Chapter 1

Sequences and Counting

This chapter is all about how to count, and the techniques that we explore here will be used again and again throughout the rest of this module. In fact, the usefulness of these basic principles is not just limited to applications in discrete mathematics; almost any branch of mathematics will involve problems that require you to count. We will begin by reviewing some useful properties of certain sequences, before exploring a range of fundamental counting problems and techniques for solving them.

1.1 Sequences

In this section we will see some results about specific sequences that we will be using later in the module. Recall that a sequence is just a list of real numbers. We will frequently use the notation u_r to represent the r^{th} term in the list; a finite sequence with n elements is denoted $(u_r)_{r=1}^{n}$, and $(u_r)_{r=1}^{\infty}$ denotes an infinite sequence.

Arithmetic Progressions

An *arithmetic progression* is a sequence in which any term differs from the previous term by some constant value d. If we denote the r^{th} element of an arithmetic progression by A_r, and set $A_1 = a$ then we have

$$A_r = a + (r-1)d.$$

It is often useful to know how to calculate the sum of the first n terms of an arithmetic progression; it can be shown that (for example, by induction)

$$\sum_{r=1}^{n} A_r = \frac{1}{2}n\big(2a + (n-1)d\big).$$

Geometric Progressions

A *geometric progression* is a sequence in which the ratio of each term to the previous term is some constant value R. Denoting the r^{th} term of a geometric progression by G_r and letting $G_1 = a$ we have

$$G_r = aR^{r-1}.$$

The sum of the first n terms of a geometric progression is given by

$$\sum_{r=1}^{n} G_r = \frac{a(R^n - 1)}{R - 1} = \frac{a(1 - R^n)}{1 - R}.$$

The Sequence $u_n = Ru_{n-1} + d$

This is an example of a sequence that is defined *recursively:* each term is expressed as a function of previous terms. It can be thought of as generalising both an arithmetic progression and a geometric progression; we will consider some closely related sequences in more detail in Chapter 3.

Letting $U_1 = U$, we can use the expression for the sum of the first r terms of a geometric progression to find an explicit formula for the r^{th} term of this sequence:

$$\begin{aligned} u_r &= \overbrace{R\Big(\dots R\big(R(RU + d) + d\big) + d \dots\Big) + d}^{r-1 \text{ times}} \\ &= R^{r-1}U + \frac{d(R^{r-1} - 1)}{R - 1}. \end{aligned}$$

The Sequence $u_r = r^k$ $(k = 1, 2, 3, 4)$

It is frequently useful to know the formula for the sum of the first n terms of this sequence for various small values of k. In the case $k = 1$, the sequence is simply an arithmetic progression whose common difference and first term are both 1, so we have the following result:

$$\sum_{r=1}^{n} r = \frac{1}{2}n\big(2 \times 1 + (n-1) \times 1\big) = \frac{1}{2}n(n+1).$$

The case of $k = 2$ is slightly trickier. One way to obtain the result is to consider the quantity $s_n = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n-1} r^3$, and evaluate it in two different ways.

Firstly, by cancelling the first $n - 1$ terms of the sums, we see that

$$s_n = n^3. \tag{1.1}$$

Next, we observe that by a change of variables[1],

$$\sum_{r=1}^{n-1} r^3 = \sum_{r=2}^{n} (r-1)^3.$$

Furthermore, since $(r-1)^3 = 0$ when $r = 1$, it follows that $\sum_{r=2}^{n}(r-1)^3 = \sum_{r=1}^{n}(r-1)^3$.

[1] Make sure that you are comfortable with changing the variables in a summation in this way -it can be an extremely useful technique.

Therefore, we have

$$\begin{aligned}
s_n &= \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n-1} r^3 \\
&= \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} (r-1)^3 \\
&= \sum_{r=1}^{n} r^3 - (r-1)^3 \\
&= \sum_{r=1}^{n} 3r^2 - 3r + 1 \\
&= 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \\
&= 3\sum_{r=1}^{n} r^2 - \frac{3}{2}n(n+1) + n \\
&= 3\sum_{r=1}^{n} r^2 - \frac{n}{2}(3n+1). \qquad (1.2)
\end{aligned}$$

Equating the expressions (1.1) and (1.2) for s_n, we obtain

$$n^3 = 3\sum_{r=1}^{n} r^2 - \frac{n}{2}(3n+1),$$

from which we derive

$$\begin{aligned}
\sum_{r=1}^{n} r^2 &= \frac{1}{3}\Big(n^3 + \frac{n}{2}(3n+1)\Big) \\
&= \frac{1}{6}n(2n^2 + 3n + 1) \\
&= \frac{1}{6}n(n+1)(2n+1).
\end{aligned}$$

Thus we have

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$

For $k = 3, 4$ it can be shown that

$$\begin{aligned}
\sum_{r=1}^{n} r^3 &= \frac{1}{4}n^2(n+1)^2, \\
\text{and } \sum_{r=1}^{n} r^4 &= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1).
\end{aligned}$$

These results could be obtained in a similar manner to the $k = 2$ case, by considering the expression $\sum_{r=1}^{n} r^l - \sum_{r=1}^{n-1} r^l$ for $l = 4, 5$. In Chapter 2 we will see another technique for computing these expressions.

