

What have we learned  
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Spatial models  
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Gaussian Random Field models  
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The SPDE approach  
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Fitting spatial  
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# Spatial modeling with INLA and inlabru

## University of Zurich, March, 2022

Instructor: Sara Martino

Department of Mathematical Science (NTNU)



Norwegian University of  
Science and Technology

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## What have we learned

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Space-time Modeling

Prediction in space

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# What have we learned

INLA in a nutshell

- many data sets these days are complex, resulting in complex models, e.g. with complex dependence structures (spatial, temporal, etc..)
  - usually Markov chain Monte Carlo (MCMC) methods have been used to fit these models
    - (realistically) complex models result in very long running times
    - often impossible (or unrealistic) to fit
  - INLA (Integrated nested Laplace approximation) is an alternative to MCMC
    - much, much **faster**
    - suitable for a specific (but very large!) class of models

# INLA in a nutshell

Three main ingredients in INLA

- Gaussian Markov random fields
- Latent Gaussian models
- Laplace approximations

which together (with a few other things) give a very nice tool for Bayesian inference

- quick
- accurate

# Models amenable to INLA: Latent Gaussian Models

$$\mathbf{y}|\mathbf{x}, \theta \sim \prod_i \pi(y_i|x_i, \theta) \quad \text{Likelihood}$$

$$\mathbf{x}|\theta \sim \exp\left(-\frac{1}{2}\mathbf{x}^T \mathbf{Q}(\theta) \mathbf{x}^T\right) \quad \text{Latent field (GMRF)}$$

$$\theta \sim \pi(\theta) \quad \text{Hyperparameter}$$

- Each data point depends on only one of the elements in the latent Gaussian field  $\mathbf{x}$ .

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INLA allows us to compute in a **fast** and **efficient** way an **accurate** approximation to the posterior marginals for the hyperparameters

$$\tilde{\pi}(\theta_j | \mathbf{y})$$

and for the latent field

$$\tilde{\pi}(x_i | \mathbf{y})$$

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All of this is implemented in the **INLA** library in R.

Yesterday we saw how to implement simple models with **INLA**

Today we are going to look at how to implement continuous spatial models.

Such model are **a lot** easier to implement in **inlabru!!!!**

## inlabru in a nutshell

`inlabru` is a friendlier version of R-INLA

- it makes INLA more accessible to the user
- makes complex features and predictions (especially for spatial data) a lot easier
- is a softer-wrapper around INLA
- in some cases it allows to release some of the constraints of R-INLA (the linearity of the predictor)

# inlabru

- Installation:

- There is a CRAN version

```
install.packages("inlabru")
```

- You can also install the development version of inlabru from GitHub (recommended)

```
# install.packages("remotes")
remotes::install_github("inlabru-org/inlabru",
                      ref="devel")
```

- Documentation

- Web site: <https://sites.google.com/inlabru.org/inlabru>
  - Github: <https://github.com/inlabru-org/inlabru>

## What have we learned

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## Gaussian Random Field models

## The SPDE approach

## Fitting spatial

## Spatial models

# Types of Spatial Data

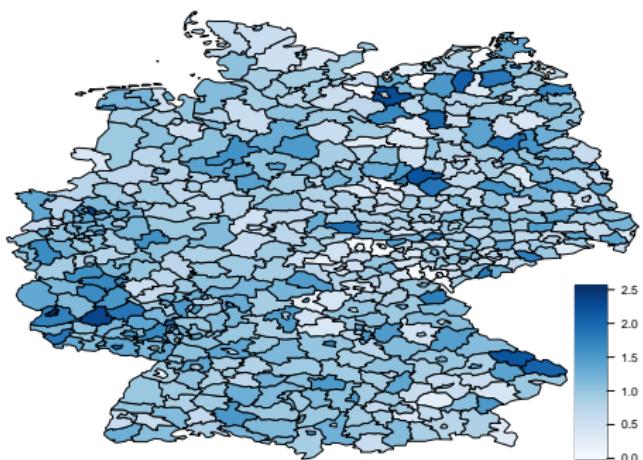
We can distinguish three types of spatial data

- Discrete space
  - data on a spatial grid
- Continuous space:
  - geostatistical data
  - spatial point data

# Discrete Counts:

## Data on a spatial grid

- examples: number of individuals in a region, average rainfall in a province
- (originally geostatistical or point data; gridded for practical reasons)

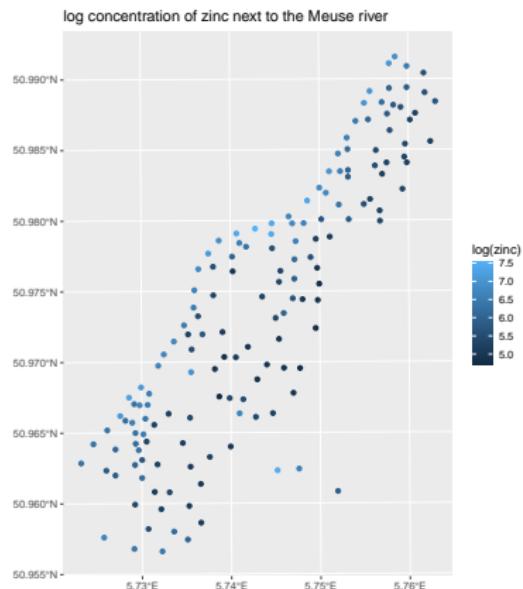


## Observed response(s):

- Measurement over each grid cell (e.g. number of individuals in cell; rainfall in province)

