Queens College, CUNY, Department of Computer Science

Computational Finance

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3 Lecture 3a

This material is important, hence put in a separate file.

3.7 Yield curve

3.7.1 General remarks

- This section is about how interest rates are determined in practice.
- In the financial markets, there are bonds with many different yields and maturities.
- You have probably seen (in the news) displays of the US Treasury yield curve, which is a graph of bond yields as a functon of maturity, typically out to 30 years.
- How do we determine interest rates in practice, from a yield curve.
- Consider a simplified model where we have only two bonds. Both are newly issued bonds, paying coupons semi-annually. The first has a maturity of 1 year and a yield y_1, and the second has a maturity of 5 years and a yield y_5, where $y_5 \neq y_1$.
- But this poses a problem: both bonds pay coupons at a date 6 months in the future, but for one coupon the discounting (present vale calculation) uses a yield of y_1 whereas for the other coupon the discounting uses a yield of $y_5 \neq y_1$. And yet both cashflows are paid on the same date. If we are told "there is a cashflow 6 months from today" how do we calculate its present value?
- If we wish to take out a loan with a term of two years (in the above interest rate environment), what is the appropriate interest rate to use to discount the (monthly?) loan payments?
- The subject of this section is to answer the above important question.

3.7.2 Bootstrap

- The procedure is called a **bootstrap**, as we shall see why as we work our way through it.
- **We use semi-annual coupons to avoid unnecessarily complicating the analysis.**
- First note that the "yield curve" is a graph of the yields of **par bonds** of different maturities. (For maturities ≥ 1 year.) A par bond is a bond whose price equals 100 (i.e. 100% of face).
- For a newly issued bond, if the price equals par, then the yield equals the coupon of the bond. Hence we do not need a separate input for the bond coupons: they equal the yields. This is why the market uses a par yield curve.
- **Although I wrote "a par bond is a bond whose price equals 100" above, to avoid needless "factor of 100" complications we set the price of a par bond to 1 and all the coupons to decimal fractions in the calculations below. Hence a coupon of 4% means $c = 0.04$, etc.**
- In practice, on any given date, there will not be bonds with maturities of exactly 6, 12, 16, 24 months, etc., and moreover even if they so exist, they will not in general trade at par. Hence every day, the yield curve is created by averaging the prices of real bonds trading in the market to calculate (or estimate) what would be the yield of the relevant bonds with maturities of exactly 6 months, 1 year, 1.5 years, etc.
- As a simple example, suppose we have four bonds, with maturities of 6, 12, 18 and 24 months. We measure time in years, so the maturities are 0.5, 1.0, 1.5 and 2.0 years. The respective yields (also called **par yields**) are $y_{0.5}$, $y_{1.0}$, $y_{1.5}$ and $y_{2.0}$. The first is a zero coupon bond, and the rest are par bonds.
- We proceed as follows. We begin with the shortest maturity and work our way to the longest. The bond with a 6 month maturity is a zero coupon bond. The discount factor of the cashflow of this bond is
$$d_{0.5} = \frac{1}{(1+\frac{1}{2}y_{0.5})^{2\times 0.5}} = \frac{1}{1+\frac{1}{2}y_{0.5}}\,. \qquad (3.7.2.1)$$
We equate this to the 6 month interest rate (continuously compounded) $r_{0.5}$ via
$$e^{-r_{0.5}\times 0.5} = d_{0.5}\,. \qquad (3.7.2.2)$$
We solve this equation using logarithms
$$r_{0.5} = -\frac{\ln(d_{0.5})}{0.5}\,. \qquad (3.7.2.3)$$
This determines the 6 month (continuously compounded) interest rate $r_{0.5}$.
- In general, for any time t, the continuously compounded interest rate r_t is related to the discount factor d_t via $e^{-r_t t} = d_t$. (We are assuming $t_0 = 0$ else r and d also depend on t_0.) Hence in general
$$r_t = -\frac{\ln(d_t)}{t}\,. \qquad (3.7.2.4)$$

- We now proceed to the 1 year bond. This has two cashflows. To avoid unnecessary conversions of percentages and decimals, say the face is 1 so the bond price is also 1. The coupon equals the yield (par bond), so the cashflows are $\frac{1}{2}y_1$ at 6 months and $1 + (\frac{1}{2}y_1)$ at 12 months. We now recall that the bond price is a sum of the present values of the cashflows. Hence we say the price is given by

$$\begin{aligned} 1 &= \mathrm{PV}(\tfrac{1}{2}y_{1.0}) + \mathrm{PV}(1 + (\tfrac{1}{2}y_{1.0})) \\ &= d_{0.5}(\tfrac{1}{2}y_{1.0}) + d_{1.0}(1 + (\tfrac{1}{2}y_{1.0})) \ . \end{aligned} \tag{3.7.2.5}$$

