Elevation Angle Determination

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Definitions

ϕ_0 – beam azimuth at 0° elevation angle
ϕ – beam azimuth at non-zero elevation angle
θ – elevation angle
$\hat{n}$ – normal unit vector in the direction of the beam
X – offset of interferometer array in $\hat{x}$ direction
Y – offset of interferometer array in $\hat{y}$ direction
Z – offset of interferometer array in $\hat{z}$ direction
f – tranmission frequency
d – distance between phase fronts in planes passing through arrays

Background

The elevation angle is a measurement that can be made with a SuperDARN radar that has both a main and interferometer array. A calibration procedure must be followed to determine the signal delays through each path from the respective array to the digitizer. This calibration procedure is describe in a separate document. In addition, the software at the radar must be configured to compute and save cross-correlation functions (XCFs).

We currently have a procedure to compute the elevation angle that is located in `elevation.c`. It was written by Kile Baker and possibly Simon Wing, and modified somewhat by Robin Barnes. Comments in the code are not sufficient to fully understand the code. It would be useful to identify a document that describes this code. Note that this code appears to work well for simple situations, i.e., interferometer array offsets that are in front of or behind the main array. However, the results do not appear consistent for more complicated offsets, i.e., those in the vertical direction or along the direction of the main array. Note that there is a special program called `elev_goose.c` that is used for the Goose Bay radar. This program is specific to the geometry of this radar and gets stuck in an infinite loop when used on other geometries.

I believe there is a simpler and more general method to determine the angle of arrival of signals from a SuperDARN radar. The basic idea is that it is possible to determine the distance between two parallel planes, one passing through the center of the main array and the other passing through the center of the inteferometer array. The separation between these two planes is a function of the beam azimuth (ϕ), the elevation angle (θ), the tranmission frequency (f) and the physical separation between the centers of the antennas (X,Y,Z). The separation between these two planes introduces a phase difference in the signals. By equating this difference (including also a constant phase offset due to the electronics and cables) to the observed phase delay in the two signals, a quadratic equation can be solved for the elevation angle. This technique is described below.

A More General Technique

To solve this quadratic equation it is first necessary to write the equations for the two planes. The normal unit vector of the planes is in the direction the signals are coming from and is given by:

$$\hat{n} = (\cos\theta\sin\phi)\,\hat{x} + (\cos\theta\cos\phi)\,\hat{y} + (\sin\theta)\,\hat{z} \tag{1}$$

or introducing $A \equiv \cos\theta\sin\phi$, $B \equiv \cos\theta\cos\phi$ and $C \equiv \sin\theta$:

$$\hat{n} = A\hat{x} + B\hat{y} + C\hat{z} \tag{2}$$

Using the convention that the center of the main array is located at the origin and the center of the interferometer array is located at position (X, Y, Z) we can write equations for planes passing through these locations and both oriented normal to $\hat{n}$.

$$\begin{aligned}
\textbf{main}: \quad & Ax + By + Cz && = 0 \equiv d_{main} && (3)\\
\textbf{int.}: \quad & A(x-X) + B(y-Y) + C(z-Z) && = 0 && (4)\\
& Ax + By + Cz && = AX + BY + CZ \equiv d_{int} && (5)
\end{aligned}$$

Note that this convention is the standard that is used in `hardware.dat` files. The distance between these two planes is given by:

$$\begin{aligned}
d &= \frac{d_{int} - d_{main}}{A^2 + B^2 + C^2} && (6)\\
&= AX + BY + CZ && (7)\\
&= (\cos\theta \sin\phi)\, X + (\cos\theta\cos\phi)\, Y + (\sin\theta)\, Z && (8)
\end{aligned}$$

since $\hat{n}$ is a unit vector the denominator is unity.

Note that the beam azimuth (ϕ is actually a function of elevation angle (θ). There is something called the *coning angle* in which the phases of the antennas interfere along a cone around the axis of the array. [**I don't really understand this bit.**] The functional form of this coning angle is given by:

$$\sin\phi = \frac{\sin\phi_0}{\cos\theta} \tag{9}$$

Equation 1 can be modified by using the coning angle equation and

$$\cos\phi = \left(1 - \sin^2\phi\right)^{\frac{1}{2}} = \left(\cos^2\theta - \sin^2\phi_0\right)^{\frac{1}{2}} / \cos\theta \tag{10}$$

giving

$$\hat{n} = \sin\phi_0 \hat{x} + \left(\cos^2\theta - \sin^2\phi_0\right)^{\frac{1}{2}} \hat{y} + \sin\theta\hat{z} \tag{11}$$

and modifying $A \equiv \sin\phi_0$, $B \equiv \left(\cos^2\theta - \sin^2\phi_0\right)^{\frac{1}{2}}$ and $C \equiv \sin\theta$ so that equation 7 is still valid and equation 8 becomes:

$$\begin{aligned}
d &= AX + BY + CZ && (12)\\
&= (\sin\phi_0)\, X + \left(\cos^2\theta - \sin^2\phi_0\right)^{\frac{1}{2}} Y + (\cos\theta\cos\phi)\, Y + (\sin\theta)\, Z && (13)
\end{aligned}$$

