BIRKBECK

(University of London)

BSc EXAMINATION
SCHOOL OF BUSINESS, ECONOMICS AND INFORMATICS

Discrete Mathematics
BUEM002S5

30 credits

Tuesday, 24 May, 2016
Morning, 10:00 a.m./p.m. - 1:00 p.m.

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 *marks each and questions in Section B are worth* 20 *marks each.*
Candidates should attempt **all** *of the questions in Section A and* **two** *questions out of the four in Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorised calculators or has been approved by the chair of the Mathematics and Statistics Examination Sub-board.

Please turn over

Section A

1. Evaluate the following sums:

   (a) $\sum_{i=0}^{20} \binom{22}{i}$; [2]

   (b) $\sum_{i=1}^{50} (i+1)(i-1)$. [3]

2. (a) How many four-letter words can be made using letters from the set $\{X, Y, Z\}$? [1]

   (b) How many of these words contain at least one X? [2]

   (c) How many of these words do not contain an X immediately followed by a Y? [2]

3. A potter has produced 20 identical bowls. He wishes to paint each bowl in a single colour, and the colours available are red, blue and yellow.

   (a) How many ways are there of painting the bowls? [1]

   (b) How many ways are there of painting the bowls if there must be at least one red bowl, at least two blue bowls and at least three yellow bowls? [1]

   (c) How many ways are there of painting the bowls so that there are more red bowls than blue bowls? [3]

4. Consider the difference equation

$$u_n = u_{n-1}^2 - 3.$$

   (a) Write down a solution to this difference equation in the case where $u_0 = 2$. [2]

   (b) Draw a cobweb diagram to show how the sequence given by this difference equation behaves as $n \to \infty$ in the case where $u_0 = 3$. [3]

Please turn over

5. A graph G has the following adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

(a) Give the degree sequence of G. [1]

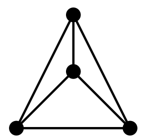
(b) How many walks of length two are there between the two vertices of G that have degree three? [2]

(c) The graph H has the following adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Determine whether G and H are isomorphic to each other, giving appropriate evidence for your answer. [2]

6. (a) What is the edge connectivity of the tetrahedron graph?



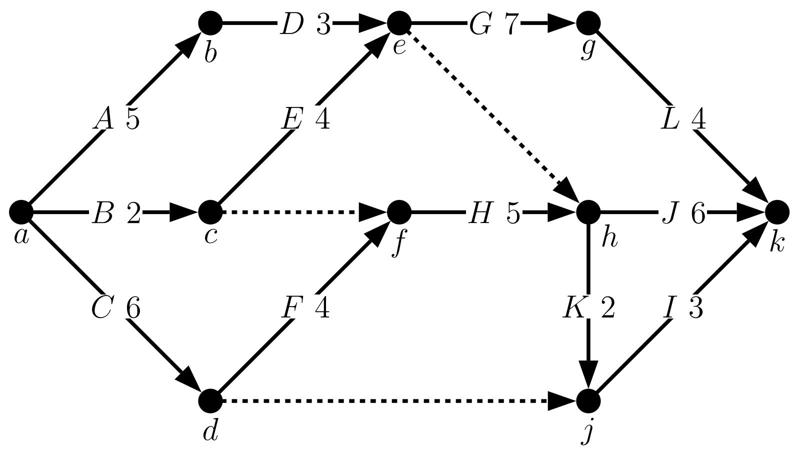
[1]

(b) What is the connectivity of the tetrahedron graph? [1]

(c) Give an example of a bipartite graph that is 3-edge-connected. [3]

Please turn over

7. Consider the following activity network:

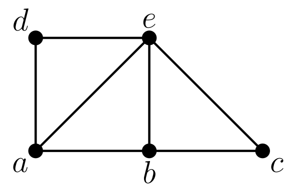


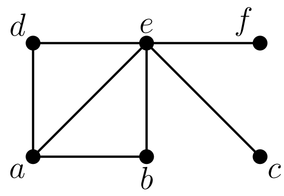
(a) Find a critical path for the network. [3]

(b) Find the earliest start time and latest start time of activity G. [2]

8. (a) Give the definition of a perfect matching in a graph. [2]

(b) Determine whether each of the following graphs has a perfect matching, giving evidence for your answer.

i) 

ii) 

[3]

Please turn over

Section B

9. (a) A baker is baking thirty pies. She can make three flavours, apple, cherry, and pumpkin. She wishes to bake at least five but at most ten apple pies, at least ten cherry pies, and at most seven pumpkin pies. The baker wishes to know how many different selections of pies are possible.

