Spatial modeling with INLA and inlabru

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What have we learned

Spatial models

Gaussian Random Field models

The SPDE approach

Fitting spatial models

Space-time Modeling

Prediction in space

What have we learned

INLA in a nurshell

- many data sets these days are complex, resulting in complex models, e.g. with complex dependence structures (spatial, temporal, etc..)
- usually Markov chain Monte Carlo (MCMC) methods have been used to fit these models
    - (realistically) complex models result in very long running times
    - often impossible (or unrealistic) to fit
- INLA (Integrated nested Laplace approximation) is an alternative to MCMC
    - much, much **faster**
    - suitable for a specific (but very large!) class of models

INLA in a nurshell

Three main ingredients in INLA

- Gaussian Markov random fields
- Latent Gaussian models
- Laplace approximations

which together (with a few other things) give a very nice tool for Bayesian inference

- quick
- accurate

Models amenable to INLA: **Latent Gaussian Models**

$$\mathbf{y}|\mathbf{x},\theta \sim \prod_i \pi(y_i|x_i,\theta) \qquad \text{Likelihood}$$

$$\mathbf{x}|\theta \sim \exp\left(-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{Q}(\theta)\mathbf{x}^{\mathbf{T}}\right) \qquad \text{Latent field (GMRF)}$$

$$\theta \sim \pi(\theta) \qquad \text{Hyperparameter}$$

- Each data point depends on only one of the elements in the latent Gaussian field $\mathbf{x}$.

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INLA allows us to compute in a **fast** and **efficient** way an **accurate** approximation to the posterior marginals for the hyperparameters

$$\tilde{\pi}(\theta_j|\mathbf{y})$$

and for the latent field

$$\tilde{\pi}(x_i|\mathbf{y})$$

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All of this is implemented in the `INLA` library in `R`.

Yesterday we saw how to implement simple models with `INLA`

Today we are going to look at how to implement continuous spatial models.

Such model are **a lot** easier to implement in `inlabru`!!!!

inlabru in a nutshell

`inlabru` is a friendlier version of `R-INLA`

- it makes `INLA` more accessible to the user
- makes complex features and predictions (especially for spatial data) a lot easier
- is a softer-wrapper around `INLA`
- in some cases it allows to release some of the contraints of `R-INLA` (the linearity of the predictor)

inlabru

- Installation:
  - There is a CRAN version

```
install.packages("inlabru")
```

- You can also install the development version of inlabru from GitHub (recommended)

```
# install.packages("remotes")
remotes::install_github("inlabru-org/inlabru",
         ref="devel")
```

- Documentation
  - Web site: https://sites.google.com/inlabru.org/inlabru
  - Github: https://github.com/inlabru-org/inlabru

Spatial models

Types of Spatial Data

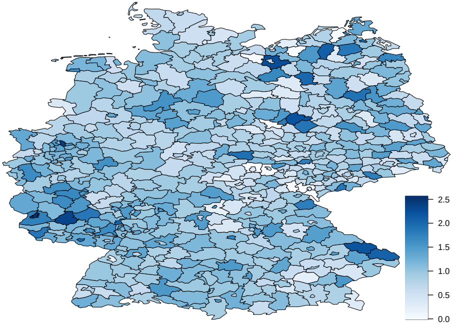
We can distinguish three types of spatial data

- Discrete space
  - data on a spatial grid
- Continuous space:
  - geostatistical data
  - spatial point data

Discrete Counts:

Data on a spatial grid

- examples: number of individuals in a region, average rainfall in a province
- (originally geostatistical or point data; gridded for practical reasons)

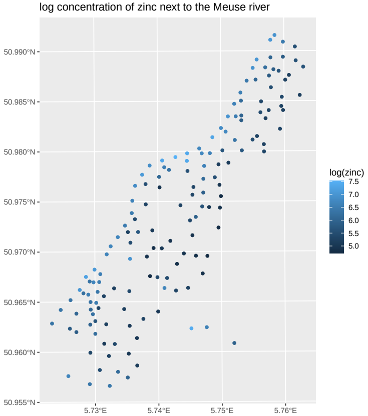


Observed response(s):

