Chapter 4

Basic Graph Theory

4.1 Definitions

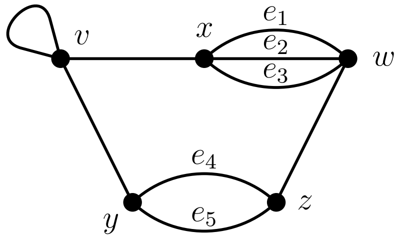
Definition 4.1. A *graph* consists of a finite set V of *vertices*, together with a collection E of (unordered) pairs of vertices, which we refer to as *edges*.

We will generally use the notation n to represent the number of vertices of a graph, and m to represent the number of edges. If u and v are vertices, then the edge $\{u, v\}$ is said to *join* u and v.

Definition 4.2. An edge that is of the form $\{u, u\}$ for some $u \in V$ is called a *loop*. If there are two vertices that are joined by more than one edge, the graph is said to have *multiple edges*. A graph with no loops or multiple edges is said to be a *simple graph.*

We will mainly be interested in simple graphs in this course.

Example 4.1. Here is a depiction of the graph $G = (V, E)$ where $V = \{v, w, x, y, z\}$ and $E = \{\{v, v\}, \{v, x\}, \{v, y\}, \{w, x\}, \{w, x\}, \{w, x\}, \{w, z\}, \{y, z\}, \{y, z\}\}$.



For this graph, we have $n = 5$ and $m = 9$. There is a loop at vertex v, and multiple edges connecting vertices y and z, and vertices x and w.

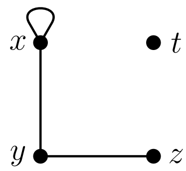
Definition 4.3. Two vertices u and v are said to be *adjacent* if they are joined by an edge $e = \{u, v\} \in E$. The edge e is said to be *incident* with the vertices v and u.

So, for the graph depicted in Example 4.1, the vertices v and x are adjacent, but the vertices y and w are not. The edge e_1 is incident with the vertices x and w, as are edges e_2 and e_3.

Definition 4.4. The *degree* of a vertex v is the number of edges that are incident with v, where each loop $\{v, v\}$ is counted twice.

Various notations are used for the degree of v, including $\deg v$, $d(v)$, $\delta(v)$ or $\rho(v)$. In these notes we will tend to use $d(v)$. The *degree sequence* of a graph is the list of the degrees of all the vertices in nondecreasing order. For example, the degree sequence of the graph from Example 4.1 is $(3, 3, 4, 4, 4)$. A vertex of degree 0 is referred to as an *isolated vertex.* A vertex of degree 1 is said to be an *endpoint.*

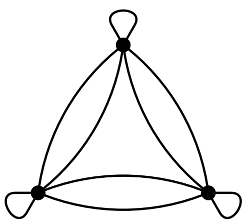
Example 4.2.



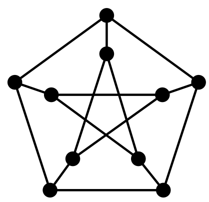
In this graph $d(x) = 3$, $d(y) = 2$, $d(z) = 1$ (so z is an endpoint), and $d(t) = 0$ (so t is an isolated vertex).

Definition 4.5. A graph G is said to be *regular of degree* r, if every vertex of G has degree r.

Example 4.3.



regular of degree 6



regular of degree 3 (Petersen's graph)

4.1.1 Digraphs, Weighted Graphs and Networks

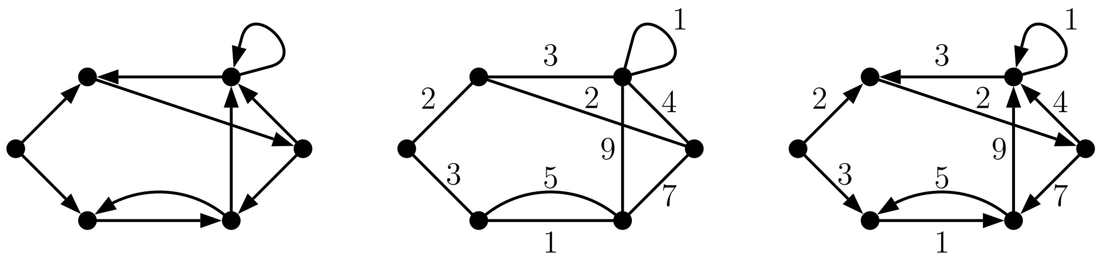
Sometimes we wish to study graphs to which extra information has been added:

Definition 4.6. A *digraph* is a graph in which each of the edges has a specified direction. In this case the edges are called *arcs.*

A *weighted graph* is a graph in which each of the edges has been assigned a real number, the *weight* of the edge.

A *network* is a weighted digraph.

Example 4.4.



digraph weighted graph network

Definition 4.7. Let v be a vertex of a digraph G. The *outdegree* of v is the number of arcs of G that are incident with v and oriented away from v. The *indegree* of v is the number of arcs of v that are incident with v and oriented towards v.

Example 4.5. In the digraph depicted in the previous example, the rightmost vertex has outdegree 2 and indegree 1. The leftmost vertex has outdegree 2 and indegree 0.

