Quantitative Methods

Serena DeStefani – Lecture 22 –8/11/2020

How did we get here?

- Statistics is about variation, specifically in data
- Types of *variables: continuous, categorical, ordinal*
- How to represent variables
  - bar graphs
  - boxplots, histograms
- How to compare apples and oranges
  - by their variation (z scores)
  - are they varying together? correlation and regression

How did we get here?

- Statistics is about variation, specifically in data
- What I am interested in?
- A larger group, or *population*
- How to get data? Three ways

How do we get data?

1. **Surveys**
   - Impossible to survey the population
   - → representative sample (SRS) – randomized!
   - But exactly what does the population think?
   - What's the population parameter?
2. **Experiments**
   - Randomized
   - Possible to infer causality
   - Is my results due to the treatment or is it due to chance?
3. **Observational studies**
   - Not possible to infer causality, or to extend findings
   - Sometimes necessary

A probability model

- Two questions:
  - What's the population parameter?
  - Is my result likely?
- To answer these questions I need a *probability model*
- → The rules of probability
- Random variable: the Bernoulli trial (our building block)
- The geometric and binomial distribution
- The normal approximation to the binomial
- The sampling distribution
- The Central Limit Theorem

Summary

- How do we build a model for our data?
- Probability rules
- Random variable
- Random variable: the Bernoulli trial (our building block)
- Probability model (or probability distribution)
- There are many different distribution we can define and use, discrete or continuous.
- Examples:
  - Discrete: geometric or binomial
  - Continuous: normal, uniform, exponential
- Focus on binomial and normal distributions

The Binomial Model

- A **Binomial probability model** describes the **number** of successes in a specified number of trials.

- It takes two parameters to specify this model: the number of trials *n* and the probability of success *p*.

Mean: $\mu = np$

Standard deviation: $\sigma = \sqrt{npq}$

The Solution for Large Sample Sizes

When the sample of our binomial model is too big, calculating probabilities becomes too complicated → use a normal model to approximate it.

First we calculate mean and SD according to the binomial model.

Then we use a **normal model** with the same mean and standard deviation as a very good approximation.

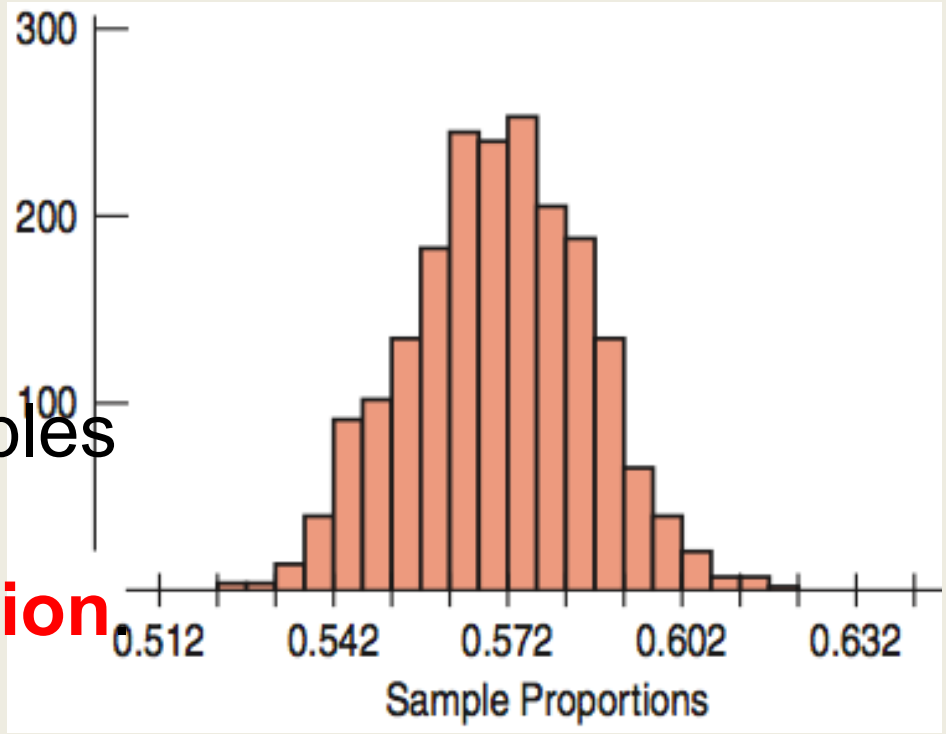
If we have a normal model, we can use z-scores to calculate probabilities.

For the approximation to work, we need at least 10 successes and 10 failures.

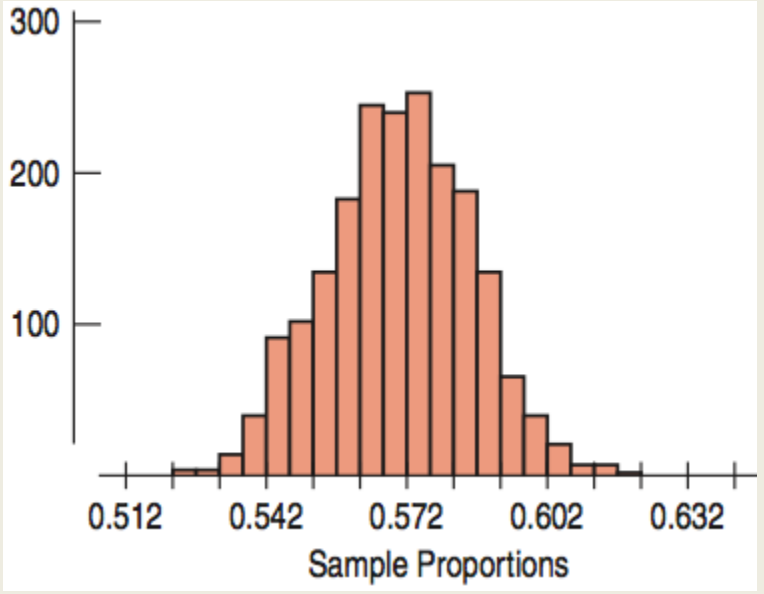
Sampling About Climate Change

According to a Gallup poll of 1022 Americans, 57% believe that climate change is due to human activity.

- If many surveys were done of 1022 Americans, we could calculate the sample proportion for each.
- The histogram shows the distribution of a simulation of 2000 sample proportions.
- The distribution of all possible sample proportions from samples with the *same sample size* is called the **sampling distribution**.



Sampling Distributions for Proportions



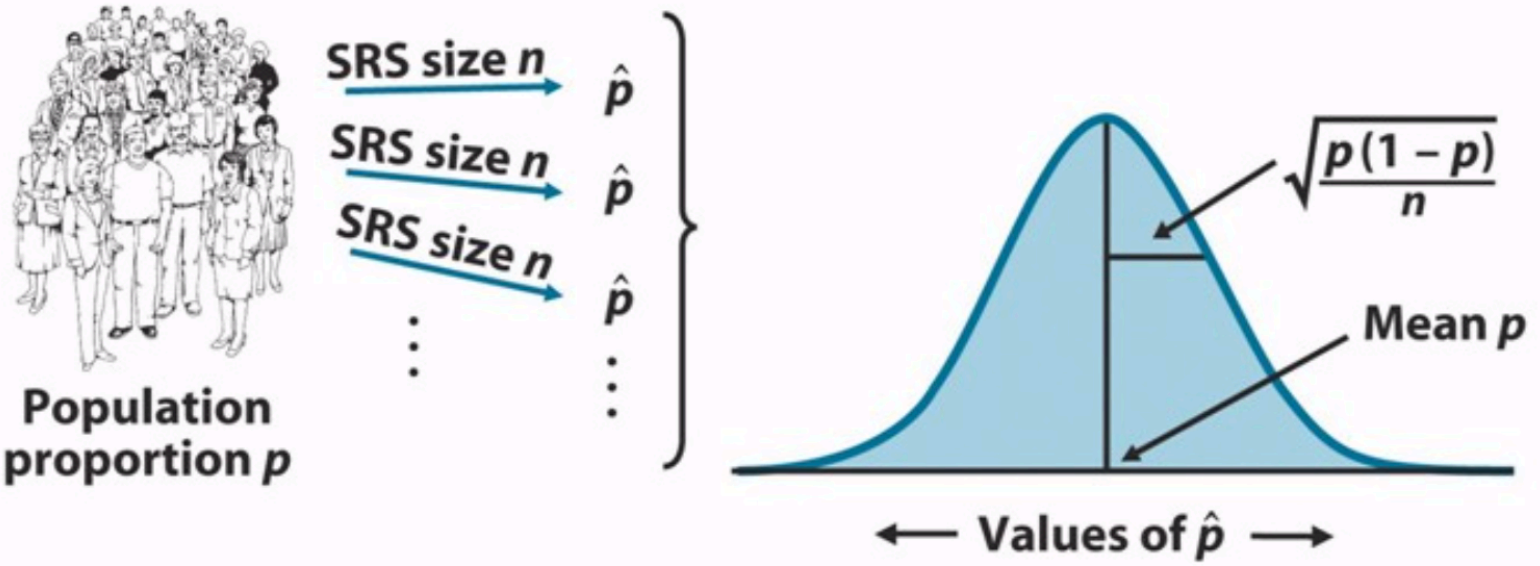
Sampling Distribution for Proportions

- Symmetric
- Unimodal
- Centered at *p*
- The sampling distribution follows the Normal model.

What does the sampling distribution tell us?

- The sampling distribution allows us to make statements about where we think the corresponding **population parameter** is and how precise these statements are likely to be.

Visual of How A Model of a Sampling Distribution of Proportions is Formed



From One Sample to Many Samples

Distribution of One Sample

- **Variable** was the *answer to the survey* question or the *result* of an experiment.
- **Proportion** is a fixed value that comes from the one sample.

