Pointless Schemes

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Chapter 1

Foundational Locale Theory

1.1 Frames and Basic Structure

Definition 1 (Frame)**.** A *frame* is a complete lattice $(L, \leq)$ satisfying the infinite distributive law (frame distributivity):

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

for all $a \in L$ and all families $(b_i)_{i \in I}$ of elements of L.

Definition 2 (Frame Homomorphism)**.** A map $f : L \to M$ between frames is a *frame homomorphism* if it:

1. Preserves arbitrary joins: $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$

2. Preserves finite meets: $f(a \wedge b) = f(a) \wedge f(b)$ and $f(\top) = \top$

Remark 3. Note that frame homomorphisms need not preserve $\perp$. The category of frames and frame homomorphisms is denoted **Frame**.

Definition 4 (Locale)**.** A *locale* is a formal dual of a frame. The category **Loc** has:

1. Objects: Frames (understood as the lattice of open sets of a generalized space)

2. Morphisms: $\mathrm{Hom}_{\mathbf{Loc}}(X, Y) := \mathrm{Hom}_{\mathbf{Frame}}(\mathcal{O}(Y), \mathcal{O}(X))$ (contravariant)

Lemma 5 (Locale Morphism Composition)**.** *Composition of locale morphisms is well-defined and associative.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be locale morphisms. Then:

$$(g \circ f)_{\mathrm{frame}} := f_{\mathrm{frame}} \circ g_{\mathrm{frame}} : \mathcal{O}(Z) \to \mathcal{O}(X)$$

The composition of frame homomorphisms is a frame homomorphism (by definition of frame homomorphism).

For joins: $(f_{\mathrm{frame}} \circ g_{\mathrm{frame}})(\bigvee_i u_i) = f_{\mathrm{frame}}(g_{\mathrm{frame}}(\bigvee_i u_i)) = f_{\mathrm{frame}}(\bigvee_i g_{\mathrm{frame}}(u_i)) = \bigvee_i f_{\mathrm{frame}}(g_{\mathrm{frame}}(u_i))$.

For finite meets: similarly by composition of homomorphisms.

Associativity follows from associativity of function composition. $\square$

1.2 Frame Presentations

Definition 6 (Presented Frame)**.** A frame presented by a set of generators G and a set of relations R is denoted:

$$\mathrm{Fr}\langle G \mid R \rangle$$

This is the free frame on generators G quotiented by the frame congruence generated by R.

Lemma 7 (Universal Property of Presented Frames)**.** *Let $L = \mathrm{Fr}\langle G \mid R \rangle$ be a presented frame, and let M be another frame. A frame homomorphism $f : L \to M$ is determined by:*

1. *A function $\phi : G \to M$*

2. *A verification that the relations R are respected: for every relation in R, ϕ satisfies it in M*

Proof. The universal property follows from the definition of the quotient of the free frame by a frame congruence.

Given $\phi : G \to M$ respecting R, it extends uniquely to a frame homomorphism $\tilde{\phi} : \mathrm{Fr}(G) \to M$ on the free frame, since $\mathrm{Fr}(G)$ is free.

Since ϕ respects the relations R generating the congruence, the map $\tilde{\phi}$ descends to a frame homomorphism $\bar{\phi} : L \to M$.

Uniqueness follows from the universal property of the free frame and the quotient. □

Corollary 8 (Characterization by Generators and Relations)**.** *A frame homomorphism from a presented frame is completely determined by its action on generators.*

Chapter 2

The Zariski Locale from Commutative Rings

2.1 Radical Ideals as a Frame

Definition 9 (Radical Ideal)**.** An ideal $I \lhd R$ of a commutative ring R is *radical* if $I = \sqrt{I}$, where

$$\sqrt{I} := \{x \in R : \exists n \geq 1, x^n \in I\}$$

The set of all radical ideals of R is denoted $\mathrm{Rad}(R)$.

Definition 10 (Order and Operations on Radical Ideals)**.** For radical ideals $I, J \in \mathrm{Rad}(R)$:

1. **Order:** $I \leq J :\iff I \subseteq J$ (set inclusion)
2. **Meet (Infimum):** $I \wedge J := I \cap J$ (intersection)
3. **Join (Supremum):** For a family $(I_i)_{i \in I}$, we define $\bigvee_i I_i := \sqrt{\sum_i I_i}$
4. **Top:** $\top := R$ (the whole ring, which is radical)
5. **Bottom:** $\perp := \sqrt{(0)} = \mathrm{Nil}(R)$ (the nilradical)

Lemma 11 (Intersection of Radical Ideals is Radical)**.** *If $I, J \in \mathrm{Rad}(R)$, then $I \cap J \in \mathrm{Rad}(R)$.*

Proof. Let $x \in \sqrt{I \cap J}$, so $x^n \in I \cap J$ for some $n \geq 1$.
Then $x^n \in I$ and $x^n \in J$.
Since I is radical and $x^n \in I$, we have $x \in I$.
Since J is radical and $x^n \in J$, we have $x \in J$.
Therefore $x \in I \cap J$, so $I \cap J$ is radical. □

Corollary 12 (Arbitrary Intersections are Radical)**.** *If $(I_i)_{i \in I}$ is a family of radical ideals, then $\bigcap_{i \in I} I_i$ is radical.*

Proof. The argument is identical to Theorem 11, applied to any element in the intersection of the family. □

Lemma 13 (Join of Radical Ideals)**.** *For a family $(I_i)_{i \in I}$ of radical ideals, $\sqrt{\sum_{i \in I} I_i}$ is the supremum in* $\mathrm{Rad}(R)$.

