Queens College, CUNY, Department of Computer Science

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7 Lecture 7c

Rational option pricing: Worked examples

- This lecture contains worked examples of arbitrage strategies for **cash settled** options.
  1. Options on stock indices are settled in cash.
  2. There is no physical delivery of the underlying asset.
  3. The option "strike" is measured in index points.
  4. To calculate the cash payment, there is a dollar multiplier M for every index point that the option is in the money.
- For a **cash settled call option** (both American and European), if the option is exercised, the option writer pays the holder cash of $M(S - K)$.
- For a **cash settled put option** (both American and European), if the option is exercised, the option writer pays the holder cash of $M(K - S)$.
- **Note: The analysis for a cash settled American call is complicated.**
- **This contents of this lecture are difficult, hence this lecture is *optional*.**

7.1 Arbitrage 1: cash settled European call

- Suppose the value of a stock index is S_0 (in index points) at time t_0.
- The dividends can usually be approximated by a continuous dividend rate q.
- We assume that q is a constant below.
- The interest rate is $r > 0$ (a constant).
- The market price of a European call is $c_{\text{Eur}} = M(S_0 + 1.5)$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **Solution for cash settled European call**
  1. At time t_0, we **sell** the call option and receive cash $= M(S_0 + 1.5)$.
  2. We save all the money in a bank.
  3. There is no underlying asset we can buy to cover the call.
  4. However, we can go **long a forward contract on the stock index, with the same expiration time as the option.**
  5. (We can also use a futures contract.)
  6. The forward price is
  $$F = S_0 e^{(r-q)(T-t_0)}\,. \tag{7.1.1}$$
  7. The forward contract costs nothing to buy at the time t_0.
  8. The forward contract is also cash settled, with the same dollar multiplier M.
- At the time t_0, our portfolio is: short one European call option, long one forwar contract, cash in bank. The total value of our portfolio is zero.
- At the expiration time T, the money in the bank compounds to $M(S_0 + 1.5)e^{r(T-t_0)}$.
- We are also long the forward contract and it has expired.
- The forward contract is cash settled and we must deal with that.

$$\text{cash settlement of forward contract} = M(S_T - F)\,. \tag{7.1.2}$$

- Note that this could be a negative amount, if $S_T < F$.
- We receive cash if $S_T > F$. We pay cash if $S_T < F$. Nothing happens if $S_T = F$.

- The sum of our cash in the bank plus the cash settlement of the forward contract is:

$$\begin{aligned}\text{cash available at expiration} &= M(S_T - F) + M(S_0 + 1.5)e^{r(T-t_0)} \\ &= M\big[\,S_T - S_0 e^{(r-q)(T-t_0)} + S_0 e^{r(T-t_0)} + 1.5e^{r(T-t_0)}\,\big] \\ &= M\big[\,S_T + S_0 e^{r(T-t_0)}(1 - e^{-q(T-t_0)}) + 1.5e^{r(T-t_0)}\,\big]\,.\end{aligned} \tag{7.1.3}$$

1. Next we must cash settle the option.
2. **Important:**
   Why do we cash settle the forward first and the option second?
   What happens if we cash settle the option first?
3. Answer: In the ideal theory all trades take place instantly, so both settlements take place at the same time.
4. In real life, all the trades take place on the same day, so it is not a problem.

- **We do not know the final value of the stock index S_T at the expiration time T.**
- Hence we must analyze all cases, i.e. all values of S_T.
- We must prove that in all cases, the profit is positive (or zero),
  but ***never negative (= loss).***
- There are two cases (i) $S_T \geq K$ or (ii) $S_T < K$.