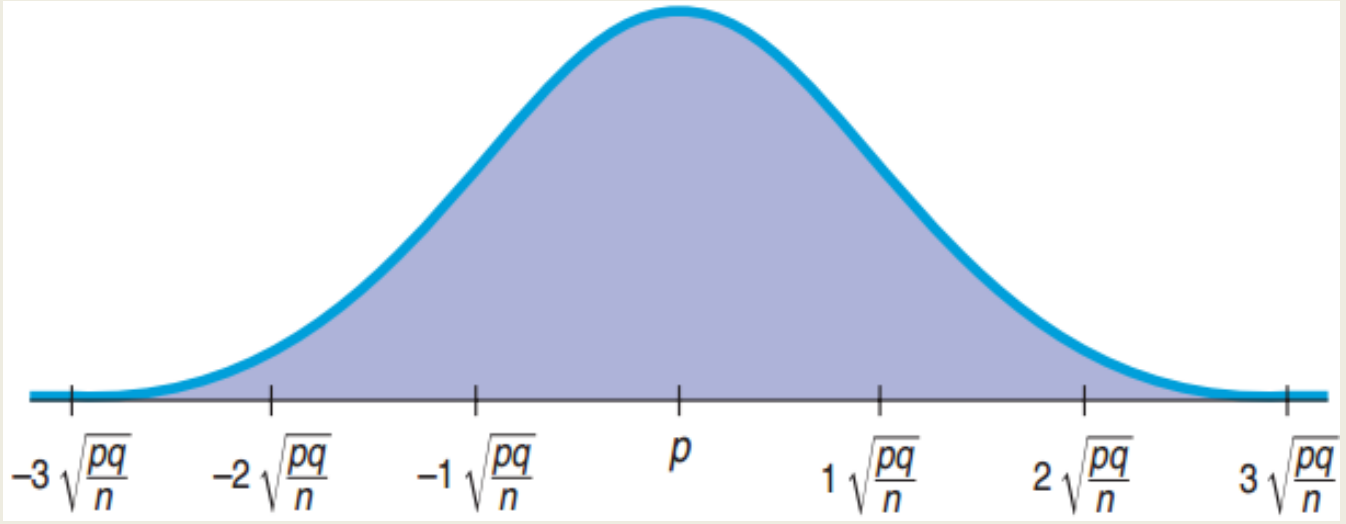
Sampling Distribution

- **Variable** is the proportion that comes from the entire sample.
- Many proportions that differ from one to another, each coming from a different sample.

Mean and Standard Deviation

Sampling Distribution for Proportions

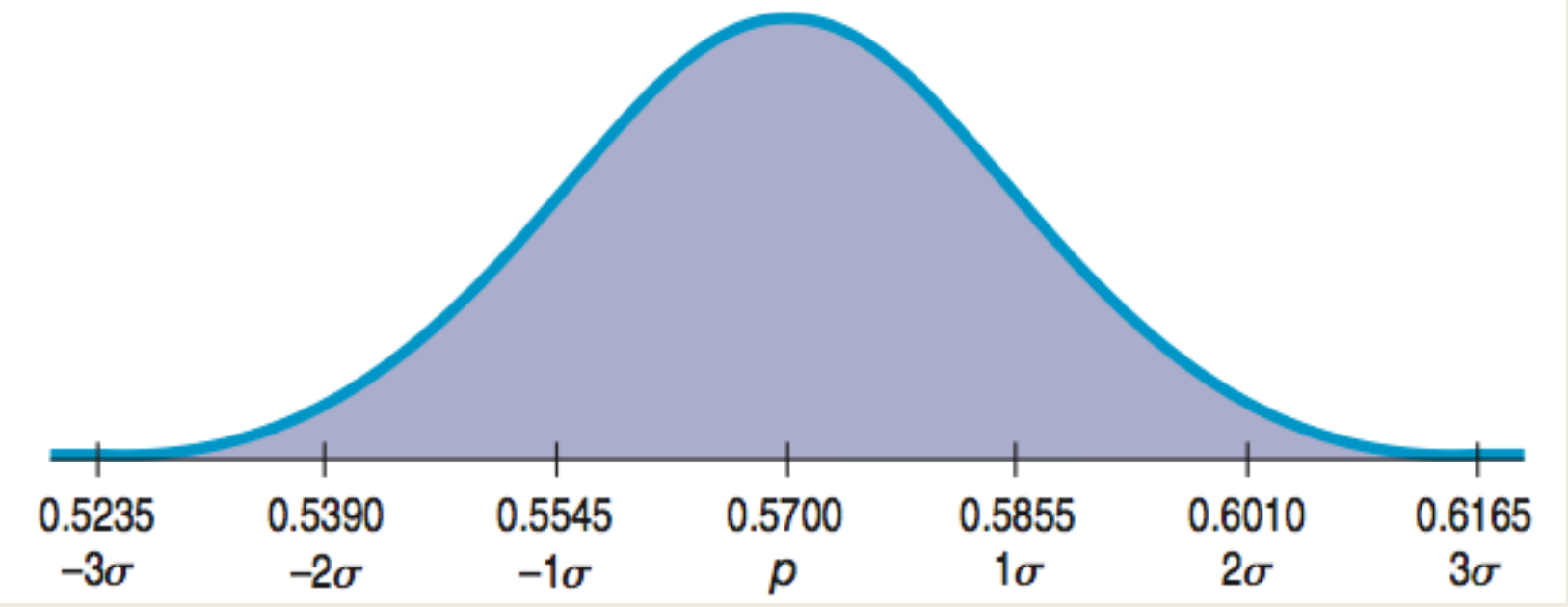
- Mean = p This p is a parameter!
- $\sigma(\hat{p}) = \frac{\sqrt{npq}}{n} = \sqrt{\frac{pq}{n}}$ Instead *p_hat* is a statistics!
- $N\left(p, \sqrt{\frac{pq}{n}}\right)$



The Normal Model for Climate Change

Population: $p = 0.57$, $n = 1022$. Sampling Distribution:

- Mean = 0.57
- Standard deviation = $SD(\hat{p}) = \sqrt{\frac{(0.57)(0.43)}{1022}} \approx 0.0155$



Standard Error

The sample-to-sample standard deviation is called the **standard error** or **sampling variability**.

- The standard error is not a “real” error, since no error has been made.

The problem is, we don’t know the population parameter.

Sampling Distribution

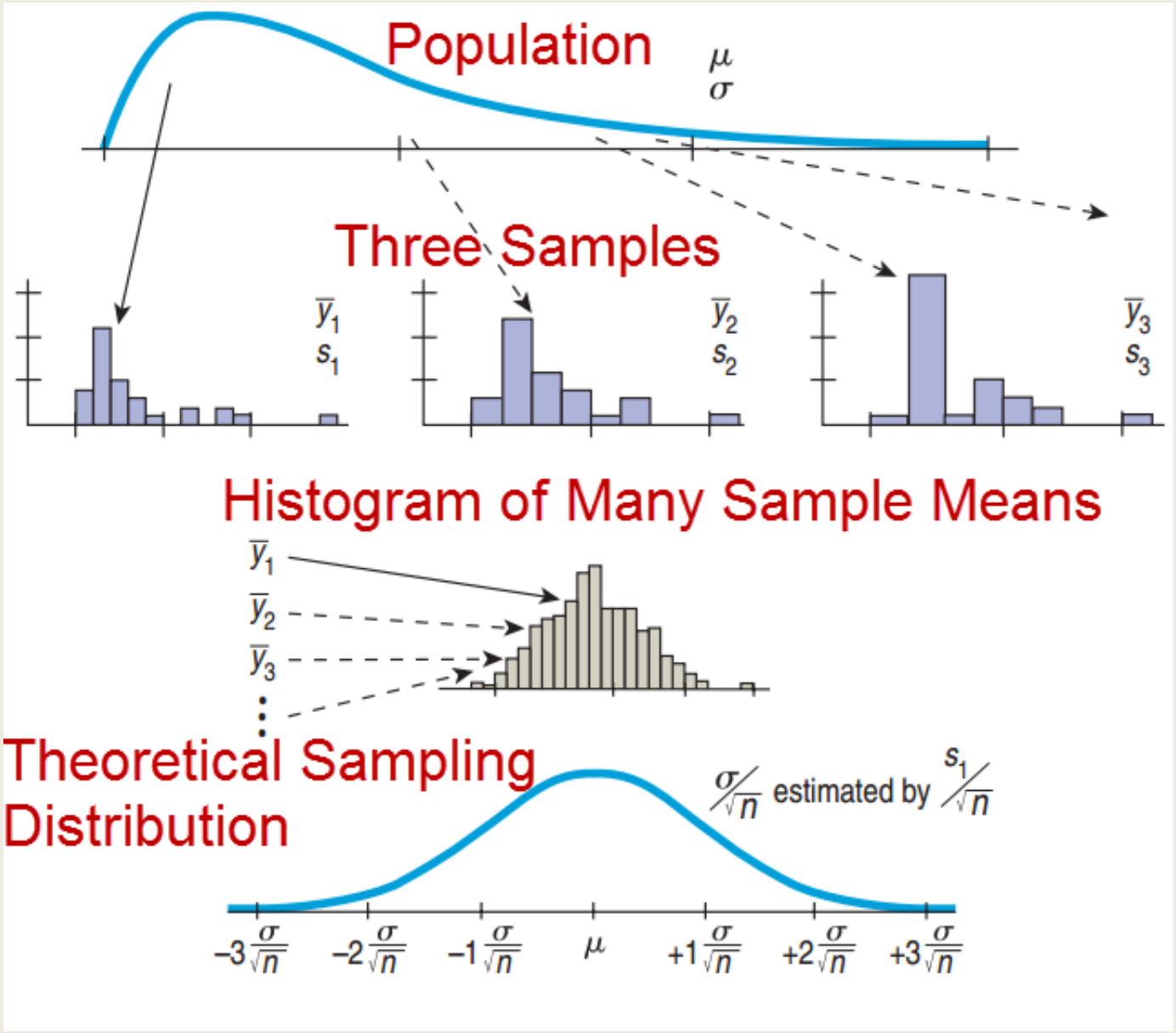
- Model a proportion with a binomial (yes/no)
- Large n: approximate with normal

- What about the mean?
- Can we build a sampling distribution for the mean?
- What will be its shape?

-

The Central Limit Theorem (CLT): sampling distribution for the mean

The mean of a random sample is a random variable whose sampling distribution can be approximated by a Normal model. The larger the sample, the better the approximation will be.


Population
μ
σ
Three Samples
$\bar{y}_1$
s_1
$\bar{y}_2$
s_2
$\bar{y}_3$
s_3
Histogram of Many Sample Means
$\bar{y}_1$
$\bar{y}_2$
$\bar{y}_3$
Theoretical Sampling Distribution
$\frac{\sigma}{\sqrt{n}}$ estimated by $\frac{s_1}{\sqrt{n}}$
$-3\frac{\sigma}{\sqrt{n}}$
$-2\frac{\sigma}{\sqrt{n}}$
$-1\frac{\sigma}{\sqrt{n}}$
μ
$+1\frac{\sigma}{\sqrt{n}}$
$+2\frac{\sigma}{\sqrt{n}}$
$+3\frac{\sigma}{\sqrt{n}}$

Population Distribution and Sampling Distribution of the Means

Population Distribution		Sampling Distribution for the Means
• Normal	→	Normal (any sample size)
• Uniform	→	Normal (large sample size)
• Bimodal	→	Normal (larger sample size)
• Skewed	→	Normal (larger sample size)

The Sampling Distribution Model for a Mean

When a random sample is drawn from a population with mean μ and standard deviation σ, the sampling distribution has:

- Mean: μ
- Standard Deviation: $\frac{\sigma}{\sqrt{n}}$
- For large sample size, the distribution is approximately normal regardless of the population the random sample comes from.
- The larger the sample size, the closer to Normal.

Our questions:

- What's the population parameter?
  → Confidence Intervals
- Is my result likely?
  → Hypothesis testing
-

Where are we going?

