3 Integration

3.1 Integration of functions of one variable

Recall that **integration** is the reverse process of differentiation, and is consequently sometimes referred to as **anti-differentiation**. More specifically, if f is a function of one variable x then an (indefinite) integral of f is a function F such that $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$. We write

$$F(x) = \int f(x)\,\mathrm{d}x.$$

For any constant c, since $\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$, we have $\frac{\mathrm{d}}{\mathrm{d}x}(F(x) + c) = f(x)$. Thus there are infinitely many possibilities for $\int f(x)\,\mathrm{d}x$. This does not usually cause a problem as we always add a constant to any integral derived.

For $a, b \in \mathbb{R}$, the (definite) integral

$$\int_a^b f(x)\,\mathrm{d}x$$

is defined to be the area enclosed by the graph of $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.[1] The fundamental theorem of calculus states the precise relation between the indefinite integral (defined as anti-differentiation of f) and the definite integral (defined as area under the graph of f). In particular, if $F(x) = \int f(x)\,\mathrm{d}x$ then

$$\int_a^b f(x)\,\mathrm{d}x = \big[F(x)\big]_a^b = F(b) - F(a).$$

The following table gives the integrals of some standard functions. Most of these follow directly from the derivatives of standard functions given in Chapter 1. Others can be checked by differentiating $\int f(x)\,\mathrm{d}x$.

$f(x)$	$\int f(x)\,\mathrm{d}x$	$f(x)$	$\int f(x)\,\mathrm{d}x$
x^n	$\frac{x^{n+1}}{n+1} + c$ (provided $n \neq -1$)	$\sec x$	$\ln\lvert\sec x + \tan x\rvert + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$	$\sec^2 x$	$\tan x + c$
$\sin x$	$-\cos x + c$	$\arcsin x$	$x \arcsin x + \sqrt{1 - x^2} + c$
$\cos x$	$\sin x + c$	$\exp(x)$ or e^x	$\exp(x) + c$
$\tan x$	$-\ln\lvert\cos x\rvert + c$	$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a} + c$
$\cot x$	$\ln\lvert\sin x\rvert + c$	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\frac{x}{a} + c$

[1] We define the area to be negative when $f(x) < 0$, and set $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$ if $b < a$.

Integration by substitution

This method of integration is based on the chain rule for differentiation. Suppose we want to compute $F(x) = \int f(x)\,\mathrm{d}x$. Since integration is the reverse of differentiation we have $F'(x) = f(x)$. Now we change the variable using the substitution $x = g(u)$. Then by the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}F(g(u)) &= F'(g(u))g'(u) \\ &= f(g(u))\frac{\mathrm{d}x}{\mathrm{d}u},\end{aligned}$$

hence $F(g(u)) = \int f(g(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$. Expressing this again as a function of x instead of u, we get $F(x)$. Thus

$$\boxed{\int f(x)\,\mathrm{d}x = \int f(g(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u.}$$

Example 3.1. Compute $\displaystyle\int \arcsin x\,\mathrm{d}x$.

With the substitution $x = \sin u$ we have $\frac{\mathrm{d}x}{\mathrm{d}u} = \cos u$. Hence

$$\begin{aligned}\int \arcsin x\,\mathrm{d}x &= \int \arcsin(\sin u)\cdot\cos u\,\mathrm{d}u \\ &= \int u\cos u\,\mathrm{d}u \qquad \text{(now use Exercise 1.(k))} \\ &= u\sin u + \cos u + c \\ &= u\sin u + \sqrt{1-\sin^2 u} + c \\ &= (\arcsin x)x + \sqrt{1-x^2} + c.\end{aligned}$$

Sometimes it easier to express u as a function of x. In this case we can compute $\frac{\mathrm{d}x}{\mathrm{d}u}$ as $\frac{\mathrm{d}x}{\mathrm{d}u} = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1}$.

Example 3.2. Compute $\displaystyle\int \frac{x}{1+x^2}\,\mathrm{d}x$.

With the substitution $u = 1+x^2$ we have $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and therefore $\frac{\mathrm{d}x}{\mathrm{d}u} = (2x)^{-1}$. Hence

$$\begin{aligned}\int \frac{x}{1+x^2}\,\mathrm{d}x &= \int \frac{x}{u}\cdot(2x)^{-1}\,\mathrm{d}u \\ &= \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u \\ &= \frac{1}{2}\ln|u| + c \\ &= \frac{1}{2}\ln(1+x^2) + c.\end{aligned}$$

When evaluating a definite integral, if we use a substitution to change the variable from x to a function u of x, it is usually easiest to change the limits of integration from x values to u values. This saves having to convert back to x again after integrating. Specifically, if $u = u(x)$, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f(g(u))\,\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u.$$

Integration by parts

If u and v are functions of x then the product rule for differentiation gives

$$\boxed{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.}$$

This is the formula for **integration by parts**.

Integration by parts is useful for integrating products where one factor will differentiate to give a simpler function, and where it is possible to integrate the other factor.

Exercises 3.1.

