Queens College, CUNY, Department of Computer Science

Computational Finance

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3 Lecture 3

3.1 Yield curve

- This lecture is about how interest rates are determined in practice.
- A set of interest rates, from short to long dated maturities, is called a **yield curve.**
- Consider a simplified model where we have only two bonds. Both are newly issued bonds, paying coupons semi-annually. The first has a maturity of 6 months and a yield y_1, and the second has a maturity of 5 years and a yield y_5, where $y_5 \neq y_1$.
- But this poses a problem: both bonds pay coupons at a date 6 months from today, ***but what is the correct discount factor to employ, to calculate the present value of a cashflow 6 months from today?***
- If we say the answer is given by using the 6 month bond, then why do we employ the yield y_5 to discount the first coupon of the 5 year bond?
- If we say "the discount factor depends on the bond" then we do not have a workable model of interest rates in the financial markets.
- Different people will disagree as to the interest rate (or discount factor) to apply to the cashflow 6 months from today.
- The subject of this lecture is to answer the above important questions.

3.2 Bootstrap

- The procedure is called a **bootstrap**, as we shall see why as we work our way through it.
- For simplicity, we only use bonds with semi-annual coupons.
- Also, all the bonds have a face $F = 100$.
- It is simplest to explain the procedure by working through an example.
- We can demonstrate the essential ideas using only two bonds.
- Let us have two newly issued bonds, with maturities of 0.5 and 1.0 years.
    1. Hence the first bond, call it $B_{0.5}$, pays only one coupon, 6 months from today.
    2. The second bond, call it $B_{1.0}$, pays a coupon 6 months from today and a second coupon 12 months from today.
- Now comes a key feature: **both bonds are par bonds**.
    1. That means, by definition, the market price of both bonds is 100.
    2. **How is it possible in practice to find par bonds with the above properties?**
    3. In fact, on any given date, there is a large variety of bonds, with different maturities, coupons and yields (and market prices).
    4. We interpolate the market data, to find hypothetical (or theoretical) bonds with maturities of 6 months and 1 year, whose prices equal par ($= 100$), and we also interpolate to determine what their yields are.
    5. The US Treasury has an interpolation formula. Other countries have their own interpolation formulas. The interpolation formula is not unique (and can be modified in times of financial crisis).
- Let the bond yields be $y_{0.5}$ and $y_{1.0}$, respectively.
    1. **The yield equals the coupon** for a newly issued par bond (with equal coupons).
    2. ***** However, after reading some homework solutions, I realized the above statement is confusing. *****
        (a) In all the bond pricing formulas, the yield appears as a decimal "$1 + \frac{1}{2}y$" whereas the coupon appears as an annualized percentage rate $\frac{1}{2}c$ or $F + \frac{1}{2}c$, where $F = 100$.
        (b) Hence I should really say **the yield <u>in percent</u> equals the annualized coupon rate <u>in percent.</u>**
        (c) Or alternatively $c = 100y$ in the bond pricing formula.
        (d) The bond pricing formula for a par bond would then be written as
$$B_{\rm par} = \frac{\frac{1}{2}(100y)}{1+\frac{1}{2}y} + \frac{\frac{1}{2}(100y)}{(1+\frac{1}{2}y)^2} + \cdots + \frac{100+\frac{1}{2}(100y)}{(1+\frac{1}{2}y)^n} = 100\,. \tag{3.2.1}$$
    3. Hence because it is a par bond, the value of the yield tells us the coupon.
    4. Therefore we have all the information to completely specify the bonds.

3.2.1 6 month point

- We begin with the discount factor for the 6 month point, say $d_{0.5}$.
- We use the 6 month bond. There is only one cashflow, hence

$$B_{0.5} = \left(F + \tfrac{1}{2}c_{0.5}\right) d_{0.5}\,. \tag{3.2.1.1}$$

- Because it is a par bond, $B_{0.5} = 100$ and also the coupon is $c_{0.5} = 100y_{0.5}$. Also $F = 100$.
- Hence the discount factor for the 6 month point is given by

$$d_{0.5} = \frac{100}{100 + \frac{1}{2}c_{0.5}} = \frac{100}{100 + \frac{1}{2}(100y_{0.5})} = \frac{1}{1 + \frac{1}{2}y_{0.5}}\,. \tag{3.2.1.2}$$

3.2.2 12 month point

- However, as noted in Sec. 3.1, we must deal with the cashflows of the 12 month bond.
- ***What do we do?***
- Let the annualized coupon rate of the 12 month bond be $c_{1.0}$.
- Because it is a par bond, we know $c_{1.0} = 100y_{1.0}$ and we know the value of $y_{1.0}$.
- By definition, the sum of the present values of the cashflows of the 12 month bond is

$$B_{1.0} = \mathrm{PV}\left(\tfrac{1}{2}c_{1.0}\right) + \mathrm{PV}\left(F + \tfrac{1}{2}c_{1.0}\right). \tag{3.2.2.1}$$

- We know $B_{1.0} = 100$ and $c_{1.0} = 100y_{1.0}$ (par bond).
- **We also already know the discount factor for the 6 month point ($= d_{0.5}$).**
- **Hence the only unknown in eq. (3.2.2.1) is the discount factor for the 1 year point.**
- Let the discount factor for the 1 year point be $d_{1.0}$.
- **We employ eq. (3.2.2.1) to calculate the value of $d_{1.0}$.**
- Using $B_{1.0} = 100$ and $c_{1.0} = 100y_{1.0}$ and $F = 100$, we obtain the following equation

$$100 = \tfrac{1}{2}(100y_{1.0}) \times d_{0.5} + (100 + \tfrac{1}{2}(100y_{1.0})) \times d_{1.0}\,. \tag{3.2.2.2}$$