7.1.1 Case $S_T \geq K$

- In this situation the option holder exercises the call option.
- The writer (= us) pays the option holder cash in the amount of $M(S_T - K)$.
- ***Do we have enough money to do this?***
- Yes, because the amount of cash on the right hand side of eq. (7.1.3) is **more that MS_T.**
- Then because $S_T > K$, the amount of cash available to us is greater than $M(S_T - K)$ to settle the call option.
- Our total cash at the end is

$$\begin{aligned}\text{total cash} &= M\big[\,S_T + S_0 e^{r(T-t_0)}(1 - e^{-q(T-t_0)}) + 1.5e^{r(T-t_0)}\,\big] - M(S_T - K) \\ &= M\big[\,K + S_0 e^{r(T-t_0)}(1 - e^{-q(T-t_0)}) + 1.5e^{r(T-t_0)}\,\big]\,.\end{aligned} \tag{7.1.4}$$

- This is a positive number.
  We started with zero and we have a positive amount of money at the end.
- *The cash settlement of the forward contract and the option "cancel out" in the sense that the cash settlement of the forward enabled us to pay what we owe to settle the forward.*
- Note that the above statement is true even if $S_T < F$ and we had to pay money to cash settle the forward.

- We do not know in advance what the value of S_T will be, so we have to accept the possibility that it might lie in the interval $K < S_T < F$. Then we have to pay cash to settle both the option and the forward.

- However, by going long the forward contract, we have a **guaranteed profit.**

7.1.2 Case $S_T < K$

- The holder does not exercise because the call option is out of the money.

- So we throw away the option (= not exercised) and do not pay the option holder.

- Our cash profit is given by the amount of money in eq. (7.1.3).

- We started with zero and we have a positive amount of money at the end.

7.1.3 Review

- We must also understand these facts:

  1. Our total profit ***depends on the value of S_T.***
  2. Our total profit ***depends whether or not the option is exercised.***
  3. But our arbitrage profit ***does not depend*** if $S_T < F$ or $S_T > F$ or $S_T = F$.
  4. It is a different amount of profit in case (i) $S_T \geq K$ or case (ii) $S_T < K$.
  5. But the profit is ***always positive.***
  6. There is ***never a loss.***

- We must also understand that we do not need to know the strike price of the call option, to formulate we arbitrage strategy.

- It does not matter if $S_0 \geq K$ or $S_0 < K$.

- That is why the value of K was not stated in the question.

7.1.4 Problems

- What happens if we cannot find an investor who is willing to trade a forward contract on the stock index?

- In the case of futures contracts (which are exchange listed securities), what happens if we cannot find a futures contract with the same expiration time as the option?

- **These are real problems in practice. Sometimes we cannot formulate a perfect arbitrage strategy.**

- In the case of *exchange listed futures and options*, the options and futures expiration dates are the same.

7.2 Arbitrage 2: cash settled European put

- Suppose the value of a stock index is S_0 (in index points) at time t_0.
- The dividends can usually be approximated by a continuous dividend rate q.
- We assume that q is a constant below.
- The interest rate is $r > 0$ (a constant).
- The market price of a European put is $p_{\mathrm{Eur}} = M(e^{-r(T-t_0)}K + 0.75)$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **Solution for cash settled European put**
  1. At time t_0, we **sell** the put option and receive cash $= M(e^{-r(T-t_0)}K + 0.75)$.
  2. We save **all the money** in a bank.
  3. The arbitrage strategy for a put **does not involve buying/selling a forward contract.**
- At the time t_0, our portfolio is: short one European put option, cash in bank.
  The total value of our portfolio is zero.
- At the expiration time T, the money in the bank compounds to $M(K + 0.75e^{r(T-t_0)})$.
- **However, we do not know the final value of the stock index S_T at the expiration time T.**
- Hence we must analyze all cases, i.e. all values of S_T.
- We must prove that in all cases, the profit is positive (or zero),
  but ***never negative (= loss).***
- There are two cases (i) $S_T \le K$ or (ii) $S_T > K$.