  In the last line we have performed two important steps:
  (i) we applied the known discount factor $d_{0.5}$ to discount the 6 month cashflow, and
  (ii) we introduced the one year discount factor $d_{1.0}$ to discount the 1 year cashflow.
  The only unknown quantity in the above equation is $d_{1.0}$.
  Hence we solve the above equation to determine $d_{1.0}$:

$$d_{1.0} = \frac{1 - d_{0.5}(y_{1.0}/2)}{1 + (y_{1.0}/2)} \,. \tag{3.7.2.6}$$

  From $d_{1.0}$ we can determine the 12 month continuously compounded interest rate $r_{1.0}$.

- The procedure should be clear now, also why it is called a bootstrap. We start from the shortest maturity and work our way to the longest. At each step we use the previously calculated discount factors to discount all the cashflows except the last one. For the last cashflow we introduce a new (unknown) interest rate and solve an equation to determine it.

- The next bond is the 18 month bond. There are three cashflows, of values $\frac{1}{2}y_{1.5}$ at 6 months, $\frac{1}{2}y_{1.5}$ at 12 months and $1 + (\frac{1}{2}y_{1.5})$ at 18 months. We calculate the present value of the 6 month cashflow using the discount factor $d_{0.5}$ and that of the 12 month cashflow using the discount factor $d_{1.0}$. For the present value of the 18 month cashflow we introduce a new discount factor $d_{1.5}$. Hence we obtain the equation

$$1 = d_{0.5}(\tfrac{1}{2}y_{1.5}) + d_{1.0}(\tfrac{1}{2}y_{1.5}) + d_{1.5}(1 + (\tfrac{1}{2}y_{1.5}))\,. \tag{3.7.2.7}$$

  We solve the above equation to determine $d_{1.5}$:

$$d_{1.5} = \frac{1 - d_{0.5}(y_{1.5}/2) - d_{1.0}(y_{1.5}/2)}{1 + (y_{1.5}/2)} \,. \tag{3.7.2.8}$$

  From $d_{1.5}$ we can determine the 18 month continuously compounded interest rate $r_{1.5}$.

- The procedure is repeated ("bootstrapped") for the 2 year bond, and so on till the end. We obtain the discount factor d_i by solving the equation (note that $i = 1.0, 1.5, 2.0, \dots$)

$$d_i = \frac{1 - (y_i/2)\sum_{j=0.5,1.0,\dots,(i-0.5)} d_j}{1 + (y_i/2)} \,. \tag{3.7.2.9}$$

3.7.3 Bootstrap: matrix formulation

- The bootstrap calculation can also be expressed as a matrix problem. The following matrix equation should be reasonably self-explanatory. The goal is to solve for the discount factors:

$$\begin{pmatrix} 1+\frac{1}{2}y_{0.5} & & & & \\ \frac{1}{2}y_{1.0} & 1+\frac{1}{2}y_{1.0} & & & \\ \frac{1}{2}y_{1.5} & \frac{1}{2}y_{1.5} & 1+\frac{1}{2}y_{1.5} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{2}y_n & \frac{1}{2}y_n & \dots & \dots & 1+\frac{1}{2}y_n \end{pmatrix} \begin{pmatrix} d_{0.5} \\ d_{1.0} \\ d_{1.5} \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{3.7.3.1}$$

- All the blank matrix entries in the upper right triangle (above the main diagonal) are zero.
- This is an example of a **lower triangular** matrix problem. All matrix elements above the main diagonal are zero.
- The bootstrap procedure described above is in fact the most efficient way to solve such a matrix problem.

3.7.4 Bootstrap: questions/problems

- **Question:** What about cashflows for dates less than 6 months from today?
  Answer: In practice there are Treasury Bills with maturities of less than one year. These are highly liquid financial instruments and help to set the short term interest rates. They are effectively zero coupon bonds because they only pay on maturity. Hence in practice the bootstrap of the yield curve does not really begin at 6 months.

- **Question:** That is not a complete answer. How to discount cashflows at 7 months, or 1.2345 years, etc., i.e. not exactly at 6, 12, 18, ... months?
  Answer: We must interpolate the spot interest rates. ***There is no universally agreed upon procedure how to do this.*** For example, some people may choose to interpolate the discount factors, whereas others choose to interpolate the spot rates. These are *not* equivalent calculations. Each company has its own proprietary algorithm how to interpolate interest rates.