- **Cancelling all the factors of 100, we solve this to obtain the value of $d_{1.0}$:**

$$d_{1.0} = \frac{1 - \frac{1}{2}y_{1.0}\,d_{0.5}}{1 + \frac{1}{2}y_{1.0}}\,. \tag{3.2.2.3}$$

3.2.3 18 month point

- We can easily extend the above procedure to the 18 month point.
- Let the yield of a newly issued 1.5 year par bond be $y_{1.5}$.
- We know the discount factors for the 0.5 year and 1.0 year points.
- Similarly to eq. (3.2.2.2), we obtain, with $c_{1.5} = 100y_{1.5}$,

$$\begin{aligned} 100 &= \tfrac{1}{2}c_{1.5} \times d_{0.5} + \tfrac{1}{2}c_{1.5} \times d_{1.0} + \big(100 + \tfrac{1}{2}c_{1.5}\big) \times d_{1.5} \\ &= \tfrac{1}{2}(100y_{1.5}) \times d_{0.5} + \tfrac{1}{2}(100y_{1.5}) \times d_{1.0} + \big(100 + \tfrac{1}{2}(100y_{1.5})\big) \times d_{1.5}\,. \end{aligned} \tag{3.2.3.1}$$

- **The value of the discount factor at the 1.5 year point $d_{1.5}$ is**

$$d_{1.5} = \frac{1 - \frac{1}{2}y_{1.5}\, d_{0.5} - \frac{1}{2}y_{1.5}\, d_{1.0}}{1 + \frac{1}{2}y_{1.5}}\,. \tag{3.2.3.2}$$

3.2.4 24 month point, etc.

- Let the yield of a newly issued 2 year par bond be $y_{2.0}$.
- We know the discount factors for the 0.5 year, 1.0 year and 1.5 year points.
- Similarly to eqs. (3.2.2.2) and (3.2.3.1), we obtain, with $c_{2.0} = 100y_{2.0}$,

$$\begin{aligned} 100 &= \tfrac{1}{2}c_{2.0} \times d_{0.5} + \tfrac{1}{2}c_{2.0} \times d_{1.0} + \tfrac{1}{2}c_{2.0} \times d_{1.0} + \big(100 + \tfrac{1}{2}c_{2.0}\big) \times d_{2.0} \\ &= \tfrac{1}{2}(100y_{2.0}) \times d_{0.5} + \tfrac{1}{2}(100y_{2.0}) \times d_{1.0} + \tfrac{1}{2}(100y_{2.0}) \times d_{1.0} + \big(100 + \tfrac{1}{2}(100y_{2.0})\big) \times d_{2.0}\,. \end{aligned} \tag{3.2.4.1}$$

- **The value of the discount factor at the 2 year point $d_{2.0}$ is**

$$d_{2.0} = \frac{1 - \frac{1}{2}y_{2.0}\, d_{0.5} - \frac{1}{2}y_{2.0}\, d_{1.0} - \frac{1}{2}y_{2.0}\, d_{1.5}}{1 + \frac{1}{2}y_{2.0}}\,. \tag{3.2.4.2}$$

- ***And so on for longer maturities!***

3.2.5 General case

For the 6 month point $T = 0.5$, let the yield be $y_{0.5}$ and the discount factor be $d_{0.5}$. Then

$$d_{0.5} = \frac{1}{1 + \frac{1}{2}y_{0.5}}\,. \tag{3.2.5.1}$$

For the general case $T = 1.0, 1.5, 2.0, \dots$, let the yield be y_T and the discount factor be d_T. Then

$$d_T = \frac{1 - \frac{1}{2}y_T\, d_{0.5} - \frac{1}{2}y_T\, d_{1.0} - \cdots - \frac{1}{2}y_T\, d_{T-0.5}}{1 + \frac{1}{2}y_T}\,. \tag{3.2.5.2}$$

3.3 Bootstrap: review

- We obtain (by interpolation of market data of real bonds) a set of theoretical par bonds with maturities of 0.5, 1.0, 1.5, 2.0, etc. years.
- We arrange the bonds in order of increasing maturity.
- This is why the procedure is called a bootstrap:
  1. At every step of the calculation, we use the information from all the previous bonds to obtain the discount factor of the last cashflow.
  2. We begin with the 0.5 year bond (only one cashflow) to obtain the 6 month discount factor $d_{0.5}$.
  3. We use that information and the 1.0 year bond (two cashflows) to obtain the 1 year discount factor $d_{1.0}$.
  4. We use all of the previous information and the 1.5 year bond (three cashflows) to obtain the 1.5 year discount factor $d_{1.5}$.
  5. *And so on.*
- **Maybe what confuses some of you is the presence of $d_{0.5}$ in the formula for the 12 month bond.**
  1. **Remember that we used the 6 month bond to calculate $d_{0.5}$.**
  2. **But now that we have it, $d_{0.5}$ applies to *any* cashflow at 6 months, in any other bond (or mortgage etc).**
  3. **It is *not* restricted to the original 6 month bond.**
- **Next we have $d_{0.5}$ and $d_{1.0}$. The value of $d_{1.0}$ will apply to *any* cashflow at 12 months. It is also not restricted to only one bond.**

3.4 Spot rates

- What we have calculated up to now is a set of discount factors, not interest rates.
- Associated with the discount factors are the so-called **spot rates**.
- They are also called **zero coupon spot rates** for reasons that will be explained in Sec. 3.5.
- In general, for any time t, the continuously compounded interest rate r_t is related to the discount factor d_t via $e^{-r_t(t-t_0)} = d_t$.
- Hence in general

$$r_t = -\frac{\ln(d_t)}{t - t_0} \,. \tag{3.4.1}$$

- Hence from the bootstrap we obtain a set of interest rates, which are called **spot rates.**

$$r_{0.5} = -\frac{\ln(d_{0.5})}{0.5} \,, \qquad r_{1.0} = -\frac{\ln(d_{1.0})}{1.0} \,, \qquad r_{1.5} = -\frac{\ln(d_{1.5})}{1.5} \qquad \dots \tag{3.4.2}$$

- **The set of spot rates, plotted as a function of maturity, constitutes a yield curve.**
- It is called the **spot rate curve**.
- It is called the **zero coupon yield curve** for reasons that will be explained in Sec. 3.5.