1. Integrate the following expressions with respect to x.

(a) $3x^5 - 2x^2 + 3x - 2$;	(b) $\frac{4}{x^3} - \frac{5}{x^5}$;	(c) $\frac{x^3+3x^2}{x+1}$;
(d) $(3x+1)^5$;	(e) $\sin(3x-2)$;	(f) $2x(x^2+7)^8$;
(g) $x\mathrm{e}^{2x^2}$;	(h) $\sin^7 x\,\cos x$;	(i) $\tan x$;
(j) $\frac{x+3}{x^2+4}$;	(k) $x\cos x$;	(l) $x(x+1)^{10}$;
(m) $x\sin 2x$;	(n) xe^{-x};	(o) x^2e^x;
(p) $\arctan x$;	(q) $\ln x$;	(r) $\arccos x$.

2. Evaluate each of the following integrals using the given substitution.

(a) $\displaystyle\int \frac{\mathrm{d}x}{x\ln x}$, $u = \ln x$.

(b) $\displaystyle\int \frac{4x}{(3-2x)^2}\,\mathrm{d}x$, $u = 3-2x$.

(c) $\displaystyle\int \sec^5 x\tan x\,\mathrm{d}x$, $u = \sec x$.

(d) $\displaystyle\int \frac{2e^{2x}}{1+e^{4x}}\,\mathrm{d}x$, $u = e^{2x}$.

(e) $\displaystyle\int 5\sin^7 x\,\mathrm{d}x$, $u = \cos x$.

(f) $\displaystyle\int 14x^2\sqrt{1+x}\,\mathrm{d}x$, $x = u^2 - 1$.

(g) $\displaystyle\int \sqrt{1-x^2}\,\mathrm{d}x$, $x = \sin u$.

3.2 Integrals with infinite limits

It is sometimes possible to evaluate a definite integral that has one, or both, of its limits equal to infinity or negative infinity. We define

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{X\to\infty}\int_a^X f(x)\,\mathrm{d}x.$$

If the limit exists then the definite integral exists and its value is the limit; otherwise we say that the definite integral does not exist.

Example 3.3. Evaluate $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^3}$.

$$\begin{aligned}
\int_1^\infty \frac{\mathrm{d}x}{x^3} &= \lim_{X\to\infty}\int_1^X \frac{\mathrm{d}x}{x^3} \\
&= \lim_{X\to\infty}\int_1^X x^{-3}\,\mathrm{d}x \\
&= \lim_{X\to\infty}\left[-\frac{1}{2}x^{-2}\right]_1^X \\
&= \lim_{X\to\infty}\left(-\frac{1}{2X^2}+\frac{1}{2}\right) \\
&= \frac{1}{2} \qquad\qquad \text{because } \lim_{X\to\infty}\frac{1}{2X^2} = 0
\end{aligned}$$

Example 3.4. Let $a \in \mathbb{R}$. Evaluate $\displaystyle\int_a^\infty e^{-x}\,\mathrm{d}x$.

$$\begin{aligned}
\int_a^\infty e^{-x}\,\mathrm{d}x &= \lim_{X\to\infty}\int_a^X e^{-x}\,\mathrm{d}x \\
&= \lim_{X\to\infty}\left[-e^{-x}\right]_a^X \\
&= \lim_{X\to\infty}\left(-e^{-X}+e^{-a}\right) \\
&= e^{-a} \qquad\qquad \text{because } \lim_{X\to\infty} e^{-X} = 0
\end{aligned}$$

Exercises 3.2.

1. Where possible, evaluate the following integrals:

   (a) $\displaystyle\int_0^\infty e^{-5x}\,\mathrm{d}x$;

(b) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}}$;

(c) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{1+x^2}$;

(d) $\displaystyle\int_0^\infty e^{-x}\cos x\,\mathrm{d}x$.

2. Determine all real numbers s for which $\displaystyle\int_1^\infty x^s\,\mathrm{d}x$ is defined.

3. Let $f : (a,b] \to \mathbb{R}$ be a continuous function with $\lim_{x\to a} f(x) = \infty$. Suggest how you could define $\displaystyle\int_a^b f(x)\,\mathrm{d}x$. (Recall that $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$.) Use your suggestion to evaluate the following integrals:

   (a) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}$;

   (b) $\displaystyle\int_1^{10} \frac{\mathrm{d}x}{(x-1)^{\frac{2}{3}}}$.

3.3 Double integrals over rectangular regions

Let $f : \mathbb{R}^2 \to \mathbb{R}$. We want to find the volume enclosed between the surface of $z = f(x,y)$, the (x,y)-plane and the vertical planes $x = a$, $x = b$, $y = c$ and $y = d$.

Example 3.5. Find the volume enclosed between the surface of

$$z = 8x^3y + 3x^2y^2,$$

the (x,y)-plane and the planes $x = 2$, $x = 3$, $y = 1$ and $y = 2$.