The delay to the signal caused by the offset between the arrays is given by:

$$\phi_{geo} = kd \tag{14}$$

where $k = 2\pi/\lambda = 2\pi c/f$ is the wavenumber, c is the speed of light and f is the transmission frequency. Note that ϕ_{geo} is a nonlinear function of the elevation angle, θ, and is the phase delay due to geometry; array separation and look-direction.

There is an additional delay in SuperDARN systems that is introduced by differences in the paths that signal make from the main array and the interferometer array. This delay is largely due to differences in cable lengths to the antenna arrays, but can also be due to differences in the electrical paths in the electronics. As part of a proper calibration the time difference between the two paths (from antenna to digitizer) is measured and entered in the `hardware.dat` file as the parameter t_{diff}. This number is typically a few hundred nanoseconds and is listed in the file as a decimal number of microseconds.
Using this measured time delay the phase delay associated with t_{diff} is given by:

$$\phi_{path} = 2\pi f t_{diff} \tag{15}$$

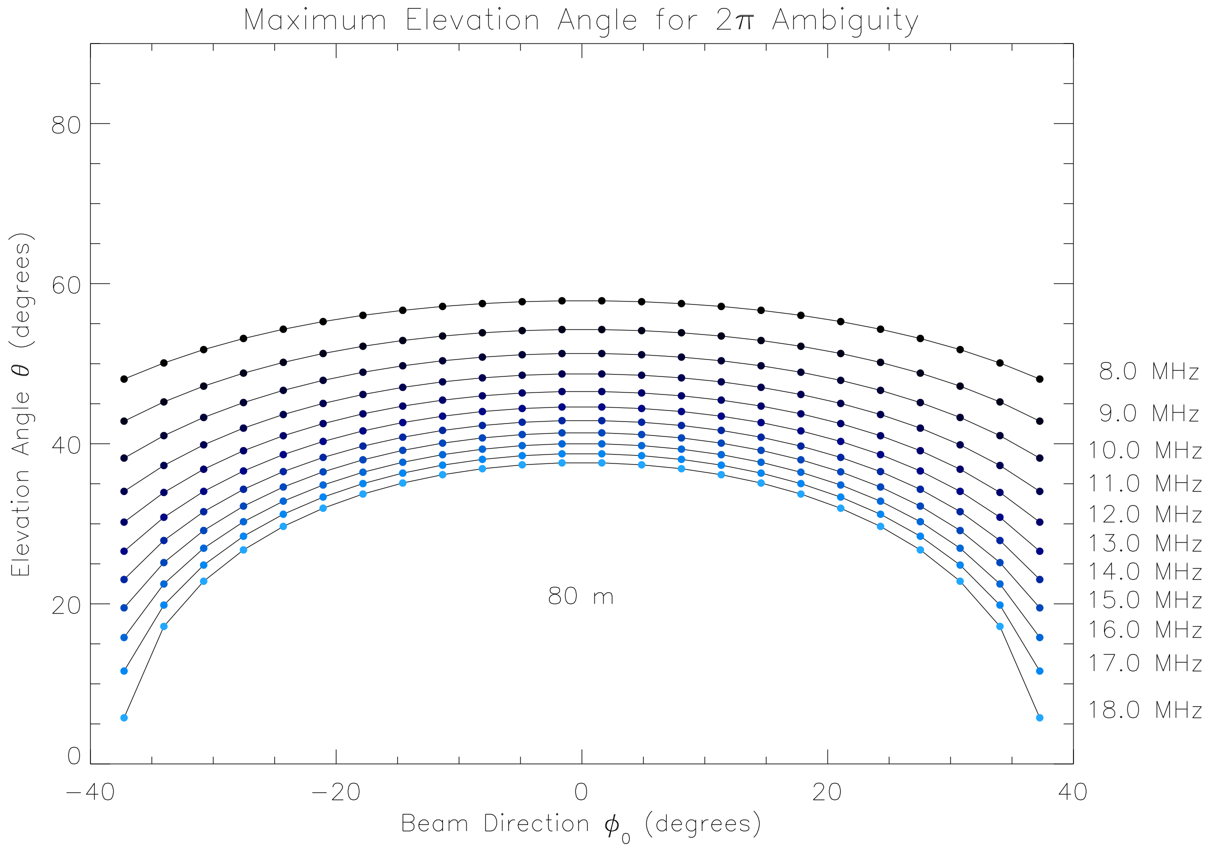

Figure 1: Maximum elevation angle before a 2π ambiguity in the observed phase is introduced. The example shown is for the MSI Oregon radar configuration in which the interferometer arrays are offset behind the main arrays by 80 m. No other offsets are present.

