Bayesian Statistics with R-INLA

University of Zurich, March, 2022

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Norwegian University of Science and Technology

Notes

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Notes

Plan for this 2-day course

Today

- **9:00-10:45**Introduction and basics concepts of INLA
- **11:00-12:30** Practical session I

Lunch

- **13:30-15:00** R-INLA: Basics
- **15:15-17:00** Practical session II

Notes

Plan for this 2-day course

Tomorrow

- **9:00-10:45** Space time models with `inlabru`
- **11:00-12:30** Practical session III

Lunch

- **13:30-15:00** Advanced topic
- **15:15-16:00** Practical session IV

Notes

Introduction

Notes

What is inla?

The short answer:

> *INLA is a fast method to do Bayesian inference with latent Gaussian models and `INLA` is an `R`-package that implements this method with a flexible and simple interface.*

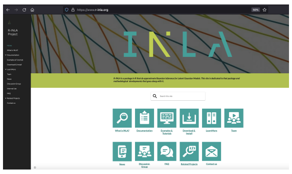
The (much) longer answer:

- Rue, Martino, and Chopin (2009) "Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations." *JRSSB*
- Rue, Riebler, Sørbye, Illian, Simpson, Lindgren (2017) "Bayesian Computing with INLA: A Review." *Annual Review of Statistics and Its Application*
- Martino, Riebler "Integrated Nested Laplace Approximations (INLA)" (2021) *arXiv:1907.01248*

Notes

Where?

The software, information, examples and help can be found at `http://www.r-inla.org`



- paper
- tutorials
- discussion group
- ...

Notes

So... Why should you use `R-INLA`?

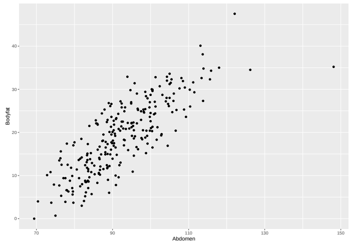
- What type of problems can we solve?
- What type of models can we use?
- When can we use it?

> *To give proper answers to these questions, we need to start at the very beginning*

Notes

The core

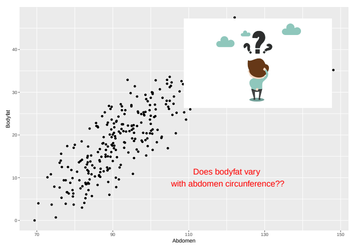
- We have observed something.



Notes

The core

- We have observed something.
- We have questions.



Notes

The core

- We have observed something.
- We have questions.
- We want answers!

Notes

How do we find answers?

We need to make choices:

- Bayesian or frequentist?
- How do we model the data?
- How do we compute the answer?

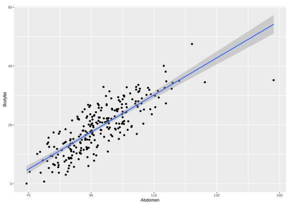
These questions are **not** independent.

Notes

A simple example

Assume a simple linear regression model with Gaussian observations $y = (y_1, \ldots, y_n)$, where

$$\mathrm{E}(y_i) = \alpha + \beta x_i, \mathrm{Var}(y_i) = \tau^{-1}, \quad i = 1, \ldots, n$$



Estimates:

	Estimate	Std.Error
(Intercept)	-39.280	2.66
Abdomen	0.631	0.029
Residual sd	4.877	

Notes

A Bayesian hyerarchical model

- Observation model

$$y \mid \underbrace{\mu, \beta}_{x}, \underbrace{\tau}_{\theta}$$

Encodes information about observed data

- Latent model x: The unobserved process
- Hyperprior for θ

From this we can compute the posterior distribution

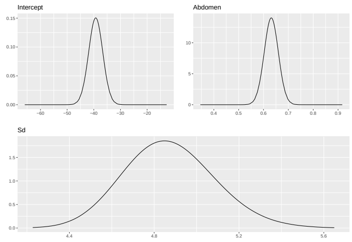
$$\pi(x, \theta | y) \propto \pi(y | x, \theta) \pi(x) \pi(\theta)$$

and then the corresponding posterior marginal distributions.

Notes

Results

- Assign priors to α, β, τ
- Use Bayes theorem to compute posterior distributions



Notes

Bayesian hierarchical models

Notes

Real-world datasets are usually much more complicated!