7.2.1 Case $S_T \le K$

- In this situation the option holder exercises the put option.
- The writer (= us) and pays the option holder cash in the amount of $M(K - S_T)$.
- The option writer (= us) **must obey** because the option writer has **no rights, only obligations.**
- However, we already have enough cash in the bank to cover the required payment
- After the cash settlement, our final cash amount is

$$\text{final cash} = M(K + 0.75e^{r(T-t_0)}) - M(K - S_T) = M(S_T + 0.75e^{r(T-t_0)})\,. \tag{7.2.1}$$

- This is a positive number.
- We started with zero and we have a positive amount of money at the end.

7.2.2 Case $S_T > K$

- The holder does not exercise because the put option is out of the money.
- So we throw away the option (= not exercised) and do not pay the option holder.
- Then we have cash $= M(K + 0.75e^{r(T-t_0)})$.
- We started with zero and we have a positive amount of money at the end.

7.2.3 Review

- We must also understand these facts:
  1. Our total profit ***depends on the value of S_T.***
  2. Our total profit ***depends whether or not the option is exercised.***
  3. It is a different amount of profit in case (i) $S_T \leq K$ or case (ii) $S_T > K$.
  4. But the profit is ***always positive.***
  5. There is ***never a loss.***
- We must also understand that in this case, we need to know the strike price of the put option, *but we do not need to know the stock index value,* to formulate we arbitrage strategy.
- It does not matter if $S_0 \leq K$ or $S_0 > K$.
- That is why the value of S_0 was not stated in the question for the put option.
- Unlike the case for a European call, we do not have to worry if there is no futures contract with the same expiration as the put option.

7.3 Arbitrage 3: cash settled American call

- Suppose the value of a stock index is S_0 (in index points) at time t_0.
- The dividends can usually be approximated by a continuous dividend rate q.
- We assume that q is a constant below.
- The interest rate is $r > 0$ (a constant).
- The market price of an American call is $C_{\text{Eur}} = M(S_0 + 2.5)$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **This is not so easy. The answer depends if $r - q \geq 0$ or $r - q < 0$.**
- **Solution for cash settled American call with $r - q \geq 0$**

  1. At time t_0, we **sell** the call option and receive cash $= M(S_0 + 2.5)$.
  2. We save all the money in a bank.
  3. We go long a futures contract on the stock index, with the same expiration time as the option.
  4. ***Unlike for a European call, in this case it is essential to use a futures contract.***
  5. The futures price at time t_0 is
  $$F(t_0) = S_0 e^{(r-q)(T-t_0)} . \tag{7.3.1}$$
  6. The futures contract costs nothing to buy at the time t_0.
  7. The futures contract is also cash settled, with the same dollar multiplier M.

- At the time t_0, our portfolio is: short one American call option, long one futures contract, cash in bank. The total value of our portfolio is zero.
- This is the same arbitrage strategy as for the European call option.
- However, we must consider the possibility that the American call will be exercised early.
- We must prove that our arbitrage strategy yields a guaranteed profit **at any time t, where $t_0 < t \leq T$,** not only at the expiration time T.
- Suppose the option holder exercises the American call at an intermediate time t, where $t_0 < t < T$.
- The stock index value at the time t is S_t, and the holder will only exercise if $S_t \geq K$.
- Hence there is only one case to analyze, which is $S_t \geq K$.
- At the time t, the money in the bank compounds to $M(S_0 + 2.5)e^{r(t-t_0)}$.
- We are also long the futures contract.

- **We sell the futures contract.**
- This is why it is essential to use a futures contract, because *if we were long a forward contract, we could not generate a cashflow with it at time t.*
- The futures price at time t is

$$F(t) = S_t e^{(r-q)(T-t)} . \tag{7.3.2}$$

- The profit by selling the futures contract is

$$M\big[\, F(t) - F(t_0) \,\big] = M\big[\, S_t e^{(r-q)(T-t)} - S_0 e^{(r-q)(T-t_0)} \,\big] . \tag{7.3.3}$$

