Numerical Solution of 2D Heat Conduction

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December 7, 2025

Problem Statement

Solution

Discretization

The standard five-point difference scheme is used to discretize the equation.

We define that:

$$-\frac{1}{h^2}\left[k_{i+\frac{1}{2},j}(u_{i+1,j}-u_{i,j})-k_{i-\frac{1}{2},j}(u_{i,j}-u_{i-1,j})\right]-\frac{1}{h^2}\left[k_{i,j+\frac{1}{2}}(u_{i,j+1}-u_{i,j})-k_{i,j-\frac{1}{2}}(u_{i,j}-u_{i,j-1})\right]=f_{i,j}$$

and get

$$a_P u_{i,j} - a_E u_{i+1,j} - a_W u_{i-1,j} - a_N u_{i,j+1} - a_S u_{i,j-1} = h^2 f_{i,j}$$

which

- $a_E = k_{i+1/2,j}$
- $a_W = k_{i-1/2,j}$
- $a_N = k_{i,j+1/2}$
- $a_S = k_{i,j-1/2}$
- $a_P = a_E + a_W + a_N + a_S$

Code

Shown in the attachment.

Final Result

Convergence speed comparison:

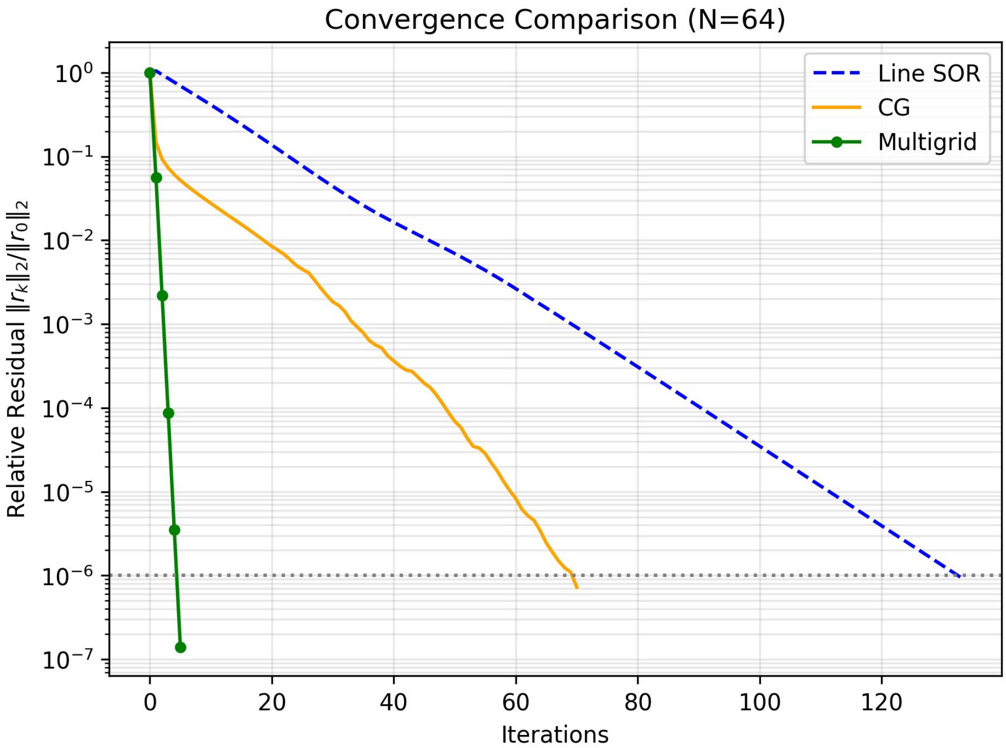

Figure 1: Convergence Comparison

Method	Iterations	Time (s)
Line SOR ($\omega = 1.75$)	133	0.5103
Conjugate Gradient (CG)	70	0.0025
Multigrid (V-Cycle)	**5**	0.5535

Table 1: Comparison of numerical methods for solving the electrostatic field problem ($N = 64$, relative tolerance $\varepsilon = 10^{-6}$).

- **Multigrid (MG)**: Exhibits the fastest convergence. For elliptic equations, MG typically demonstrates a grid-independent convergence rate (requiring $O(1)$ iterations). It generally achieves a relative error of 10^{-6} within 5 iterations.
- **Conjugate Gradient (CG):** Exhibits moderate convergence speed. For a grid with $N = 64$, the condition number is $\kappa(A) \approx O(N^2)$, and the number of iterations for CG is roughly proportional to N (specifically $\sqrt{\kappa}$). While significantly faster than SOR, it is slower than MG.
- **Line Gauss-Seidel SOR:** Exhibits the slowest convergence (despite utilizing an optimized relaxation factor ω). As the iteration count increases, once high-frequency errors are eliminated, the removal of low-frequency errors becomes extremely slow.