Queens College, CUNY, Department of Computer Science

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6 Lecture 6

6.1 Options: basic notions

- We now come to options. Options are probably the most famous of all financial derivatives. We shall study them in some detail in these lectures.

- One can create an option on almost anything.

  1. There are options on stocks and stock indices (we shall begin with these).
  2. Options on futures contracts.
  3. Options on currencies (FX options).
  4. There are options on interest rate instruments such as bonds.
  5. ***There are even options on options.*** These are called **compound options**. We shall also study them.

- However, before we do all that, the first question to answer is:
  What is an option?

- The word "option" indicates a choice, and that is at the heart of the definition. In the case of a forward or a futures contract, on the expiration date the owner of the forward or a futures contract is *obligated to buy the stock (or commodity, etc).* That is not the case with an option: the owner of an option (also known as the "holder" of the option) can ***choose*** whether he/she wants to buy the stock or not.

- There is a stock S, with a price S_0 today. We wish to buy this stock, not today but at a fixed time T in the future, and at a fixed price K (to be paid at the time T). This is a forward contract: at the time T we pay the price K and take delivery of the stock. In the case of an option, the situation is different.

  1. At the time T **we can decide if we wish to buy the stock or not.**
  2. Let the stock price at time T be S_T.
  3. The price K is called the **strike price** of the option.
  4. The time T is called the **expiration date** of the option.
  5. If $S_T < K$, we can decide not to buy the stock (we decide "not to exercise the option").
  6. If $S_T \geq K$, we can decide to buy the stock (we decide to "exercise the option").
  7. If we own a forward or futures contract, we have no choice: we have to pay and take delivery of the stock/commodity/etc. (or engage in cash settlement).
  8. If we own an option, we can ***choose*** what to do (hence the name "option").
     We are not forced to do anything.

6.2 Options: formal definition

- Hence this is the formal definition of an option on a stock: A **call option** is a financial instrument which gives the owner ***the right, but not the obligation***, to buy one share of a stock at a specified date in the future, at the strike price.

  1. Let us now go over some technical details. There are different types of options.
  2. What I have described above is a **call option**.
  3. There are also **put options**. On the expiration date, the holder of a put option can decide to **sell** one share of stock, at the price K. If $S_T \leq K$, the holder can decide to exercise the put option. If $S_T > K$, the holder can decide not to exercise the put option. This is the opposite of a call option.
  4. A person who buys an option is called the **holder** of the option.
     An option holder has a **long position** in the option.
  5. A person who sells an option is called the **writer** of the option.
     An option writer has a **short position** in the option.
  6. Some types of options allow the holder to exercise the option **at any time prior to expiration,** not only on the expiration date.
  7. A **European option** allows the holder to exercise **only on the expiration date.**
  8. A **American option** allows the holder to exercise **at any time on or before the expiration date.**

- Hence in total there are four major kinds of options on a stock:

  1. European call option.
  2. European put option.
  3. American call option.
  4. American put option.

- Note that in the context of options, the names "American" and "European" have no connection to geography. They simply describe different types of business contracts. Both American and European options are traded in the USA, also in Europe and other countries.

- Nevertheless, geography has led to the invention of many other names. A **Bermudan option** can be exercised at selected dates prior to and including the expiration date. There are also **Asian options** and other variants. In the case of an Asian option, instead of using the terminal stock price S_T, we use the *average stock price over the lifetime of the option.* One can also say "the average stock price over the ten days prior to and including expiration" and other variations. These are all classified as **exotic options**.

6.3 Options: holders have rights, writers have obligations

- It is important to emphasize that **only the holder of an option has rights.**
- **The writer of an option has obligations but no rights.**
  1. By the above we mean that, on the option expiration date, **only the holder** can decide whether or not to exercise the option. (The type of option does not matter: American/European, call/put. All that matters is that only the holder can make the decision to exercise the option.)
  2. The ***writer*** of an option is **obligated to accept the decision of the option holder.** Clearly, the option holder will exercise when it to the holder's advantage. This means the option writer will suffer a loss if the option holder exercises the option.
- Let us see how this works out for the option holder and writer, first for a call, then a put.