Exercise 1.1. Simplify the following sums:

1. $1 - \gamma + \gamma^2 - \gamma^3 + \cdots + (-1)^n\gamma^n \quad$ (where $\gamma \in \mathbb{R}$, $\gamma \neq -1$)
2. $\sum_{r=1}^{n} r(r+3)(r-1)$

1.2 Counting

This section underpins this entire module, as the results we discuss here will be used repeatedly in the later chapters. The principles we explore here are fairly straightforward in themselves; the important thing is to be able to recognise when and how to apply them. Because of this, we will devote considerable time to examining examples of problems where these techniques can be used, in order to develop a helpful overall approach to thinking about counting problems.

1.2.1 Basic Counting Principles

Here we examine two simple, yet powerful, principles that will be used in the development of almost all subsequent results in this module.

The Rule of Sum

Theorem 1.1 (The Rule of Sum). *If S and T are disjoint sets such that S contains n elements and T contains m elements, then the number of ways of selecting an element from either S or T is $m + n$.*

Another way of stating this simple rule is that if S and T are disjoint finite sets, then the number of elements in their union is the sum of the number of elements in both sets. This rule can be generalised to any finite number of disjoint sets.

Example 1.2. How many ways are there to pick

1. an ace or queen from a standard pack of cards?[2]
2. a queen or a red card from a standard pack of cards?

Solution. The presence of the word *or* suggests that the Rule of Sum might be used.

1. There are 4 ways to pick an ace and 4 ways to pick a queen. Thus by the Rule of Sum there are $4 + 4 = 8$ ways of picking an ace or a queen.
2. Here the two options are not disjoint since the number of choices for the red card differs depending on whether the queen chosen is red or black. However the problem is equivalent to picking a black queen or a red card. There are 2 ways to pick a black queen and 26 ways to pick a red card. Thus by the Rule of Sum there are $2 + 26 = 28$ ways to pick a queen or a red card.

On its own the rule of sum is of limited use. Its real strength lies in its use in combination with the following counting principle.

[2] A standard pack of cards consists of 52 cards divided equally into 4 suits: $\spadesuit, \heartsuit, \clubsuit, \diamondsuit$. The 13 cards of each suit are labelled: $\text{Ace}, 2, 3, \ldots, 10, \text{Jack}, \text{Queen}, \text{King}$. $\spadesuit$ and $\clubsuit$ are black cards and $\heartsuit$ and $\diamondsuit$ are red cards.

The Rule of Product

Theorem 1.2 (The Rule of Product)**.** *If S is a set containing n elements and T is a set containing m elements, then the number of ways of selecting one element from S and one element from T is mn (provided that the choice of the element from S does not affect the choice of the element from T).*

Another way of stating this rule is that if S and T are finite sets, then the number of elements in $S \times T = \{(s,t) : s \in S, t \in T\}$ is the product of the number of elements in S and T.

Example 1.3. A bookshelf contains 8 books of Discrete Mathematics, 5 books on Algebra and 4 books on Calculus. How many ways are there of choosing two books that cover different subject areas?

Solution. The possible choices are a DM book and an A book, a DM book and a C book or an A book and a C book. By the Rule of Product there are $8 \times 5 = 40$ choices for the first option, $8 \times 4 = 32$ choices for the second option and $5 \times 4 = 20$ choices for the third option. Thus by the Rule of Sum, there are

$$40 + 32 + 20 = 92$$

ways of selecting the two books.

Example 1.4. A committee consists of four men and three women. How many ways can a subcommittee of three people be chosen if it must contain at least one man and at least one woman?

Solution. The subcommittee either consists of 2 men and 1 woman or 1 man and two women. There are 6 ways of choosing 2 men and 3 ways of choosing 2 women. Thus by the Rule of Product 2 men and 1 woman can be chosen in $6 \times 3 = 18$ ways and 1 man and 2 women can be chosen in $4 \times 3 = 12$ ways. Thus by the Rule of Sum there are

$$18 + 12 = 30$$

ways of choosing the subcommittee.

Example 1.5. How many factors does 64800 have?

Solution. $64800 = 2^5 \times 3^4 \times 5^2$. Thus any factor of 64800 is of the form $2^i \times 3^j \times 5^k$ where $0 \le i \le 5$, $0 \le j \le 4$ and $0 \le k \le 2$. So there are 6 choices for i, 5 for j and 3 for k. Hence 64800 has

$$6 \times 5 \times 3 = 90$$

factors.

Example 1.6. How many words with five letter are there?[3] How many of these are palindromes? How many of these contain

1. exactly two Cs?

[3] In this course a word is just a finite string of letters from a given alphabet.

2. at most two Cs?

Solution. There are 26 different possibilities for each letter. Thus by the Rule of Product there are 26^5 different words. In a five letter palindrome the last two letters have to be the same as the first two letters. Thus 26^3 of them are palindromes.

1. In a five letter word with exactly 2 Cs, there are 10 possible positions in the word for the Cs. The remaining 3 letters can be chosen in 25^3 ways. Thus by the Rule of Product 10×25^3 of the words contain exactly 2 Cs.

2. As in part (1), there are 25^5 five letter words containing no Cs, and 5×25^4 five letter words containing one C. Thus by the Rule of Sum there are

$$25^5 + 5 \times 25^4 + 10 \times 25^3$$

five letter words with at most 2 Cs.

