Chap 2 行列式

§2.2 高階行列式

$$\begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{3} & \cancel{4} \\ 0 & 1 & 0 & 3 \\ 2 & 5 & 4 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 3 \\ 5 & 4 & 1 \\ 0 & 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 0 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$+3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & 1 \\ 3 & 0 & 2 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 & 0 \\ 2 & 5 & 4 \\ 3 & 0 & 1 \end{vmatrix}$$

Def.

$A: n\times n$, A 之 determinant, $\det(A)$ 可遞迴定義如下

(1) $n=1$. $\det(A) = a_{11}$

(2) $n\geq 2$. $\det(A) = a_{11}\det(A_{11}) - a_{12}(A_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n})$

其中, A_{ij} 表 A 中去掉第 i 列, 第 j 行的 $(n-1)\times(n-1)$ submatrix

Thm

$A: n\times n$

(1) $\det(A) = \sum_{j=1}^{n} a_{ij}\,\mathrm{cof}(a_{ij})$, $\forall i = 1, 2, \cdots, n$

(2) $\det(A) = \sum_{i=1}^{n} a_{ij}\,\mathrm{cof}(a_{ij})$, $\forall j = 1, 2, \cdots, n$

Thm:

$A: n\times n$. 則 $\det(A) = \det(A^T)$

pf. (By induction on n)

$n=1$. , 成立

設 $n=k$, 成立

consider, $n=k+1$. 令 $B=A^T$

$$\det(A) = a_{11}\det(\underbrace{A_{11}}_{k\times k}) - a_{12}\det(\underbrace{A_{12}}_{k\times k}) + \cdots + (-1)^{1+n} a_{1n}\det(\underbrace{A_{1n}}_{k\times k})$$

$$= a_{11}\det(A_{11}^T) - a_{12}\det(A_{12}^T) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n}^T)$$

$$= b_{11}\det(B_{11}) - b_{21}\det(B_{21}) + \cdots + (-1)^{n+1} b_{n1}\det(B_{n1})$$

$$= \det(B) = \det(A^T)$$