   (i) Express this problem as an integer equation. [2]

   (ii) Write the generating function corresponding to this integer equation. [2]

   (iii) Use the generating function to find the solution to the counting problem. [5]

   (b) Prove that $(1-x)^{-1}$ is the generating function associated with the sequence $(u_i)_{i=0}^{\infty}$ with $u_i = 1$ for all i. [3]

   (c) Find the first five terms of the sequence whose generating function is $(1-5x+6x^2)^{-1}$. [2]

   (d) The sequence of *triangular numbers* is the sequence whose n^{th} term is the sum of all the integers from 0 up to n. Find the generating function of the sequence of triangular numbers. [2]

   (e) Find the generating functions of the following sequences:

   (i) $0, 0, 1, 0, 2, 0, 3, 0, 4, 0, \ldots$ [2]

   (ii) $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ [2]

Please turn over

10. (a) Solve the following difference equations:

(i)

$$u_n = 3u_{n-1} + 10u_{n-2} - 36n + 9, \qquad u_0 = 11, \quad u_1 = 10.$$

[4]

(ii)

$$u_n = 2u_{n-1} - u_{n-2} + 8, \qquad u_0 = 6, \quad u_1 = 13.$$

[4]

(iii)

$$u_n = \frac{n+1}{n} u_{n-1} + n + 1, \qquad u_0 = 1.$$

[4]

(b) Consider the following linear, second-order, homogeneous difference equation with constant coefficients:

$$u_n = Au_{n-1} + Bu_{n-2}. \tag{1}$$

(i) Define the *characteristic polynomial* of the equation (1). [1]

(ii) Show that ω^n is a nonzero solution of (1) if and only if ω is a zero of the characteristic polynomial of (1). [2]

(c) (i) Give an expression for the n^{th} Catalan number C_n in terms of a binomial coefficient. [2]

(ii) Hence, or otherwise, show that the n^{th} Catalan number C_n satisfies

$$C_n = \prod_{k=2}^{n} \frac{n+k}{k}.$$

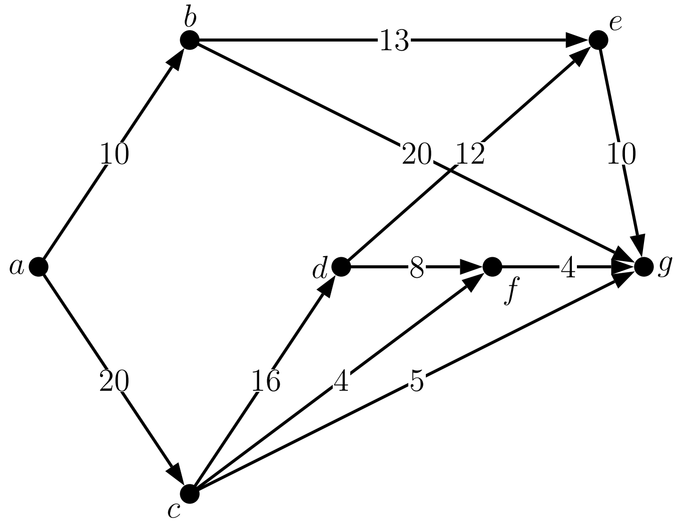
[3]

Please turn over

11. (a) (i) State the maximum-flow/minimum-cut theorem. [2]

(ii) Prove that in a network with integer weights, the value of the maximum flow is an integer. [2]

(iii) Use the Ford-Fulkerson algorithm to find a maximum flow in the following network:



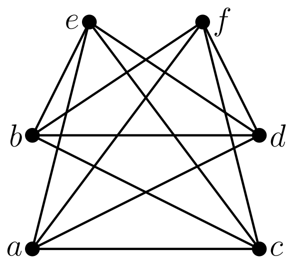
[6]

(b) (i) State Euler's polyhedral formula. [2]

(ii) A three-regular planar graph has twenty vertices. Into how many distinct regions does it divide the plane? [3]

(iii) For what values of m and n is the complete bipartite graph $K_{m,n}$ planar? [2]

(iv) The following graph is a complete tripartite graph:



Does it contain a subgraph isomorphic to a subdivision of $K_{3,3}$? (Justify your answer.) [3]

Please turn over

12. (a) (i) State Ore's Theorem. [2]

(ii) Give an example of a Hamiltonian graph that does not satisfy the conditions of Ore's Theorem. [2]

(iii) Give an example of a 3-regular Hamiltonian graph. [2]

(iv) How many distinct Hamiltonian cycles does K_n have when $n \geq 3$? [2]

(b) State the König-Egerváry Theorem. [2]

(c) A manager wishes to have five tasks completed, Task A, Task B, Task C, Task D and Task E. He approaches five contractors, Contractor 1, Contractor 2, Contractor 3, Contractor 4 and Contractor 5. The amounts they quote in pounds to perform each task are given in the following table:

	A	B	C	D	E
1	4	7	5	6	9
2	6	8	10	8	9
3	7	7	9	7	7
4	6	9	5	10	6
5	8	9	6	9	8

(i) Find an assignment of contractors to tasks that minimises the total cost. [5]

(ii) How would your approach to answering question 12(c)i change if the table entries represented values for which you wished to maximise the total? [1]

(iii) How would your approach answering question 12(c)i if you also had the option of considering Contractor 6 who charges £7 for performing each task? [2]

(iv) Find an assignment of contractors to tasks in which the smallest table entry is as large as possible. [2]

End of examination paper