6.4 Terminal payoff: call option

The terminal payoff for a **long position in a call option** is shown in Fig. 1. The payoff function is the same for both American and European call options. If the stock price is S_T at time T, and $S_T \geq K$, we pay the strike price K and we sell the stock immediately and make a profit $S_T - K$. If $S_T < K$, we do not exercise the option and it expires worthless. The payoff is zero. Hence the terminal payoff for a long position in a call option is

$$\text{Payoff call option} = \max(S_T - K, 0)\,. \tag{6.4.1}$$

Notice that the payoff function has zero downside and unlimited upside.

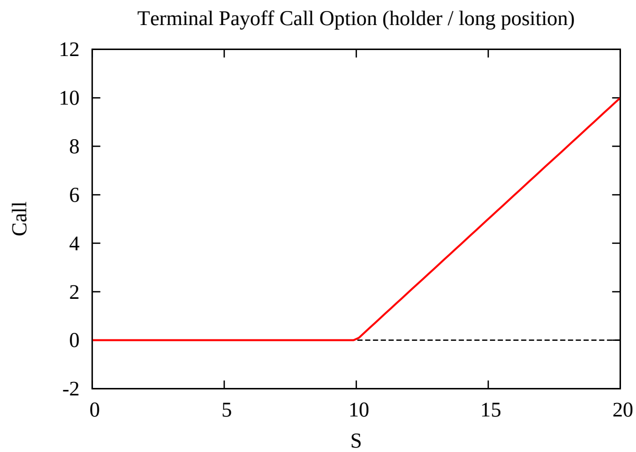

Figure 1: Graph of the payoff for a **long position** in a call option on a stock S, for a strike $K = 10$.

The terminal payoff for a **short position in a call option** is shown in Fig. 2. It is simply the negative of that in Fig. 1. Consequently, Fig. 2 shows that the writer of a call option has **zero upside** and **unlimited downside.** This fact is very important. The writer of a call option carries a lot of risk of loss.

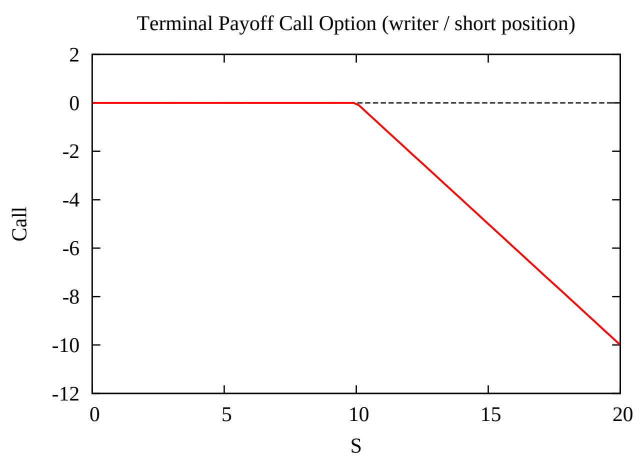

Figure 2: Graph of the payoff for a **short position** in a call option on a stock S, for a strike $K = 10$.

6.5 Terminal payoff of put option

The terminal payoff of a put option is shown in Fig. 3. The payoff function is the same for both American and European put options. For a put, the decision making process is reversed: we wish to *sell* a share of stock, at the strike price K. Hence if $S_T \leq K$, we exercise the put option. We receive cash K, which we can use to buy the stock back at the lower price S_T, yielding a net profit of $K - S_T$. If $S_T > K$, we do not exercise the option and it expires worthless. The payoff is zero. Hence the terminal payoff of a put option is

$$\text{Payoff put option} = \max(K - S_T, 0)\,. \tag{6.5.1}$$

Unlike a call, the put payoff function has zero downside but it also has **limited upside.**

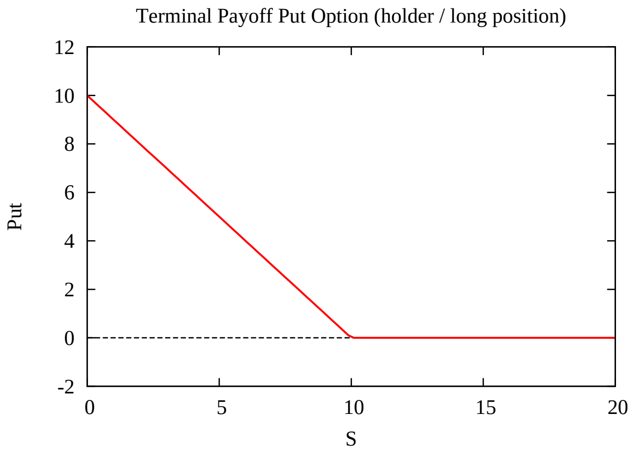

Figure 3: Graph of the payoff for a **long position** in a put option on a stock S, for a strike $K = 10$.