Note that these delays are in the opposite sense. For example, in the MSI radars the interferometer arrays is behind the main array, so $X = Z = 0$ and $Y < 0$ so that $\phi_{geo} < 0$ for any beam direction coming from in front of the arrays. In this situation signals arrive at the main array before the interferomter array. On the other hand, the interferometer cables are much shorter than the main array cables (350 ft compared to 650 ft) and $t_{diff} < 0$, by convention. In this case the signals arrive at the interferometer array because the electrical path is short. The total delay is then given by:

$$\phi_{total} = \phi_{geo} - \phi_{path} \tag{16}$$

In the SuperDARN system the phase difference between the main and interferometer array is calculated from the auto-correlation (ACF) and cross-correlation (XCF) functions. One complication is that the phase delay obtained from the ACF and XCF is only the residual phase and is between $-\pi$ and π. It is likely that the observed delay is actually several multiples of 2π, however, it is not possible to definitively determine the actual phase delay from the ACF and XCF measurements alone. The implications of this ambiguity in the observed phase difference is two-fold. First, there is an inherent limitation in the maximum elevation angle that can be uniquely determined. This maximum elevation angle depends on the separation between the arrays and the beam direction (ϕ_0).

Figure 1 shows the maximum elevation angle for the MSI Oregon radars in which the interferometer arrays are offset behind their respective main arrays by a distance of 80 m, i.e., (X,Y,Z) = (0,-80,0). As shown, the maximum elevation angle decreases from a maximum value of $\sim 60°$, for a transmission frequency of 8.0 MHz and looking normal to the array, for both higher frequencies and for beams looking off the boresight direction.

The second impact is that the residual phase must shifted by the appropriate number of 2π factors in order to match the delays that are introduced by the other factors; geometry and electrical path. The need for this shifting introduces the possibilty that the elevation angle determined can be significantly from the actual elevation angle.

Once the appropriate number of 2π factors have been added to the observed residual phase, the resulting phase difference (ϕ_{obs}) is used to determine an equation that depends on the elevation angle. Equating the observed and expected phase delays gives:

$$\phi_{total} = \phi_{geo} - \phi_{path} = \phi_{obs} \tag{17}$$

which can be written as:

$$\begin{aligned} \phi_{obs} + \phi_{path} &= \phi_{geo} && (18) \\ &= kd && (19) \\ &= k\left[\sin\phi_0 X + \left(\cos^2\theta - \sin^2\phi_0\right)^{\frac{1}{2}} Y + \sin\theta Z\right] && (20) \end{aligned}$$

Rearranging and using the identity $\sin^2\theta + \cos^2\theta = 1$:

$$\left[1 - \sin^2\theta - \sin^2\phi_0\right] Y^2 = \left[\frac{\phi_{obs} + \phi_{path}}{k} - \sin\phi_0 X - \sin\theta Z\right]^2 \tag{21}$$

and simplifying using the following factors:

$$\begin{aligned} D &\equiv 1 - \sin^2\phi_0 && (22) \\ E &\equiv \frac{\phi_{obs} + \phi_{path}}{k} - \sin\phi_0 X && (23) \end{aligned}$$

We can write this equation as:

$$\begin{aligned} \left[D - \sin^2\theta\right] Y^2 &= \left[E - \sin\theta Z\right]^2 && (24) \\ &= E^2 - 2EZ\sin\theta + \sin^2\theta Z^2 && (25) \end{aligned}$$

and finally as:

$$\left(Z^2 + Y^2\right)\sin^2\theta - 2EZ\sin\theta + \left(E^2 - DY^2\right) = 0 \tag{26}$$

Equation 26 is a quadratic equation in $\sin\theta$ and can be easily solved:

$$\begin{aligned} \sin\theta &= \frac{EZ \pm \left[E^2Z^2 - \left(Z^2 + Y^2\right)\left(E^2 - DY^2\right)\right]^{\frac{1}{2}}}{Z^2 + Y^2} && (27) \\ &= \frac{EZ \pm |Y|\left[D\left(Z^2 + Y^2\right) - E^2\right]^{\frac{1}{2}}}{Z^2 + Y^2} && (28) \end{aligned}$$

The solution to equation 28 gives the elevation angle θ in terms of known quantities.
Figure 2 shows an example of solving equation 28 for two different beams (12 and 21) with the MSI Oregon geometry. The dark blue curves are the total delay ϕ_{total} from equation 16 for the two different beams. Note that for simplicity t_{diff} is assumed to be zero so that ϕ_{path} is also zero. A non-zero value would simply shift the blue curves vertically by the appropriate constant phase delay.