Counting what you do not want

Look again at part (2) of the last example. We know there are 26^5 possible words. So the number of words containing at most two Cs is the same as

$$26^5 - (\text{the number of words containing three or more Cs}).$$

Now there are 10×25^2 words containing exactly 3 Cs, 5×25 words containing exactly 4 Cs and 1 word containing exactly 5 Cs. Thus there are $10 \times 25^2 + 5 \times 25 + 1$ words with 3 or more Cs. Hence there are $26^5 - (10 \times 25^2 + 5 \times 25 + 1)$ words containing at most two Cs.

Notice that we have counted what we do not want and subtracted this from the total number of possibilities. In this case it is no quicker to do the problem by this method, but sometimes this approach can save a lot of effort, as the following examples illustrate.

Example 1.7. How many five letter words are there with at least one letter repeated?

Solution. There are 26^5 different five letter words. Of these $26 \times 25 \times 24 \times 23 \times 22$ have no letter repeated. Thus there are

$$26^5 - 26 \times 25 \times 24 \times 23 \times 22$$

five letter words with at least one letter repeated.

Example 1.8. How many seven digit telephone numbers are there with at least one digit repeated?

Solution. There are 10^7 possible telephone numbers. Of these $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$ have no digit repeated. Thus there are

$$10^7 - 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 9395200$$

seven digit telephone numbers with at least one digit repeated.

1.2.2 Sampling

How many ways are there to select a sample of r elements from a set of n distinct elements? This seems like a straightforward question, but before we can try to answer it, we need to clarify some aspects of how the sampling is carried out, namely:

1. Does the order in which the elements are sampled matter?

2. Are elements allowed to occur more than once in the sample?

Example 1.9. How many ways are there of sampling two elements from the set $\{a, b, c\}$? The answer depends on how we do the sampling. The possibilities are listed in the following table.

order matters		order doesn't matter	
repetition	no repetition	repetition	no repetition
aa	ab	aa	ab
ab	ac	ab	ac
ac	ba	ac	bc
ba	bc	bb	
bb	ca	bc	
bc	cb	cc	
ca			
cb			
cc			
9 possibilities	6 possibilities	6 possibilities	3 possibilities

The answer to the two questions depends on what it is we are sampling:

Example 1.10. *Suppose a company makes five flavours of canned soup: tomato, mushroom, lentil, beef, and chicken. They want to package the soup in boxes with three different flavoured cans of soup in each box. How many different boxes can they produce?*

In this case it doesn't matter in which order we select the soup flavours to go in to each box, but we are not allowed to repeat a flavour.

My grandmother knits black socks with a coloured toe, a coloured heel, and a coloured stripe around the top. If she has red wool, yellow wool and green wool, how many different kinds of sock can she knit?

For this problem the order does matter, but repetition is allowed.

Exercise 1.11. For each of the scenarios described below, determine whether or not order is important and whether repetition is allowed.

What are we counting?	**Order matters?**	**Repetition allowed?**
How many seven digit telephone numbers are there?		
In how many ways can a football coach distribute one of eight shirts numbered 7 to 14 inclusive to each of his five forwards?		
How many ways are there to choose 12 identical cups available in white, green or blue?		
How many ways are there of selecting a subcommittee of 5 people from a committee of 16?		
I wish to place a bet on which horses will come first, second and third in a race with 10 horses. How many ways can I select the horses for my bet?		
A bakery makes cupcakes with 5 smarties on top. If smarties come in eight colours, how many different ways are there to pick the smarties that go on a cake?		
A library has 20 DVDs available for borrowing. Fred wishes to borrow 4 DVDs to watch on the weekend; in how many ways can he make his selection?		
The conductor of an orchestra is selecting the program of music for a concert. How many ways are there for her to do this if the orchestra knows how to play six pieces of music and she wants them to play 3 pieces in the concert?		
The boardgame Mastermind requires one player to fill four hidden holes with coloured pegs; the other player then has to guess which colour peg is in each hole. There is a large supply of pegs in six different colours –how many ways are there to guess which colours have been chosen?		
If there are 20 people in a swim squad, how many ways are there of choosing 4 swimmers for a medley relay? (The medley relay involves backstroke, breaststroke, butterfly and freestyle swimming, with a different swimmer performing each stroke.)		

There are four different cases we need to consider. We will now look separately at how to count the number of samples possible in each of these cases; the results are summarised in Table 1.1.

r-Sequences

An *r-sequence* is a sample for which order does matter, and repetition is allowed. For example, $[1, 3, 5, 3, 6]$ is a 5-sequence sampled from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, and $[3, 6, 5, 1, 3]$ is a different 5-sequence sampled from that set.

We can use the Rule of Product to determine the number of possible ways of sampling an r-sequence from a set of size n:
there are

$$\begin{array}{c} n \text{ choices for the first element in the sequence,} \\ n \text{ choices for the second element in the sequence,} \\ \vdots \\ n \text{ choices for the } r^{th} \text{ element in the sequence,} \end{array}$$

thus the total number of possible r-sequences is

$$\underbrace{n \times n \times \cdots \times n}_{r \text{ times}} = n^r.$$

Example 1.12. A multiple choice test has ten questions. In how many ways can it be answered if

1. each question has four possible answers?
2. the first five questions each have three possible answers and the last five questions each have six possible answers?

Solution.