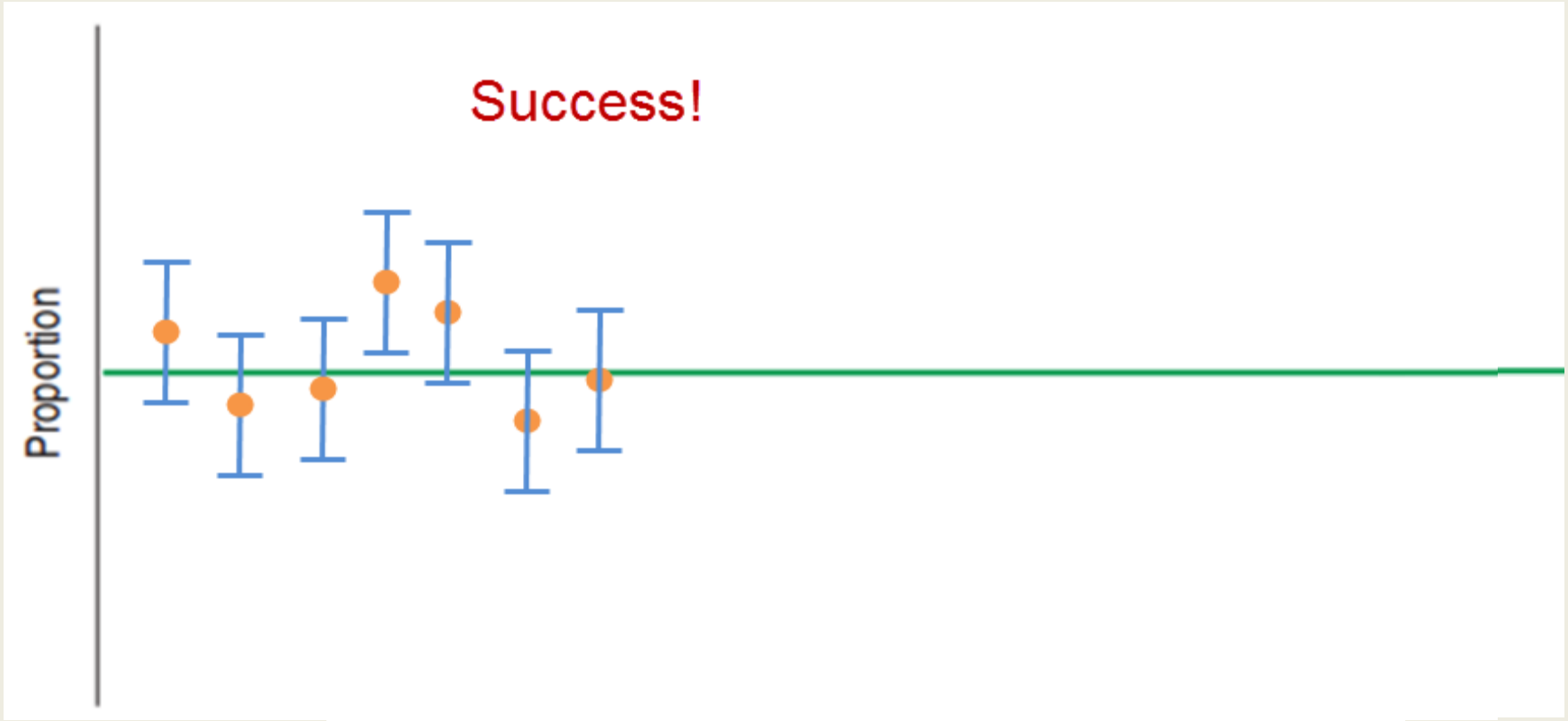
- Confidence intervals
- Hypothesis testing for proportions
- Hypothesis testing for means
- What if I have counts? → Chi-square tests
- What if I want to test an association? → hypothesis testing for regression
- What if I have more than two groups? → Analysis of Variance (ANOVA)

Confidence intervals for proportions

Capturing a Proportion

- The confidence interval may or may not contain the true population proportion.
- Consider repeating the study over an over again, each time with the same sample size.
  - Each time we would get a different $\hat{p}$.
  - From each $\hat{p}$, a different confidence interval could be computed.
  - About 95% of these confidence intervals will capture the true proportion.
  - 5% will not.

Simulating Confidence Intervals



Confidence Intervals

There are a huge number of confidence intervals that could be drawn.

- In theory, all the confidence intervals could be listed.
- 95% will "work" (capture the true proportion).
- 5% will not capture the true proportion.

What about our confidence interval (0.234, 0.382)?

- We will never know whether it captures the population proportion.

"Statistics Means Never Having to Say You Are Certain"

Margin of Error

- Confidence interval for a population proportion (95%):

$$\hat{p} \pm 2SE(\hat{p})$$

- The distance, $2SE(\hat{p})$, from $\hat{p}$ is called the margin of error.

- Confidence intervals also work for means, regression slopes, and others. In general, the confidence interval has the form

$$Estimate \pm ME$$

Critical Values

- For a 95% confidence interval, the margin of error was $2SE$.
  - The 2 comes from the normal curve.
  - 95% of the area is within about $2SE$ from the mean.

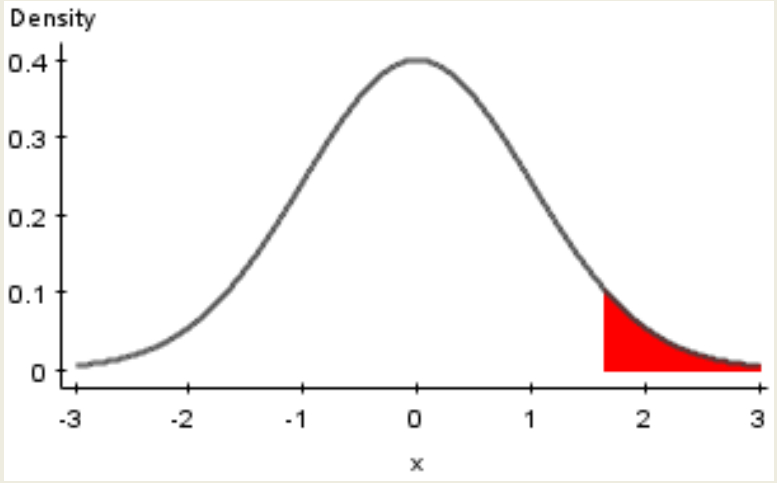
- In general the *number* of SE is called the critical value. Since we use the normal distribution here we denote it z^*
- To be more precise, z* for 95%CI is 1.96

Finding the Critical Value

Find the critical value corresponding to 90% confidence.

- 90% inside gives 10% outside.
- 2 tails outside with 10% means 1 tail with 5% or 0.05.

- The critical value is about $z^* = 1.645$.



Finding the Margin of Error (Take 2)

Yale/George Mason Poll: 1010 US adults, 40% think scientists disagree about global warming. At 95% confidence *ME* = 3%.

- Find the margin of error at 90% confidence.

Finding the Margin of Error (Take 2)

Yale/George Mason Poll: 1010 US adults, 40% think scientists disagree about global warming. At 95% confidence *ME* = 3%.

- Find the margin of error at 90% confidence.

$$SE(\hat{p}) = \sqrt{\frac{(0.4)(0.6)}{1010}} \approx 0.0154$$

- For 90%, $z^* \approx 1.645$: $ME = (1.645)(0.0154) = 0.025$.
- This gives a smaller margin of error which is *good*.
- **Drawback:** lower level of confidence which is *bad*

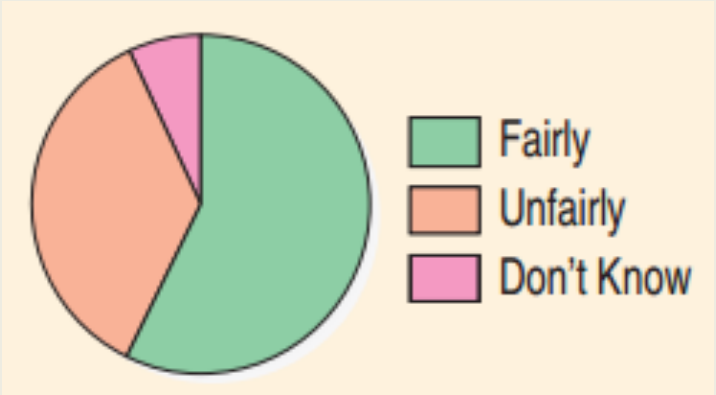
One-Proportion z-Interval

- First check for randomization, independence, 10%, and conditions on sample size.
- Confidence level C, sample size n, proportion $\hat{p}$.
- Confidence interval: $\hat{p} \pm z^* SE(\hat{p})$
- $SE(\hat{p}) = \sqrt{\frac{(\hat{p})(\hat{q})}{n}}$
- z^*: the critical value that specifies the number of SE's needed for C% of random samples to yield confidence intervals that capture the population proportion.

Do You Believe the Death Penalty is Applied Fairly?

- Sample size: 510

- Answers:
  - 58% “Fairly”
  - 36% “Unfairly”
  - 7% “Don’t Know”



- Construct a confidence interval for the population proportion that would reply “Fairly.”

Do You Believe the Death Penalty is Applied Fairly?

- **Plan:** Find a 95% confidence interval for the population proportion.
- **Model:**
  - ✓ Randomization: Randomly selected by Gallup Poll
  - ✓ 10% Condition: Population is all Americans
  - ✓ Success/Failure Condition
    - ✓ $(510)(0.58) = 296 \geq 10$, $(510)(0.42) = 214 \geq 10$
  - Use the Normal Model to find a one-proportion z-interval.

Do You Believe the Death Penalty is Applied Fairly?

- **Mechanics:** $n = 510$, $\hat{p} = 0.58$
-
-
-
-

Do You Believe the Death Penalty is Applied Fairly?

- **Mechanics:** $n = 510$, $\hat{p} = 0.58$
- $SE(\hat{p}) = \sqrt{\frac{(0.58)(0.42)}{510}} \approx 0.022$
- $z^* \approx 1.96$
- $ME \approx (1.96)(0.022) \approx 0.043$
- The 95% Confidence Interval is:
  0.58 ± 0.043 or $(0.537, 0.623)$

Do You Believe the Death Penalty is Applied Fairly?

- **Conclusion:** I am 95% confident that between 57.3% and 62.3% of all US adults think that the death penalty is applied fairly.

Inference about proportions

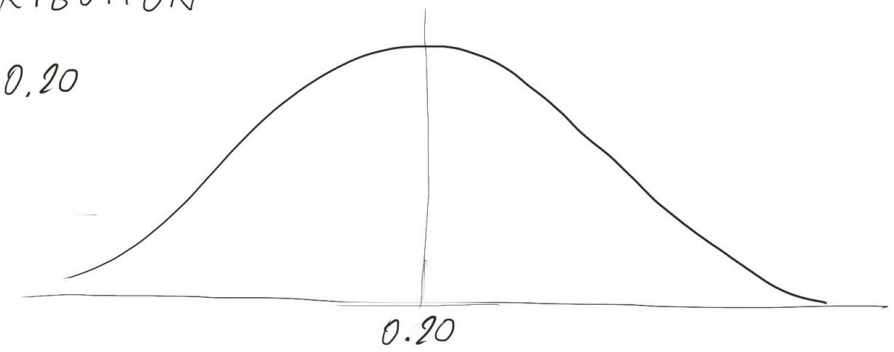
Cracking Rate < 20%?