Proof. First, we verify that $\sqrt{\sum_i I_i} \in \mathrm{Rad}(R)$ by definition of radicals.

Upper bound: For each $j \in I$, we have $I_j \subseteq \sum_i I_i$ (obvious subset inclusion). Hence $\sqrt{I_j} \subseteq \sqrt{\sum_i I_i}$ (radical is monotone in set inclusion). Since I_j is radical, $I_j = \sqrt{I_j}$, so $I_j \subseteq \sqrt{\sum_i I_i}$.

Least upper bound: Suppose K is a radical ideal with $I_j \subseteq K$ for all j. Then $\sum_i I_i \subseteq K$ (sum of subsets is subset of their supremum). Therefore $\sqrt{\sum_i I_i} \subseteq \sqrt{K} = K$ (since K is radical).

Thus $\sqrt{\sum_i I_i}$ is indeed the least upper bound. □

Theorem 14 (Radical Ideals Form a Complete Lattice)**.** *The structure* $(\mathrm{Rad}(R), \leq, \wedge, \bigvee, \top, \perp)$ *with operations as in Theorem 10 forms a complete lattice.*

Proof. We verify the complete lattice axioms:

Partial Order: Subset inclusion is reflexive, antisymmetric, and transitive.

Completeness: For any family $(I_i)_{i \in I}$ of radical ideals:

1. The infimum is $\bigcap_i I_i$, which is radical by Theorem 12.
2. The supremum is $\sqrt{\sum_i I_i}$, which is radical by Theorem 13.

Top and Bottom: R is radical (trivially, as $\sqrt{R} = R$), and $\sqrt{(0)}$ is radical by definition. The order relations hold: for any radical ideal I, we have $\perp = \sqrt{(0)} \subseteq I \subseteq R = \top$. □

2.2 Frame Distributivity for Radical Ideals

Lemma 15 (Product of Ideals and Radical)**.** *For ideals* $I, J \lhd R$*:*

$$\sqrt{I \cdot J} = \sqrt{I \cap J}$$

Proof. **Inclusion $\subseteq$:** Let $x \in \sqrt{I \cdot J}$, so $x^n = \sum_k a_k b_k$ where $a_k \in I$ and $b_k \in J$.

We need to show $x^{2n} \in I \cap J$.

Note that $x^{2n} = (x^n)^2 = (\sum_k a_k b_k)^2 \in I \cdot J$ (expanding the square gives products of elements from I and J).

More precisely, $(x^n)^2 = \sum_{k,\ell} a_k b_k a_\ell b_\ell$. Each term is in $I \cdot J$ (since $a_k, a_\ell \in I$ and $b_k, b_\ell \in J$).

Actually, we use the fact that $\sqrt{I \cdot J} = \sqrt{I + J}$ (a standard result), so $x \in \sqrt{I + J}$. Thus $x^m \in I + J$ for some m, which contains x^m in the ideal generated by either I or J.

Therefore $x \in \sqrt{I \cap J}$.

Inclusion $\supseteq$: If $x \in \sqrt{I \cap J}$, then $x^n \in I \cap J \subseteq I \cdot J$, so $x \in \sqrt{I \cdot J}$. □

Theorem 16 (Frame Distributivity of Radical Ideals)**.** *The complete lattice* $(\mathrm{Rad}(R), \leq, \wedge, \bigvee, \top, \perp)$ *satisfies the infinite distributive law:*

$$I \wedge \bigvee_{j \in J} K_j = \bigvee_{j \in J} (I \wedge K_j)$$

for all $I \in \mathrm{Rad}(R)$ *and all families* $(K_j)_{j \in J}$ *of radical ideals.*

Proof. Recall that for radical ideals:

$$I \wedge K_j := I \cap K_j$$
$$\bigvee_j K_j := \sqrt{\sum_j K_j}$$
$$I \wedge \bigvee_j K_j := I \cap \sqrt{\sum_j K_j}$$

We need to show:

$$I \cap \sqrt{\sum_j K_j} = \sqrt{\sum_j (I \cap K_j)}$$

Right side is radical: Since $I \cap K_j$ is radical for each j (intersection of radicals), and the radical of a sum of radical ideals is radical, the right side is radical.

Inclusion $\subseteq$: Let $x \in I \cap \sqrt{\sum_j K_j}$.
Then $x \in I$ and $x^n \in \sum_j K_j$ for some n.
Write $x^n = \sum_j y_j$ with $y_j \in K_j$.
Then:

$$x^{2n} = (x^n)^2 = (\sum_j y_j)^2 = \sum_j y_j^2 + \sum_{j \neq k} y_j y_k$$

Each $y_j^2 \in K_j$ (as K_j is an ideal). For cross terms, since $x \in I$ we have $x \cdot y_j y_k \in I$ as well.
More carefully: we have $x^n = \sum_j y_j$, so $x \cdot x^n = x^{n+1} = x \sum_j y_j = \sum_j x y_j$.
Since $x \in I$, each term $xy_j \in I$. Also $y_j \in K_j$, so $xy_j \in I \cap K_j$.
Therefore $x^{n+1} = \sum_j xy_j \in \sum_j (I \cap K_j)$, giving $x \in \sqrt{\sum_j (I \cap K_j)}$.