The required volume is given by the double integral

$$\begin{aligned}
\int_1^2\int_2^3 \left(8x^3y + 3x^2y^2\right)\mathrm{d}x\,\mathrm{d}y &= \int_1^2 \left[2x^4y + x^3y^2\right]_2^3 \mathrm{d}y \\
&= \int_1^2 \left(162y + 27y^2 - 32y - 8y^2\right)\mathrm{d}y \\
&= \int_1^2 \left(130y + 19y^2\right)\mathrm{d}y \\
&= \left[65y^2 + \frac{19}{3}y^3\right]_1^2 \\
&= 260 + \frac{152}{3} - 65 - \frac{19}{3} = 239\tfrac{1}{3}.
\end{aligned}$$

Exercises 3.3.

1. Evaluate $\displaystyle\int_1^3 \int_0^2 x^2 y \,\mathrm{d}x\,\mathrm{d}y$.

2. Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \sin(x+y)\,\mathrm{d}x\,\mathrm{d}y$.

3.4 Non-rectangular regions of integration

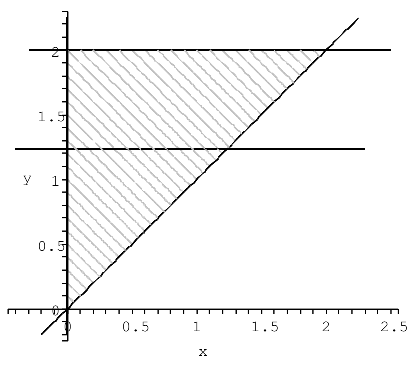
We now consider the problem of evaluating a double integral over a non-rectangular region D. We write this as

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

As given, we are integrating first with respect to x. In this case the limits of the inner integral are (possibly constant) functions of y and the limits of the outer integral are constants. When evaluating a double integral the final answer represents a volume and so should not involve either of the variables of integration; if your final answer does involve either variable then you should check your limits.

Example 3.6. Find the volume beneath the surface of $z = xy$ over the region of the (x,y)-plane enclosed by the y-axis, $y = x$ and $y = 2$.

Here we require $\displaystyle\iint_D xy\,\mathrm{d}x\,\mathrm{d}y$ where D is the region of the (x,y)-plane enclosed by the y-axis, $y = x$ and $y = 2$. We start by sketching the region of integration.



The region D

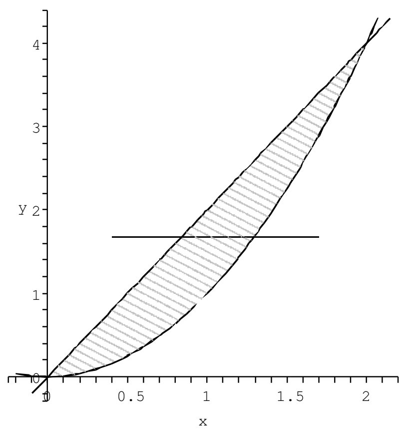
Since we are integrating with respect to x first we need to find the limits of the inner integral as functions of y. To do this we draw a line parallel to the x-axis passing through D. Where the line crosses the left hand side of D gives the lower limit ($x = 0$) and where it crosses the right hand side of D gives the upper limit ($x = y$).

The limits of the outer integral are then just the minimum ($y = 0$) and maximum ($y = 2$) values of y over D. Thus we get

$$\begin{aligned}\iint_D xy \, \mathrm{d}x \, \mathrm{d}y &= \int_0^2 \int_0^y xy \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^2 \left[\frac{1}{2}x^2 y\right]_0^y \mathrm{d}y \\ &= \int_0^2 \frac{1}{2}y^3 \, \mathrm{d}y \\ &= \left[\frac{1}{8}y^4\right]_0^2 \\ &= 2.\end{aligned}$$

Example 3.7. Evaluate $\iint_D (xy+2) \, \mathrm{d}x \, \mathrm{d}y$, where D is the bounded region enclosed between $y = 2x$ and $y = x^2$.

First we sketch the region D.



The region D

From the line parallel to the x-axis we see that for the inner integral the lower limit is $x = \frac{1}{2}y$ and the upper limit is $x = \sqrt{y}$. For the outer integral, the lower limit is

$y = 0$ and the upper limit is $y = 4$. Thus

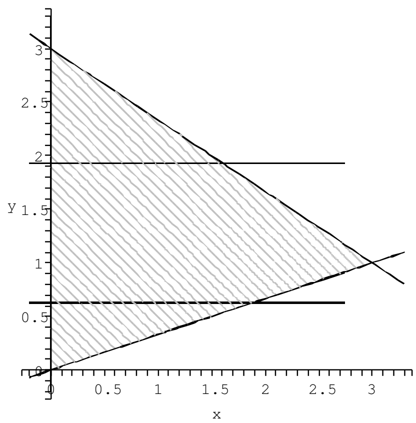
$$\begin{aligned}
\iint_D (xy+2)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (xy+2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^4 \left[\frac{1}{2}x^2 y + 2x\right]_{\frac{1}{2}y}^{\sqrt{y}} \mathrm{d}y \\
&= \int_0^4 \left(\frac{1}{2}y^2 + 2\sqrt{y} - \frac{1}{8}y^3 - y\right) \mathrm{d}y \\
&= \left[\frac{1}{6}y^3 + \frac{4}{3}y^{\frac{3}{2}} - \frac{1}{32}y^4 - \frac{1}{2}y^2\right]_0^4 \\
&= \frac{1}{6} \times 64 + \frac{4}{3} \times 8 - \frac{1}{32} \times 256 - \frac{1}{2} \times 16 = \frac{16}{3}.
\end{aligned}$$

Splitting the integral

So far we have been able to determine the limits of the inner integral of a double integral as simple functions of y. However, this is not always possible. Such cases can sometimes be resolved by splitting the integral into two, or more, integrals.

