon bond is less than par.
- **The Macaulay duration of a zero coupon bond is equal to its time to maturity.**
- **This is a very important fact.**
- Proof:

$$D_{\text{Mac}} = \frac{1}{B} \frac{(T - t_0)F}{(1 + y/f)^{f(T-t_0)}} = \frac{(T - t_0)B}{B} = T - t_0 . \quad (2.7.2)$$

- If there is only one cashflow then the “weighted average” is obviously the time to that cashflow.

2.8 Worked example: Bond price

- Consider a bond with two years to maturity.
- For simplicity, we consider only semiannual coupons (two coupons per year).
- Hence there are four coupons, paid at times $t_1 = 0.5$, $t_2 = 1.0$, $t_3 = 1.5$ and $t_4 = 2.0$.
- Let $F = 100$ and $c_1 = 4.1$, $c_2 = 4.2$, $c_3 = 4.3$, $c_4 = 4.4$ and the yield be $y = 6.0\%$.
- Suppose $t_0 < t_1$ (date of first coupon). Then the bond price is (see eq. (2.1.2))

$$B = \frac{\frac{1}{2}c_1}{(1 + \frac{1}{2}y)^{2(t_1-t_0)}} + \frac{\frac{1}{2}c_2}{(1 + \frac{1}{2}y)^{2(t_2-t_0)}} + \frac{\frac{1}{2}c_3}{(1 + \frac{1}{2}y)^{2(t_3-t_0)}} + \frac{F + \frac{1}{2}c_4}{(1 + \frac{1}{2}y)^{2(t_4-t_0)}}. \quad (2.8.1)$$

- **Let $t_0 = 0.0$.** Then the bond price is

$$B = \frac{2.05}{(1.03)^{1.0}} + \frac{2.1}{(1.03)^{2.0}} + \frac{2.15}{(1.03)^{3.0}} + \frac{102.2}{(1.03)^{4.0}} \\ \simeq 96.74067. \quad (2.8.2)$$

- **Let $t_0 = 0.1$.** Then the bond price is

$$B = \frac{2.05}{(1.03)^{0.8}} + \frac{2.1}{(1.03)^{1.8}} + \frac{2.15}{(1.03)^{2.8}} + \frac{102.2}{(1.03)^{3.8}} \\ \simeq 97.31428. \quad (2.8.3)$$

- **Let $t_0 = 0.55$.** Then $t_1 < t_0$ so we skip the first coupon. The bond price is

$$B = \frac{\frac{1}{2}c_2}{(1 + \frac{1}{2}y)^{2(t_2-t_0)}} + \frac{\frac{1}{2}c_3}{(1 + \frac{1}{2}y)^{2(t_3-t_0)}} + \frac{F + \frac{1}{2}c_4}{(1 + \frac{1}{2}y)^{2(t_4-t_0)}} \\ = \frac{2.1}{(1.03)^{0.9}} + \frac{2.15}{(1.03)^{1.9}} + \frac{102.2}{(1.03)^{2.9}} \\ \simeq 97.88179. \quad (2.8.4)$$

2.9 Worked example: Macaulay and modified duration

- We use the same bond as in Sec. 2.8.
- **Let $t_0 = 0.0$.** The Macaulay duration is

$$\begin{aligned} D_{\text{Mac}} &= \frac{1}{B} \left[0.5 \times \frac{2.05}{(1.03)^{1.0}} + 1.0 \times \frac{2.1}{(1.03)^{2.0}} + 1.5 \times \frac{2.15}{(1.03)^{3.0}} + 2.0 \times \frac{102.2}{(1.03)^{4.0}} \right] \\ &\simeq \frac{187.5327}{96.74067} \\ &\simeq 1.938509. \end{aligned} \tag{2.9.1}$$

- The modified duration is

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1 + \frac{1}{2}y} \simeq \frac{1.938509}{1.03} \simeq 1.882048. \tag{2.9.2}$$

- **Let $t_0 = 0.1$.** The Macaulay duration is

$$\begin{aligned} D_{\text{Mac}} &= \frac{1}{B} \left[0.4 \times \frac{2.05}{(1.03)^{0.8}} + 0.9 \times \frac{2.1}{(1.03)^{1.8}} + 1.4 \times \frac{2.15}{(1.03)^{2.8}} + 1.9 \times \frac{102.2}{(1.03)^{3.8}} \right] \\ &\simeq \frac{178.9132}{97.31428} \\ &\simeq 1.838509. \end{aligned} \tag{2.9.3}$$

- The modified duration is

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1 + \frac{1}{2}y} \simeq \frac{1.838509}{1.03} \simeq 1.78496. \tag{2.9.4}$$

- **Let $t_0 = 0.55$.** The Macaulay duration is

$$\begin{aligned} D_{\text{Mac}} &= \frac{1}{B} \left[0.45 \times \frac{2.1}{(1.03)^{0.9}} + 0.95 \times \frac{2.15}{(1.03)^{1.9}} + 1.45 \times \frac{102.2}{(1.03)^{2.9}} \right] \\ &\simeq \frac{138.8674}{97.88179} \\ &\simeq 1.418726. \end{aligned} \tag{2.9.5}$$

- The modified duration is

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1 + \frac{1}{2}y} \simeq \frac{1.418726}{1.03} \simeq 1.377404. \tag{2.9.6}$$

2.10 Worked example: Bond yield

- We use the same bond as in Sec. 2.8.
- Let the market price of the bond be $B_{\text{market}} = 99.5$.
- For simplicity, we analyze only the case $t_0 = 0.0$.
- We already know that if the yield is $y = 6\%$, the bond price is $B \simeq 96.74067$.
- Next try a guess $y = 4\%$. Using the formulas in Sec. 2.8, the bond price is $B \simeq 100.4713$.
- Hence we have found a bracket because $B(4\%) > B_{\text{market}} > B(6\%)$.
- **Hence we know the true yield lies somewhere between $y_{\text{low}} = 4\%$ and $y_{\text{high}} = 6\%$.**
- Use the midpoint $y_{\text{mid}} = (y_{\text{low}} + y_{\text{high}})/2.0 = 5.0\%$. The bond price is $B \simeq 98.58345$.
- Hence $B(5\%)$ and $B(6\%)$ are **on the same side of B_{market} .**
- **Hence we update $y_{\text{high}} := y_{\text{mid}} = 5\%$ and iterate again.**
- The updated midpoint is $y_{\text{mid}} = (4.0 + 5.0)/2.0 = 4.5\%$. The bond price is $B \simeq 99.52164$.
- We stop here.
- This is not a very accurate “convergence” but is enough for the purposes of a worked example.
- We say that for a market price of $B_{\text{market}} = 99.5$, the yield is approximately 4.5%.