

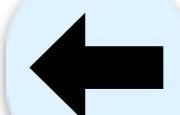
# Variance



# Distribution Properties

Deterministic functions of distribution

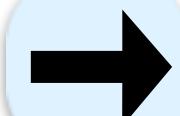
Not random



Long-term average

Expectation  $E(X, \mu)$

Die:  $\mu=3.5$



Consistency

Variation from the mean

# Money Matters

Two companies, each with 1,000 employees

Both same mean salary: \$100K But

C1: Every employee makes \$100K 100M total

C2: Every employee makes \$1, CEO \$99,999,001

Which will you join?

Same mean Very different distributions

Mean ain't all Variation matters!



# Difference from Mean

$X$  r.v. with mean  $\mu$

How much  $X$  differs from  $\mu$  on average?

Candidate

$E|X - \mu|$

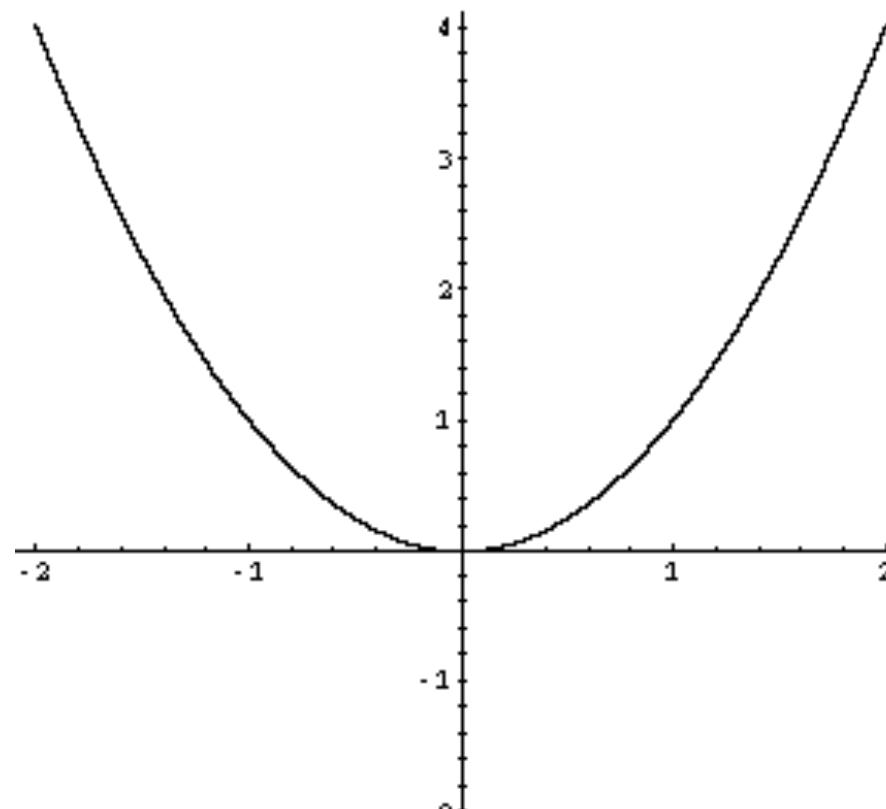
Mean absolute difference

Not commonly used

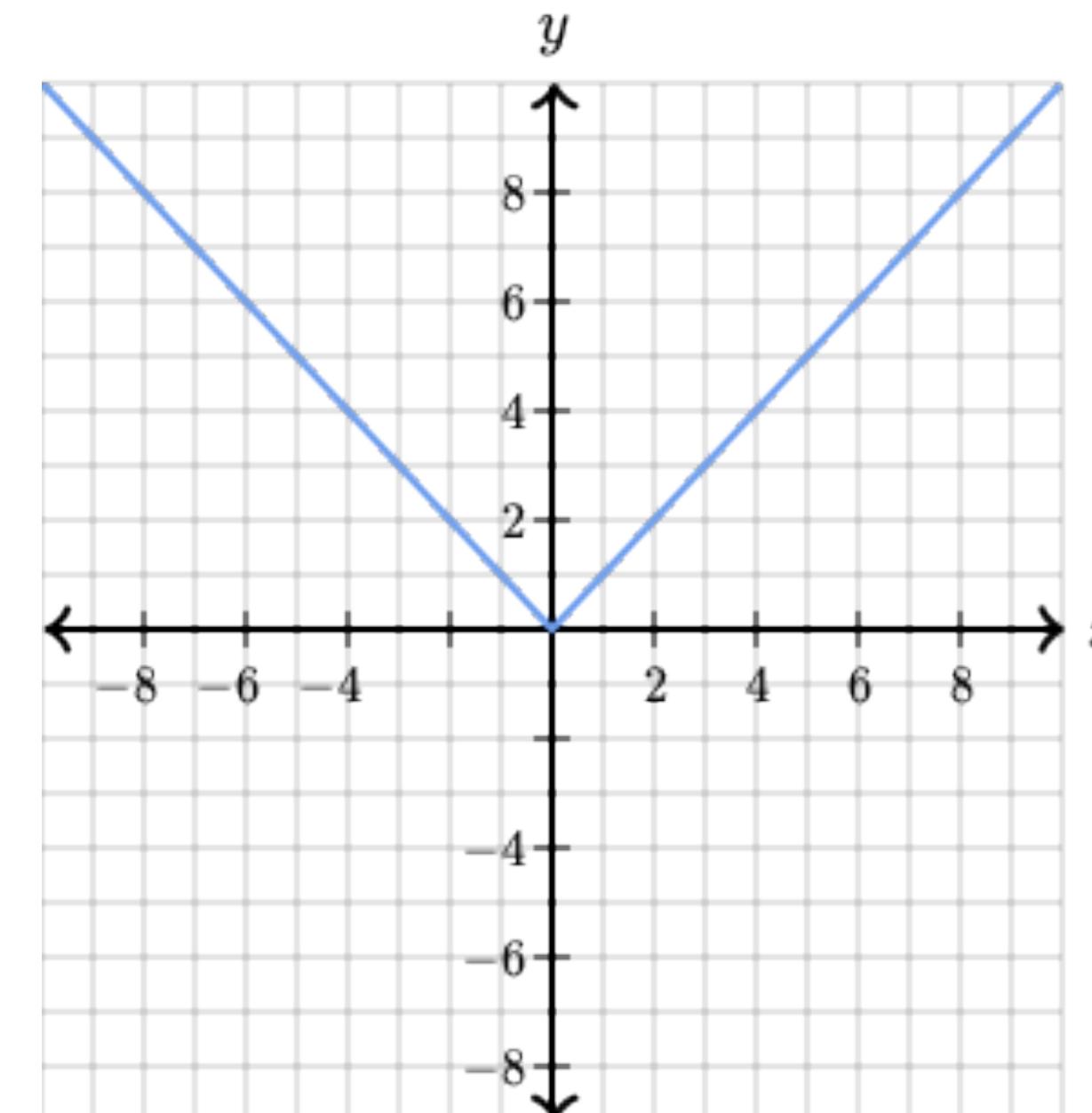
Absolute value function hard to analyze

Instead

$E(X - \mu)^2$



$$y = x^2$$



# Variance

Expected squared difference between X and its mean

$$V(X) = E [ (X - \mu)^2 ]$$

$$V(X) = E (X - \mu)^2$$

Standard deviation

$$\sigma_x = \sqrt{V(X)}$$

(positive)

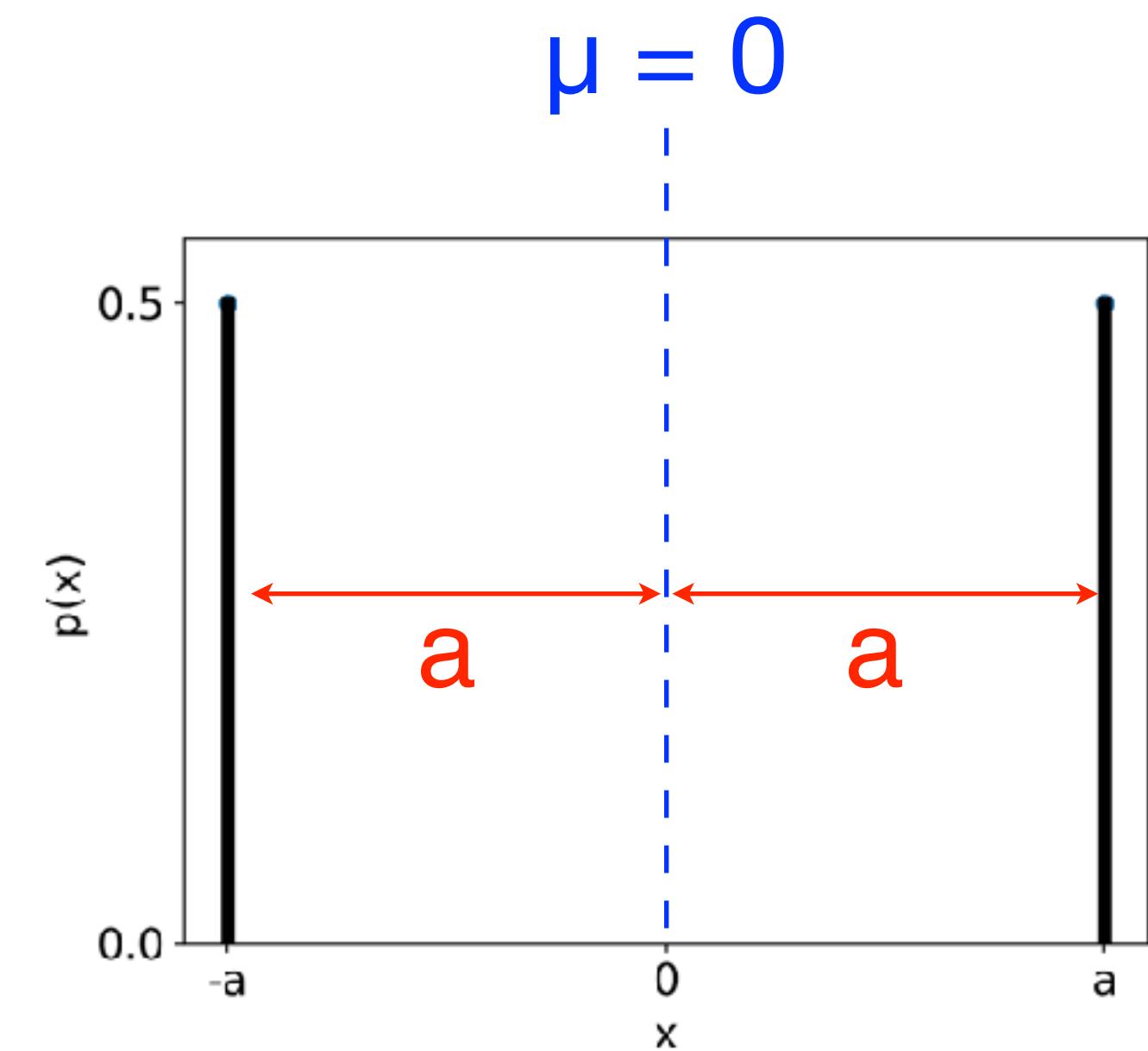
Constants

Properties of distribution

# Examples

$x$	$p_x$	$x - \mu$	$(x - \mu)^2$
$-a$	$\frac{1}{2}$	$-a$	$a^2$
$a$	$\frac{1}{2}$	$a$	$a^2$