3.5 Spot curve & par curve

- The original yield curve is composed of the yields of par bonds ("par yields").
- It is usually called the **par yield curve** or simply the **par curve**.
- When the bootstrap is completed, we obtain a set of (continuously compounded) interest rates (or equivalently, discount factors) for all the maturities in the original set of par bonds.
- The interest rates $r_{0.5}$, $r_{1.0}$, etc. are called (continuously compounded) **spot rates** or **zero coupon rates** or simply **zero rates**. Note that $d_{0.5}$, $d_{1.0}$, $d_{1.5}$, etc., are simply the discount factors for zero coupon bonds with the respective maturities of 0.5, 1.0, 1.5 years, etc. The interest rates $r_{0.5}$, $r_{1.0}$, etc. are the corresponding (continuously compounded) interest rates for those zero coupon bonds.
- The spot rates can also be used to make a yield curve. It is called the **spot rate curve** or simply the **spot curve** or the **zero coupon yield curve**.
- Real bonds which trade in the market typically pay coupons. Hence one deduces their yields (from their market prices). From this one can construct a par yield curve.
- The spot curve must be calculated mathematically from the par curve. The spot curve is effectively the yield curve of zero coupon bonds, but typically there are not enough zero coupon bonds to supply enough data to construct the spot curve directly from market data.
- Financial academics prefer to work with the spot curve because it is easier to employ in mathematical calculations.
- A spot rate directly tells us the discount factor (present value) of a cashflow at a particular point in time. A bond yield does not directly tell us the present value of a cashflow at a particular point in time, because to calculate the yield of a bond, we must sum over multiple cashflows which are paid at different times.
- When we analyze the pricing of financial securities such as optiona and other derivatives, we shall employ spot rates. The mathematics is much easier.
- A sketch of some par and spot yield curves is shown in Fig. 1.
  1. Usually the yield curve slopes upwards. This is called a **normal yield curve.**
  2. When the par curve slopes upwards, the spot curve also slopes upwards.
  3. When the par curve slopes downwards, the spot curve also slopes downwards.
  4. A downward sloping yield curve is called an **inverted yield curve.**
     (a) An inverted yield curve is usually bad news.
     (b) An inverted yield curve has historically been a reliable predictor of a recession.
  5. The yield curve can exhibit a peak (maximum or minimum) in the middle.
     (a) This is called a **humped yield curve.**
     (b) A humped yield curve is rare but can exist.