- Measurement over each grid cell (e.g. number of individuals in cell; rainfall in province)

Continuous Space: Geostatistics

- phenomenon that is continuous in space
- examples: nutrient levels in soil, salinity in the sea
- measurements at a given set of locations that are determined by surveyor

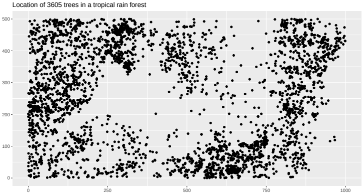


Observed response(s):

- measurement(s) taken at given locations

Continuous Space: Point Process

- locations of objects (individuals) in space (typically 2D)
- examples: locations of trees in a forest, groups of animals



Observed response(s):

- x, y coordinates of points (individuals/groups)
- maybe also properties of individuals/groups ("marks")

Gaussian Random Field models

Gaussian Random fields

Definition:A random function $u(x) : R^d \to R$ is a Gaussian random field if for any finite collection of locations, $(x_1, \ldots, x_n)$, $x_i \in R^d$, the joint distribution of $\mathbf{u} = (u(x_1), \ldots, u(x_n))$ is $\mathbf{u} \sim N(0, \Sigma)$, and

$$
\begin{gathered}
E(u(x)) = 0, \\
Cov(u(x), u(x')) = R(x, x'), \quad \Sigma_{ij} = R(x_i, x_j)
\end{gathered}
$$

for some expectation function $\mu(\cdot)$ and positive definite covariance function $R(\cdot, \cdot)$. Σ is the covariance matrix for the specific location collection.

Matern covariance function

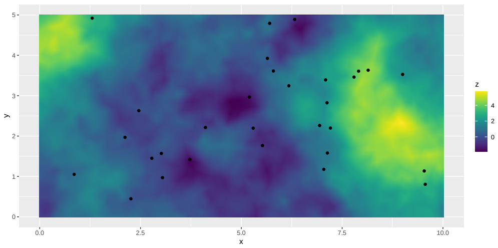
The Matern covariance function is a popular model in spatial problems:

$$c_\nu(d;\sigma,\rho) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\sqrt{8\nu}\frac{d}{\rho}\right)^\nu K_\nu \left(\sqrt{8\nu}\frac{d}{\rho}\right)$$

* ρ and σ are the range and the marginal standard deviation

- K_ν modified Bessel function of the second kind, order ν.

Example



Gaussian Random field

GRF are a very popular model

- Flexible and easy to use
- Can be part of the latent Gaussian field in a LGM

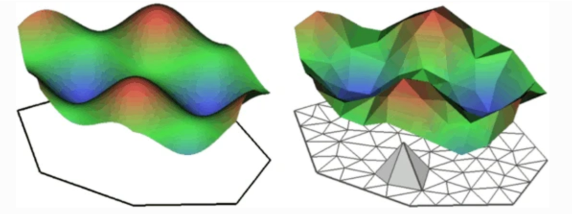
However:

- **computationally inefficient** (the precision matrix is dense)
- not flexible enough (complicated boundary, barrier,...)

The SPDE approach

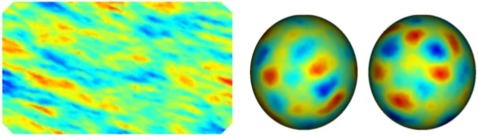
The SPDE approach

- Matern fields can be seen as solution to a PDE
- Using finite element methods such solution can be represented using a GRMF



Advantages of the SPDE approach

- Computationally fast
- Allows for flexible modeling
  - non-stationary models (anisotropy)
  - models on a sphere
  - non separable models