Using a Bayesian framework:

- Build (hierarchical) models to account for potentially complicated dependency structures in the data.
- Attribute uncertainty to model parameters and latent variables using priors.

Two main challenges:

1. Need computationally efficient methods to calculate posteriors.
2. Select priors in a sensible way (see tomorrow)

Notes

Bayesian hierarchical models

INLA can be used with Bayesian hierarchical models where we model in different stages or levels:

- **Stage 1:** What is the distribution of the responses?
- **Stage 2:** What is the distribution of the underlying unobserved (latent) components?
- **Stage 3:** What are our prior beliefs about the parameters controlling the components in the model?

Notes

Stage 1: The data generating process

How is our data ($\boldsymbol{y}$) generated from the underlying components ($\boldsymbol{x}$) and hyperparameters ($\boldsymbol{\theta}$) in the model:

- Gaussian response? (temperature, rainfall, fish weight . . .)
- Count data? (people infected with a disease in each area)
- Point pattern? (locations of trees in a forest)
- Binary data? (yes/no response, binary image)
- Survival data? (recovery time, time to death)

It is also important how data are collected!

This information is placed into our likelihood $\pi(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta})$

Notes

Stage 1: The data generating process

We assume that *given* the underlying components ($\boldsymbol{x}$) and hyperparameters ($\boldsymbol{\theta}$) the data are independent on each other

$$\pi(y|x, \theta) = \prod_{i \in \mathcal{I}} \pi(y_i | x_{\mathcal{I}_i}, \theta)$$

This implies that all the dependence structure in the data is explained in Stage II !!

Notes

Stage 2: The dependence structure

The underlying unobserved components $\boldsymbol{x}$ are called **latent components** and can be:

- Fixed effects for covariates
- Unstructured random effects (individual effects, group effects)
- Structured random effects (AR(1), regional effects, ...)

These are linked to the responses in the likelihood through linear predictors.

Notes

Introduction Bayesian hierarchical models Latent Gaussian models Deterministic inference

Stage 3: The hyperparameters

The likelihood and the latent model typically have hyperparameters that control their behavior.

The hyperparameters $\boldsymbol{\theta}$ can include:

Examples likelihood:

- Variance of observation noise
- Dispersion parameter in the negative binomial model
- Probability of a zero (zero-inflated models)

Examples latent model:

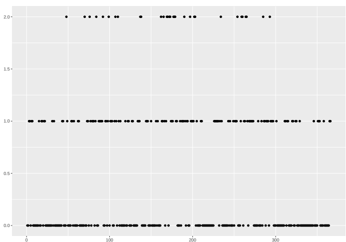
- Variance of unstructured effects
- Correlation of multivariate effects
- Range and variance of spatial effects
- Autocorrelation parameter

Notes

Introduction Bayesian hierarchical models Latent Gaussian models Deterministic inference

Example: Tokyo rainfall data

Rainfall over 1 mm in the Tokyo area for each calendar day during two years (1983-84) are registered.



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Tokyo rainfall data

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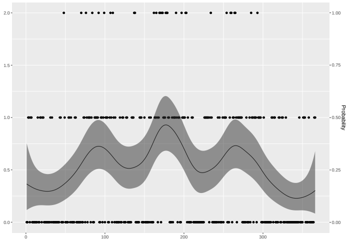


Notes

Stage 1: The data

$$y_i \mid p_i \sim \text{Binomial}(n_i, p_i),$$

for $i = 1, 2, ..., 366$

$$n_i = \begin{cases} 1, & \text{for 29 February} \\ 2, & \text{other days} \end{cases}$$

$$y_i = \begin{cases} \{0,1\}, & \text{for 29 February} \\ \{0,1,2\}, & \text{other days} \end{cases}$$

Linear predictor

$$logit(p_i) = x_i \quad \Leftrightarrow \quad p_i = \frac{1}{1 + exp(-x_i)}$$

- probability of rain on day i depends on x_i
- the likelihood has no hyperparameters θ

Notes

Stage 2: The latent model

It seems natural borrow strength over time and assume a cyclic smooth random effect, e.g. a cyclic random walk of first or second order. A random walk of first order (CRW1) is defined as:

$$\pi(x|\theta) \propto \exp\left\{-\frac{\theta}{2}\left[(x_1 - x_{366})^2 + \sum_{i=2}^{366}(x_i - x_{i-1})^2\right]\right\}$$
$$= \exp\left\{-\frac{\theta}{2}x^T R x\right\}$$

$$R = \begin{bmatrix} 2 & -1 & & & & & & -1 \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & & & \ddots & & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ -1 & & & & & & -1 & 2 \end{bmatrix}$$

Notes

Stage 3: Hyperparameters

The structured time effect is controlled by one precision (inverse variance) parameter θ.

