Quantitative Methods

Serena DeStefani – Lecture 20 –8/6/2020

Announcements

- HW8 due today
- HW9 due Monday

- Today: Analysis of Variance (ANOVA)
- After ANOVA → general review (Mon and Tue)
- Next Wednesday: Final exam

Announcements: Final Exam

- Non-cumulative, covers CH 17- 28 (no CH 15/16)
- for CH 26, 27,28, focus on material covered in slides
- Same format as HWs, focus on inference

Announcements: Final Exam

- Inference problems (can be either hypothesis tests or confidence intervals) on:

1. Proportions (one-sample / two-samples)
2. Means (one-sample / two-samples / paired data)
3. Regression

- one chi-square test
- questions about Analysis of Variance
- (I will give you the ANOVA table)
- A few conceptual questions incorporated into the problems
- Total of 40/45 questions

Review

Inference about?	One sample or two?	Procedure	Model	Parameter	Estimate	SE	Chapter
Proportions	One sample	1-Proportion z-Interval	z	p	$\hat{p}$	$\sqrt{\frac{\hat{p}\hat{q}}{n}}$	19
		1-Proportion z-Test				$\sqrt{\frac{p_0 q_0}{n}}$	20, 21
	Two independent groups	2-Proportion z-Interval	z	$p_1 - p_2$	$\hat{p}_1 - \hat{p}_2$	$\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$	22
		2-Proportion z-Test				$\sqrt{\frac{\hat{p}\hat{q}}{n_1} + \frac{\hat{p}\hat{q}}{n_2}}, \hat{p} = \frac{y_1 + y_2}{n_1 + n_2}$	22
Means	One sample	t-Interval t-Test	t df $= n - 1$	μ	$\bar{y}$	$\frac{s}{\sqrt{n}}$	23
	Two independent groups	2-Sample t-Test 2-Sample t-Interval	t df from technology	$\mu_1 - \mu_2$	$\bar{y}_1 - \bar{y}_2$	$\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$	24
	n Matched pairs	Paired t-Test Paired t-Interval	t df $= n - 1$	μ_d	$\bar{d}$	$\frac{s_d}{\sqrt{n}}$	25

Would being part of a support group that meets regularly help people who are wearing the nicotine patch actually quit smoking? A county health department tries an experiment using several hundred volunteers who were planning to use the patch to help them quit smoking. The subjects were randomly divided into two groups. People in Group 1 were given the patch and attended a weekly discussion meeting with counselors and others trying to quit. People in Group 2 also used the patch but did not participate in the counseling groups. After six months 46 of the 143 smokers in Group 1 and 30 of 151 smokers in Group 2 had successfully stopped smoking.

Do these results suggest that such support groups could be an effective way to help people stop smoking?

Inference about?
One sample or two?
Procedure?
Model?
Parameter?
Estimate?
SE?

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Do these results suggest that such support groups could be an effective way to help people stop smoking?

Now that we've concluded the support program is beneficial, can we convince the government to fund it? That might depend on *how* effective it is.

Inference about?

One sample or two?

Procedure?

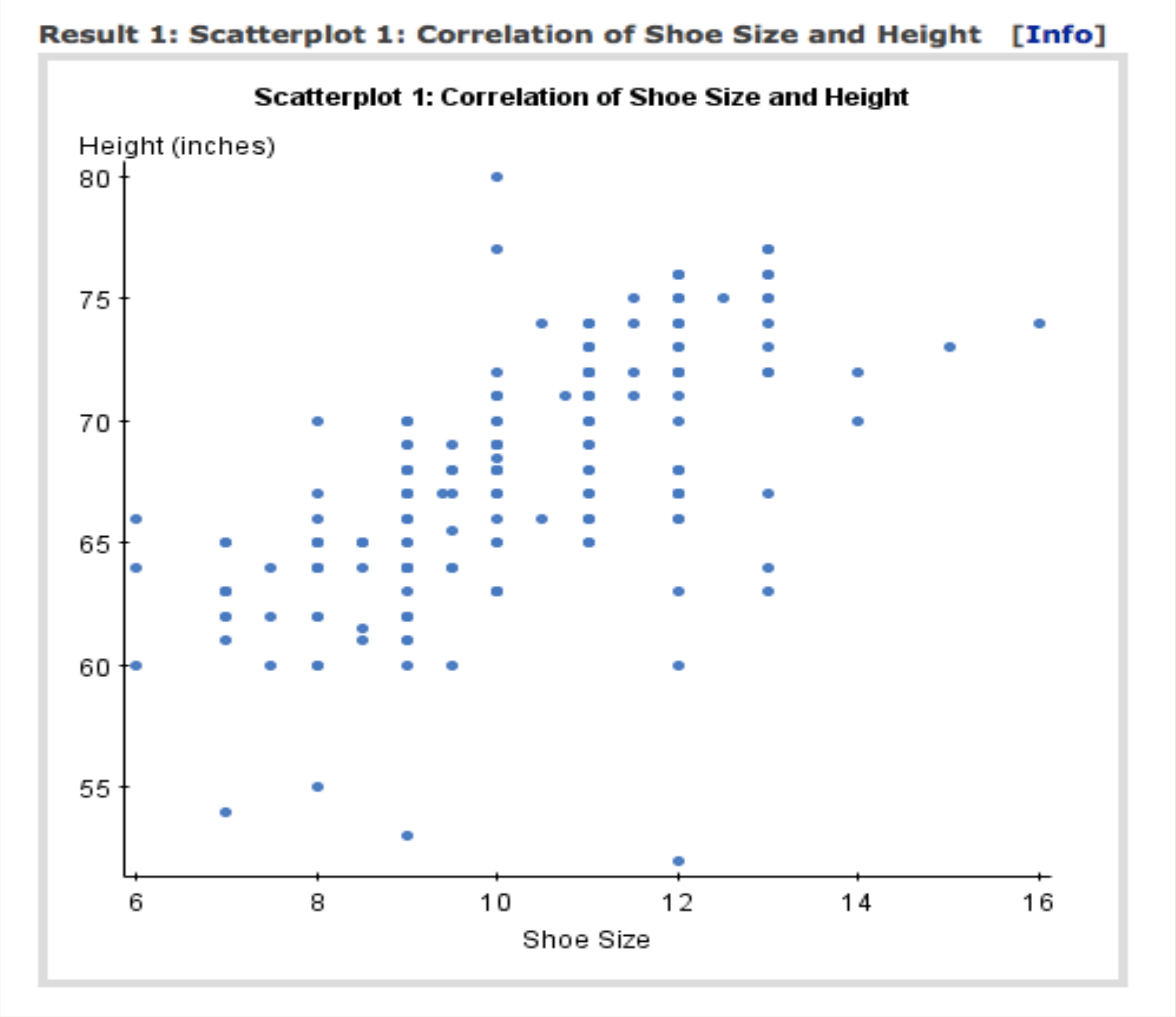
Model?

Parameter?

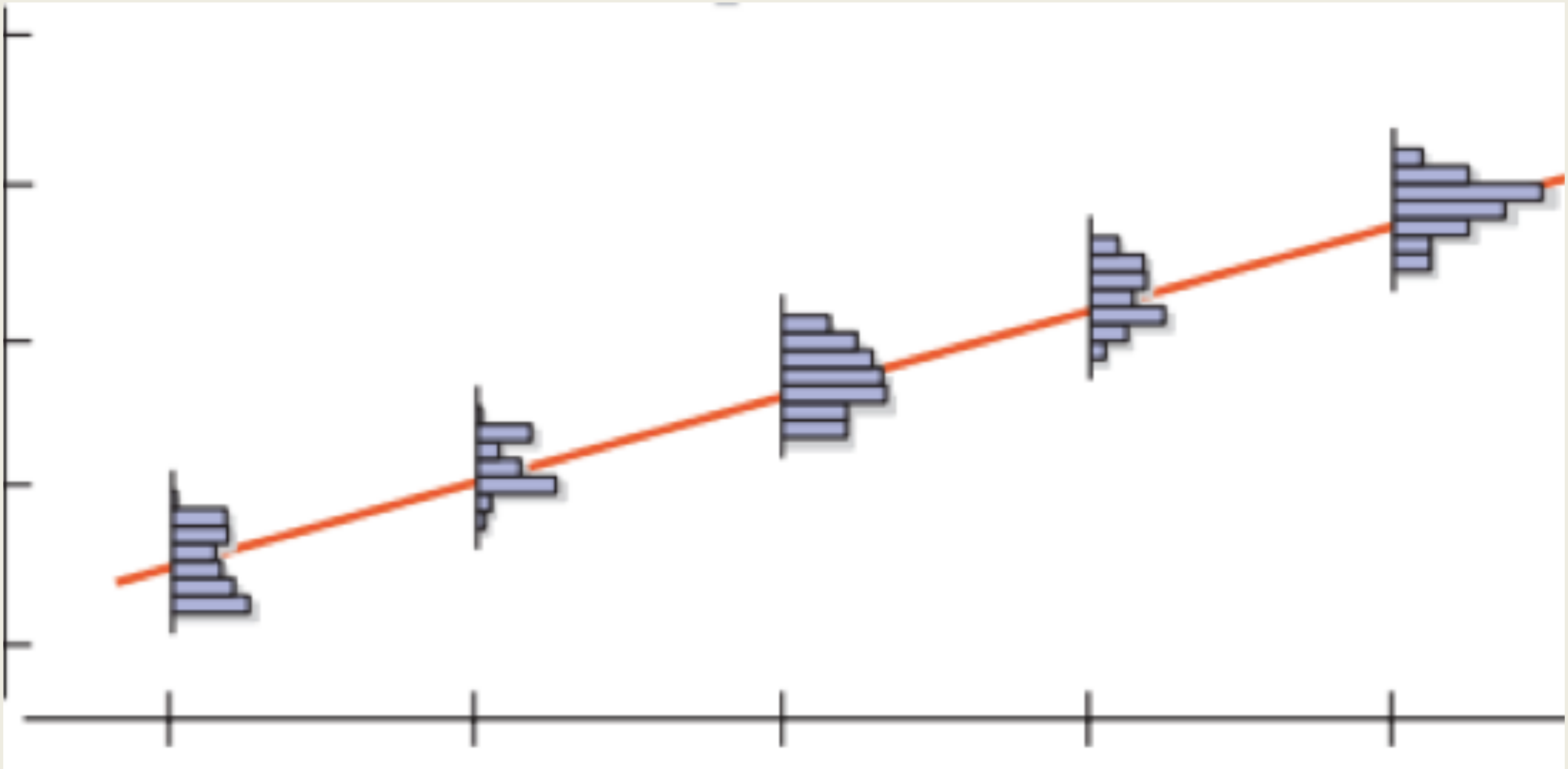
Estimate?

SE?

Shoe size and height: a small sample



Shoe size and height
A bigger sample: <u>many</u> y values for each x



Sample vs. <u>Model</u> Regression

Sample: $\hat{y} = b_0 + b_1 x$

- This gives a prediction for *y* based on the sample.

Model: $\mu_y = \beta_0 + \beta_1 x$

- β_0 = *y*-intercept for the model
- β_1 = slope for the model
- The model assumes that for every value of *x*, the <u>mean of all the *y*'s</u> lies on the line.

$$y = \beta_0 + \beta_1 x + \varepsilon$$

How Good is the Model?
Use sample values as estimates

- The least squares regression line $\hat{y} = b_0 + b_1 x$ obtained from the sample gives estimates for the model.
  - b_0 is an estimate for β_0.
  - b_1 is an estimate for β_1.

- Challenge: How good are these estimates?

Sampling Distribution for Regression Slopes

- When the conditions are met, $t = \dfrac{b_1 - \beta_1}{SE(b_1)}$

  follows Student's t-model with df = n – 2.

- Estimate of the standard error:

$$SE(b_1) = \frac{s_e}{\sqrt{n-1}\, s_x}, \quad s_e\sqrt{\frac{\sum (y-\hat{y})^2}{n-2}}$$

Testing for β_1

- If no linear association, $\beta_1 = 0$
- H_0: $\beta_1 = 0$
- $t_{n-2} = \dfrac{b_1 - 0}{SE(b_1)}$

Dependent variable is %BF
R-squared = 67.8%
s = 4.713 with 250 − 2 = 248 degrees of freedom

Variable	Coeff	SE(Coeff)	t-Ratio	P-Value
Intercept	−42.734	2.717	−15.7	<0.0001
Waist	1.70	0.0743	22.9	<0.0001

- For the *%Body Fat* and *Waist* data:

$$\frac{1.7 - 0}{0.0743} \approx 22.9 \qquad \text{P-value} < 0.0001$$

- Very unlikely to have such a high b_1 if $\beta_1 = 0$
- IF no association in population, sample is unlikely

Confidence Interval for β_1

- The hypothesis test for *%Body Fat* and *Weight* told us what we already know.
- A confidence interval is needed.

$$b_1 \pm t^*_{n-2} \times SE(b_1)$$

- For %Body Fat and Weight:

$$1.7 \pm 1.97 \times 0.074 = (1.55\%, 1.85\%)$$

- With 95% confidence the slope of the line for *%Body Fat* and *Weight* is between 1.55% and 1.85%.