Example 3.8. Evaluate $\displaystyle\iint_D xy^2\,\mathrm{d}x\,\mathrm{d}y$, where D is triangle with vertices at $(0,0)$, $(3,1)$ and $(0,3)$.

First we sketch the region D.



The region D

From the lines parallel to the x-axis we see that the limits of the inner integral depend on whether $y < 1$ or $y > 1$. Thus we split the integral into two integrals where the limits of the outer integrals are 0 and 1, and 1 and 3, respectively. Doing this, and working out the limits of the inner integrals as before gives:

$$\begin{aligned}
\iint_D xy^2 \,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_0^{3y} xy^2 \,\mathrm{d}x\,\mathrm{d}y + \int_1^3 \int_0^{\frac{3}{2}(3-y)} xy^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left[\frac{1}{2}x^2y^2\right]_0^{3y} \mathrm{d}y + \int_1^3 \left[\frac{1}{2}x^2y^2\right]_0^{\frac{3}{2}(3-y)} \mathrm{d}y \\
&= \int_0^1 \frac{9}{2}y^4 \,\mathrm{d}y + \int_1^3 \frac{9}{8}(3-y)^2y^2 \,\mathrm{d}y \\
&= \int_0^1 \frac{9}{2}y^4 \,\mathrm{d}y + \int_1^3 \frac{9}{8}(y^4 - 6y^3 + 9y^2) \,\mathrm{d}y \\
&= \frac{9}{2}\left[\frac{1}{5}y^5\right]_0^1 + \frac{9}{8}\left[\frac{1}{5}y^5 - \frac{3}{2}y^4 + 3y^3\right]_1^3 \\
&= \frac{9}{10} + \frac{9}{8}\left(\frac{1}{5} \times 243 - \frac{3}{2} \times 81 + 3 \times 27 - \frac{1}{5} + \frac{3}{2} - 3\right) \\
&= \frac{9}{10} + \frac{9}{8} \times \frac{32}{5} = 8\tfrac{1}{10}.
\end{aligned}$$

Exercises 3.4.

1. Evaluate $\displaystyle\iint_D (x+y)\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) : 0 \le y \le 1, y \le x \le 1 + 2y\}$.

2. Evaluate $\displaystyle\iint_D (x-y)^2\,\mathrm{d}x\,\mathrm{d}y$, where D is the bounded region enclosed between $x = 3$, $y = 1$ and $y = x^2$.

3. Evaluate $\displaystyle\iint_D xy\,\mathrm{d}x\,\mathrm{d}y$, where D is the square with vertices at $(1,0)$, $(0,1)$, $(1,2)$ and $(2,1)$.

4. Evaluate $\displaystyle\iint_P xy\,\mathrm{d}x\,\mathrm{d}y$, where P is the pentagon with vertices at $(0,0)$, $(0,2)$, $(1,2)$, $(2,1)$ and $(2,0)$.

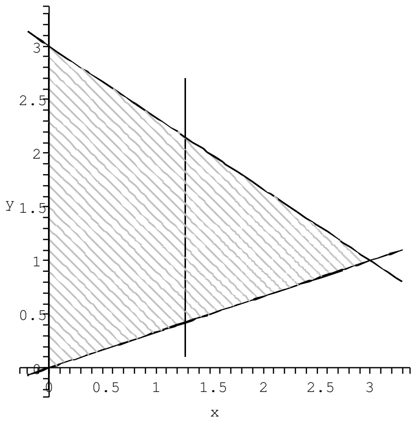
5. Instead of $\displaystyle\iint_D 1\,\mathrm{d}x\,\mathrm{d}y$ we usually write $\displaystyle\iint_D \mathrm{d}x\,\mathrm{d}y$. Explain why $\displaystyle\iint_D \mathrm{d}x\,\mathrm{d}y$ gives the area of D. Use this result to find the area of the region enclosed between the curves $x = y^2$ and $y = x - 2$.

3.5 Changing the order of integration

In the examples considered so far, we have always integrated first with respect to x. This is not necessary, and there is no reason why we should not choose to integrate first with respect to y. In some cases it can be advantageous, or even necessary, to change the order of integration, for example

- to avoid having to split the integral into two, or more, regions;
- when it is difficult, or not possible, to integrate $f(x,y)$ with respect to x.

Consider again Example 3.8, but this time we integrate first with respect to y.