- A larger value of θ means less variation in x, i.e. a smoother effect.
- θ is related to the variation in p_i.
- $\theta > 0$: people commonly assume

$$\theta \sim \text{Ga}(\text{shape} = a, \text{rate} = b)$$

- However, θ depends on R, so it is hard to define values for a and b. You could do this by defining reasonable lower and upper quantiles. (We talk about this tomorrow)

Notes

Latent Gaussian models

Notes

Latent Gaussian models

This was just one example of a very useful class of models called **Latent Gaussian models**.

- The characteristic property is that the latent part of the hierarchical model is Gaussian, $\boldsymbol{x}|\boldsymbol{\theta} \sim N(0, Q^{-1})$
- The expected value is $\mathbf{0}$
- The *precision* matrix (inverse covariance matrix) is Q

Notes

The general set-up

The set up contains GLMs, GLMMs, GAMs, GAMMs, and more. The mean of the observation i, μ_i, is connected to the linear predictor, η_i, through a link function g,

$$\eta_i = g(\mu_i) = \mu + \boldsymbol{z}_i^\top \boldsymbol{\beta} + \sum_\gamma w_{\gamma,i} f_\gamma(c_{\gamma,i}) + v_i, \quad i = 1, 2, \ldots, n$$

where

μ : Intercept

$\boldsymbol{\beta}$: Fixed effects of covariates $\boldsymbol{z}$

$\{f_\gamma(\cdot)\}$: Non-linear/smooth effects of covariates $\boldsymbol{c}$

$\{w_{\gamma,i}\}$: Known weights defined for each observed data point

$\boldsymbol{v}$: Unstructured error terms

Notes

Loads of examples

- Generalized linear and additive (mixed) models
- Disease mapping
- Survival analysis
- Log-Gaussian Cox-processes
- Geostatistics
- Spatio and spatio-temporal models
- Stochastic volatility models
- Measurement error models
- And more!

Notes

Specification of the latent field

- Collect all parameters (random variables) in the latent field $x = \{\mu, \beta, \{f_\gamma(\cdot)\}, \eta\}$.
- A latent Gaussian model is obtained by assigning Gaussian priors to all elements of x.
- Very flexible due to many different forms of the unknown functions $\{f_\gamma(\cdot)\}$:
- Hyperparameters account for variability and length/strength of dependence

Notes

Flexibility through f-functions

The functions $\{f_\gamma\}$ in the linear predictor make it possible to capture very different types of random effects in the same framework:

- $f(\texttt{time})$:, For example, an AR(1) process, RW1 or RW2
- $f(\texttt{spatial location})$:, For example, a Mat'ern field
- $f(\texttt{covariate})$:, For example, a RW1 or RW2 on the covariate values
- $f(\texttt{time}, \texttt{spatial location})$ can be a spatio-temporal effect
- And much more

Notes

Additivity

- One of the most useful features of the framework is the additivity.
- Effects can easily be removed and added without difficulty.
- Each component might add a new latent part and might add new hyperparameters, but the modelling framework and computations stay the same.

OBS: The *linear* predictor needs to stay linear!! So effects can be added but not multiplied (will say more tomorrow..)

Notes

A small point to think about

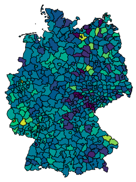
From a Bayesian point of view fixed effects and random effects are all the same.

- Fixed effects are also random
- They only differ in the prior we put on them

Notes

Example: disease mapping

We observed larynx cancer mortality counts for males in 544 district of Germany from 1986 to 1990 and want to make a model.



- y_i: The count at location i.
- E_i: An offset; expected number of cases in district i.
- c_i: A covariate (level of smoking consumption) at i
- $\boldsymbol{s}_i$: spatial location i .