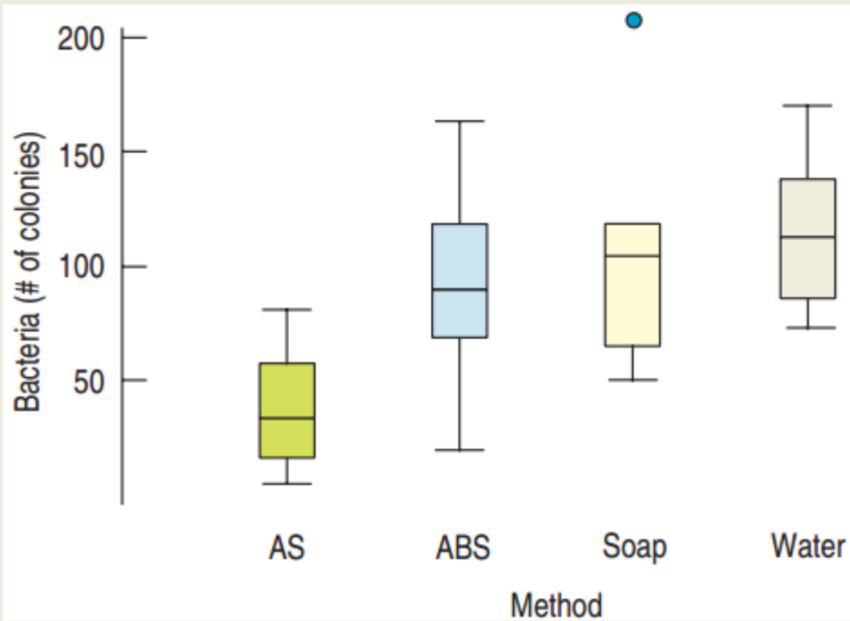
Chapter 26

Analysis of Variance

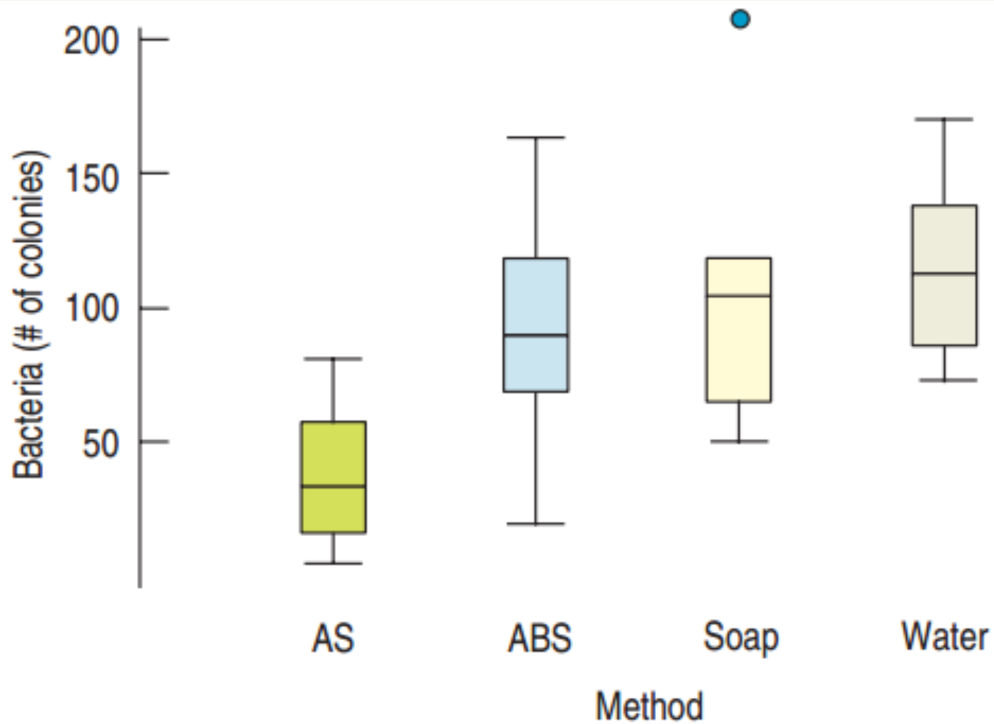
Hand Washing Comparison

Is there a difference between washing hands with: water only, regular soap, antibacterial soap (ABS), and antibacterial spray (AS)?

- Each tested with 8 replications
- Treatments randomly assigned



Hand Washing



- The means all differ.

- Is this just natural variability? Is this happening by chance alone?

- Null hypothesis: All the means are the same.
- Alternative hypothesis: The means are not all the same.

26.1

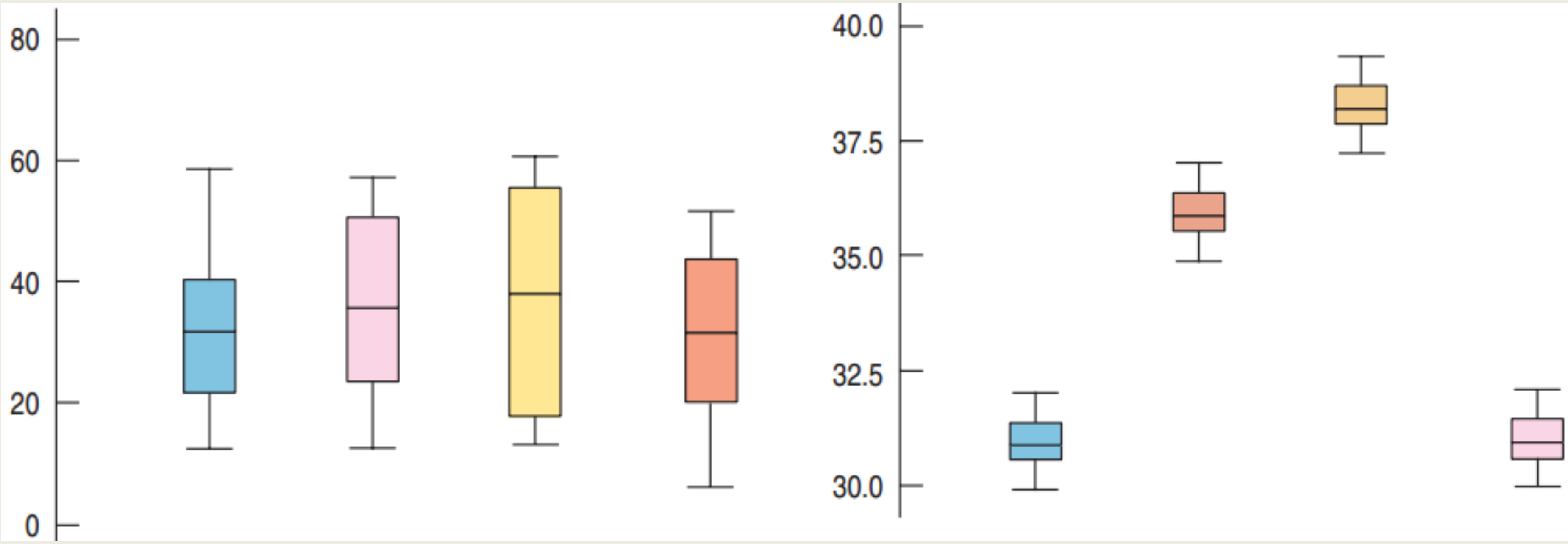
Testing Whether the Means of Several Groups Are Equal

Comparing Many Means

- H_0: All means are the same (= from same underlying population)
- H_A: Not all the means are the same.

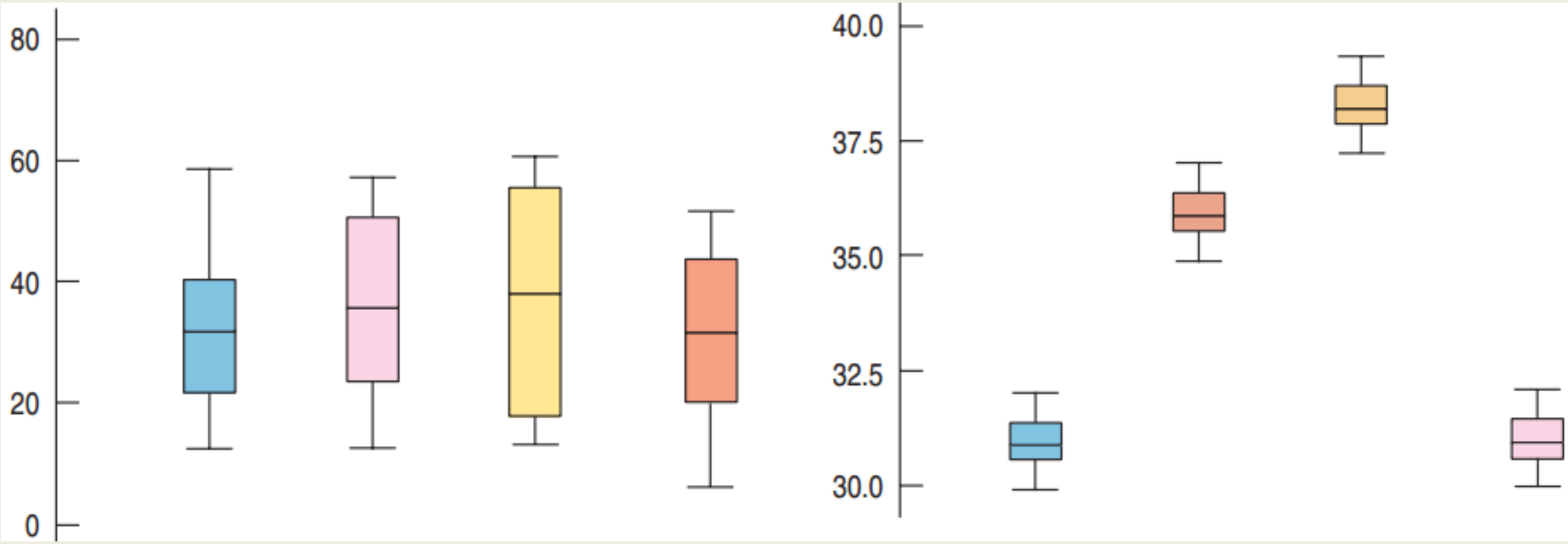
- If all the same, then they should not vary too much. Maybe a little bit, because the samples are different (→ ”sampling variability”)
- Considering the means as data values, we measure the variance of this data.
- If the means of the underlying population are different, the variance will be larger.

Variation *within* and *between*



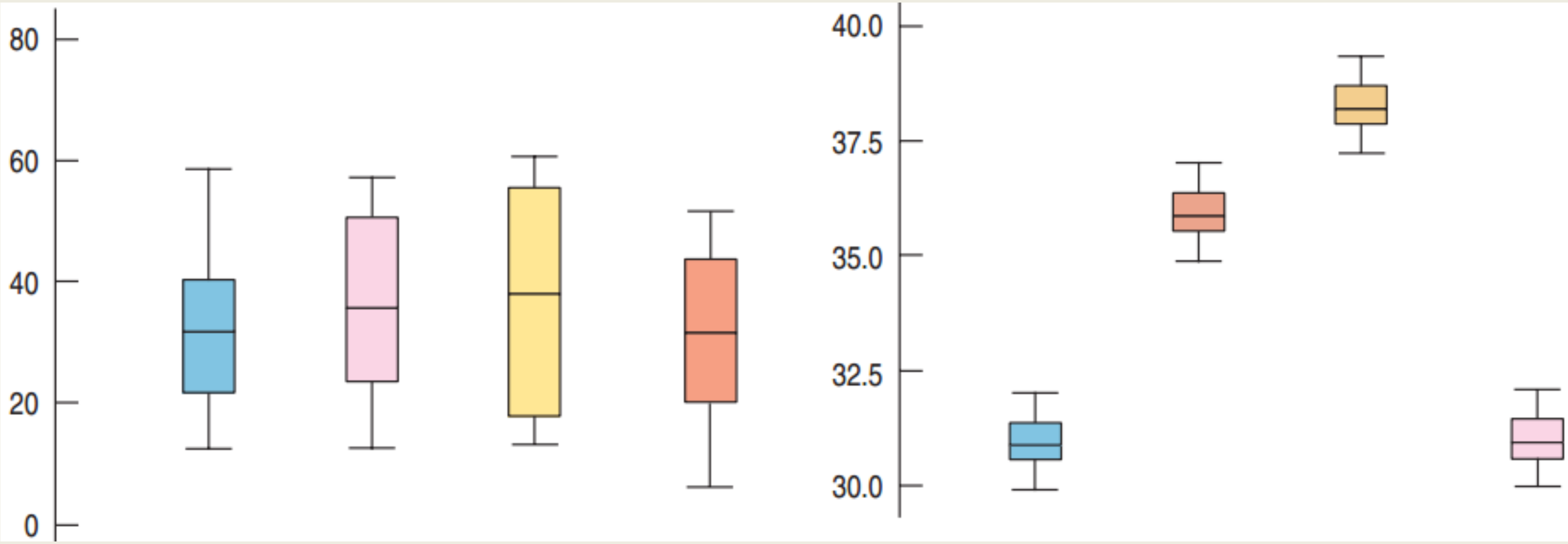
- The *means* in the two examples are the same!
- The variances are different

