Solutions Chapter 1

Solutions to Exercises 1.1.

1. (a) $2x \sin x + x^2 \cos x$

   (b) $\dfrac{1}{(2x+3)^2}$

   (c) $3x^2 \ln x + x^2$

   (d) $-\dfrac{1}{\sin^2 x} = -\csc^2 x$

   (e) $\dfrac{2x}{x^2+2}$

   (f) $\cos(3x-5) \cdot 3$

   (g) Applying the quotient rule gives $\dfrac{2\cos 2x \cos x + \sin 2x \sin x}{\cos^2 x}$, which (with some effort) can be simplified to $2\cos x$. Alternatively, one can first simplify the function y and then compute the derivative more easily: $y = \dfrac{\sin 2x}{\cos x} = \dfrac{2 \sin x \cos x}{\cos x} = 2 \sin x$, hence $y' = 2\cos x$.

   (h) $4x^3 e^{2x} + 2x^4 e^{2x}$

   (i) $\dfrac{1}{1+4x} \cdot \dfrac{1}{\sqrt{x}}$

   (j) $e^{\sin x} \cos x$

2. (a) $(x+1)^{10} + 10x(x+1)^9$

   (b) $(2x+2)\exp(x^2+2x+3)$

   (c) $\ln(3x+2) + \dfrac{x}{3x+2} \cdot 3$

   (d) $\dfrac{(\cos x - x\sin x)(x+1) - x\cos x}{(x+1)^2} = \dfrac{\cos x - x(x+1)\sin x}{(x+1)^2}$

3. We compute the derivatives of y:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 \ln x + x^3 \qquad \text{and} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 \ln x + 7x^2.$$

   Substituting y, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into the equation gives

$$\begin{aligned} x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7x\frac{\mathrm{d}y}{\mathrm{d}x} + 16y &= x^2(12x^2\ln x + 7x^2) - 7x(4x^3\ln x + x^3) + 16(x^4 \ln x) \\ &= 0 \end{aligned}$$

   as desired.

Solutions to Exercises 1.2.

1. (a) 4

   (b) $= \lim_{x\to 3} \dfrac{(x+3)(x-3)}{x-3} = \lim_{x\to 3} x+3 = 6$

   (c) $= \lim_{x\to 0} \dfrac{5-x}{3x+2} = \dfrac{\lim_{x\to 0} 5-x}{\lim_{x\to 0} 3x+2} = \dfrac{5}{2}$

   (d) rationalise to obtain

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{2-h}-\sqrt{2}}{h} &= \lim_{h\to 0} \frac{\sqrt{2-h}-\sqrt{2}}{h} \cdot \frac{\sqrt{2-h}+\sqrt{2}}{\sqrt{2-h}+\sqrt{2}} \\
&= \lim_{h\to 0} \frac{2-h-2}{h(\sqrt{2-h}+\sqrt{2})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{2-h}+\sqrt{2}} \\
&= \frac{-1}{2\sqrt{2}}
\end{aligned}$$

   (e) $= \lim_{h\to\infty} \dfrac{3-4/h+6/h^2}{5+2/h^2} = \dfrac{3}{5}$

   (f) $= \lim_{h\to 0}(h+5) = 5$

2. In all cases, you should verify that the hypotheses of L'Hôpital's rule apply.

   (a) $= \lim_{x\to 0} \dfrac{4\cos 4x}{5\sec^2 5x} = \dfrac{4\cdot 1}{5\cdot 1} = \dfrac{4}{5}$

   (b) $= \lim_{x\to\frac{\pi}{2}} \dfrac{-2}{-\sin x} = \dfrac{-2}{-1} = 2$

   (c) $= \lim_{x\to 2} \dfrac{3x^2-3}{2x-3} = \dfrac{9}{1} = 9$

   (d) $= \lim_{x\to 1} \dfrac{1}{1/x} = 1$

   (e) $= \lim_{x\to 1} \dfrac{1-\exp(x-1)}{2(x-1)} = \lim_{x\to 1} \dfrac{-\exp(x-1)}{2} = \dfrac{-1}{2}$ (Note this required two applications of L'Hôpital's Rule.)

3. Note that $|\sin\alpha| \le 1$ for all $\alpha\in\mathbb{R}$, and therefore $|\sin\frac{1}{x}| \le 1$ for all $x\in\mathbb{R}-\{0\}$. Hence $|x\sin\frac{1}{x}| \le |x|$ and so

$$-|x| \le x\sin\frac{1}{x} \le |x| \quad \text{for all } x\in\mathbb{R}-\{0\}.$$

   Since $\lim_{x\to 0} -|x| = 0$ and $\lim_{x\to 0}|x| = 0$, the squeeze rule implies

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

4. (a) $= \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x+2}{x+3} = 5/6$

   (b) The assumptions of L'Hôpital's rule are satisfied, hence $\lim_{x\to 0} \frac{\sin 3x}{e^{2x}-1} = \lim_{x\to 0} \frac{3\cos 3x}{2e^{2x}} = 3/2$.

Solutions to Exercises 1.3.