# Continuous Space: Geostatistics

- phenomenon that is continuous in space
- examples: nutrient levels in soil, salinity in the sea
- measurements at a given set of locations that are determined by surveyor



## Observed response(s):

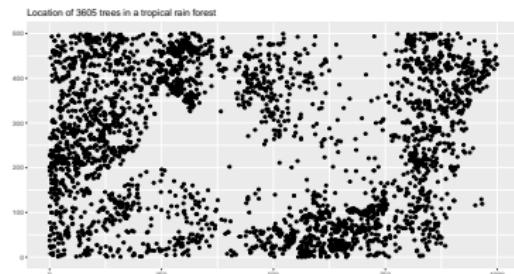
- measurement(s) taken at given locations

# Continuous Space: Point Process

- locations of objects (individuals) in space (typically 2D)
- examples: locations of trees in a forest, groups of animals

**Observed response(s):**

- $x, y$  coordinates of points (individuals/groups)
- maybe also properties of individuals/groups (“marks”)



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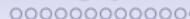
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## Gaussian Random Field models



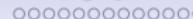
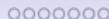
## Gaussian Random fields

**Definition:** A random function  $u(x) : R^d \rightarrow R$  is a Gaussian random field if for any finite collection of locations,  $(x_1, \dots, x_n)$ ,  $x_i \in R^d$ , the joint distribution of  $\mathbf{u} = (u(x_1), \dots, u(x_n))$  is  $\mathbf{u} \sim N(0, \Sigma)$ , and

$$E(u(x)) = 0,$$

$$Cov(u(x), u(x')) = R(x, x'), \quad \Sigma_{ij} = R(x_i, x_j)$$

for some expectation function  $\mu(\cdot)$  and positive definite covariance function  $R(\cdot, \cdot)$ .  $\Sigma$  is the covariance matrix for the specific location collection.



## Matern covariance function

The Matern covariance function is a popular model in spatial problems:

$$c_\nu(d; \sigma, \rho) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \sqrt{8\nu} \frac{d}{\rho} \right)^\nu K_\nu \left( \sqrt{8\nu} \frac{d}{\rho} \right)$$

\*  $\rho$  and  $\sigma$  are the range and the marginal standard deviation

- $K_\nu$  modified Bessel function of the second kind, order  $\nu$ .

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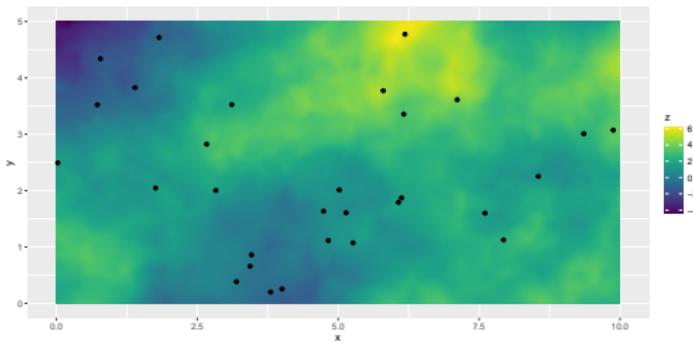
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## Example



# Gaussian Random field

GRF are a very popular model

- Flexible and easy to use
- Can be part of the latent Gaussian field in a LGM

However:

- **computationally inefficient** (the precision matrix is dense)
- not flexible enough (complicated boundary, barrier, . . . )

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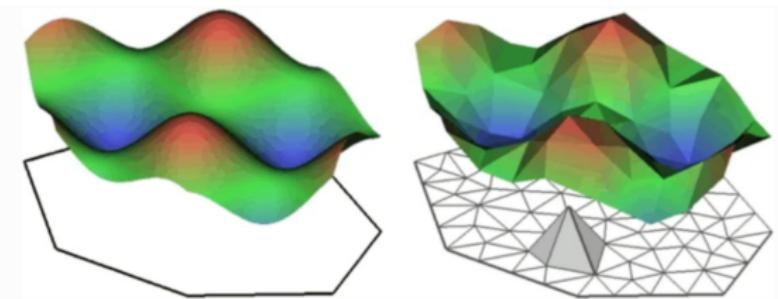
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## The SPDE approach

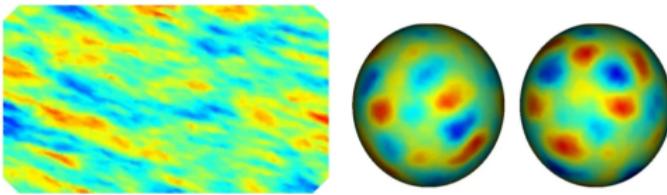
## The SPDE approach

- Matern fields can be seen as solution to a PDE
- Using finite element methods such solution can be represented using a GRMF



## Advantages of the SPDE approach

- Computationally fast
- Allows for flexible modeling
  - non-stationary models (anisotropy)
  - models on a sphere
  - non separable models