$$\mu = 0$$



$$V(X) = \frac{1}{2} \cdot a^2 + \frac{1}{2} \cdot a^2 = a^2$$

$X^2$  is always  $a^2$

$(X - \mu)^2 = a^2$  always

$$\sigma_x = a$$

“average” distance from mean

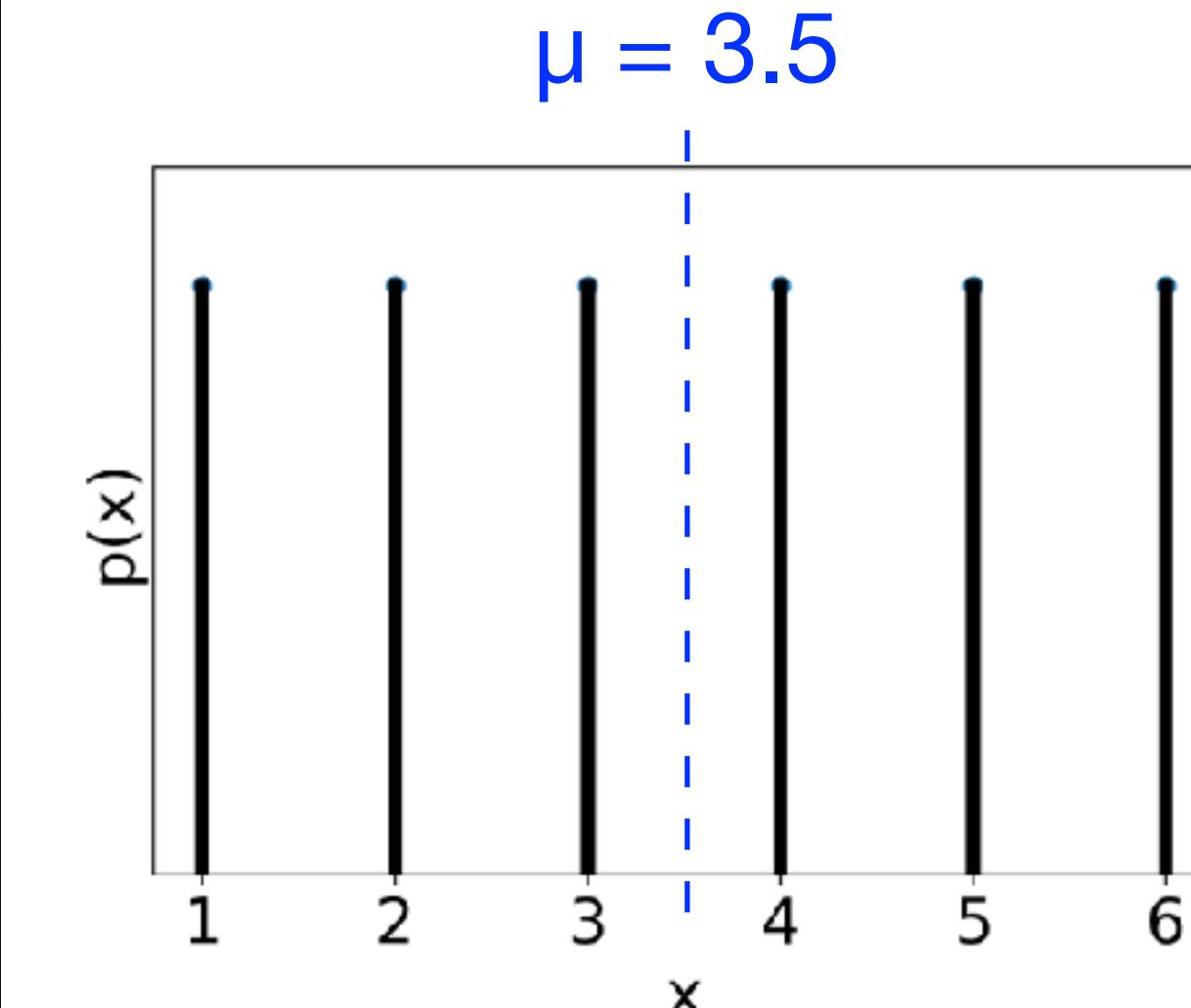
# Fair Die

$$\mu = 3.5$$

$$V(X) = E(X - \mu)^2 = \frac{2(6.25 + 2.25 + 0.25)}{6} = \frac{8.75}{3} = 2.92..$$

x	p <sub>x</sub>	x - μ	(x - μ) <sup>2</sup>
1	1/6	-2.5	6.25
2	1/6	-1.5	2.25
3	1/6	-0.5	0.25
4	1/6	0.5	0.25
5	1/6	1.5	2.25
6	1/6	2.5	6.25