General cracking rate: 20%
After a new engineering process, the cracking rate of 400 casts fell to 17%.
Is this due to the new engineering or just random chance?

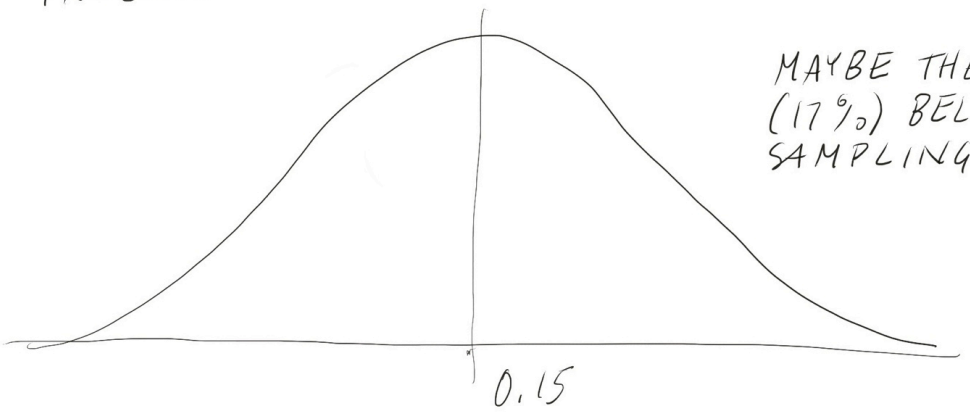
- Null Hypothesis: Nothing has changed
  - H_0: parameter = hypothesized value
  - H_0: $p = 0.20$
- Alternative Hypothesis:
  - H_A: $p < 0.20$

WE CAN THINK OF A SAMPLING DISTRIBUTION CENTERED AT $p=0.20$ ALL THE RANDOM FLUCTUATIONS FROM THIS PROPORTION WILL BELONG TO THIS DISTRIBUTION

$H_0: p = 0.20$

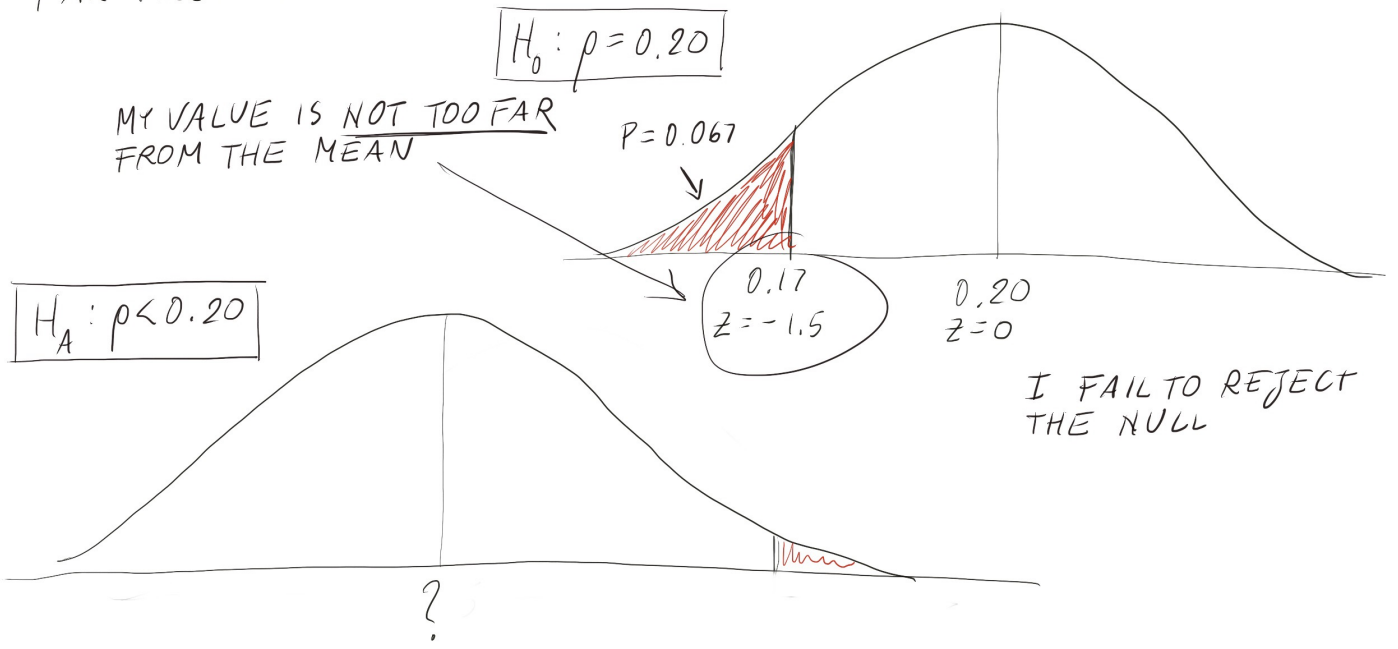


BUT MAYBE SOMETHING HAPPENED: THE DIFFERENCE BETWEEN THE SAMPLE PROPORTION $\hat{p}$ AND p IS DUE TO THE NEW ENGINEERING PROCESS



MAYBE THE SAMPLE PROPORITION (17%) BELONG TO A DIFFERENT SAMPLING DISTRIBUTION?

WHENEVER I RUN AN HYPOTHESIS TEST, I AM TRYING TO ANSWER THE QUESTION:
DOES THE SAMPLE PROPORTION (0.17) BELONG TO THE SAMPLING DISTRIBUTION SPECIFIED BY THE NULL HYPOTHESIS?
TO ANSWER THE QUESTION, I USE Z-SCORES TO SEE HOW FAR FROM THE MEAN THE SAMPLE PROPORTION IS.



How Small to Convince Us?

- Had the new cracking rate been 1%, it would clearly indicate a change from 20%.
  - Extremely unlikely that this could happen just by random chance
- Had the new cracking rate been 19.8%, we would be skeptical.
  - Not so unlikely to be just random chance
- How about 17%?
  - How likely is it that a random sample would have a cracking rate 17% or less?

Checking Conditions and Finding the Standard Error

Checking Conditions: $n = 400$, $p = 0.20$

- ✓ $np = (400)(0.20) = 80 \geq 10$
  $nq = (400)(0.80) = 320 \geq 10$
- ✓ Independence plausible
- ✓ The Normal model applies.

Find the standard deviation of the model.

- $$SD(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(0.20)(0.80)}{400}} = 0.02$$
- **Note:** Use p and not $\hat{p}$ to find standard deviation.

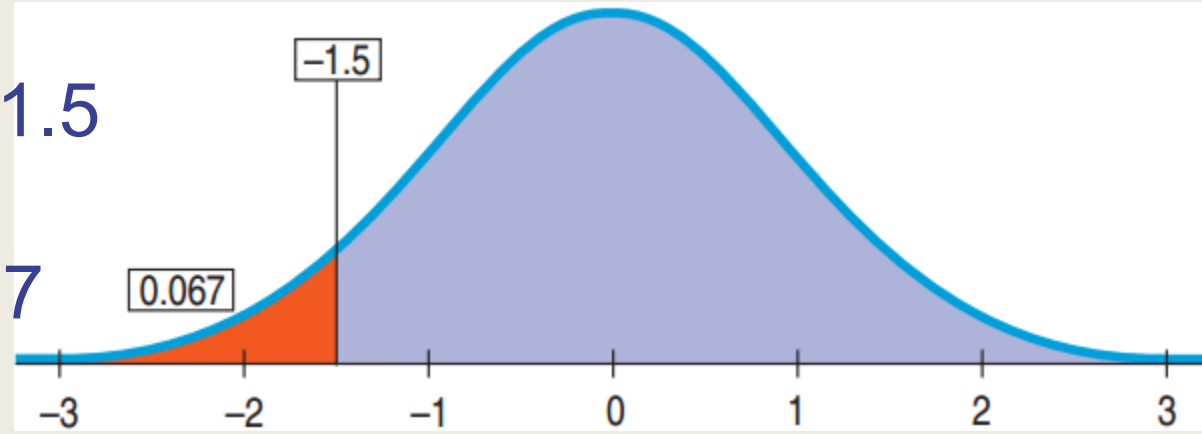
Using the Normal Model

$p = 0.20,\ \hat{p} = 0.17, SD(\hat{p}) = 0.02$

Using the Normal Model

$p = 0.20,\ \hat{p} = 0.17, SD(\hat{p}) = 0.02$

- $z = \dfrac{0.17 - 0.20}{0.02} = -1.5$
- $P(z < -1.5) \approx 0.067$



- If the null hypothesis is true that the cracking rate is still equal to 20%, then the probability of observing a cracking rate of 17% in a random sample of 400 is 6.7%.

When the P-Value is Not Small

Wrong

- Accept H_0.
- We have proven H_0.

Right

- **Fail to reject H_0**
- There is insufficient evidence to reject H_0.
- H_0 may or may not be true.

Example: H_0: All swans are white.

- If we sample 100 swans that are all white, there could still be a black swan.

Step 1: State the Hypotheses

H_0:

- H_0 usually states that there's nothing different.
- H_0: parameter = hypothesized value
- Note the parameter describes the population not the sample.
- H_0 is called the null hypothesis.

H_A:

- H_A is a statement that something has changed, gotten bigger or smaller or just different
- H_A is called the alterative hypothesis.

Hypotheses About the DMV

The DMV claims 80% of all drivers pass the driving test.
In a survey of 90 teens, only 61 passed.

- Is there evidence that teen pass rates are below 80%?
  - H_0: $p = 0.80$
  - H_A: $p < 0.80$

1-Proportion z-Test

Conditions

- Same as a 1-Proportion *z*-Interval

Null Hypothesis

- H_0: $p = p_0$

Test Statistic:

- $$z = \frac{\hat{p} - p_0}{SD(\hat{p})} \qquad SD(\hat{p}) = \sqrt{\frac{p_0 q_0}{n}}$$

DMV Study: Mechanics

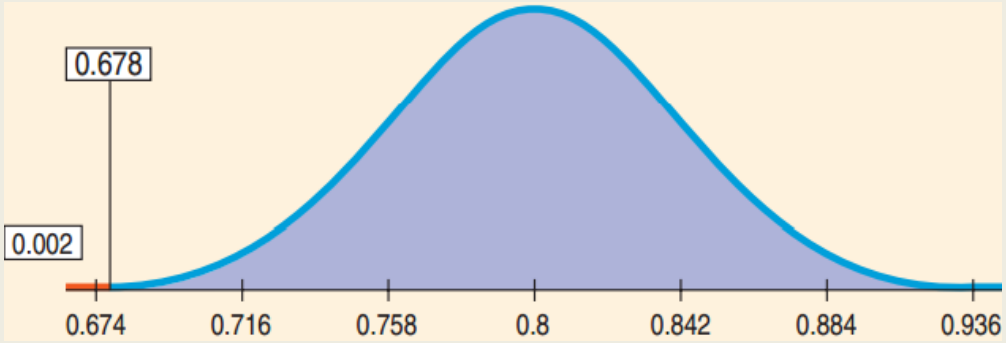
Claim: 80% pass. 61 of 90 teens tested passed.

