Assignment 3 - Solutions

1(a) $$x^2 + y^2 \frac{dy}{dx} = 0$$

<u>separable</u>

$$y^2 \frac{dy}{dx} = -x^2$$

$$\int y^2\, dy = -\int x^2\, dx$$

$$y^3/3 = -x^3/3 + C$$

$$\Rightarrow \quad y^3 = -x^3 + D$$

$$\Rightarrow \quad y = \sqrt{-x^3 + D} \qquad C, D \text{ constants.}$$

(b) $$\underbrace{x^2}_{M(x,y)} + \underbrace{y^2}_{N(x,y)} \frac{dy}{dx}$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\left.\begin{aligned} M(x,y) &= x^2 \Rightarrow M_y = 0 \\ N(x,y) &= y^2 \Rightarrow N_x = 0 \end{aligned}\right\} \text{Same so D.E. is } \underline{\text{exact}}$$

$\therefore$ solution is of form $f(x,y) = \text{const.}$

$$f_x = M(x,y) = x^2 \Rightarrow f = \int x^2\, dx$$

$$\cancel{f = \frac{x^3}{3} + g(x), g\ \text{const}}$$

$= \frac{x^3}{3} + g(y)$, $g(y)$ some function of y.

$f_y = g'(y) = N(x,y) = y^2 \Rightarrow g(y) = \int y^2 dy$
$= y^3/_3 + const$

$\therefore f(x,y) = \frac{x^3}{3} + y^3/_3 = const$

$\Rightarrow x^3 + y^3 = const$

$\therefore$ same as part (a)

(c) $x^2 + y^2 \frac{dy}{dx} = 0$

again: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

$M(\lambda x, \lambda y) = (\lambda x)^2 = \lambda^2 x^2 = \lambda^2 M(x,y)$

similarly $N(\lambda x, \lambda y) = \lambda^2$

$\therefore$ M and N both homogeneous of degree 2.

So D.E. is <u>homogeneous</u>

make subst. $y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$:

$$\cancel{x^2} + \cancel{(xv)^2}\left(v + x\frac{dv}{dx}\right) = 0.$$

$$1 + v^3 + v^2 x\frac{dv}{dx} = 0.$$

$$v^2 x\frac{dv}{dx} = -(1 + v^3)$$

$$\int \frac{v^2}{1+v^3}\,dv = -\int \frac{1}{x}\,dx$$

$$\frac{1}{3}\int \frac{3v^2\,dv}{1+v^3} = -\ln x + c$$

$$\frac{1}{3}\ln(1+v^3) = -\ln x + c$$

$$\ln(1+v^3) = -3\ln x + D, \quad D = 3c$$

$$\ln(1+v^3) = -\ln x^3 + D$$

$$\ln\left[(1+v^3)x^3\right] = D$$

$$(1+v^3)x^3 = E, \quad E = e^D$$

$v = \frac{y}{x}$: $\quad x^3 + \frac{y^3}{x^3}x^3 = E \Rightarrow x^3 + y^3 = E$, const.

$\therefore$ Same as parts (a) & (b)

2(a) $$y^2 + x^2 \frac{dy}{dx} = 0$$

separable: $$x^2 \frac{dy}{dx} = -y^2$$

$$\int \frac{1}{y^2}\, dy = -\int \frac{1}{x^2}\, dx$$

$$-\frac{1}{y} = +\frac{1}{x} + C, \quad C \text{ const.}$$

$$\frac{1}{y} = -\left(\frac{1}{x} + C\right)$$

$$y = -\frac{1}{\frac{1}{x} + C} \cdot \frac{x}{x}$$

$$= \frac{-x}{1 + Cx}$$

(b) $$\underbrace{y^2}_{M(x,y)} + \underbrace{x^2}_{N(x,y)} \frac{dy}{dx} = 0$$

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0$$

$M(\lambda x, \lambda y) = (\lambda y)^2 = \lambda^2 y^2 = \lambda^2 M(x,y)$

similarly $N(\lambda x, \lambda y) = \lambda^2 N(x,y)$

$\Bigg\}$ homogeneous of degree 2.

$\therefore$ D.E. is <u>homogeneous</u>.

subst: $y = xv$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$(xv)^2 + \cancel{x^2}\left(v + x\frac{dv}{dx}\right) = 0$$

$$v^2 + v + x\frac{dv}{dx} = 0$$

$$x\frac{dv}{dx} = -(v^2+v)$$

$$\int \frac{dv}{v^2+v} = -\int \frac{1}{x}\,dx$$

(partial fractions)

$$\int \frac{1}{v} - \frac{1}{v+1}\,dv = -\ln x + C$$

(const)

$$\ln v - \ln(v+1) = -\ln x + C$$

$$\ln\left(\frac{vx}{v+1}\right) = C$$

$$\frac{vx}{v+1} = D \quad , \quad D = e^C$$

$$vx = D(v+1)$$

solving for v:

$$v(x - D) = D$$

$$\Rightarrow \quad v = \frac{D}{x - D}$$

subst $v = \frac{y}{x}$, $\quad \frac{y}{x} = \frac{D}{x - D}$

$$\Rightarrow \quad y = \frac{Dx}{x - D} \cdot \frac{-\frac{1}{D}}{-\frac{1}{D}}$$

$$= \frac{-x}{1 - \frac{x}{D}} = \frac{-x}{1 + Cx}$$

where $C = -\frac{1}{D}$.

