CALCULUS 3: TRANSFORMS & MODELS
Problem sheet 4: Models

Due 24th April 2018

Answer all questions. A total mark out of 20 will be awarded, with individual marks for each question being given in square brackets. This work is worth 5% of the marks for this module. Late submissions will be awarded at most 8/20; work that is more than 14 days late will receive 0.

Coursework should be neatly written or typed on A4 paper. You should submit your work by placing it, together with a signed cover sheet, inside the Assignment Box opposite the lifts on the 7th floor. Full coursework regulations are given in the handbook.

1. Consider a battle between two armies, X and Y, of total size x and y respectively. Army X is composed of x_1 chariots and x_2 spearmen; army Y is composed of y archers. The battle evolves according to

$$\begin{aligned}\dot{x}_1 &= -\alpha y \frac{x_1}{x}\\ \dot{x}_2 &= -\alpha y \frac{x_2}{x}\\ \dot{y} &= -\beta_1 x_1 - \beta_2 x_2,\end{aligned}$$

where α, β_1 and β_2 are time-independent positive parameters.

   (a) Define $\bar{\beta} = \frac{\beta_1 x_1 + \beta_2 x_2}{x}$. Given that $x = x_1 + x_2$, show that $\bar{\beta}$ is independent of time and hence verify that $\alpha y^2 - \bar{\beta} x^2$ is constant throughout the battle. What is the interpretation of $\bar{\beta}$? [4]

   (b) At the beginning of the battle X has 100 chariots and 500 spearmen, and Y has 1000 archers. Suppose that $\alpha = 2$, $\beta_1 = 4$ and $\beta_2 = 1$. What is the outcome of the battle? How many soldiers does the victor have left at the end? [2]

2. In the SI model for an epidemic we suppose that, once infected, a member of the population always remains infected (i.e. there is no removed population). The dynamical system is

$$\begin{aligned}\dot{S} &= -\beta S I\\ \dot{I} &= \beta S I,\end{aligned}$$

where S and I give the number of susceptible and infected people respectively, and β is a positive constant.

   (a) Establish that the total population $N = S + I$ is a constant. [1]

   (b) Sketch a portrait of the system in the phase plane. [2]

   (c) Given that at $t = 0$ half the population is infected, solve the dynamical system to derive the following forms for the trajectories:

$$\begin{aligned}S &= \frac{N}{1+e^{\tau}}\\ I &= \frac{N}{1+e^{-\tau}},\end{aligned}$$

   where $\tau = \beta N t$ is a dimensionless time-like variable. [5]

Turn over for question 3

3. The predator-prey relationship for the number of rabbits and foxes on an island is given by

$$\dot{x} = 2x - \tfrac{1}{2}xy$$
$$\dot{y} = xy - \tfrac{1}{2}y,$$

where x and y give the number, in thousands, of rabbits and foxes respectively.

(a) Ignoring the trivial solution that $x = y = 0$, what is the steady state number of rabbits and foxes on the island? [2]

(b) Due to hunting, both species are on the verge of extinction: there are only 10 rabbits and 10 foxes left on the island. Find the Jacobian at $(0, 0)$ and hence give expressions for how the population of both species evolves while close to extinction. Why should this behaviour be expected given the initial assumptions of the model? [4]