- $n = 90$, $x = 61$, $p_0 = 0.80$,

DMV Study: Mechanics

Claim: 80% pass. 61 of 90 teens tested passed.

- Find P-value.
  - $n = 90$, $x = 61$, $p_0 = 0.80$, $\hat{p} = \dfrac{61}{90} \approx 0.678$
  - $SD(\hat{p}) = \sqrt{\dfrac{p_0 q_0}{n}} \approx \sqrt{\dfrac{(0.80)(0.20)}{90}} \approx 0.042$
  - $z = \dfrac{0.678 - 0.80}{0.042} \approx -2.90$



  - $\text{P-value} = P(z < -2.90) \approx 0.002$

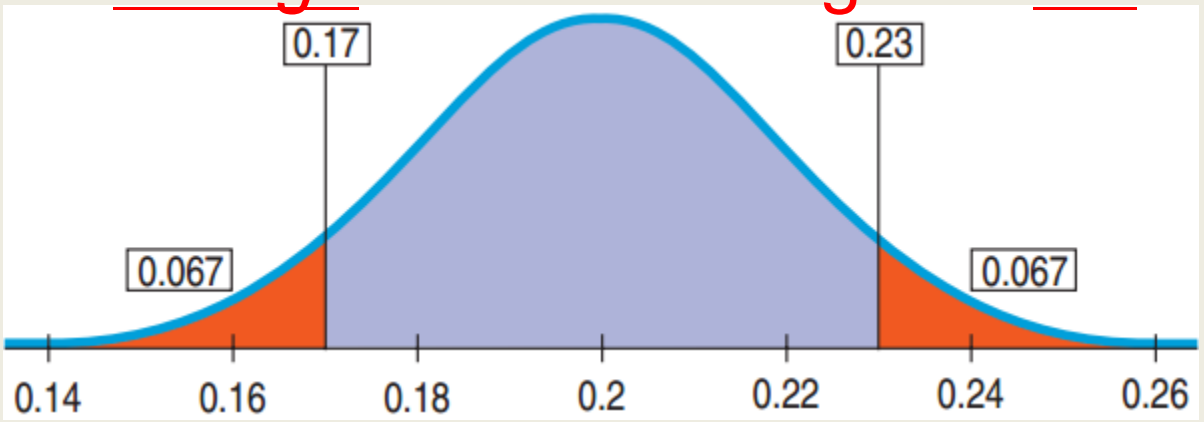
DMV Study: Conclusion

Is the teen pass rate less than 80%? P-value = 0.002

- What can be concluded? What does the P-value mean?
  - P-value = 0.002 is very small → Reject H_0
  - The survey data provide strong evidence that the pass rate for teens is less than 80%.
  - This should not be the end of the conversation.
  - The next step would be to see if the pass rate is low enough to take further action.

Two-Sided Alternative

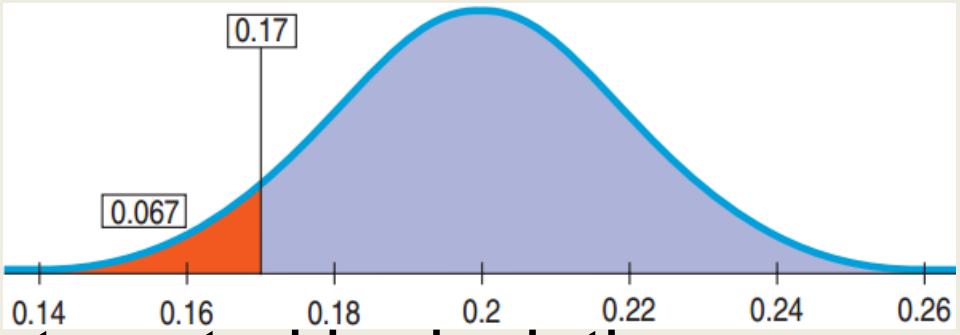
For the new process the engineer may be interested in whether there has been a *change* in the cracking rate not just a decrease.



- H_0: $p = 0.20$
- H_A: $p \neq 0.20$
- An alternative hypothesis where we are interested in deviation on either side is called a two-sided alternative.
- The P-value is the probability of deviating from *either* direction from the null hypothesis.

One-Sided Alternative

The engineer may be interested in whether there has been a decrease in the cracking rate.



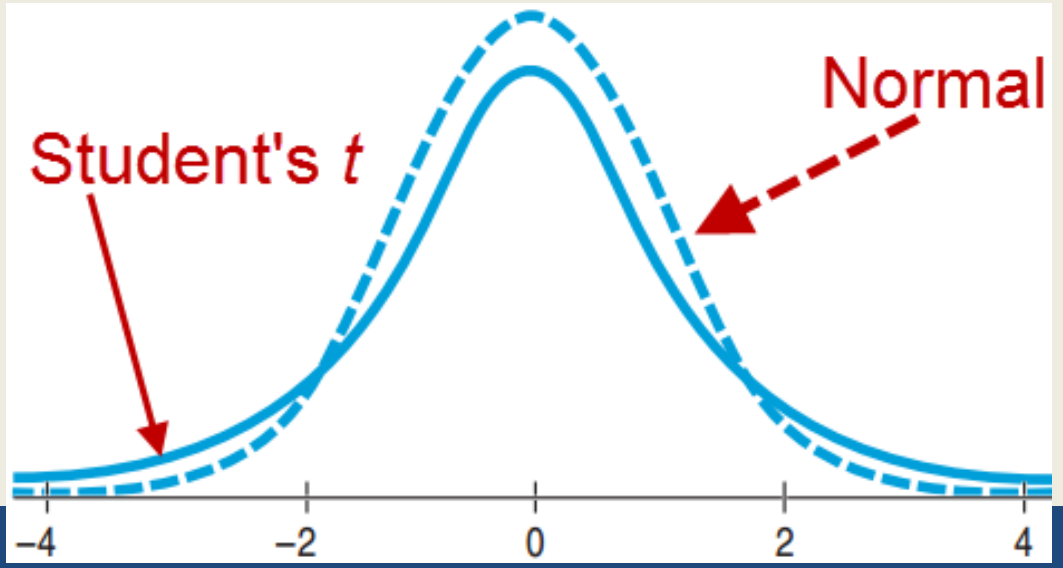
- H_0: $p = 0.20$
- H_A: $p < 0.20$
- An alternative hypothesis where we are interested in deviation on only one side is called a one-sided alternative.
- The P-value for a one-sided alternative is always half the P-value for the two-sided alternative.
- The P-value for a two-sided alternative is always double the P-value for the one-sided alternative.

Inference about means

Gosset at Guinness

At Guinness, Gosset experimented with beer.

- The Normal Model was not right <u>for small samples</u>.
- Right model still bell shaped, but details differed, depending on *n*
- Came up with the "Student's *t* Distribution" as the correct model



Degrees of Freedom

- For every sample size n there is a different Student's t distribution.
- Degrees of freedom: $df = n - 1$.
- Similar to the "$n - 1$" in the formula for sample standard deviation
- It is the number of independent quantities left after we've estimated the parameters.

Confidence Interval for Means

Sampling Distribution Model for Means

- With certain conditions (seen later), the standardized sample mean follows the Student's *t* model with $n - 1$ degrees of freedom.

$$t = \frac{\bar{y} - \mu}{SE(\bar{y})}$$

- We estimate the standard deviation with

$$SE(\bar{y}) = \frac{s}{\sqrt{n}}$$

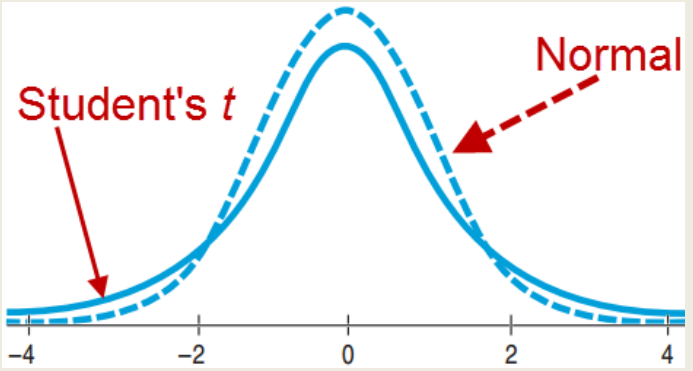
One Sample *t*-Interval for the Mean

- When the assumptions are met (seen later), the confidence interval for the mean is

$$\bar{y} \pm t^*_{n-1} \times SE(\bar{y})$$

- The critical value t^*_{n-1} depends on the confidence level, C, and the degrees of freedom $n - 1$.

Thoughts about *z* and *t*



- The Student's *t* distribution:
  - Is unimodal.
  - Is symmetric about its mean.
  - Has higher tails than Normal.
  - Is very close to Normal for large *df*.
  - Is needed because we are using *s* as an estimate for σ.

- If you happen to know σ, which almost never happens, use the Normal model and not Student's *t*.

One-Sample *t*-Test for the Mean

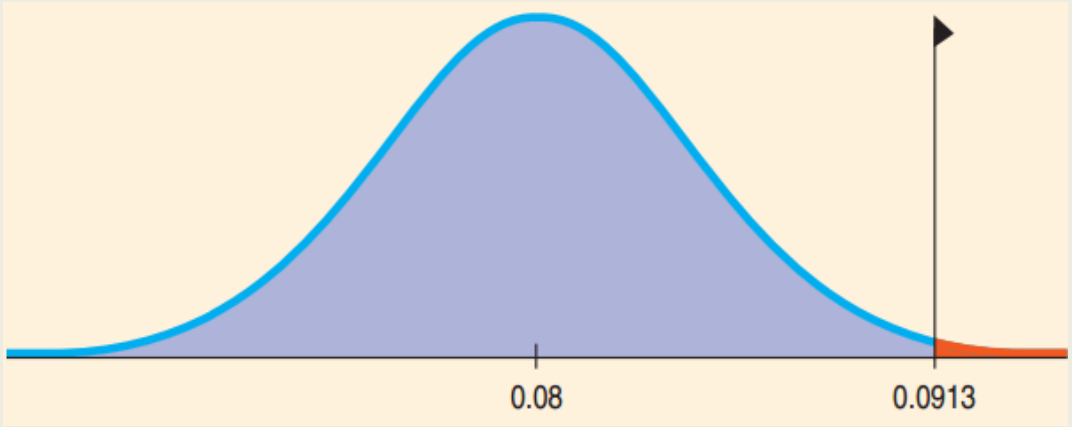
- Assumptions are the same.
- H_0: $\mu = \mu_0$
- $t_{n-1} = \dfrac{\bar{y} - \mu_0}{SE(\bar{y})}$
- Standard Error of $\bar{y}$: $SE(\bar{y}) = \dfrac{\sigma}{\sqrt{n}}$
- When the conditions are met and H_0 is true, the statistic follows the Student's *t* Model.
- Use this model to find the P-value.