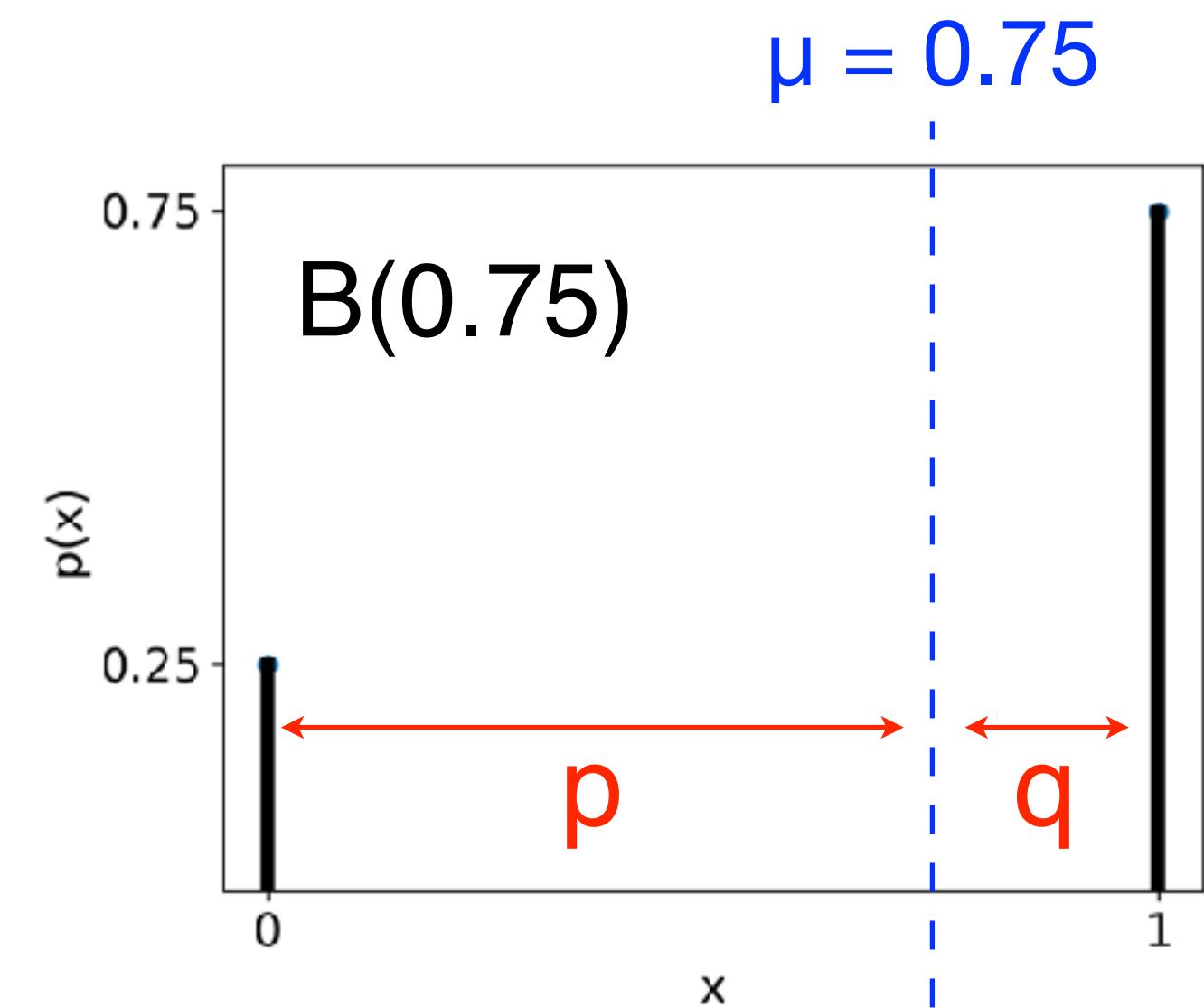
$$\sigma = \sqrt{2.92..} = 1.71...$$



# Bernoulli p

$x$	$p_x$	$x - \mu$	$(x - \mu)^2$
0	q	$0 - p = p$	$p^2$
1	p	$1 - p = q$	$q^2$

$$\mu = p$$



$$V(X) = q \cdot p^2 + p \cdot q^2 = pq(p+q) = pq$$