- **Question:** The yield curve extends out to 30 years, but it does not have points every 6 months out to 30 years. There are yields for 5 and 10 years: how to obtain the par yields for maturities between 5 and 10 years?
  Answer: Bootstrapping a real yield curve is an example of an **undetermined problem**. There is not enough market data, hence is also necessary to interpolate the original yield curve to estimate the par yields every 6 months out to 30 years, to perform the bootstrap successfully. There is also no universally agreed upon procedure how to interpolate par yields.

- **Question:** The bootstrap procedure returns a set of discount factors. But the above procedure does not guarantee the output values will lie between 0 and 1. What happens if $d_t \le 0$ or $d_t > 1$ for some value of t?
  Answer: This is a very important detail. From the definition of the interest rate $e^{-r_t t} = d_t$, we that if $d_t \le 0$, the value of r_t is mathematically ill-defined. Also if $d_t > 1$, the value of r_t is negative. There is nothing we can do if the market produces a yield curve which leads to bad discount factors. Typically it signifies chaos or a crisis in the financial markets.

3.7.5 Spot curve & par curve

- When the bootstrap is completed, we will have a set of (continuously compounded) interest rates (or equivalently, discount factors) for all the maturities in the original yield curve.

- The interest rates $r_{0.5}$, $r_{1.0}$, etc. are called (continuously compounded) **spot rates** or **zero coupon rates** or simply **zero rates**. Note that $d_{0.5}$, $d_{1.0}$, $d_{1.5}$, etc., are simply the discount factors for zero coupon bonds with the respective maturities of 0.5, 1.0, 1.5 years, etc. The interest rates $r_{0.5}$, $r_{1.0}$, etc. are the corresponding (continuously compounded) interest rates for those zero coupon bonds.

- The original yield curve is composed of the yields of par bonds ("par yields"). It is usually called the **par yield curve** or simply the **par curve**.

- The spot rates can also be used to make a yield curve. It is called the **spot rate curve** or simply the **spot curve**.

- Real bonds which trade in the market typically pay coupons. Hence one deduces their yields (from their market prices). From this one can construct a par yield curve.

- The spot curve must be calculated mathematically from the par curve. The spot curve is effectively the yield curve of zero coupon bonds, but typically there are not enough zero coupon bonds to supply enough data to construct the spot curve directly from market data.

- Financial academics prefer to work with the spot curve because it is easier to employ in mathematical calculations.

- A spot rate directly tells us the discount factor (present value) of a cashflow at a particular point in time. A yield does not directly tell us the present value of a cashflow at a particular point in time, because it complicates the calculation with a set of other intermediate cashflows (coupons).

- When we analyze the pricing of financial securities such as derivatives, we shall employ spot rates. The mathematics is much easier.

3.7.6 Term structure of interest rates

- The **term structure of interest rates** is effectively the same concept as the yield curve.
- The spot rates (and par yields) are obviously not all equal, in general.
- The expression “term structure of interest rates” is a way of expressing the fact that the applicable interest rate for a cashflow depends on the time at which it is paid.

3.7.7 Spot rates and forward rates

- What is the effective interest rate for a financial transaction will last for one year, but it will begin 6 months from now, not today?
- More generally, the transaction will begin at time t_i and end at time t_f. What is the effective interest rate over that time interval?
- This is known as a **forward rate**.
- A forward rate is easy to calculate from spot rates. Discount \$1 to today in two ways:
(i) from t_f to t_0 (today), and
(ii) from t_f to t_i, then from t_i to t_0 (today).
Equate the two sets of discount factors
$$d(t_f, t_0) = d(t_f, t_i) d(t_i, t_0) \,. \tag{3.7.7.1}$$
The spot rates for times t_i and t_f are r_i and r_f, respectively, They are known, or can be interpolated from the spot curve. Let the unknown forward rate be r_{fi}. From the above equation we obtain
$$d(t_f, t_i) = \frac{d(t_f, t_0)}{d(t_i, t_0)} \,. \tag{3.7.7.2}$$
Expressing this in terms of interest rates, we obtain
$$e^{-r_{fi}(t_f - t_i)} = \frac{e^{-r_f(t_f - t_0)}}{e^{-r_i(t_i - t_0)}} \,. \tag{3.7.7.3}$$
This can be solved for the forward rate r_{fi} by taking logarithms
$$r_{fi} = \frac{r_f(t_f - t_0) - r_i(t_i - t_0)}{t_f - t_i} \,. \tag{3.7.7.4}$$
- This is our first example of an **no-arbitrage** calculation.
- The term "**arbitrage**" means a financial transaction which incurs no loss in any future scenario, and a positive probability of profit in some scenarios. Hence it is a "no lose" transaction, and will yield a positive profit in some cases. Essentially a riskless way to make a profit.
- The above value for the forward rate is the *only* value which is consistent with "no arbitrage" in the interest rate environment of today's spot rates. If the forward rate had any other value, the two paths of discounting would not have equal present value. Then formulate an arbitrage strategy as follows:
(i) sell the transaction which has the higher present value, and
(ii) use the money to buy the transaction which has the lower present value, and
(iii) invest the money left over in a bank.
There must be investors who will buy the more expensive transaction in (i), else the forward rate would not have a mismatched value. At time t_f, both transactions (i) and (ii) will be worth \$1 (by definition). Net them against each other and they will cancel to zero. Pocket the money in (iii) as profit.