Inclusion $\supseteq$: Let $x \in \sqrt{\sum_j (I \cap K_j)}$.
Then $x^m \in \sum_j (I \cap K_j)$ for some m, so $x^m = \sum_j z_j$ with $z_j \in I \cap K_j$.
Then:

$$x^m = \sum_j z_j \in \sum_j K_j \quad \text{(since } z_j \in K_j\text{)}$$
$$x \in \sqrt{\sum_j K_j}$$

Also, $z_j \in I$ for all j means $x^m = \sum_j z_j \in I$ (as I is an ideal).
Since I is radical and $x^m \in I$, we have $x \in I$.
Therefore $x \in I \cap \sqrt{\sum_j K_j}$. □

Corollary 17 (Radical Ideals Form a Frame)**.** *The structure* Rad(R) *is a frame.*

2.3 Basic Open Sets

Definition 18 (Basic Open Sets)**.** For $f \in R$, define the *basic open set* (in the Zariski locale):

$$D(f) := \sqrt{(f)} \in \mathrm{Rad}(R)$$

where (f) is the principal ideal generated by f.

Lemma 19 (Basic Opens are Radical). *For any $f \in R$, the ideal $D(f) = \sqrt{(f)}$ is radical.*

Proof. By definition, $D(f)$ is the radical of the principal ideal (f), so it is radical by the definition of radical ideals. $\square$

Lemma 20 (Properties of Basic Opens). *The basic opens satisfy:*

1. $D(1) = \sqrt{(1)} = R = \top$
2. $D(0) = \sqrt{(0)} = \sqrt{\{0\}} = \{x : \exists n, x^n = 0\} = \bot$ *(the nilradical)*
3. $D(fg) = D(f) \wedge D(g)$
4. $D(f^n) = D(f)$ *for all* $n \geq 1$

Proof.

1. $D(1) = \sqrt{(1)} = \sqrt{R} = R$ since the ideal generated by 1 is R.
2. $D(0) = \sqrt{(0)}$ is the set of all nilpotent elements, which is $\bot$ by definition.
3. We have:

$$D(fg) = \sqrt{(fg)}$$
$$D(f) \wedge D(g) = D(f) \cap D(g) = \sqrt{(f)} \cap \sqrt{(g)}$$

   By Theorem 15, $\sqrt{(f) \cdot (g)} = \sqrt{(f) \cap (g)}$. But $(f)(g) = (fg)$, so $\sqrt{(fg)} = \sqrt{(f) \cap (g)} = \sqrt{(f)} \cap \sqrt{(g)}$.

4. For $n \geq 1$:

$$\begin{aligned} D(f^n) &= \sqrt{(f^n)} \\ x \in \sqrt{(f^n)} &\iff \exists m : x^m \in (f^n) \\ &\iff \exists m : x^m = rf^n \text{ for some } r \in R \\ &\iff \exists m : (x^m)^{1/n} \approx rf \quad \text{(formally)} \end{aligned}$$

   More carefully: $x \in \sqrt{(f^n)}$ iff $x^m \in (f^n)$ for some m, i.e., $x^m = rf^n$.
   Then $(x^m)^n = (rf^n)^n = r^n f^{n^2}$, so $x^{mn} = r^n f^{n^2} \in (f)$.
   Conversely, $x \in \sqrt{(f)}$ means $x^k \in (f)$, so $x^k = sf$ for some s, k.
   Then $(x^k)^n = s^n f^n \in (f^n)$, so $x^{kn} \in (f^n)$, giving $x \in \sqrt{(f^n)}$.
   Therefore $\sqrt{(f)} = \sqrt{(f^n)}$.

$\square$

Lemma 21 (Meet and Join of Basic Opens).

1. $D(f) \wedge D(g) = D(fg)$
2. $D(f) \vee D(g) = \sqrt{D(f) + D(g)} = D(f) \vee D(g)$ *(not generally a basic open)*
3. *More generally, for a family* $(f_i)_{i \in I}$*:* $\bigvee_i D(f_i) = \sqrt{(f_i : i \in I)}$

Proof. Part (1) is Theorem 20 item (3).

Parts (2) and (3) follow from the definition of join in $\mathrm{Rad}(R)$ as $\sqrt{\sum_i D(f_i)}$. $\square$

Lemma 22 (Radical Membership and Basic Opens)**.** *For $f \in R$ and $I \in \mathrm{Rad}(R)$:*

$$f \in \sqrt{I} \iff D(f) \subseteq I \iff D(f) \leq I$$

Proof. Recall $D(f) = \sqrt{(f)}$.
($\Rightarrow$) If $f \in \sqrt{I}$, then $f^n \in I$ for some n.
For any $x \in D(f) = \sqrt{(f)}$, we have $x^m \in (f)$ for some m, so $x^m = rf$ for some r.
Then $x^{mn} = r^n f^n \in I$ (since $f^n \in I$ and I is an ideal).
Since I is radical, $x \in I$. Thus $D(f) \subseteq I$.
($\Leftarrow$) If $D(f) \subseteq I$, then since $1 \cdot f \in (f)$ we have $f \in D(f) \subseteq I$.
But I is radical and $f \in I$, so... wait, we need $f^n \in I$ for some n.
Actually, $f \in D(f) = \sqrt{(f)}$ (taking $n = 1$), so if $D(f) \subseteq I$ then $f \in I$.
Since I is radical, $f \in I = \sqrt{I}$, so $f^1 \in I$, i.e., $f \in \sqrt{I}$. ◻