All these models (and many more) can be fitted with **R-INLA** and **inlabru**

## Learning about the SPDE approach

- F. Lindgren, H. Rue, and J. Lindström. *An explicit link between Gaussian fields and Gaussian Markov random fields: The SPDE approach (with discussion)*. In: Journal of the Royal Statistical Society, Series B 73.4 (2011), pp. 423–498.
- H. Bakka, H. Rue, G. A. Fuglstad, A. Riebler, D. Bolin, J. Illian, E. Krainski, D. Simpson, and F. Lindgren. *Spatial modelling with R-INLA: A review*. In: WIREs Computational Statistics 10:e1443.6 (2018). (Invited extended review). DOI: 10.1002/wics.1443.
- E. T. Krainski, V. Gómez-Rubio, H. Bakka, A. Lenzi, D. Castro-Camilo, D. Simpson, F. Lindgren, and H. Rue. *Advanced Spatial Modeling with Stochastic Partial Differential Equations using R and INLA*. Github version [www.r-inla.org/spde-book](http://www.r-inla.org/spde-book). CRC press, Dec. 20

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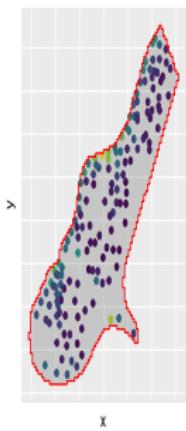
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## Fitting spatial models

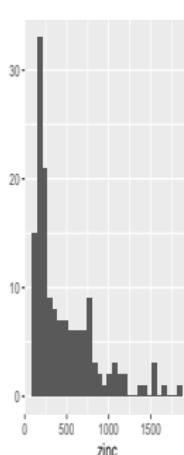
# Example: Meuse Data

Measures of zinc concentration.

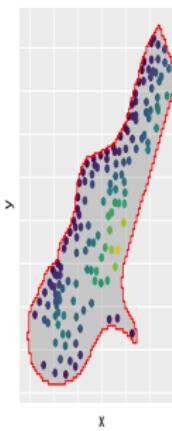
Zinc concentration



zinc



Dist from river



dist

## The model

$$\log(Y(s)) \sim \mathcal{N}(\eta(s), \sigma_y^2)$$

$$\eta(s) = \alpha + \beta x(s) + u(s)$$

where

- $Y(s)$  is the measure of zinc in location  $s$
- $\alpha$  a common intercept
- $\beta$  a model parameter
- $x(s)$  distance from the river at location  $s$
- $u(s)$  the Matern Gaussian field

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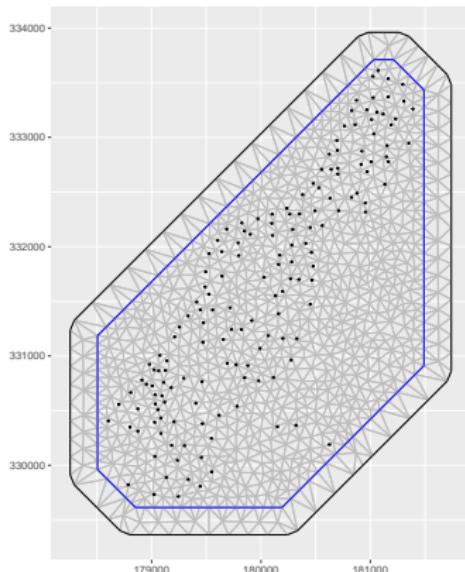
## Step 1: Define the SPDE representation

- The mesh
- The SPDE model

# Define the SPDE representation: The mesh

## 1. Define the mesh

```
mesh <- inla.mesh.2d(loc.domain = cbind(meuse$x, meuse$y),  
                      max.edge = c(150, 500),  
                      offset = c(100, 250) )
```



## Define the SPDE representation: The mesh

- All random field models need to be discretised for practical calculations.
- The SPDE models were developed to provide a consistent model definition across a range of discretisations.
- We use finite element methods with local, piecewise linear basis functions defined on a triangulation of a region of space containing the domain of interest.
- Deviation from stationarity is generated near the boundary of the region.
- The choice of region and choice of triangulation affects the numerical accuracy.

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Two separate issues:

- Continuous space with bounded domain: Boundary effect
- Discretised model: Numerical accuracy

Sometimes the boundary effect may be desireable.

## Define the SPDE representation: The mesh

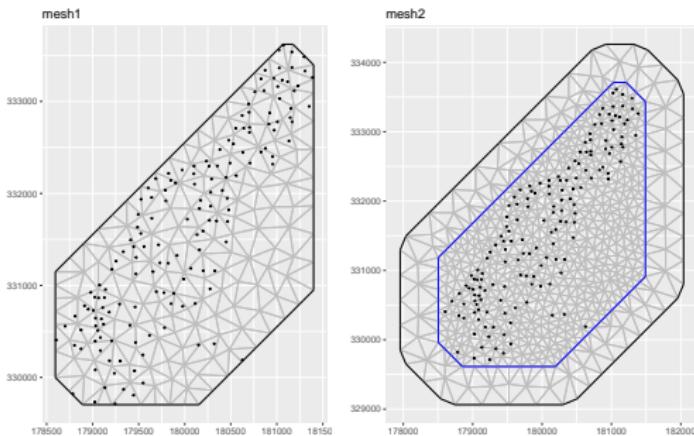
- Too fine meshes → heavy computation
- Too coarse mesh → not accurate enough