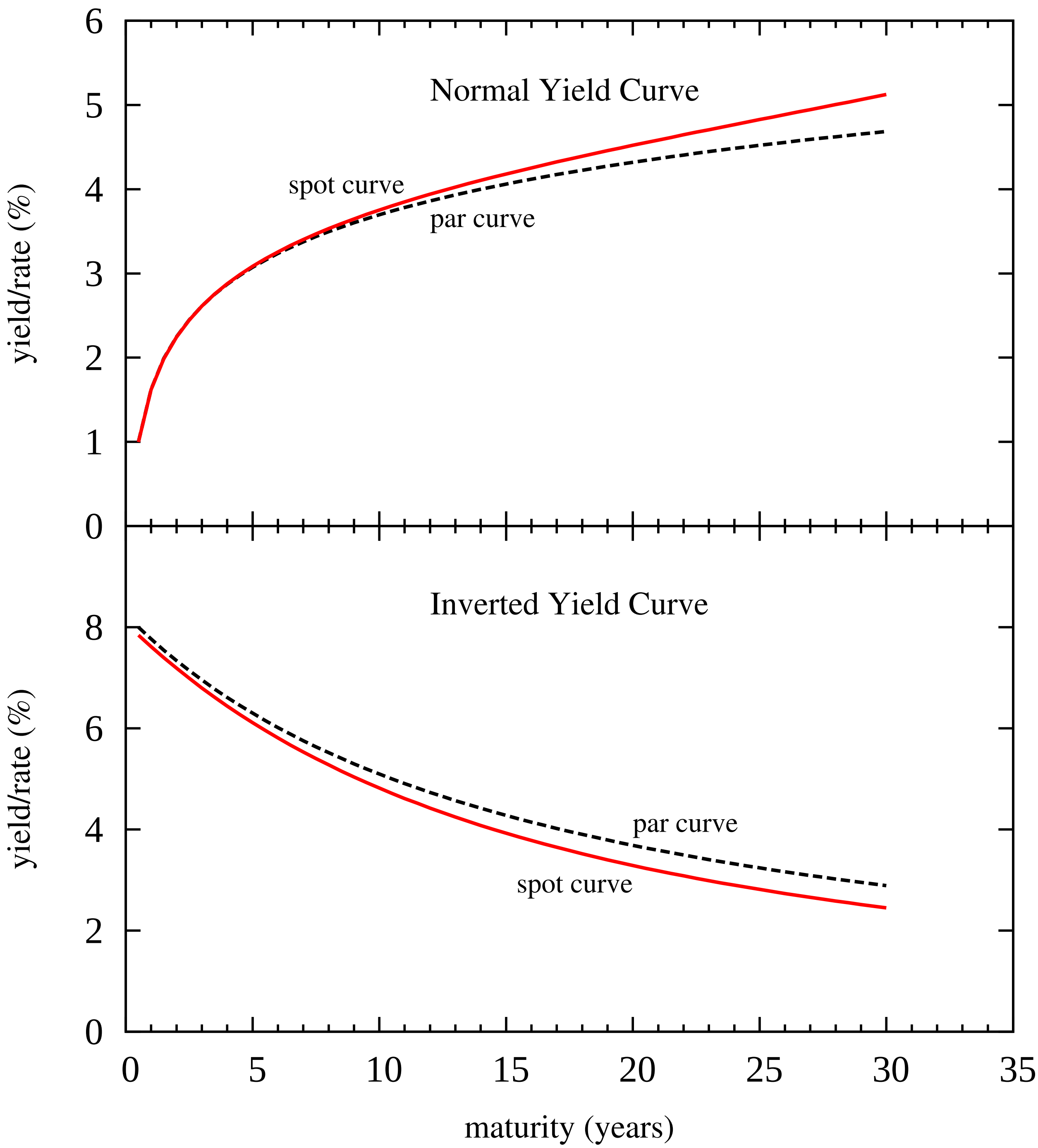

Figure 1: Sketch of normal (upward sloping) and inverted (downward sloping) yield curves.

3.6 ***Why par bonds?***

- ***Why use par bonds?***
- Most bonds trading in the financial markets are not par bonds, after all.
- Actually, we can construct a yield curve from any set of bonds in the marketplace.
- However, the obvious question arises: ***which set of bonds should we use?***
- Ultimately, that is the real answer: *we wish to avoid confusion and ambiguity.* The concept of par bonds and a par yield curve is something everyone can agree on. It is a matter of choice, which the traders in the financial markets have agreed on.
- Then, from the par yield curve, one can construct the spot rate curve.

3.7 Crackpot history

- The par yield curve is a graph of par yields against bond maturities.
- However, the finance academics advocated, for theoretical calculations, the use of a graph of bond yields vs. duration.
- In practice, as one can imagine, the "duration based yield curve" never proved popular with traders in the financial markets. The maturity of a bond is a known parameter, from the terms of its issue. However, how to measure the duration of a bond? The duration changes with market conditions and is not an easily determined parameter.
- *However*, there *is* a special case where the duration of a bond is easy to determine. That is the case of zero coupon bonds. The (Macaulay) duration of a zero coupon bond equals its maturity. Hence a graph based on the yields of zero coupon bonds satisfies both the academics (who want duration) and market practitioners (who want maturity). This is the spot curve.

3.8 Term structure of interest rates

- The **term structure of interest rates** is effectively the same concept as the yield curve.
- The spot rates (and par yields) are obviously not all equal, in general.
- The expression "term structure of interest rates" is a way of expressing the fact that the applicable interest rate for a cashflow depends on the time at which it is paid.

3.9 Bootstrap: matrix formulation (not for examination)

- The bootstrap calculation can also be expressed as a matrix problem. The following matrix equation should be reasonably self-explanatory. The goal is to solve for the discount factors:

$$\begin{pmatrix} 1+\frac{1}{2}y_{0.5} & & & & \\ \frac{1}{2}y_{1.0} & 1+\frac{1}{2}y_{1.0} & & & \\ \frac{1}{2}y_{1.5} & \frac{1}{2}y_{1.5} & 1+\frac{1}{2}y_{1.5} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{2}y_n & \frac{1}{2}y_n & \dots & \dots & 100+\frac{1}{2}y_n \end{pmatrix} \begin{pmatrix} d_{0.5} \\ d_{1.0} \\ d_{1.5} \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} . \tag{3.9.1}$$

- All the blank matrix entries in the upper right triangle (above the main diagonal) are zero.

- This is an example of a **lower triangular** matrix problem. All matrix elements above the main diagonal are zero.

- The bootstrap procedure described above is in fact the most efficient way to solve such a matrix problem.

3.10 Bootstrap: questions/problems

- **Question:** What about cashflows for dates less than 6 months from today?
  Answer: In practice there are Treasury Bills with maturities of less than one year. These are highly liquid financial instruments and help to set the short term interest rates. They are effectively zero coupon bonds because they only pay on maturity. Hence in practice the bootstrap of the yield curve does not really begin at 6 months.

- **Question:** How do we interpolate a yield curve? How to discount cashflows at 7 months, or 1.2345 years, etc., i.e. not exactly at 6, 12, 18, ... months?
  Answer: We must interpolate the spot interest rates.

  1. ***There is no universally agreed upon procedure how to do this.***
  2. Every finance company I have worked at has its own proprietary algorithm.
  3. They all claim that their algorithm is the "best" way to interpolate a yield curve.

- **Question:** The yield curve extends out to 30 years, but it does not have points every 6 months out to 30 years. There are yields for 5 and 10 years: how to obtain the par yields for maturities between 5 and 10 years?
  Answer: Bootstrapping a real yield curve is an example of an **undetermined problem**. There is not enough market data, hence is also necessary to interpolate the original yield curve to estimate the par yields every 6 months out to 30 years, to perform the bootstrap successfully. There is also no universally agreed upon procedure how to interpolate par yields.

- **Question:** The bootstrap procedure returns a set of discount factors. But the above procedure does not guarantee the output values will lie between 0 and 1. What happens if $d_t \le 0$ or $d_t > 1$ for some value of t?
  Answer: This is a very important detail. From the definition of the interest rate $e^{-r_t(t-t_0)} = d_t$, we that if $d_t \le 0$, the value of r_t is mathematically ill-defined. Also if $d_t > 1$, the value of r_t is negative. *There is nothing we can do if the market produces a yield curve which leads to bad discount factors.* Typically it signifies chaos or a crisis in the financial markets.