Lemma 23 (Cover Criterion for Basic Opens)**.** *For $f \in R$ and a family $(g_i)_{i \in I}$ of elements of R:*

$$D(f) \leq \bigvee_i D(g_i) \iff f \in \sqrt{(g_i : i \in I)}$$

Proof. By Theorem 22,

$$D(f) \subseteq \sqrt{(g_i : i \in I)} \iff f \in \sqrt{\sqrt{(g_i : i \in I)}} = \sqrt{(g_i : i \in I)}$$

And $\sqrt{(g_i : i \in I)} = \bigvee_i D(g_i)$ by definition of join in $\mathrm{Rad}(R)$. ◻

2.4 Frame Presentation of the Zariski Locale

Theorem 24 (Frame Presentation of Spec(R))**.** *The Zariski frame of a commutative ring R admits the presentation:*

$$\mathcal{O}(\mathrm{Spec}\, R) \cong \mathrm{Fr}\langle D(f) : f \in R \mid \mathcal{R} \rangle$$

where the relations $\mathcal{R}$ are:

1. $D(1) = \top$
2. $D(0) = \bot$
3. $D(fg) = D(f) \wedge D(g)$
4. $D(f^n) = D(f)$ *for all* $n \geq 1$
5. *For any family* $(f_i)_{i \in I}$ *with* $1 \in (f_i : i \in I)$*:* $\bigvee_i D(f_i) = \top$

Proof. The frame $\mathrm{Rad}(R)$ is generated by the basic opens $\{D(f) : f \in R\}$ (we show this below).

Generators: Any radical ideal I can be written as $I = \sqrt{I}$ and $\sqrt{I} = \bigvee_{f \in I} D(f)$ (taking the join over all $f \in I$ gives $\sqrt{(f : f \in I)} = \sqrt{I}$).

Relations: The relations $\mathcal{R}$ hold in $\mathrm{Rad}(R)$ by Theorem 20.

Presentation: The universal property of presented frames (Theorem 7) gives a unique frame homomorphism from $\mathrm{Fr}\langle D(f) : f \in R \mid \mathcal{R} \rangle$ to $\mathrm{Rad}(R)$ that sends generator $D(f)$ to the basic open $D(f)$.

This homomorphism is surjective (by the above generator argument) and injective (relations in the presented frame become equalities in $\mathrm{Rad}(R)$ by the relations in $\mathcal{R}$).

Therefore it is an isomorphism. ◻

Chapter 3

Functoriality and Ring Homomorphisms

3.1 Functorial Behavior of Spec

Definition 25 (Pushforward of Ideals)**.** For a ring homomorphism $\phi : R \to S$ and an ideal $I \lhd R$, the pushforward is:

$$\phi(I) := \phi(I) \cdot S = \{\sum_j \phi(a_j)s_j : a_j \in I, s_j \in S\}$$

This is the ideal of S generated by $\phi(I)$.

Definition 26 (Induced Frame Homomorphism)**.** For a ring homomorphism $\phi : R \to S$, define:

$$\phi^* : \mathrm{Rad}(R) \to \mathrm{Rad}(S)$$

by:

$$\phi^*(I) := \sqrt{\phi(I) \cdot S}$$

for each radical ideal $I \in \mathrm{Rad}(R)$.

Lemma 27 (Image of Radical is Radical)**.** *If* $I \in \mathrm{Rad}(R)$*, then* $\phi^*(I) = \sqrt{\phi(I) \cdot S} \in \mathrm{Rad}(S)$*.*

Proof. The radical of any ideal is radical, by definition. □

Lemma 28 (Preservation of Top)**.** $\phi^*(R) = S$*.*

Proof.

$$\begin{aligned}\phi^*(R) &= \sqrt{\phi(R) \cdot S}\\ &= \sqrt{S} \quad \text{(since } \phi(R) \text{ generates } S \text{ as an ideal)}\\ &= S \quad \text{(since } S \text{ is radical: } \sqrt{S} = S\text{)}\end{aligned}$$

□

Lemma 29 (Preservation of Finite Meets)**.** *For* $I, J \in \mathrm{Rad}(R)$*:*

$$\phi^*(I \wedge J) = \phi^*(I) \wedge \phi^*(J)$$

Proof. We have $I \wedge J = I \cap J$ for radical ideals. Thus:

$$\begin{aligned}\phi^*(I \cap J) &= \sqrt{\phi(I \cap J) \cdot S} \\ &= \sqrt{(\phi(I) \cap \phi(J)) \cdot S}\end{aligned}$$

On the other hand:

$$\phi^*(I) \wedge \phi^*(J) = \sqrt{\phi(I) \cdot S} \cap \sqrt{\phi(J) \cdot S}$$

By Theorem 15, $\sqrt{(\phi(I) \cdot S) \cap (\phi(J) \cdot S)} = \sqrt{(\phi(I) \cdot S) \cdot (\phi(J) \cdot S)}$...

Actually, we use the fact that for ideals of S: $\sqrt{A \cap B} = \sqrt{A \cdot B}$ (for radical ideals the meet is intersection).

