Solutions Chapter 7

Solutions to Exercises 7.1.

1. This can be shown using the sum formulas for sinh and cosh.

$$\begin{aligned}\tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)}\\ &= \frac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y+\sinh x\sinh y}\\ &= \frac{\frac{1}{\cosh x\cosh y}(\sinh x\cosh y+\cosh x\sinh y)}{\frac{1}{\cosh x\cosh y}(\cosh x\cosh y+\sinh x\sinh y)}\\ &= \frac{\tanh x+\tanh y}{1+\tanh x\tanh y}\end{aligned}$$

2. This is an easy computation using the definition of $\cosh x$ and $\sinh x$.

$$\begin{aligned}(\cosh x+\sinh x)^n &= \left(\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}\right)^n\\ &= (e^x)^n\\ &= e^{nx}\\ &= \frac{e^{nx}+e^{-nx}}{2}+\frac{e^{nx}-e^{-nx}}{2}\\ &= \cosh nx+\sinh nx\end{aligned}$$

3. Note that $\cosh x = 5/3$ has two solutions for x, so we will find two values for $\sinh x$ and $\tanh x$. We have $\cosh^2 x - \sinh^2 x = 1$, hence

$$\sinh x = \pm\sqrt{\cosh^2 x-1} = \pm\sqrt{\left(\frac{5}{3}\right)^2-1} = \pm\frac{4}{3},$$

and

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\pm 4/3}{5/3} = \pm\frac{4}{5}.$$

4. (a) If $y = A\sinh mx + B\cosh mx$, then $y'' = Am^2\sinh mx + Bm^2\cosh mx$, hence $y'' = m^2 y$. This shows that the given function is a solution for any constants A and B.

   If we solve the differential equation with the methods from Chapter 4, we get the auxiliary equation $r^2 - m^2 = 0$, so $r = \pm m$. Hence the general solution is

$$y = ae^{mx} + be^{-mx}$$

for constants a and b. However $A\sinh mx + B\cosh mx$ and $ae^{mx} + be^{-mx}$ are just different ways of writing the same solution because

$$\begin{aligned} A\sinh mx + B\cosh mx &= A\frac{e^{mx} - e^{-mx}}{2} + B\frac{e^{mx} + e^{-mx}}{2} \\ &= \left(\frac{A}{2} + \frac{B}{2}\right)e^{mx} + \left(-\frac{A}{2} + \frac{B}{2}\right)e^{-mx} \\ &= ae^{mx} + be^{-mx} \end{aligned}$$

with $a = \frac{A}{2} + \frac{B}{2}$ and $b = -\frac{A}{2} + \frac{B}{2}$.

(b) From part (a), the general solution to $y'' = 9y$ is $y = A\sinh 3x + B\cosh 3x$. Then $y' = 3A\cosh 3x + 3B\sinh 3x$. Hence the initial conditions $y(0) = -4$ and $y'(0) = 6$ imply

$$\begin{aligned} A\sinh 0 + B\cosh 0 &= -4, \\ 3A\cosh 0 + 3B\sinh 0 &= 6. \end{aligned}$$

Now $\sinh 0 = 0$ and $\cosh 0 = 1$, hence we find $B = -4$ and $A = 2$. Thus the solution is $y = 2\sinh 3x - 4\cosh 3x$.

Solutions to Exercises 7.2.

1. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\arctan(\sin x) &= \frac{1}{1 - \sin^2 x} \cdot \cos x \\ &= \frac{1}{\cos^2 x} \cdot \cos x \\ &= \frac{1}{\cos x}. \end{aligned}$$

2. (a)

$$\begin{aligned} \int \frac{\sqrt{x^2+1} + \sqrt{x^2-1}}{\sqrt{x^4-1}}\,\mathrm{d}x &= \int \left(\frac{1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2+1}}\right)\mathrm{d}x \\ &= \operatorname{arccosh} x + \operatorname{arcsinh} x + c, \end{aligned}$$

c an arbitrary constant.

(b)

$$\begin{aligned} \int \frac{\sqrt{x^2-19} + x^2 - 19}{\left(x^2-19\right)^{3/2}}\,\mathrm{d}x &= \int \left(\frac{1}{x^2-19} + \frac{1}{\sqrt{x^2-19}}\right)\mathrm{d}x \\ &= -\frac{1}{\sqrt{19}}\operatorname{arctanh}\left(\frac{x}{\sqrt{19}}\right) + \operatorname{arccosh}\left(\frac{x}{\sqrt{19}}\right) + c, \end{aligned}$$

c an arbitrary constant.