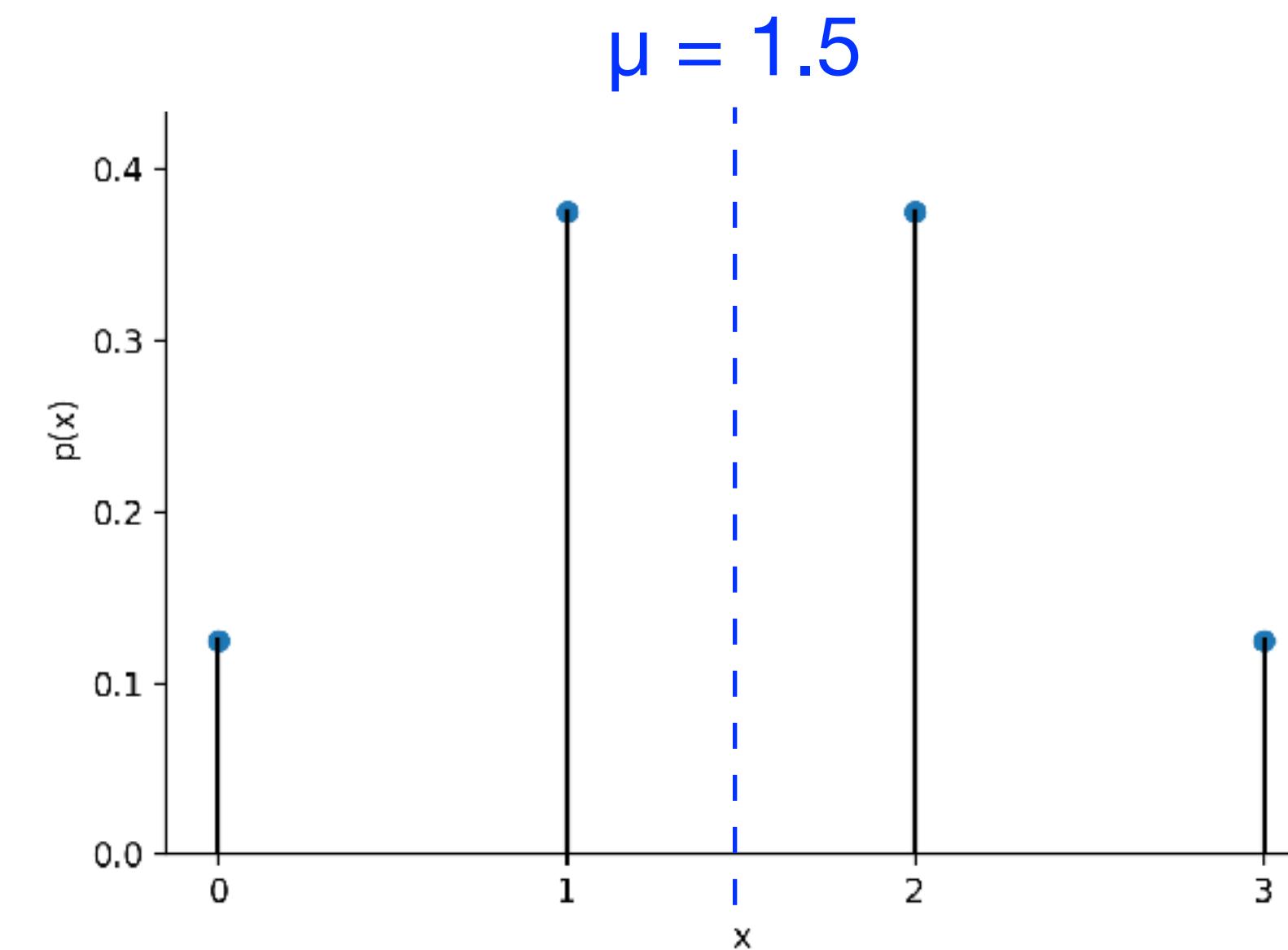
# 3 Coins

Toss 3 fair coins

$X = \# \text{ heads}$

$X$	$p_X$	$X - \mu$	$(X - \mu)^2$
0	$\frac{1}{8}$	-1.5	2.25
1	$\frac{3}{8}$	-0.5	0.25
2	$\frac{3}{8}$	0.5	0.25
3	$\frac{1}{8}$	1.5	2.25