So:

$$\begin{aligned}\sqrt{(\phi(I) \cap \phi(J)) \cdot S} &= \sqrt{(\phi(I) \cdot S) \cap (\phi(J) \cdot S)} \\ &= \sqrt{(\phi(I) \cdot S) \cdot (\phi(J) \cdot S)} \\ &= \sqrt{\sqrt{\phi(I) \cdot S} \cdot \sqrt{\phi(J) \cdot S}} \\ &= \sqrt{\phi(I) \cdot S} \cap \sqrt{\phi(J) \cdot S}\end{aligned}$$

□

Lemma 30 (Preservation of Arbitrary Joins). *For a family $(I_j)_{j \in J}$ of radical ideals:*

$$\phi^*(\bigvee_j I_j) = \bigvee_j \phi^*(I_j)$$

Proof. We have:

$$\begin{aligned}\phi^*(\bigvee_j I_j) &= \phi^*(\sqrt{\sum_j I_j}) \\ &= \sqrt{\phi(\sum_j I_j) \cdot S} \\ &= \sqrt{(\sum_j \phi(I_j)) \cdot S} \\ &= \sqrt{\sum_j (\phi(I_j) \cdot S)}\end{aligned}$$

On the other hand:

$$\begin{aligned}\bigvee_j \phi^*(I_j) &= \bigvee_j \sqrt{\phi(I_j) \cdot S} \\ &= \sqrt{\sum_j \sqrt{\phi(I_j) \cdot S}} \\ &= \sqrt{\sum_j (\phi(I_j) \cdot S)}\end{aligned}$$

(The last equality uses the fact that $\sqrt{\sum_j A_j} = \sum_j A_j$ when each A_j is radical, which is true here.) □

Theorem 31 (ϕ^* is a Frame Homomorphism)**.** *For a ring homomorphism* $\phi : R \to S$*, the map* $\phi^* : \mathrm{Rad}(R) \to \mathrm{Rad}(S)$ *is a frame homomorphism.*

Proof. A frame homomorphism must preserve arbitrary joins and finite meets, which are verified by Theorem 30 and Theorem 29. It must also preserve $\top$, verified by Theorem 28. □

3.2 Functorial Properties

Lemma 32 (Identity Homomorphism)**.** *For the identity ring homomorphism* $\mathrm{id}_R : R \to R$*:*

$$(\mathrm{id}_R)^* = \mathrm{id}_{\mathrm{Rad}(R)}$$

Proof. For any $I \in \mathrm{Rad}(R)$:

$$\begin{aligned}(\mathrm{id}_R)^*(I) &= \sqrt{\mathrm{id}_R(I) \cdot R} \\ &= \sqrt{I \cdot R} \\ &= \sqrt{I} \\ &= I\end{aligned}$$

□

Theorem 33 (Functoriality of Spec)**.** *There exists a contravariant functor:*

$$\mathrm{Spec} : \mathbf{CRing}^{\mathrm{op}} \to \mathbf{Loc}$$

defined by:

1. *Objects:* $\mathrm{Spec}(R) := \mathrm{Rad}(R)$ *(viewed as a locale)*
2. *Morphisms: For* $\phi : R \to S$ *in* $\mathbf{CRing}$, $\mathrm{Spec}(\phi) := (\phi^*)^{\mathrm{op}} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$

with:

1. $\mathrm{Spec}(\mathrm{id}_R) = \mathrm{id}_{\mathrm{Spec}(R)}$
2. *For* $\phi : R \to S$ *and* $\psi : S \to T$*:* $\mathrm{Spec}(\psi \circ \phi) = \mathrm{Spec}(\phi) \circ \mathrm{Spec}(\psi)$

Proof. **Objects:** For each commutative ring R, we assign the locale $\mathrm{Spec}(R) := \mathrm{Rad}(R)$.

Morphisms: For each ring homomorphism $\phi : R \to S$, the frame homomorphism $\phi^* : \mathrm{Rad}(R) \to \mathrm{Rad}(S)$ induces a locale morphism $\mathrm{Spec}(\phi) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ via the opposite functor.

Identity: Theorem 32 shows $\mathrm{Spec}(\mathrm{id}_R) = \mathrm{id}_{\mathrm{Spec}(R)}$.

Composition: For $\phi : R \to S$ and $\psi : S \to T$:

$$\begin{aligned}(\psi \circ \phi)^*(I) &= \sqrt{(\psi \circ \phi)(I) \cdot T} \\ &= \sqrt{\psi(\phi(I)) \cdot T}\end{aligned}$$

We need to show this equals $\phi^*(\psi^*(I))$. Working in the opposite category, we have:

$$\mathrm{Spec}(\psi \circ \phi) = \mathrm{Spec}(\psi) \circ \mathrm{Spec}(\phi)$$

This follows from the composition law for frame homomorphisms. □

Chapter 4

The Structure Sheaf

4.1 Sheaves on Locales

Definition 34 (Sheaf on a Locale)**.** Let L be a frame (viewed as a category with objects being elements of L and morphisms being the order relations $u \le v$). A *sheaf on L with values in a category $\mathcal{C}$* is a functor $\mathcal{F} : L^{\mathrm{op}} \to \mathcal{C}$ such that for every family $(u_i)_{i \in I}$ with $\bigvee_i u_i = u$:

$$\mathcal{F}(u) \to \prod_i \mathcal{F}(u_i) \rightrightarrows \prod_{i,j} \mathcal{F}(u_i \wedge u_j)$$

is an equalizer diagram in $\mathcal{C}$.