The terminal payoff for a **short position in a put option** is shown in Fig. 4. It is simply the negative of that in Fig. 3. Unlike a call, Fig. 4 shows that the writer of a put option has zero upside and **limited downside.**

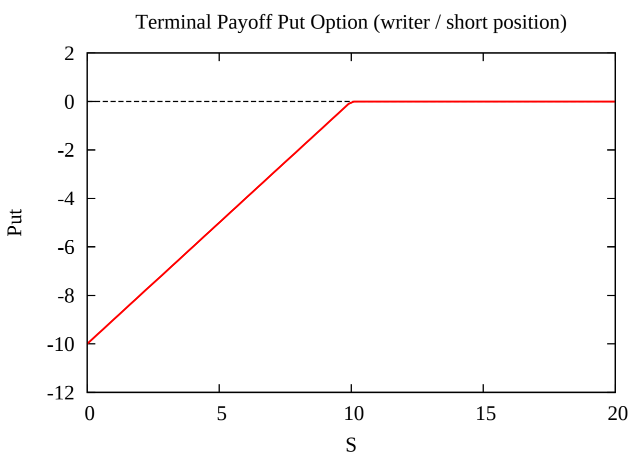

Figure 4: Graph of the payoff for a **short position** in a put option on a stock S, for a strike $K = 10$.

6.6 Options: basic notions (continued)

- Options introduce a new concept into these lectures: **probability.**
- We do not know, today, if we will exercise the option or not, on the expiration date.
- Let the stock price be S and the current time be t, where $t \leq T$.
  1. A call option is **in the money** if $S > K$.
     A put option is in the money if $S < K$.
  2. A call option is **out of the money** if $S < K$.
     A put option is out of the money if $S > K$.
  3. Both a call and put option are **at the money** if $S = K$.
  4. Both a call and put option are **at the money forward** if $Se^{r(T-t)} = K$. Here r is the interest rate, and the above formula is only for a stock which does not pay dividends.
- In the case of a forward contract with a forward price F, we could say that the present value of F is $Fe^{-r(T-t)}$, and to avoid arbitrage this must equal S. Hence $S = Fe^{-r(T-t)}$ or $F = e^{r(T-t)}S$. (To keep things simple, I assumed the stock pays no dividends in the time interval from t to T.) We were able to formulate a simple no-arbitrage argument because we know, in all scenarios at the time T, that the price we must pay is F.
- Let us consider a European option, because the analysis is simpler. A European option also has a payoff only on the expiration date. However, unlike a forward contract, an option has a nonzero payoff ***only if the option is in the money on its expiration date.***
- **We do not know if the option will be in the money on its expiration date.**
  How shall we determine the option's fair value?

6.7 Chasing the market

- Would the holder of a call option exercise an option if it expired out of the money?
- The formal idealized theory says no.
- But in real like the answer could be yes.
- Suppose an investor wishes to buy a large number of shares of the stock of a company.
- If the investor placed a large order to buy, in real life it would not be possible to fill the order instantly. Then the news would spread in the market and the stock price would go up. The investors action (of placing a large order to buy) would drive up the stock price, and cause the investor to pay more.
- This is called **chasing the market.**
- The investor's own actions drive up the stock price that the investor wants to buy.
- However, if the investor bought options instead, the stock proce would not go up in response. The option expiration date is in the future, the options could expire out of the money, and no one knows what the investor will do.
- If the stock price on the expiration date were out of the money, but only slightly less than the strike price, the investor might choose to exercise anyway.
- If this were to happen, the stock price would go up, ***but the investor only pays the strike price, not the market price of the stock***. It is the option writer's problem to deliver the stock, and absorb the consequences of the increase in the stock price. Remember that the option writer has obligations but no rights. Hence the investor avoids chasing the market. This is a possible trading strategy.
- In our idealized theory, chasing the market is impossible. We assume that there is an infinite supply of shares of stock available, and we assume that all trades are performed instantly (regardless of the size of the trade) and that all orders to buy/sell stock (regardless of the size of the order) do not change the stock price.
- Hence in our idealized theory, if an option expires out of the money, it is not exercised. This may or may not be true in real life.

