Queens College, CUNY, Department of Computer Science

Numerical Methods

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Instructor: Dr. Sateesh Mane

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25 Lecture 25

Complex numbers

- We shall study Fourier transforms and Fourier series later in this course.
- To do so, we shall require the use of **complex numbers.**
- In this lecture we shall review some basic definitions and properties of complex numbers.

25.1 Complex numbers: general definitions

- Let x and y br real numbers.
- Also define i to be the square root of -1, so $i^2 = -1$. Clearly $(-i)^2 = -1$ also.
- A **complex number** z is defined via

$$z = x + iy\,. \tag{25.1.1}$$

- The number x is called the **real part** of z, denoted by $\Re\{z\}$.
- The number y is called the **imaginary part** of z, denoted by $\Im\{z\}$.
  1. If $y = 0$ we say that z is **pure real**, or just real.
  2. If $x = 0$ we say that z is **pure imaginary**.
- Complex numbers have a close connection to (x, y) coordinates in a Cartesian plane.
  1. The x axis of the plane is called the **real axis.**
  2. The y axis of the plane is called the **imaginary axis.**
- The **complex conjugate** is denoted by z^* and given by reversing the imaginary part of z:

$$z^* = \bar{z} = x - iy\,. \tag{25.1.2}$$

- The complex conjugate of z^* is the original number z itself: $(z^*)^* = z$.
- Many authors also denote the complex conjugate by $\bar{z}$. We shall employ the notation z^*.
- In terms of points in a Cartesian plane, z^* corresponds to the point $(x, -y)$.
- The complex conjugate z^* is the reflection of z in the real axis.
- The reflection of z in the imaginary axis is obviously given by $-(z^*) = -(x - iy) = -x + iy$.
- **See Fig. 1.**
- Given z and z^*, the real and imaginary parts of z can be obtained via

$$\Re\{z\} = \frac{z + z^*}{2}\,, \qquad \Im\{z\} = \frac{z - z^*}{2i}\,. \tag{25.1.3}$$