Notes

Bayesian disease mapping

- **Stage 1:** We choose a Poisson distribution for the responses, so that

$$y_i \mid \eta_i \sim \text{Poisson}(E_i \exp(\eta_i)))$$

- **Stage 2:** η_i is a linear function of the latent components: a covariate c_i, a spatially structured effect f_u, an unstructured effect v likelihood by

$$\eta_i = \mu + \beta c_i + f_u(s_i) + v_i$$

- **Stage 3:**
  - τ_f: Precision parameter for the structured effect
  - τ_v: Precision parameter for the unstructured effect

The latent field is $\boldsymbol{x} = (\mu, \beta, \{f_u(\cdot)\}, v_1, v_2, \dots, v_n)$, the hyperparameters are $\boldsymbol{\theta} = (\tau_f, \tau_v)$, and must be given a prior.

Notes

So. . . which model fit the INLA framework??

1. Latent **Gaussian** model
2. The latent field has a sparse precision matrix (Markov properties)
3. The data are conditionally independente given the latent field
4. The predictor is linear

Notes

Quiz!

Assume that, given $\eta = (\eta_1, \ldots, \eta_n)$ the observations $y = (y_1, \ldots, y_n)$ are independent and Poisson distributed with parameter $\lambda_i = \exp(\eta_i)$ i.e.

$$y_i | \eta_i = \text{Poisson}(\lambda_i); i = 1, \ldots, n$$

1. $\eta_i = \alpha + \beta x_i + U_i$ where
$$\alpha, \beta \sim \mathcal{N}(0,1)$$
$$U_i \sim \mathcal{N}(0,1) \text{ for } i = 1, \ldots, n$$
2. $\eta_i = \alpha + \beta x_i + V_i$ where
$$\alpha, \beta \sim \mathcal{N}(0,1)$$
$$U_i \sim \text{Bernoulli}(0.4) \text{ for } i = 1, \ldots, n$$
3. $\eta_i = \alpha + \beta x_i$ where
$$\alpha, \beta \sim \mathcal{N}(0,1)$$
4. $\eta_i = \alpha + \beta x_i + U_i V_i$ where
$$\alpha, \beta \sim \mathcal{N}(0,1)$$
$$U_i \sim \mathcal{N}(0,1) \text{ for } i = 1, \ldots, n$$
$$V_i \sim \mathcal{N}(0,1) \text{ for } i = 1, \ldots, n$$

Notes

Deterministic inference

Notes

Computations

So...

Now we have a modelling framework...

But how do we get our answers?

Notes

What do we care about?

It depends on the problem!

- A single element of the latent field (e.g.the sign or quantiles of a fixed effect)
- A linear combination of elements from the latent field (the average over an area of a spatial effect, the difference of two effects)
- A single hyperparameter (the correlation)
- A non-linear combination of hyper parameters (animal models)
- Predictions at unobserved locations

Notes

What do we care about?

The most important quantity in Bayesian statistics is the posterior distribution:

$$\overbrace{\pi(x,\theta \mid y)}^{\text{Posterior}} \propto \overbrace{\pi(\theta)\pi(x \mid \theta)}^{\text{Prior}} \overbrace{\prod_{i \in \mathcal{I}} \pi(y_i \mid x_i, \theta)}^{\text{Likelihood}}$$

from which we can derive the quantities of interest, such as

$$\begin{aligned}\pi(x_i \mid y) &\propto \int\int \pi(x,\theta \mid y) dx_{-i} d\theta \\ &= \int \pi(x_i \mid \theta, y)\pi(\theta \mid y) d\theta\end{aligned}$$

or $\pi(\theta_j \mid y)$.

These are very high dimensional integrals and are typically not analytically tractable.

Notes

Traditional approach: MCMC

MCMC is based on sampling with the goal to construct a Markov chain with the target posterior as stationary distribution.

- Extensively used within Bayesian inference since the 1980's.
- Flexible and general, sometimes the only thing we can do!
- A generic tool is available with `JAGS`/`OpenBUGS`.
- Tools for specific models are of course available, e.g.~`BayesX` and `stan`.
- Standard MCMC sampler are generally easy-ish to program and are in fact implemented in readily available software
- However, depending on the complexity of the problem, their efficiency might be limited.

Notes

Approximate inference

Bayesian inference can (almost) never be done exactly. Some form of approximation must always be done.

- MCMC "works" for everything, but it can be incredibly slow
- Is it possible to make a quicker, more specialized inference scheme which only needs to work for this limited class of models? (specifically LGM)

Notes

Recall: What is our model framework?