6.8 Options: covered call

- There is a trading strategy known as a **covered call**.
- Recall that the writer of an option has obligations but no rights, and is therefore exposed to risk.
- The writer of a call option has unlimited downside risk, as we can see from Fig. 2. There is no upper bound as to how high the stock price can increase, therefore there is no maximum bound on how much the option writer can lose.
- In a **covered call**, the writer of a call option already owns the stock. Alternatively, the option writer buys the underlying stock at the same time as selling the call option. In this way, if the option is exercised, the option writer can deliver the underlying stock to the option holder.
- Sometimes, an investor who owns stock tries to earn extra income by writing options on some of the shares of that stock. Here is the reasoning behind the plan.
    1. The investor owns some shares of stock and thinks that the stock price will go down.
    2. Hence the investor writes some call options because the investor thinks *the options will expire out of the money*, hence the call options will not be exercised.
    3. The investor earns some income by selling the call options, and hopes that there will be no loss because the options will not be exercised.
    4. Of course, if the stock price goes up and the options expire in the money, the options will be exercised. In that case, the investor will deliver shares for the number of options the investor wrote. Hence the investor will experience some loss, but not unlimited loss (because the investor already owns the shares).
- Hence writing covered calls is a way to earn some extra income on shares of stock that one already owns.
- Investors who buy and sell options on an exchange are not required to cover any call options they write. The options clearing house will make sure that all parties honor their debt obligations.
- However, for options which are sold privately, not on an exchange, there are laws which say that a person who wishes to sell call options must write covered calls, i.e. the option writer must own the stock. This is to guarantee that if the options are exercised, the option writer can deliver the stock.

6.9 Options: covered put

- There is also a **covered put.**

1. In a covered put, an investor sells a put option and also shorts the underlying stock.
2. In this case, the option writer thinks that the stock price will go down, and so the investor will make a profit by writing a put. If the stock price goes down, the put is exercised, the option writer pays the strike price K and takes delivery of the stock (and by definition the stock price is less than K). The stock is used to close out the investors short sale of the stock, yielding a net profit.
3. However, if the stock price goes up, the put will not be exercised. The investor has to cover his short sale of the stock by buying back the stock at a high price (higher than the strike price K), and may experience unlimited loss.

6.10 Options: married put and protective put

- There is also a trading strategy called a **married put** (see also **protective put** below).

  1. In a **married put**, an investor buys a put option and also buys the underlying stock.
  2. In this case, if the stock price goes up, the investor benefits from the increase in the stock price. The put option expires worthless.
  3. If the stock price goes down below the strike price K, the investor exercises the put and sells the stock (which the investor already owns). This reduces the loss to the investor due to the decline in the stock price.

- A **protective put** is the same as a married put except that the investor already owns the stock before the investor buys the put option.

- In a married put, the investor buys the stock and the put option at the same time.

6.11 Options: formal pricing formula

- Before we proceed any further, there is one important lesson you must always remember:

Never confuse a business product with a mathematical pricing formula.

- An option is a **business product.** Note ***very carefully*** that the definition of an option does ***not*** depend on any model of random walks or probability distribution for the stock price.
- What we are trying to do now is to find a mathematical formula for the fair value of an option. However, that mathematical formula is based on a theoretical model. Real stock prices may or may not agree with that model. Nobody knows the actual probability model that real stock prices follow.
- ***The theoretical models are approximations.***
- Although we do not know exactly what the terminal stock price S_T will be at time T, we can argue that, as the stock price evolves (in a random walk, etc.), then on the option expiration date, the best estimate of the cashflow we will receive is the average value of the terminal payoff of the option. The mathematically rigorous term for "average" is **expectation value**. The expectation value is calculated using a theoretical model.
- The fair value of a European option today is given by the
present value of the expectation value of the terminal payoff of the option.
We say "European option" because an American option could be exercised prior to expiration. It might not live to the expiration date. We write the fair value formula symbolically as

$$(\text{fair value of European option}) = \text{PV}\big(\mathbb{E}[\text{terminal payoff of option}]\big)\,. \tag{6.11.1}$$

The symbol "$\mathbb{E}$" denotes an expectation value.