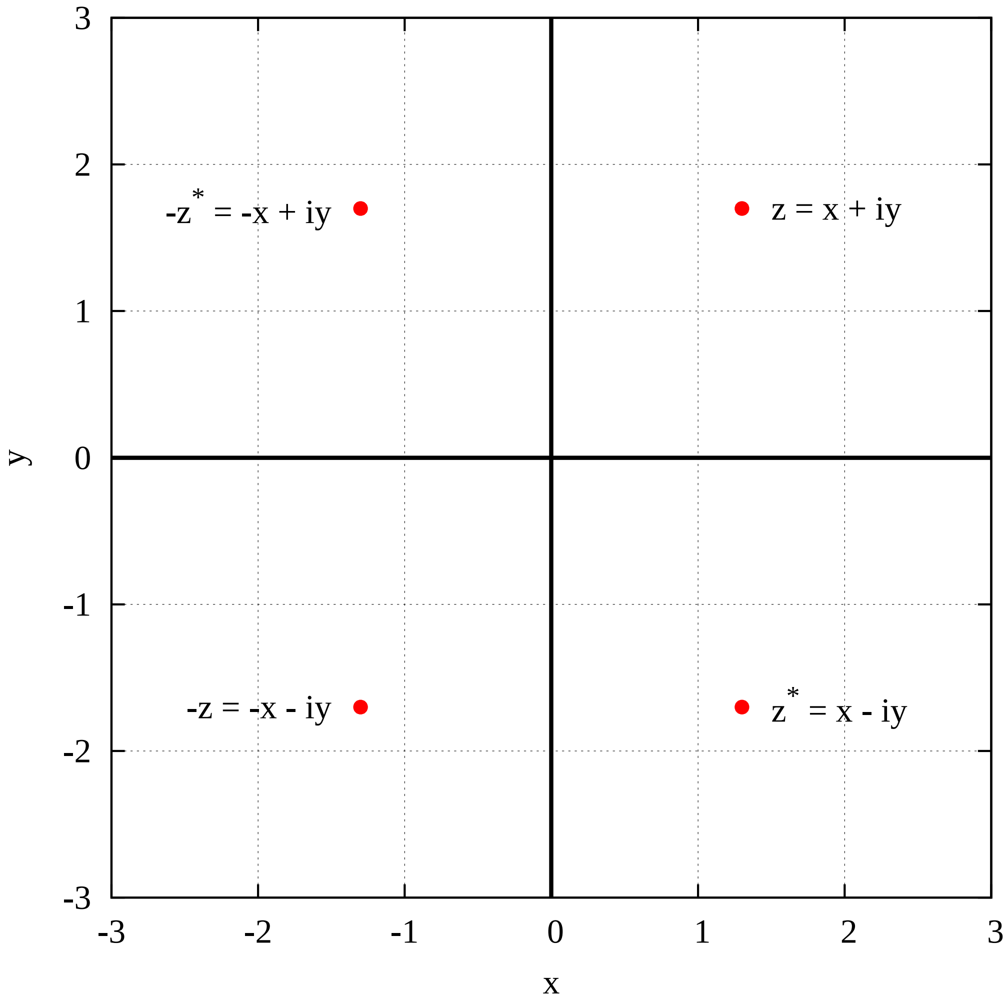

Figure 1: Complex plane with four points $z = x + iy$, $z^* = x - iy$, $-z = -x - iy$ and $-z^* = -x + iy$.

25.2 $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

- Mathematicians usually denote various sets of numbers by the following notations.
    1. $\mathbb{N}$: Natural numbers $n = 0, 1, 2, \dots$.
    2. $\mathbb{Z}$: Integers $m = 0, \pm 1, \pm 2, \dots$.
    3. $\mathbb{Q}$: Rational numbers m_1/m_2 $(m_2 \neq 0)$.
    4. $\mathbb{R}$: Real numbers x.
    5. $\mathbb{C}$: Complex numbers $z = x + iy$.
- Using this notation, $x, y \in \mathbb{R}$ and $z, z^* \in \mathbb{C}$.
- **Do not be frightened by the above notation. We shall not use it.**

25.3 Complex numbers: basic operations

- The product of a complex number $z = x + iy$ by a real number t is

$$tz = zt = tx + ity \,. \tag{25.3.1}$$

- We saw this above when I wrote $-(z^*) = -(x - iy) = -x + iy$.
- Obviously therefore $-z = -x - iy$.
- Let $w = u + iv$ be a complex number.
- The sum of two complex numbers $z + w$ is given by

$$z + w = w + z = (x + u) + i(y + v) \,. \tag{25.3.2}$$

- The product of two complex numbers zw is given by

$$zw = wz = (x + iy)(u + iv) = xu - yv + i(xv + yu) \,. \tag{25.3.3}$$

- The product of z and its complex conjugate z^* is always a nonnegative real number:

$$zz^* = z^*z = (x + iy)(x - iy) = x^2 + y^2 \,. \tag{25.3.4}$$

- The value of zz^* is zero if and only if $x = y = 0$, i.e. $z = 0$.
- The **absolute value** or **magnitude** or **amplitude** of z is

$$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2} \,. \tag{25.3.5}$$

- Obviously $|z^*| = |z|$ and $|zw| = |z||w|$ and $|zt| = |z||t|$.
- Therefore if $z \neq 0$ the value of $z^{-1} = 1/z$ is given by

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{x - iy}{x^2 + y^2} \,. \tag{25.3.6}$$

- Therefore the division of two complex numbers w/z is given by the product wz^{-1}:

$$\frac{w}{z} = wz^{-1} = \frac{wz^*}{zz^*} = \frac{(u + iv)(x - iy)}{x^2 + y^2} = \frac{xu + yv + i(xv - yu)}{x^2 + y^2} \,. \tag{25.3.7}$$

25.4 Complex numbers: amplitude–argument or polar form

- We can also represent complex numbers using polar coordinates (r, θ).
- Recall that in the Cartesian plane, $x = r\cos\theta$ and $y = r\sin\theta$.
- Then we obtain

$$z = x + iy = r(\cos\theta + i\sin\theta)\,. \tag{25.4.1}$$

- We need to process this some more.
- The **exponential of a complex number** w is *defined* as the sum of the power series

$$e^w = \exp(w) = \sum_{n=0}^{\infty} \frac{w^n}{n!} = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots \tag{25.4.2}$$

- The series in eq. (25.4.2) converges for all values of w.
- Set $w = i\theta$ and we obtain

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\,\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\,\frac{\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta\,. \end{aligned} \tag{25.4.3}$$