All these models (and my more) can be fitted with `R-INLA` and `inlabru`

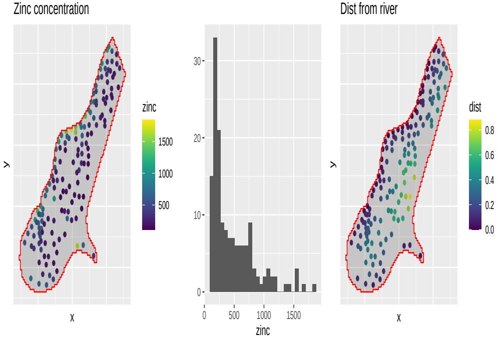
Learning about the SPDE approach

- F. Lindgren, H. Rue, and J. Lindström. *An explicit link between Gaussian fields and Gaussian Markov random fields: The SPDE approach (with discussion)*. In: Journal of the Royal Statistical Society, Series B 73.4 (2011), pp. 423–498.
- H. Bakka, H. Rue, G. A. Fuglstad, A. Riebler, D. Bolin, J. Illian, E. Krainski, D. Simpson, and F. Lindgren. *Spatial modelling with R-INLA: A review*. In: WIREs Computational Statistics 10:e1443.6 (2018). (Invited extended review). DOI: 10.1002/wics.1443.
- E. T. Krainski, V. Gómez-Rubio, H. Bakka, A. Lenzi, D. Castro-Camilio, D. Simpson, F. Lindgren, and H. Rue. *Advanced Spatial Modeling with Stochastic Partial Differential Equations using R and INLA*. Github version www.r-inla.org/spde-book. CRC press, Dec. 20

Fitting spatial models

Example: Meuse Data

Measures of zinc concentration.



The model

$$\log(Y(s)) \sim \mathcal{N}(\eta(s), \sigma_y^2)$$
$$\eta(s) = \alpha + \beta\ x(s) + u(s)$$

where

- $Y(s)$ is the measure of zinc in location s
- α a common intercept
- β a model parameter
- $x(s)$ distance from the river at location s
- $u(s)$ the Matern Gaussian field

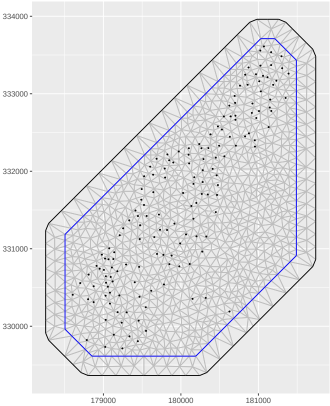
Step 1: Define the SPDE representation

- The mesh
- The SPDE model

Define the SPDE representation: The mesh

1. Define the mesh

```
mesh <- inla.mesh.2d(loc.domain = cbind(meuse$x, meuse$y),
                     max.edge = c(150, 500),
                     offset = c(100, 250) )
```



Define the SPDE representation: The mesh

- All random field models need to be discretised for practical calculations.
- The SPDE models were developed to provide a consistent model definition across a range of discretisations.
- We use finite element methods with local, piecewise linear basis functions defined on a triangulation of a region of space containing the domain of interest.
- Deviation from stationarity is generated near the boundary of the region.
- The choice of region and choice of triangulation affects the numerical accuracy.

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Two separate issues:

- Continuous space with bounded domain: Boundary effect
- Discretised model: Numerical accuracy

Sometimes the boundary effect may be desireable.

Define the SPDE representation: The mesh

- Too fine meshes → heavy computation
- Too coarse mesh → not accurate enought

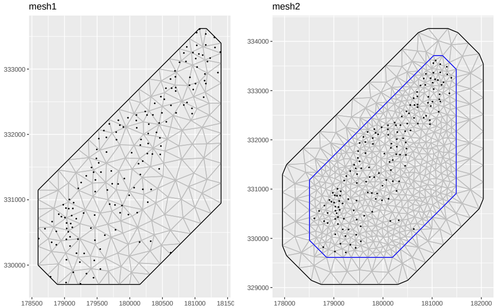
Some guidelines

- Create triangulation meshes with `inla.mesh.2d()`
- Move undesired boundary effects away from the domain of interest by extending to a smooth external boundary (`inla.nonconvex.hull(loc, convex)`, `convex` $\geq$ correlation range)
- Use a coarser resolution in the extension to reduce computational cost (`max.edge=c(inner, outer)`)
- Use a fine resolution (subject to available computational resources) for the domain of interest (inner correlation range) and filter out small input point clusters (`0 < cutoff < inner`)
- Coastlines and similar can be added to the domain specification in `inla.mesh.2d()`

