Example 3.1.2 For the complex numbers $z_1 = 3 + 5i$ and $z_2 = -8 + 3i$ find $\frac{z_1}{z_2}$.

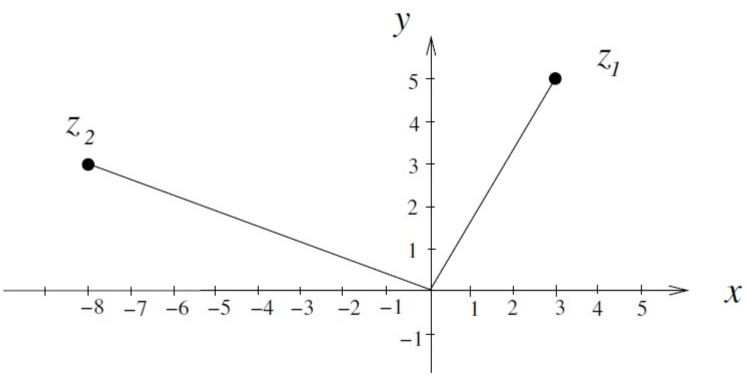
$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} \\
&= \frac{(3+5i)}{(-8+3i)}\frac{(-8-3i)}{(-8-3i)} \\
&= \frac{-24-40i-9i-15i^2}{8^2+3^2} \\
&= \frac{-9-49i}{73} \\
&= -\frac{9}{73} - \frac{49}{73}i.
\end{aligned}$$

Exercise 3.2 For the complex numbers $z_1 = 4 - 3i$ and $z_2 = 1 + 6i$ find $\frac{z_1}{z_2}$.

3.1.3 The polar form of a complex number

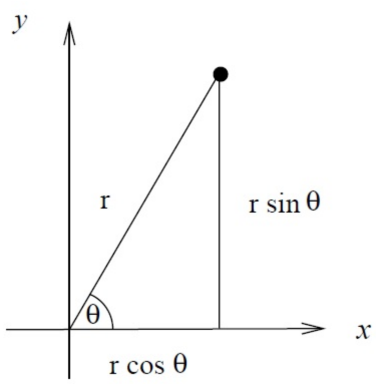
Given a complex number $z = x + iy$ we can express it as the point (x, y) in the plane. That is, the x coordinate is the real part of z and the y coordinate is the imaginary part of z. This pictorial representation of the complex numbers is called an **Argand diagram**.

For example, the complex numbers $z_1 = 3 + 5i$ and $z_2 = -8 + 3i$ would appear as below on the Argand diagram.



Consider the line segment joining the point representing the number $z = x + iy$ to the origin $(0, 0)$. Using Pythagoras's Theorem, the length r of this line segment is given by $r = \sqrt{x^2 + y^2}$. (Note that this is the same as $\sqrt{z\overline{z}}$.)

Let the angle measured anticlockwise from the x-axis to the line segment be θ.



Using the definitions of the trigonometric functions sin and cos, we have

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta).$$

Thus if we know the values of r and θ we can determine z uniquely since

$$z = r(\cos(\theta) + \sin(\theta)i).$$

This is called the **polar form** of z. The length r is called the **modulus** of z, and is denoted by $|z|$, and θ is called the **argument** of z and is denoted by $\arg(z)$.[1]

We have already seen how to determine the value of r from x and y. To determine θ, note that, if $x \neq 0$, then

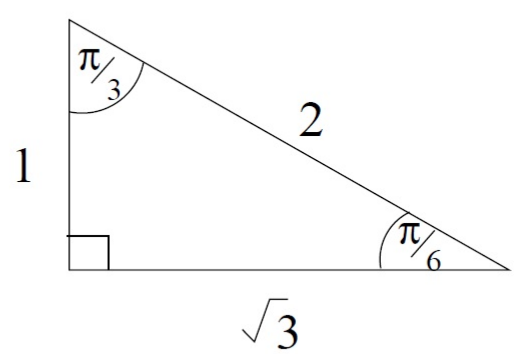
$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan(\theta).$$

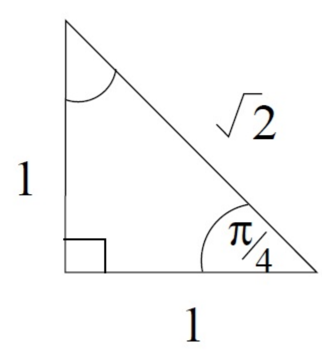
Thus $\theta = \tan^{-1}(\frac{y}{x})$. However, since $\tan(\theta) = \tan(\theta + \pi)$ for any θ, there are two possibilities for θ in the interval $[0, 2\pi)$ and so we have to choose the one which is in the same quadrant as z.