- Hence eq. (25.4.1) can be reexpressed as

$$z = x + iy = r\,e^{i\theta}\,. \tag{25.4.4}$$

- Let $z_1 = r_1\,e^{i\theta_1}$ and $z_2 = r_2\,e^{i\theta_2}$ be two complex numbers. The product $z_1 z_2$ is given by

$$z_1 z_2 = z_2 z_1 = r_1 r_2\,e^{i(\theta_1+\theta_2)}\,. \tag{25.4.5}$$

- Note that $z^* = r\,e^{-i\theta}$. Set $z_1 = z = r\,e^{i\theta}$ and $z_2 = z^* = r\,e^{-i\theta}$, then

$$zz^* = r^2\,e^{i(\theta-\theta)} = r^2\,. \tag{25.4.6}$$

- Then r is the amplitude of z. Note that $r \ge 0$.
- We call θ the **argument** of z.
- We call eq. (25.4.4) the **polar form** of a complex number.
- We also call eq. (25.4.4) the **amplitude–argument** representation of a complex number.
- Note that $\cos(\theta + 2j\pi) = \cos\theta$ and $\sin(\theta + 2j\pi) = \sin\theta$ for any integer $j = 0, \pm1, \pm2, \dots$.
- Hence

$$r\,e^{i(\theta+2j\pi)} = r\big[\cos(\theta+2j\pi) + i\sin(\theta+2j\pi)\big] = r\big[\cos\theta + i\sin\theta\big] = r\,e^{i\theta} = z\,. \tag{25.4.7}$$

- Hence $r\,e^{i(\theta+2j\pi)}$ represents the *same* complex number z, for any integer j.
- We shall see below in Sec. 25.8 that this leads to some problems.

25.5 Complex numbers: multiplication using polar form

- We saw above that if $z_1 = r_1\,e^{i\theta_1}$ and $z_2 = r_2\,e^{i\theta_2}$, the product $z_1 z_2$ is given by

$$z_1 z_2 = z_2 z_1 = r_1 r_2\,e^{i(\theta_1+\theta_2)}\,. \tag{25.5.1}$$

- We **multiply the amplitudes** and **add the arguments.**
- This has a simple interpretation when $r_2 = 1$.
- Let $z = r\,e^{i\theta}$ and $w = e^{i\phi}$. Then

$$zw = r\,e^{i(\theta+\phi)}\,. \tag{25.5.2}$$

- The amplitude does not change $(= r)$ but the argument changes from θ to $\theta + \phi$.
- In terms of points in a Cartesian plane, multiplication by $e^{i\phi}$ corresponds to a **counterclockwise rotation around the origin through an angle ϕ.**
- See Fig. 2.

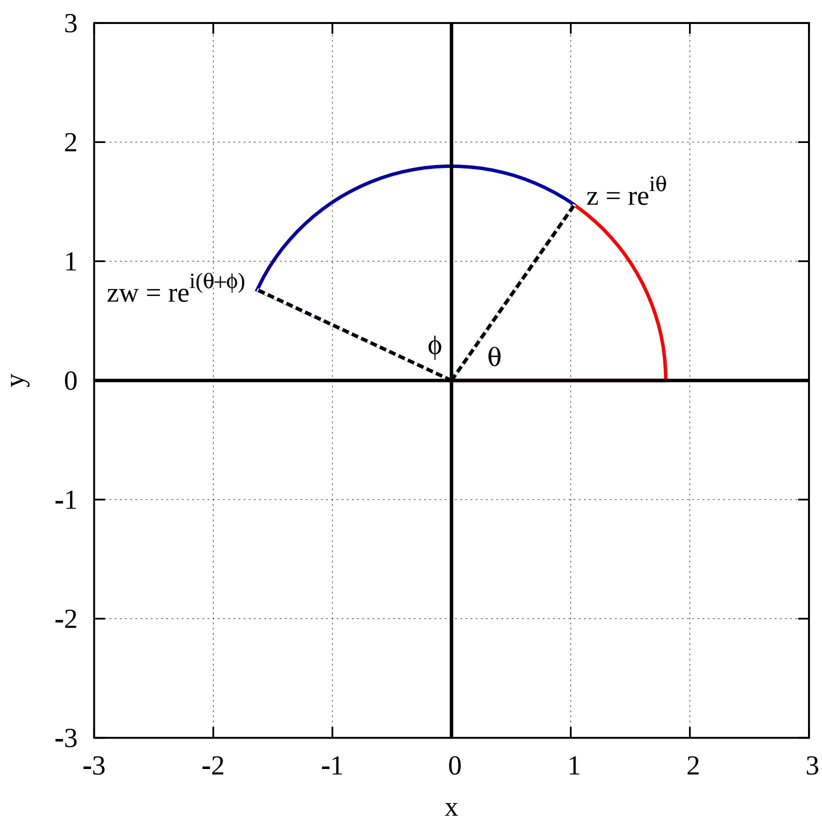

Figure 2: Complex multiplication using polar form, to demonstrate rotation in the complex plane.