- Note that this could be a negative amount, but that does affect the arbitrage strategy.
- The sum of our cash in the bank plus the cash settlement of the futures contract is:

$$\begin{aligned} \text{cash available at time } t &= M(F(t) - F(t_0)) + M(S_0 + 2.5)e^{r(t-t_0)} \\ &= M\big[\, S_t e^{(r-q)(T-t)} - S_0 e^{(r-q)(T-t_0)} + S_0 e^{r(t-t_0)} + 2.5e^{r(t-t_0)} \,\big] \\ &= M\big[\, S_t e^{(r-q)(T-t)} + S_0 e^{r(T-t_0)}(1 - e^{-q(T-t_0)}) + 2.5e^{r(t-t_0)} \,\big] . \end{aligned} \tag{7.3.4}$$

- To cash settle the option we must pay the holder a cash amount $M(S_t - K)$.
- Hence overall we have cash at time t in the amount

$$\begin{aligned} \text{cash at time } t &= M\big[\, S_t e^{(r-q)(T-t)} + S_0 e^{r(T-t_0)}(1 - e^{-q(T-t_0)}) + 2.5e^{r(t-t_0)} \,\big] - M(S_t - K) \\ &= M\big[\, K + S_t(e^{(r-q)(T-t)} - 1) + S_0 e^{r(T-t_0)}(1 - e^{-q(T-t_0)}) + 2.5e^{r(t-t_0)} \,\big] . \end{aligned} \tag{7.3.5}$$

- This is a positive number because $r - q \geq 0$ hence $S_t(e^{(r-q)(T-t)} - 1) \geq 0$. Hence all the terms on the right hand side in eq. (7.3.5) are positive or ≥ 0 and the sum is positive.
- If the option is held to expiration, the analysis is the same as for a European call.
- Therefore the above arbitrage strategy yields a positive guaranteed profit at any time $t_0 < t \leq T$.
- **Solution for cash settled American call with $r - q < 0$**
  1. If $r - q < 0$ then $S_t(e^{(r-q)(T-t)} - 1) < 0$ and the cash amount on the right hand side in eq. (7.3.5) may not be positive.
  2. Hence we require a different arbitrage strategy.
  3. At time t_0, we **sell** the call option and receive cash $= M(S_0 + 2.5)$.
  4. We save all the money in a bank.
  5. We define a parameter

$$N_F = \frac{1}{e^{(r-q)(T-t_0)}} . \tag{7.3.6}$$

6. Note that N_F is a constant, whose value is known at the time t_0.
7. We go long N_F futures contract on the stock index, with the same expiration time as the option.
8. The futures price at time t_0 is
$$F(t_0) = S_0 e^{(r-q)(T-t_0)}\,. \tag{7.3.7}$$
9. The futures contract costs nothing to buy at the time t_0.
10. The futures contract is also cash settled, with the same dollar multiplier M.