What if $x = 0$? Then $z = iy$, so z lies on the y-axis (and $r = |y|$). If $y > 0$, then we have $\theta = \frac{\pi}{2}$. If $y < 0$, then $\theta = \frac{3\pi}{2}$. If we also have that $y = 0$, then $z = 0$, so $r = 0$ and we could pick any value of θ. In this case we have the convention that $\theta = 0$.

Warning: It is important that the argument θ be an angle in the range $0 \leq \theta < 2\pi$. This is the standard way of writing complex numbers in polar form and **you would lose marks on an exam for giving an argument θ which was not in this range.**

Most of the time, the argument will not be a nice fraction of π, and you will need to use a calculator to get an approximation to it. However, there are some important angles whose tangents you are expected to know. The best way to learn them is by drawing a couple of triangles.





You also need to remember that $\tan(\theta) = \tan(\pi + \theta)$ and $-\tan(\theta) = \tan(\pi - \theta)$, for any angle θ. You can then construct the following table:

[1] Although most current meanings of argument indicate a disagreement, the Latin root *arguere* was closer to such present meanings as declare or prove. When we talk about the argument of a function, we are declaring a value for the independent variable of the function, in this case the function $f(\theta) = r(\cos(\theta) + \sin(\theta)i)$.

$\tan(\theta)$	Possibilities for θ with $0 \leq \theta < 2\pi$
1	$\frac{\pi}{4}, \frac{5\pi}{4}$
-1	$\frac{3\pi}{4}, \frac{7\pi}{4}$
$\sqrt{3}$	$\frac{\pi}{3}, \frac{4\pi}{3}$
$-\sqrt{3}$	$\frac{2\pi}{3}, \frac{5\pi}{3}$
$\frac{1}{\sqrt{3}}$	$\frac{\pi}{6}, \frac{7\pi}{6}$
$-\frac{1}{\sqrt{3}}$	$\frac{5\pi}{6}, \frac{11\pi}{6}$

Example 3.1.3 Find the modulus and argument of (i) $2-2i$; (ii) $2\sqrt{3}+2i$; (iii) $-8+6i$.

In each case we denote the argument of the complex number by θ.

(i) $|2-2i| = \sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$.
$\tan\theta = \frac{-2}{2} = -1$. Thus $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$, Since $2-2i$ is in the bottom right hand quadrant it follows that $\theta = \frac{7\pi}{4}$. That is, $\arg(2-2i) = \frac{7\pi}{4}$. In particular we have

$$2-2i = 2\sqrt{2}\left(\cos\left(\tfrac{7\pi}{4}\right) + i\sin\left(\tfrac{7\pi}{4}\right)\right).$$

(ii) $|2\sqrt{3}+2i| = \sqrt{(2\sqrt{3})^2+2^2} = \sqrt{16} = 4$.
$\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$. Thus $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. Since $2\sqrt{3}+2i$ is in the top right hand quadrant it follows that $\theta = \frac{\pi}{6}$. That is, $\arg(2\sqrt{3}+2i) = \frac{\pi}{6}$. In particular we have

$$2\sqrt{3}+2i = 4\left(\cos\left(\tfrac{\pi}{6}\right) + i\sin\left(\tfrac{\pi}{6}\right)\right).$$

(iii) $|-8+6i| = \sqrt{(8)^2+6^2} = \sqrt{100} = 10$.
$\tan\theta = \frac{6}{-8} = -\frac{3}{4}$. Thus $\theta \approx 2\cdot 498$ or $5\cdot 640$. Since $-8+6i$ is in the top left hand quadrant it follows that $\theta \approx 2.498$. That is, $\arg(-8+6i) \approx 2.498$. In particular we have

$$-8+6i \approx 10\left(\cos(2\cdot 498) + i\sin(2\cdot 498)\right).$$

Exercise 3.3 Find the modulus and argument of (i) $1+i$; (ii) $2-4i$.

3.1.4 De Moivre's Theorem[2]

We now consider finding powers of complex numbers. Let $z = r(\cos(\theta) + i\sin(\theta))$. Using the standard rules for indices we have $z^t = r^t(\cos(\theta) + i\sin(\theta))^t$ for all real numbers t. Thus to determine z^t we need to be able to find $(\cos(\theta) + i\sin(\theta))^t$.

In general we have the following result.

Theorem 3.1.4 (De Moivre's Theorem) *For all $\theta \in \mathbb{R}$ and all $n \in \mathbb{N}$*

$$(\cos(\theta) + \sin(\theta)i)^n = \cos(n\theta) + i\sin(n\theta).$$

[2]Abraham de Moivre (May 26, 1667–November 27, 1754). In later life, Newton used to put off questions on the Principia with the words "Go to Mr de Moivre; he knows these things better than I do."