We will consider these structures in greater detail in later chapters.

4.1.2 The Adjacency Matrix of a Graph

The structure of a graph can be recorded in a convenient way by the use of an *adjacency matrix*:

Definition 4.8. Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$. The *adjacency matrix* of G is the $n \times n$ matrix whose ij^{th} entry is the number of edges joining v_i and v_j (where a loop $\{v_i, v_i\}$ counts twice towards the ii^{th} entry).

Example 4.6. Consider again the graph G from Example 4.1. Setting $v_1 = v, v_2 = w \ldots, v_5 = z$ we have that the adjacency matrix for this graph is

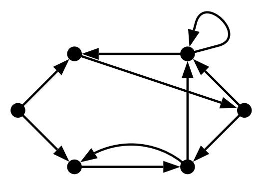
$$\begin{pmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \end{pmatrix}$$

There is a one-to-one relationship between (labelled) graphs and their adjacency matrices. Several properties of the adjacency matrix can be deduced directly by considering the properties of the corresponding graph:

- The adjacency matrix of a graph is a symmetric matrix.
- The entries in the adjacency matrix of a simple graph are all 0 or 1, and the entries on the diagonal are all 0.
- The sum of the entries in the i^{th} row or column of the adjacency matrix of a graph is the degree of vertex v_i.

The adjacency matrix of a digraph with n vertices is defined to be the $n \times n$ matrix whose ij^{th} entry is the number of arcs of the digraph that go from vertex v_i to vertex v_j.

Example 4.7.



If we label the vertices of this digraph $v_1, v_2, \ldots, v_6$ starting with the leftmost vertex and proceeding clockwise then we obtain the following adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We note that the adjacency matrix of a digraph is not symmetric in general. The sum of the entries in the i^{th} row gives the outdegree of vertex v_i, whereas the sum of the entries in the i^{th} column gives the indegree of v_i.

4.2 Basic Properties and Isomorphism

We now look at some properties common to all graphs, and we consider what it means for two graphs to be "the same".

Lemma 4.1 (The Handshaking Lemma)**.** *In any graph $G = (V, E)$, the sum of the degrees of the vertices of G is equal to twice the number of edges:*

$$\sum_{v \in V} d(v) = 2|E|.$$

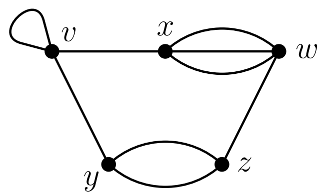
Proof. In summing the degrees, each loop is counted twice by definition; every other edge $\{u, v\}$, $u \neq v$, is counted twice, once in $d(u)$ and once in $d(v)$. ☐

This lemma gets its name from the fact that if there are n people in a room and some of them shake hands, the total number of hands that are shaken is even, as every handshake involves two hands. This scenario can be translated into a graph by representing each person by a node and connecting two nodes if the corresponding people shake hands. The following result follows directly from the lemma:

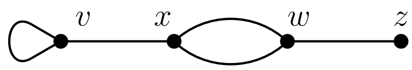
Corollary 4.2. *In any graph there is an even number of vertices with odd degree.*

Definition 4.9. A *subgraph* of a graph G is a graph whose vertices and edges are all vertices and edges in G. A subgraph of G is said to be a *spanning subgraph* if it contains all the vertices of G.

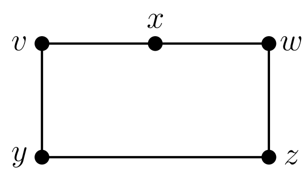
Example 4.8. Let G be the graph



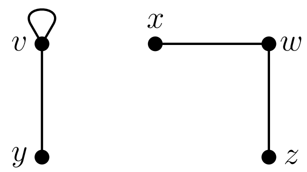
Then



is a subgraph of G, and

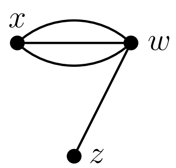


is a spanning subgraph of G, as is

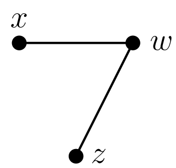


Definition 4.10. A subgraph $H = (U, F)$ of a graph $G = (V, E)$ is said to be an *induced subgraph* if for any $x, y \in U$ with $\{x, y\} \in E$ then $\{x, y\} \in F$. That is, F consists of the set of all edges of G that join vertices in the set U.

Example 4.9.



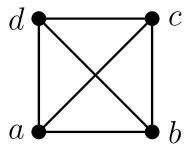
H_1

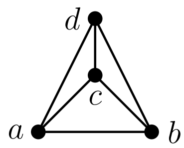


H_2

H_1 is an induced subgraph of the graph G of the previous example, but H_2 is not an induced subgraph, as it is missing two of the edges between x and w that occur in G.

What does it mean for two graphs to be the same? By moving around the vertices, we can redraw a graph so that it looks quite different:





However, the properties of the graph in which we are interested are not affected by how it is represented; rather, we are particularly concerned with the overall structure in terms of which vertices are adjacent to each other. This is captured in the following definition.