- Let the interest rate be r over the interval from the current time t to the expiration date T. Then the discount factor is $e^{-r(T-t)}$. Let the fair value of a European call option be c, and p for a put. Then, using eqs. (6.4.1) and (6.5.1), we obtain the following expressions. They may look frightening but do not be afraid of them:

$$c(S,t) = e^{-r(T-t)}\mathbb{E}[\max(S_T - K, 0)]\,, \tag{6.11.2a}$$

$$p(S,t) = e^{-r(T-t)}\mathbb{E}[\max(K - S_T, 0)]\,. \tag{6.11.2b}$$

1. We write "$c(S,t)$" and $p(S,t)$" because S and t are the two most important variables, but the call and put fair values depend on other parameters as well (such as r).
2. As they stand, eqs. (6.11.2a) and (6.11.2b) are really just symbolic (to us, anyway). Nevertheless, they embody two important concepts.
   (a) First, if we have multiple cashflows, we must always calculate their present values to sum them meaningfully.
   (b) Second, when the cashflows depend on probability, we calculate the average, or expectation value, of the cashflows.

3. Both eqs. (6.11.2a) and (6.11.2b) disguise complicated mathematical details that we shall not discuss in these lectures. The "expectation value" in eqs. (6.11.2a) and (6.11.2b) is actually not well-defined as written above. The correct statement is the **expectation value under the martingale measure**, say Q. To be rigorous, we must write

$$c(S,t) = e^{-r(T-t)}\mathbb{E}^Q[\max(S_T - K, 0)]\,, \tag{6.11.3a}$$

$$p(S,t) = e^{-r(T-t)}\mathbb{E}^Q[\max(K - S_T, 0)]\,. \tag{6.11.3b}$$

4. This looks terrible, and it is. This requires complicated probability theory, useless to us, which we shall not discuss.

5. Another fact to note is that the expectation values in eqs. (6.11.2a) and (6.11.2b) depend on S and t. Suppose the time to expiration $T - t$ is very small. If the option is deeply in the money, it is reasonable to guess the option will be exercised. Conversely, if the option is deeply out of the money, it is reasonable to guess the option will expire worthless. The "expectation value under martingale measure" depends on both S and t.

- The formulas in eqs. (6.11.3a) and (6.11.3b) are formal expressions. To proceed further with eqs. (6.11.3a) and (6.11.3b) we require a theoretical model for the probability distribution for the evolution of the stock price. We need to do that now.

- The fair values of American options cannot be expressed using eqs. (6.11.2a) and (6.11.2b) (or eqs. (6.11.3a) and (6.11.3b)). Because American options can be exercised at any time, their cashflows are not expressible as a simple expectation (and present value).

6.12 Stock prices: random paths

- In the mathematical theory of options pricing, it is typical to model the movement of stock prices as random walks. There are many different models of random walks, but the one which is the most commonly used in the financial markets is called **geometric Brownian motion**.

- First we discuss **standard Brownian motion (SBM).** It is the most popular (or widely used) model for random walks. This is the model used in physics, for example, to analyze the movement of molecules in a gas. It is an extensively studied theoretical model.

- Let us consider a real-valued random variable $x(t)$, which is a function of time t. The initial condition is $x(0) = 0$ at $t = 0$. For $t > 0$ the motion of x is a random walk, which is a continuous but zig-zag path. We shall skip the mathematical probability theory to make the concept of "zig-zag" rigorous. What we need to know is, if the motion of x exhibits a standard Brownian motion (SBM), then for $t > 0$, the expectation value of $x(t)$ is zero and the expectation value of $x^2(t)$ equals t:

$$\mathbb{E}[x(t)] = 0\,, \qquad \mathbb{E}[x^2(t)] = t\,. \tag{6.12.1}$$

  These expectation values are calculated using a martingale measure (not important to us).