## Some guidelines

- Create triangulation meshes with `inla.mesh.2d()`
- Move undesired boundary effects away from the domain of interest by extending to a smooth external boundary (`inla.nonconvex.hull(loc, convex)`, `convex ≥ correlation range`)
- Use a coarser resolution in the extension to reduce computational cost (`max.edge=c(inner, outer)`)
- Use a fine resolution (subject to available computational resources) for the domain of interest (inner correlation range) and filter out small input point clusters (`0 < cutoff < inner`)
- Coastlines and similar can be added to the domain specification in `inla.mesh.2d()`

## Define the SPDE representation: The mesh

```
mesh1 = inla.mesh.2d(loc.domain = cbind(meuse$x, meuse$y),  
                      max.edge = 350,  
                      offset = 10)  
  
mesh2 = inla.mesh.2d(loc.domain = cbind(meuse$x, meuse$y),  
                      max.edge = c(150, 500),  
                      cutoff = 100,  
                      offset = c(100, 550) )
```



## Define the SPDE representation: The SPDE model

```
meuse.spde <- inla.spde2.pcmatern(mesh = mesh,  
                                     prior.sigma = c(1, 0.1),  
                                     prior.range = c(1000, 0.5))
```

PC-priors for the range  $\rho$  and the standard deviarion  $\sigma$

- Define the prior for the range `prior.range` = `(range0,Prange)`  $\text{Prob}(\rho < \rho_0) = p_\rho$
- Define the prior for the range `prior.sigma` = `(sigma0,Psigma)`  $\text{Prob}(\sigma > \sigma_0) = p_\sigma$

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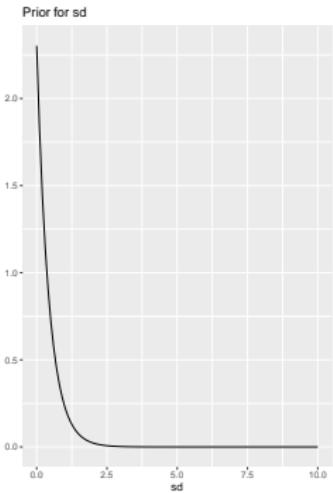
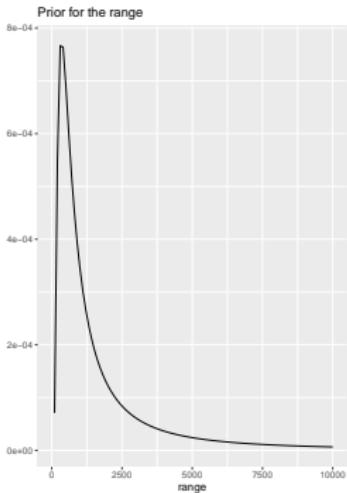
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## Prior for range $\rho$ and sd $\sigma$



## Run the model inlabru

```
# create a spatial object
coordinates(meuse) = c("x","y")
# covariate values
dist_SPDE = SpatialPixelsDataFrame(data$dist_raster[,c(1,2)],
                                     data = data.frame(dist = data$dist_raster[,3]))
# model components
cmp = ~ Intercept(1) + dist(dist_SPDE, model = "linear") +
      spde(coordinates, model = meuse.spde)
# define likelihood
lik = like(formula = Y ~ Intercept + dist + spde,
           family = "gaussian",
           data = meuse)
#fit the model
fit <- bru(cmp, lik)
# define prediction area
pix <- pixels(mesh, nx = 200, ny = 200, mask = boundary)
# generate predictions
pred = predict(fit, pix, ~ data.frame(
  spde = spde,
  logscale = Intercept + dist + spde,
  naturalscale = exp(Intercept + dist + spde)))
```

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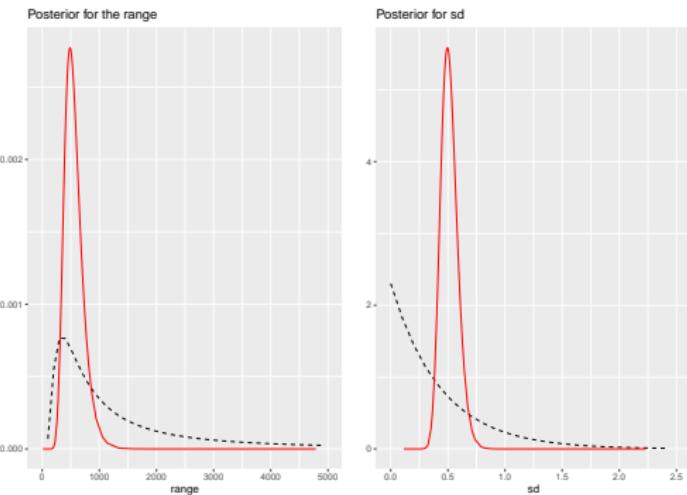
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# Notes!