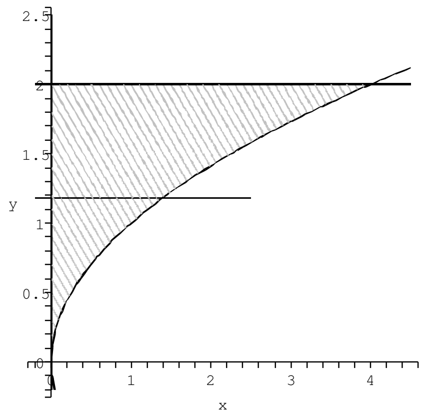
To find the limits of the inner integral we draw a line through the region parallel to the y-axis; where the line crosses the region at the lowest point ($y = \frac{1}{3}x$) gives the lower limit, and where it crosses the highest point ($y = 3 - \frac{2}{3}x$) gives the upper limit. The limits of the outer integral are just the minimum ($x = 0$) and maximum ($x = 3$) values of x. Thus we get:

$$\begin{aligned}
\iint_D xy^2 \, \mathrm{d}x \, \mathrm{d}y &= \iint_D xy^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^3 \int_{\frac{1}{3}x}^{3-\frac{2}{3}x} xy^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^3 \left[\frac{1}{3} xy^3 \right]_{\frac{1}{3}x}^{3-\frac{2}{3}x} \mathrm{d}x \\
&= \int_0^3 \left(\frac{1}{3} x \big(3 - \frac{2}{3}x\big)^3 - \frac{1}{81} x^4 \right) \mathrm{d}x \\
&= \int_0^3 \left(\frac{1}{3} x \big(27 - 18x + 4x^2 - \frac{8}{27}x^3\big) - \frac{1}{81} x^4 \right) \mathrm{d}x \\
&= \int_0^3 \left(9x - 6x^2 + \frac{4}{3}x^3 - \frac{1}{9}x^4 \right) \mathrm{d}x \\
&= \left[\frac{9}{2}x^2 - 2x^3 + \frac{1}{3}x^4 - \frac{1}{45}x^5 \right]_0^3 \\
&= \frac{9}{2} \times 9 - 2 \times 27 + \frac{1}{3} \times 81 - \frac{1}{45} \times 243 \\
&= 8\tfrac{1}{10}.
\end{aligned}$$

The following two examples illustrate the case when we change the order of integration because it is difficult, or impossible, to evaluate the integral in the order given.

Example 3.9. Evaluate $I = \displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\,\mathrm{d}y\,\mathrm{d}x.$

Here I is expressed so that first we have to integrate with respect to y. Since it is difficult to integrate $\frac{1}{1+y^5}$ with respect to y, we change the order of integration. To do this we need to find the new limits, and so we sketch the region of integration. Thus we sketch $y = \sqrt{x}$ and $y = 2$ (the limits of the inner integral of I) between $x = 0$ and $x = 4$ (the limits of the outer integral of I). Thus the region of integration is as follows.



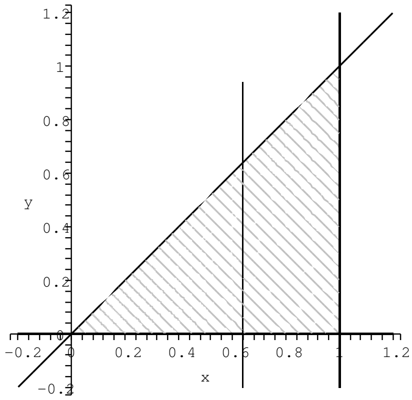
The region D

From the line parallel to the x-axis, we see that the lower limit for x is $x = 0$, and the upper limit is $x = y^2$. The corresponding limits for y are $y = 0$ and $y = 2$, respectively. Thus

$$\begin{aligned} I &= \int_0^2 \int_0^{y^2} \frac{x}{1+y^5}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^2 \left[\frac{\frac{1}{2}x^2}{1+y^5}\right]_0^{y^2} \mathrm{d}y \\ &= \int_0^2 \frac{\frac{1}{2}y^4}{1+y^5}\,\mathrm{d}y \\ &= \frac{1}{10}\int_0^2 \frac{5y^4}{1+y^5}\,\mathrm{d}y \\ &= \frac{1}{10}\left[\ln(1+y^5)\right]_0^2 \\ &= \frac{1}{10}\ln 33. \end{aligned}$$

Example 3.10. Evaluate $I = \displaystyle\int_0^1 \int_y^1 3y^2 e^{-x^2}\,\mathrm{d}x\,\mathrm{d}y.$

Here I is expressed so that first we have to integrate with respect to x. Since it is not possible to express $\displaystyle\int e^{-x^2}\,\mathrm{d}x$ in terms of simple functions, we change the order of integration. To find the new limits, we sketch the region of integration. So we sketch $x = y$ and $x = 1$ (the limits of the inner integral of I) between $y = 0$ and $y = 1$ (the limits of the outer integral of I). Thus the region of integration is as follows.