- At the time t_0, our portfolio is: short one American call option, long N_F futures contract, cash in bank. The total value of our portfolio is zero.
- Suppose the option holder exercises the American call at an intermediate time t, where $t_0 < t < T$.
- The stock index value at the time t is S_t, and the holder will only exercise if $S_t \ge K$.
- At the time t, the money in the bank compounds to $M(S_0 + 2.5)e^{r(t-t_0)}$.
- We are also long N_F futures contracts.
- The futures price at time t is
$$F(t) = S_t e^{(r-q)(T-t)}\,. \tag{7.3.8}$$
- The profit by selling the futures contract is
$$N_F M\big[\,F(t) - F(t_0)\,\big] = N_F M\big[\,S_t e^{(r-q)(T-t)} - S_0 e^{(r-q)(T-t_0)}\,\big]\,. \tag{7.3.9}$$
- Note that this could be a negative amount, but that does affect the arbitrage strategy.
- The sum of our cash in the bank plus the cash settlement of the forward contract is:
$$\begin{aligned}\text{cash available at time } t &= N_F M(F(t) - F(t_0)) + M(S_0 + 2.5)e^{r(t-t_0)} \\ &= M\big[\,N_F S_t e^{(r-q)(T-t)} - N_F S_0 e^{(r-q)(T-t_0)} + S_0 e^{r(t-t_0)} + 2.5 e^{r(t-t_0)}\,\big] \\ &= M\Big[\,S_t \frac{e^{(r-q)(T-t)}}{e^{(r-q)(T-t_0)}} + S_0(e^{r(T-t_0)} - 1) + 2.5 e^{r(t-t_0)}\,\Big] \\ &= M\big[\,S_t e^{-(r-q)(t-t_0)} + S_0(e^{r(T-t_0)} - 1) + 2.5 e^{r(t-t_0)}\,\big]\,.\end{aligned} \tag{7.3.10}$$
- To cash settle the option we must pay the holder a cash amount $M(S_t - K)$.
- Hence overall we have cash at time t in the amount
$$\begin{aligned}\text{cash at time } t &= M\big[\,S_t e^{-(r-q)(t-t_0)} + S_0(e^{r(T-t_0)} - 1) + 2.5 e^{r(t-t_0)}\,\big] - M(S_t - K) \\ &= M\big[\,K + S_t(e^{-(r-q)(t-t_0)} - 1) + S_0(e^{r(T-t_0)} - 1) + 2.5 e^{r(t-t_0)}\,\big]\,.\end{aligned} \tag{7.3.11}$$

- Now because $r-q<0$ hence $S_t(e^{-(r-q)(t-t_0)}-1)>0$, hence all the terms on the right hand side in eq. (7.3.11) are positive and the sum is positive.
- If the option is held to expiration and is exercised, we substitute $t=T$ and the cash amount in eq. (7.3.11) is positive.
- If the option is held to expiration and is not exercised, our final cash amount is given by eq. (7.3.10) (with $t=T$) and is positive.
- Therefore the above arbitrage strategy yields a positive guaranteed profit at any time $t_0 < t \le T$.

7.4 Arbitrage 4: cash settled American put

- Suppose the value of a stock index is S_0 (in index points) at time t_0.
- The dividends can usually be approximated by a continuous dividend rate q.
- We assume that q is a constant below.
- The interest rate is $r > 0$ (a constant).
- The market price of an American put is $P_{\mathrm{Am}} = M(K + 1.75)$.
- **Formulate an arbitrage strategy to take advantage of the mispriced option.**
- **Solution for cash settled American put**
  1. At time t_0, we **sell** the put option and receive cash $= M(K + 0.75)$.
  2. We save **all the money** in a bank.
- At the time t_0, our portfolio is: short one American put option, cash in bank.
  The total value of our portfolio is zero.
- This is the same arbitrage strategy as for the European put option.
- Suppose the option holder exercises the American put at an intermediate time t, where $t_0 < t < T$.
- The stock price at the time t is S_t, and the holder will only exercise if $S_t \le K$.
- At the time t, the money in the bank compounds to $M(K + 1.75)e^{r(t-t_0)}$.
- **However, we do not know the final value of the stock index S_T at the expiration time T.**
- The writer (= us) pay the holder cash in the amount of $M(K - S_t)$.
- We have enough cash available to pay the option holder.
- We pay $M(K - S_t)$ to the option holdre and our final cash amount is

$$\begin{aligned} \text{cash at time } t &= M(K + 1.75)e^{r(t-t_0)}) - M(K - S_t) \\ &= M\big[\, S_t + K(e^{r(t-t_0)} - 1) + 1.75e^{r(t-t_0)} \,\big]\,. \end{aligned} \tag{7.4.1}$$

- This is a positive number.
- If the option is held to expiration and exercised, we put $t = T$ in eq. (7.4.1) and the value is still positive.
- If the option is held to expiration and not exercised, our final cash amount is $M(K + 1.75)e^{r(T-t_0)}$, also positive.
- Therefore the above arbitrage strategy yields a positive guaranteed profit at any time $t_0 < t \le T$.