Comparing Means



- The *means* in the two examples are the same!
- The variances are different
- Second case: the variation *within* is so small that the differences *between* stands out

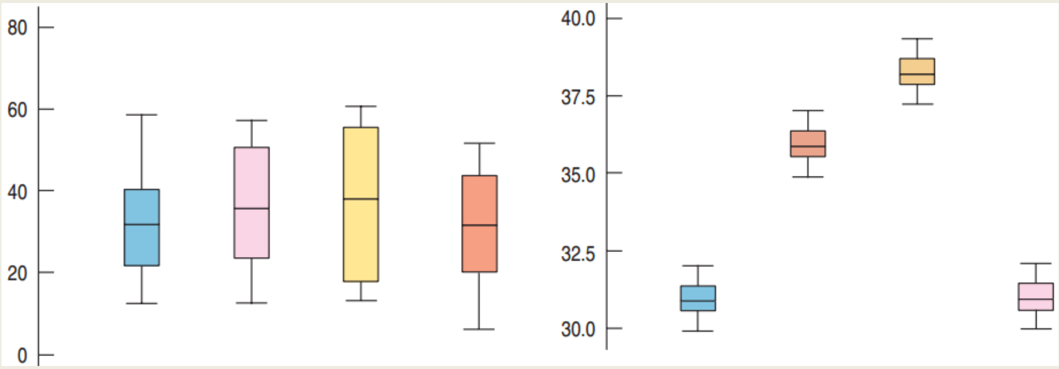
Comparing Means



- Note scales.
- **Left:** Plausible that the means are equal. Large variance *within* each sample, not *between* samples.
- **Right:** Means not equal. Small variance *within* samples, large variance *between*

Comparing Many Means: the F-test

- Second case: the variation within is so small that the differences between stands out



- This is the idea behind the F-test: we compare the variation within the groups with the variation between the groups
- If Var Between> Var Within → we reject the null
- If Var Between < Var Within → we fail to reject the null

Comparing Many Means: the variance BETWEEN

- To compare two means: *t*-test on the difference.
- To compare many means?
  - If H_0 is true, there is a common mean.
  - Look at the differences from this common mean.
  - How do we do that?
  - The variance of <u>all the means</u> does just that.
  - This variance of the sample means is a measure of how much the sample means differ from each other.
  - If H_0 is true, this variance should be small.

Washing Type All the Same?

Level	n	Mean
Alcohol Spray	8	37.5
Antibacterial Soap	8	92.5
Soap	8	106.0
Water	8	117.0

- H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$

- We will calculate the variance between the means
- It is an estimate of σ^2, the variance of the observations.

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- 1) First, by CLT: $Var(\bar{y}) = \dfrac{\sigma^2}{n} = \dfrac{\sigma^2}{8}$
- This is the variance of the sampling distribution of the means

Variance of {37.5, 92.5, 106.0, 117.0} is 1245.08.

Washing Type All the Same?

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- We will calculate the <u>variance between the means</u>
- It is an <u>estimate</u> of σ^2, the variance of the observations.
- 1) First, by CLT: $Var(\bar{y}) = \frac{\sigma^2}{n} = \frac{\sigma^2}{8}$ Variance of the SAMPLING DISTRIBUTION OF THE MEANS

  Variance of {37.5, 92.5, 106.0, 117.0} is 1245.08.

- 2) $\frac{\sigma^2}{8} = 1245.08 \rightarrow \sigma^2 = 9960.64$ I estimate the variance of OBSERVATIONS

This <u>estimate for σ^2</u> is called the Treatment Mean Square or MS_T ,

Also: Between Mean Square, Regression Mean Square

Washing Type All the Same?

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Alcohol Spray	8	37.5
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$$\frac{\sigma^2}{8} = 1245.08 \rightarrow \sigma^2 = 9960.64$$

Estimates

Variance of OBSERVATIONS

This estimate for σ^2 is called MS_T .

MS_T depends on whether the means are equal or not

If the null were true, and the means were basically equal (belonging to the same distribution), then this variance is due only to sampling, only to the observations, so that MS_T perfectly estimates σ^2

Washing Type All the Same?

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- H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$

$$\frac{\sigma^2}{8} = 1245.08 \rightarrow \sigma^2 = 9960.64$$

Variance of OBSERVATIONS

This estimate for σ^2 is called MS_T

Depends on whether the means are equal or not
Means are closer → MS_T gets smaller
Means are farther apart → MS_T gets bigger → reject null
Is this very high compared to what we would expect?
To know, we need to compare it to another value.

The Variance WITHIN

Level	n	Mean	Std Dev	Variance
Alcohol Spray	8	37.5	26.56	705.43
Antibacterial Soap	8	92.5	41.96	1760.64
Soap	8	106.0	46.96	2205.24
Water	8	117.0	31.13	969.08

- There is another way to estimate the variance of observations σ^2: the variance <u>within</u>
- Assume each washing method has the same sample variance.
- Then we can pool them all together to get the pooled variance s_p^2.
- <u>Does not depends on whether means are equal</u> or not
- Since the sample sizes are all equal, we can average the four variances: $s_p^2 = 1410.10$.
- Other names for s_p^2: <u>Error Mean Square</u> or MS_E
- Within Mean Square,. It estimates σ^2 directly.

Comparing MS_T and MS_E

MS_T (between)

- Estimates σ^2 if H_0 is true and the means are equal
- Should be larger than σ^2 if H_0 is false

MS_E (within)

- Estimates σ^2 whether H_0 is true or not
- If H_0 is true, both close to σ^2, so MS_T is close to MS_E

Comparing

- If H_0 is true, MS_T / MS_E should be close to 1
- If H_0 is false, MS_T / MS_E tends to be > 1
- But how do we tell whether MS_T/MS_E is larger enough?

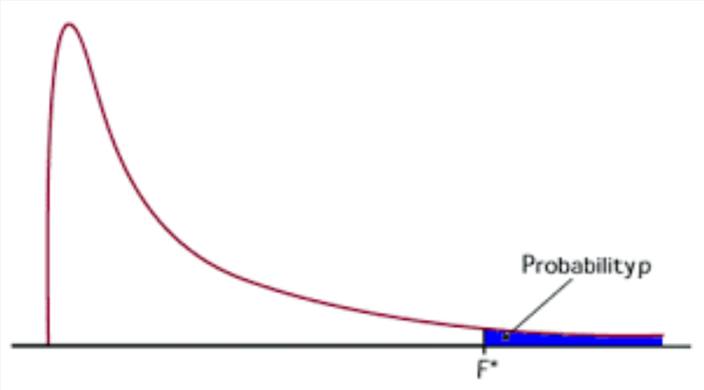
Sir Ronald Fisher Saves the Day!

- How do we tell whether MS_T/MS_E is larger enough to not be due just to random chance?

- MS_T/MS_E follows the *F*-Distribution
  - Numerator df: $k - 1$ (k = number of groups)
  - Denominator df: $k(n - 1)$
  - n = # observations in each group

- $F = MS_T/MS_E$ is called the *F*-Statistic.

Analysis of Variance



- The area to the right of the *F*-statistic represents the P-Value
  → probability I can get a more extreme value of F
- If the P-value is small, we reject H_0 and conclude that there is evidence that the means are not all equal to each other.
- Since both MS_T and MS_E come from looking at variances, we did an ANalysis Of Variance.
- ANOVA

Back to Bacteria

Level	n	Mean	Std Dev	Variance
Alcohol Spray	8	37.5	26.56	705.43
Antibacterial Soap	8	92.5	41.96	1760.64
Soap	8	106.0	46.96	2205.24
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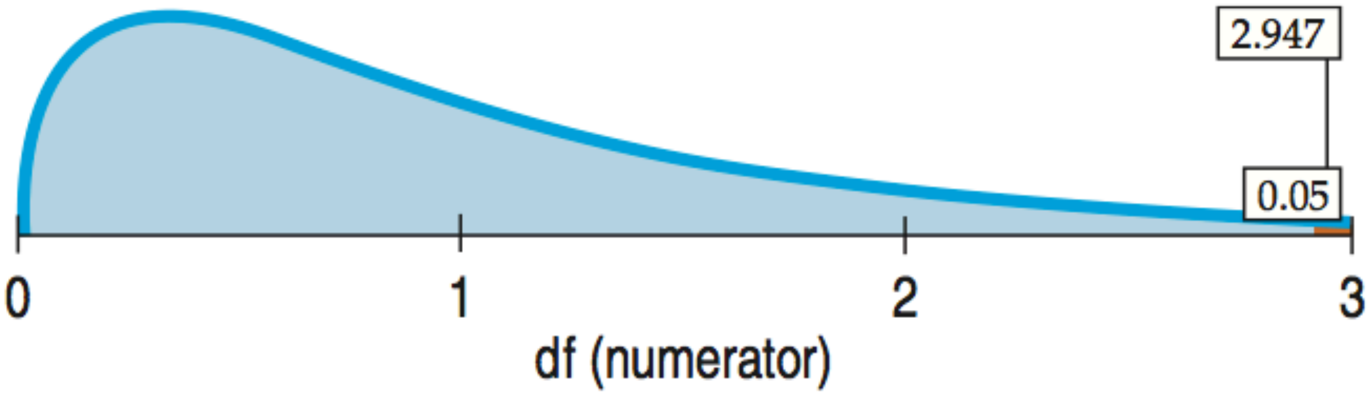
- $MS_T = 9960.64$, $MS_E = 1410.14$
- $F = \dfrac{MS_T}{MS_E} = \dfrac{9960.64}{1410.14} \approx 7.06$
- Num: df = 4 – 1 = 3, Den: df = 4(8 – 1) = 28.
- Technology gives P-value for $F_{3,28} = 0.0011$.

Analysis of Variance Table

Source	Sum of Squares	DF	Mean Square	F-ratio	P-value
Method	29882	3	9960.64	7.0636	0.0011
Error	39484	28	1410.14		
Total	69366	31			

The *F*-Table



df (denominator)	1	2	3	4	5	6	7
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388
27	4.210	3.354	2.960	2.728	2.572	2.459	2.373
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334
31	4.160	3.305	2.911	2.679	2.523	2.409	2.323
32	4.149	3.295	2.901	2.668	2.512	2.399	2.313

Conclusions About Hand Washing

- If the means were all equal then only 11 in 10,000 experiments such as these would have MS_T/MS_E at least as large as 7.06.
- We have strong evidence that the four different methods of hand washing are not equally effective at eliminating germs.

Residual Standard Deviation

- To have the response variable's units, we should work with the standard deviation instead of the variance:

$$s_p = \sqrt{MS_E} = \sqrt{\frac{\sum e^2}{N-k}}$$

- s_p is the residual standard deviation.
- For the hand washing data: $s_p = \sqrt{1410.14} \approx 37.6$
- This is a reasonable compromise standard deviation for all four groups.

26.2

The ANOVA Table

Contrast Baths

Contrast baths: To reduce hand swelling and stiffness in patients that had carpal tunnel release surgery, hands are alternately immersed in cold and hot water.

- 59 patients randomly assigned to one of:
  - Contrast bath with no exercise
  - Contrast bath with exercise
  - Exercise but no contrast bath
- Specify the subjects, sample size, experimental factor, treatment levels, response, null hypothesis, and research methods.