$$\mu = 1.5$$



$$V = 2\left(\frac{1}{8} \cdot 2.25 + \frac{3}{8} \cdot 0.25\right) = \frac{1}{4}(2.25 + 0.75) = \frac{3}{4}$$

$$\sigma = \sqrt{3}/2$$

Shortly: simpler derivation

# Different Formula

$$V(X)$$

$$= E(X - \mu)^2$$

$$E(X) = \mu$$

$$= E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - E(2\mu X) + E(\mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

2,  $\mu$  - constants

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - (E X)^2$$



# Bernoulli p Again

$X \sim B(p)$

Recall:  $EX = p$

$V(X) = pq$

Re-derive using

$$V(X) = E X^2 - (EX)^2$$

$$E(X^2) = (1 - p) \cdot 0^2 + p \cdot 1^2 = p$$

Even simpler

$$0^2=0, 1^2=1$$

$$\rightarrow X^2=X$$

$$\rightarrow EX^2 = EX = p$$

$$V(X) = E X^2 - (EX)^2 = p - p^2 = p(1 - p) = pq$$



# Observations

$$V(X) = E(X - \mu)^2$$

$$0 \leq V \leq \max (X-\mu)^2$$

=  
X is a  
constant

=  
X constant or  
takes two values  
with equal prob.

$$0 \leq \sigma \leq \max |X-\mu|$$

$$V(X) = EX^2 - \mu^2$$

$$V(X) \leq E(X^2)$$

# Properties

# How simple modification affect $V$ and $\sigma$

# Addition (translation) $x + b$

Multiplication (scaling)  $a \cdot X$

$+ \& x$  (affine transformation)  $aX + b$

# Addition

X - random variable

b - constant (e.g. 2)

$$\mu_{x+b} = \mu_x + b$$

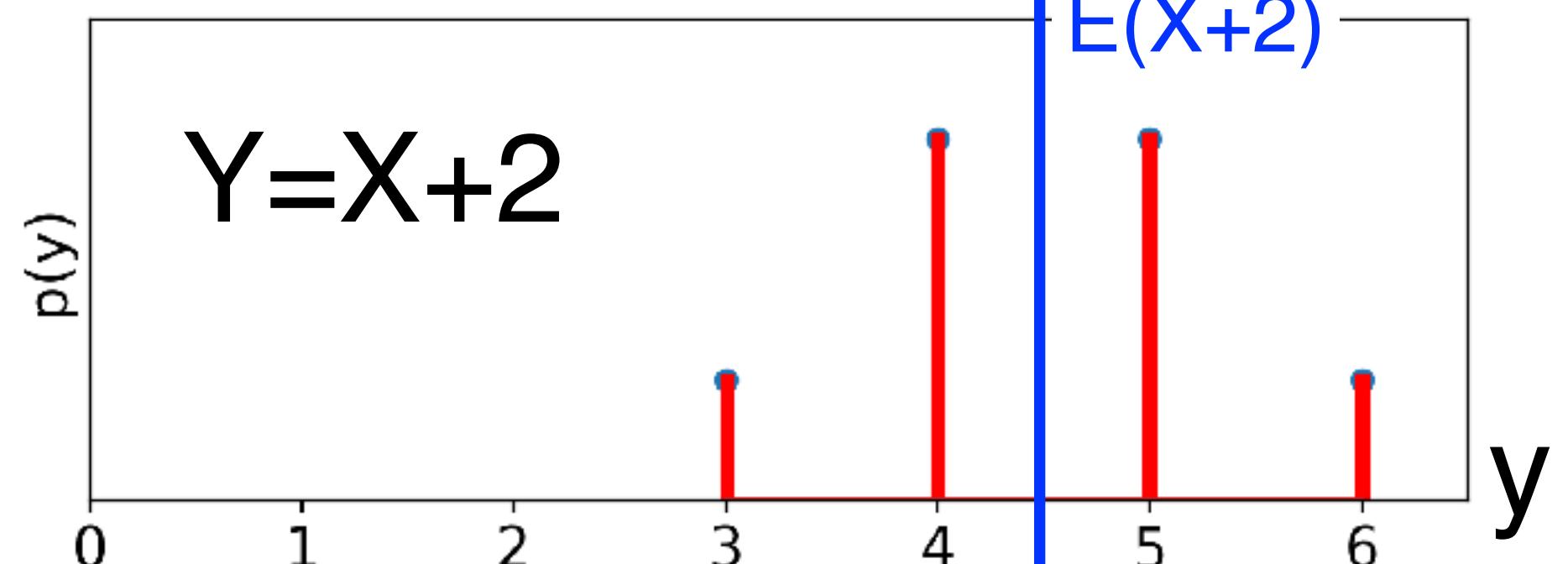
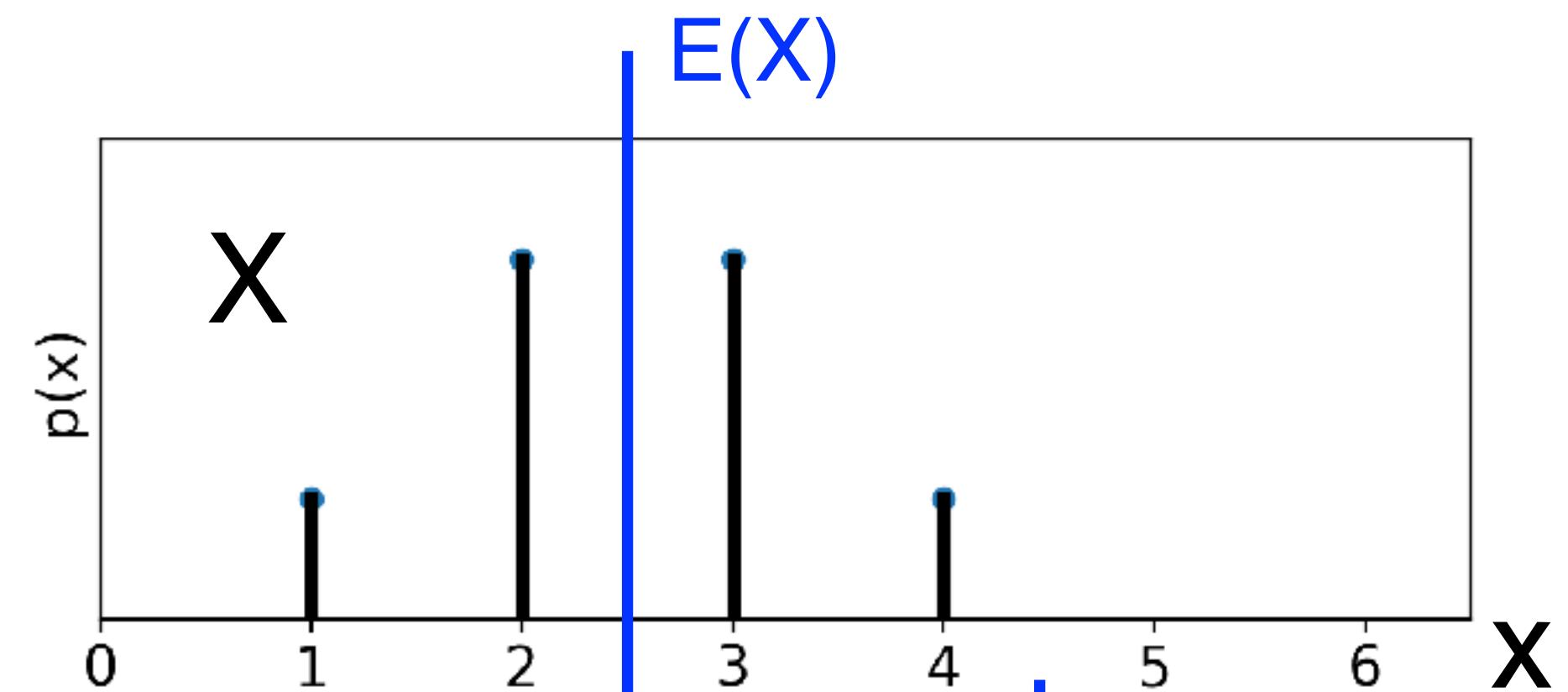
Linearity of expectation