1. This is just the number of 10-sequences from a set with 4 elements. Thus there are $4^{10} = 1048576$ ways of answering the test.
2. There are 3^5 ways of answering the first five questions and 6^5 ways of answering the last five questions. Thus, by the rule of Product, there are $3^5 \times 6^5 = 1889568$ ways of answering the test.

r-Permutations

An *r-permutation* is a sample in which order does matter, but repetition is not allowed (note that for this to be possible we require $r \leq n$). The number of possible r-permutations can also be determined using the Rule of Product. As in the case of r-sequences, there are n possible choices for the first element of the permutation. However, there are only $n - 1$ possibilities for the second element, as we are not allowed to repeat the element that has already been chosen for the first position. Similarly, there are

$$n-2 \text{ choices for the third element,}$$
$$\vdots$$
$$n-r+1 \text{ choices for the } r^{th} \text{ element,}$$

and so the total number of r-permutations is

$$n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}.$$

The number of r-permutations sampled from a set of size n is frequently denoted by the notation $P(n,r)$.

Example 1.13. How many six letter words can be formed from the letters of QUESTION, if no letter is to be repeated?

Solution. There are eight different letters in QUESTION, so $P(8,6) = 20160$ six letter words can be formed with no letter repeated.

r-Combinations

An *r-combination* is a sample for which order does not matter, and repetition is not allowed. If we have an r-combination we can turn it into an r-permutation by specifying an ordering for the elements; as the elements are all distinct there are $r!$ possible ways to do this. Thus the number of possible r-combinations is equal to $P(n,r)$ divided by $r!$, which is equal to

$$\frac{n!}{r!(n-r)!}.$$

Many different notations are used for this quantity, including $C(n,r)$, $\binom{n}{r}$, C_r^n and ${}^nC_r$. We will most frequently use the second of these notations in these notes.

Example 1.14. How many subsets of 5 integers can be chosen from the set $\{x \in \mathbb{Z} | 1 \leq x \leq 14\}$? How many of these subsets contain

1. the number 3?

2. at least one multiple of 3?

Solution. There are $\binom{14}{5} = 2002$ different possible subsets.

1. Since the number 3 has to be in the set, there are $\binom{13}{4} = 715$ ways of choosing the other four elements. Thus 715 subsets contain the number 3.

2. Ther multiples of 3 in the given set are 3,6,9 and 12. Thus there are $\binom{10}{5} = 252$ subsets containing no multiple of 3. Thus there are $2002 - 252 = 1750$ subsets containing at least one multiple of 3.

r-Multisets

Finally, an *r-multiset* is a sample in which order does not matter, and repetition is allowed. These are a little trickier to count. One way of doing this is to turn each r-multiset into a table with n columns. Each column represents one of the elements of the set being sampled, and for each element in the r-multiset, a star is placed in the column that represents the corresponding element of the set. So, for example, if the 9-multiset $x_1\ x_2\ x_2\ x_4\ x_4\ x_4\ x_4\ x_5\ x_5$ is sampled from the set $\{x_1, x_2, x_3, x_4, x_5\}$ then the resulting table would be as follows.

x_1	x_2	x_3	x_4	x_5
★	★★		★★★★	★★

The table can be thought of as a sequence of r stars and $n-1$ vertical bars, where the bars separate the various columns. For the above example, we obtain the following sequence.

$$\bigstar \mid \bigstar\bigstar \mid \quad \mid \bigstar\bigstar\bigstar\bigstar \mid \bigstar\bigstar$$

There are a total of $n+r-1$ elements in such a sequence, and each different table can be generated by choosing which r elements of the sequence will be occupied by the vertical bars, the rest being filled by stars. There are $\binom{n-1+r}{r}$ ways of doing this, and hence the total number of possible r-multisets is

$$\binom{n-1+r}{r}.$$

Example 1.15. A bag of fruit gums contains 15 sweets. There are four different flavours of fruit gum: orange, blackcurrant, lemon and strawberry. How many possible different bags are there? How many of these contain at least one fruit gum of each flavour?

Solution. The number of different bags is just the number of 15-multisets from 4 objects (the different flavours). Thus there are

$$\binom{4-1+15}{15} = \binom{18}{15} = 816$$

different bags.

If there is at least one fruit gum of each flavour then we can choose 11 of the sweets to be any of the four flavours. Thus there are

$$\binom{4-1+11}{11} = \binom{14}{11} = 364$$

different bags containing at least one fruit gum of each flavour.

Thus we have seen how to count the number of ways of sampling r elements out of n in each of the four possible cases. The results are summarised in Table 1.1.

	order matters	order does not matter
repetition allowed	**r-sequence**	**r-multiset**
	n^r	$\binom{n-1+r}{r}$
repetition not allowed	**r-permutation**	**r-combination**
	$P(n,r) = \frac{n!}{(n-r)!}$	$\binom{n}{r}$

Table 1.1: Summary of the different ways of sampling r objects from a set of n objects

Exercise 1.16.

1. How many 6 digit telephone numbers are there? How many of them have no repeated digits? How many of them contain at least two zeros?

2. A garden contains bushes with red, yellow, pink, and white roses. How many different bouquets of four roses can be selected? How many different bouquets are possible if there is only one white rose blossom in the entire garden?

3. How many 5 digit telephone numbers have no repeated digits? How many of these numbers have their digits in ascending order?

4. A class of 30 school students is electing members for the student-staff committee. In how many ways can they elect a president, secretary, and 3 committee members?

5. How many ways are there of putting 15 pennies into three piles (each with at least one penny) such that each pile contains an odd number of pennies?