1. On each of the intervals $(-\infty,-1),(-1,2)$ and $(2,\infty)$ the function f agrees with a polynomial. Since polynomials are continuous, f is continuous on those intervals. For $x=-1$ we find

$$\lim_{\substack{x\to -1\\ \text{from the left}}} f(x) = \lim_{\substack{x\to -1\\ \text{from the left}}} 1-x = 2,$$
$$\lim_{\substack{x\to -1\\ \text{from the right}}} f(x) = \lim_{\substack{x\to -1\\ \text{from the right}}} 3+x = 2.$$

   Hence $\lim_{x\to -1} f(x)$ exists and is equal to 2. Since

$$\lim_{x\to -1} f(x) = 2 \neq 3 = f(-1),$$

   the function f is not continuous at $x=-1$. Finally for $x=2$ we find

$$\lim_{\substack{x\to 2\\ \text{from the left}}} f(x) = \lim_{\substack{x\to 2\\ \text{from the left}}} 3+x = 5,$$
$$\lim_{\substack{x\to 2\\ \text{from the right}}} f(x) = \lim_{\substack{x\to 2\\ \text{from the right}}} 3x-1 = 5.$$

   Hence $\lim_{x\to -1} f(x)$ exists and is equal to 5. Since

$$\lim_{x\to 2} f(x) = 5 = f(2),$$

   the function f is continuous at $x=2$.

   To summarize, f is continuous at a for all $a \in \mathbb{R} - \{-1\}$.

Solutions to Exercises 1.4.

1. (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{3(x+h)-3x}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3 \end{aligned}$$

(b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + 3h^3 - x^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + 3h^2 \\ &= 3x^2 \end{aligned}$$

(c)

$$\begin{aligned} h'(x) &= \lim_{h\to 0} \frac{1/(x+h) - 1/x}{h} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} \\ &= \frac{-1}{x^2} \end{aligned}$$

(d) Recall that for a positive integer n the binomial theorem says

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \cdots + \binom{n}{n-1} xh^{n-1} + h^n.$$

Hence

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \cdots + h^n - x^n}{h} \\ &= \lim_{h\to 0} nx^{n-1} + \binom{n}{2} x^{n-2}h + \cdots + h^{n-1} \\ &= nx^{n-1}. \end{aligned}$$

The derivative $k'(x) = nx^{n-1}$ is correct for all $n \in \mathbb{R}$, however the computation using the binomial theorem works only for positive integers n.

(e) Using $\cos(x+h) = \cos x \cos h - \sin x \sin h$ and the limits $\lim_{h\to 0} \dfrac{\cos h - 1}{h} = 0$

and $\lim_{h\to 0} \frac{\sin h}{h} = 1$ from the course, we find

$$\begin{aligned}
l'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h} \\
&= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

2. Let f, g be two real valued functions and assume that for both functions the derivative at x exists. Then

$$\begin{aligned}
\frac{\mathrm{d}(fg)}{\mathrm{d}x}(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(g(x+h)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x+h) - g(x)}{h} \right).
\end{aligned}$$

Since all relevant limits exist, the previous line becomes

$$\begin{aligned}
&= \lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x).
\end{aligned}$$

Notice that $\lim_{h\to 0} g(x+h) = g(x)$ is simply stating that g is continuous at x. Since we are assuming that g is differentiable at x, we can conclude g is continuous at x.

Solutions to Exercises 1.5.

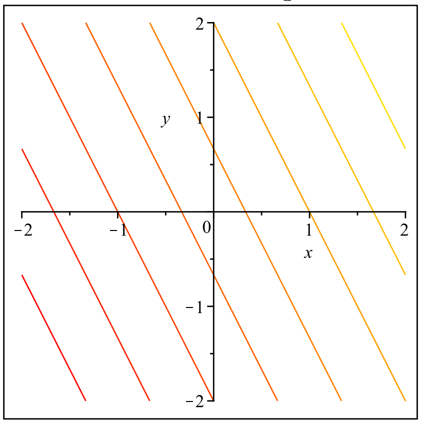
1. The domain is given by the set U indicated.

   (a) $U = \{(x, y) : y \neq -x\}$

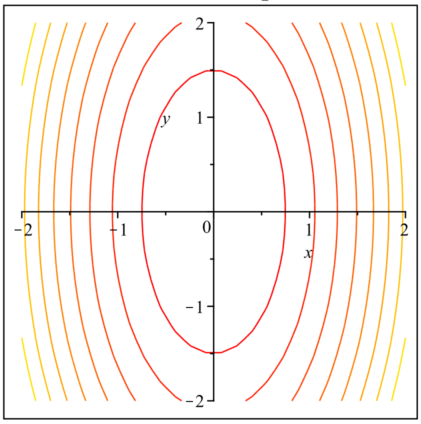
   (b) $U = \{(x, y) : x \neq 0 \text{ and } y \neq 0\}$

2. For any function, notice from the definition of contour plot that contours don't cross (since each contour corresponds to different values of $f(x, y)$.