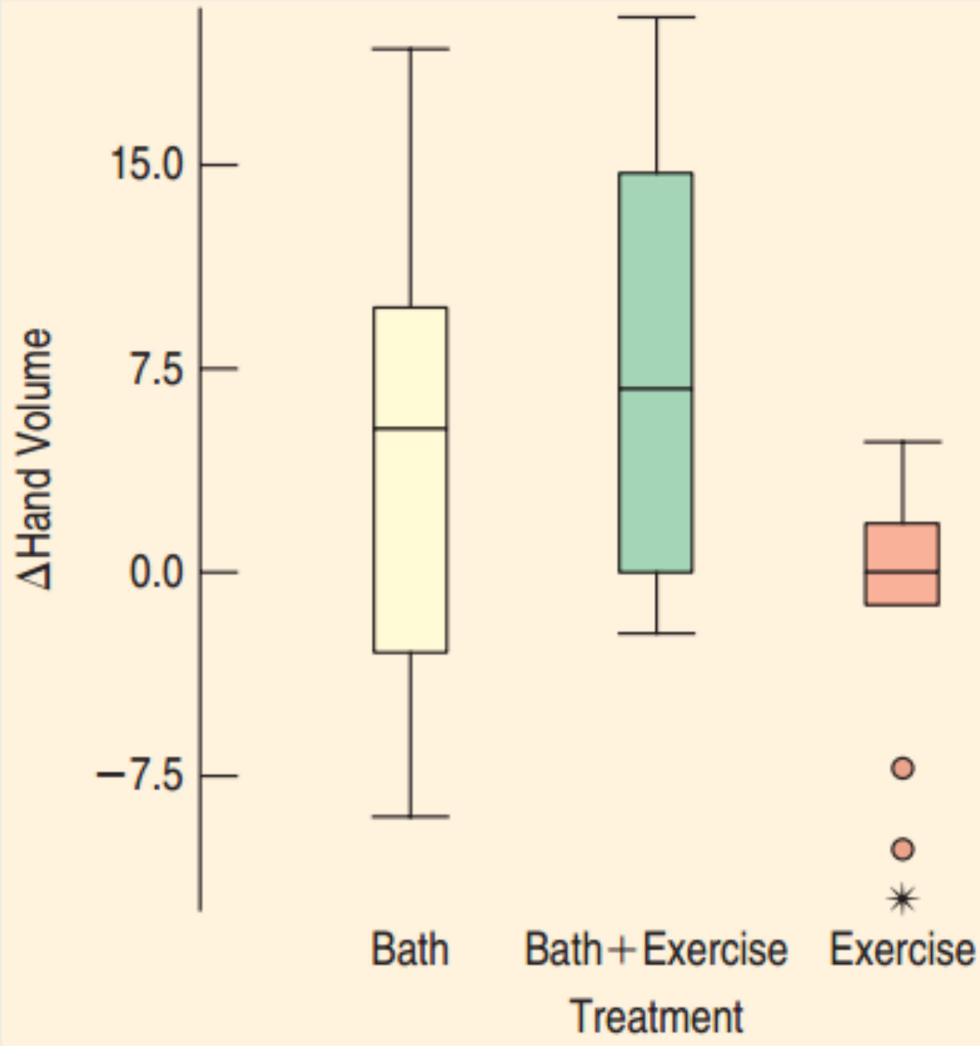
Contrast Baths

- Subjects: Patients receiving carpal tunnel release surgery
- Sample Size: 59
- Factor: Contrast bath treatment
- Levels: Contrast bath alone, Contrast with exercise, Exercise only.
- Response: Change in hand volume
- Null Hypothesis: Mean change in hand volume **same** for all three treatments
- Methods: Single blind (patients but not the researchers aware of treatments)

Bath Treatment

- What does the boxplot tell us about the means for the three treatments?

- Not much difference between the two contrast bath treatments.
- The exercise only treatment may result in less swelling.



Hand Washing

Analysis of Variance for Hand Volume Change

Source	df	Sum of Squares	Mean Square	F-Ratio	P-Value
Treatment	2	716.159	358.080	7.4148	0.0014
Error	56	2704.38	48.2926		
Total	58	3420.54			

- What does the ANOVA table say about the differences in the mean swelling for the three different types of contrast bath treatments?
  - F = 7.4148 and the P-value is small.
  - We can reject the null hypothesis that the mean change in hand volume is the same for all three treatments.

Hand Washing

Analysis of Variance for Hand Volume Change

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Error	56	2704.38	48.2926		
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Divide sum of squares by degrees of freedom to obtain mean squares

The mean squares are formed by dividing the sum of squares by the associated degrees of freedom.

Let $N = \sum n_i$. Then, the degrees of freedom for treatment are

$$DFT = k - 1 \,,$$

and the degrees of freedom for error are

$$DFE = N - k \,.$$

The corresponding *mean squares* are:

$$MST = SST/DFT$$

$$MSE = SSE/DFE \,.$$

The F-test

The test statistic, used in testing the equality of treatment means is: $F = MST/MSE$.

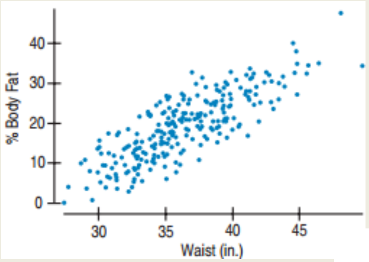
The critical value is the tabular value of the F distribution, based on the chosen α level and the degrees of freedom DFT and DFE.

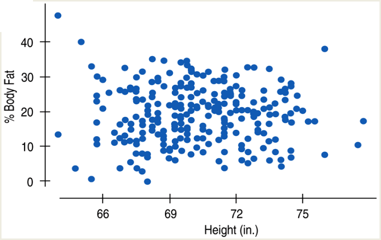
The calculations are displayed in an ANOVA table, as follows:

ANOVA table

Source	SS	DF	MS	F
Treatments	SST	$k-1$	$SST/(k-1)$	MST/MSE
Error	SSE	$N-k$	$SSE/(N-k)$	
Total (corrected)	SS	$N-1$		

Review - Multiple Regression: Waist, Height and %BodyFat





Computer output:

Variable	Coefficient	SE(Coeff)	t-ratio	P-value
Intercept	−3.10088	7.686	−0.403	0.6870
Waist	1.77309	0.0716	24.8	<0.0001
Height	−0.60154	0.1099	−5.47	<0.0001

The estimated regression equation is

$$\widehat{\%Body\ Fat} = -3.10 + 1.77\,Waist - 0.60\,Height.$$

Example: Multiple Regression

Dependent variable is: %Body Fat
R-squared = 71.3% R-squared (adjusted) = 71.1%
s = 4.460 with 250 − 3 = 247 degrees of freedom

Source	Sum of Squares	DF	Mean Square	F-ratio	P-value
Regression	12216.6	2	6108.28	307	<0.0001
Residual	4912.26	247	19.8877		

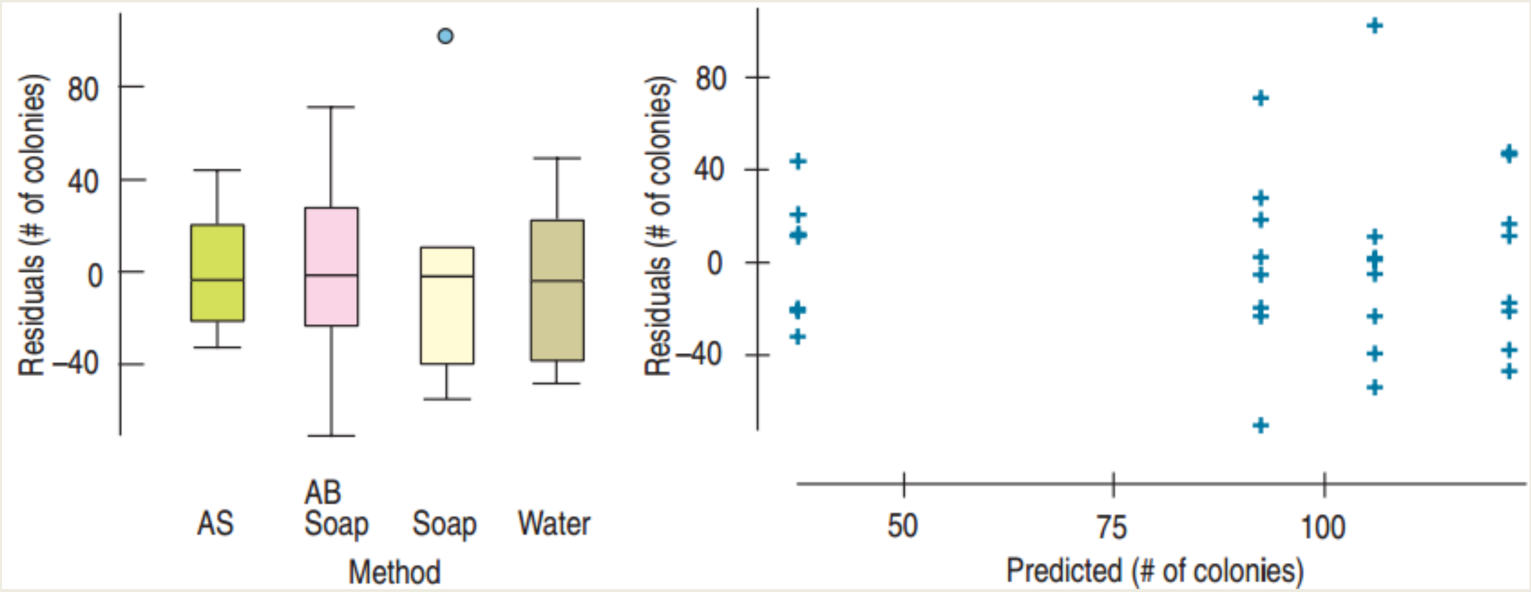
26.3

Plot the Data

Assumptions and Conditions

- To check the assumptions and conditions for ANOVA, always look at the side-by-side boxplots.
  - Check for outliers within any group.
  - Check for similar spreads.
  - Look for skewness.
  - Consider re-expressing.

Checking Conditions for Hand Washing



- IQR's are about the same for all groups.
- "The Plot Does Not Thicken."
- One outlier, not too extreme
- $s_p = 36.7$ reasonable for all four groups

Normal Population Assumption

Nearly Normal Condition

- Must be satisfied for each group.
- Use side-by-side boxplot to check for skewness and outliers.
- Equal Variance Condition allows us to look at all the residuals together.
- Use a histogram and Normal probability plot of all the data to check for Normality.

One Way ANOVA *F*-Test

- H_0: $\mu_1 = \mu_2 = \ldots \mu_k$
- $F = \dfrac{MS_T}{MS_E}$
- MS_T is found FROM the variance BETWEEN means of the treatment groups.
- MS_E is found by POOLING the variances WITHIN each of the treatment groups.
- If F is large, reject H_0.

26.4

Comparing Means

What Next?

ANOVA:

- P-value large → Nothing left to say
- P-value small → Which means are large and which means are small?
- We can perform a *t*-test to compare two of them.
- We assumed the standard deviations are all equal.
- Use s_p, for pooled standard deviations.
- Use the Students *t*-model, df = $N - k$ (N = total number of trials)

t-Test for Hand Washing

Level	n	Mean	Std Dev	Variance
Alcohol Spray	8	37.5	26.56	705.43
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Soap	8	106.0	46.96	2205.24
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- H_0: $\mu_W - \mu_{ABS} = 0$
- H_A: $\mu_W - \mu_{ABS} \neq 0$

- Diff. in observed means: 117.0 – 92.5 = 24.5.

- $SE\left(\mu_W - \mu_{ABS}\right) = s_p\sqrt{\frac{1}{n_W} + \frac{1}{n_{ABS}}} \approx 18.775$
- $t = \frac{24.5}{18.775} \approx 1.31, \;\; \text{df} = 32 - 4 = 28, \;\; \text{P-value} = 0.2$

- We cannot discern a difference between washing with antibacterial soap and just using water.

Bonferroni Multiple Comparisons

- If we wanted to do a *t*-test for each pair:
- P(Type I Error) = 0.05 for each test.
- Good chance at least one will have a Type I error.
- Bonferroni devised a correction:
  - Adjust α to α/J where J is the number of comparisons.
  - 95% confidence (1 – 0.05) with 3 comparisons adjusts to (1 – 0.05/3) ≈ 0.98333.
  - Use this adjusted value to find t^{**}.

26.5

ANOVA on Observational Studies

The Trouble with Observational Studies

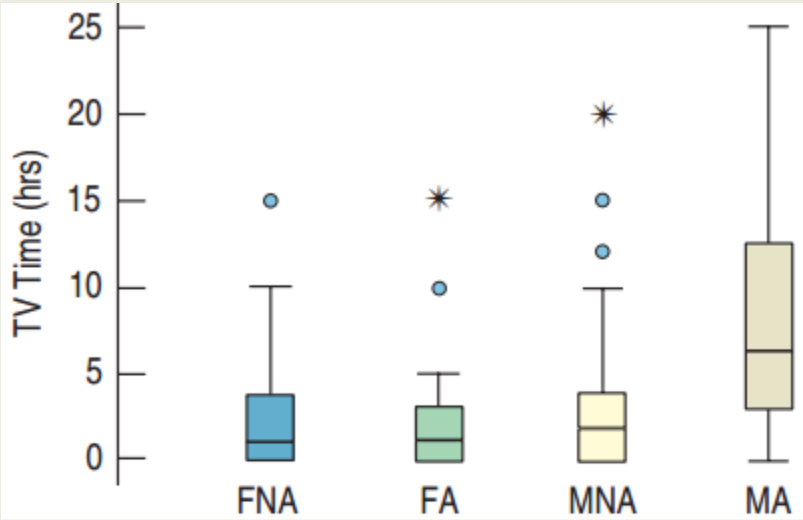
- Difficult to verify equal variances.
- May not same sample size for each treatment.
- Without randomization, causation cannot be concluded even with a small P-value.
- Cannot avoid lurking variables.
- Since the subjects are not randomly assigned, the confidence interval may not represent the general population.

Who Watches More TV?

Four groups: male athletes (MA), male non-athletes (MNA), female athletes (FA), and female non-athletes (FNA) were surveyed. Do these four groups spend about the same amount of time watching TV?