- The expressions in eq. (6.12.1) state that, as the random walk moves forward in time, the mean $\langle x(t)\rangle$ equals zero, which is obvious, but the variance $\langle x^2(t)\rangle$ increases proportionally to t. The random walk "spreads out" and the standard deviation is $\sigma_x(t) = \sqrt{t}$.

- A sample of 1000 particles obeying standard Brownian motion was tracked, for $0 \le t \le 10$. All the paths started with $x = 0$ at $t = 0$. The paths of the first three particles are displayed in Fig. 5 ("zig-zag" lines). In addition, the mean $\langle x(t)\rangle$ is also plotted and marked in the figure (dashed curve). It is almost zero, for all t. The standard deviation $\sigma_x(t)$ is also plotted and marked in the figure (red solid curve). It is almost a parabola $\sigma_x(t) = \sqrt{t}$, for all t.

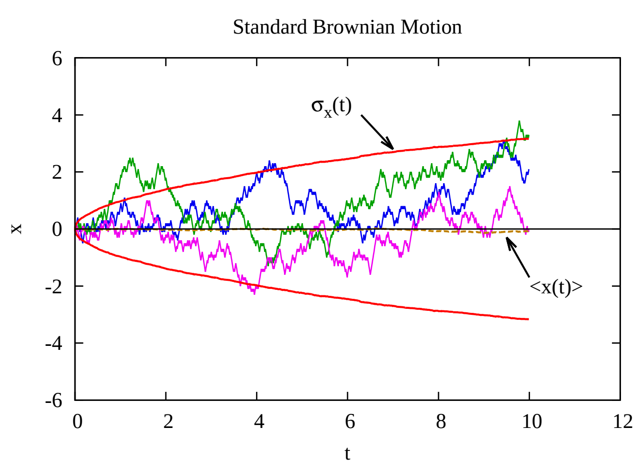

Figure 5: Plots of three random walks of Standard Brownian Motion (SBM). The mean $\langle x(t)\rangle$ and standard deviation $\sigma_x(t)$ for a sample of 1000 particles are also plotted.

- However, standard Brownian motion takes negative as well as positive values, whereas stock prices are always positive (or zero if the company goes bankrupt).

- To overcome this difficulty, we model the motion of stock prices as random walks obeying **geometric Brownian motion (GBM)**.

- Define $x = \ln(S)$, where S is the stock price. The $0 < S < \infty$ and $-\infty < x < \infty$, because $\ln(S) \to -\infty$ as $S \to 0$. Geometric Brownian motion (GBM) says that the **logarithm of the stock price** obeys standard Brownian motion. This is the random walk model that is employed in the financial markets.

- ***Do real stock prices actually obey geometric Brownian motion (GBM)?***

- **There are numerous studies of stock prices which say that they do *not*.**

- Hence why do we use geometric Brownian motion to model the movements of stock prices? The answer is simple. Brownian motion (standard or geometric) is mathematically simpler to treat than other models of random walks. It has been extensively studied in mathematics and theoretical physics. Many fundamental results are known about Brownian motion. There are many theorems about it, and many numerical algorithms to process it.

- Hence the financial industry uses geometric Brownian motion because it can take advantage of the extensive knowledge that already exists about Brownian motion (including numerical algorithms). It is not because real stock prices actually obey geometric Brownian motion.

- ***Never confuse a business product with a mathematical model.***

6.13 Stock prices: volatility

- Suppose that the current time is t_0 and the current stock price is S_0. For simplicity, say the interest rate is zero. What will happen at a future time $t_1 > t_0$? (Also for simplicity, the stock pays no dividends in the time interval from t_0 to t_1.)

  1. We do not know exactly what the stock price will be at the time t_1.
  2. The stock price will exhibit a random walk, from t_0 to t_1.
  3. ***We model the evolution of the stock price using geometric Brownian motion.***
  4. Note that we are making a theoretical model, as I have explained previously.

- We introduce a new parameter called **volatility**. Volatility is basically a measure of the standard deviation of the percentage changes in the stock price movements.

- Volatility is conventionally denoted by the Greek letter **$\boldsymbol{\sigma}$ (sigma)**.