3.11 Spot rates and forward rates

- What is the effective interest rate for a financial transaction will last for one year, but it will begin 6 months from now, not today?
- More generally, the transaction will begin at time t_i and end at time t_f. What is the effective interest rate over that time interval?
- This is known as a **forward rate**.
- A forward rate is easy to calculate from spot rates. Discount \$1 to today in two ways:
  (i) from t_f to t_0 (today), and
  (ii) from t_f to t_i, then from t_i to t_0 (today).
  Equate the two sets of discount factors
$$d(t_f, t_0) = d(t_f, t_i)d(t_i, t_0)\,. \tag{3.11.1}$$
- The spot rates for times t_i and t_f are r_i and r_f, respectively, They are known, or can be interpolated from the spot curve. Let the unknown forward rate be r_{fi}. From the above equation we obtain
$$d(t_f, t_i) = \frac{d(t_f, t_0)}{d(t_i, t_0)}\,. \tag{3.11.2}$$
  Expressing this in terms of interest rates, we obtain
$$e^{-r_{fi}(t_f - t_i)} = \frac{e^{-r_f(t_f - t_0)}}{e^{-r_i(t_i - t_0)}}\,. \tag{3.11.3}$$
  This can be solved for the forward rate r_{fi} by taking logarithms
$$r_{fi} = \frac{r_f(t_f - t_0) - r_i(t_i - t_0)}{t_f - t_i}\,. \tag{3.11.4}$$
- This is our first example of an **no-arbitrage** calculation.
- The term "**arbitrage**" means a financial transaction which incurs no loss in any future scenario, and a positive probability of profit in some scenarios. Hence it is a "no lose" transaction, and will yield a positive profit in some cases. Essentially a riskless way to make a profit.
- The above value for the forward rate is the *only* value which is consistent with "no arbitrage" in the interest rate environment of today's spot rates. If the forward rate had any other value, the two paths of discounting would not have equal present value. Then formulate an arbitrage strategy as follows:
  (i) sell the transaction which has the higher present value, and
  (ii) use the money to buy the transaction which has the lower present value, and
  (iii) invest the money left over in a bank.
  There must be investors who will buy the more expensive transaction in (i), else the forward rate would not have a mismatched value. At time t_f, both transactions (i) and (ii) will be worth \$1 (by definition). Net them against each other and they will cancel to zero. Pocket the money in (iii) as profit.

3.12 Interpolation of the spot curve

- The par and spot curves have data only at a discrete set of dates (tenors).
- In general we wish to obtain information at an arbitrary date.
- Hence we must interpolate the yield curve.
- **We shall consider only the interpolation of the spot curve.**
- ***There is no unique way to interpolate the spot curve.***
- Every financial company I have worked for has employed a different interpolation scheme.
- Hence we must discuss interpolation algorithms.
- Suppose we have times t_i and t_j (and $t_i < t_j$) with corresponding spot rates r_i and r_j.
  1. We want to interpolate the spot curve at the time t, where $t_i \le t \le t_j$.
  2. Let the fraction of the time interval be λ, where
  $$\lambda = \frac{t - t_i}{t_j - t_i}\,. \tag{3.12.1}$$
  3. Then $0 \le \lambda \le 1$ and $\lambda = 0$ at $t = t_i$ and $\lambda = 1$ at $t = t_j$.
  4. We shall use the above notation in the analysis below.
- We shall consider three interpolation algorithms below.

3.12.1 Interpolation of discount factors

There *is* one method that *nobody* uses, and that is to interpolate the discount factors. It is easy to see why this is so. Suppose the spot curve is flat, and suppose $r = 10\%$ at all points on the spot curve. What is the interpolated spot rate for 0.75 years? The discount factor for $t_{0.5} = 0.5$ is $d_{0.5} = e^{-0.05} \simeq 0.951229$ and for $t_{1.0} = 1.0$ is $d_{1.0} = e^{-0.1} \simeq 0.904837$. Then the interpolated value for $t_{0.75} = 0.75$ is the average

$$d_{0.75} = \frac{d_{0.5} + d_{1.0}}{2} \simeq 0.928033\,. \tag{3.12.1.1}$$

From this we obtain the interpolated spot rate

$$r_{0.75} = -\frac{\ln(d_{0.75})}{0.75} \simeq 9.958\%\,. \tag{3.12.1.2}$$

This is no good. For a flat spot the answer should be $r_{0.75} = 10\%$. Hence interpolating the discount factors is a bad idea.

3.12.2 Interpolation of spot rates

- It makes more sense to interpolate the spot rates themselves. In the above example of a flat spot curve, the interpolated spot rate would have the correct value of 10%:

$$r_{0.75} = \frac{r_{0.5} + r_{1.0}}{2} = \frac{10 + 10}{2} = 10\% \,. \tag{3.12.2.1}$$

- Many people do interpolate the spot rates. It makes a lot of sense.

- But even then the interpolation algorithm is not unique. The simplest is to employ linear interpolation. Others use polynomial fits or (better) cubic splines to fit a smooth curve through the spot rates. The interpolated spot rates are obtained from the fitted curve. But there is no unique cubic spline or polynomial formula to fit a spot curve.

- The formula for linear interpolation of the spot rates is easy. Using eq. (3.12.1), we obtain

$$r_{\rm lin} = (1-\lambda) r_i + \lambda\, r_j \,. \tag{3.12.2.2}$$

- From this we obtain the interpolated discount factor $d_{\rm lin} = e^{-r_{\rm lin}(t-t_0)}$.

3.12.3 Constant forward rate

- There is a different algorithm, also widely employed.
- ***Just to teach you something new***.
- We shall employ the forward rate.