3(a) Recall that an integrating factor μ is a function which, when multiplied by the D.E. makes it <u>exact</u>.

$$\mu(x,y)\left[f_1(x)\cdot g_1(y) + f_2(x)\cdot g_2(y)\frac{dy}{dx}\right] = 0$$

$$= \frac{f_1(x)\cdot g_1(y)}{g_1(y)\cdot f_2(x)} + \frac{f_2(x)\cdot g_2(y)}{g_1(y)\cdot f_2(x)}\frac{dy}{dx} = 0$$

$$= \frac{f_1(x)}{f_2(x)} + \frac{g_2(y)}{g_1(y)}\frac{dy}{dx} = 0 \qquad (*)$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Then $M_y = \frac{\partial}{\partial y}\left(\frac{f_1(x)}{f_2(x)}\right) = 0$ since f_1, f_2 are functions of x only

$N_x = \frac{\partial}{\partial x}\left(\frac{g_2(y)}{g_1(y)}\right) = 0$ since g_1, g_2 are functions of y only

Then $M_y = N_x$ and so $(*)$ is now <u>exact</u>

(b) $$\underbrace{y^2}_{f_1(x)\,g_1(y)} + \underbrace{x^2}_{f_2(x),\,g_2(y)} \frac{dy}{dx} = 0$$

So in this case $f_1(x) = 1$, ~~$g(x)$~~ $g_1(y) = y^2$
$f_2(x) = x^2$, $g_2(y) = 1$

and $$\mu(x,y) = \frac{1}{g_1(y) f_2(x)} = \frac{1}{y^2 x^2}$$

and equation becomes

$$\frac{y^2}{y^2x^2} + \frac{x^2}{y^2x^2}\frac{dy}{dx} = 0$$

Exact by 3(a)

$$x^{-2} + y^{-2}\frac{dy}{dx} = 0 \quad \text{———} \quad (*)$$

Solution is of the form $f(x,y) = \text{const.}$

$$f_x = x^{-2} \Rightarrow f(x,y) = \int x^{-2}\,dx$$
$$= -x^{-1} + g(y)$$

$$\cancel{g(y) \Rightarrow}\; f_y = g'(y) = y^{-2} \Rightarrow g(y) = \int y^{-2}\,dy$$

$$\Rightarrow g(y) = -y^{-1}.$$

$\therefore$ solution is $-\frac{1}{x} - \frac{1}{y} = C$

$$\Rightarrow \frac{1}{y} = -\frac{1}{x} - C = -\left(\frac{1}{x} + C\right)$$

$$\Rightarrow y = \frac{-1}{\frac{1}{x} + C} \cdot \frac{x}{x} = \frac{-x}{1 + Cx}$$

4 $$x^4\frac{d^2y}{dx^2} + 2x^3\frac{dy}{dx} - 4y = 4 \qquad — (*)$$

$x = t^{-1}$

<u>Always</u> use the hint!

$$\left.\begin{aligned} &x = t^{-1} \Rightarrow \frac{dx}{dt} = -t^{-2} = -x^2 \\ &\text{similarly } \frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1} = -t^2 \text{ or } -\frac{1}{x^2} \end{aligned}\right\}$$ ingredients needed for subst.

We need to make $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to transform the D.E.

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -t^2\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\cdot -t^2$$ (minus signs cancel.)

$$= \left(2t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right)t^2$$

subst. into (*) to get

$$x^4\left(2t\frac{dy}{dt}+t^2\frac{d^2y}{dt^2}\right)t^2+2x^3\left(-t^2\frac{dy}{dt}\right)-4y=4$$

but $x=\frac{1}{t}$ so subst this and expand:

$$\frac{2t^3}{t^4}\frac{dy}{dt}+\frac{t^4}{t^4}\frac{d^2y}{dt^2}-\frac{2t^2}{t^3}\frac{dy}{dt}-4y=4$$

simplifies to $\frac{d^2y}{dt^2}-4y=4$

This is 2nd order linear with constant coefficients.

$$y=c(t)+p(t)$$

$c(t)$ is solution to homogeneous part

$$\frac{d^2y}{dt^2}-4y=0$$

characteristic equation $= \quad r^2-4=0$

$$r=\pm 2$$

$$c(t)=Ae^{-2t}+Be^{2t}$$

Easy to see particular integral is $y=-1$

$$\therefore \quad y(t)=Ae^{-2t}+Be^{2t}-1$$

$$\Rightarrow y(x)=Ae^{-\frac{2}{x}}+Be^{\frac{2}{x}}-1.$$

5. $$y = \underbrace{e^{2x}\left(A\cos(3x) + B\sin(3x)\right)}_{} + \underbrace{4x^2 + x - 1}_{}$$

$$y = C(x) \qquad + \qquad p(x)$$

$$C(x) = e^{\underset{a}{2}x}\left(A\cos(\underset{b}{3}x) + B\sin(3x)\right)$$

This corresponds to complex conjugate roots of characteristic equation.

$$\alpha = a + bi = 2 + 3i$$
$$\beta = a - bi = 2 - 3i$$

So we rebuild the characteristic equation from its roots:

$$\left(t - (2+3i)\right)\left(t - (2-3i)\right) = 0$$

$$t^2 - (2-3i)t - (2+3i)t + (2-3i)(2+3i) = 0$$

$$t^2 - 4t + 4 + 9 = 0$$

$$t^2 - 4t + 13 = 0$$

So the homogeneous part must have been

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

To find $S(x)$? $p(x)$ solves

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = S(x)$$

So if we subst in for $p(x)$ we should get $s(x)$

$$p(x) = 4x^2 + x - 1$$
$$p'(x) = 8x + 1$$
$$p''(x) = 8$$

Then $p''(x) - 4p'(x) + 13p(x)$

$$= 8 - 4(8x+1) + 13(4x^2 + x - 1)$$

$$= 8 - 32x - 4 + 52x^2 + 13x - 13$$

$$= 52x^2 - 19x - 9$$

So the D.E. which the student solved way

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 52x^2 - 19x - 9.$$