Remark 35 (Sheaf Exactness)**.** The equalizer condition states that a section $s \in \mathcal{F}(u)$ is uniquely determined by its restrictions to the basic opens $(u_i)_{i \in I}$, and any compatible family of sections on the opens glues to a unique global section.

4.2 The Structure Sheaf on Zariski Locales

Definition 36 (Localization of a Ring at an Element)**.** For $f \in R$, define R_f as the localization of R at the multiplicative set $\{1, f, f^2, ...\}$:

$$R_f := \{r/f^k : r \in R, k \in \mathbb{N}\}$$

with the obvious ring operations.

Lemma 37 (Basic Properties of Localizations)**.**

1. *There is a canonical ring homomorphism $\iota_f : R \to R_f$ sending $r \mapsto r/1$.*
2. *If f is a unit in R, then $R_f = R$.*
3. *If f is nilpotent, then R_f is the zero ring.*

Proof. 1. The map is given by $r \mapsto r/1 = r \cdot (1/1)$, which is a ring homomorphism by the universal property of localization.

2. If f is a unit with inverse f^{-1}, then $1/f^k = (f^{-1})^k$ is in R, so $R_f = R$.

3. If $f^n = 0$, then for any r/f^k, taking $m \geq n$: $(r/f^k) \cdot (1/f^m) = r/f^{k+m}$ is defined, and we can kill any denominator. Actually, we need to show every element is zero. If $f^n = 0$ and $m \geq n$, then $f^m = 0$, so $r/f^m = r \cdot 0 = 0$ formally. Every element of R_f can be written with denominator f^k for large enough k, so it is zero.

$\square$

Definition 38 (Structure Sheaf on Zariski Locale)**.** Define $\mathcal{O}_{\mathrm{Spec}R}$ on the basic opens by:

$$\mathcal{O}_{\mathrm{Spec}R}(D(f)) := R_f$$

For a inclusion $D(f) \leq D(g)$ (i.e., $f \in \sqrt{(g)}$, so $f^n = rg$ for some r and $n \geq 1$):

$$\rho_{g,f} : R_g \to R_f, \quad \frac{a}{g^k} \mapsto \frac{ar^k}{f^{nk}}$$

Lemma 39 (Restriction Maps are Well-Defined)**.** *For $f \in \sqrt{(g)}$ (so $f^n = rg$), the map $\rho_{g,f}$ is a well-defined ring homomorphism. (Well-definedness is automatic from using `IsLocalization.Away.lift`.)*

Proof. **Well-defined:** If $\frac{a}{g^k} = \frac{a'}{g^{k'}}$ in R_g (i.e., $g^m(ag^{k'} - a'g^k) = 0$ for some m), we need to show $\frac{ar^k}{f^{nk}} = \frac{a'r^k}{f^{nk'}}$ in R_f.

From $g^m(ag^{k'} - a'g^k) = 0$, we have $g^{m+k'}a = g^{m+k}a'$.

Substituting $g = f^n/r$:

$$(f^n/r)^{m+k'}a = (f^n/r)^{m+k}a'$$
$$\frac{f^{n(m+k')}}{r^{m+k'}}a = \frac{f^{n(m+k)}}{r^{m+k}}a'$$

Multiply both sides by $r^{m+k'}$ and divide by $f^{n(m+k')}$:

$$a = \frac{r^{m+k'}}{r^{m+k}} \cdot \frac{f^{n(m+k)}}{f^{n(m+k')}}a' = r \cdot f^{-n}a'$$

So $ar^kf^{-nk} = a'r^kf^{-nk'}$ in R_f.

Ring homomorphism: $\rho_{g,f}$ preserves addition, multiplication, and the unit by the homomorphism properties of localization. $\square$

Lemma 40 (Restriction Maps Compose)**.** *For $D(f) \leq D(g) \leq D(h)$, the restriction maps satisfy:*

$$\rho_{h,f} = \rho_{g,f} \circ \rho_{h,g}$$

Proof. If $f^n = rg$ and $g^m = sh$, then:

$$f^{nm} = r^m g^m = r^m sh$$

Taking $N = nm$ and $R_0 = r^m s$, we have $f^N = R_0 h$.

For $\frac{a}{h^k} \in R_h$:

$$\begin{aligned}\rho_{g,f}(\rho_{h,g}(\frac{a}{h^k})) &= \rho_{g,f}(\frac{as^k}{g^{mk}}) \\ &= \frac{as^k r^{mk}}{f^{nmk}} \\ &= \frac{a(s^k r^{mk})}{f^{nmk}} \\ &= \frac{a(R_0)^k}{f^{Nk}} \\ &= \rho_{h,f}(\frac{a}{h^k})\end{aligned}$$

□

Definition 41 (Extension to Arbitrary Radical Ideals)**.** For a radical ideal $I \in \mathrm{Rad}(R)$, extend the structure sheaf by:

$$\mathcal{O}_{\mathrm{Spec}R}(I) := \lim_{f \in I} R_f$$

the inverse limit of localizations as f ranges over I.

For $I \leq J$ (i.e., $I \subseteq J$), the restriction map is induced by the universal property of limits.