Latent Gaussian models

$$
\begin{aligned}
y|x,\theta &\sim \prod \pi(y_i|x_i,\theta) \\
x|\theta &\sim \mathcal{N}(0, Q(\theta)) \qquad \text{Gaussian!!!} \\
\theta &\sim \pi(\theta) \qquad \text{Not Gaussian}
\end{aligned}
$$

where the precision matrix $Q(\theta)$ is sparse. Generally these "sparse" Gaussian distributions are called Gaussian Markov random fields (GMRFs).

The sparseness can be exploited for very quick computations for the Gaussian part of the model through numerical algorithms for sparse matrices.

Notes

The INLA idea

Use the properties of the LGM we have defined to approximate the posterior marginals

$$\pi(x_i \mid \boldsymbol{y}) \quad \text{and} \quad \pi(\theta_j \mid \boldsymbol{y})$$

directly.

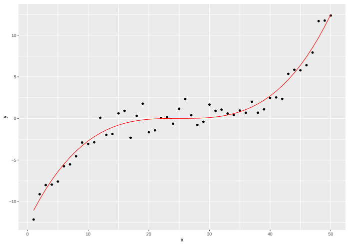
Let us consider a toy example to illustrate the ideas.

Notes

How does INLA work? A toy example

Smoothing noisy observations - Data

We observe some smooth function but our measures are noisy (but we know the size of such noise!)



Goal: Recover the smooth function observed with noise!

Notes

Smoothing noisy observations - Model

Assume:

$$y_i = f(i) + \epsilon_i; i = 1, \ldots, n$$
$$\epsilon_i \sim N(0, 1)$$
$$f(i) = x_i \text{ smooth function of } i$$

- Only one hyperparameter
- Gaussian likelihood

Is this a Latent Gaussian model?

Notes

Smoothing noisy observations - LGM

- **Data** Gaussian Observations with known precision

$$y_i | x_i \sim \mathcal{N}(x_i, 1)$$

- **Latent Model**: A Gaussian model for the smooth function (RW2 model)

$$\pi(\mathbf{x}|\theta) \propto \theta^{(n-2)/n} \exp\left\{ -\frac{\theta}{2} \sum_{i=2}^{n} (x_i - 2x_{i-1} + x_{i-2})^2 \right\}$$

- **Hyperparameter** The precision of the smooth function θ. We assign a Gamma prior

$$\pi(\theta) \propto \theta^{a-1} \exp(-b\theta)$$

Notes

Smoothing noisy observations - Goal

Find approximations for:

1. The posterior marginal for the hyperparameter $\pi(\theta|\mathbf{y})$
2. The posterior marginals for the elements of the latent field $\pi(x_i|\mathbf{y})$

Notes

Approximating $\pi(\theta|\mathbf{y})$

We have that

$$\pi(\mathbf{x}, \theta, \mathbf{y}) = \pi(\mathbf{x}|\theta, \mathbf{y})\pi(\theta|\mathbf{y})\pi(\mathbf{y})$$

so

$$\pi(\theta|\mathbf{y}) = \frac{\pi(\mathbf{x}, \theta, \mathbf{y})}{\pi(\mathbf{x}|\theta, \mathbf{y})\pi(\mathbf{y})} \propto \frac{\pi(\mathbf{y}, \mathbf{x}|\theta)\ \pi(\theta)}{\pi(\mathbf{x}|\theta, \mathbf{y})}$$

Since the likelihood is Gaussian, then $\pi(\mathbf{y}, \mathbf{x}|\theta)$ is also Gaussian. We have then:

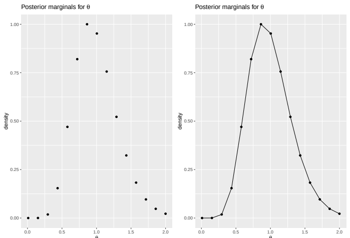
$$\pi(\theta|\mathbf{y}) \propto \frac{\overbrace{\pi(\mathbf{y}, \mathbf{x}|\theta)}^{\text{Gaussian}}\ \pi(\theta)}{\underbrace{\pi(\mathbf{x}|\theta, \mathbf{y})}_{\text{Gaussian}}}$$