Are the Salmon Unsafe?

EPA recommended mirex screening is 0.08 ppm.

- Are farmed salmon contaminated beyond the permitted EPA level?
- Recap: Sampled 150 salmon. Mean 0.0913 ppm, Standard Deviation 0.0495 ppm.

- H_0: $\mu = 0.08$
- H_A: $\mu > 0.08$



Are the Salmon Unsafe?

One-Sample *t*-Test for the Mean

- $n = 150$, $df = 149$, $\bar{y} = 0.0913$, $s = 0.0495$

Are the Salmon Unsafe?

One-Sample *t*-Test for the Mean

- $n = 150$, $df = 149$, $\bar{y} = 0.0913$, $s = 0.0495$
- $SE(\bar{y}) = \frac{0.0495}{\sqrt{150}} \approx 0.0040$, $t_{149} = \frac{0.0913 - 0.08}{0.0040} = 2.825$
- $P(t_{149} > 2.825) = 0.0027$
  or P < 0.005 using the t-table
- Since the P-value is so low, reject H_0 and conclude that the population mean mirex level does exceed the EPA screening value.

Intervals and Tests

Confidence Intervals

- Start with data and find plausible values for the parameter.
- Always 2-sided

Hypothesis Tests

- Start with a proposed parameter value and then use the data to see if that value is not plausible.

The Special Case with Proportions!

Confidence Intervals

- Use $\hat{p}$ to calculate $SE(\hat{p})$.

Hypothesis Tests

- Use p to calculate $SD(p)$.

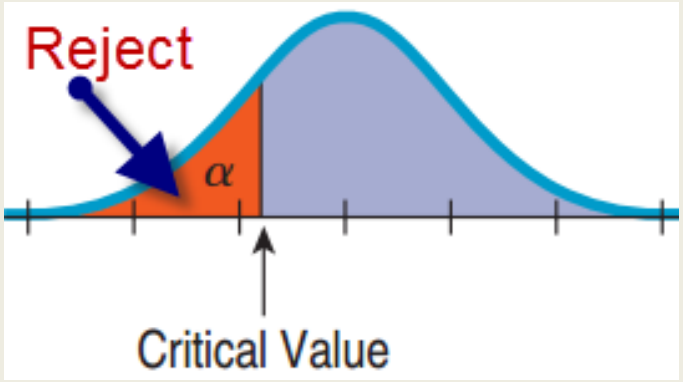
Considerations on α and type I & II errors

Choosing an α

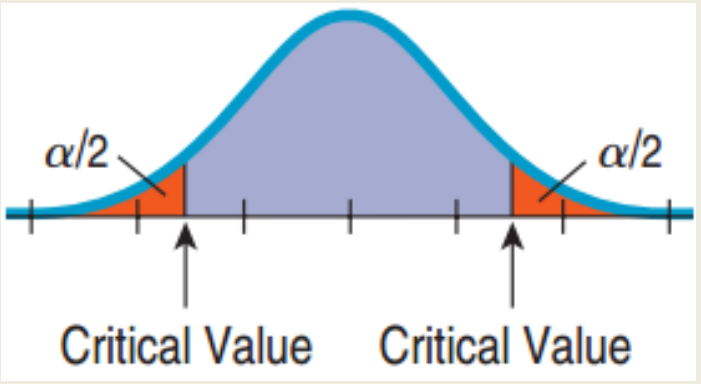
- $\alpha = 0.05$ is most common .
  - (1 in 20 chances is pretty rare)
- Other levels of significance commonly used:
  - 0.001, 0.01, 0.1
- Are the air bags safe?
  - Low α such as 0.001.
- Do students like pepperoni or sausage?
  - High α such as 0.1.

Critical Values: An Alternative to P-Values.

- For a hypothesis test look at the *z*-axis (or *t*-axis) to decide on whether to reject H_0.
- Find the value of *z* (or *t*) that corresponds to α.



- 2-tails: use $\alpha/2$



Confidence Intervals and Hypothesis Tests

- A confidence interval contains all plausible values.

- **Two Tailed Test:** Any hypothesized value outside will be rejected. $\alpha = 100 - C$.
  - $C = 95\% \rightarrow \alpha = 5\%$

- **One Sided Test:** $\alpha = \frac{1}{2}(100 - C)$
  - $C = 95\% \rightarrow \alpha = \frac{1}{2}(100 - 95) = 2.5\%$

Type I and II Errors

Type I Error

- Reject H_0 when H_0 is true.

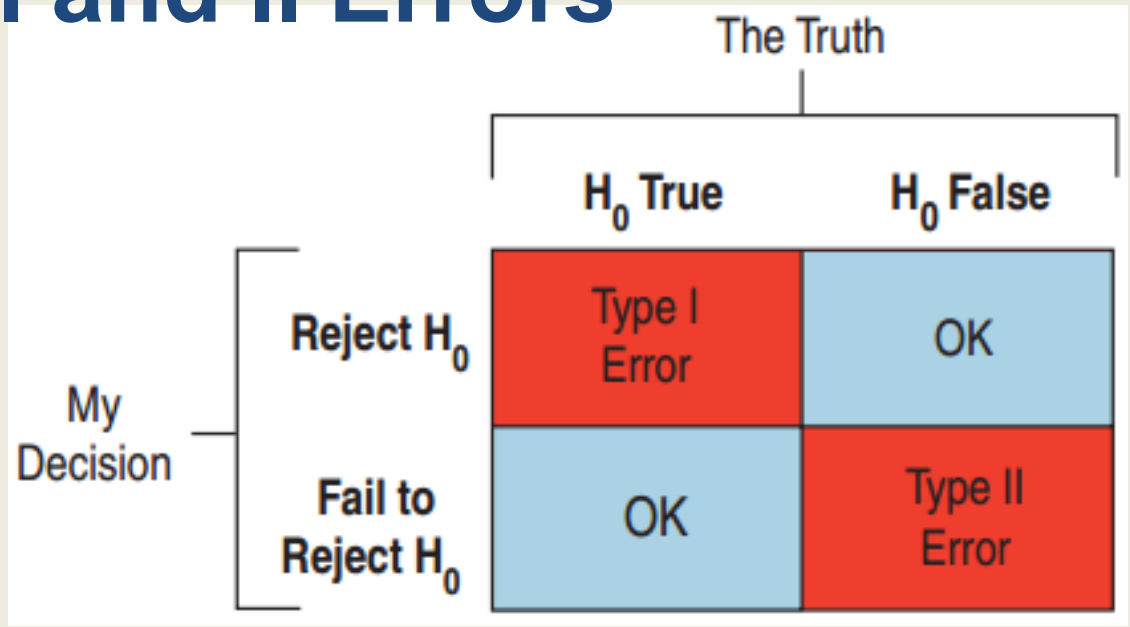
Type II Error

- Fail to reject H_0 when H_0 is false.

Medicine: Such as an AIDS test

- Type I Error → False positive: Healthy person is diagnosed with the disease.
- Type II Error → False negative: Infected person is diagnosed as disease free.

Type I and II Errors



Jury Decisions

- **Type I:** Found guilty when the defendant is innocent. Put an innocent person in jail.
- **Type II:** Not enough evidence to convict, but was guilty. A murderer goes free.

Inference for difference between proportions

Review: Two-Proportion *z*-Interval

If the conditions are met, the confidence interval for $p_1 - p_2$ is

$$\left(\hat{p}_1 - \hat{p}_2\right) \pm z^* \times SE\left(\hat{p}_1 - \hat{p}_2\right)$$

$$SE\left(\hat{p}_1 - \hat{p}_2\right) = \sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

- z^* is the critical value that corresponds to the confidence level C.

Review: Hypothesis Test - Pooling the Proportions

- H_0 comparing two proportions: proportions are equal.
- To do an hypothesis test, we *assume* that the proportions are equal
- We should have only *one* value for proportion
- Total successes: 205 + 235 = 440
- Total trials: 293 + 469 = 762
- Pooled Proportion:

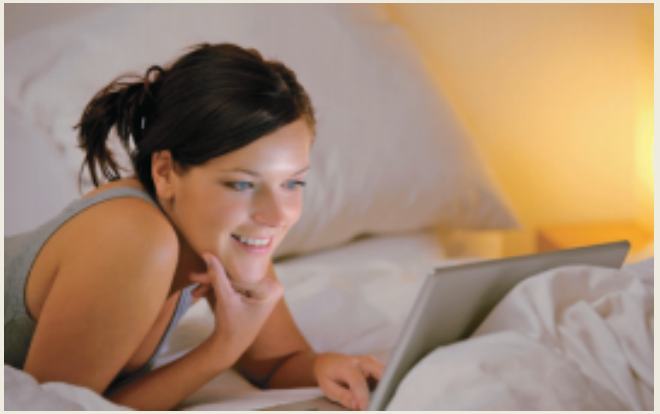
$$\hat{p}_{pooled} \frac{Success_1 + Success_2}{n_1 + n_2} = \frac{440}{762} \approx 0.5774$$

Two-Proportion z-Test

- Conditions same as 2-proportion CI.
- H_0: $p_1 - p_2 = 0$
- $\hat{p}_{pooled} = \dfrac{Success_1 + Success_2}{n_1 + n_2}$
- $SE_{pooled}\left(\hat{p}_1 - \hat{p}_2\right) = \sqrt{\dfrac{\hat{p}_{pooled}\hat{q}_{pooled}}{n_1} + \dfrac{\hat{p}_{pooled}\hat{q}_{pooled}}{n_2}}$
- $z = \dfrac{\hat{p}_1 - \hat{p}_2 - 0}{SE_{pooled}\left(\hat{p}_1 - \hat{p}_2\right)}$
- The statistic follows the Normal model.

Pre-Sleep Internet Rates Differ?

- **Plan:** Random sample of adults:
- 293 Gen Y (19-29 years old)
- 469 Gen X (30-45 years old)
- Percentage reporting using internet before sleep
- **Hypotheses:**
  - H_0: $p_{GenY} - p_{GenX} = 0$
  - H_A: $p_{GenY} - p_{GenX} \neq 0$



- **Model**
  - ✓ **Randomization Condition:** Randomly selected and stratified by sex.

Pre-Sleep Internet Rates Differ?

- **Model** (Continued)
  - ✓ **10% Condition:** Samples less than 10% of all Gen X and Gen Y.
  - ✓ **Independent Groups Assumption:** The samples were selected at random so independent.
  - ✓ **Success/Failure Condition:** Observed numbers of successes and failures for both groups ≥ 10.
- The conditions are all met. Use the Normal model and perform a two-proportion *z*-test.