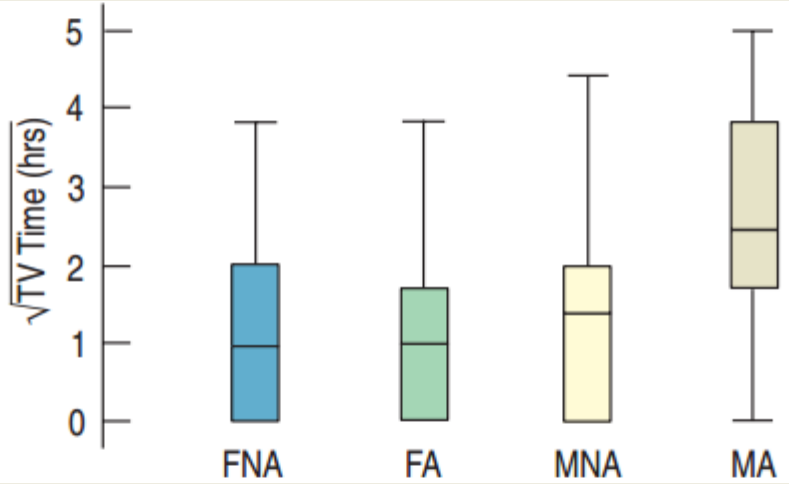
- **Variables:** Number of hours watching TV, the viewer's gender and athlete status. Total of 197 randomly selected respondents.

Who Watches More TV?

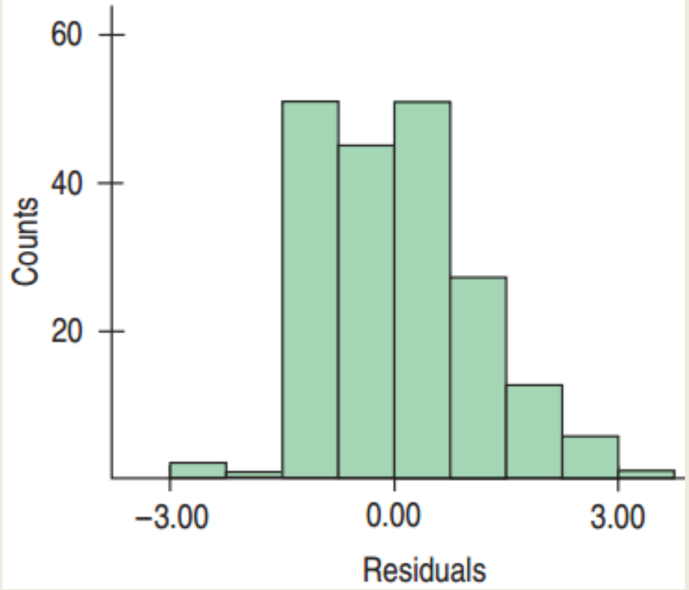


- **Plot:** Trouble! Each skewed, outliers (some extreme), category with highest center has a large spread.

- Re-express with a root.
  - ✓**Randomization Condition:** Students selected randomly.
  - ✓**Similar Spread Condition:** New boxplot much better.



Who Watches More TV?



- ✓ **Nearly Normal Condition:** Histogram of the residuals looks reasonably normal.
- It is appropriate to use ANOVA.

Source	DF	Sum of Squares	Mean Square	F-Ratio	P-Value
Group	3	47.24733	15.7491	12.8111	<0.0001
Error	193	237.26114	1.2293		
Total	196	284.50847			

- **Conclusion:** The *F*-statistic is large and the corresponding P-value small. I conclude that the TV-watching behavior is not the same among these groups.

So Do Male Athletes Watch More TV?

	Difference	Std. Err.	P-Value
FA–FNA	0.049	0.270	0.9999
MNA–FNA	0.205	0.182	0.8383
MNA–FA	0.156	0.268	0.9929
MA–FNA	1.497	0.250	<0.0001
MA–FA	1.449	0.318	<0.0001
MA–MNA	1.292	0.248	<0.0001

- Bonferroni comparison of all pairs of groups.
- Three significant differences:
  - Male athletes watch more TV than both female athletes and non-athletes.
  - Male athletes watch more than male non-athletes.
- Does this generalize to all weeks and all colleges?
  - The study was conducted during the NCAA men's basketball tournament.

What Can Go Wrong?

Watch out for outliers.

- Outliers affect most ANOVA statistics: mean, spread, *F*, P-value.
- Increase *P*(Type II Error).

Watch out for changing variances.

- ANOVA depends on independence, constant variance, and Normality.
- If conditions on the residuals are violated, try re-expressing.

What Can Go Wrong?

Be wary of drawing conclusions about causality from observational studies.

- Causality can only be concluded by ANOVA from randomized experiments.

Be wary of generalizing.

- Don’t generalize if the data do not represent the general population.

Watch for multiple comparisons.

- Use a multiple comparison method such as Bonferroni to test many pairs.

Chapter 27

Multifactor Analysis of Variance

27.1

A Two-Factor ANOVA Model

Two Factors at Once?!

- Unlike experiments we have seen thus far, some studies have not one but two factors.

- Example: study on accuracy throwing darts
- Depends on which **hand** used and **distance**

Two Factors at Once?!

- Two factors do not confuse things, but rather improve the experiment and analysis.

- With two factors we have two hypotheses tests.
- Each of those hypotheses asks whether the mean of the response variable is the same for each of the treatment levels.
- Earlier, we considered ways to remove or avoid extra variation in designing experiments. A two-factor experiment does just that.

An ANOVA Model (cont)

- Our null hypothesis on each factor is that the effects of that treatment are all zero (the means are equal)
- The alternative hypotheses are that the treatment effects are not all equal (the means are different)
- We want to compare the differences BETWEEN the treatment effects with the underlying variability WITHIN the treatment.

27.2

Assumptions and Conditions

Partial boxplots

Plot the Data …

- Start by using side-by-side boxplots across levels of *each factor.*

- Look for outliers and correct them or omit them if you can't correct them.

- The problem with looking at the boxplots is that the responses at each level of the factor contain *all levels of the other factor.*

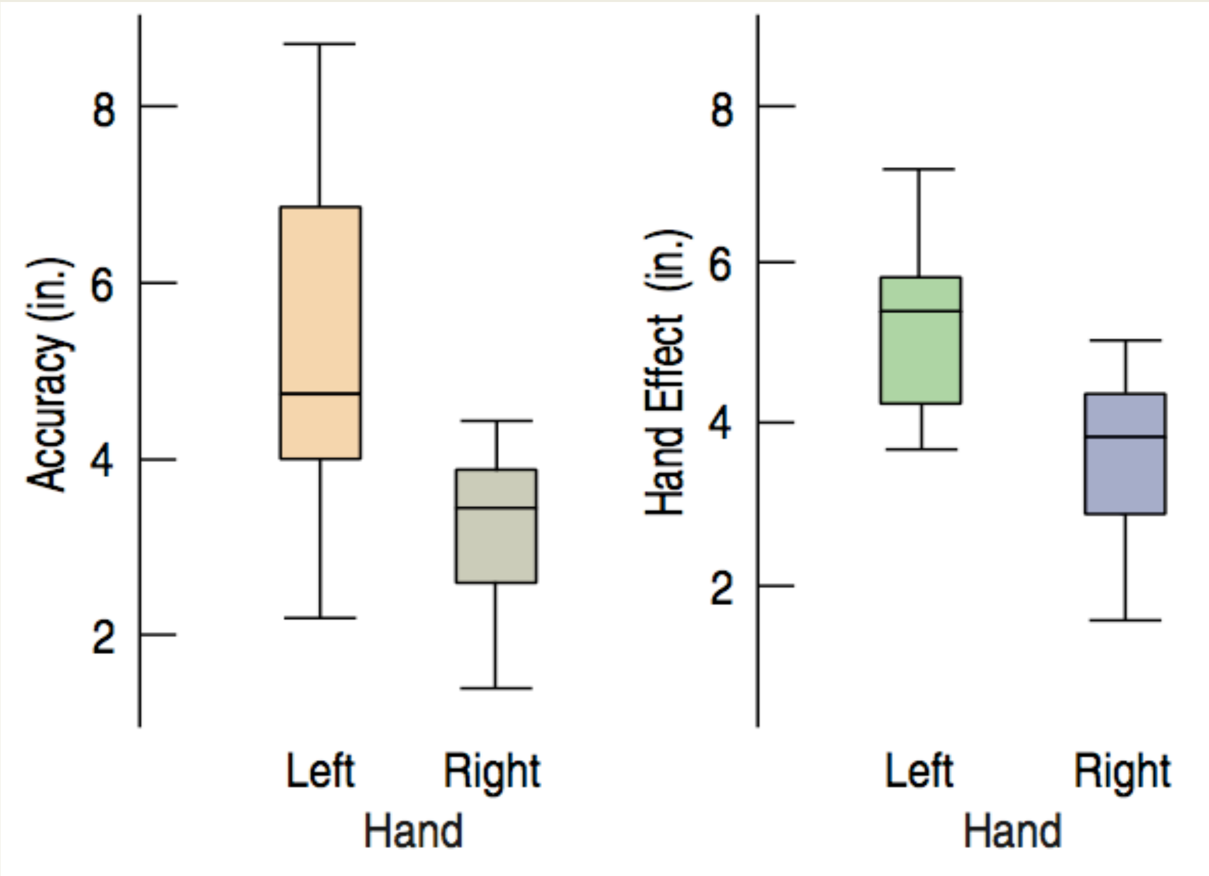
Partial boxplots

Plot the Data …

- A better alternative would be to make boxplots for each factor level *after removing the effects of the other factor.*

- We could compute a one-way ANOVA on one factor and find the residuals. Then, make boxplots of those *residuals* for each level of the other factor. We might call this display a partial boxplot.

Dart-throwing study: partial boxplots



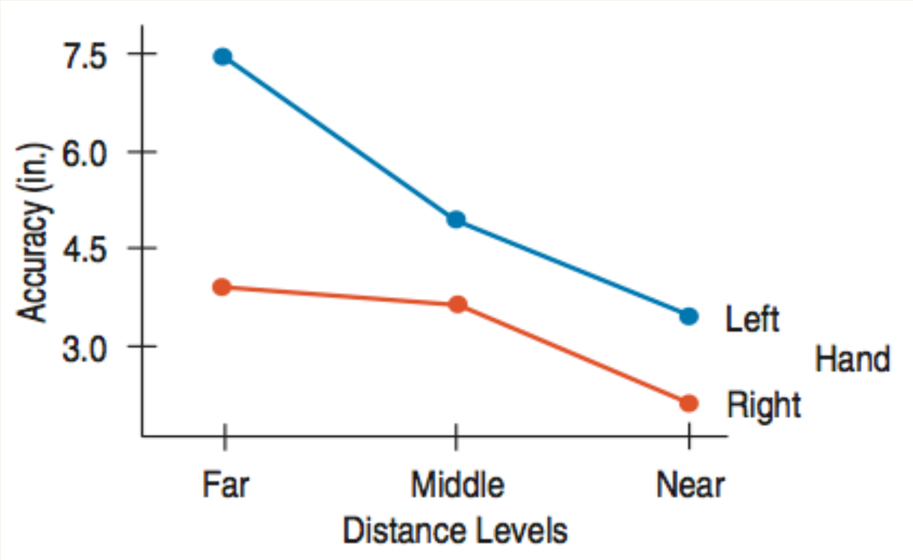
Boxplots of *Accuracy* by *Hand (1)* show the effect of changing *Hand* less clearly

Boxplots on the right (2) shows the effects of *Hand* after the effect of *Distance* has been removed.

The effect of changing hands is much easier to see.

Interaction Plot

When the effects of one factor change for different levels of another factor, we say there is an interaction. To show the interaction, we use an interaction plot.



This plots shows the averages of the observations at each level of one factor broken up by the levels of the other factors.

Assumptions and Conditions

Independence Assumptions

- An Independence Assumption for the two-factor model is the same as in a one-way ANOVA. Hence, the observations *within each treatment group* must be independent of each other. However, no test can verify that assumption.
- Check the Randomization Condition. Were the data collected with suitable randomization? This is true for both surveys and experiments.