This is valid for any $\mathbf{x}$

Notes

Posterior marginal for the hyperparameter

Select a grid of points to represent the density $\pi(\theta|\mathbf{x})$



Notes

Approximating $\pi(x_i|y, \theta)$

Again we have that

$$\mathbf{x}, \mathbf{y}|\theta \sim \mathbf{N}(\cdot, \cdot)$$

so also $\pi(x_i|\theta, \mathbf{y})$ is Gaussian!!
We compute

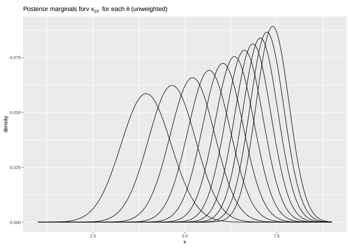
$$\begin{aligned}\pi(x_i|\mathbf{y}) &= \int \pi(x_i|\theta, \mathbf{y})\pi(\theta|\mathbf{y})d\theta \\ &\approx \sum_k \pi(x_i|\theta_k, \mathbf{y})\pi(\theta_k|\mathbf{y})\Delta_k\end{aligned}$$

where $\theta_k, k = 1, \ldots, K$ are the representative points of $\pi(\theta|\mathbf{y})$ and Δ_k are the corresponding weights

Notes

Posterior marginals for latent field I

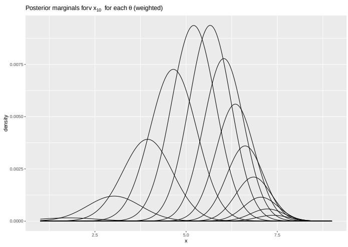
Compute the conditional posterior marginal for x_i given each θ_k



Notes

Posterior marginals for latent field II

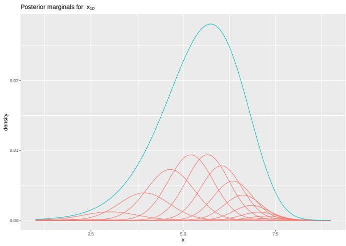
Weight the conditional posterior marginal for $\pi(x_i|\theta_k, \mathbf{y})$ by $\pi(\theta_k|\mathbf{y})\mathbf{\Delta_k}$



Notes

Posterior marginals for latent field III

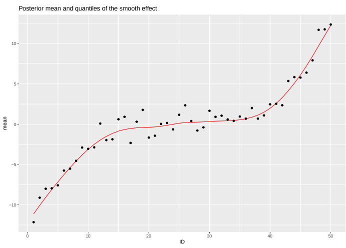
Sum to get the posterior marginal for $x_i|\mathbf{y}$



Notes

Fitted Spline

The posterior marginals are used to calculate summary statistics, like means, variances and credible intervals:



Notes

R-INLA code

```
formula = y ~ -1 + f(idx, model="rw2", constr=FALSE,
   hyper=list(prec=list(prior="loggamma", param=c(a,b))))

result = inla(formula,
      data = data.frame(y=y, idx=idx),
      control.family = list(initial = log(tau_0), fixed=TRUE))
```

Notes

Extending the method

This is the basic idea behind INLA. It is quite simple.

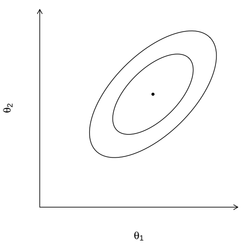
However, we need to extend this basic idea so we can deal with

1 Non-Gaussian observations 2 More than one hyperparameter

Notes

1. More than one hyperparameter

Main use: Select good evaluation points θ_k for the numerical integration when approximating $\widetilde{\pi}(x_i|y)$



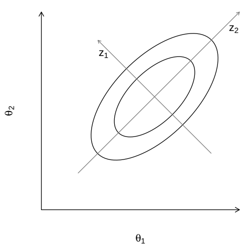
- Locate the mode

Notes

1. More than one hyperparameter

Main use: Select good evaluation points θ_k for the numerical integration when approximating $\widetilde{\pi}(x_i|y)$

- Locate the mode
- Compute the Hessian to construct principal components

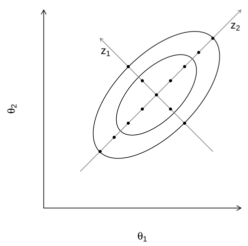


Notes

1. More than one hyperparameter

Main use: Select good evaluation points θ_k for the numerical integration when approximating $\widetilde{\pi}(x_i|y)$

- Locate the mode
- Compute the Hessian to construct principal components
- Grid-search to locate bulk of the probability mass