Definition 4.11. Two graphs G and H are said to be *isomorphic* if

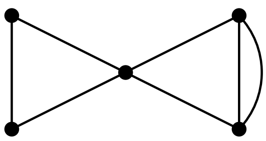
1. G and H have the same number n of vertices;
2. The vertices of G and H can be labelled[1] with the integers $1, 2, \ldots, n$ such that for all $i, j \in \{1, 2, \ldots, n\}$ the vertices i and j are joined by exactly d edges in G if and only if they are joined by exactly d edges in H.

The term "isomorphic" occurs in many different places in mathematics, describing mathematical objects that have the same structure. Two graphs that are isomorphic have essentially the same properties. In particular,

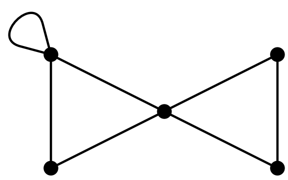
Theorem 4.3. *If $G = (V, E)$ and $H = (U, F)$ are isomorphic, then*

1. $|G| = |H|$
2. *If G has s vertices of degree r, then H has s vertices of degree r.*
3. *If G is simple then H is simple.*
4. *If K is a subgraph of G, then H contains a subgraph isomorphic to K.*

Exercise 4.10.



G



H

G and H are not isomorphic because:

1.

2.

[1]'labelled' means there are no repeats

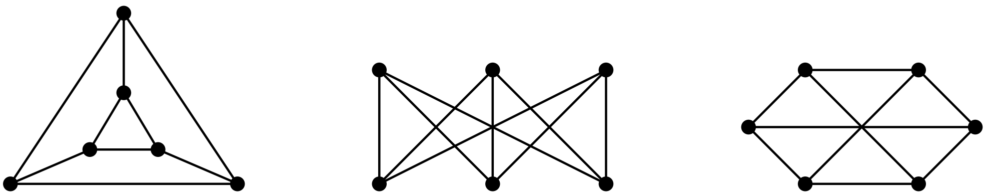
3.

4.

Example 4.11.

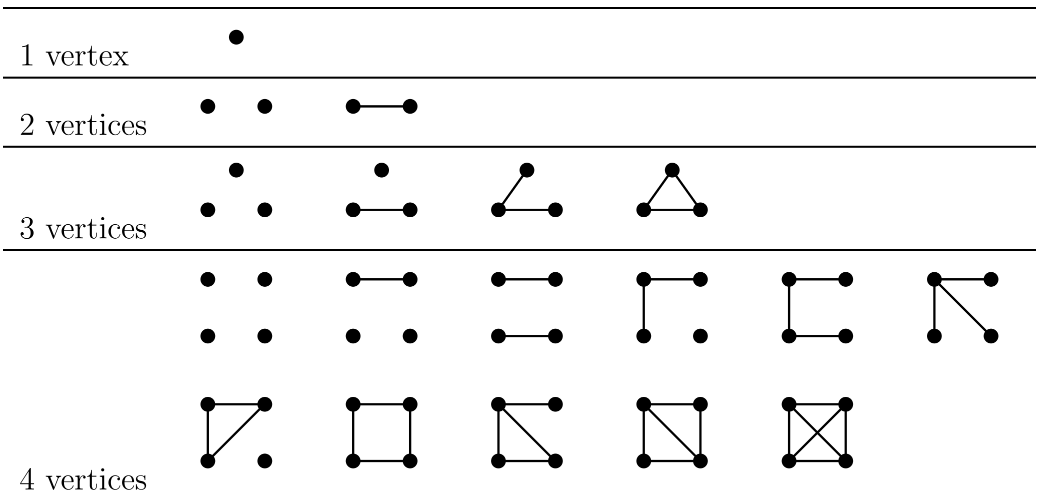
These two graphs are isomorphic.

Exercise 4.12. Which (if any) of the following graphs are isomorphic?



G H K

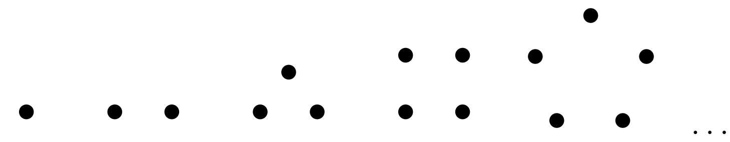
Example 4.13. Here is a list of all the simple graphs with 1, 2, 3 or 4 vertices, up to isomorphism.



4.3 Some Families of Graphs

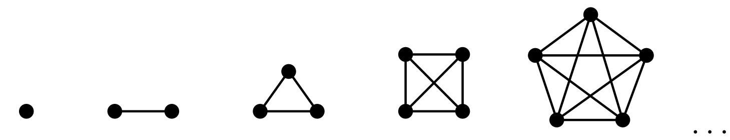
Here we mention some important families of graphs.

Null Graphs A graph with n vertices but no edges is called a *null graph*, and is denoted by N_n. The graph N_n is regular of degree 0.