An additional curve is shown in a lighter shade of blue. This is the total delay ignoring the coning angle equation and is shown for context only. Note that for beams that are close to the boresight (e.g., beam

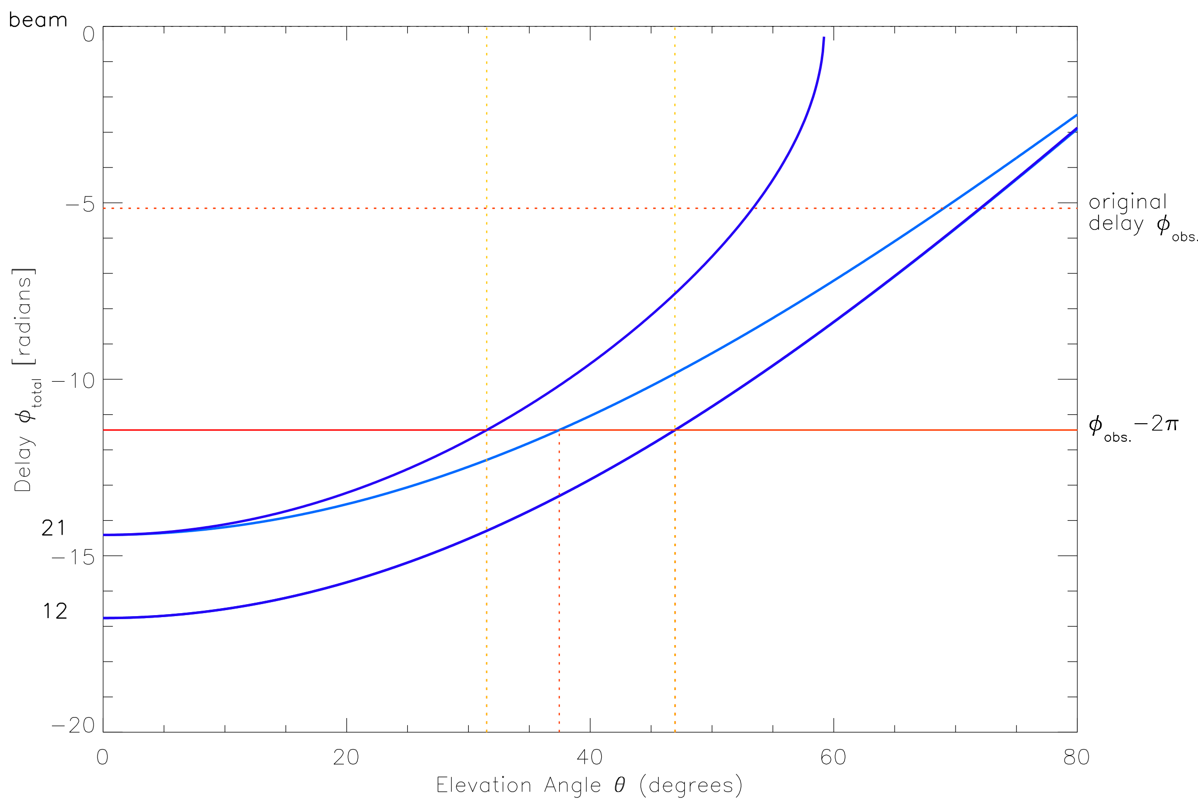

Figure 2: Dark blue curves show the phase delay introduced by the separation of the main and interferometer arrays for two different beam directions. The intersection of these curves with the observed delay, indicated by the hozrizonal solid red line, gives the elevation angle from equation 28. Horizontal vertical dotted lines show the good agreement with the existing methods currently used to determine elevation angles.

12 in a 24 beam array) the two curves are nearly identical.

The horizontal solid red line is the observed residual phase delay adjusted with the appropriate number of 2π factors. The intersection of the red and blue curves gives the elevation angle θ from equation 28. Vertical dotted lines extending downward from the horizontal red line show this solution for the two beams (and the uncorrected lighter blue curves). The solution given by our current algorithm (`elevation.c`) is indicated by vertical dotted lines that extend upward from the horizontal red line and show very good agreement with our computed values.

Figures 3 and 4 show examples of the agreement between the two techniques for two different beams; 12 and 21. The errors between the two techniques are around 10^{-5} degrees. Note the shift in elevation angle around -2 radians.