Notes

1. More than one hyperparameter

- Locate the mode
- Compute the Hessian to construct principal components
- Grid-search to locate bulk of the probability mass
- For each point k in the grid compute:
  - $\widetilde{\pi}(\theta^k|y)$
  - $\widetilde{\pi}(x_i|\theta^k, y)$
  - Δ_k

Notes

2. Non-Gaussian observations

In application we may choose likelihoods other than a Gaussian. How does this change things?

$$\pi(\theta \mid \mathbf{y}) \propto \frac{\overbrace{\pi(\mathbf{x}, \mathbf{y} \mid \theta)}^{\text{Non-Gaussian, BUT KNOWN}} \qquad \pi(\theta)}{\underbrace{\pi(\mathbf{x} \mid \mathbf{y}, \theta)}_{\text{Non-Gaussian and UNKNOWN}}}$$

- In many cases $\pi(\boldsymbol{x} \mid \boldsymbol{y}, \boldsymbol{\theta})$is very close to a Gaussian distribution, and can be replaced with a Laplace approximation.

Notes

The GMRF (Laplace) approximation

Let $\mathbf{x}$ denote a GMRF with precision matrix $\mathbf{Q}$ and mean μ.

Approximate

$$\pi(\mathbf{x}|\theta, \mathbf{y}) \propto \exp\left(-\frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \sum_{\mathbf{i=1}}^{\mathbf{n}} \log \pi(\mathbf{y_i}|\mathbf{x_i})\right)$$

by using a second-order Taylor expansion of $\log \pi(y_i|x_i)$ around $\mu_\mathbf{0}$, say.

- Recall

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 = a + bx - \frac{1}{2}cx^2$$

with $b = f'(x_0) - f''(x_0)x_0$ and $c = -f''(x_0)$. (Note: a is not relevant).

Notes

The GMRF approximation (II)

Thus,

$$\begin{aligned}\widetilde{\pi}(\mathbf{x}|\theta, \mathbf{y}) &\propto \exp\left(-\frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \sum_{\mathbf{i=1}}^{\mathbf{n}} (\mathbf{a_i} + \mathbf{b_i x_i} - \mathbf{0.5 c_i x_i^2})\right) \\ &\propto \exp\left(-\frac{1}{2}\mathbf{x^T}(\mathbf{Q} + \mathrm{diag}(\mathbf{c}))\mathbf{x} + \mathbf{b^T x}\right)\end{aligned}$$

which is Gaussian with precision matrix $\mathbf{Q} + \mathrm{diag}(\mathbf{c})$ and mean given by the solution of $(\mathbf{Q} + \mathrm{diag}(\mathbf{c}))\mu = \mathbf{b}$

The canonical parameterisation is

$$\mathcal{N}_C(\mathbf{b}, \mathbf{Q} + \mathrm{diag}(\mathbf{c}))$$

which corresponds to

$$\mathcal{N}((\mathbf{Q} + \mathrm{diag}(\mathbf{c}))^{-\mathbf{1}}\mathbf{b}, (\mathbf{Q} + \mathrm{diag}(\mathbf{c}))^{-\mathbf{1}}).$$

Notes

The GMFR approximation - One dimensional example

Assume

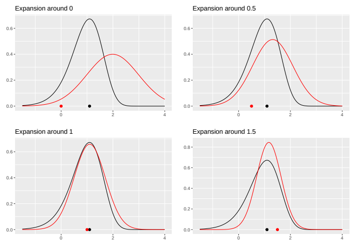
$$\begin{aligned} y|\lambda &\sim \mathrm{Poisson}(\lambda) \text{ Likelihood} \\ \lambda &= \exp(x) \text{ Likelihood} \\ x &\sim \mathcal{N}(0,1) \text{ Latent Model} \end{aligned}$$

we have that

$$\pi(x|y) \propto \pi(y|x)\pi(x) \propto \exp\{-\frac{1}{2}x^2 + \underbrace{xy - \exp(x)}_{\text{non-gaussian part}}\}$$

Notes

The GMRF approximation



If $\mathbf{y} \mid \mathbf{x}, \theta$ is Gaussian "the approximation" is exact! }

Notes

What do we get ...

$$\widetilde{\pi}(\theta \mid \mathbf{y}) \propto \frac{\pi(\mathbf{x}, \mathbf{y} \mid \theta)\ \pi(\theta)}{\widetilde{\pi}_{\mathbf{G}}(\mathbf{x} \mid \mathbf{y}, \theta)}\Bigg|_{\mathbf{x}=\mathbf{x}^{\star}(\theta)}$$

* find the mode of $\widetilde{\pi}(\theta \mid \mathbf{y})$ (optimization) * explore $\widetilde{\pi}(\theta \mid \mathbf{y})$ to find grid points t_k for numerical integration.