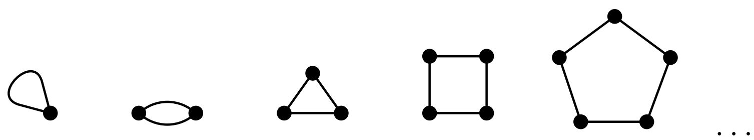
N_1 N_2 N_3 N_4 N_5

Complete Graphs A simple graph with n vertices in which every pair of vertices is joined by an edge is called a *complete graph*, denoted by K_n. The graph K_n has $\binom{n}{2} = \frac{1}{2}n(n-1)$ edges, and is regular of degree $n-1$.



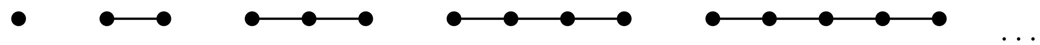
K_1 K_2 K_3 K_4 K_5

Cycles The graphs depicted here are known as n-cycles, denoted by C_n. The graph C_n has n edges, and is regular of degree 2. C_3 is called a *triangle*.



C_1 C_2 C_3 C_4 C_5

Paths A *path* P_n is obtained by removing an edge from an n-cycle.



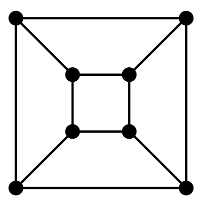
P_1 P_2 P_3 P_4 P_5

Bipartite Graphs A graph $G = (V, E)$ is said to be *bipartite* if the set of vertices can be split into two nonempty subsets V_1 and V_2 such that

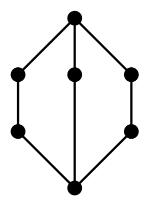
1. V_1 and V_2 have no vertices in common.
2. Each edge of G joins a vertex in V_1 to a vertex in V_2.

We often denote a bipartite graph by $G = (V_1, V_2, E)$.

Example 4.14.



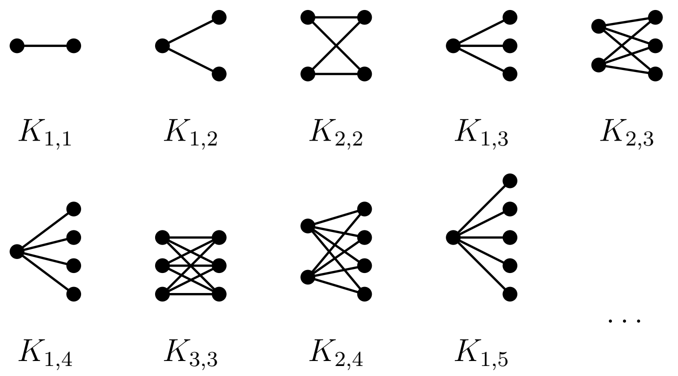
G



H

G is bipartite, but H is not bipartite.

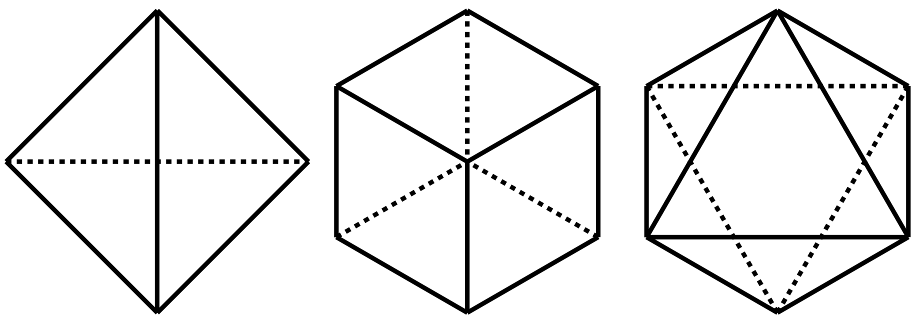
A graph $G = (V_1, V_2, E)$ is a *complete bipartite graph* if G is simple and every vertex of V_1 is adjacent to every vertex of V_2. If $|V_1| = m$ and $|V_2| = n$ then the complete bipartite graph $G = (V_1, V_2, E)$ is denoted by $K_{m,n}$.



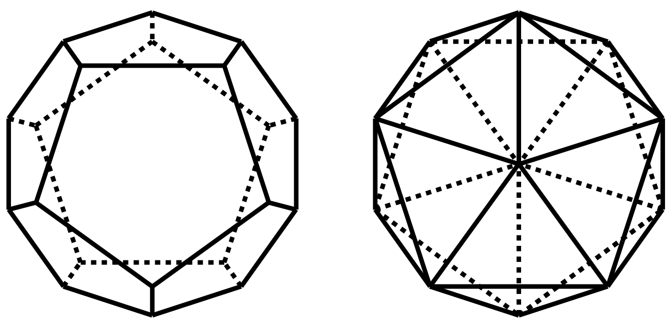
$K_{1,n}$ is called a *star* graph.

Note that $K_{m,n}$ is isomorphic to $K_{n,m}$. The graph $K_{m,n}$ has mn edges, and $K_{n,n}$ is regular of degree n.

The Platonic Graphs The following five polyhedra are known as the *platonic solids*. They are convex polyhedra with faces that are regular polygons, and with the same number of faces surrounding each vertex.