Pre-Sleep Internet Rates Differ?

- **Mechanics:**

$$n_{GenY} = 293,\ y_{GenY} = 205,\ \hat{p}_{GenY} = 0.700$$
$$n_{GenX} = 469,\ y_{GenX} = 235,\ \hat{p}_{GenX} = 0.501$$

Pre-Sleep Internet Rates Differ?

- **Mechanics:**

$$n_{GenY} = 293,\ y_{GenY} = 205,\ \hat{p}_{GenY} = 0.700$$

$$n_{GenX} = 469,\ y_{GenX} = 235,\ \hat{p}_{GenX} = 0.501$$

$$\hat{p}_{pooled} = \frac{y_{GenY} + y_{GenX}}{n_{GenY} + n_{GenX}} = \frac{205 + 235}{293 + 469} \approx 0.5774$$

$$SE_{Pooled}\left(\hat{p}_{GenY} - \hat{p}_{GenX}\right) = \sqrt{\frac{\hat{p}_{pooled}\left(1 - \hat{p}_{pooled}\right)}{n_{GenY}} + \frac{\hat{p}_{pooled}\left(1 - \hat{p}_{pooled}\right)}{n_{GenX}}}$$

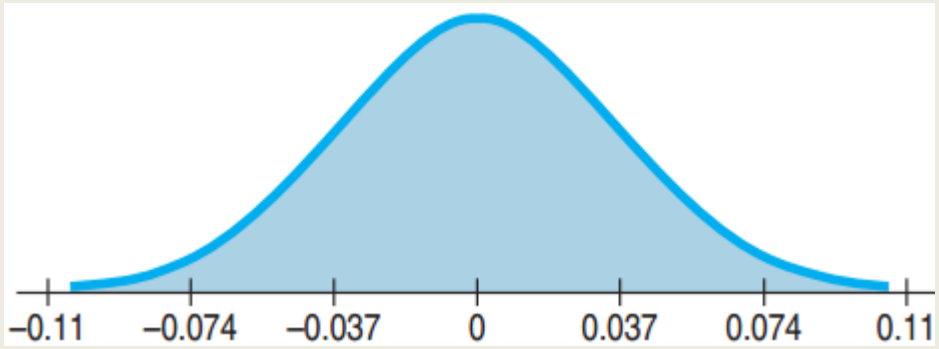
$$= \sqrt{\frac{(0.5774)(0.4226)}{293} + \frac{(0.5774)(0.4226)}{469}} \approx 0.0368$$

Pre-Sleep Internet Rates Differ?

- **Mechanics:** (Continued)

$$\hat{p}_{GenY} - \hat{p}_{GenX} =$$

$$z =$$



- P-value =

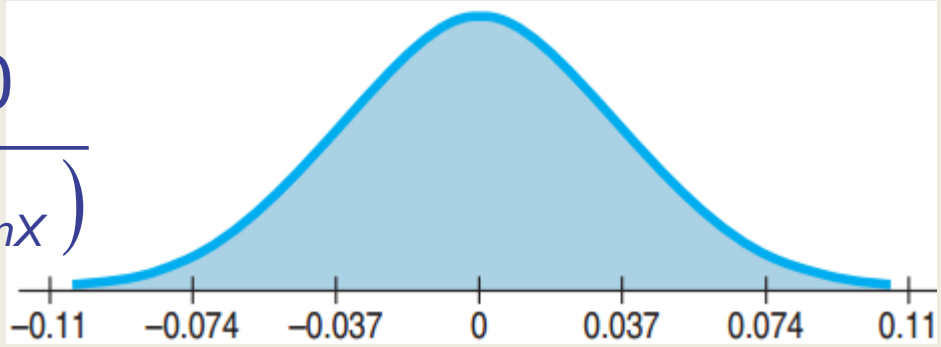
Pre-Sleep Internet Rates Differ?

- **Mechanics:** (Continued)

$$\hat{p}_{GenY} - \hat{p}_{GenX} = 0.700 - 0.501 = 0.199$$

$$z = \frac{\left(\hat{p}_{GenY} - \hat{p}_{GenX}\right) - 0}{SE_{pooled}\left(\hat{p}_{GenY} - \hat{p}_{GenX}\right)}$$

$$= \frac{0.199}{0.0368} \approx 5.41$$



- P-value = $2 \times P(z > 5.41) \leq 0.0001$

Pre-Sleep Internet Rates Differ?

- **Conclusion:**
  - P-value ≤ 0.0001: If there really was no difference in surfing rates between the two groups, then the difference observed in this study would be very rare indeed.
  - We can conclude that there is, in fact, a difference in the rate of surfing between GenY and GenX adults.

Inference for difference between means

Sampling Distribution for the Difference Between Two Means

- When the assumptions are met, the sampling distribution for the difference between two independent means:

$$t = \frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{SE(\bar{y}_1 - \bar{y}_2)} \qquad SE(\bar{y}_1 - \bar{y}_2) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

- Uses the Student's *t*-model
- The degrees of freedom are complicated, so just use a computer.

Two-Sample *t*-Interval for the Difference Between Two Means

- When the conditions are met, the confidence interval for the difference between means from two independent groups is

$$\left(\bar{y}_1 - \bar{y}_2\right) \pm t^*_{df} \times SE\left(\bar{y}_1 - \bar{y}_2\right)$$

$$SE(\bar{y}_1 - \bar{y}_2) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Two-Sample *t*-Test for the Difference Between Means

- Conditions same as two-sample *t*-interval
- H_0: $\mu_1 - \mu_2 = \Delta_0$ (Δ_0 **usually 0**)
- $$t = \frac{(\bar{y}_1 - \bar{y}_2) - \Delta_0}{SE(\bar{y}_1 - \bar{y}_2)} \qquad SE(\bar{y}_1 - \bar{y}_2) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$
- When the conditions are met and the null hypothesis is true, use the **Student's *t*-model** to find the **P-value**.

Do People Offer a Lower Price to Friends than to Strangers?

Is there a difference in the average price for a used camera people would offer a friend and a stranger?

- **Plan:** I have bid prices from 8 subjects buying from a friend and 7 buying from a stranger, found in a randomized experiment.
- **Hypotheses**
  - H_0: $\mu_F - \mu_S = 0$
  - H_A: $\mu_F - \mu_S \neq 0$

Friends vs. Strangers

- **Mechanics:**

$$n_F = 8,\ n_S = 7$$

$$\bar{y}_F = \$281.88,\ \bar{y}_S = \$211.43$$

$$s_F = \$18.31,\ s_S = \$46.43$$

Friends vs. Strangers

- **Mechanics:**

$$n_F = 8,\ n_S = 7$$

$$\bar{y}_F = \$281.88,\ \bar{y}_S = \$211.43$$

$$s_F = \$18.31,\ s_S = \$46.43$$

$$SE(\bar{y}_F - \bar{y}_S) = \sqrt{\frac{18.31^2}{8} + \frac{46.43^2}{7}} \approx 18.70$$

$$\bar{y}_F - \bar{y}_S = 281.88 - 211.43 = \$70.45$$

$$t = \frac{70.45}{18.70} \approx 3.77$$

$$\text{P-value} = 2 \times P\left(t > 3.77\right) = 0.006$$

Friends vs. Strangers

- **Conclusion:** The P-value = 0.006 is very small.
- If there were no difference in the mean prices, then a difference this large would occur 6 times in 1000.
- Too rare to believe
- Reject H_0.
- Conclude that people are likely, on average, to offer a friend more than they'd offer a stranger for a used camera.

Paired samples

Pairing

Inner Lane		Outer Lane	
Name	**Time**	**Name**	**Time**
ZHANG Xiaolei	125.75	NEMOTO Nami	122.34
ABRAMOVA Yekaterina	121.63	LAMB Maria	122.12
REMPEL Shannon	122.24	NOH Seon Yeong	123.35
LEE Ju-Youn	120.85	TIMMER Marianne	120.45
ROKITA Anna Natalia	122.19	MARRA Adelia	123.07
YAKSHINA Valentina	122.15	OPITZ Lucille	122.75

- Speed-skating races run in pairs. One starts in the inner lane, the other in the outer lane.
- Above are some of the randomly assigned pairs.
- The data are paired rather than independent.
- Blocking involves pairing arising from an experiment.
- Matching involves pairing arising from an observational study.
- With pairing, we look at the differences.

Differences for Speed-Skater Pairs

- For paired data, create a new data set of the differences.
- We can now look only at the differences.
- Ignoring the original data, we now have a single data set.

Skating Pair	Inner Time	Outer Time	Inner – Outer
1	125.75	122.34	3.41
2	121.63	122.12	−0.49
3	122.24	123.35	−1.11
4	120.85	120.45	0.40
5	122.19	123.07	−0.88
6	122.15	122.75	−0.60

- Proceed with a one-sample *t*-test. This process is called a paired *t*-test.
- Only use pairing if there is a natural matching

The Paired *t*-Test

When the conditions are met, we can test whether the mean differences significantly differ from 0.

- H_0: $\mu_d = \Delta_0$ (Δ_0 is usually 0)
- $t = \dfrac{\bar{d} - \Delta_0}{SE(\bar{d})}$ $\qquad SE(\bar{d}) = \dfrac{s_d}{\sqrt{n}}$
- $\bar{d}$ and s_d are the mean and standard deviation of the pairwise differences and n is the number of pairs.
- Use the Student's *t*-model with $n - 1$ degrees of freedom and find the P-value.

Speed-Skating Comparisons

Was there a difference between speeds between inner and outer speed-skating lanes?

- **Plan:** I have data for 17 pairs of racers.
- **Hypotheses:**
  - H_0: $\mu_d = 0$
  - H_A: $\mu_d \neq 0$

Speed-Skating Comparisons

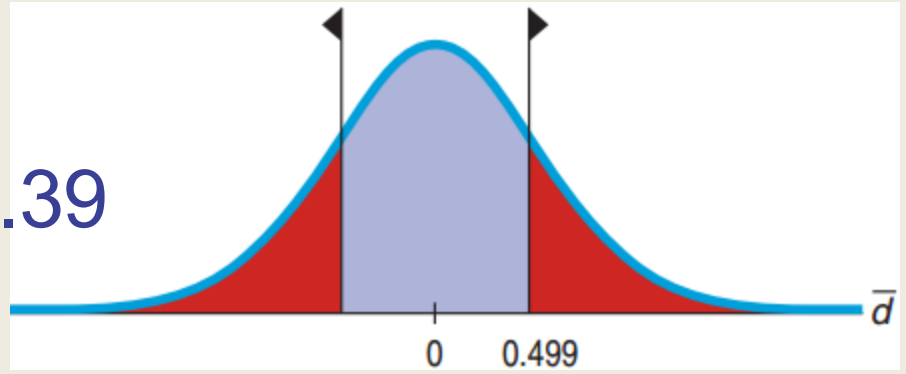
- **Mechanics:**

$n = 17$ pairs, $\bar{d} = 0.499$ sec., $s_d = 2.333$ sec.

$$SE(\bar{d}) = \frac{s_d}{\sqrt{n}} = \frac{2.333}{\sqrt{17}} = 0.5658$$

$$t_{16} = \frac{\bar{d} - 0}{SE(\bar{d})} = \frac{0.499}{0.5658} \approx 0.882$$

$$\text{P-value} = 2P(t_{16} > 0.882) = 0.39$$



Speed-Skating Comparisons

- **Conclusion:**
  - Events that occur more than a third of the time (P-value = 0.39) are not remarkable.
  - I can't conclude that the observed difference isn't due simply to random chance. It appears the fans may have interpreted a random fluctuation in the data as favoring one lane. There's insufficient evidence to declare any lack of fairness.