The region D

From the line parallel to the y-axis, we see that the lower limit for y is $y = 0$, and the upper limit is $y = x$. The corresponding limits for x are $x = 0$ and $x = 1$, respectively. Thus

$$
\begin{aligned}
I &= \int_0^1 \int_0^x 3y^2 e^{-x^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[y^3 e^{-x^2}\right]_0^x \mathrm{d}x \\
&= \int_0^1 x^3 e^{-x^2}\,\mathrm{d}x \qquad \text{(Put } w = x^2 \text{, so } \tfrac{\mathrm{d}w}{\mathrm{d}x} = 2x\text{. The lower limit } x = 0 \text{ becomes} \\
&\qquad\qquad\qquad\quad w = 0\text{, and the upper limit } x = 1 \text{ becomes } w = 1.) \\
&= \int_0^1 x w e^{-w}\,\frac{\mathrm{d}w}{2x} \\
&= \frac{1}{2}\int_0^1 w e^{-w}\,\mathrm{d}w \qquad \left(\text{Use } \int w e^{-w}\mathrm{d}w = -(1+w)e^{-w} \text{ from Exercise 3.1.(n)}\right) \\
&= \frac{1}{2}\left[-(1+w)e^{-w}\right]_0^1 \\
&= \frac{1}{2}(-2e^{-1} + 1) \\
&= \frac{1}{2} - e^{-1}.
\end{aligned}
$$

Exercises 3.5.

1. For each of the following double integrals:

   (i) evaluate the integral as given;

   (ii) sketch the region of integration;

   (iii) change the order of integration, and re-evaluate the integral.

   (a) $\displaystyle\int_0^3 \int_0^y (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$

   (b) $\displaystyle\int_0^2 \int_0^{4-2x} xy\,\mathrm{d}y\,\mathrm{d}x$

   (c) $\displaystyle\int_0^1 \int_y^{\sqrt{y}} y^2 x\,\mathrm{d}x\,\mathrm{d}y$

2. Evaluate the following double integrals.

   (a) $\displaystyle\int_0^1 \int_{2y}^2 6y\sqrt{1+x^3}\,\mathrm{d}x\,\mathrm{d}y$

   (b) $\displaystyle\int_1^2 \int_{\frac{1}{2}}^{\frac{1}{y}} x^3\cos(x^2 y)\,\mathrm{d}x\,\mathrm{d}y$

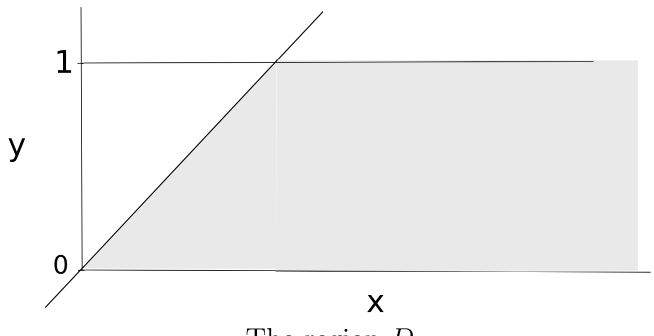
3. Evaluate $\displaystyle\int_0^2 \int_y^{2y} \sin\left(\frac{\pi y}{x}\right)\,\mathrm{d}x\,\mathrm{d}y + \int_2^4 \int_y^4 \sin\left(\frac{\pi y}{x}\right)\,\mathrm{d}x\,\mathrm{d}y$.

3.6 Unbounded regions of integration

It is possible to evaluate some double integrals over an unbounded region. In such cases, one or more of the limits is infinite.

Example 3.11. Evaluate $\displaystyle\iint_D e^{-(x+y)}\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1, y \le x\}$.

First we sketch the region of integration.



The region D

From the line parallel to the x-axis, we see that the lower limit for x is $x = y$, and the upper limit is $x = \infty$. The corresponding limits for y are $y = 0$ and $y = 1$, respectively. Thus

$$
\begin{aligned}
\iint_D e^{-(x+y)}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_y^\infty e^{-(x+y)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \lim_{X\to\infty} \left[-e^{-(x+y)}\right]_y^X \mathrm{d}y \\
&= \int_0^1 \lim_{X\to\infty} \left(-e^{-(X+y)}\right) + e^{-2y}\,\mathrm{d}y \\
&= \int_0^1 e^{-2y}\,\mathrm{d}y \\
&= \left[-\frac{1}{2}e^{-2y}\right]_0^1 \\
&= -\frac{1}{2}e^{-2} + \frac{1}{2} \\
&= \frac{1}{2}(1 - e^{-2}).
\end{aligned}
$$

Exercises 3.6.

1. Evaluate $\displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2y^2}$, where $D = \{(x,y) \in \mathbb{R}^2 : 1 \le y \le 2, x \ge y^2\}$.
2. Evaluate $\displaystyle\iint_D xe^{-y^2}\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 : x \ge 0, y \ge x^2\}$.