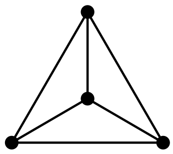


tetrahedron cube octahedron

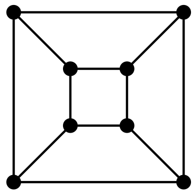


dodecahedron icosahedron

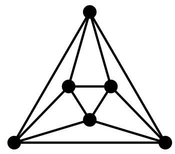
We can construct a graph from each of these polyhedra (or indeed, from any polyhedron) by associating a vertex of the graph with each vertex of the polyhedron, and joining two vertices of the graph if the corresponding vertices of the polyhedron are connected by an edge of the polyhedron. The resulting graphs are known as the *Platonic graphs*.



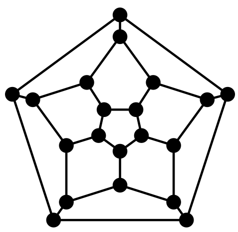
tetrahedron
(regular of degree 3)



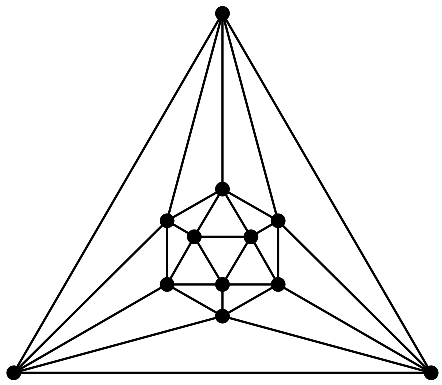
cube
(regular of degree 3)



octahedron
(regular of degree 4)



dodecahedron
(regular of degree 3)



icosahedron
(regular of degree 5)

The symmetry of the Platonic solids means that the resulting graphs are all regular.

4.4 Connectedness

Definition 4.12. Let $G = (V, E)$ be a graph. A *walk* is a sequence

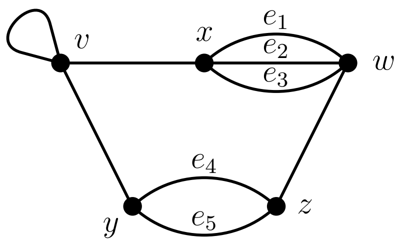
$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{p-1}, e_p, v_p$$

such that $v_i \in V$, $e_i \in E$ and $e_i = \{v_{i-1}, v_i\}$. The vertices v_0 and v_p are known as the *endpoints* of the walk, and the walk is said to be *closed* if $v_0 = v_p$.

A walk is known as a *trail* if it contains no repeated edges, and a *path* if it contains no repeated vertices. A *cycle* is a closed trail in which all of the intermediate vertices $v_1, v_2, \ldots, v_{p-1}$ are different. The number p is called the *length* of the walk, trail, path or cycle.

Note that in a simple graph we can omit the edges in the description of a walk, trail, path or cycle, since they are uniquely determined by the incident vertices.

Example 4.15.



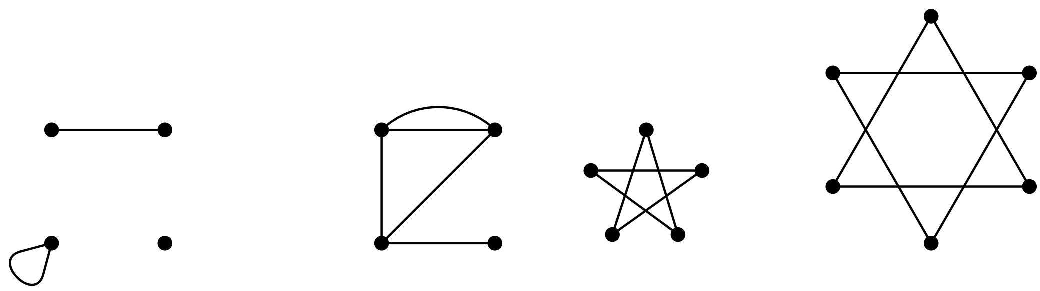
A walk: $v, x, (e_1), w, (e_2), x, v, y, (e_4), z$ (The edge $\{v, x\}$ is repeated.)
A trail: $v, v, x, (e_2), w, (e_1), x, (e_3), w, z$ (No edges repeated)
A path: $v, x, (e_3), w, z$ (No vertices repeated)
A closed trail: $v, v, x, (e_1), w, (e_2), x, (e_3), w, z, (e_5), y, v$.
A cycle: $v, y, (e_4), z, w, (e_1), x, v$

Definition 4.13. A graph is *connected* if for every pair of vertices u and v there is a walk with u and v as the endpoints. A graph is *disconnected* if it is not connected.

Definition 4.14. A *component* of a graph is a maximal connected induced subgraph.

Each vertex v of a graph is contained in a single component, which consists of all vertices and edges of the graph that can be reached by walks starting at v. If a graph is connected then it has a single component.