1. Consider the following example.
2. Once again, $t_{0.5} = 0.5$ and $t_{1.0} = 1.0$ and the interpolation time is $t_{0.75} = 0.75$.
3. Let us say the spot rates are $r_{0.5} = 10\%$ and $r_{1.0} = 12\%$ (not a flat spot curve).
4. The values of the discount factors are $d_{0.5} = e^{-0.05}$ and $d_{1.0} = e^{-0.12}$.
5. The linearly interpolated spot rate for $t_{0.75} = 0.75$ is obviously $r_{0.75} = 11\%$.
6. Let us calculate the forward rate from $t_{0.5}$ to $t_{1.0}$.
7. Using the formula from Sec. 3.11, we obtain

$$r_{\rm fwd} = \frac{r_{1.0}t_{1.0} - r_{0.5}t_{0.5}}{t_{1.0} - t_{0.5}} = \frac{12 - 5}{0.5} = 14\% \,. \tag{3.12.3.1}$$

8. **We demand that the interpolated spot rate $r_{0.75}$ is such that the forward rate from $t_{0.5}$ to $t_{0.75}$ is also 14%.**
9. In other words, the forward rate is constant in the interval from $t_{0.5}$ to $t_{1.0}$.
10. This is called the **constant forward rate** interpolation algorithm.
11. The mathematical formula for the discount factor $d_{0.75}$ is then

$$d_{0.75} = e^{-r_{\rm fwd}\times(0.75-0.5)} d_{0.5} = e^{-0.14\times0.25} e^{-0.05} = e^{-0.085} \,. \tag{3.12.3.2}$$

12. Then the interpolated spot rate is given by

$$r_{0.75} = -\frac{\ln(d_{0.75})}{0.75} = \frac{0.085}{0.75} \simeq 0.11333 = 11.3333\% \,. \tag{3.12.3.3}$$

- More analysis reveals the constant forward rate algorithm is equivalent to interpolating the **logarithm of the discount factors**.

1. Let us verify this.
2. We know $\ln(d_{0.5}) = -0.05$ and $\ln(d_{1.0}) = -0.12$. Also by definition $\ln(d_{0.75}) = -0.75\, r_{0.75}$.
3. Then we obtain

$$\begin{aligned} -0.75 r_{0.75} &= \frac{-0.05 - 0.12}{2} = -0.085 \\ r_{0.75} &= \frac{0.085}{0.75} \simeq 0.11333 = 11.3333\% \,. \end{aligned} \tag{3.12.3.4}$$

4. This is the same answer derived above.

- To interpolate between t_i and t_j, the constant forward rate algorithm ***interpolates the values of $r_i(t_i - t_0)$ and $r_j(t_j - t_0)$***, as opposed to interpolating the spot rates r_i and r_j.

- The constant forward rate interpolation formula is given as follows (using eq. (3.12.1))

$$r_{\rm cfr}(t-t_0) = (1-\lambda)\, r_i(t_i-t_0) + \lambda\, r_j(t_j-t_0)\,. \tag{3.12.3.5}$$

- Divide by $t-t_0$ to obtain the interpolated spot rate

$$r_{\rm cfr} = \frac{(1-\lambda)\, r_i(t_i-t_0) + \lambda\, r_j(t_j-t_0)}{t-t_0}\,. \tag{3.12.3.6}$$

- From this we obtain the interpolated discount factor $d_{\rm cfr} = e^{-r_{\rm cfr}(t-t_0)}$.
- The constant forward rate algorithm is different from interpolating the spot rates.
- *Who is to say which is the "better" answer?*
- **If the spot curve is flat, both methods return the same answer.**
- Both methods are used in practice. Then there are cubic splines, etc.

3.12.4 Example: present value

- Let us consider the example of the interest rates of 10% at 0.5 years and 12% at 1 year.
- **What is the present value of \$100 payable at $t = 0.75$ years?**
- **Linear interpolation of spot rates**
  1. Using linear interpolation of the spot rates, the interpolated spot rate is $r_{\rm lin} = 11\%$.
  2. Then the present value of \$100 at $t = 0.75$ is

$$\bigl[\mathrm{PV}(\$100)\bigr]_{\rm lin} = \$100\, e^{-0.11\times 0.75} \simeq \$92.08\,. \tag{3.12.4.1}$$

- **Constant forward rate**
  1. Using the constant forward rate algorithm, the interpolated spot rate is $r_{\rm cfr} = 11.3333\%$.
  2. Then the present value of \$100 at $t = 0.75$ is

$$\bigl[\mathrm{PV}(\$100)\bigr]_{\rm cfr} = \$100\, e^{-0.113333\times 0.75} \simeq \$91.85\,. \tag{3.12.4.2}$$

- **You see that the answer is not unique.**
- *There is a lot of mathematical theory, with fancy terminology, but when people in finance calculate "practical numbers" there are many ambiguities.*
- Admittedly, the above is an extreme example. Interest rates do not normally climb from 10% to 12% from 6 months to one year. The above example exaggerates the numbers, to illustrate the differences in the algorithms.

3.13 Semi-annual and continuous compounding of spot rates

- The spot rates as defined above are compounded continuously. The bond yields are compounded semi-annually. There is a slight disconnect here.
- Many textbooks (and webpages) about spot rates also employ semi-annual compounding for the spot rates.
- Let the semi-annually compounded spot rate be r_{semi} and let the continuously compounded spot rate be r_c. The relationship between them is

  $$\frac{1}{(1+\frac{1}{2}r_{\text{semi}})^{2t}} = e^{-r_c t}\,. \tag{3.13.1}$$

  From this we derive

  $$1+\tfrac{1}{2}r_{\text{semi}} = e^{r_c/2}\,. \tag{3.13.2}$$

  Solve to obtain

  $$r_{\text{semi}} = 2(e^{r_c/2}-1)\,. \tag{3.13.3}$$

  Alternatively

  $$r_c = 2\ln(1+\tfrac{1}{2}r_{\text{semi}})\,. \tag{3.13.4}$$

- **The semi-annually compounded rate r_{semi} is higher than the continuously compounded rate r_c.** (Technically, only if the interest rates are positive, which is almost always the case.)
- In this course we shall employ only the continuously compounded spot rate r_c. It is much simpler for mathematical calculations.
- **Be careful to check the definition of the spot rate in any source you consult.**

3.14 Force of interest (not for examination)

- Consider that the discount factor for a time t is e^{-rt} (if $t = 0$ is today). We have continuously compounded spot rates: for $t_{0.5} = 0.5$ years the discounrt factor is $r^{-r_{0.5}t_{0.5}}$, for $t_{1.0} = 1.0$ years the discounrt factor is $r^{-r_{1.0}t_{1.0}}$, etc. The spot rates are all (usually) different, i.e. they depend on the time.