- To make things more precise, if the stock makes a maximum of a one standard deviation move (up or down), then the stock price S_1 at time t_1 will lie in the interval

$$S_0 e^{-\sigma\sqrt{t_1-t_0}} \quad \le \quad S_1 \quad \le \quad S_0 e^{\sigma\sqrt{t_1-t_0}}\,. \tag{6.13.1}$$

  Alternatively, we can take logarithms

$$-\sigma\sqrt{t_1-t_0} \quad \le \quad \ln\Bigl(\frac{S_1}{S_0}\Bigr) \quad \le \quad \sigma\sqrt{t_1-t_0}\,. \tag{6.13.2}$$

- Note the following:

  1. **Volatility is non-directional.**
  2. If the volatility is large, the stock price can go up a lot (in the time interval $t_1 - t_0$), but it can also go down a lot.
  3. ***The probability that the stock price will go down by a factor of 2 (for example) is equal to the probability that the stock price will go up by a factor of 2.***
  4. It is surprising how many people forget this fundamental property of the volatility. They confuse the contribution of the volatility with the compounding from the interest rate.
  5. The compounding due to the interest rate is directional. Let the interest rate be r. If the volatility were zero (no random walk), the stock price at time t_1 would be

$$S_1 = S_0 e^{r(t_1-t_0)} \qquad\qquad \text{(zero volatility)}\,. \tag{6.13.3}$$

  6. Combine the two contributions of interest rates and volatility and eq. (6.13.1) changes to

$$S_0 e^{r(t_1-t_0)-\sigma\sqrt{t_1-t_0}} \quad \le \quad S_1 \quad \le \quad S_0 e^{r(t_1-t_0)+\sigma\sqrt{t_1-t_0}}\,. \tag{6.13.4}$$

6.14 Black-Scholes equation

- Consider a financial derivative on a stock S. Let the current time be t. Denote fair value of the derivative by $V(S,t)$.

- The value of V will also depend on other parameters, such as the interest rate r, etc., but the values of all such parameters are constants, so we do not list them explicitly.

- The stock is called the **underlying asset**, or simply the **underlying**.

- A derivative is also frequently called a **contingent claim**.
  The word "contingent" denotes that the payoff depends on some future event which might or might not happen.

- ***We now suppose that the stock price follows geometric Brownian motion.***
  Remember as always that this is an assumption, a theoretical model.

- For simplicity, suppose the stock does not pay dividends. We shall treat dividends later.

- Then, if the stock price S follows geometric Brownian motion (with volatility σ, a constant) and the stock pays no dividends and the interest rate is r (also a constant), the fair value of a derivative $V(S,t)$ which depends on the stock price S is given by the following partial differential equation (pde):

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\,\frac{\partial V}{\partial S} - rV = 0\,. \tag{6.14.1}$$

- This is the **Black-Scholes equation**, named after Fischer Black and Myron Scholes, and published in 1973. It led to a revolution in the theory of financial options pricing, and by extension to many other derivatives.

- The Black-Scholes equation was originally derived to price options on a stock (which does not pay dividends). However, it is applicable in general to ***any financial derivative*** which is based on a stock.

- It is a partial differential equation in two variables S and t.

- *There is no escaping partial differential equations if we wish to study options.*

- The Black-Scholes equation has also been adapted to price derivatives on other underlying assets, for example currency options. If time permits, we shall study some of these variants.

- Note that I did *not* derive eq. (6.14.1). I simply presented it as "the equation" for derivatives on a stock. Many textbooks derive the Black-Scholes equation, but the derivation requires stochastic calculus or at least a more detailed mathematical theory of Brownian motion.

6.15 Black-Scholes equation: validation checks

- Now that we have the Black-Scholes equation eq. (6.14.1), what shall we do with it?
- *We validate it!*
- Recall that our two fundamental building blocks are stocks and bonds. The stock is like the "identity function" in mathematics $f(x) = x$. It is the simplest derivative of all: it is an "identity derivative" $V(S,t) = S$.
- Hence $V(S,t) = S$ must satisfy eq. (6.14.1). This is our first validation check. Putting $V = S$ yields

$$\frac{\partial V}{\partial t} = 0\,, \qquad \frac{\partial V}{\partial S} = 1\,, \qquad \frac{\partial^2 V}{\partial S^2} = 0\,. \tag{6.15.1}$$

  Substituting into eq. (6.14.1) yields

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + rS\,\frac{\partial V}{\partial S} - rV = 0 + 0 + rS - rS = 0\,. \tag{6.15.2}$$

  Hence eq. (6.14.1) is satisfied and $V(S,t) = S$ is a valid solution.