Example 4.16.



disconnected, 3 components connected connected disconnected, 2 components

Theorem 4.4. *Let G be a simple graph with n vertices, m edges and k components. Then*

$$n - k \leq m \leq \frac{1}{2}(n-k)(n-k+1).$$

Proof. Label the components of the graph $C_1, C_2, \ldots, C_k$, and denote the number of vertices in component C_i by n_i. As G is a simple graph, the maximum number of edges possible in component C_i is $\frac{1}{2}n_i(n_i - 1)$. Thus we have $m \leq \sum_{i=1}^{k} \frac{1}{2}n_i(n_i - 1)$. We observe that this quantity is maximised if $k - 1$ of the components consist of a single vertex, and the remaining component has $n - k + 1$ vertices; this gives the upper bound of the theorem.

To obtain the lower bound we note that for component C_i to be connected, it must contain at least $n_i - 1$ edges. Hence we have that $\sum_{i=1}^{k}(n_i - 1) \leq m$. As $\sum_{i=1}^{k} n_i = n$ by definition, the lower bound of the theorem follows directly. $\square$

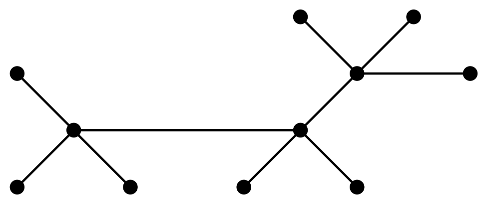
Corollary 4.5. *If G is a simple connected graph with n vertices and m edges then*

$$n-1 \leq m \leq \frac{1}{2}n(n-1).$$

Corollary 4.6. *Any simple graph with n vertices and more than $\frac{1}{2}(n-1)(n-2)$ edges is connected.*

Definition 4.15. A simple connected graph with n vertices and $n-1$ edges is known as a *tree*.

Example 4.17.

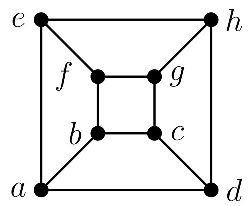


Trees are precisely those simple connected graphs that contain no cycles. We will devote considerable attention to them in Chapter 5.

4.4.1 Counting Walks in a Graph

Given a graph $G = (V, E)$, one question we might ask is: *how many walks of length k are there from vertex $v \in V$ to vertex $w \in V$?*

Example 4.18.



How many walks of length 3 are there from vertex a to vertex g?
Solution. There are six such walks:

$$a, b, f, g$$
$$a, b, c, g$$
$$a, e, f, g$$
$$a, e, h, g$$
$$a, d, c, g$$
$$a, d, h, g$$

It turns out that this problem can be easily solved with the use of the adjacency matrix:

Theorem 4.7. *Let M be the adjacency matrix of a graph G. Then the number of walks of length k from vertex v_i to vertex v_j is given by the ij^{th} entry of the matrix M^k.*

Proof. We can prove this result by induction on k. We know that the ij^{th} entry of the adjacency matrix gives the number of paths of length 1 from vertex v_i to v_j, so the result holds for $k = 1$.

Assume that the result is true for $k = r$, *i.e.* that the ij^{th} entry of M^r gives the number of walks of length r from vertex v_i to vertex v_j.

To obtain a walk of length $r+1$ from vertex v_i to vertex v_j, we simply take a walk of length r from v_i to some vertex v_l that is adjacent to v_j, then complete the length $r + 1$ walk by the addition of an edge joining v_l and v_j. Thus, the number of walks of length $r + 1$ from v_i to v_j is given by

$$\begin{aligned}
&\sum_{l=1}^{n}(\text{number of walks of length } r \text{ from } v_i \text{ to } v_l) \times (\text{number of edges between } v_l \text{ and } v_j) \\
&\qquad = \sum_{l=1}^{n} M_{il}^{r} M_{lj} \\
&\qquad = (M^r \cdot M)_{ij} \\
&\qquad = M_{ij}^{r+1}.
\end{aligned}$$

Hence the result is true for all integers $k \geq 1$. □

Example 4.19. The graph of the previous example has the adjacency matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

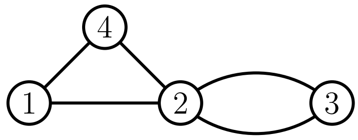
To count walks of length 3 we calculate

$$M^3 = \begin{pmatrix} 0 & 7 & 0 & 7 & 7 & 0 & 6 & 0 \\ 7 & 0 & 7 & 0 & 0 & 7 & 0 & 6 \\ 0 & 7 & 0 & 7 & 6 & 0 & 7 & 0 \\ 7 & 0 & 7 & 0 & 0 & 6 & 0 & 7 \\ 7 & 0 & 6 & 0 & 0 & 7 & 0 & 7 \\ 0 & 7 & 0 & 6 & 7 & 0 & 7 & 0 \\ 6 & 0 & 7 & 0 & 0 & 7 & 0 & 7 \\ 0 & 6 & 0 & 7 & 7 & 0 & 7 & 0 \end{pmatrix}$$

From this we can read off directly that the number of walks of length 3 from vertex a to vertex g is 6.