3.7 Change of variables

Suppose we want to evaluate $\displaystyle\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, and we change the variables (x,y) to $(u,v) = (h_1(x,y), h_2(x,y))$, where (h_1,h_2) is invertible on D. Such a change of variables transforms the integral over D in the (x,y)-plane to an integral over Δ in the (u,v)-plane. It remains to transform the function to a function in u and v, being integrated with respect to u and v. To do this, first note that since (h_1,h_2) is invertible on D, there are functions g_1 and g_2 on Δ such that $x = g_1(u,v)$ and $y = g_2(u,v)$. Now we define the **Jacobian** by

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Then

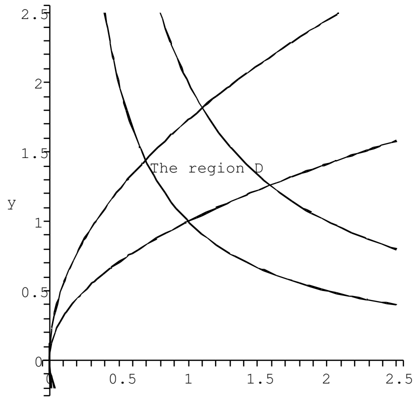
$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_\Delta f(g_1(u,v), g_2(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

If it is difficult to express x and y in terms of u and v then we can use the fact that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}.$$

Example 3.12. Evaluate $\displaystyle\iint_D x^2y^2\,\mathrm{d}x\,\mathrm{d}y$, where D is the region enclosed by the curves $y=\sqrt{x}$, $y=\sqrt{3x}$, $y=\frac{1}{x}$ and $y=\frac{2}{x}$.

The region D is illustrated in the following diagram.



The region D

Let $u=\frac{y^2}{x}$ and $v=xy$. This change of variables transforms the curves $y=\sqrt{cx}$ and $y=\frac{c}{x}$, for some constant c, to $u=c$ and $v=c$, respectively. Thus the region D is transformed into the region Δ that is enclosed between the lines $u=1$, $u=3$, $v=1$ and $v=2$. Now

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \det\begin{pmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ y & x \end{pmatrix} = -\frac{y^2}{x} - \frac{2y^2}{x} = -\frac{3y^2}{x},$$

and hence

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{x}{3y^2}.$$

Thus

$$\begin{aligned}
\iint_D x^2y^2\,\mathrm{d}x\,\mathrm{d}y &= \iint_\Delta x^2y^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \\
&= \int_1^2\int_1^3 x^2y^2\cdot\frac{x}{3y^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_1^2\int_1^3 \frac{1}{3}\frac{v^2}{u}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_1^2 \frac{1}{3}\left[v^2\ln u\right]_1^3\,\mathrm{d}v \\
&= \int_1^2 \frac{1}{3}v^2\ln 3\,\mathrm{d}v \\
&= \left(\frac{1}{3}\ln 3\right)\left[\frac{1}{3}v^3\right]_1^2 \\
&= \frac{7}{9}\ln 3.
\end{aligned}$$

Example 3.13. Evaluate $\displaystyle\iint_D y^2\cos\left(\frac{y}{x}\right)\,\mathrm{d}x\,\mathrm{d}y$, where D is the region in the first quadrant enclosed by the curves $y = ax$, $y = bx$, $y = \frac{a}{x}$ and $y = \frac{b}{x}$, where $a, b \in \mathbb{R}$ with $a > b > 0$.

Let $u = \frac{y}{x}$ and $v = xy$. This change of variables transforms the curves $y = cx$ and $y = \frac{c}{x}$, for some constant c, to $u = c$ and $v = c$, respectively. Thus the region D is transformed into the region Δ that is enclosed between the lines $u = b$, $u = a$, $v = b$ and $v = a$. Now

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix} = \det\begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x\end{pmatrix} = -\frac{y}{x^2}x - \frac{y}{x} = -\frac{2y}{x},$$

and hence

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{x}{2y}.$$

Thus

$$\begin{aligned}
\iint_D y^2 \cos\left(\frac{y}{x}\right) \mathrm{d}x\,\mathrm{d}y &= \iint_\Delta y^2 \cos\left(\frac{y}{x}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \\
&= \int_b^a \int_b^a y^2 \cos\left(\frac{y}{x}\right) \cdot \frac{x}{2y}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_b^a \int_b^a \frac{1}{2} xy \cos\left(\frac{y}{x}\right) \mathrm{d}u\,\mathrm{d}v \\
&= \int_b^a \int_b^a \frac{1}{2} v \cos u\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_b^a \frac{1}{2} v \left[\sin u\right]_b^a \mathrm{d}v \\
&= \int_b^a \frac{1}{2} (\sin a - \sin b) v\,\mathrm{d}v \\
&= \frac{1}{2}(\sin a - \sin b)\left[\frac{1}{2}v^2\right]_b^a \\
&= \frac{1}{4}(\sin a - \sin b)(a^2 - b^2).
\end{aligned}$$