- It makes sense to define a function $\delta(t)$ such that the **accumulation function** (the inverse of the discount factor) is given by

$$a(t) = e^{\int_0^t \delta(\tau)\,d\tau} . \tag{3.14.1}$$

  Conversely, if we know $a(t)$, then

$$\delta(t) = \frac{1}{a(t)}\,\frac{da}{dt} . \tag{3.14.2}$$

- The functon $\delta(t)$ is called the **force of interest**. It is a continuously compounded interest rate. If the spot rate r were constant (the same at all times), then $\delta(t) = r$.

- We shall not make any real use of the force of interest in this course. Essentially just some terminology for you to know if/when you see it in books or web pages.

- We have a spot curve of spot rates, and from that we can derive forward rates, and those are the quantities we shall work with. In fact, for many examples of derivatives that we shall study in this course, there are more important concepts to learn and so we shall simply assume the interest rate is constant over the lifetime of the derivative.

3.15 Worked example 1

- Let us go out to 2 years. Suppose the yields of the par bonds are

$$
\begin{aligned}
y_{0.5} &= 5.0\,\% &&= 0.05\,, && (3.15.1a)\\
y_{1.0} &= 5.1\,\% &&= 0.051\,, && (3.15.1b)\\
y_{1.5} &= 5.2\,\% &&= 0.052\,, && (3.15.1c)\\
y_{2.0} &= 5.3\,\% &&= 0.053\,. && (3.15.1d)
\end{aligned}
$$

- Using eqs. (3.2.5.1) and eq. (3.2.5.2), the discount factors are

$$
\begin{aligned}
d_{0.5} &= \frac{1}{1+\frac{1}{2}y_{0.5}} &&= \frac{1}{1.025} &&\simeq 0.97561\,, && (3.15.2a)\\
d_{1.0} &= \frac{1-\frac{1}{2}y_{1.0}d_{0.5}}{1+\frac{1}{2}y_{1.0}} &&\simeq \frac{0.975122}{1.0255} &&\simeq 0.95088\,, && (3.15.2b)\\
d_{1.5} &= \frac{1-\frac{1}{2}y_{1.5}d_{0.5}-\frac{1}{2}y_{1.5}d_{1.0}}{1+\frac{1}{2}y_{1.5}} &&\simeq \frac{0.9499114}{1.026} &&\simeq 0.92584\,, && (3.15.2c)\\
d_{2.0} &= \frac{1-\frac{1}{2}y_{2.0}d_{0.5}-\frac{1}{2}y_{2.0}d_{1.0}-\frac{1}{2}y_{2.0}d_{1.5}}{1+\frac{1}{2}y_{2.0}} &&\simeq \frac{0.9244134}{1.0265} &&\simeq 0.90055\,. && (3.15.2d)
\end{aligned}
$$

- The associated spot rates are (in percent)

$$
\begin{aligned}
r_{0.5} &= -\frac{\ln(d_{0.5})}{0.5} &&\simeq 4.9385\,\%\,, && (3.15.3a)\\
r_{1.0} &= -\frac{\ln(d_{1.0})}{1.0} &&\simeq 5.0373\,\%\,, && (3.15.3b)\\
r_{1.5} &= -\frac{\ln(d_{1.5})}{1.5} &&\simeq 5.1370\,\%\,, && (3.15.3c)\\
r_{2.0} &= -\frac{\ln(d_{2.0})}{2.0} &&\simeq 5.2375\,\%\,. && (3.15.3d)
\end{aligned}
$$

- **If the par yield curve slopes upwards, the spot curve also slopes upwards.**
- **Interpolation of spot curve**
  1. Use $t = 1.6$.
  2. Fraction: $t_i = 1.5$, $t_j = 2.0$, $\lambda = (1.6 - 1.5)/(2.0 - 1.5) = 0.2$.
  3. Linear interpolation of spot rates:

$$
\begin{aligned}
r_{\text{lin}} &= 0.8\times 5.1370 + 0.2\times 5.2375 = 5.1571\%\,,\\
d_{\text{lin}} &= e^{-0.051571\times 1.6} \simeq 0.920799\,.
\end{aligned}
\qquad (3.15.4)
$$

  4. Constant forward rate:

$$
\begin{aligned}
r_{\text{cfr}} &= \frac{0.8\times 5.1370\times 1.5 + 0.2\times 5.2375\times 2.0}{1.6} = 5.1621\%\,,\\
d_{\text{cfr}} &= e^{-0.051621\times 1.6} \simeq 0.920725\,.
\end{aligned}
\qquad (3.15.5)
$$

3.16 Worked example 2

- Suppose the par yield curve is flat:

$$y_{0.5} = y_{1.0} = y_{1.5} = y_{2.0} = 5.0\,\% = 0.05\,. \tag{3.16.1}$$

- Using eqs. (3.2.5.1) and eq. (3.2.5.2), the discount factors are

$$d_{0.5} = \frac{1}{1+\frac{1}{2}y_{0.5}} = \frac{1}{1.025} \simeq 0.97561\,, \tag{3.16.2a}$$

$$d_{1.0} = \frac{1-\frac{1}{2}y_{1.0}d_{0.5}}{1+\frac{1}{2}y_{1.0}} \simeq \frac{0.9756098}{1.025} \simeq 0.95181\,, \tag{3.16.2b}$$

$$d_{1.5} = \frac{1-\frac{1}{2}y_{1.5}d_{0.5}-\frac{1}{2}y_{1.5}d_{1.0}}{1+\frac{1}{2}y_{1.5}} \simeq \frac{0.9518144}{1.025} \simeq 0.92860\,, \tag{3.16.2c}$$

$$d_{2.0} = \frac{1-\frac{1}{2}y_{2.0}d_{0.5}-\frac{1}{2}y_{2.0}d_{1.0}-\frac{1}{2}y_{2.0}d_{1.5}}{1+\frac{1}{2}y_{2.0}} \simeq \frac{0.9285994}{1.025} \simeq 0.90595\,. \tag{3.16.2d}$$