$$V(X + b) = E[(X + b - \mu_{x+b})^2]$$

$$= E[(X + b - \mu_x - b)^2]$$

$$= E(X - \mu_x)^2$$

$$= V(X)$$



# Translated B(p)

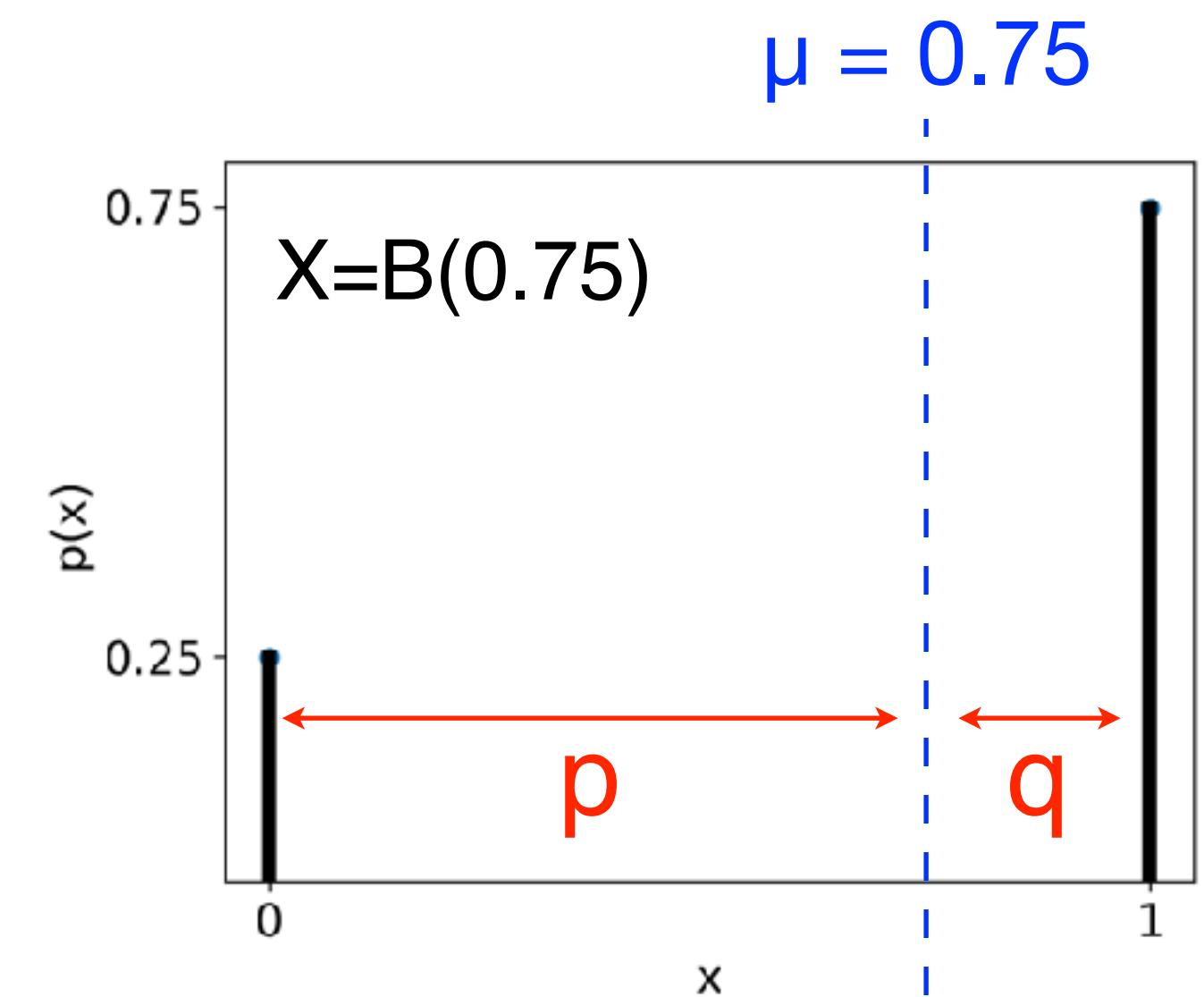
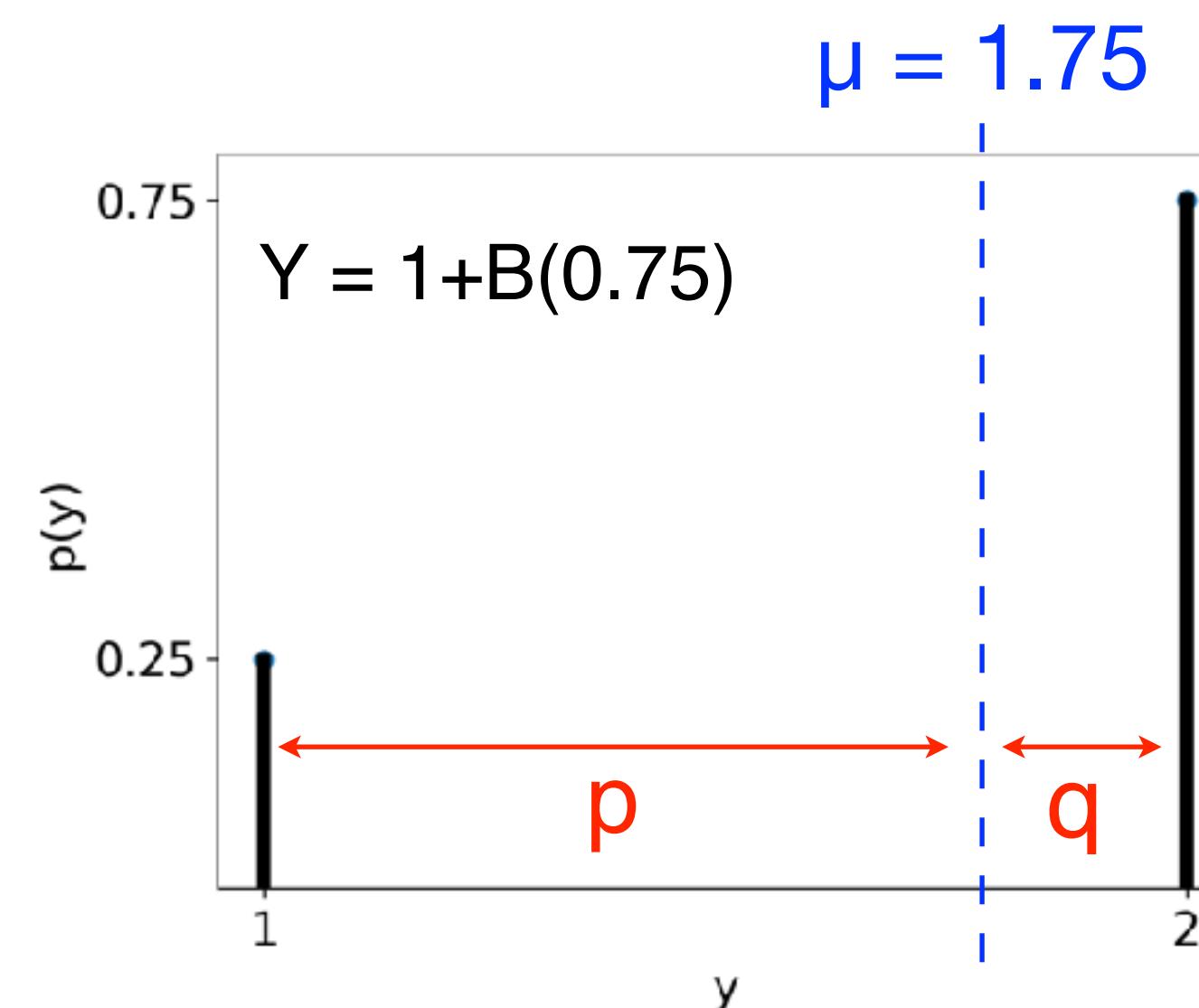
$$X \sim B(p)$$

$$V(X) = p(1-p)$$

$$Y = X + 1$$

$y$	$p_y$
1	$1-p$
2	$p$

$$\mu_y = 1 + p \quad (\text{linearity of expectations})$$



$$\begin{aligned}
 V(Y) &= E(Y - \mu_y)^2 = (1 - p)(1 - 1 - p)^2 + p(2 - 1 - p)^2 \\
 &= (1 - p)p^2 + p(1 - p)^2 = p(1 - p)(p + 1 - p) = p(1 - p)
 \end{aligned}$$