6. A juggler can perform 9 different tricks. One of the tricks is extremely difficult, so he only ever performs it at most once during any routine, but all the other tricks he can repeat as many times as he likes. How many different routines of 6 tricks can he perform?

7. My (strange!) local cafe sells sandwiches with three fillings for £2. The fillings they offer are ham, jam, spam, lamb, cheese or peas. How many different types of sandwiches can they make if they refuse to put cheese and peas on the same sandwich?

8. A polynomial in n variables $x_1, x_2, \ldots, x_n$ has terms of the form $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where the degree of a term is $i_1 + i_2 + \cdots + i_n$. How many different terms of degree d are there? How many of these terms are divisible by $x_1x_2x_3$ (supposing $d \geq 3$)?

1.2.3 Distribution Problems

One technique that can be useful for transforming a counting problem into something that we know how to solve is to translate it into a problem about placing balls into boxes. A *distribution problem* asks *given r balls, in how many ways can we place them into n distinct boxes?* As was the case for sampling problems, before we can solve a distribution problem we first have to ask

1. Are the balls distinct?

2. Can we place more than one ball in each box?

The number of ways of distributing the balls in each of the four cases is given in the following table.

	distinct balls	identical balls
nonexclusive occupancy	n^r	$\binom{n-1+r}{r}$
exclusive occupancy ($r \leq n$)	$P(n,r)$	$\binom{n}{r}$

Example 1.17. How many ways can ten identical balls be distributed into five boxes? How many of these have

1. at least one ball in each box?

2. at least two empty boxes?

Solution. The ten balls can be distributed in

$$\binom{5-1+10}{10} = \binom{14}{10} = 1001$$

ways.

1. Put one ball in each of the boxes. There are five balls left to distribute and this can be done in $\binom{9}{5} = 126$ ways. Thus 126 of the distributions have at least one ball in each box.

2. We first count the number of distributions with exactly one empty box. There are 5 ways of choosing the empty box. Put one ball in each of the remaining four boxes. There are six balls left to distribute amongst the four boxes. This can be done in

$$\binom{4-1+6}{6} = \binom{9}{6} = 84$$

ways. Thus there are $5 \times 84 = 420$ distributions with exactly one empty box. Hence, $1001 - 126 - 420 = 455$ of the distributions have at least two empty boxes.

Example 1.18. How many ways are there to get from the point $(0,0)$ to the point $(7,4)$ by a sequence of moves of one unit in either the positive x or y direction?

Solution. Here we have to distribute 11 balls in 11 boxes, 7 marked x and 4 marked y, with exclusive occupancy. This is just the number of ways of distributing the balls marked x (or those marked y) amongst the 11 boxes. Thus there are $\binom{11}{7}$ (or $\binom{11}{4}$) ways to get from $(0,0)$ to $(7,4)$.

Example 1.19. Find the number of solutions in non-negative integers of

1. $X_1 + X_2 + X_3 = 10$;

2. $2X_1 + X_2 + X_3 = 10$;

3. $X_1 + X_2 + X_3 \leq 10$;

if $0 \le X_1, X_2, X_3 \le 10$.

Solution. Each of these integer equations can be interpreted as a problem of distributing balls into three boxes.

1. This is equivalent to distributing ten balls into three boxes. Thus there are $\binom{3-1+10}{10} = \binom{12}{10} = 66$ integer solutions in this case.

2. This equation can be rewritten as $Y_1 + X_2 + X_3 = 10$ where $Y_1 \in \{0, 2, 4, 6, 8, 10\}$. So to solve the problem we split into cases depending on the value of Y_1, the resulting problem in each case being just the number of ways of distributing $10 - Y_1$ balls amongst two boxes. The cases are enumerated in the following table.

Y_1	number of solutions	Y_1	number of solutions
0	$\binom{11}{10} = 11$	6	$\binom{5}{4} = 5$
2	$\binom{9}{8} = 9$	8	$\binom{3}{2} = 3$
4	$\binom{7}{6} = 7$	10	$\binom{1}{0} = 1$

Thus the total number of solutions is $11 + 9 + 7 + 5 + 3 + 1 = 36$.

3. There are two ways of interpreting this inequality as an integer equation.

Method 1 Each solution of the inequality is a solution of

$$X_1 + X_2 + X_3 = r$$

for some value of r between 0 and 10 inclusive. For each value of r, there are

$$\binom{3-1+r}{r} = \binom{r+2}{r} = \frac{1}{2}(r+1)(r+2)$$

integer solutions of the corresponding equation. Thus the inequality has

$$\begin{aligned}\sum_{r=0}^{10} \frac{1}{2}(r+1)(r+2) &= \frac{1}{2}\sum_{r=1}^{11} r(r+1) \\ &= \frac{1}{2}\left(\sum_{r=1}^{11} r^2 + \sum_{r=1}^{11} r\right) \\ &= \frac{1}{2}\left(\frac{1}{6} \times 11 \times 12 \times 23 + \frac{1}{2} \times 11 \times 12\right) \\ &= 286.\end{aligned}$$

integer solutions.

Method 2 For each solution of the inequality, if we put

$$X_4 = 10 - (X_1 + X_2 + X_3)$$

we get a solution of the integer equation

$$X_1 + X_2 + X_3 + X_4 = 10,$$

where $0 \le X_1, X_2, X_3, X_4 \le 10$. Thus the inequality has $\binom{4-1+10}{10} = \binom{13}{10} = 286$ integer solutions.