Define the SPDE representation: The mesh

```
mesh1 = inla.mesh.2d(loc.domain = cbind(meuse$x, meuse$y),
                     max.edge = 350,
                     offset = 10)

mesh2 = inla.mesh.2d(loc.domain = cbind(meuse$x, meuse$y),
                     max.edge = c(150, 500),
                     cutoff = 100,
                     offset = c(100, 550) )
```



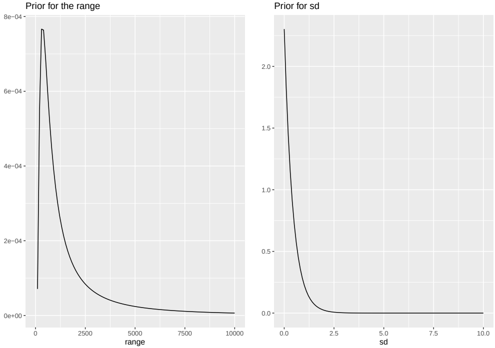
Define the SPDE representation: The SPDE model

```
meuse.spde <- inla.spde2.pcmatern(mesh = mesh,
                                  prior.sigma = c(1, 0.1),
                                  prior.range = c(1000, 0.5))
```

PC-priors for the range ρ and the standard deviarion σ

- Define the prior for the range `prior.range = (range0,Prange)` $\text{Prob}(\rho < \rho_0) = p_\rho$
- Define the prior for the range `prior.sigma = (sigma0,Psigma)` $\text{Prob}(\sigma > \sigma_0) = p_\sigma$

Prior for range ρ and sd σ



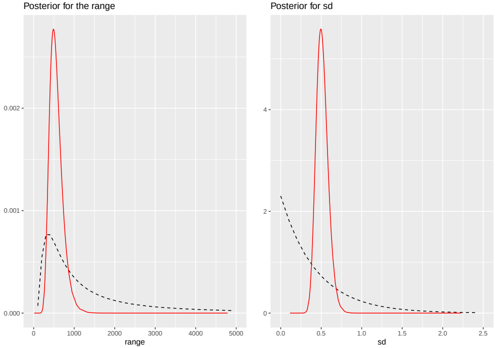
Run the model inlabru

```
# create a spatial object
coordinates(meuse) = c("x","y")
#  covariate values
dist_SPDE = SpatialPixelsDataFrame(data$dist_raster[,c(1,2)],
                                   data = data.frame(dist = data$dist_raster[,3]))
# model components
cmp =  ~ Intercept(1) + dist(dist_SPDE, model = "linear") +
  spde(coordinates, model = meuse.spde)
# define likelihood
lik = like(formula = Y ~ Intercept + dist +   spde,
           family = "gaussian",
           data  = meuse)
#fit the model
fit <- bru(cmp, lik)
# define prediction area
pix <- pixels(mesh, nx = 200, ny = 200, mask = boundary)
# generate predictions
pred = predict(fit, pix, ~ data.frame(
                                       spde = spde,
                                       logscale = Intercept + dist + spde,
                                       naturalscale = exp(Intercept + dist + spde)))
```

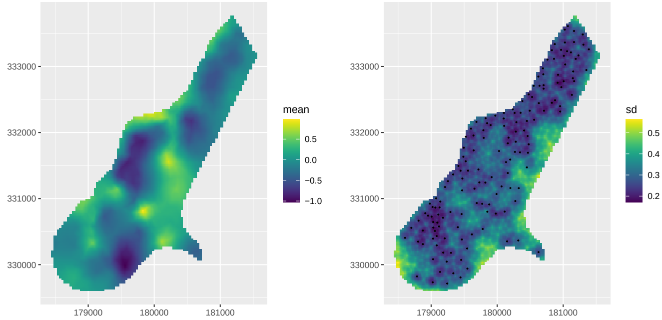
Notes!