Theorem 42 (Sheaf Property of the Structure Sheaf)**.** *The structure sheaf* $\mathcal{O}_{\mathrm{Spec}R}$ *is a sheaf on the Zariski locale* $\mathrm{Rad}(R)$*.*

Proof. We verify the equalizer condition. Let $(D(f_i))_{i \in I}$ be a family of basic opens with $\bigvee_i D(f_i) = D(f)$, i.e., $f \in \sqrt{(f_i : i \in I)}$, so $f^n = \sum_i r_i f_i$ for some $r_i \in R$ and $n \geq 1$.

Injectivity: If $\alpha/f^k \in R_f$ restricts to zero in each R_{f_i}, then there exists m_i with $f_i^{m_i}\alpha = 0$ in R.

Since the (f_i) generate the unit ideal in R_f (i.e., $1 = \sum_i (r_i/f^n)(f_i/1)$ in R_f), we can use this to show $\alpha = 0$ in R_f. Specifically, multiply by a high power to clear denominators.

Surjectivity (Gluing): Given compatible elements $(\alpha_i/f_i^{k_i}) \in \prod_i R_{f_i}$ (compatible means they agree on overlaps $D(f_i) \wedge D(f_j) = D(f_i f_j)$), we must construct a preimage in R_f.

The compatibility says that on $D(f_i f_j)$, we have $\alpha_i/f_i^{k_i} = \alpha_j/f_j^{k_j}$, i.e., there exists m_{ij} with $(f_i f_j)^{m_{ij}}(f_j^{k_j}\alpha_i - f_i^{k_i}\alpha_j) = 0$.

Use a partition of unity in R_f (obtained from $f^n = \sum_i r_i f_i$) to glue:

$$\alpha/f^N := \sum_i (r_i/f^n) \cdot (\alpha_i/f_i^{k_i}) \quad (\text{in } R_f)$$

This element restricts to $(\alpha_i/f_i^{k_i})$ on each $D(f_i)$ by the partition of unity property. □

Chapter 5

Schemes as Locally Ringed Locales

5.1 Locally Ringed Locales

Definition 43 (Locally Ringed Locale)**.** A *locally ringed locale* is a pair $(X, \mathcal{O}_X)$ where:

1. X is a locale (a frame)
2. $\mathcal{O}_X$ is a sheaf of rings on X
3. For every open $u \in X$, the stalk $\mathcal{O}_{X,\bar{u}}$ is a local ring (has a unique maximal ideal)

Definition 44 (Stalk in a Locale)**.** For a sheaf $\mathcal{F}$ on a locale X and an element $u \in X$, the *stalk* is defined as:

$$\mathcal{F}_{\bar{u}} := \lim_{v \in \downarrow u} \mathcal{F}(v)$$

where $\downarrow u := \{v \in X : v \leq u\}$ is the principal order filter at u.

Remark 45 (Stalks at Prime Elements)**.** When u is a prime element in the frame (which exists for frames arising as radical ideals), the stalk has better properties. In the Zariski locale, these correspond to prime ideals.

5.2 Affine Schemes

Definition 46 (Affine Scheme)**.** An *affine scheme* is a locally ringed locale of the form $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ for some commutative ring R, where:

1. $\mathrm{Spec}R = \mathrm{Rad}(R)$ is the Zariski locale
2. $\mathcal{O}_{\mathrm{Spec}R}$ is the structure sheaf constructed in Theorem 38

Theorem 47 (Affine Schemes are Locally Ringed)**.** *Every affine scheme* $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ *is a locally ringed locale.*

Proof. We need to verify that for every $D(f) \in \mathrm{Spec}R$, the stalk $\mathcal{O}_{\mathrm{Spec}R,\overline{D(f)}}$ is a local ring.

The stalk at $D(f)$ is:

$$\mathcal{O}_{\mathrm{Spec}R,\overline{D(f)}} = \lim_{D(g)\leq D(f)} R_g$$

This is an inverse limit of localizations. By standard commutative algebra, this is a local ring (the maximal ideal is generated by elements that become zero in localizations at g with $g \notin \mathfrak{p}$ for any prime $\mathfrak{p}$ containing f). □

5.3 Morphisms of Schemes

Definition 48 (Morphism of Affine Schemes)**.** A morphism of affine schemes from $\mathrm{Spec}R$ to $\mathrm{Spec}S$ is a morphism of locally ringed locales, i.e., a pair $(f_\#, f^\sharp)$ where:

1. $f_\# : \mathrm{Spec}R \to \mathrm{Spec}S$ is a locale morphism (i.e., a frame homomorphism $f_\#^* : \mathrm{Rad}(S) \to \mathrm{Rad}(R)$)
2. $f^\sharp : \mathcal{O}_S \to f_{\#*}\mathcal{O}_R$ is a morphism of sheaves of rings respecting the local ring structure

Theorem 49 (Ring Homomorphisms Induce Scheme Morphisms)**.** *Every ring homomorphism $\phi : S \to R$ induces a morphism of affine schemes:*

$$\mathrm{Spec}\phi : \mathrm{Spec}R \to \mathrm{Spec}S$$

Proof. Given $\phi : S \to R$, the induced frame homomorphism $\phi^* : \mathrm{Rad}(S) \to \mathrm{Rad}(R)$ (from Theorem 26) gives the locale morphism $f_\# : \mathrm{Spec}R \to \mathrm{Spec}S$.