=  $V(X)$

# Scaling

$$V(aX) = E(aX - \mu_{ax})^2$$

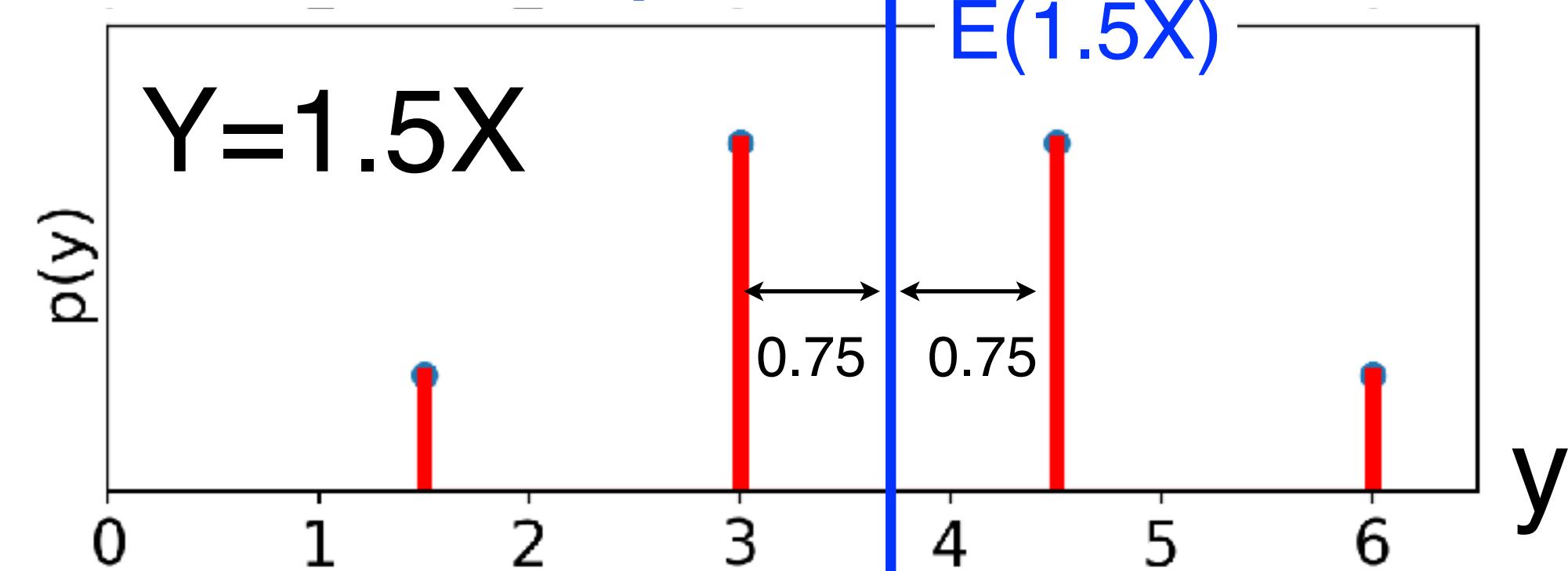
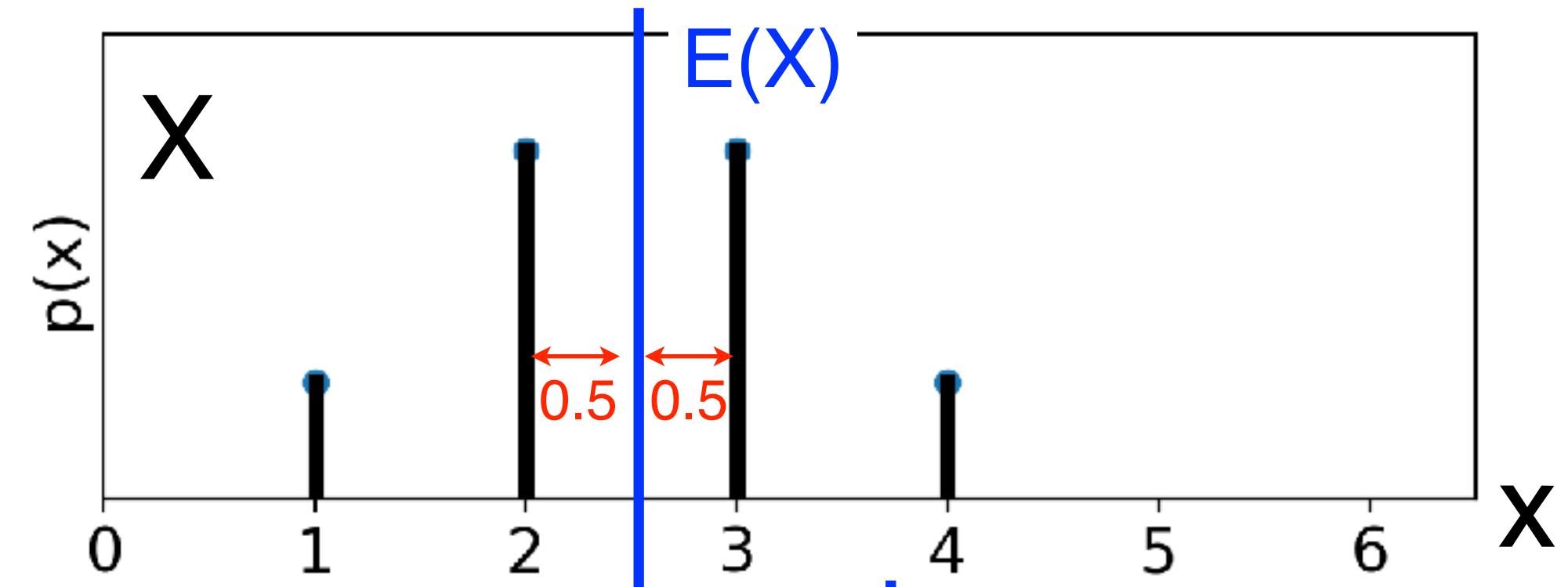
$$= E(aX - a\mu_x)^2$$

$$= E[a^2(X - \mu_x)^2]$$

$$= a^2 E(X - \mu_x)^2$$

$$= a^2 V(X)$$

$$\mu_{ax} = a\mu_x$$



Difference from mean grew by  $a^2$

$$\sigma_{ax} = \sqrt{V(aX)} = \sqrt{a^2 V(X)} = |a| \sigma_x$$

“Average” difference  
from mean grew by  
a factor of  $|a|$

# Affine Transformation

$$V(aX + b) = V(aX) = a^2 V(X)$$

$$\sigma_{ax+b} = |a| \sigma_x$$

**This Lecture: Variance**

**Next: Two Variables**