- The associated spot rates are all equal

$$r_{0.5} = -\frac{\ln(d_{0.5})}{0.5} \simeq 4.9385\,\%\,, \tag{3.16.3a}$$

$$r_{1.0} = -\frac{\ln(d_{1.0})}{1.0} \simeq 4.9385\,\%\,, \tag{3.16.3b}$$

$$r_{1.5} = -\frac{\ln(d_{1.5})}{1.5} \simeq 4.9385\,\%\,, \tag{3.16.3c}$$

$$r_{2.0} = -\frac{\ln(d_{2.0})}{2.0} \simeq 4.9385\,\%\,. \tag{3.16.3d}$$

- **If the par yield curve is flat, the spot rate curve is also flat.**

- **Interpolation of spot curve**

  1. Use $t = 1.6$.
  2. Fraction: $t_i = 1.5$, $t_j = 2.0$, $\lambda = (1.6 - 1.5)/(2.0 - 1.5) = 0.2$.
  3. Linear interpolation of spot rates:

$$\begin{aligned} r_{\rm lin} &= 0.8 \times 4.9385 + 0.2 \times 4.9385 = 4.9385\%\,, \\ d_{\rm lin} &= e^{-0.049385\times 1.6} \simeq 0.924025\,. \end{aligned} \tag{3.16.4}$$

  4. Constant forward rate:

$$\begin{aligned} r_{\rm cfr} &= \frac{0.8 \times 4.9385 \times 1.5 + 0.2 \times 4.9385 \times 2.0}{1.6} = 4.9385\%\,, \\ d_{\rm cfr} &= e^{-0.049385\times 1.6} \simeq 0.924025\,. \end{aligned} \tag{3.16.5}$$

3.17 Worked example 3

- Suppose the par yield curve slopes downwards (this sometimes happens).

$$y_{0.5} = 5.0\,\% \qquad =0.05\,, \tag{3.17.1a}$$
$$y_{1.0} = 4.9\,\% \qquad =0.049\,, \tag{3.17.1b}$$
$$y_{1.5} = 4.8\,\% \qquad =0.048\,, \tag{3.17.1c}$$
$$y_{2.0} = 4.7\,\% \qquad =0.047\,. \tag{3.17.1d}$$

- Using eqs. (3.2.5.1) and eq. (3.2.5.2), the discount factors are

$$d_{0.5} = \frac{1}{1+\frac{1}{2}y_{0.5}} \qquad = \frac{1}{1.025} \qquad \simeq 0.97561\,, \tag{3.17.2a}$$
$$d_{1.0} = \frac{1-\frac{1}{2}y_{1.0}d_{0.5}}{1+\frac{1}{2}y_{1.0}} \qquad \simeq \frac{0.9760976}{1.0245} \qquad \simeq 0.95276\,, \tag{3.17.2b}$$
$$d_{1.5} = \frac{1-\frac{1}{2}y_{1.5}d_{0.5}-\frac{1}{2}y_{1.5}d_{1.0}}{1+\frac{1}{2}y_{1.5}} \qquad \simeq \frac{0.9537192}{1.024} \qquad \simeq 0.93137\,, \tag{3.17.2c}$$
$$d_{2.0} = \frac{1-\frac{1}{2}y_{2.0}d_{0.5}-\frac{1}{2}y_{2.0}d_{1.0}-\frac{1}{2}y_{2.0}d_{1.5}}{1+\frac{1}{2}y_{2.0}} \qquad \simeq \frac{0.9327963}{1.0235} \qquad \simeq 0.91138\,. \tag{3.17.2d}$$

- The associated spot rates are (in percent)

$$r_{0.5} = -\frac{\ln(d_{0.5})}{0.5} \qquad \simeq 4.9385\,\%\,, \tag{3.17.3a}$$
$$r_{1.0} = -\frac{\ln(d_{1.0})}{1.0} \qquad \simeq 4.8397\,\%\,, \tag{3.17.3b}$$
$$r_{1.5} = -\frac{\ln(d_{1.5})}{1.5} \qquad \simeq 4.7402\,\%\,, \tag{3.17.3c}$$
$$r_{2.0} = -\frac{\ln(d_{2.0})}{2.0} \qquad \simeq 4.6398\,\%\,. \tag{3.17.3d}$$

- **If the par yield curve slopes downwards, the spot curve also slopes downwards.**
- **Interpolation of spot curve**
  1. Use $t = 1.6$.
  2. Fraction: $t_i = 1.5$, $t_j = 2.0$, $\lambda = (1.6 - 1.5)/(2.0 - 1.5) = 0.2$.
  3. Linear interpolation of spot rates:

$$\begin{aligned} r_{\text{lin}} &= 0.8 \times 4.7402 + 0.2 \times 4.6398 = 4.7121\%\,, \\ d_{\text{lin}} &= e^{-0.047121\times 1.6} \simeq 0.927378\,. \end{aligned} \tag{3.17.4}$$

  4. Constant forward rate:

$$\begin{aligned} r_{\text{cfr}} &= \frac{0.8 \times 4.7402 \times 1.5 + 0.2 \times 4.6398 \times 2.0}{1.6} = 4.7076\%\,, \\ d_{\text{cfr}} &= e^{-0.047076\times 1.6} \simeq 0.927445\,. \end{aligned} \tag{3.17.5}$$