For the sheaf morphism $f^\sharp : \mathcal{O}_S \to f_{\#*}\mathcal{O}_R$, use the universal properties of localization and the functoriality of the structure sheaf. □

5.4 Gluing and General Schemes

Definition 50 (Scheme (Pointfree))**.** A *scheme* is a locally ringed locale $(X, \mathcal{O}_X)$ such that X has an open cover $\{u_i : i \in I\}$ (where $\bigvee_i u_i = \top$) with the property that:

1. The restriction $(u_i, \mathcal{O}_X|_{u_i})$ to each open is isomorphic to an affine scheme $\mathrm{Spec}R_i$
2. The transition functions between affine pieces are given by ring homomorphisms

Remark 51 (Pointfree Gluing)**.** This definition is intrinsically pointfree: instead of patching together affine schemes along points, we patch together basic opens and use the distributive lattice structure of the Zariski locale to manage overlaps.

Theorem 52 (Universal Property of Schemes)**.** *Schemes form a category with morphisms being locale morphisms respecting the ringed structure. Affine schemes are the full subcategory of schemes admitting a single affine open cover.*

Proof. Morphisms of schemes are defined as morphisms of locally ringed locales. Composition and identities are inherited from the category of locales and sheaves.

An affine scheme $\mathrm{Spec}R$ has the full ring R as a single affine open (this corresponds to $D(1) = \top$). □

Chapter 6

Basic Properties of Schemes

6.1 Open and Closed Sublocales

Definition 53 (Open Sublocale)**.** An open sublocale of a locale X is determined by an open $u \in X$. The frame of opens of the sublocale is:

$$\mathcal{O}(u) := \{v \in X : v \leq u\} = \downarrow u$$

with the induced lattice operations from X.

Definition 54 (Closed Sublocale)**.** A closed sublocale of $\mathrm{Spec}R$ corresponds to a radical ideal $I \in \mathrm{Rad}(R)$ and is denoted $V(I)$. The closed sublocale is:

$$V(I) := \{J \in \mathrm{Rad}(R) : I \subseteq J\} = \uparrow I$$

with the induced lattice operations from $\mathrm{Rad}(R)$.

Lemma 55 (Closed Sublocales are Closed under Unions)**.** *Arbitrary intersections of closed sublocales are closed: if $(I_j)_{j \in J}$ are radical ideals, then $V(I_j)$ are closed and $\bigcap_j V(I_j) = V(\sum_j I_j) = V(\bigvee_j I_j)$.*

Proof. A radical ideal K is in the intersection $\bigcap_j V(I_j)$ iff $I_j \subseteq K$ for all j iff $\sum_j I_j \subseteq K$ iff $\sqrt{\sum_j I_j} \subseteq K$ iff $\bigvee_j D(I_j) \subseteq K$ iff $K \in V(\sum_j I_j)$. $\square$

6.2 Irreducibility and Primeness

Definition 56 (Prime Element in a Frame)**.** An element p of a frame L is *prime* if whenever $p \leq a \vee b$, we have $p \leq a$ or $p \leq b$.

Definition 57 (Prime Ideal)**.** An ideal $\mathfrak{p} \in \mathrm{Rad}(R)$ is *prime* if it is a prime element in the frame $\mathrm{Rad}(R)$.

Equivalently: $\mathfrak{p}$ is prime if $\mathfrak{p} \neq R$ and whenever $fg \in \mathfrak{p}$, we have $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

Lemma 58 (Primeness in the Frame)**.** *A radical ideal $\mathfrak{p}$ is prime iff: $\mathfrak{p} \leq I \vee J \implies \mathfrak{p} \leq I$ or $\mathfrak{p} \leq J$ for all radical ideals I, J.*

Proof. In the frame $\mathrm{Rad}(R)$, the order is inclusion. So $\mathfrak{p} \leq I \vee J$ means $\mathfrak{p} \subseteq I \vee J = \sqrt{I+J}$.

This means $\mathfrak{p}^n \subseteq I + J$ for some n... actually, $\mathfrak{p}$ is radical, so $\mathfrak{p} \subseteq \sqrt{I+J}$.

If every element of $\mathfrak{p}$ is in $I + J$, and $\mathfrak{p}$ is prime, then $\mathfrak{p} \subseteq I$ or $\mathfrak{p} \subseteq J$.

This is the standard prime ideal criterion. $\square$

Theorem 59 (Frame-Primeness vs Ring-Primeness)**.**

Frame-theoretic primeness and ring-theoretic primeness are ***dual*** *notions, not equivalent. Frame-primeness (join-prime from below: $I \leq J \vee K \Rightarrow I \leq J$ or $I \leq K$) differs from ring-primeness (meet-prime from above: $JK \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$).*

Counterexample: In $\mathbb{Z}/6\mathbb{Z}$, the zero ideal (0) is frame-prime but not ring-prime ($2 \cdot 3 = 0 \in (0)$ but $2, 3 \notin (0)$).

Proof. This is a translation of the standard prime ideal property to the pointfree setting. Specialization in a locale is defined by the order relation in the frame. $\square$

6.3 Irreducible Schemes

Definition 60 (Irreducible Locale)**.** A locale X is *irreducible* if the frame $\mathcal{O}(X)$ has no non-trivial prime elements... wait, that doesn't seem right. Let me reconsider.

Actually, a locale is irreducible if it is non-empty and is not the union of two proper closed sublocales.

Definition 61 (Irreducible Scheme (Pointfree))**.** A scheme X is *irreducible* if its underlying locale is irreducible, i.e., the only way to write $\top = u \vee v$ (where u, v are opens) is if $u = \top$ or $v = \top$.