Polar coordinates

Every non-zero point $(x,y) \in \mathbb{R}^2$ can be defined uniquely by its distance r from the origin and the angle θ between the positive x-axis and the line joining the the origin to (x,y). The pair (r,θ) are the **polar coordinates** of (x,y). (For completeness, the polar coordinates of the origin can be taken to be $(0,0)$.) From this, simple trigonometry gives

$$\begin{aligned}
x &= r\cos\theta, \\
y &= r\sin\theta.
\end{aligned}$$

It is often convenient to use polar coordinates instead of rectangular coordinates. Here we consider only their use in a change of variables in a double integral. This is usually advantageous when:

- the region of integration involves an area enclosed by a circle, or parts of a circle;
- $f(x,y)$ contains terms involving x^2+y^2.[2]

[2]This is because

$$\begin{aligned}
x^2+y^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\
&= r^2(\cos^2\theta + \sin^2\theta) \\
&= r^2.
\end{aligned}$$

If we change to polar coordinates then

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\
&= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\
&= r(\cos^2\theta + \sin^2\theta) = r.
\end{aligned}$$

When changing variables from rectangular to polar coordinates you may quote the fact that $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ without going through the above derivation.

Example 3.14. Let $a > 0$ and $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 \leq a^2\}$. Find $\displaystyle\iint_D x^2 y \, \mathrm{d}x \, \mathrm{d}y$.

Changing variables to polar coordinates we get:

$$\begin{aligned}
\iint_D x^2 y \, \mathrm{d}x \, \mathrm{d}y &= \int_0^{\frac{\pi}{2}} \int_0^a (r^2\cos^2\theta)(r\sin\theta) r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^a r^4 \cos^2\theta \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{r^5}{5}\right]_0^a \cos^2\theta \sin\theta \, \mathrm{d}\theta \\
&= \frac{a^5}{5} \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, \mathrm{d}\theta \\
&= \frac{a^5}{5} \left[-\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} \\
&= \frac{a^5}{5}\frac{1}{3} = \frac{1}{15}a^5.
\end{aligned}$$

Example 3.15. Evaluate $\displaystyle\iint_D xy \, \mathrm{d}x \, \mathrm{d}y$, where D is the interior of the circle of radius 1, centre $(1,1)$.

Here we first make the change of variable $u = x - 1$, $v = y - 1$. It is easy to check that $\dfrac{\partial(u,v)}{\partial(x,y)} = 1$ and the region of integration Δ is the interior of the circle of radius 1, centre $(0,0)$. Thus

$$\iint_D xy \, \mathrm{d}x \, \mathrm{d}y = \iint_\Delta (u+1)(v+1) \, \mathrm{d}u \, \mathrm{d}v.$$

Now we change to polar coordinates with $u = r\cos\theta$ and $v = r\sin\theta$. This gives:

$$\begin{aligned}
\iint_\Delta (u+1)(v+1)\,\mathrm{d}u\,\mathrm{d}v &= \int_0^{2\pi}\int_0^1 (r\cos\theta+1)(r\sin\theta+1)r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 \Big(r^3\cos\theta\sin\theta + r^2(\cos\theta+\sin\theta)+r\Big)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\left[\frac{r^4}{4}\cos\theta\sin\theta + \frac{r^3}{3}(\cos\theta+\sin\theta)+\frac{r^2}{2}\right]_0^1\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\frac{1}{4}\cos\theta\sin\theta+\frac{1}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right)\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\frac{1}{8}\sin 2\theta+\frac{1}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right)\mathrm{d}\theta \\
&= \left[-\frac{1}{16}\cos 2\theta+\frac{1}{3}(\sin\theta-\cos\theta)+\frac{1}{2}\theta\right]_0^{2\pi} \\
&= \left(-\frac{1}{16}+\frac{1}{3}(0-1)+\pi\right)-\left(-\frac{1}{16}+\frac{1}{3}(0-1)+0\right) \\
&= \pi.
\end{aligned}$$

Note that in the process of evaluating the above double integral we used the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$.

Exercises 3.7.

1. Evaluate $\displaystyle\iint_D (x+y)(2x+1)\,\mathrm{d}x\,\mathrm{d}y$, where D is the region in the upper right quadrant of the (x,y)-plane that is enclosed between the curves $y = x^2$, $y = x^2+2$, $x+y=4$ and $x+y=6$. [Hint: Use the change of variables $u = x+y$, $v = y-x^2$.]

2. Evaluate $\displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{xy}$, where D is the region enclosed between the curves $y = \frac{1}{x^2}$, $y = \frac{3}{x^2}$, $y = \sqrt{x^3}$ and $y = \sqrt{5x^3}$.

3. Evaluate $\displaystyle\iint_D x^2\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) : x^2+y^2 \le a^2, y \ge 0\}$, for some nonzero real number a.

4. Evaluate $\displaystyle\iint_D xy\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, where D is the region
$$D = \big\{(x,y) : 1 \le x^2+y^2 \le 4, x \ge 0, y \ge 0\big\}.$$

5. Evaluate the integral given in Example 3.15 using the change of variable $x = 1+r\cos\theta$, $y = 1+r\sin\theta$.