Discrete Assignment 2

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1. Difference equations

   (a) Here, u_n is an inhomogeneous first order difference equation, of the general form

   $$u_n = f(n)u_{n-1} + g(n) = U \cdot \prod_{i=1}^{n} f(i) + \sum_{i=1}^{n} \left(g(i) \cdot \prod_{j=i+1}^{n} f(j) \right)$$

   where

   $$U = 1$$
   $$f(n) = 16^{n^3}$$
   $$g(n) = 2^{n^2(n+1)^2}$$

   hence

   $$u_n = \prod_{i=1}^{n} 16^{i^3} + \sum_{i=1}^{n} \left(2^{n^2(i+1)^2} \cdot \prod_{j=i+1}^{n} 16^{j^3} \right)$$

   (b) Let $k = n + 2$, then a_{n+2} can be written

   $$a_k = 4a_{k-2} + 10 \cdot 3^{k-2}$$

   Here, a_k is an inhomogeneous second order difference equation with constant coefficients and can be written in the general form

   $$u_n = au_{n-1} + bu_{n-2} + f(n)$$

   where

   $$a = 0$$
   $$b = 4$$
   $$f(n) = 10 \cdot 3^{n-2}$$

   and

   $$u_n = P(n) + G(n)$$

where $P(n)$ is a particular solution to the inhomogeneous difference equation and $G(n)$ is a general solution to the homogenous part of the inhomogeneous difference equation.

The homogenous part of the difference equation has characteristic polynomial $\lambda^2 - 4\lambda$ with distinct real zeros $w_1 = 2$ and $w_2 = -2$. There must be values c_1 and c_2 such that the initial conditions are satisfied as follows

$$\begin{aligned} c_1 + c_2 &= 9 \\ c_2 &= 9 - c_1 \end{aligned}$$

$$\begin{aligned} c_1 w_1 + c_2 w_2 &= 4 \\ 2c_1 - 2c_2 &= \\ 2c_1 - 2(9 - c_1) &= \\ 2c_1 - 18 + 2c_1 &= \\ 4c_1 &= 22 \\ c_1 &= 11 \end{aligned}$$

$$\begin{aligned} c_2 &= 9 - 11 \\ c_2 &= -2 \end{aligned}$$

so the general solution to the homogenous part is $G(n) = A \cdot 11^n + B \cdot 2^n$.

$f(n)$ has the form $c\alpha^n$ where $\alpha = 3$ and α is not a zero of the characteristic polynomial, therefore we can try $u_n = M \cdot 3^n$ as a particular solution, hence

$$\begin{aligned} M \cdot 3^n &= 4M \cdot 3^{n-2} + 10 \cdot 3^{n-2} \\ M \cdot 3^2 &= 4M + 10 \\ 9M &= 4M + 10 \\ 5M &= 10 \\ M &= 2 \end{aligned}$$

and our particular solution is $P(n) = 2 \cdot 3^n$. Now we can write a general solution for the inhomogeneous difference equation u_n as

$$\begin{aligned} u_n &= P(n) + G(n) \\ &= A \cdot 2^n + B \cdot 2^n + 2 \cdot 3^n. \end{aligned}$$

The initial conditions $u_0 = 9$, $u_1 = 4$ imply that

$$\begin{aligned} A + B + 2 &= 9 \\ A &= 7 - B \end{aligned}$$

$$\begin{aligned} 11A + 2B + 6 &= 4 \\ 11(7 - B) + 2B &= 4 \\ 9B &= 81 \\ B &= 9 \end{aligned}$$

$$\begin{aligned} A &= 7 - 9 \\ A &= 2 \end{aligned}$$

and therefore our general solution to the inhomogeneous difference equation $u_n = 2 \cdot 2^n + 9 \cdot 2^n + 2 \cdot 3^n$.

2. The reproduction of flora on planet Zod can be described as a homogenous second order difference equation with constant coefficients and the general form

$$u_n = au_{n-1} + bu_{n-2}$$

where $a = 1$, $b = 6$ and $u_0 = U = 1$, $u_1 = V = 1$. Let $u_n = w^n$. We can now write

$$\begin{aligned} w^n &= aw^{n-1} + bw^{n-2} \\ w^2 &= aw + b \\ w^2 - aw - b &= 0. \end{aligned}$$

For the above to be true, w must be a zero of the characteristic polynomial $\lambda^2 - a\lambda - b$. Since we know that $a^2 + 4b > 0$, this polynomial has distinct real zeros w_1 and w_2. Substituting a and b for their values and factorising gives the values of these zeros: $\lambda^2 - a\lambda - b = 0 = (\lambda - 3)(\lambda + 2)$

We also know that for $c_1, c_2 \in \mathbb{R}$ we have $u_n = c_1 w_1^n + c_2 w_2^n$ with c_1 and c_2 taking values such that they satisfy the initial conditions U and V.

So, for $n = 0$

$$\begin{aligned} c_1 + c_2 &= U = 1 \\ c_2 &= 1 - c_1 \end{aligned}$$

and for $n = 1$

$$\begin{aligned} c_1 w_1 + c_2 w_2 &= V \\ 3c_1 - 2c_2 &= 1 \\ 3c_1 - 2(1 - c_1) &= 1 \\ c_1 &= \frac{3}{5} \iff c_2 = \frac{2}{5} \end{aligned}$$

then $u_n = 3(\frac{3}{5})^n - 2(\frac{2}{5})^n$.

3. (a) When $u_0 = 0$, $u_n = 0$. Similarly, when $u_0 = 2$, $u_n = 2$.
   (b)