1.2.4 Binomial Coefficients

The numbers $\binom{n}{r}$ are frequently referred to as the *binomial coefficients*, because of their role in the binomial theorem.

Theorem 1.3 (The Binomial Theorem)**.**

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

Proof. By definition, $(x+y)^n = \overbrace{(x+y)(x+y)\cdots(x+y)}^{n \text{ times}}$. When we expand the brackets, each term of the form $x^r y^{n-r}$ arises from multiplying the x term in r of the brackets with the y term in $n-r$ of the brackets. There are $\binom{n}{r}$ ways of choosing the brackets from which the x term is selected, and hence the coefficient of $x^r y^{n-r}$ in the resulting sum is $\binom{n}{r}$. $\square$

The binomial theorem can be used to obtain many results about the binomial coefficients.

Example 1.20. Use the binomial theorem to show that

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Find an alternative proof of this result by considering the number of subsets of a set of size n.

Solution. The result follows immediately from the binomial theorem by considering $x = 1$ and $y = 1$.

Alternatively, we can provide a purely combinatorial proof as follows. For each subset of a set of n elements, every element is either in the subset, or it is not in the subset. Therefore by the rule of product, the total number of possible subsets is 2^n. Furthermore, the number of subsets containing r elements is equal to $\binom{n}{r}$ for each $r = 0, 1, \ldots, n$ and hence the total number of subsets is equal to $\sum_{r=0}^{n} \binom{n}{r}$, from which we deduce the desired result.

Exercise 1.21.

1. What is the coefficient of a^2b^3 in $(a+b)^5$?

2. What is the coefficient of x^7 in $(1+x)^{10}$?

3. Use the binomial theorem to show that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$.

4. (harder!) Use the binomial theorem to prove that

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}.$$

(Hint: consider the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$ and use the binomial theorem to find the coefficient of x^n on each side of the identity.)
Can you find a counting argument that gives the same result?

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Figure 1.1: Pascal's Triangle (row 0 up to row 7)

One place where the binomial coefficients occur is in *Pascal's triangle.* Row 0 of the triangle contains the number 1. Row n contains $n+1$ elements for $n \geq 0$, the first and last of which are 1. The remaining elements are generated by taking the r^{th} element of row n to be the sum of the $(r-1)^{th}$ and r^{th} elements of row $n-1$ for $1 \leq r \leq n-1$ and $n \geq 2$ (so, when laid out as in Figure 1.1, each entry is the sum of the two entries immediately above it).

Example 1.22. Show that

1. $\binom{n}{r} = \binom{n}{n-r}$
2. $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$
3. the r^{th} element of row $n+1$ of Pascal's triangle is given by $\binom{n+1}{r}$.

Solution.

1. This follows immediately from the formula for $\binom{n}{r}$, or can be shown by observing that the number of ways of choosing r elements from a set of size n is the same as the number of ways of choosing the $n-r$ elements that are left behind.

2. This can be proved by manipulating the formulas for the relevant binomial coefficients, although this is slightly more involved than in the previous example. Alternatively, observe that $\binom{n+1}{r}$ is the number of ways of choosing r elements from a set of $n+1$ elements. If we distinguish one of the elements of this set, then the total number of ways of choosing r elements is equal to the number of ways of choosing r elements other than the distinguished one (which is just $\binom{n}{r}$) plus the number of ways of choosing r elements one of which is the distinguished one (which is equivalent to choosing $r-1$ of the undistinguished elements, which can be done in $\binom{n}{r-1}$ ways).

3. Compare result (2) to the description of how Pascal's triangle is generated. This could be proved formally using induction.

Exercise 1.23. Simplify the following sum: $\sum_{r=0}^{n} \binom{n}{r} 2^r$.

1.2.5 The Pigeonhole Principle

The pigeonhole principle (also known as the *schubfachprinzip* if you want to sound fancy!) is a simple, yet surprisingly useful idea that can be stated as follows:

Theorem 1.4 (The Pigeonhole Principle)**.** *If $n+1$ or more objects are placed into n pigeonholes, then at least one hole contains more than one object.*

Example 1.24. If a university runs 4 courses, but there are only 3 lecturers, then at least one of the lecturers has to teach two (or more) courses. In fact, it may be the case that one of the lecturers is away on sabbatical and so the remaining lecturers each teach two courses. Or one lecturer may teach all four of the courses. In any case, no matter how the teaching load is assigned, some lecturer will end up teaching at least two of the courses.

There is also an extended form of this principle that can give more detailed information.

Theorem 1.5 (Extended Pigeonhole Principle)**.** *If n objects are placed into m pigeonholes, then at least one hole contains at least $\left\lceil \frac{n}{m} \right\rceil$ objects. Furthermore, there is at least one hole that contains at most $\left\lfloor \frac{n}{m} \right\rfloor$ objects*[4]*.*

Example 1.25. If 8 lollipops are distributed among 3 children, then there will be some child who receives at least $\left\lceil \frac{8}{3} \right\rceil = 3$ lollipops. Furthermore, there will be some other child who receives at most $\left\lfloor \frac{8}{3} \right\rfloor = 2$ lollipops.

Exercise 1.26. Alice has a drawer full of socks that are identical except that some are red and some are green. Alice asks Bob, who is colourblind, to go and fetch her a pair of socks from the drawer. How many socks must Bob bring back in order to guarantee that he has a matching pair?