- The data are a spatial object!
- For prediction, the covariates are stored in a `SpatialPixelsDataFrame` and need to cover all the mesh nodes

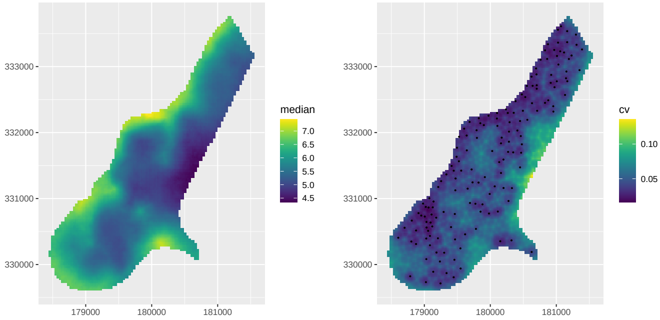
Estimation: Posterior for the hyperparameters



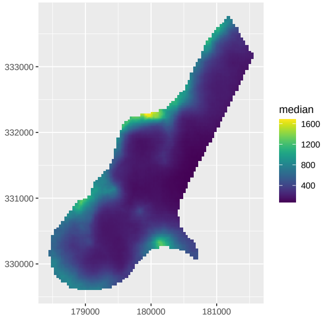
Predictions: The SPDE field

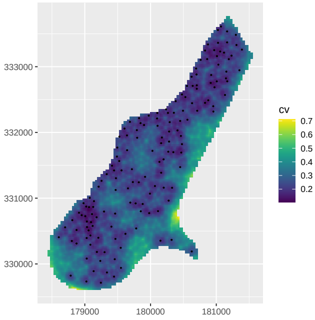


Predictions: The log concentrations



Predictions: The concentrations





Same in plain INLA (1)

```
A.meuse <- inla.spde.make.A(mesh = mesh, loc = coordinates(meuse))
s.index <- inla.spde.make.index(name = "spatial.field",
  n.spde = meuse.spde$n.spde)

#Create data structure
meuse.stack <- inla.stack(data  = list(zinc = meuse$zinc),
  A = list(A.meuse, 1),
  effects = list(c(s.index, list(Intercept = 1)),
    list(dist = meuse$dist)),
  tag = "meuse.data")

data(meuse.grid)
coordinates(meuse.grid) = ~x+y
gridded(meuse.grid) = TRUE

#Create data structure for prediction
A.pred <- inla.spde.make.A(mesh = mesh, loc = coordinates(meuse.grid))
meuse.stack.pred <- inla.stack(data = list(zinc = NA),
  A = list(A.pred, 1),
  effects = list(c(s.index, list (Intercept = 1)),
    list(dist = meuse.grid$dist)),
  tag = "meuse.pred")

#Join stack
join.stack <- inla.stack(meuse.stack, meuse.stack.pred)
```

Same in plain INLA (2)

```
#Fit model
form <- log(zinc) ~ -1 + Intercept + dist + f(spatial.field, model = spde)

m1 <- inla(form, data = inla.stack.data(join.stack, spde = meuse.spde),
  family = "gaussian",
  control.predictor = list(A = inla.stack.A(join.stack), compute = TRUE))
```

Note: We still have not compute predictions. . . and this is not too easy in plain INLA!!