Paired *t*-interval

When the conditions are met, the confidence interval for the mean paired difference is

$$\bar{d} \pm t^*_{n-1} \times SE\left(\bar{d}\right) \qquad SE\left(\bar{d}\right) = \frac{s_d}{\sqrt{n}}$$

- The critical value t^* from the Student's *t*-model depends on the confidence level C and df = $n - 1$. (n is the number of pairs.)

Inference for counts

Review: Chi-Square test for count data

- A single basis of classification: **Goodness of fit**
- df: N-1, where n is the number of categories
- I have a model and I am checking it's validity
- Two variables: **Homogeneity**
- The counts are the results of some process. Are the counts the same for every cell?
- Two variables: **Independence**
- If factors independent I should have same proportion across cells

Example 1
Are CEO Zodiac Signs Uniform?

- A survey of 256 randomly selected CEOs Zodiac signs
- If uniform, each sign expected to have 256/12 ≈ 21.3 births

Births	Sign	Births	Sign
23	Aries	18	Libra
20	Taurus	21	Scorpio
18	Gemini	19	Sagittarius
23	Cancer	22	Capricorn
20	Leo	24	Aquarius
19	Virgo	29	Pisces

- Pisces has more.
- Others have fewer.
- Is the distribution far enough from uniform to conclude the data do not come from a uniform distribution?

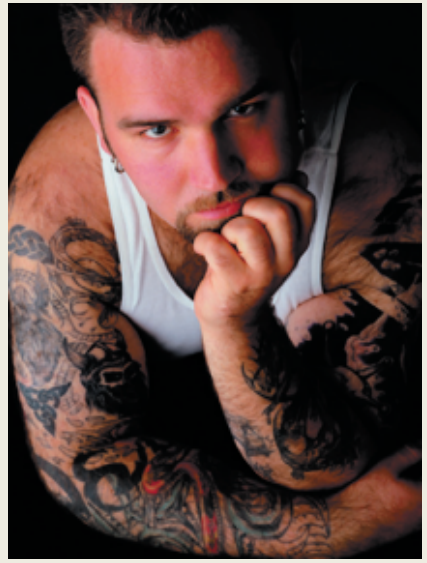
Example 2
Activities of Graduates at Different Colleges

	Agriculture	Arts & Sciences	Engineering	ILR	Total
Employed	209	198	177	101	**685**
Grad School	104	171	158	33	**466**
Other	135	115	39	16	**305**
Total	**448**	**484**	**374**	**150**	**1456**

- Are graduates' activities **the same** at different colleges within the same university?

Example 3
Tattoos and Hepatitis

Are *Tattoo Status* and *Hepatitis Status* independent?



- Plan: Test for independence. I have a contingency table of 626 patients.
- Hypotheses:
  - H_0: *Tattoo Status* and *Hepatitis Status* are independent.
  - H_A: *Tattoo Status* and *Hepatitis Status* are not independent.

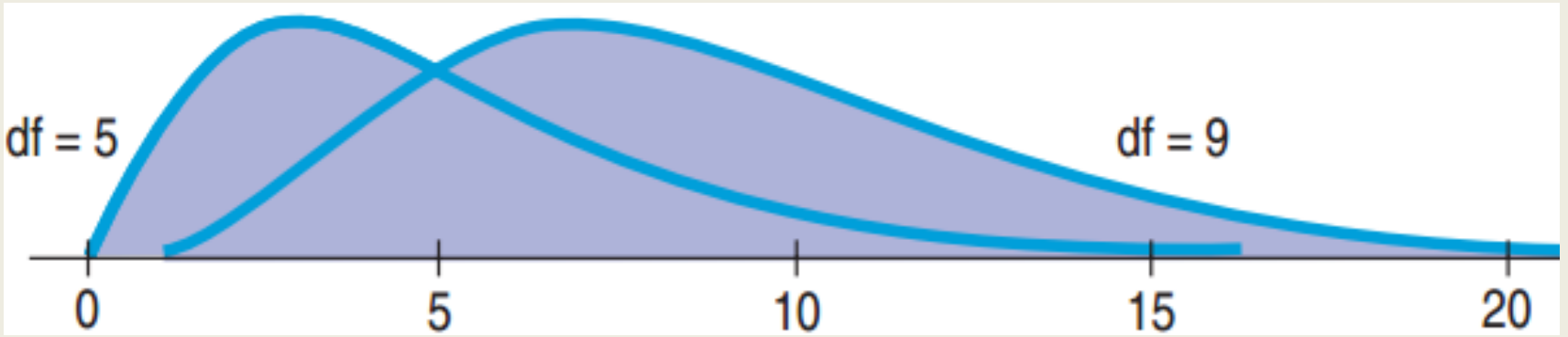
Chi-Square Calculations

- Interested in the differences between observed and expected counts: called residuals.
- Make positive by squaring them all.
- Get relative sizes of the residuals by dividing them by the expected counts.

$$\chi^2 = \sum \frac{(Obs - Exp)^2}{Exp}$$

- This is a Chi-Square Model with df = $n - 1$.
- n is the number of categories, not the sample size.

Why so big? The Shape of χ^2



- With df large (more cells), the weighted residuals add up quickly → numerator gets bigger
- Unlike *z* or *t*, a larger χ^2 is more common. It all depends on the df: look at value 10
- The mode of χ^2 is df – 2.
- The expected value of χ^2 is df, to the right of the mode due to the right skewed distribution.
- The CEO curve had df=11, peaks at 9 and mean=11

$$\chi^2 = \sum \frac{(Obs - Exp)^2}{Exp}$$

Inference for regression

Body Fat and Waist Size

- **Mechanics:** Computer Output:

Dependent variable is %BF
R-squared = 67.8%
s = 4.713 with 250 − 2 = 248 degrees of freedom

Variable	Coeff	SE(Coeff)	t-Ratio	P-Value
Intercept	−42.734	2.717	−15.7	<0.0001
Waist	1.70	0.0743	22.9	<0.0001

- The estimated regression equation is:

$$\widehat{\%Body\ Fat} = -42.73 + 1.70\ Waist$$

Testing for β_1

- With no linear association, $\beta_1 = 0$
- H_0: $\beta_1 = 0$
- $t_{n-2} = \dfrac{b_1 - 0}{SE(b_1)}$

Dependent variable is %BF
R-squared = 67.8%
s = 4.713 with 250 − 2 = 248 degrees of freedom

Variable	Coeff	SE(Coeff)	t-Ratio	P-Value
Intercept	−42.734	2.717	−15.7	<0.0001
Waist	1.70	0.0743	22.9	<0.0001

- For the *%Body Fat* and *Waist* data:

$$\frac{1.7 - 0}{0.0743} \approx 22.9 \qquad \text{P-value} < 0.0001$$

- Very unlikely to have such a high b_1 if $\beta_1 = 0$

Confidence Interval for β_1

- The hypothesis test for *%Body Fat* and *Weight* told us what we already know.
- A confidence interval is needed.

$$b_1 \pm t^*_{n-2} \times SE(b_1)$$

- For %Body Fat and Weight:

$$1.7 \pm 1.97 \times 0.074 = (1.55\%, 1.85\%)$$

- With 95% confidence the slope of the line for *%Body Fat* and *Weight* is between 1.55% and 1.85%.

Analysis of Variance

Back to Bacteria

Level	*n*	Mean	Std Dev	Variance
Alcohol Spray	8	37.5	26.56	705.43
Antibacterial Soap	8	92.5	41.96	1760.64
Soap	8	106.0	46.96	2205.24
Water	8	117.0	31.13	969.08

- $MS_T = 9960.64$, $MS_E = 1410.14$
- $F = \dfrac{MS_T}{MS_E} = \dfrac{9960.64}{1410.14} \approx 7.06$
- Num df = 4 – 1 = 3, Den df = 4(8 – 1) = 28.
- Technology gives P-value for $F_{3,28} = 0.0011$.

Analysis of Variance Table

Source	Sum of Squares	DF	Mean Square	F-ratio	P-value
Method	29882	3	9960.64	7.0636	0.0011
Error	39484	28	1410.14		
Total	69366	31			

Example: Two-Factor ANOVA with Interaction

Step-By-Step

Plan

Show the ANOVAtable.

Analysis of Variance for gpa

Source	DF	Sum of Squares	Mean Square	F-ratio	P-value
Sex	1	0.3040	0.3040	1.7681	0.1852
Varsity	1	2.4345	2.4345	14.1871	0.0002
Sex × Varsity	1	1.0678	1.0678	6.2226	0.0134
Error	196	33.6397	0.1716		
Total	199	37.4898			

This experiment is modeled after one reported by Laird (1974) . The experiment asks whether an induced smile actually has the effect of making a person feel happier. In the experimental condition, participants were instructed to move facial muscles to mimic a smile while looking at a picture of children playing. In a second control condition, participants simply looked at the picture with a neutral facial expression. No mention was made of smiling to any of the participants; participants believed the purpose of the experiment was to evaluate the effect of facial muscles on perception. A total of 40 volunteers participated in the experiment. They were assigned randomly to the two conditions, 20 participants in each. The response measure was a "happiness score" obtained by administering a mood questionnaire.

Calculate the 95% and the 99% confidence intervals.

-

Data Set 6.2 Happiness scores from the induced happiness experiment

Control condition: $n = 20, \overline{Y}_1 = 12.0, SS_Y = 686$

2	3	20	10	15	8	14	6	11	17
18	5	11	5	16	13	12	24	19	11

Experimental condition: $n = 20, \overline{Y}_2 = 16.0, SS_Y = 500$

15	10	14	19	16	26	23	12	20	13
17	3	13	17	23	14	20	15	13	17

- Suppose a simple random sample of 150 students is drawn from a population of 3000 college students. Among sampled students, the average IQ score is 115 with a standard deviation of 10. What is the 99% confidence interval for the students' IQ score?

For boys, the average number of absences in the first grade is 15 with a standard deviation of 7; for girls, the average number of absences is 10 with a standard deviation of 6.

In a nationwide survey, suppose 100 boys and 50 girls are sampled. What is the probability that the male sample will have *at most* three more days of absences than the female sample?