- The data are a spatial object!
- For prediction, the covariates are stored in a `SpatialPixelsDataFrame` and need to cover all the mesh nodes

# Estimation: Posterior for the hyperparameters



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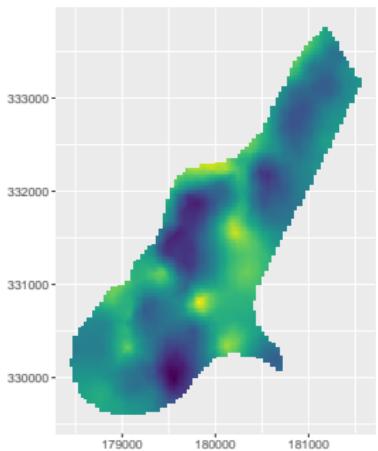
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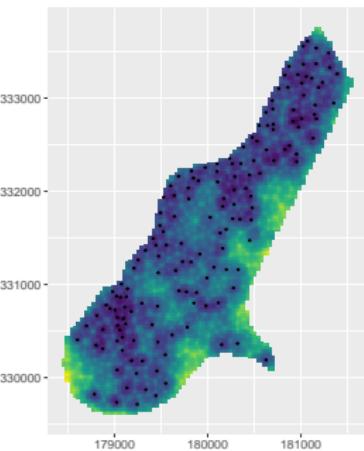
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## Predictions: The SPDE field



mean

0.5  
0.0  
-0.5  
-1.0



sd

0.5  
0.4  
0.3  
0.2

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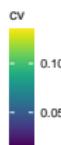
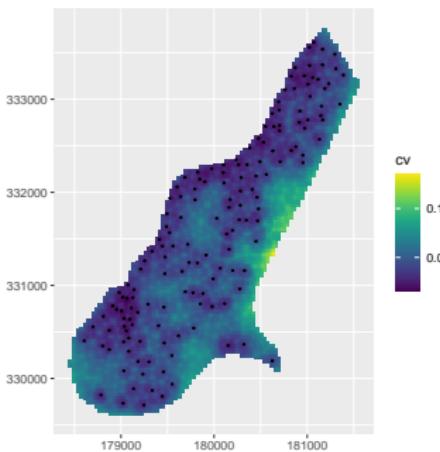
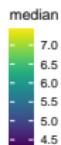
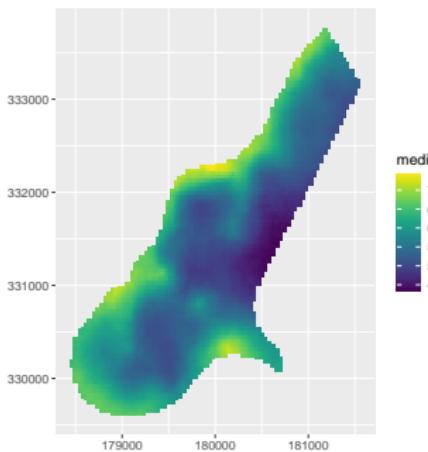
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## Predictions: The log concentrations



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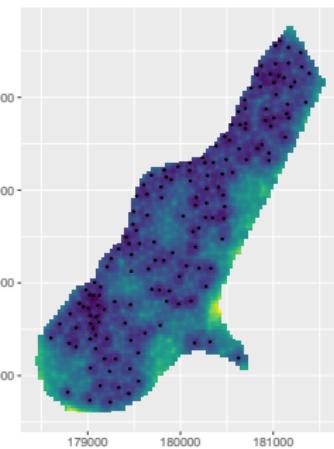
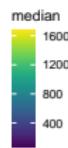
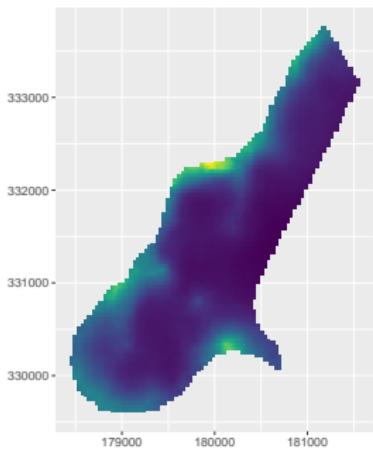
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## Predictions: The concentrations



# Same in plain INLA (1)

```
A.meuse <- inla.spde.make.A(mesh = mesh, loc = coordinates(meuse))
s.index <- inla.spde.make.index(name = "spatial.field",
  n.spde = meuse.spde$n.spde)

#Create data structure
meuse.stack <- inla.stack(data = list(zinc = meuse$zinc),
  A = list(A.meuse, 1),
  effects = list(c(s.index, list(Intercept = 1)),
    list(dist = meuse$dist)),
  tag = "meuse.data")

data(meuse.grid)
coordinates(meuse.grid) = ~x+y
gridded(meuse.grid) = TRUE

#Create data structure for prediction
A.pred <- inla.spde.make.A(mesh = mesh, loc = coordinates(meuse.grid))
meuse.stack.pred <- inla.stack(data = list(zinc = NA),
  A = list(A.pred, 1),
  effects = list(c(s.index, list (Intercept = 1)),
    list(dist = meuse.grid$dist)),
  tag = "meuse.pred")

#Join stack
join.stack <- inla.stack(meuse.stack, meuse.stack.pred)
```

## Same in plain INLA (2)

```
#Fit model
form <- log(zinc) ~ -1 + Intercept + dist + f(spatial.field, model = spde)

m1 <- inla(form, data = inla.stack.data(join.stack, spde = meuse.spde),
           family = "gaussian",
           control.predictor = list(A = inla.stack.A(join.stack), compute = TRUE))
```

**Note:** We still have not compute predictions... and this is not too easy in plain INLA!!

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## When is `inlabru` easier to use

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- multiple likelihoods
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  - born for ecological models (for example transect sampling) but used also in other fields