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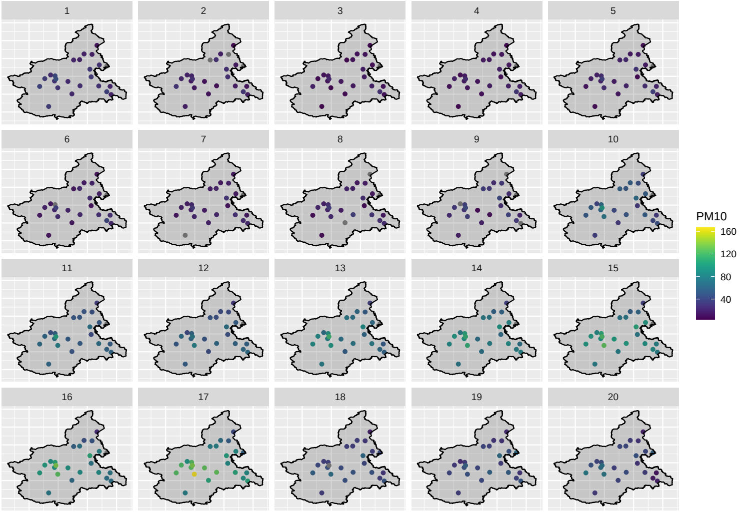
What cannot (at the moment) be done with `inlabru`

- Survival models

Space-time Modeling

Modeling PM10 concentration in Piemonte (Italy)



Modeling PM10 concentration in Piemonte (Italy)

Example from the Blangiardo & Cameletti book but simplified!

```
##   Station.ID       Date   dem      x       y   WS   temp   HMIX PREC   EMI PM10
## 1          1 2005-10-01  95.2 469.45 4972.85 0.90 288.81 1294.6    0 26.05   28
## 2          2 2005-10-01 164.1 423.48 4950.69 0.82 288.67 1139.8    0 18.74   22
## 3          3 2005-10-01 242.9 490.71 4948.86 0.96 287.44 1404.0    0  6.28   17
## 4          4 2005-10-01 149.9 437.36 4973.34 1.17 288.63 1042.4    0 29.35   25
## 5          5 2005-10-01 405.0 426.44 5045.66 0.60 287.63 1038.7    0 32.19   20
##   time  logPM10
## 1    1 3.332205
## 2    1 3.091042
## 3    1 2.833213
## 4    1 3.218876
## 5    1 2.995732
```

The model

We model the log PM10 concentration:

$$
\begin{aligned}
y_{it} &\sim \mathcal{N}(\eta_i t, \sigma_e^2) \\
\eta_{it} &= \alpha + \beta_1 \text{dem}_i + \beta_2 \text{temp}_{it} + \omega_{it}
\end{aligned}
$$

- y_{it} is the log-concentration at location i in time t
- α is an intercept
- β_1 and β_2 parameters of altitude and temperature
- ω_{it} is the space-time residual

Space time residual model

A first order autoregressive process with spatially colored innovations

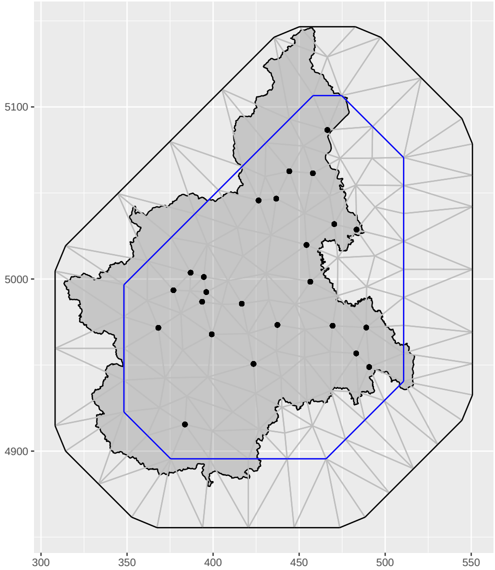
$$\omega_{it} = a\omega_{i(t-1)} + \xi_{it}$$

- $|a| < 1$ parameter of the AR1 process
- ξ_{it} is a zero mean, temporally independent, Gaussian field with

$$\mathrm{Cov}(\xi_{it}, \xi_{ju}) = \begin{cases} 0, & \text{if } t \neq u \\ \mathcal{C}(h), & \text{if } t = u \end{cases}$$

- h distance between locations i and j - $\mathcal{C}(h)$ is a Matern correlation function