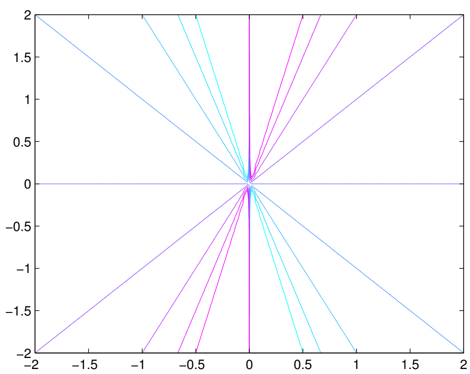
(a) The contours are straight lines with slope -2.



(b) The contours are ellipses.



(c) Each contour is given by different constants $k = f(x, y)$, which means that $y = kx, x \neq 0$. So the contours are straight lines going through the origin with slope k, but the origin on each line is omitted (since $x \neq 0$). Great, because otherwise the lines would cross! (Note that the y-axis is not a contour, even though on the picture it looks like one.)



Solutions to Exercises 1.6.

1. $f_x(x,y) = 3x^2y - 4xy^2 + 3y$ and $f_y(x,y) = x^3 - 4x^2y + 3x - 3y^2$

2. $$\frac{\partial z}{\partial x} = \exp(x+y^2)\cdot\sin(xy) + \exp(x+y^2)y\cos(xy),$$
$$\frac{\partial z}{\partial y} = 2y\exp(x+y^2)\sin(xy) + \exp(x+y^2)x\cos(xy)$$

Solutions to Exercises 1.7.

1. We have
$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix},$$
$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix},$$
$$\mathbf{v}\times\mathbf{w} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix},$$
$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \begin{pmatrix} 13 \\ -2 \\ -3 \end{pmatrix}.$$
Since $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} \neq \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$, the cross product is not associative.

2. We have
$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} = -\begin{pmatrix} v_2u_3 - v_3u_2 \\ v_3u_1 - v_1u_3 \\ v_1u_2 - v_2u_1 \end{pmatrix} = -(\mathbf{v}\times\mathbf{u}).$$

Solutions to Exercises 1.8.

1. $\begin{pmatrix} x-3 \\ y-1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0$, which can also be rewritten as $y = 2x - 5$

2. $\begin{pmatrix} x-2 \\ y-4 \\ z-8 \end{pmatrix}\cdot\begin{pmatrix} 32 \\ -2 \\ -1 \end{pmatrix} = 0$, which can also be written as $32x - 2y - z - 48 = 0$

3. The tangent plane is $\begin{pmatrix} x+2 \\ y-1 \\ z-1 \end{pmatrix}\cdot\begin{pmatrix} 1/2 \\ 2 \\ -1 \end{pmatrix} = 0$. Solving this for z gives
$$z = \frac{1}{2}x + 2y.$$

The tangent plane touches the surface $z = f(x, y)$ at the point $(-2, 1)$, and for points that are near $(-2, 1)$ this tangent plane will still be close to the surface. Therefore setting $x = -1.9$ and $y = 1.1$ in the equation of the tangent plane gives an estimate for $f(-1.9, 1.1)$. Thus $\frac{1}{2}\cdot(-1.9)+2\cdot 1.1 = 1.25$ is an estimate for $f(-1.9, 1.1)$.

Solutions to Exercises 1.9.

1. $\nabla f = -\frac{1}{4}\begin{pmatrix} 3y^2 - 15x^2y + 8x^3 \\ 6xy - 5x^3 \end{pmatrix}$

Solutions to Exercises 1.10. When computing the directional derivative, remember to normalise u if needed.

1. The vector $\mathbf{u}$ is already normalised. We have $\nabla f(2, 4) = \begin{pmatrix} 32 \\ -2 \end{pmatrix}$ and hence $f_{\mathbf{u}}(2, 4) = \nabla f(2, 4).\mathbf{u} = \dfrac{226}{17}$.
2. The normalisation of $\mathbf{u}$ is $\hat{\mathbf{u}} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$. We have $\nabla f(1, 2) = \begin{pmatrix} -9 \\ 7 \end{pmatrix}$ and hence $f_{\mathbf{u}}(1, 2) = \nabla f(1, 2).\hat{\mathbf{u}} = 11$.
3. (Exam question, 2005)
   (a) $\nabla f(x, y) = \begin{pmatrix} -5y + 4xy - 3x^2 \\ 2y - 5x + 2x^2 \end{pmatrix}$
   (b) The vector u is already normalised. We have $\nabla f(3, 4) = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$ and therefore $f_u(3, 4) = \nabla f(3, 4) \cdot u = \dfrac{43}{13}$.

Solutions to Exercises 1.11.

1. The gradient is $\nabla f = \begin{pmatrix} 2\cos(2x + y - z^2) \\ \cos(2x + y - z^2) \\ -2z\cos(2x + y - z^2) \end{pmatrix}$, hence $\nabla f(2\pi, \pi, 0) = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$.
   The vector $\mathbf{u}$ is already normalised. The directional derivative is $f_{\mathbf{u}}(2\pi, \pi, 0) = \nabla f(2\pi, \pi, 0).\mathbf{u} = -\dfrac{4}{3}$.