Example 4.20. In the movie Good Will Hunting, a professor at MIT leaves a problem written on a blackboard as a challenge to his students. It is solved by Will Hunting (played by Matt Damon), who is the janitor at MIT and happens to be a maths genius.

The problem he solves is the following:



1. Find the adjacency matrix A of the graph G.
2. Find the matrix giving the number of 3 step walks in G.
3. Find the generating function for walks from point i to j.
4. Find the generating function for walks from points 1 to 3.

Solution.

1. $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$
2. $A^3 = \begin{pmatrix} 2 & 7 & 2 & 3 \\ 7 & 2 & 12 & 7 \\ 2 & 12 & 0 & 2 \\ 3 & 7 & 2 & 2 \end{pmatrix}$
3. We know that the number of walks of length r from point i to point j is given by the ij^{th} entry of A^r. Thus the generating function is

$$\begin{aligned} \sum_{r=0}^{\infty} (A^r)_{ij} x^r &= \sum_{r=0}^{\infty} (xA)^r_{ij} \\ &= \left(\sum_{r=0}^{\infty} (xA)^r \right)_{ij} \end{aligned}$$

We are familiar with the identity $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ -it turns out that an analogous rule applies for matrices, so $\sum_{r=0}^{\infty} A^r = (\mathbb{I} - A)^{-1}$, where $\mathbb{I}$ is the 4×4 identity matrix. So our generating function can be written

$$(\mathbb{I} - xA)^{-1}_{ij}.$$

4. From the result to the previous part, we know the answer to this problem is the element in row 1, column 3 of the inverse of the matrix $(\mathbb{I} - xA)$. That is, the element in row 1, column 3 of

$$\begin{pmatrix} 1 & -x & 0 & -x \\ -x & 1 & -2x & -x \\ 0 & -2x & 1 & 0 \\ -x & -x & 0 & 1 \end{pmatrix}^{-1}$$

By Cramer's rule, this element is $(\det A)^{-1}$ multiplied by the appropriate cofactor

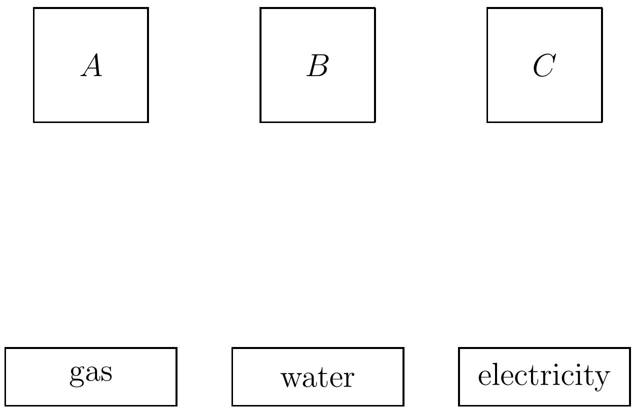
$$\left| \begin{pmatrix} -x & 1 & -x \\ 0 & -2x & 0 \\ -x & -x & 1 \end{pmatrix} \right| = 2x^2 + 2x^3.$$

The determinant of A is $4x^4 - 2x^3 - 7x^2 + 1$, thus the desired generating function is

$$\frac{2x^2 + 2x^3}{4x^4 - 2x^3 - 7x^2 + 1}.$$

4.5 Planar Graphs

Exercise 4.21 (The Three Utilities Problem)**.** Three neighbouring cottages (cottage A, cottage B and cottage C) require connection to the gas, electricity and water companies. Can you draw a (not necessarily straight) line from each cottage to each of the utilities such that none of the lines cross over each other?



This is a very old problem that leads us to consider the following definition.

Definition 4.16. A graph $G = (V, E)$ is said to be *planar* if it can be drawn in the plane such that no two of its edges cross (other than where they meet at the vertices). Otherwise it is *nonplanar*.

Example 4.22. Any tree is a planar graph.

Example 4.23. The platonic graphs are all planar. More generally, the graph corresponding to any convex polyhedron is planar.

Example 4.24. Our consideration of the three utilities problem leads us to the conclusion that $K_{3,3}$ is nonplanar.

Example 4.25. The graphs K_i for $i = 1, 2, 3, 4$ are planar, but K_5 is nonplanar. We will see a proof of this shortly.

It was observed by Euler that if you take any convex polyhedron and add the number of faces to the number of vertices, then the result is two more than the number of edges. If we consider the planar graph associated with a convex polyhedron then we see that the vertices of the polyhedron correspond to vertices of the graph, the edges of the polyhedron correspond to the edges of the graph, and the faces of the polyhedron correspond to regions of the plane that are separated from each other by edges of the graph. In fact Euler's result can be applied to any connected planar graph, not just those arising from polydedra:

Theorem 4.8 (Euler's Polyhedral Formula)**.** *Let G be a connected planar graph with n vertices and m edges. If G divides the plane into f regions, then it holds that*

$$n + f = m + 2.$$

Proof. We can prove this result by induction on the number of regions. The case where there is only one region corresponds to a graph that is a tree. We have observed that trees have n vertices and $n - 1$ edges, hence the result is true in this case. Now suppose the result is true for all connected planar graphs with k or fewer regions. If we have a connected planar graph G with n vertices, m edges and $k + 1$ regions, then by removing an edge that is on the boundary between two of the regions we obtain a graph with n vertices, $m - 1$ edges and k regions. By our inductive assumption Euler's formula holds for this new graph, hence $n + k = (m - 1) + 2$. From this we deduce that $n+(k+1) = m+2$, and hence the formula holds for the original graph G. Thus, by induction, the result is true for graphs with any number $f \geq 1$ of regions. □

Example 4.26. Consider the octahedron graph. It has 8 faces, 12 edges, and 6 vertices, and $8 + 6 = 12 + 2$, as expected.