Assumptions and Conditions (cont)

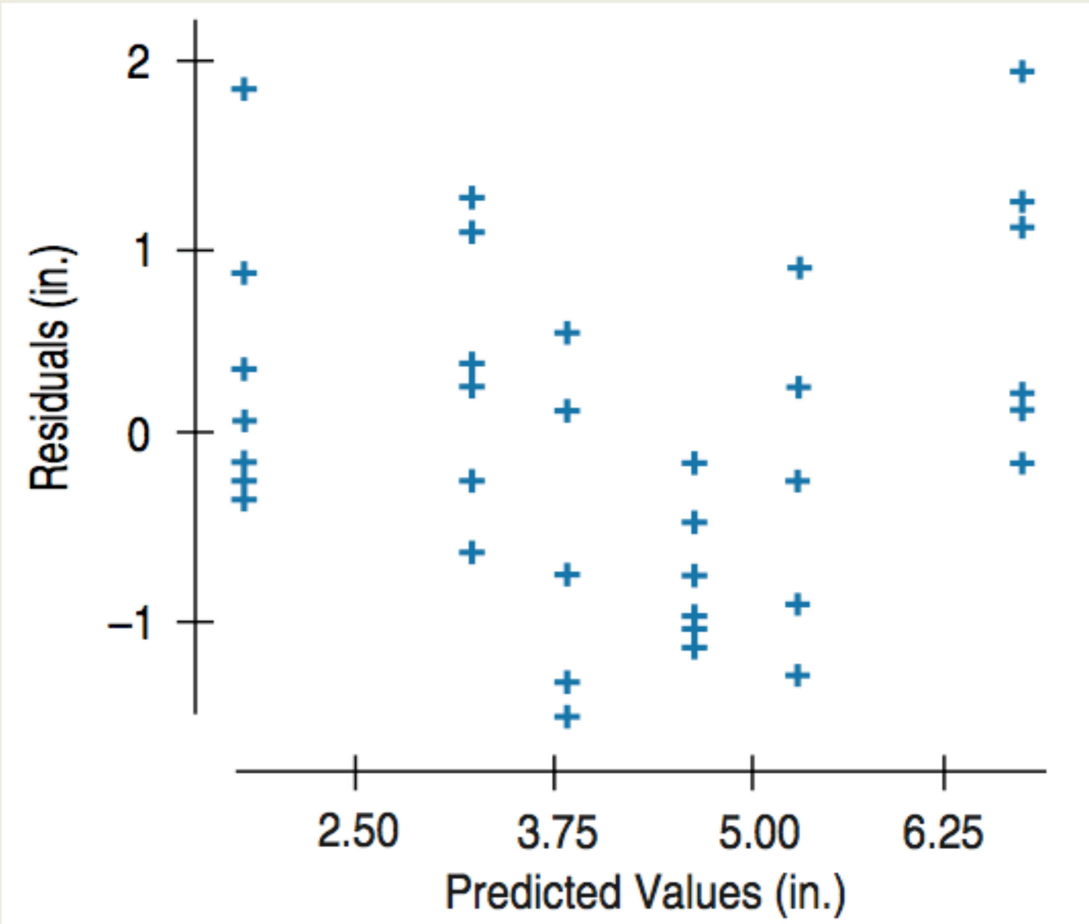
Equal Variance Assumption

- Like the one-way ANOVA, the two-factor ANOVA requires that the variances of all treatment groups be equal. We need to check for equal spread across all treatment groups.

- Look at the residuals plotted against the predicted values. If the plot thickens (to one side or the other), it's a sign that the variance is changing systematically.

- Consider re-expressing the response variable.

Assumptions and Conditions (cont)

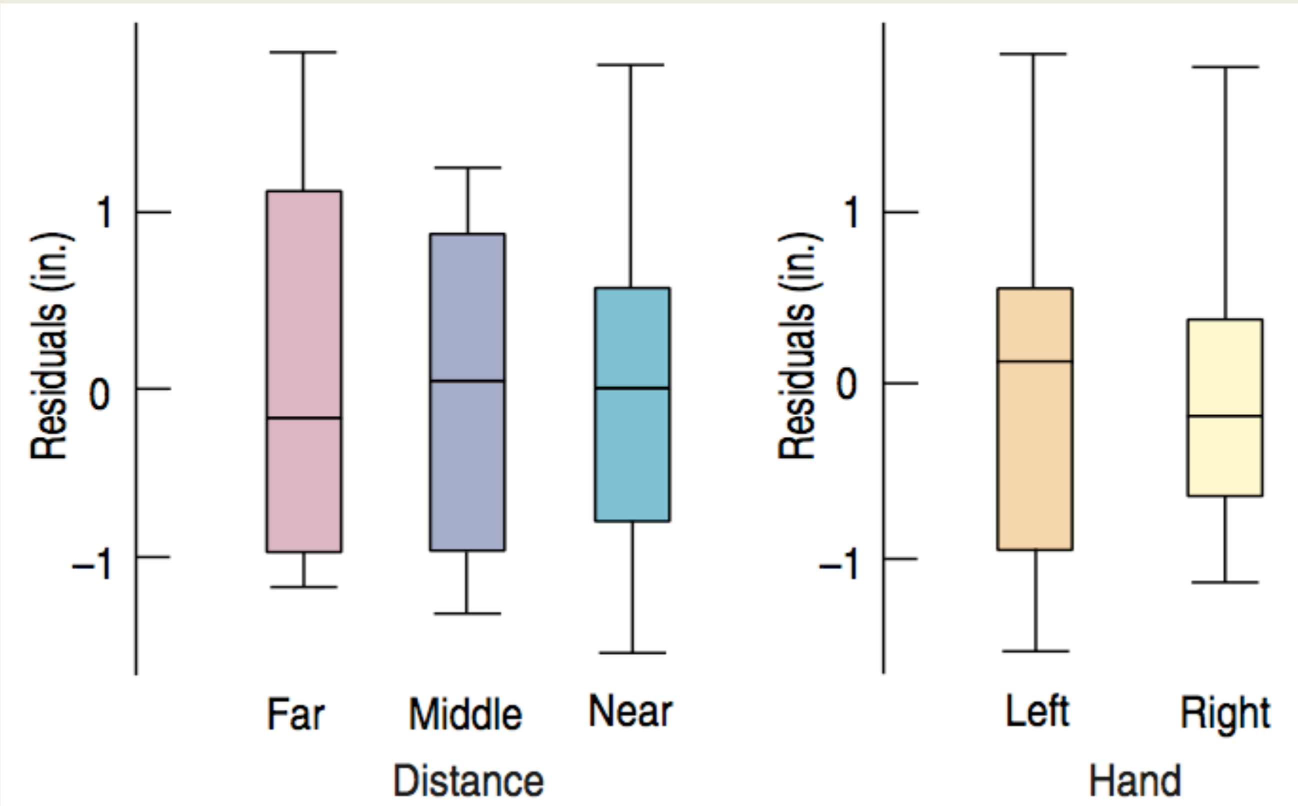


Scatterplot of residuals vs. predicted values from the two-way ANOVA model.

This figure doesn't show changing variance, but it's certainly not patternless. It shows a U-shaped pattern. This suggests the condition is violated.

Assumptions and Conditions (cont)

You can also plot the residuals grouped by each factor.

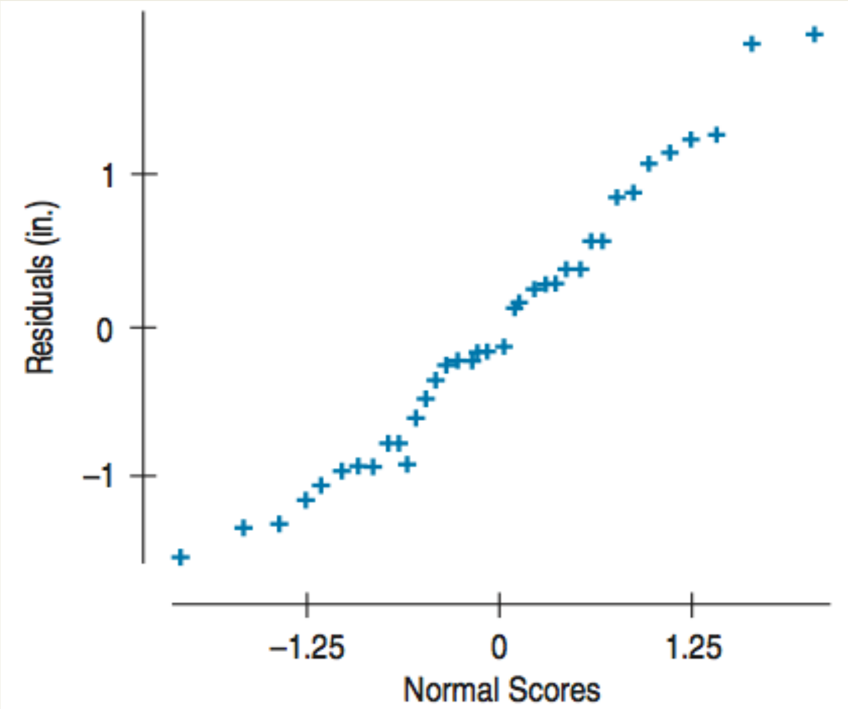


Boxplots of the residuals from the two-way ANOVA of the dart accuracy experiment show roughly equal variability when plotted for each factor.

Assumptions and Conditions (cont)

Normal Error Assumption

- As with one-way ANOVA, the *F*-tests require that the underlying errors follow a Normal model. We'll check a corresponding Nearly Normal Condition with a Normal probability plot or histogram of the residuals.



A Normal probability plot of the residuals from the two-way ANOVA model for dart accuracy seems reasonably straight.

Example: Two-Factor Analysis of Variance Tennis balls' bounce

Step-by Step

Perform a two-factor experiment on tennis balls *Brand* and *Temperature,* using two *Brands* of tennis balls and three levels of *Temperature*.

The subject bounced three balls under each of the six treatment conditions by first randomly selecting a *Brand* and for that ball, randomly selecting whether to leave it at room temperature or to put it in the refrigerator or the freezer. After 8 hours she dropped the balls from a height of 1 meter to a concrete floor, recording the Height of the bounce in centimeters.

Example: Two-Factor Analysis of Variance
Step-by Step

This is a completely randomized replicated design in two factors.

The factor *Brand* has two levels: *Premium* and *Standard*.
The factor *Temperature* has three levels: *Room, Fridge, and Freezer*.
The null hypotheses are that *Brand* has no effect and that *Temperature* has no effect.

Brand	Temperature	Bounce Height
Standard	Freezer	37
Standard	Fridge	59
Standard	Room	59
Premium	Freezer	45
Premium	Fridge	60
Premium	Room	63
Standard	Freezer	37
Standard	Fridge	58
Standard	Room	60
Premium	Freezer	39
Premium	Fridge	64
Premium	Room	62
Standard	Freezer	41
Standard	Fridge	60
Standard	Room	61
Premium	Freezer	37
Premium	Fridge	63
Premium	Room	61

Example: Two-Factor Analysis of Variance

Step-by Step

State what we want to know and the null hypotheses we wish to test.

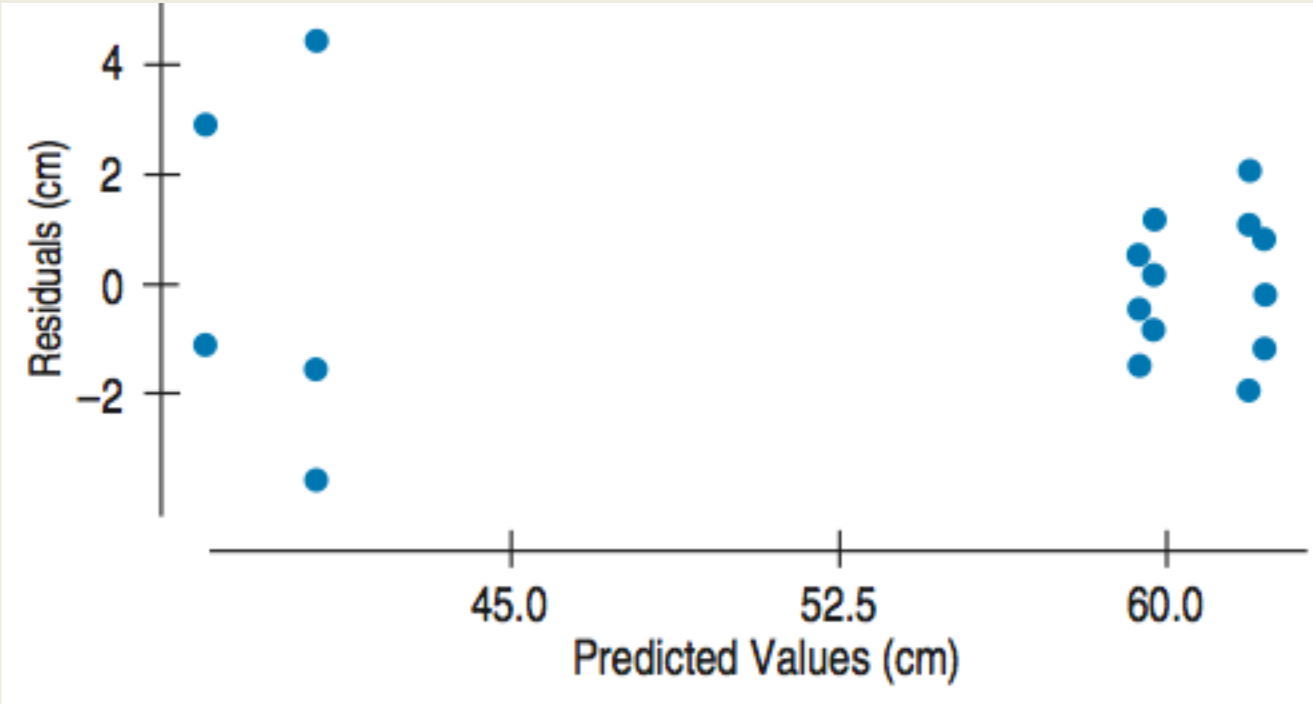
For two-factor ANOVA, the null hypotheses are that all the treatment groups have the same mean for each factor. The alternatives are that the means are not all equal.