While the agreement is good for the simplest situation (and it should be noted that most SuperDARN radars fall into this category) there are difference between the two techniques for more complicated array separations, i.e., those with offsets in more than just the $\hat{y}$ direction.

Figure 5 shows the elevation angles calculated using equation 28 in solid blue and the current algorithm in dotted red for a range of offsets in the $\hat{x}$ direction, i.e., along the direction of the arrays. Note that the Goose Bay radar and a few others falls into this category, but the offset in the $\hat{x}$ direction is small compared to the offset in the $\hat{y}$ direction.

Note that the result using our current technique cannot be correct. While there is some dependence of the elevation angle on the amount of offset in the $\hat{x}$ direction, it is the same for symmetric beams on either side of the boresight. This result is clearly not correct as the distance between planes for beams oriented toward the direction of the offset must be smaller than in the direction of the boresight. The opposite must be true for beams oriented away from the direction of the offest. The blue curves show this behavior.

Figure 6 shows that the elevation angles computed using the two techniques is more complicated for offsets in the $\hat{z}$ direction. Here there is good agreement for some offsets, but there are significant jumps in the elevation angles computed using the current algorithm (dotted red curves) and negative angles that are reported for positive offsets. Note that in practice offsets in the $\hat{z}$ direction are limited to ±5–10 m, but there are still measureable difference in this limited range.

Conclusions

The existing technique works well and gives the expected results for the simple situation in which the interferometer array is offset only in the boresight direction ($\hat{y}$). For more complicated offsets the results obtained from our current technique are incorrect. A better, more intuitive and more general method for determining the elevation angle has been descibed. Note that more testing is necessary to refine the algorithm so that it can deal with the following situations:

- non-zero t_{diff} – testing is necessary to make sure that the appropriate number of 2π factors are always used.
- ϕ_{diff} – this is parameter that was introduced to account for errors in connecting the main and interferometer cables. It has not been incorporated in the analysis, but should be.
- offsets – more testing for offsets in the $\hat{x}$ and $\hat{z}$ directions is needed to be sure that the jumps in elevation angle are real and that the separation between the planes in 3D is consistent with the computed elevation angle.
- coning angle – a more complete description of this phenomenon is needed for this document.