There are a couple of simple corollaries that can be drawn from this result.

Corollary 4.9. *Let G be a connected planar simple graph with $n \geq 3$ vertices and m edges. Then $m \leq 3n - 6$.*

Proof. Suppose G divides the plane into f regions. As G is simple, there must be at least three edges surrounding each region. Furthermore, each edge lies on the boundary of at most two regions. From this we conclude that $2m \geq 3f$, and so $2m - 3f \geq 0$. Euler's formula tells us that $n - 2 = m - f$. Multiplying both sides by three we get

$$\begin{aligned} 3n - 6 &= 3(m - f) \\ &= 3m - 3f \\ &= m + (2m - 3f) \\ &\geq m. \end{aligned}$$

□

Example 4.27. This corollary allows us to prove that K_5 is nonplanar: it has $m = 10$ and $n = 5$, but $3n - 6 = 9 < 10$ and so it cannot be planar.

This corollary provides a necessary but not sufficient condition for determining whether a graph is planar: we know that $K_{3,3}$ is nonplanar, and yet it has $n = 6$ and $m = 9$, which does not contradict $3n - 6 > m$. However, we can deduce a second corollary that will help demonstrate the nonplanarity of $K_{3,3}$.

Corollary 4.10. *Let G be a connected planar simple graph with $n \geq 3$ vertices and m edges and no triangles (where a triangle is simply a cycle of length 3). Then $m \leq 2n - 4$.*

Proof. The proof of this result is similar to the previous proof, except that we know there are at least four edges surrounding each region, and so $2m \geq 4f$ and hence $m - 2f \geq 0$. From Euler's formula we have $n - 2 = m - f$ and so

$$\begin{aligned} 2n - 4 &= 2m - 2f \\ &= m + (m - 2f) \\ &\geq m. \end{aligned}$$

□

Example 4.28. This corollary allows us to show the nonplanarity of $K_{3,3}$ (and hence prove definitively that the utilities problem has no solution). We observe that since $K_{3,3}$ is bipartite it contains no cycles of odd length, and hence no triangles. However, it has $n = 6$ and $m = 9$, so $2n - 4 = 8 < 9$ and thus it cannot be planar, by the above corollary.

These two corollaries give necessary (but not sufficient) conditions for determining whether a graph is planar. So far we have seen two examples of nonplanar graphs, namely K_5 and $K_{3,3}$. We can generalise this by noting that any graph that has K_5 or $K_{3,3}$ as a subgraph cannot be planar either. In fact this idea has been strengthened by the mathematician Kazimierz Kuratowski to give a necessary and sufficient condition for a graph to be planar. Before we state his result, we first give a definition.

Definition 4.17. A graph H is a *subdivision* of a graph G if it can be obtained from G by inserting new vertices into edges of G (essentially replacing edges by paths of length > 1).

Example 4.29.

G H

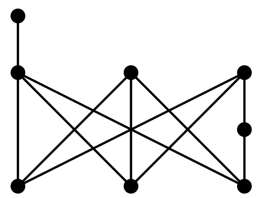
Here graph H is a subdivision of graph G.

We are now in a position to state Kuratowski's theorem:

Theorem 4.11 (Kuratowski's Theorem)**.** *A graph G is planar if and only if it does not contain a subgraph that is a subdivision of either K_5 or $K_{3,3}$.*

(The proof of this result goes beyond the scope of this course.)

Example 4.30. The following graph has a subgraph isomorphic to a subdivision of $K_{3,3}$ and hence is nonplanar.



Learning Outcomes

After completing this chapter and the related problems you should be able to:

- recall basic terminology relating to graphs, including *vertices, edges, simple graphs, the degree of a vertex, digraphs, weighted graphs, networks, subgraphs;*
- construct the adjacency matrix of a graph, and recover a graph from its adjacency matrix;
- relate properties of the adjacency matrix of a graph to properties of the corresponding graph;
- be familiar with common families of graphs and the notation used to describe them, include *null graphs, complete graphs, paths, cycles, bipartite graphs, the platonic graphs;*
- understand the notation of connectedness, and determine the connected components of a graph;
- use the adjacency matrix of a graph to count the number of walks of a specified length between designated vertices of the graph;
- understand the concept of a planar graph, and be able to determine whether or not a graph is planar;
- recall Euler's polyhedral formula for planar graphs;
- appreciate that K_5 and $K_{3,3}$ are nonplanar;
- state and apply Kuratowski's theorem.