3.7.8 Crackpot history

- The par yield curve contains a plot of par yields against bond maturities.
- However, the finance academics advocated, for theoretical calculations, the use of a graph of bond yields vs. duration.
- In practice, as one can imagine, the “duration based yield curve” never proved popular with traders in the financial markets. The maturity of a bond is a known parameter, from the terms of its issue. However, how to measure the duration of a bond? The duration changes with market conditions and is not an easily determined parameter.
- *However*, there *is* a special case where the duration of a bond is easy to determine. That is the case of zero coupon bonds. The (Macaulay) duration of a zero coupon bond equals its maturity. Hence a graph based on the yields of zero coupon bonds satisfies both the academics (who want duration) and market practitioners (who want maturity). This is the spot curve.

3.7.9 Semi-annual and continuous compounding of spot rates

- The spot rates as defined above are compounded continuously. The bond yields are compounded semi-annually. There is a slight disconnect here.
- Many textbooks (and webpages) about spot rates also employ semi-annual compounding for the spot rates.
- Let the semi-annually compounded spot rate be r_{sa} and let the continuously compounded spot rate be r_c. The relationship between them is

$$\frac{1}{(1+\frac{1}{2}r_{sa})^{2t}} = e^{-r_c t}\,. \tag{3.7.9.1}$$

From this we derive

$$1+\tfrac{1}{2}r_{sa} = e^{r_c/2}\,. \tag{3.7.9.2}$$

Solve to obtain

$$r_{sa} = 2(e^{r_c/2}-1)\,. \tag{3.7.9.3}$$

Alternatively

$$r_c = 2\ln(1+\tfrac{1}{2}r_{sa})\,. \tag{3.7.9.4}$$

- **The semi-annually compounded rate r_{sa} is higher than the continuously compounded rate r_c.** (Technically, only if the interest rates are positive, which is almost always the case.)
- In this course we shall employ only the continuously compounded spot rate r_c. It is much simpler for mathematical calculations.
- However, in practice, be careful to check the definition of the spot rate in any source you reference.
- ***What about quarterly compounding?*** **Don't ask.**

3.7.10 Force of interest

- Consider that the discount factor for a time t is e^{-rt} (if $t = 0$ is today). We have continuously compounded spot rates: for $t_{0.5} = 0.5$ years the discounrt factor is $r^{-r_{0.5}t_{0.5}}$, for $t_{1.0} = 1.0$ years the discounrt factor is $r^{-r_{1.0}t_{1.0}}$, etc. The spot rates are all (usually) different, i.e. they depend on the time.

- It makes sense to define a function $\delta(t)$ such that the **accumulation function** (the inverse of the discount factor) is given by

$$a(t) = e^{\int_0^t \delta(\tau)\,d\tau} . \tag{3.7.10.1}$$

  Conversely, if we know $a(t)$, then

$$\delta(t) = \frac{1}{a(t)} \frac{da}{dt} . \tag{3.7.10.2}$$

- The functon $\delta(t)$ is called the **force of interest**. It is a continuously compounded interest rate. If the spot rate r were constant (the same at all times), then $\delta(t) = r$.

- We shall not make any real use of the force of interest in this course. Essentially just some terminology for you to know if/when you see it in books or web pages.

- We have a spot curve of spot rates, and from that we can derive forward rates, and those are the quantities we shall work with. In fact, for many examples of derivatives that we shall study in this course, there are more important concepts to learn and so we shall simply assume the interest rate is constant over the lifetime of the derivative.