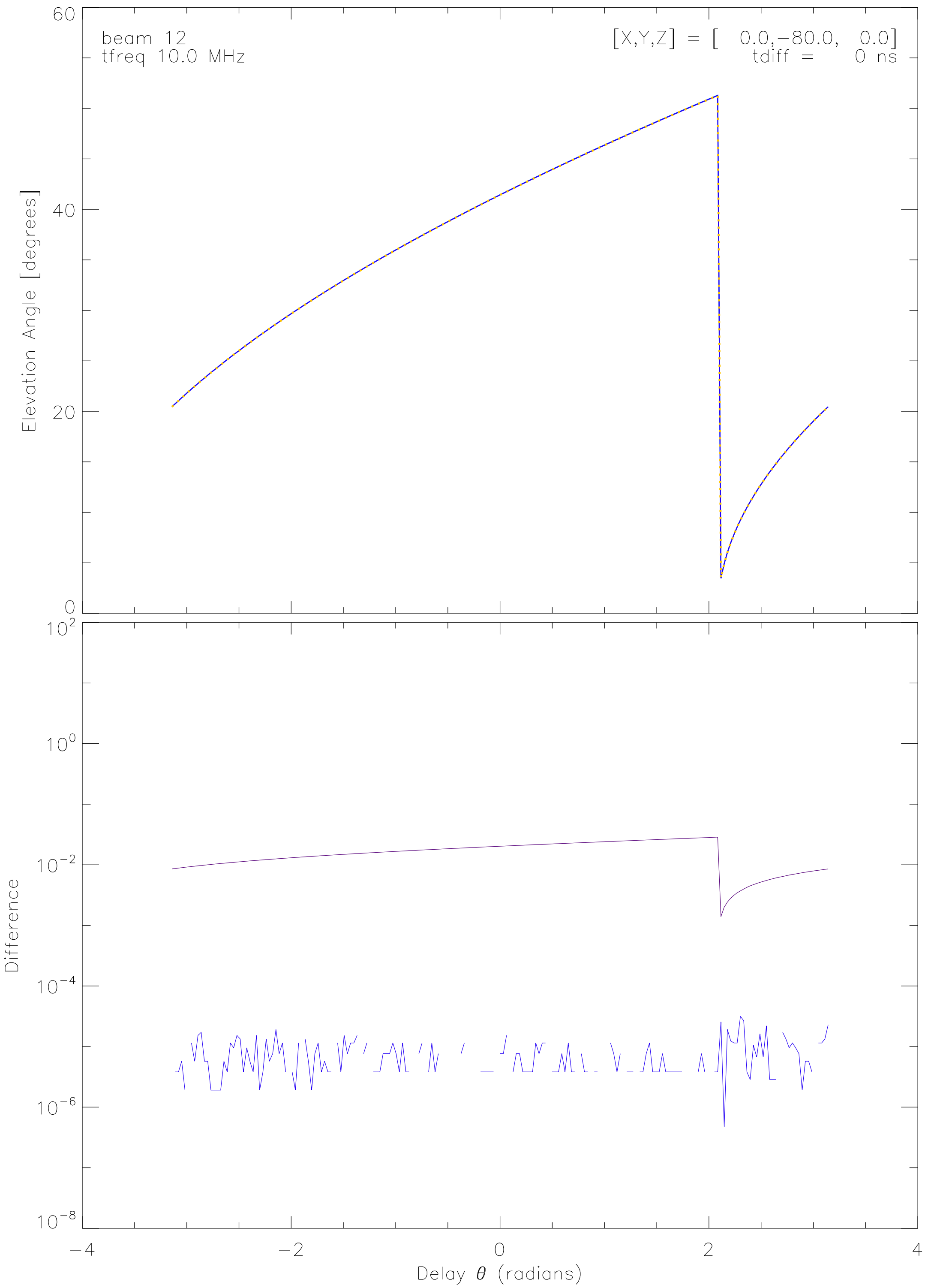

Figure 3: Elevation angles for observed residual phase delays ranging from -π to π. The solid dark blue curve shows the result of equation 28. The purple curve is for reference only and shows the result if the coning angle equation is not used. A dotted orange curve shows the result of our current algorithm and lies on top of the dark blue curve. The lower figure shows the difference between the new method and the current technique.