Exercise 1.27. Suppose Bob makes n marks on a ruler so that the distances between marks are always a whole number of centimetres, and the distances between any two pairs of marks are different. Show that the distance between the first mark and the last mark is at least $\frac{n(n-1)}{2}$ centimetres.

Exercise 1.28. Show that for any function f mapping the set $\{0, 1, 2, \ldots, 47\}$ to the set $\{0, 1, 2, 3, 4\}$ it is possible to find ten numbers from the first set that are mapped to the same element of the second set.

Exercise 1.29. My local pet shop sells goldfish in four colours: orange, white, black and yellow.

1. How many ways are there for me to choose ten fish for my tank?

2. How many ways are there for me to choose ten fish if I want the number of black fish to be a (nonzero) multiple of three?

3. In the end I decided to buy fourteen fish, so that I could give four to my friend Fred. When I arrived home from the pet shop with the fish, Fred phoned to say that he didn't mind what colour fish I gave him, as long as they were all the same colour. Will I be able to satisfy his request?

[4]Recall that for any real number x, the expression $\lceil x \rceil$ (pronounced "ceiling of x") represents the smallest integer that is greater than or equal to x, *i.e.* the quantity you obtain by rounding x up to the nearest whole number. Similarly, $\lfloor x \rfloor$ (the "floor" of x) is the largest integer that is less than or equal to x.

1.2.6 The Inclusion-Exclusion Theorem

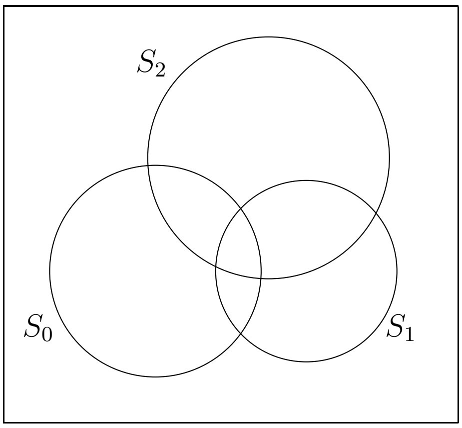
The inclusion-exclusion theorem allows us to express the number of elements in a union of sets in terms of the number of elements in the sets, and in the various intersections between the sets. Symbolically, it can be expressed as follows:

Theorem 1.6 (The Inclusion-Exclusion Theorem)**.**

$$\left|\bigcup_{i=0}^{n} S_i\right| = \sum_{i=0}^{n}|S_i| - \sum_{i,j=0}^{n}|S_i \cap S_j| + \sum_{i,j,k=0}^{n}|S_i \cap S_j \cap S_k| - \cdots + (-1)^n \left|\bigcap_{i=0}^{n} S_i\right|$$

That may look like a complicated expression, but the intuition behind it is relatively straightforward. The goal is to count the total number of elements that lie in at least one of the sets S_0 up to S_n. We start by adding the number of elements in each set, but then each element that lies in two of the sets has been counted twice and so we substract the number of elements in each of the pairwise intersections of the sets. Any elements that lie in three of the sets were counted three times initially, but have been subtracted three times as well, so we need to add back on the number of elements that lie in each intersection of three of the sets, then we proceed in a similar manner until we have accounted for the elements that lie in $4, 5, \ldots, n+1$ of the sets. It's called the inclusion-exclusion theorem because we include the number of elements in each set, then exclude the number of elements in pairs of sets, and so on.

Example 1.30. Consider the case of $n = 2$. (You should have encountered the cases with $n = 1$ and $n = 2$ in Algebra 1.)



Then the inclusion-exclusion theorem has the form

$$|S_0 \cup S_1 \cup S_2| = |S_0| + |S_1| + |S_2| - |S_0 \cap S_1| - |S_0 \cap S_2| - |S_1 \cap S_2| + |S_0 \cap S_1 \cap S_2|.$$

It is useful in solving problems like the following:
In a music class there is a total of 11 children who play the flute, 15 children who play the drums and 13 children who play the banjo. If 5 of the children play both flute and drums, 6 children play drums and banjo, 7 children play banjo and flute, and 2 children play all three instruments, how many children are there in the class (assuming that all the children play at least one of flute, drums, or banjo)?

Solution: If we let S_0 be the set of children who play flute, S_1 be the set of children who play the drums, and S_2 be the set of children who play the banjo, then the number we wish to calculate is precisely $|S_0 \cup S_1 \cup S_2|$, which we can obtain by plugging the given information into the above formula. We obtain the answer

$$11 + 15 + 13 - 5 - 6 - 7 + 2 = 23$$

children.

Example 1.31. Thirty-three people go to a restaurant that serves lasagne, salad, and chips. Three people decide not to eat anything, four people order all three dishes, 5 people order lasagne with chips, and 9 people order lasagne with salad. If there are 10 people who order chips, 18 who order lasagne, 19 who order salad, how many people have ordered both salad and chips?

Solution: Let S_0 be the set of people who order chips, S_1 be the set of people who order lasagne, and S_1 be the set of people who order salad. We observe that since three people do not eat we have that $|S_0 \cup S_1 \cup S_2| = 33 - 3 = 30$. Then the inclusion-exclusion principle tells us

$$30 = 10 + 18 + 19 - 5 - 9 - |S_1 \cap S_2| + 4.$$

From this we conclude that $|S_1 \cap S_2| = 7$, *i.e.* 7 people order both salad and chips.