However, why is it called integrated nested Laplace approximation?

There is another step that changes:

$$\pi(x_i \mid \mathbf{y}) \approx \sum_{\mathbf{k}} \underbrace{\pi(x_i \mid \mathbf{y}, \theta^{\mathbf{k}})}_{\text{Not Gaussian!}} \widetilde{\pi}_{\mathbf{G}}(\theta^{\mathbf{k}} \mid \mathbf{y}) \boldsymbol{\Delta}_{\mathbf{k}}$$

Notes

Approximating $\pi(x_i|\mathbf{y}, \theta)$

Three possible approximations:

- 1. Gaussian distribution derived from $\widetilde{\pi}_G(\mathbf{x}|\theta, \mathbf{y})$, i.e.

$$\widetilde{\pi}(x_i|\theta, \mathbf{y}) = \mathcal{N}(\mathbf{x_i}; \mu_{\mathbf{i}}(\theta), \sigma_{\mathbf{i}}^{\mathbf{2}}(\theta))$$

with mean $\mu_i(\theta)$ and marginal variance $\sigma_i^2(\theta)$.

However, errors in location and/or lack of skewness possible

- 2. Laplace approximation
- 3. Simplified Laplace approximation

Notes

Laplace approximation of $\pi(x_i|\theta, \mathbf{y})$

$$\widetilde{\pi}_{\text{LA}}(x_i|\theta, \mathbf{y}) \propto \left.\frac{\pi(\mathbf{x}, \theta, \mathbf{y})}{\widetilde{\pi}_{\text{GG}}(\mathbf{x_{-i}}|\mathbf{x_i}, \theta, \mathbf{y})}\right|_{\mathbf{x_{-i}}=\mathbf{x^{\star}}_{-\mathbf{i}}(\mathbf{x_i},\theta)}$$

The approximation is very good but expensive as n factorizations of $(n-1) \times (n-1)$ matrices are required to get the n marginals.

Computational modifications exist:

1. Approximate the modal configuration of the GMRF approximation.
2. `\item Reduce the size n by only involving the ``neighbors''.`

Notes

Simplified Laplace approximation

Faster alternative to the Laplace approximation

- based on a series expansion up to third order of the numerator and denominator of $\widetilde{\pi}_{\text{LA}}(x_i|\theta, \mathbf{y})$
- corrects the Gaussian approximation for error in location and lack of skewness.

This is default option when using INLA but this choice can be modified.

Notes

INLA: Overview

- **Step I** Approximate $\pi(\theta|y)$ using the Laplace approximation and select good evaluation points θ_k.
- **Step II** For each θ_k and i approximate $\pi(x_i|\theta_k, y)$ using the Laplace or simplified Laplace approximation for selected values of x_i
- **Step III** For each i, sum out θ_k

$$\widetilde{\pi}(x_i|y) = \sum_k \widetilde{\pi}(x_i|\theta_k, y) \times \widetilde{\pi}(\theta_k|y) \times \Delta_k.$$

Build a log spline corrected Gaussian to represent $\widetilde{\pi}(x_i|y)$.

Notes

INLA: Why does it work?

- The full conditional $\pi(x|y,\theta)$ is “almost” Gaussian
- The latent field x is a GMRF
  - GMRF $\rightarrow$ sparse precision matrix!!
  - Easy to solve and store
- Smart numerical methods
- Parallel implementation

Notes

Limitations

- The dimension of the latent field x can be large ($10^2 - 10^6$)
- The dimension of the hyperparameters θ must be small (≤ 9)

In other words, each random effect can be big, but there cannot be too many random effects unless they share parameters.

Notes

INLA features

INLA fully incorporates posterior uncertainty with respect to hyperparameters $\Rightarrow$ tool for full Bayesian inference

- Marginal posterior densities of all (hyper-)parameters
- Posterior mean, median, quantiles, std.~deviation, etc.
- The approach can be used for predictions, model assessment, ...
- Joint posterior marginal not available... but it is possible to sample from $\widetilde{\pi}(x,\theta|y)$

Notes

Thank you for your attention!

If you have any doubts or questions, please write :
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