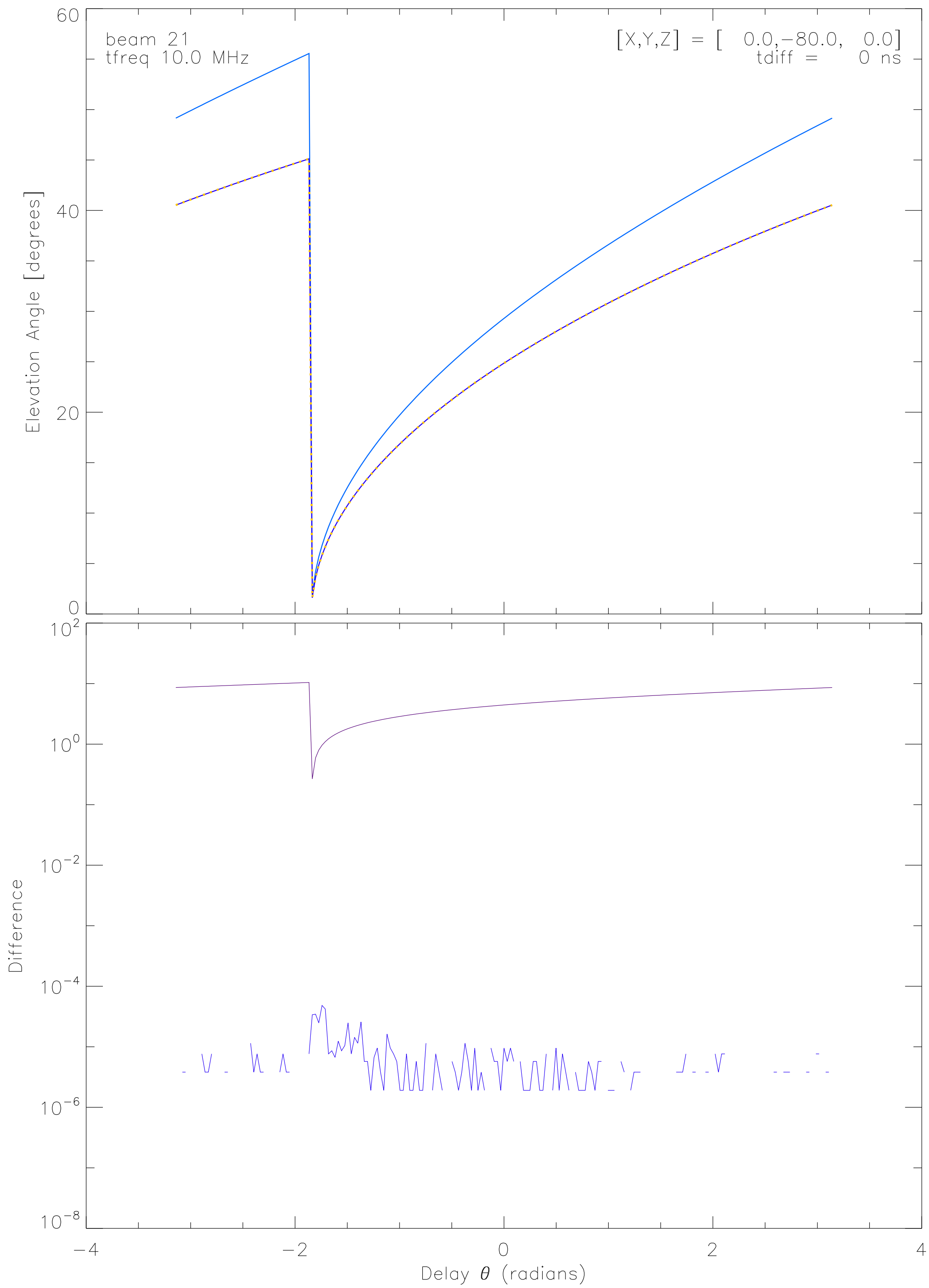

Figure 4: The same format as Figure 3 but for beam 21.

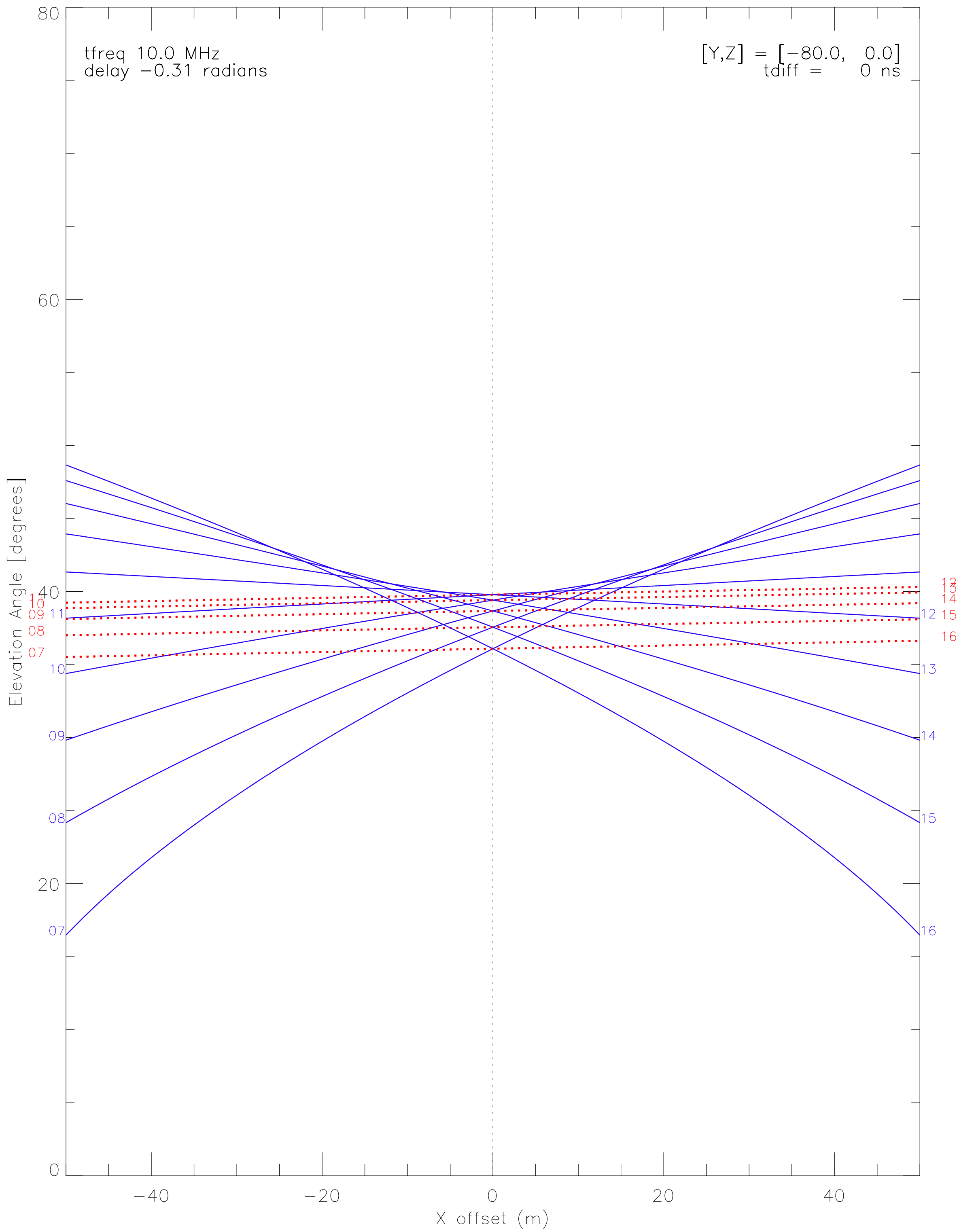

Figure 5: Elevation angles for a range of offsets in the $\hat{x}$ direction. Blue curves show the result for various beams using equation 28 while the dotted red lines show the result using the existing algorithm. The two solutions agree for zero offset in the $\hat{x}$ direction, but differ increasingly for larger offsets.