Building the model: mesh



Building the model: Prior for the spde model

```
spde = inla.spde2.pcmatern(mesh = mesh,
                           prior.range=c(100,0.5),
                           prior.sigma=c(1,0.1))
```

- $\text{Prob}(\rho < 100 \text{ Km}) = 0.5$
- $\text{Prob}(\sigma > 1) = 0.1$

Implement the model

```r
# make the data a spatial object
coordinates(df) = c("x","y")

# model component
cmp  = ~ Intercept(1) +
  SPDE(coordinates, model = spde,
       group = time, control.group = list(model = "ar1")) +
  dem(dem, model = "linear") +
  temp(temp, model = "linear")

# likelihood
lik = like(formula = logPM10 ~ Intercept + SPDE + dem + temp ,
           family = "gaussian",
           data = df)

# fit the model
fit = bru(cmp, lik,
          options = list(verbose = F,
                         bru_max_iter = 1,
                         inla.mode  = "experimental"))
```

Model Parameters

Fixed effects:

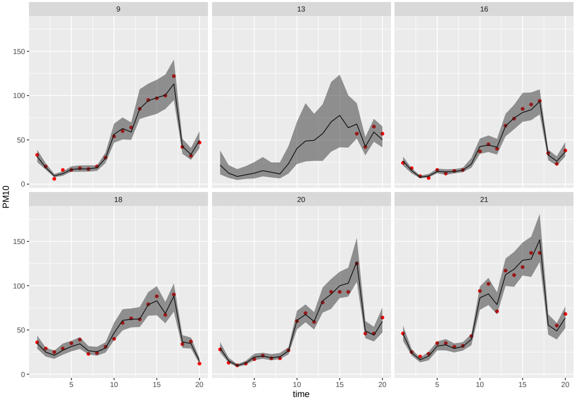
```
##                   mean           sd    0.025quant   0.975quant
## Intercept -1.438103e+01 9.2183351517 -32.139029373 4.0476156436
## dem       -7.283307e-04 0.0008432533  -0.002398673 0.0009354501
## temp       6.250902e-02 0.0320760645  -0.001679595 0.1243716481
```

Hyperparameters of the random field:

```
##                                             mean          sd  0.025quant
## Precision for the Gaussian observations  41.8579575  7.23518895  29.3185249
## Range for SPDE                          129.1799687 16.96397904 101.0510932
## Stdev for SPDE                            0.7511733  0.09963733   0.5913784
## GroupRho for SPDE                         0.9105294  0.02594638   0.8555876
##                                         0.975quant
## Precision for the Gaussian observations  57.6625010
## Range for SPDE                          167.2757328
## Stdev for SPDE                            0.9796968
## GroupRho for SPDE                         0.9557794
```

Prediction at station locations

```
pred_at_station = predict(fit, df,
                          ~exp(Intercept + SPDE + dem + temp ))
```



Prediction is space

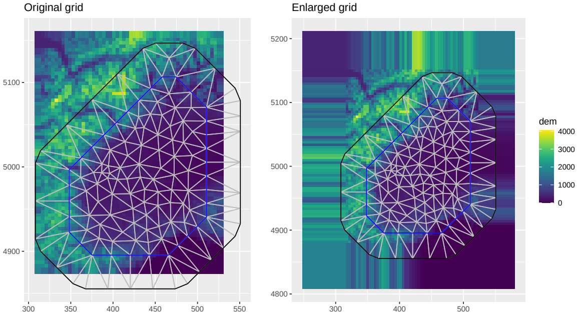
- To do this we need to have the covariates over the space of interest.
- The should cover the whole mesh

Expanding the covariates: the bru_fill_missing

```
# read the covariate values
alt = read.table("Data/Piemonte_data/Altitude.dat")
## object with altitude, this does not cover the whole mesh!
dem = SpatialPixelsDataFrame(points = cbind(alt[,1],
                                            alt[,2]),
                             data = data.frame(dem = alt[,3]))
## create another object which  does cover the whole mesh
large_grid = expand_grid(x = seq(250, 580,4),
                         y = seq(4810, 5210,4))
# expand dem ------------------------------------------------------------
dem_large = SpatialPixelsDataFrame(points = large_grid,
                                   data = data.frame(dem =
                                                    rep(NA,length(large_grid$x))))

# Use bru_fill_missing to fill the new object
dem_large$dem <- bru_fill_missing(data = dem,
                       where = dem_large,
                       values = dem_large$dem)
```

Expanding the covariates: the `bru_fill_missing`



Note: You can do this with your favourite method!