Example 1.32. How many numbers between 1 and 100 are coprime to 30? (Two numbers are said to be coprime if their greatest common divisor is 1.)

Solution. The prime factors of 30 are 2,3 and 5. There are 100 numbers between 1 and 100. Fifty of them are divisible by 2, 33 of them are divisible by 3, and 20 of them are divisible by 5. The numbers divisible by both 2 and 3 are precisely those divisible by 6; there are 16 of these. Similarly, there are 6 divisible by both 3 and 5, and 10 divisible by both 2 and 5. Finally, there are 3 divisible by 2,3 and 5. Therefore the number of numbers between 1 and 100 that are divisible by at least one out of 2,3 or 5 is

$$50 + 33 + 20 - 16 - 6 - 10 + 3 = 74.$$

From this we conclude that there are $100 - 74 = 26$ numbers between 1 and 100 that are coprime to 30.

Exercise 1.33. The original version of the children's game Guess Who involves a board with pictures of 24 people's faces. Five of them are women, five of them wear hats, and five of them have glasses. Claire is the only woman to wear glasses, and she also wears a hat. Maria is the only other woman who has a hat. None of the men has both a hat and glasses. How many men on the board have neither hats nor glasses?

For completeness, we provide a proof of the inclusion-exclusion theorem, which involves an application of the binomial theorem.

Proof of the Inclusion-Exclusion Theorem. Suppose an element is contained in t of the sets. We will consider its contribution to each of the summation terms in the righthand side of the statement of the theorem. The r^{th} summation counts the number of times each element occurs in the intersection of r sets. An object that lies in a total of t sets can be considered to lie in the intersection of r sets in $\binom{t}{r}$ different ways. Therefore the overall amount that that particular element counts to the total is $\sum_{r=1}^{t}(-1)^{r+1}\binom{t}{r}$. We observe that $\sum_{r=1}^{t}(-1)^{r+1}\binom{t}{r} = 1 - \sum_{r=0}^{t}(-1)^{r}\binom{t}{r}$ and by the binomial theorem, this is equal to $1-(1-1)^t = 1$. Hence each item is counted precisely once, and the theorem holds.

Further Exercises

Exercise 1.34. Simplify the following sums:

1. $0.2 + 0.04 + 0.008 + 0.0016 + \cdots + 0.0000001024$
2. $\sum_{r=1}^{n}(r+2)^3$

Exercise 1.35.

1. How many five letter words can be made from the 26 letters of the English alphabet?
2. How many of these contain the letter j three times?
3. How many words can be made if it is forbidden to have four consecutive vowels? (The vowels in the English alphabet are a, e, i, o and u.)

Exercise 1.36. How many integers between 0 and 250 inclusive are neither squares, cubes, nor divisible by 6?

Exercise 1.37. A physical education teacher is buying new balls for her school. The supplier stocks basketballs, soccer balls, netballs and volleyball balls. How many ways are there for the teacher to buy 21 balls if:

1. there are no further conditions on the balls purchased?
2. she wants to purchase at most five soccer balls?
3. she wants to purchase between two and five basketballs, at least three netballs, at least one soccer ball, and at most three volleyball balls?
4. During a physical education lesson, the children are split into six groups, and each group is given one ball to play with. Is it possible to choose a single sport for all the groups to play so that each group can have an appropriate ball for that sport?

Exercise 1.38. Evaluate the following sums, showing relevant working:

1. $\sum_{r=1}^{50} 2^{-r}$
2. $\sum_{r=1}^{50} \binom{r}{2}$
3. $\sum_{r=1}^{50} \binom{50}{r} 2^{-r}$

Exercise 1.39. A company has 40 employees, 17 of whom are female.

1. How many ways are there of choosing 10 employees to form a committee?
2. How many ways are there of choosing 10 employees to form a committee if the committee must contain at least two women?
3. After the committee has been formed, each of the 10 members selects a person who is not on the committee to be their partner in the company's annual bridge tournament. How many possible outcomes are there if we assume that each employee is the partner of at most one committee member?

Exercise 1.40. I have written down a set of 4 distinct positive integers.

1. Prove that it is possible to find two numbers in my set whose difference is divisible by 3.
2. If all my numbers are less than 100, prove that it is possible to find two numbers in my set whose difference is less than 33.

Exercise 1.41. A carpenter has built 15 identical tables. Her assistants Pat, Terry and Chris will be delivering them. How many ways are there for her to divide the tables among the assistants for delivery if

1. there are no further conditions on the number of tables each assistant delivers?
2. she wants to ensure that Terry and Chris each deliver the same number of tables?
3. she wants Chris to deliver at most 5 tables, Terry to deliver between 2 and 7 tables, and Pat to deliver at least one table?

Learning Outcomes

After completing this chapter and the related problems you should be able to:

- know expressions for the r^{th} term and the sum of the first n terms of an arithmetic progression and a geometric progression;
- be able to express $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ in terms of n;
- understand the Rule of Sum and the Rule of Product and know how to apply them;
- know the different ways of sampling r elements from a set of n elements and, in each case, be able to express the number of such samples in terms of r and n;
- understand the connection between sampling and distribution problems;
- be able to apply your knowledge to solve sequence and counting problems;
- be able to state the binomial theorem and use it directly to compute coefficients of polynomials of the form $(x+y)^n$;
- be able to state and apply both the pigeonhole principle and the extended pigeonhole principle;
- be able to understand and apply the inclusion-exclusion theorem.