Example: Two-Factor Analysis of Variance

Step-by Step

Plan Before Testing, Check Conditions.

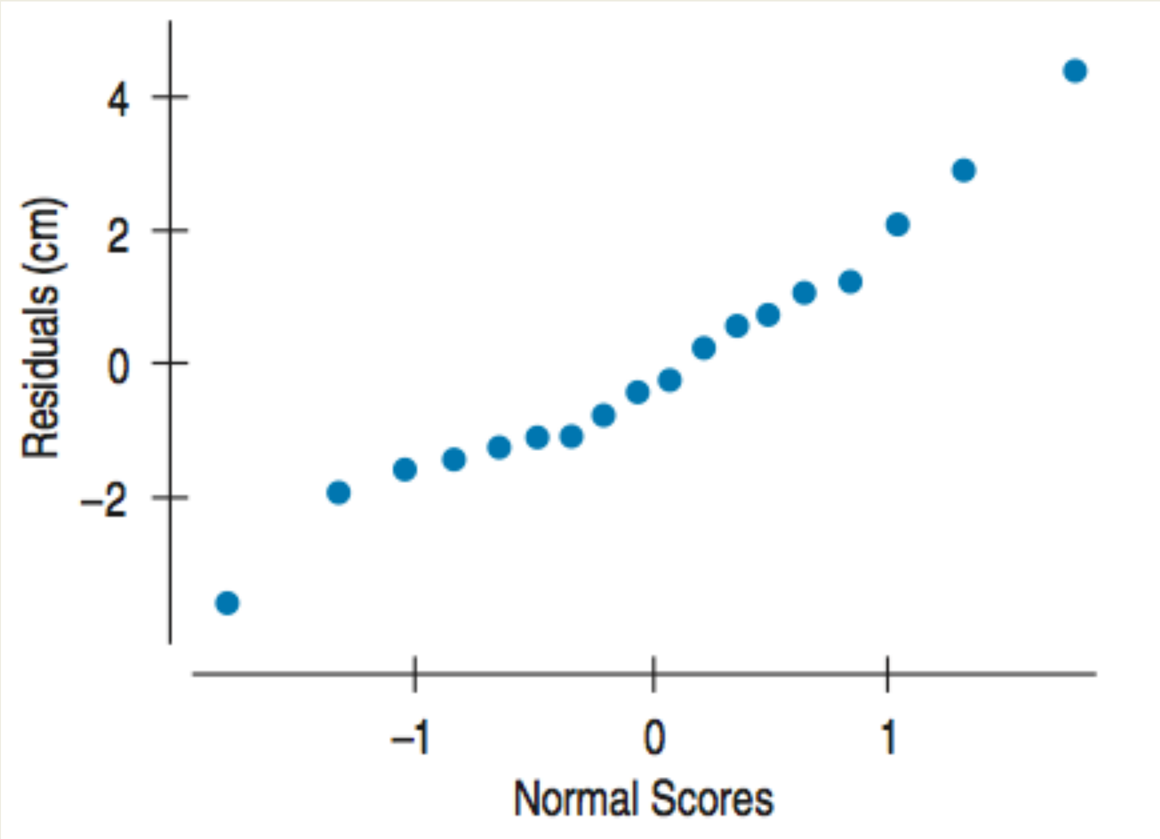
Similar Variance Condition.



Example: Two-Factor Analysis of Variance

Step-by Step

Nearly Normal Condition, Outlier Condition.



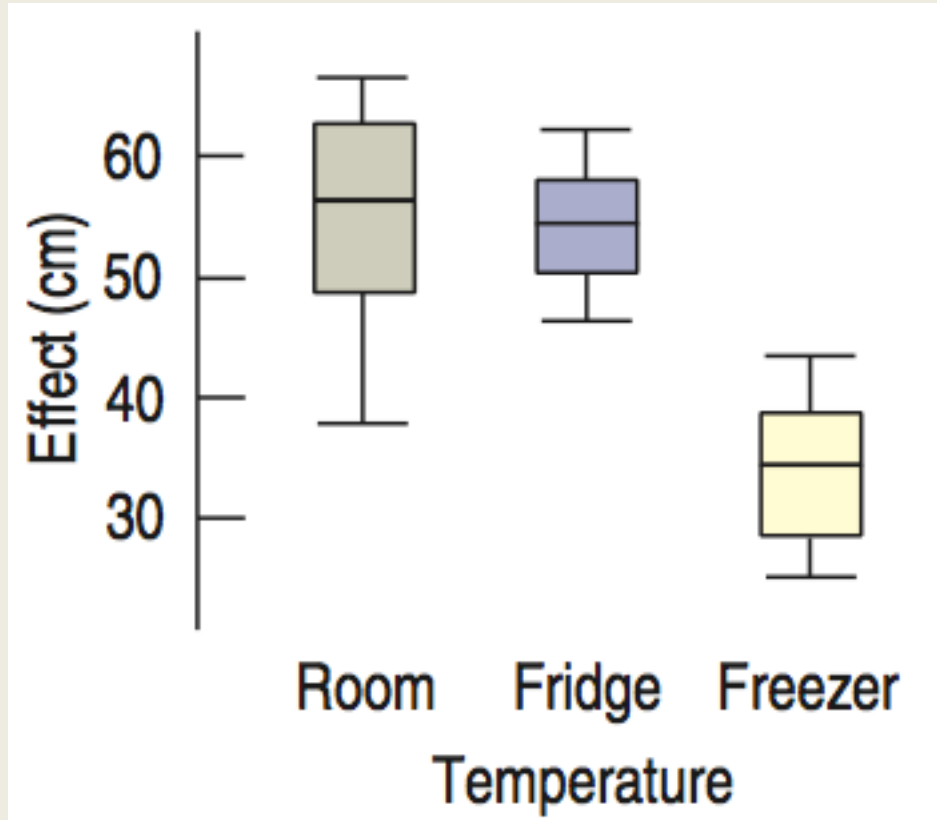
Example: Two-Factor Analysis of Variance

Step-by Step

Plot — Examine the side-by-side partial boxplots of the data.

Temperature has an effect.

Can’t be sure about Brand from the plots.

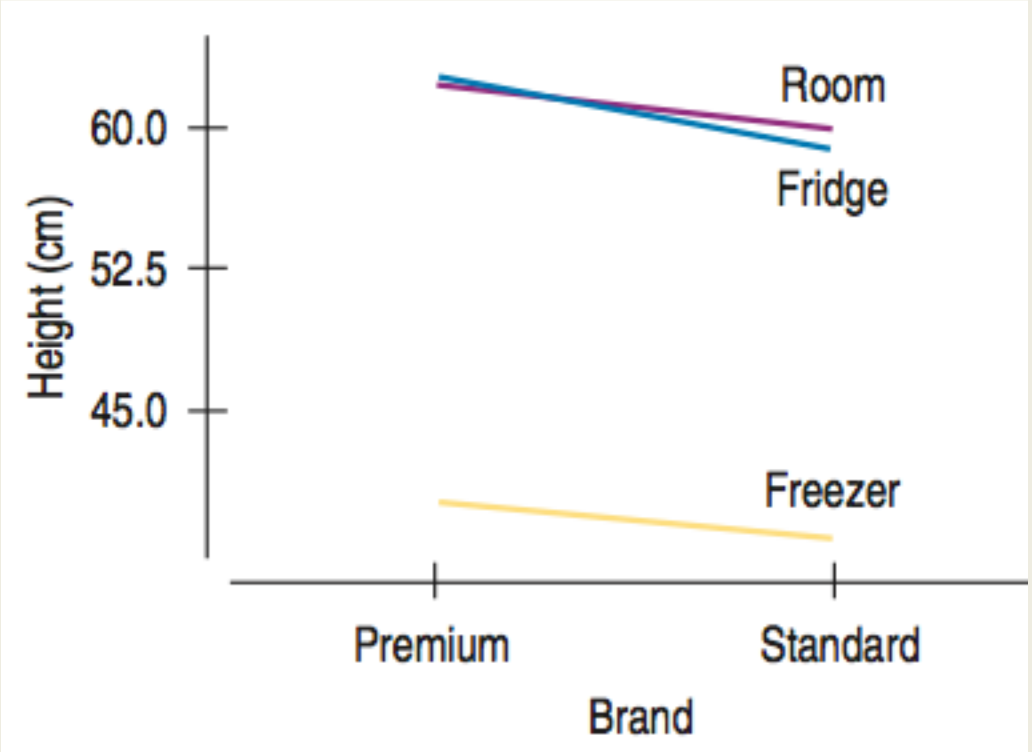


Example: Two-Factor Analysis of Variance

Step-by Step

Plan

Think about the assumptions and check the appropriate conditions.



Example: Two-Factor Analysis of Variance

Step-by Step

Plan — Show the ANOVA table.

Analysis of Variance For Height

Source	DF	Sum of Squares	Mean Square	F-ratio	P-value
Temp	2	1849.33	924.665	209.55	<0.0001
Brand	1	26.8889	26.8889	6.0935	0.0271
Error	14	61.7778	4.41270		
Total	17	1938.00			

Example: Two-Factor Analysis of Variance
Step-by Step

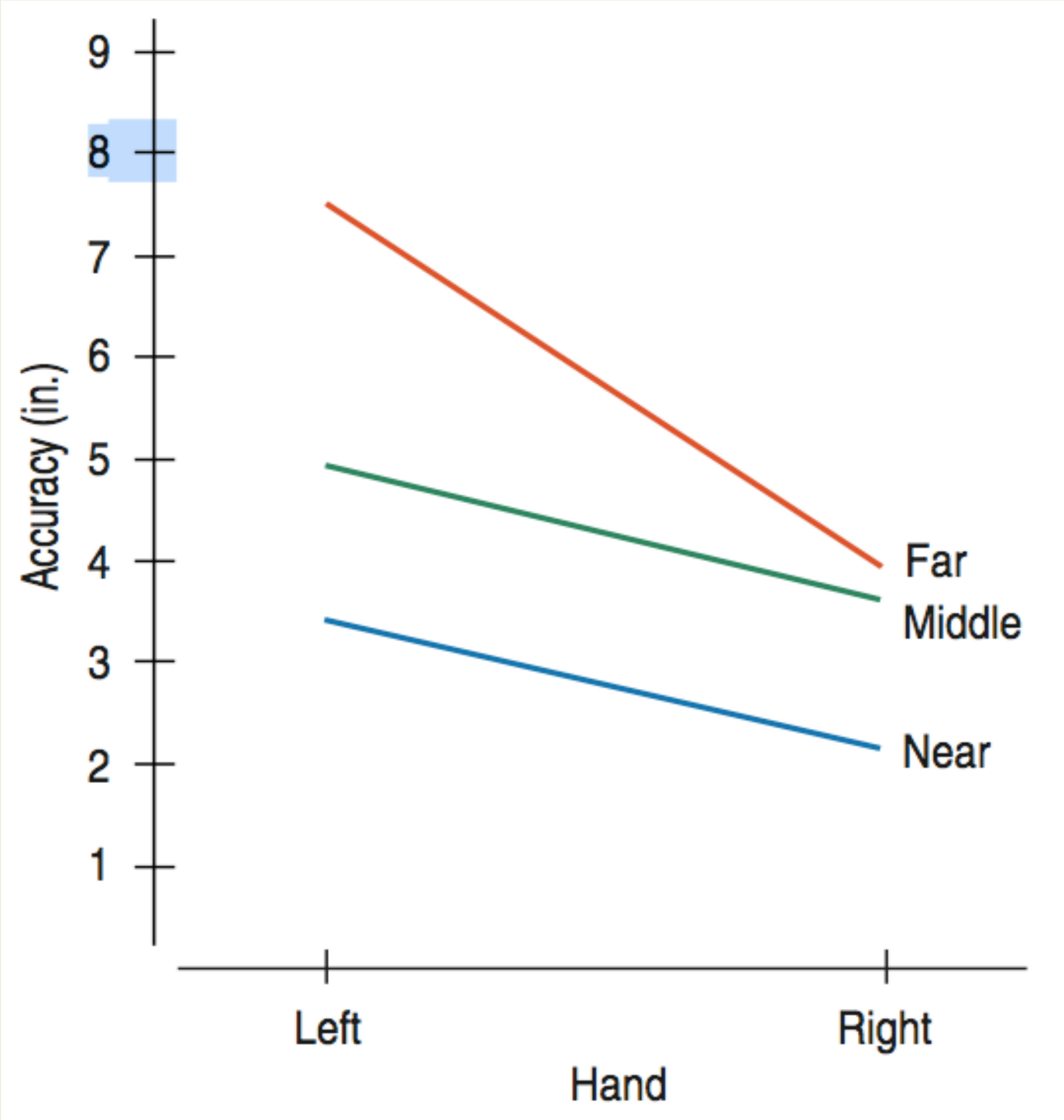
Interpretation Tell what the *F*-tests mean.

