Algebra 2 Assignment 3

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1. (a) Let $A, B \in T$ with $A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 \\ d & c \end{pmatrix}$. Using the subring criterion

   (i) $A + B = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} + \begin{pmatrix} c & 0 \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & 0 \\ b+d & a+c \end{pmatrix} \in T$,

   (ii) $-A = \begin{pmatrix} -a & 0 \\ -b & -a \end{pmatrix} \in T$,

   (iii) $AB = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \begin{pmatrix} c & 0 \\ d & c \end{pmatrix} = \begin{pmatrix} ac & 0 \\ bc+ad & ac \end{pmatrix} \in T$.

   Hence T is a subring of $\mathcal{M}_2(\mathbb{R})$.

   (b) For T to be an ideal of $\mathcal{M}_2(\mathbb{R})$, for all $r \in \mathcal{M}_2(\mathbb{R})$ and $s \in T$ it must hold that $rs \in T$ and $sr \in T$. Let's test this

$$rs = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} = \begin{pmatrix} ax+yb & bx \\ cx+yd & dx \end{pmatrix} \notin T.$$

   As such, T is not an ideal of $\mathcal{M}_2(\mathbb{R})$.

   (c) As shown in 1.(a)(iii) the product of any two elements of T has the form $AB = \left(\begin{smallmatrix} a & 0 \\ b & a \end{smallmatrix}\right)\left(\begin{smallmatrix} c & 0 \\ d & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} ac & 0 \\ bc+ad & ac \end{smallmatrix}\right)$. As such the zero divisors of T are those elements for which $ac = 0$ and $bc + ad = 0$ with at least one of a, b nonzero and at least one of c, d nonzero.

   Let $a = c = 0$ and b, d be nonzero, now we have $AB = 0$. Neither of A or B are zero and we have that A is a zero divisor. Therefore the zero divisors of T are $\{\left(\begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix}\right) \in T : x > 0\}$.

(d) Let $A, B \in T$;

(i) T is commutative, we have

$$\begin{aligned}
AB &= \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \begin{pmatrix} c & 0 \\ d & c \end{pmatrix} \\
&= \begin{pmatrix} ac & 0 \\ bc + ad & ac \end{pmatrix} \\
&= \begin{pmatrix} ca & 0 \\ da + cb & ca \end{pmatrix} \\
&= \begin{pmatrix} c & 0 \\ d & c \end{pmatrix} \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \\
&= BA,
\end{aligned}$$

(ii) T is is a ring with identity, we have I_2 the 2×2 identity matrix in T and therefore $I_2 A = A = A I_2$ for all A in T,

(iii) T is not a division ring. To see this, let B be A's inverse and write $AB = \left(\begin{smallmatrix} ac & 0 \\ bc+ad & ac \end{smallmatrix}\right) = I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, giving $ac = 1$ and $bc + ad = 0$. These equations hold when $c = \frac{1}{a}$ and $d = -\frac{b}{a^2}$. Now suppose $a = 0$. In this case B is undefined and A does not have an inverse. QED
Lemma, zero divisors of T have no inverse.

2. (a)

$$\begin{aligned}
p(x) &= f(x) - xg(x) \\
&= -x^3 + x^2 - x + 1 \\
q(x) &= g(x) + p(x) \\
&= 3x^2 + 3 \\
r(x) &= p(x) + \tfrac{1}{3} x q(x) \\
&= x^2 + 1 \\
s(x) &= q(x) - 3r(x) \\
&= 0
\end{aligned}$$

Hence $\gcd(f(x), g(x)) = r(x) = x^2 + 1$.

(b) By long division we know that, $\frac{f(x)}{x^2+1} = x^2 + x + 1$ and $f(x) = (x^2 + x + 1)(x^2 + 1)$. These two quadratic factors are irreducible because they have complex roots and therefore factors of the form $(x - z)$ where $z \in \mathbb{C}$ so $(x - z) \notin \mathbb{R}[x]$.

3. (a) *Proof.* If a is a zero divisor of R, there must be some element $z \in R$ such that $az = 0$. Since a is nilpotent, we know there exists some positive integer n such that $a^n = 0$ Let $z = a^{n-1}$, it is easy to see that $az = aa^{n-1} = a^n = 0$, therefore a is a nilpotent element of R if and only if a is a zero divisor of R. $\square$

   (b) Clearly, 0 is a nilpotent element of $\mathbb{Z}_{12}$ because $0^n = 0$ for all n. 6 is also a nilpotent element of $\mathbb{Z}_{12}$, because $6^2 = 36 = 0$. By inspection, there are no other nilpotent elements of $\mathbb{Z}_{12}$.

   (c) *Proof.* $0^n = 0$, so certainly 0 is a nilpotent element of $\mathbb{Z}_{967}$. Let $a \neq 0$ be a nilpotent element of $\mathbb{Z}_{967}$, by the proof in 3.(a), a is a zero divisor. Because 967 is prime, by Lemma 3.3.4, $\mathbb{Z}_{967}$ is a field and therefore every element is a unit. Now by Lemma 3.3.12 we have a contradiction. Hence 0 is the only nilpotent element $\mathbb{Z}_{967}$. $\square$