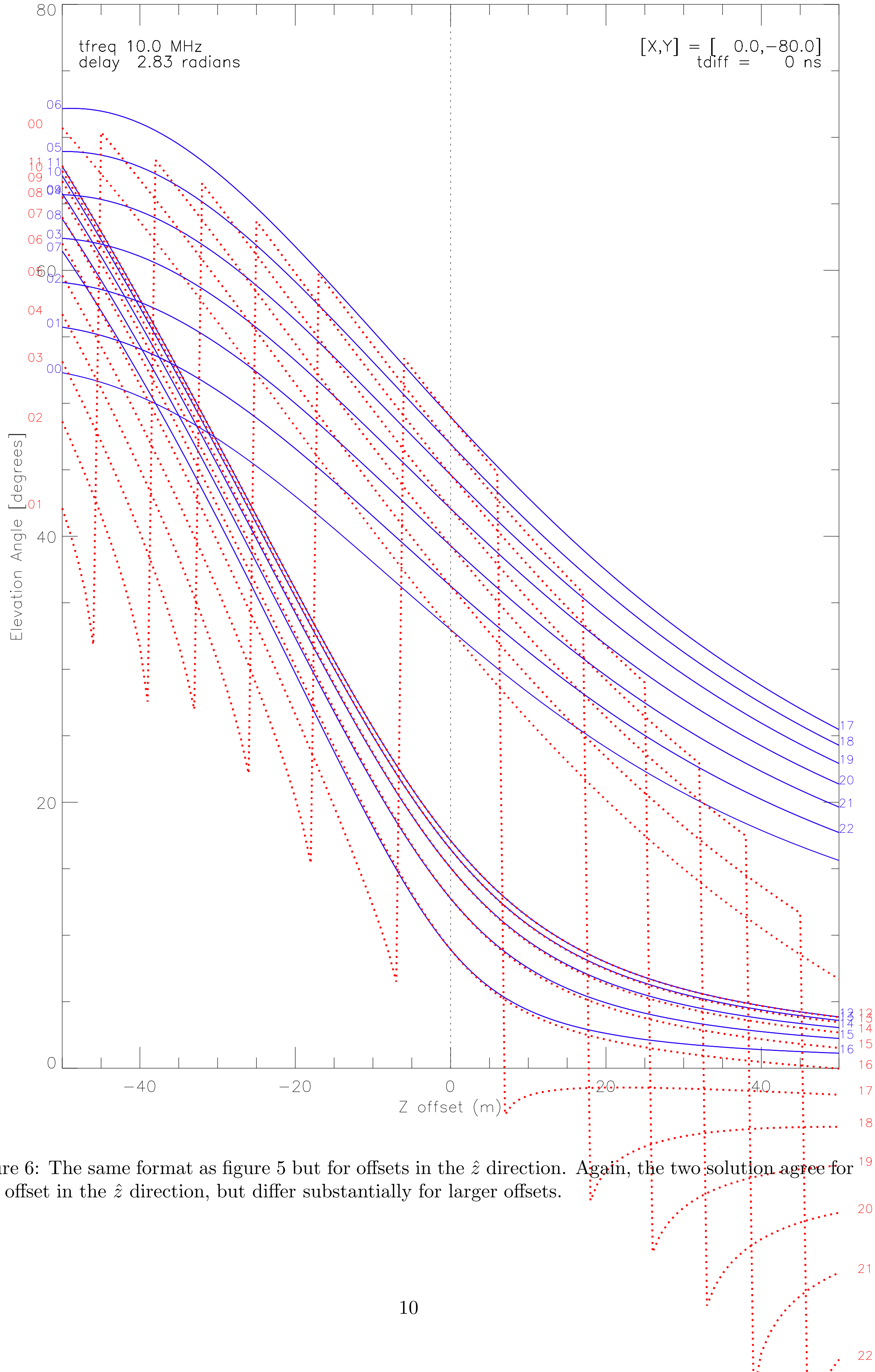

Figure 6: The same format as figure 5 but for offsets in the $\hat{z}$ direction. Again, the two solution agree for zero offset in the $\hat{z}$ direction, but differ substantially for larger offsets.