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What cannot (at the moment ) be done with `inlabru`

- Survival models

What have we learned

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## Space-time Modeling

What have we learned  
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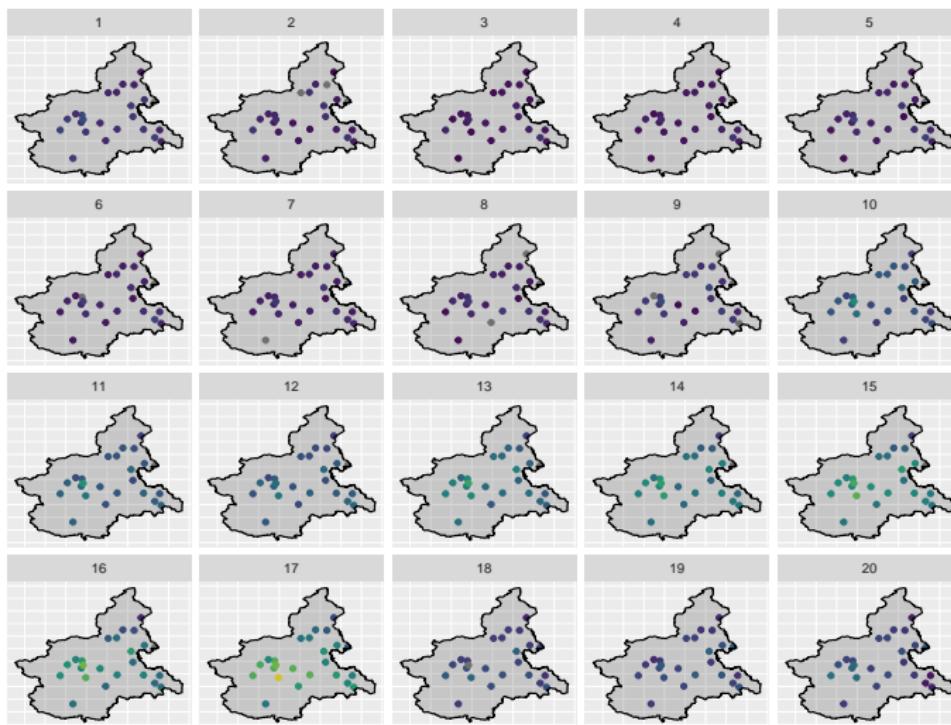
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# Modeling PM10 concentration in Piemonte (Italy)



# Modeling PM10 concentration in Piemonte (Italy)

Example from the Blangiardo & Cameletti book but simplified!

```
##   Station.ID      Date    dem      x      y    WS    temp    HMIX PREC    EMI PM10
## 1           1 2005-10-01  95.2 469.45 4972.85 0.90 288.81 1294.6     0 26.05   28
## 2           2 2005-10-01 164.1 423.48 4950.69 0.82 288.67 1139.8     0 18.74   22
## 3           3 2005-10-01 242.9 490.71 4948.86 0.96 287.44 1404.0     0  6.28   17
## 4           4 2005-10-01 149.9 437.36 4973.34 1.17 288.63 1042.4     0 29.35   25
## 5           5 2005-10-01 405.0 426.44 5045.66 0.60 287.63 1038.7     0 32.19   20
##   time logPM10
## 1     1 3.332205
## 2     1 3.091042
## 3     1 2.833213
## 4     1 3.218876
## 5     1 2.995732
```

## The model

We model the log PM10 concentration:

$$y_{it} \sim \mathcal{N}(\eta_{it}, \sigma_e^2)$$

$$\eta_{it} = \alpha + \beta_1 \text{dem}_i + \beta_2 \text{temp}_{it} + \omega_{it}$$

- $y_{it}$  is the log-concentration at location  $i$  in time  $t$
- $\alpha$  is an intercept
- $\beta_1$  and  $\beta_2$  parameters of altitude and temperature
- $\omega_{it}$  is the space-time residual

## Space time residual model

A first order autoregressive process with spatially colored innovations

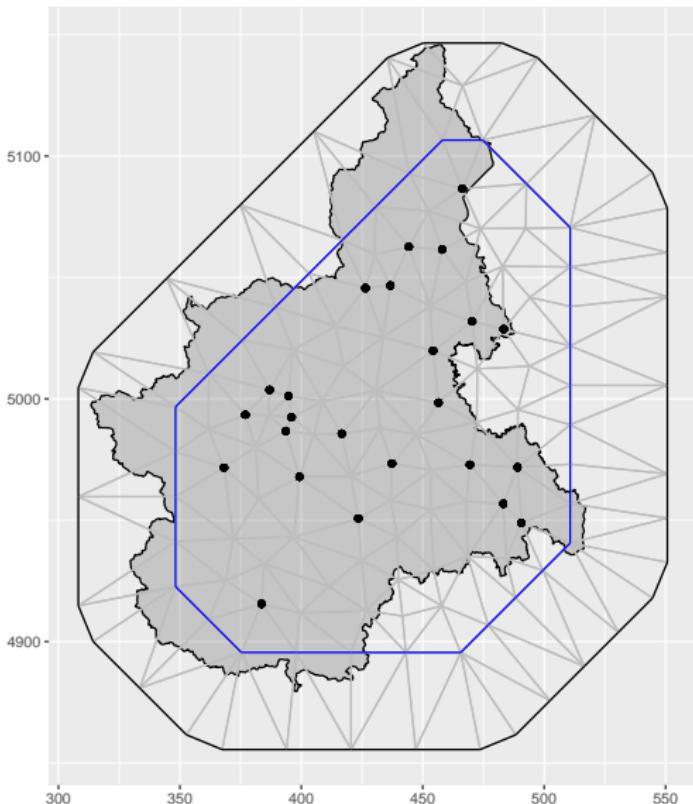
$$\omega_{it} = a\omega_{i(t-1)} + \xi_{it}$$