F-ratios this large would be very unlikely if the factors had no effect. I conclude that the means of the two *Brand* levels are not equal.

27.3

Interactions

Throwing darts: Interactions



Interaction plot of *Accuracy* by *Hand* and *Distance*. The effect of *Distance* appears to be greater for the left hand.

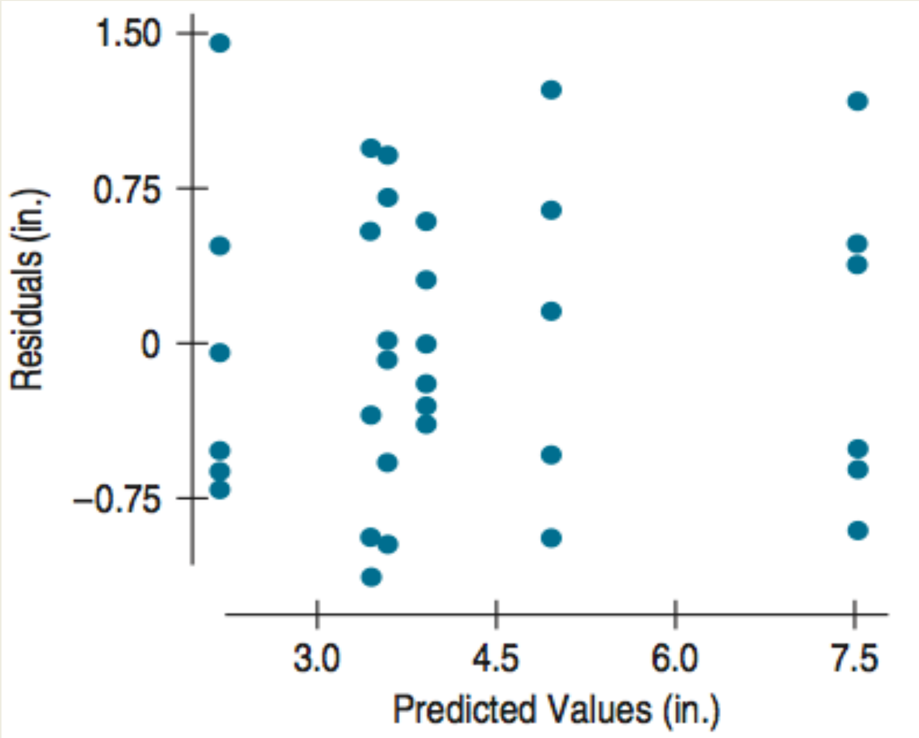
Inference When Variables are Related

- When the effect of one factor *changes* depending on the levels of the other factor, the factors are said to *interact*.

- When significant interaction is present, the best way to display the results is with an interaction plot.

- When we have the interaction term in our model, we include it in the equation and recalculate the residuals.

Inference When Variables are Related (cont)



Residuals for the two-way ANOVA of dart accuracy with an interaction term included. Now there is no U-shaped pattern.

Fitting the interaction term succeeded in removing structure from the error. The new model seems to satisfy the assumptions more successfully and so our inferences are likely to be closer to the truth.

Example: Two-Factor ANOVA with Interaction

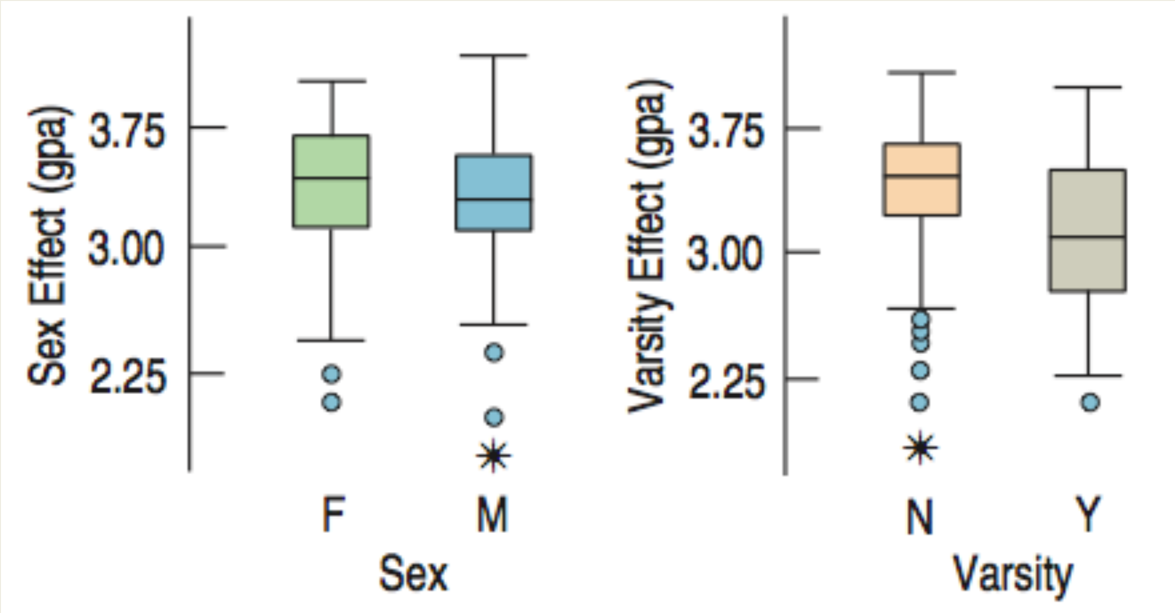
Step-By-Step

Earlier, we looked at how much TV four groups of students watched on average.
Let’s look at their grade point averages (GPAs).
We treated the four groups (male athletes, female athletes, male non- athletes, and female non-athletes) as four levels of the factor *Group*.
Now we recognize that there are really two factors: the factor *Sex* with levels *male* and *female* and the factor *Varsity* with levels *yes* and *no*. Let’s analyze the GPA data with a two-factor ANOVA.

Example: Two-Factor ANOVA with Interaction
Step-By-Step

Plot

Examine the side-by-side partial boxplots of the data.

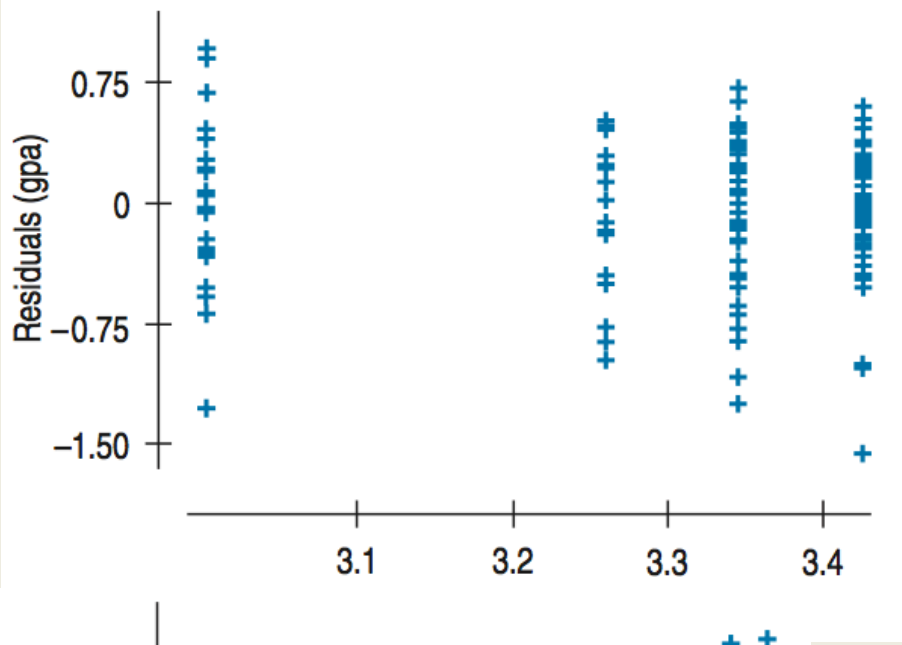


Example: Two-Factor ANOVA with Interaction

Step-By-Step

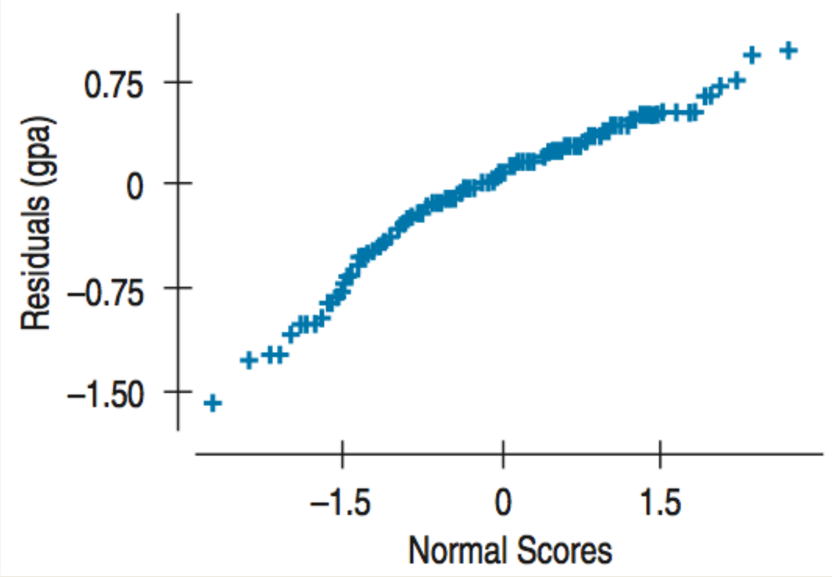
Similar Variance Condition:

no thickening, no pattern



Nearly Normal Condition, Outlier Condition:

plot is reasonable



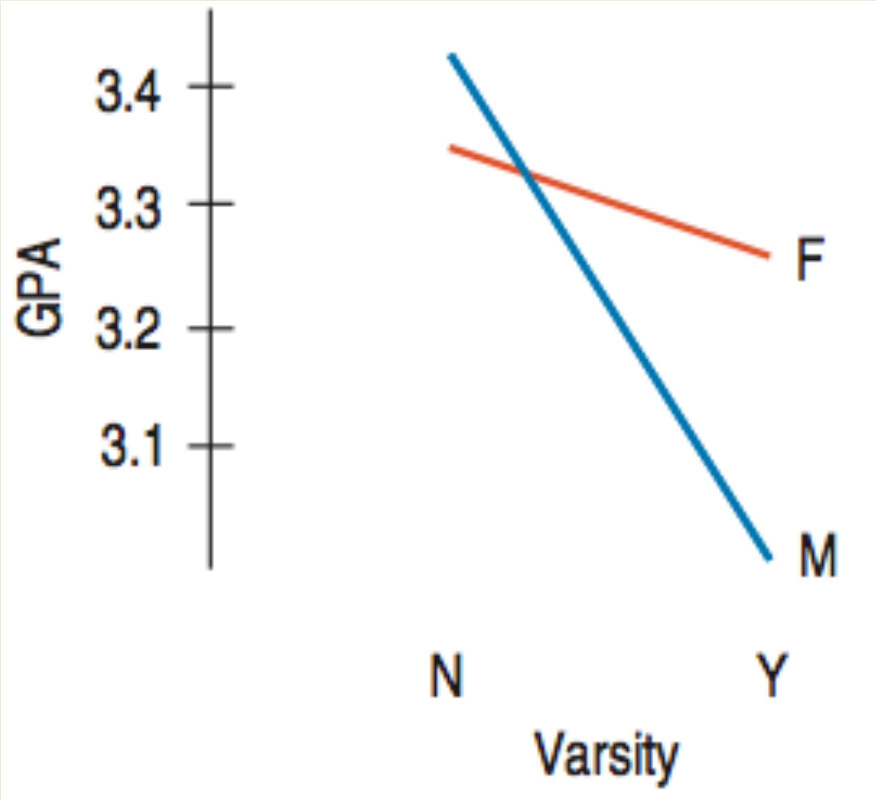
Example: Two-Factor ANOVA with Interaction
Step-By-Step

Plan Show the ANOVAtable.

Analysis of Variance for gpa

Source	DF	Sum of Squares	Mean Square	F-ratio	P-value
Sex	1	0.3040	0.3040	1.7681	0.1852
Varsity	1	2.4345	2.4345	14.1871	0.0002
Sex × Varsity	1	1.0678	1.0678	6.2226	0.0134
Error	196	33.6397	0.1716		
Total	199	37.4898			

Example: Two-Factor ANOVA with Interaction
Step-By-Step



What Can Go Wrong?

Always check for outliers.

Outliers can distort your conclusions.

Check for skewness.

If the underlying data distributions are skewed, consider a transformation to make them more symmetric.

Beware of unbalanced designs and designs with empty cells.

Empty cells and other more serious violations of balance require different methods and additional assumptions.

What Have We Learned?

- We can extend the Analysis of Variance to designs with more than one factor.
- Partial boxplots are a good way to examine the effect of each factor on the response.
- Sometimes factors can interact with each other and we can add an interaction term to our model to account for the possible interaction.
- We need to check the appropriate assumptions and conditions as we did for the simple ANOVA.