

Functions of Random Variables

X - income, $Y = 0.3X$ - income tax

X - driving speed, $Y = 2^x$ - speeding ticket

$X \sim f_X$ f_X known distribution

$Y = g(X)$ g known deterministic function

f_Y ?

What is the distribution of Y ?

Power-law pdf's

$$a > -1$$

$$\underbrace{\int_0^1 (a+1)x^a dx}_{\geq 0} = x^{a+1} \Big|_0^1 = 1$$

$$a < -1 \text{ later}$$

$$f(x) = \begin{cases} (a+1)x^a & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

pdf!

$$F(x) = \int_0^x (a+1)u^a du = u^{a+1} \Big|_0^x = x^{a+1}$$

$$a \leq x \leq 1$$

Use $f(x) = 3x^2$

$$F(x) = x^3$$

$$0 \leq x \leq 1$$

$g \nearrow$

$$X \sim f_X(x) = 3x^2$$

$$F_X(x) = x^3$$

$$0 \leq x \leq 1$$

$$Y = g(X) \triangleq X^{\frac{3}{2}}$$

$$F_Y(y) = P(Y \leq y) \quad 0 \leq y \leq 1$$

$$= P(X^{\frac{3}{2}} \leq y)$$

$$= P(X \leq y^{\frac{2}{3}})$$

$$= F_X(y^{\frac{2}{3}})$$

$$= y^2$$

$$F_Y(0) = 0$$

$$F_Y(1) = 1 \quad \checkmark$$

$$f_Y(y) = F'_Y(y) = 2y$$

$g \searrow$

$$Y = g(X) \triangleq X^{-3}$$

$$F_Y(y) = P(Y \leq y) \quad y \geq 1$$

$$= P(X^{-3} \leq y)$$

$$= P(X \geq y^{-\frac{1}{3}})$$

$$= 1 - P(X \leq y^{-\frac{1}{3}})$$

$$= 1 - F_X(y^{-\frac{1}{3}})$$

$$= 1 - y^{-1}$$

$$\checkmark F_Y(\infty) = 1$$

$$f_Y(y) = F'_Y(y) = y^{-2}$$

g ↗

Generalized form for increasing $Y=g(X)$

$$f_X(x) = 3x^2$$

$$F_X(x) = x^3$$

$$g(x) = x^{\frac{3}{2}}$$

$$h(y) = y^{\frac{2}{3}}$$

$$f_Y(y) = 2y$$

$$F_Y(y) \triangleq P(Y \leq y)$$

$$f_Y(y) = F'_Y(y)$$

$$= P(g(X) \leq y)$$

$$P(X^{\frac{3}{2}} \leq y)$$

$$= [F_X(h(y))]'$$

$$[(y^{\frac{2}{3}})^3]'$$

$$= P(X \leq g^{-1}(y))$$

$$P(X \leq y^{\frac{2}{3}})$$

$$= F'_X(h(y)) \cdot h'(y)$$

$$3(y^{\frac{2}{3}})^2 \cdot \frac{2}{3}y^{-\frac{1}{3}}$$

$$= F_X(g^{-1}(y))$$

$$F_X(y^{\frac{2}{3}})$$

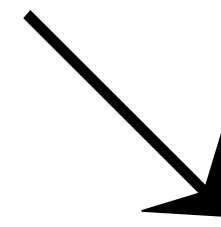
$$= f_X(h(y)) \cdot h'(y)$$

$$2y$$

$$= F_X(h(y))$$

$$h(y) \stackrel{\text{def}}{=} g^{-1}(y)$$

g



Generalized form for decreasing $Y=g(X)$

$$F_Y(y) \triangleq P(Y \leq y)$$

$$= P(g(X) \leq y)$$

$$= P(X \geq g^{-1}(y))$$

$$= 1 - P(X \leq g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(h(y))$$

$$f_Y(y) = F'_Y(y)$$

$$= [1 - F_X(h(y))]'$$

$$= -F'_X(h(y)) \cdot h'(y)$$

$$= -f_X(h(y)) \cdot h'(y)$$

Combining

$g \nearrow$

$$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

$g \searrow$

$$f_Y(y) = -f_X(h(y)) \cdot h'(y)$$

For both

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

Alternative formulation

$$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) \Big|_{y=g(x)}$$

Functions of Random Variables

Uniform Distributions