- $|a| < 1$  parameter of the AR1 process
- $\xi_{it}$  is a zero mean, temporally independent, Gaussian field with

$$\text{Cov}(\xi_{it}, \xi_{ju}) = \begin{cases} 0, & \text{if } t \neq u \\ \mathcal{C}(h), & \text{if } t = u \end{cases}$$

- $h$  distance between locations  $i$  and  $j$  -  $\mathcal{C}(h)$  is a Matern correlation function

## Building the model: mesh



## Building the model: Prior for the spde model

```
spde = inla.spde2.pcmatern(mesh = mesh,  
                           prior.range=c(100,0.5),  
                           prior.sigma=c(1,0.1))
```

- $\text{Prob}(\rho < 100 \text{ Km}) = 0.5$
- $\text{Prob}(\sigma > 1) = 0.1$

# Implement the model

```
# make the data a spatial object
coordinates(df) = c("x", "y")

# model component
cmp = ~ Intercept(1) +
  SPDE(coordinates, model = spde,
        group = time, control.group = list(model = "ari")) +
  dem(dem, model = "linear") +
  temp(temp, model = "linear")

# likelihood
lik = like(formula = logPM10 ~ Intercept + SPDE + dem + temp ,
           family = "gaussian",
           data = df)

# fit the model
fit = bru(cmp, lik,
          options = list(verbose = F,
                         bru_max_iter = 1,
                         inla.mode = "experimental"))
```

# Model Parameters

Fixed effects:

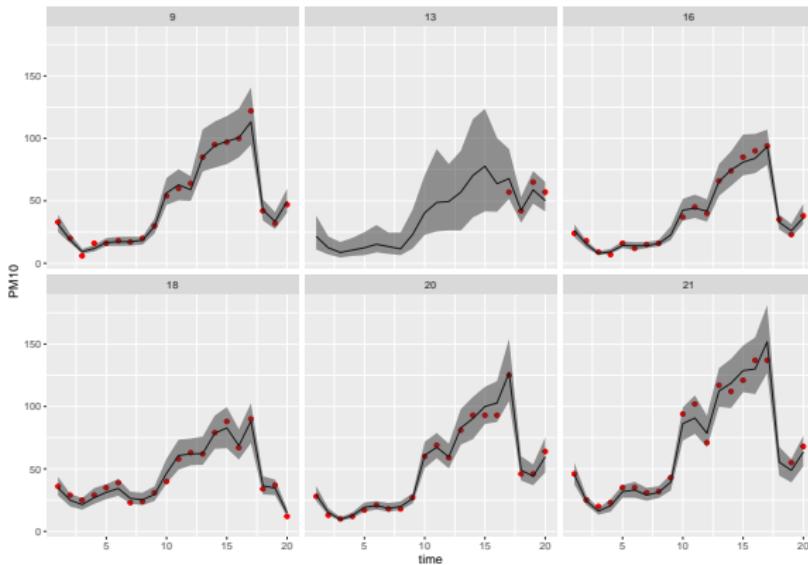
```
##               mean        sd 0.025quant 0.975quant
## Intercept -1.438103e+01 9.2183351517 -32.139029373 4.0476156436
## dem       -7.283307e-04 0.0008432533 -0.002398673 0.0009354501
## temp      6.250902e-02 0.0320760645 -0.001679595 0.1243716481
```

Hyperparameters of the random field:

```
##                               mean        sd 0.025quant 0.975quant
## Precision for the Gaussian observations 41.8579575 7.23518895 29.3185249
## Range for SPDE                      129.1799687 16.96397904 101.0510932
## Stdev for SPDE                      0.7511733 0.09963733 0.5913784
## GroupRho for SPDE                   0.9105294 0.02594638 0.85555876
##                                         0.975quant
## Precision for the Gaussian observations 57.6625010
## Range for SPDE                      167.2757328
## Stdev for SPDE                      0.9796968
## GroupRho for SPDE                   0.9557794
```

# Prediction at station locations

```
pred_at_station = predict(fit, df,
                           ~exp(Intercept + SPDE + dem + temp ))
```

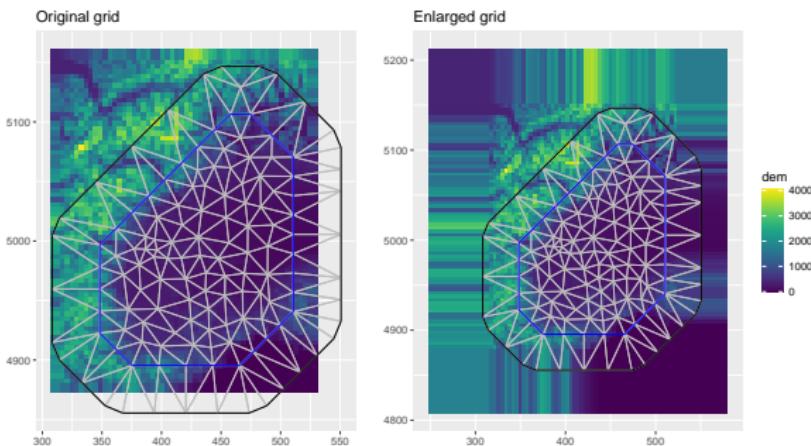


# Prediction is space

- To do this we need to have the covariates over the space of interest.
- They should cover the whole mesh

## Expanding the covariates: the `bru` `fill` `missing`

# Expanding the covariates: the bru\_fill\_missing



Note: You can do this with your favourite method!

