

# Algebra 2 Assignment 1

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1. (a) Let  $\alpha$  represent a clockwise rotation of  $\frac{\pi}{2}$  about the origin and  $\beta$  represent a particular reflection. Now the Cayley table of  $\text{Dih}(8)$  can be written

	$e$	$\alpha$	$\alpha^2$	$\alpha^3$	$\beta$	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$
$e$	$e$	$\alpha$	$\alpha^2$	$\alpha^3$	$\beta$	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$
$\alpha$	$\alpha$	$\alpha^2$	$\alpha^3$	$e$	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\beta$
$\alpha^2$	$\alpha^2$	$\alpha^3$	$e$	$\alpha$	$\alpha^2\beta$	$\alpha^3\beta$	$\beta$	$\alpha\beta$
$\alpha^3$	$\alpha^3$	$e$	$\alpha$	$\alpha^2$	$\alpha^3\beta$	$\beta$	$\alpha\beta$	$\alpha^2\beta$
$\beta$	$\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	$e$	$\alpha^3$	$\alpha^2$	$\alpha$
$\alpha\beta$	$\alpha\beta$	$\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha$	$e$	$\alpha^3$	$\alpha^2$
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	$\beta$	$\alpha^3\beta$	$\alpha^2$	$\alpha$	$e$	$\alpha^3$
$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	$\beta$	$\alpha^3$	$\alpha^2$	$\alpha$	$e$

- (b) The set of elements that commute in  $\text{Dih}(8)$  is  $\{\alpha^2, e\}$  this can be observed in the Cayley table above by noticing that for each of these elements, both the row and column for that element contain the same elements in the same order.
- (c) The subgroups of  $\text{Dih}(8)$  are the trivial and identity subgroups  $\{e\}$  and  $\text{Dih}(8)$ , the four 2-subgroups formed by every reflecting element and the identity,  $\{e, \alpha^n\beta\}$  for  $0 \leq n \leq 3$ , the commutative 2-subgroup  $\{e, \alpha^2\}$ , the 4-subgroup containing the identity and all rotation elements  $\{e, \alpha, \alpha^2, \alpha^3\}$  and finally the 4-subgroup containing  $\{e, \alpha^2, \alpha\beta, \alpha^3\beta\}$ .

By Lagrange, the order of any subgroup  $H \leq \text{Dih}(8)$ ,  $|H|$  must divide  $|\text{Dih}(8)|$ . So there aren't any 3- or 5-subgroups because neither 3 or 5 divide 8.

Now, a subgroup must use a subset of  $\text{Dih}(8)$  and we needn't consider subsets of size 3 or 5.

There can only be one 1-subgroup,  $\{e\}$ , so there aren't any more of those.

There aren't any 2-subgroups beyond those already mentioned, because none of the non-reflecting 2-subsets  $\{e, \alpha^n\}$  with  $1 \leq n \leq 3$  have closure.

There aren't any more 4-subgroups because (apart from the last subgroup identified above) any subgroup that has a combination of reflections and rotations doesn't have closure.

There can only be one 8-subgroup,  $\text{Dih}(8)$ , so there aren't any more of those either.

2. Let  $k, l \in K$  and  $1_K$  be the identity element of  $K$ . Since  $K \leq G$ , we know

$$k, l, kl \in G.$$

Because  $f : G \rightarrow H$ , certainly  $f(k), f(l), f(kl) \in H$  and  $f(K) \subseteq H$ .

Since  $f$  is a homomorphism, we know that  $f(ab) = f(a)f(b)$  and therefore

$$f(k)f(l) \in H, \quad (1)$$

and, if  $1_H$  is the identity element of  $H$ ,

$$f(1_K) = 1_H. \quad (2)$$

The fact that  $f$  is a homomorphism also tells us that  $f(a^{-1}) = f(a)^{-1}$  and since  $k^{-1} \in K$  it is also true that

$$f(k)^{-1} \in H. \quad (3)$$

Due to the facts that  $K \leq G$  and that  $f$  is a homomorphism, it is also true that if  $j \in K$ ,

$$f((jk)l) = f(jk)f(l) = f(j)f(k)f(l) = f(j)f(kl) = f(j(kl)). \quad (4)$$

Therefore  $f(K)$  is closed (1), has an identity element  $1_H$  (2), has inverses (3) and is associative (4). Hence,  $f(K) \leq H$ .

3. (a)  $V_4 = \{e, a, b, c\}$  such that

$$ab = ba = c,$$

$$ac = ca = b,$$

$$bc = cb = a \text{ and}$$

$$a^2 = b^2 = c^2 = e.$$

Let  $f$  be an automorphism on  $V_4$ . Because  $f$  is a homomorphism,  $f(e) = e$ , that is,  $f$  fixes  $e$ .

Because of this and because of the definition of  $V_4$ , any permutation on  $\{a, b, c\}$  is an automorphism of which there are  $|\{a, b, c\}|! = 6$ .

- (b) There is clearly one automorphism on  $\mathbb{Z}_4$ , that is the identity automorphism  $\theta : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4$  where  $\theta(n) = n$  for all  $n \in \mathbb{Z}_4$ . There is also the automorphism  $\vartheta : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4$ , which for  $n \in \mathbb{Z}_4$

$$\vartheta(n) = \begin{cases} \vartheta(0) = 0 \\ \vartheta(1) = 3 \\ \vartheta(2) = 2 \\ \vartheta(3) = 1 \end{cases}$$