3. We write $z = x + iy$ with $x, y \in \mathbb{R}$.

   (a) We have $\sinh z = i$ if and only if

   $$\begin{aligned} \sinh x \cos y &= 0, \\ \cosh x \sin y &= 1. \end{aligned}$$

   The first of these equations implies $\sinh x = 0$ or $\cos y = 0$.
   If $\sinh x = 0$ then we obtain $x = 0$ and $y = \frac{\pi}{2} + 2\pi k$ where k is an integer.
   If $\cos y = 0$ then we again obtain $x = 0$ and $y = \frac{\pi}{2} + 2\pi k$ where k is an integer.
   Thus in both cases we obtain the same solutions

   $$z = x + iy = 0 + i\left(\tfrac{\pi}{2} + 2\pi k\right),$$

   where k is an integer.

   (b) We have $\sin z = 5$ if and only if

   $$\begin{aligned} \sin x \cosh y &= 5, \\ \cos x \sinh y &= 0. \end{aligned}$$

   The second of these equations implies $\cos x = 0$ or $\sinh y = 0$.
   If $\cos x = 0$ then we obtain $x = \frac{\pi}{2} + 2\pi k$ where k is an integer and $y = \pm \ln\left(5 + 2\sqrt{6}\right)$.
   If $\sinh y = 0$ we don't obtain any solutions.
   Hence the solutions are

   $$z = x + iy = \tfrac{\pi}{2} + 2\pi k \pm i \ln\left(5 + 2\sqrt{6}\right),$$

   where k is an integer.

   (c) We have $\sin z = 1$ if and only if

   $$\begin{aligned} \sin x \cosh y &= 1, \\ \cos x \sinh y &= 0. \end{aligned}$$

   The second of these equations implies $\cos x = 0$ or $\sinh y = 0$.
   If $\cos x = 0$ then we obtain $x = \frac{\pi}{2} + 2\pi k$ where k is an integer and $y = 0$.
   If $\sinh y = 0$ then we again obtain $x = \frac{\pi}{2} + 2\pi k$ where k is an integer and $y = 0$.
   Thus in both cases we obtain the same solutions

   $$z = x + iy = \tfrac{\pi}{2} + 2\pi k + 0i,$$

   where k is an integer. Note that all solutions are real!

Solutions to Exercises 7.3.

1. (a) $\displaystyle\int_0^\infty x^3 e^{-x}\,dx = \Gamma(4) = 3! = 6$

   (b) Use the substitution $t = 2x$. Then

$$\int_0^\infty x^6 e^{-2x}\,dx = \int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t}\cdot\frac{1}{2}\,dt = \frac{1}{2^7}\Gamma(7) = \frac{45}{8}.$$

2. Correction: As it stands, the integral does not exist (it is ∞). We need the exponent of e to be negative, i.e. we evaluate the integral $\int_0^\infty x^m e^{-ax^n}\,dx$ where m, n and a are positive integers.

   Using the substitution $t = ax^n$ (so $x = (t/a)^{1/n}$) the integral becomes

$$\begin{aligned}\int_0^\infty x^m e^{-ax^n}\,dx &= \int_0^\infty x^m e^{-ax^n}\cdot\frac{1}{anx^{n-1}}\,dt\\ &= \frac{1}{an}\int_0^\infty x^{m-n+1}e^{-t}\,dt\\ &= \frac{1}{an}\int_0^\infty (t/a)^{\frac{m+1}{n}-1}e^{-t}\,dt\\ &= \frac{1}{na^{\frac{m+1}{n}}}\int_0^\infty t^{\frac{m+1}{n}-1}e^{-t}\,dt\\ &= \frac{1}{na^{\frac{m+1}{n}}}\Gamma\left(\frac{m+1}{n}\right).\end{aligned}$$

3. Use the substitution $t = \ln\left(\frac{a}{x}\right)$. Then $\frac{\mathrm{d}x}{\mathrm{d}t} = -x$. Hence

$$\begin{aligned}\int_0^a \frac{1}{\sqrt{\ln\left(\frac{a}{x}\right)}}\,dx &= \int_\infty^0 \frac{1}{\sqrt{t}}\cdot(-x)\,dt\\ &= \int_0^\infty \frac{1}{\sqrt{t}}\cdot x\,dt\\ &= \int_0^\infty t^{-1/2}\cdot ae^{-t}\,dt\\ &= a\Gamma\left(\frac{1}{2}\right)\\ &= a\sqrt{\pi}.\end{aligned}$$

4. Use the substitution $t = y^3$.

$$\begin{aligned}\int_0^\infty \sqrt{y}e^{-y^3}\,dy &= \frac{1}{3}\int_0^\infty t^{-1/2}e^{-t}\,dt\\ &= \frac{1}{3}\Gamma\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{\pi}}{3}\end{aligned}$$

5. With the substitution $x = e^{-y}$ the integral $\int_0^1 x^m (\ln x)^n \, dx$ becomes

$$\int_\infty^0 e^{-my}(-y)^n \cdot (-e^{-y}) \, dy = (-1)^n \int_0^\infty y^n e^{-(m+1)y} \, dy.$$

Now setting $t = (m+1)y$ this becomes

$$\begin{aligned}(-1)^n \int_0^\infty \left(\frac{t}{m+1}\right)^n e^{-t} \cdot \frac{1}{m+1} \, dt &= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1) \\ &= \frac{(-1)^n n!}{(m+1)^{n+1}}.\end{aligned}$$

Solutions to Exercises 7.4.