- A bond compounds interest $B(t) = B_0 e^{rt}$ so setting $V(S,t) = B(t) = B_0 e^{rt}$ yields

$$\frac{\partial V}{\partial t} = rB_0 e^{r(T-t)} = rB\,, \qquad \frac{\partial V}{\partial S} = 0\,, \qquad \frac{\partial^2 V}{\partial S^2} = 0\,. \tag{6.15.3}$$

  Substituting into eq. (6.14.1) yields

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + rS\,\frac{\partial V}{\partial S} - rV = rB + 0 + 0 - rB = 0\,. \tag{6.15.4}$$

  Hence eq. (6.14.1) is satisfied and $V(S,t) = B(t)$ is a valid solution.

6.16 Black-Scholes-Merton equation

- Also in 1973, Robert Merton published a paper with the title "Theory of Rational Option Pricing" which has become a very influential and highly cited paper. In this paper, Merton published many important theorems about the theory of option pricing.

- One of Merton's achievements in his 1973 paper was to extend the equation derived by Black and Scholes to handle a stock paying continuous dividends at a rate q. Recall the formula for the forward stock price changes to $F = e^{(r-q)(T-t)}S$.

- The **Black-Scholes-Merton equation** generalizes eq. (6.14.1) to include continuous dividends, and is
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0\,. \tag{6.16.1}$$
This is the equation that most people actually use in the financial industry.

- Robert Merton and Myron Scholes shared the 1997 Nobel Memorial Prize in Economic Sciences for their work in option pricing theory. (Fischer Black died in 1995.)

- As always, perform a validation check. This time, we set $V(S,t) = Se^{qt}$. Then
$$\frac{\partial V}{\partial t} = qSe^{qt}\,, \qquad \frac{\partial V}{\partial S} = e^{qt}\,, \qquad \frac{\partial^2 V}{\partial S^2} = 0\,. \tag{6.16.2}$$
Substituting into eq. (6.14.1) yields
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = qSe^{qt} + 0 + (r-q)Se^{qt} - rSe^{qt} = 0\,. \tag{6.16.3}$$
Hence eq. (6.16.1) is satisfied and $V(S,t) = Se^{qt}$ is a valid solution.

- We shall employ eqs. (6.14.1) and (6.16.1) to calculate the fair values of European put and call options.

- Both eqs. (6.14.1) and (6.16.1) can be used to calculate the fair values of American options also. Some additional details are required, to handle the "early exercise" feature (the fact that American options can be exercised at any time, not just at expiration).

- In his 1973 paper, Merton showed that the price of an American call option on a stock which pays no dividends (during the lifetime of the option) is the same as the price of a European call option on the same stock. (This is not the case if the stock pays dividends.)

- Merton also showed that the price of an American put option on a stock is not equal to the price of a European put option on the same stock, even if the stock pays no dividends.

6.17 Risk-free interest rate

- Once again, I have been casual about what I mean by the "interest rate" r.
- If the derivative is traded on an exchange (an **exchange-listed derivative**), then the clearing house will enforce that both sides of the trade will honor their obligations. In that case, the interest rate r will be the risk-free interest rate.
- Most options are listed on exchanges. Then we use the risk-free rate $r_{\rm rf}$ in the Black-Scholes equation eq. (6.14.1) and the Black-Scholes-Merton equation eq. (6.16.1). This is the standard assumption made in option pricing theory.
- However, there are also OTC (over the counter) options, which are private trades not conducted on an options exchange. For OTC options, the value of the interest rate depends on ther creditworthiness of the counterparties involved. The same option will be valued using different values for the interest rate, for different counterparties. It can be difficult to determine a suitable value for the interest rate r.
- In most of our lectures, we shall use the risk-free interest rate to value options and other derivatives.