25.6 Complex numbers: useful numbers

- The following are useful expressions:

$$e^{i\pi} = -1\,, \qquad e^{i\pi/2} = i\,, \qquad e^{-i\pi/2} = -i\,. \tag{25.6.1}$$

- Multiplication by i yields a counterclockwise rotation through 90° in the complex plane.
- Multiplication by $-i$ yields a clockwise rotation through 90° in the complex plane.
- The following equation is called **Euler's equation** or **Euler's identity**

$$e^{i\pi} + 1 = 0\,. \tag{25.6.2}$$

- It is frequently advertised as an example of mathematical beauty.
- It relates five of the most important numbers in mathematics: 0, 1, e, π and i.

25.7 Complex numbers: primitive roots of unity

- For any positive integer $n > 1$, a **primitive n^{th} root of unity** is a complex number $\omega_n \neq 1$ such that $\omega_n^n = 1$ and $\omega_n^j \neq 1$ for any smaller positive integer j, where $1 \leq j < n$.
- Hence by definition, a primitive n^{th} root of unity is a root of the equation $z^n - 1 = 0$ and is *not a root of the equation* $z^j - 1 = 0$ *for any smaller positive integer* j, where $1 \leq j < n$.
- Obviously $e^{i2\pi/n}$ is a primitive n^{th} root of unity, but there are others.
- The following are examples of primitive roots of unity:
  1. $\boldsymbol{n = 2}$. The primitive root is -1.
  2. $\boldsymbol{n = 3}$. The primitive roots are $e^{i2\pi/3}$ and $e^{i4\pi/3}$.
  3. $\boldsymbol{n = 4}$. The primitive roots are i and $-i$.
     (a) Note that -1 is not a primitive fourth root of unity because $(-1)^2 = 1$.
  4. $\boldsymbol{n = 5}$. The primitive roots are $e^{i2\pi/5}$, $e^{i4\pi/5}$, $e^{i6\pi/5}$ and $e^{i8\pi/5}$.
  5. $\boldsymbol{n = 6}$. The primitive roots are $e^{i2\pi/6} = e^{i\pi/3}$ and $e^{i5\pi/3}$.
     (a) Note that $e^{i2\pi/3}$ is not a primitive sixth root of unity because $(e^{i2\pi/3})^3 = 1$.
     (b) Note that $e^{i3\pi/3} = -1$ is not a primitive sixth root of unity because $(-1)^2 = 1$.
     (c) Note that $e^{i4\pi/3}$ is not a primitive sixth root of unity because $(e^{i4\pi/3})^3 = 1$.
- Clearly $e^{i2\pi m/n}$ is a primitive n^{th} root of unity if and only if $\gcd(m, n) = 1$.
- Hence finding all the primitive n^{th} roots of unity is equivalent to finding all the prime divisors of n, which is a computationally hard problem.
- **The complex conjugate of a primitive root is also a primitive root, hence we can also write $e^{-i2\pi m/n}$, where $\gcd(m, n) = 1$.**
- If ω_n is a primitive n^{th} root of unity, then the following sums of powers all add up to zero:

$$\begin{aligned} 1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} &= 0\,, \\ 1 + \omega_n^2 + \omega_n^4 + \cdots + \omega_n^{2(n-1)} &= 0\,, \\ &\vdots \\ 1 + \omega_n^j + \omega_n^{2j} + \cdots + \omega_n^{j(n-1)} &= 0 \qquad (1 \leq j < n)\,. \end{aligned} \tag{25.7.1}$$

- However, if $j = n$, then the following sum adds up to n:

$$1 + \omega_n^n + \omega_n^{2n} + \cdots + \omega_n^{(n-1)n} = 1 + 1 + \cdots + 1 = n\,. \tag{25.7.2}$$

- What this means, in general, is that if j is a multiple of n (including zero), then the sum of powers adds up to n, else the sum adds up to zero. ***Note that j can be zero or negative in the above statement.***

$$\sum_{k=0}^{n-1} \omega^{jk} = \begin{cases} n & j \text{ divides } n\text{, including } j \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{25.7.3}$$

- **The primitive roots of unity will be important for later use in Fourier series.**