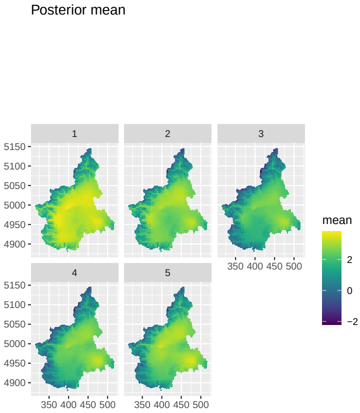
Prediction in space

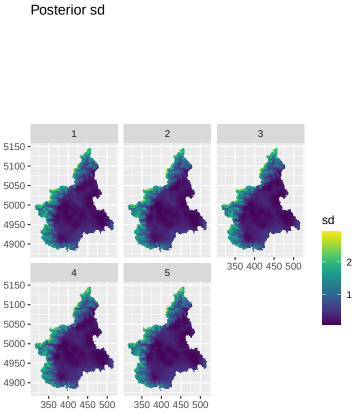
Prediction in space

```
# Create a space time grid
pxl = pixels(mesh, nx = 200, ny = 200, mask = shape)
ips2 <- ipoints(domain = c(1:5), name = "time")
pxl_time <- cbind(cprod(pxl,ips2), data.frame(temp = 0))

pred_space = predict(fit, pxl_time,
                      ~ Intercept + SPDE +
                       dem_eval(inlabru:::eval_SpatialDF(dem_large,
                                                          .data.)) +
                       temp_eval(inlabru:::eval_SpatialDF(temp_large,
                                                           .data.,
                                                           selector = "time")))
```

Prediction in space





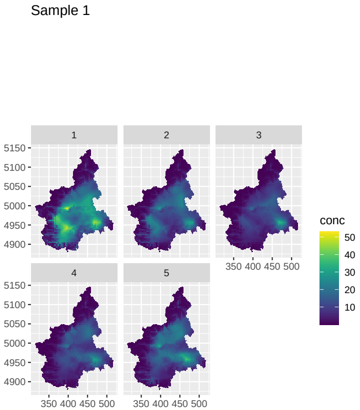
Samples form the fitted model

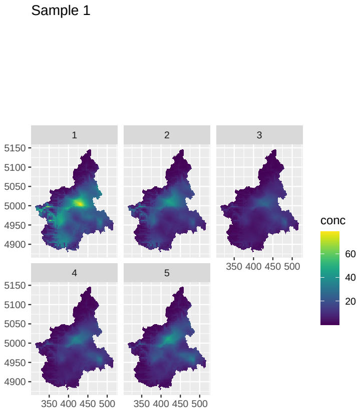
It is also possible to generate from the fitted model

```
sim_space = generate(fit, pxl_time,
                     ~ exp(Intercept + SPDE +
                      dem_eval(inlabru:::eval_SpatialDF(dem_large,
                                                        .data.)) +
                      temp_eval(inlabru:::eval_SpatialDF(temp_large,
                                                         .data.,
                                                         selector = "time"))),
                    n.samples = 2)
```

This can be usefull as the posterior means are always smoother than the “real” field

Samples form the fitted model





Some concluding remark

- The functions `predict()` and `generate()` in `inlabru` use the `INLA` function `inla.posterior.sample()` internally.
- `inlabru` is a wrapper around `INLA` so all the internal computations are identical!
- `inlabru` makes handling of spatial object a lot easier
- Transition from `sp` to `sf`/`terra` some changing can be expected

Thank you for your attention!

If you have any doubts or questions, please write :

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