1. (a)

$$\int_0^1 x^4 (1-x)^3 \, dx = B(5,4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{4! \cdot 3!}{8!} = \frac{1}{280}$$

(b) We have

$$\int_0^a y^4 \sqrt{a^2 - y^2} \, dy = a \int_0^a y^4 \sqrt{1 - \frac{y^2}{a^2}} \, dy.$$

With the substitution $t = \frac{y^2}{a^2}$ this becomes

$$\begin{aligned} a \int_0^1 y^4 \sqrt{1-t} \cdot \frac{a^2}{2y} \, dt &= \frac{a^3}{2} \int_0^1 \left(a^2 t\right)^{3/2} \sqrt{1-t} \, dt \\ &= \frac{a^6}{2} \int_0^1 t^{3/2} (1-t)^{1/2} \, dt \\ &= \frac{a^6}{2} B\left(\frac{5}{2}, \frac{3}{2}\right) \\ &= \frac{\pi a^6}{32}. \end{aligned}$$

2. (a) We use Proposition 7.17.

$$\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{1}{2} \cdot 2 \int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{1}{2} \cdot B\left(\frac{5}{2}, 3\right) = \frac{8}{315}$$

(b) First we split the integral into two integrals,

$$\int_0^\pi \cos^4 x \, dx = \int_0^{\pi/2} \cos^4 x \, dx + \int_{\pi/2}^\pi \cos^4 x \, dx.$$

Then we use Proposition 7.17 to evaluate each integral separately, using the substitution $u = x - \pi/2$ for the second.

$$\begin{aligned}
\int_0^{\pi/2} \cos^4 x\, dx &= \frac{1}{2} \cdot B\left(\frac{1}{2}, \frac{5}{2}\right), \\
\int_{\pi/2}^{\pi} \cos^4 x\, dx &= \int_0^{\pi/2} (-\sin u)^4\, du \\
&= \frac{1}{2} \cdot B\left(\frac{5}{2}, \frac{1}{2}\right).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\int_0^{\pi} \cos^4 x\, dx &= \frac{1}{2} \cdot B\left(\frac{1}{2}, \frac{5}{2}\right) + \frac{1}{2} \cdot B\left(\frac{5}{2}, \frac{1}{2}\right) \\
&= B\left(\frac{1}{2}, \frac{5}{2}\right) \\
&= \frac{3\pi}{8}.
\end{aligned}$$

(c) We split the integral into four integrals with limits $[0, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \pi]$, $[\pi, \frac{3\pi}{2}]$ and $[\frac{3\pi}{2}, 2\pi]$, and evaluate each of these separately.

$$\begin{aligned}
\int_0^{\pi/2} \sin^8 x\, dx &= \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right), \\
\int_{\pi/2}^{\pi} \sin^8 x\, dx &= \int_0^{\pi/2} \cos^8 u\, du = \frac{1}{2} B\left(\frac{1}{2}, \frac{9}{2}\right), \\
\int_{\pi}^{3\pi/2} \sin^8 x\, dx &= \int_0^{\pi/2} (-\sin u)^8\, du = \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right), \\
\int_{3\pi/2}^{2\pi} \sin^8 x\, dx &= \int_0^{\pi/2} (-\cos u)^8\, du = \frac{1}{2} B\left(\frac{1}{2}, \frac{9}{2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^{2\pi} \sin^8 x\, dx &= \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right) + \frac{1}{2} B\left(\frac{1}{2}, \frac{9}{2}\right) + \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right) + \frac{1}{2} B\left(\frac{1}{2}, \frac{9}{2}\right) \\
&= 2B\left(\frac{9}{2}, \frac{1}{2}\right) \\
&= \frac{35\pi}{64}.
\end{aligned}$$

3. We have

$$\begin{aligned}
\Gamma(x)\Gamma(1-x) &= B(x, 1-x) && \text{(by Theorem 7.19)} \\
&= \int_0^{\infty} \frac{u^{x-1}}{1+u}\, du && \text{(by Proposition 7.18)} \\
&= \frac{\pi}{\sin(x\pi)} && \text{(given in question).}
\end{aligned}$$