25.8 Complex numbers: square roots and branch cuts (advanced topic)

- Let $z = r\,e^{i\theta}$ and let $w = \sqrt{z}$. How do we calculate w?
- By definition $w^2 = z$, hence logically
$$w = r^{1/2}\,e^{i\theta/2}\,. \tag{25.8.1}$$
- This works, but it hides some difficulties.
- Suppose $r = 4$ and $\theta = 4\pi/3$, hence $z = 4\,e^{i4\pi/3}$.
- Then using eq. (25.8.1) we obtain the square root
$$w = 2\,e^{i2\pi/3} = 2\Bigl(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\Bigr) = 2\Bigl(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\Bigr) = -1 + i\sqrt{3}\,. \tag{25.8.2}$$
- Now consider $r\,e^{i(\theta-2\pi)} = 4\,e^{i(4\pi/3-2\pi)} = 4\,e^{-i2\pi/3}$, which is the *same* number.
- Then using eq. (25.8.1) we obtain the square root
$$w = 2\,e^{-i\pi/3} = 2\Bigl(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\Bigr) = 2\Bigl(\frac{1}{2} - i\frac{\sqrt{3}}{2}\Bigr) = 1 - i\sqrt{3}\,. \tag{25.8.3}$$
- This is the negative of the square root in eq. (25.8.2).
- Of course, we know that every number has two square roots (except zero), but how do we know which root eq. (25.8.1) will yield?
- For a real number, we can say 'select the positive root' or 'choose the negative root' but for a complex number, eq. (25.8.1) is ambiguous.
- How do we specify a rule to say which square root do we want from eq. (25.8.1), i.e. eq. (25.8.2) or eq. (25.8.3)?
- There is no simple way to constrain eq. (25.8.1) to yield a unique answer.
- We must make a **branch cut** in the complex plane.
- That is to say, we must restrict the value of θ to an interval of length 2π.
- One popular choice is to restrict $0 \le \theta < 2\pi$. Another popular choice is to restrict $-\pi < \theta \le \pi$.
- **There are infinitely many choices of branch cuts.**
- **The results from eq. (25.8.1) depend on the choice of the branch cut.**
- When performing calculations with complex numbers, we must select a branch cut and stick with it consistently in all our calculations.
- Another possibility, which mathematicians also use, is to allow $-\infty < \theta < \infty$ and to say that every nonzero complex number has **infinitely many square roots.**
- You can see that the problem has no simple answer.
- **We shall avoid calculations which require branch cuts in the complex plane.**

25.9 Complex numbers: logarithm (advanced topic)

- Perhaps the nastiest of the standard mathematical functions is the complex logarithm.
- Write $z = r\,e^{i\theta}$. Go one step further and write $z = r\,e^{i(\theta+2j\pi)}$ for $j = 0, \pm1, \pm2, \dots$.
- Then the complex logarithm $\ln(z)$ is given by

$$\ln(z) = \ln\Bigl(r\,e^{i(\theta+2j\pi)}\Bigr) = \ln(r) + i(\theta + 2j\pi) \qquad (j = 0, \pm1, \pm2, \dots)\,. \tag{25.9.1}$$

- The imaginary part of $\ln(z)$ has an infinity of values, uniformly spaced at intervals of 2π.
- **We shall avoid the complex logarithm in this class.**

25.10 Complex numbers: equations (advanced topic)

- One can write polynomial and differential equations, etc. using complex numbers.
- The binomial theorem and Taylor series, etc. are all applicable for complex numbers.
- For example the following are applications of the binomial theorem

$$\begin{aligned} (1+z)^2 &= 1 + 2z + z^2\,, \\ \frac{1}{1-z} &= 1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n\,. \end{aligned} \tag{25.10.1}$$

- We can also write

$$\frac{1}{1-z} = \frac{1-z^*}{(1-z)(1-z^*)} = \frac{1-z^*}{1-z-z^*+zz^*} = \frac{1-z^*}{1-2\Re\{z\}+|z|^2}\,. \tag{25.10.2}$$

- The denominator on the right hand side of eq. (25.10.2) is real.
- The following is a complex differential equation for a complex function $f(z)$

$$\frac{df}{dz} = f\,. \tag{25.10.3}$$

- The solution of eq. (25.10.3) is $f(z) = \exp(z)$, for the initial condition $f(0) = 1$.
- The following is a complex quadratic equation (a, b, c and z are all complex)

$$az^2 + bz + c = 0\,. \tag{25.10.4}$$

- The solution of eq. (25.10.4) is

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\,. \tag{25.10.5}$$

- ***How do we calculate the square root in eq.* (25.10.5)*?*** We require a branch cut.
- Hence even the solutions of simple equations such as quadratics can get messy.