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## Prediction in space

# Prediction in space

```
# Create a space time grid
pxl = pixels(mesh, nx = 200, ny = 200, mask = shape)
ips2 <- ipoints(domain = c(1:5), name = "time")
pxl_time <- cbind(cprod(pxl,ips2), data.frame(temp = 0))

pred_space = predict(fit, p xl_time,
                     ~ Intercept + SPDE +
                     dem_eval(inlabru:::eval_SpatialDF(dem_large,
                     .data.)) +
                     temp_eval(inlabru:::eval_SpatialDF(temp_large,
                     .data.,
                     selector = "time")))
```

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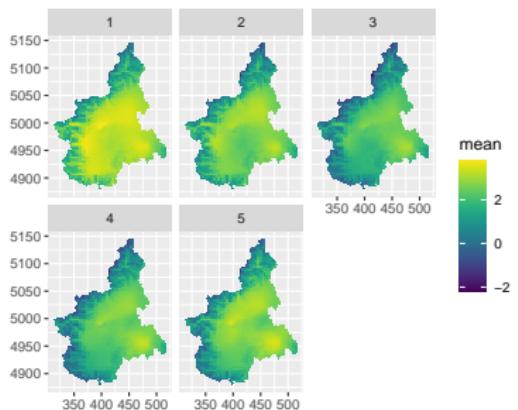
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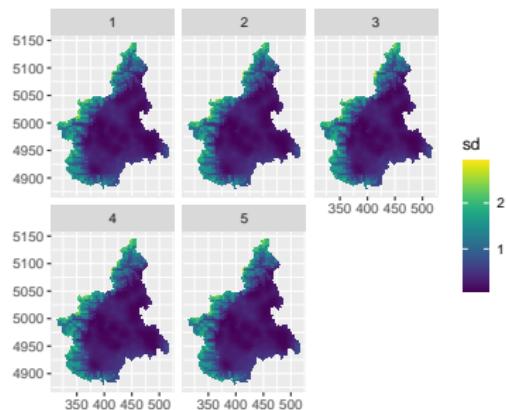
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# Prediction in space

Posterior mean



Posterior sd



## Samples form the fitted model

It is also possible to generate from the fitted model

```
sim_space = generate(fit, pxi_time,
                     ~ exp(Intercept + SPDE +
                           dem_eval(inlabru:::eval_SpatialDF(dem_large,
                                                               .data.)) +
                           temp_eval(inlabru:::eval_SpatialDF(temp_large,
                                                               .data.,
                                                               selector = "time"))),
                     n.samples = 2)
```

This can be useful as the posterior means are always smoother than the “real” field

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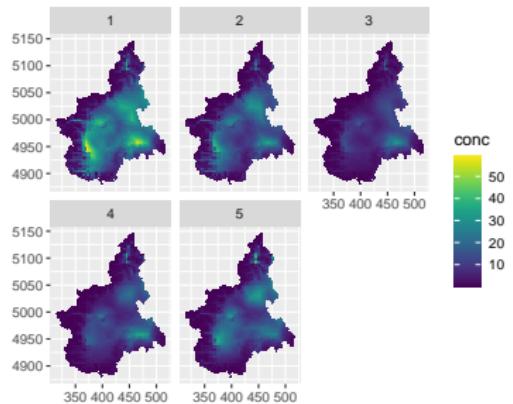
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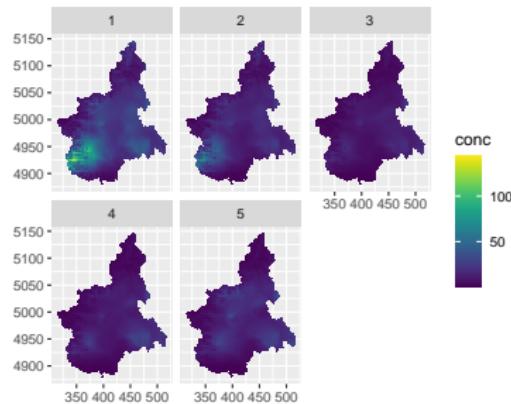
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## Samples form the fitted model

Sample 1



Sample 1



## Some concluding remark

- The functions `predict()` and `generate()` in `inlabru` use the INLA function `inla.posterior.sample()` internally.
- `inlabru` is a wrapper around INLA so all the internal computations are identical!
- `inlabru` makes handling of spatial object a lot easier
- Transition from `sp` to `sf/terra` some changing can be expected

Thank you for your attention!

If you have any doubts or questions, please write :  
[sara.martino@math.ntnu.no](mailto:sara.martino@math.ntnu.no)

