6 Applications of differential equations

As mentioned, differential equations have many real world applications. In fact, much of the work and theory in this particular area of mathematics has been motivated by scientific applications. While many of these applications have traditionally come from physics, the fields of biology, economics and chemistry have also given rise to many interesting studies in the field. We briefly touch on the subject of applications here. In our applications, we will often pay close attention to the effect of initial conditions on the resulting solution of the differential equation. We shall see that the behaviour of the system changes dramatically depending on the initial conditions and the various parameters in our system.

6.1 Exponential growth and decay

In this section we give real world examples of systems that exhibit behaviours of exponential growth and decay.

In what follows, our variable is often time t. Suppose that we have a quantity x that varies with time. It is customary to use the notation $\dot{x}$ to mean $\frac{dx}{dt}$ just as we use $y' = \frac{dy}{dx}$.

Suppose that we have a population of, say, bacteria. Let $P(t)$ be the number of bacteria at time t. Let's suppose that the rate the population grows is proportional to its size. That is, suppose there is some number α such that between time t and time $t + \Delta t$ the population has grown by $\Delta P \simeq P(t)(\alpha \Delta t)$. In this case, since the population is growing, $\alpha > 0$. Taking the limit as $\Delta t \to 0$, we get the differential equation

$$\dot{P} = \alpha P.$$

This equation is separable and one of the simplest differential equations we have encountered. We saw in Example 6, Chapter 4 that the solution to this equation is

$$P(t) = Ae^{\alpha t}, \tag{1}$$

for some constant A, which can be determined from the initial conditions. Notice that A can be interpreted as the initial population size since $P_0 = P(0) = A$. Thus, we write

$$P(t) = P_0 e^{\alpha t},$$

where P_0 is the initial population. This equation is the classic example of *exponential growth.*

A similar example is that of *exponential decay*, which can be seen in nuclear decay. Suppose that we have a mass of radioactive substance whose rate of change in mass is proportional to the present mass. Let $M(t)$ be the mass of our radioactive substance at time t and let $M_0 = M(0)$ be the initial mass. We therefore have the differential equation,

$$\frac{dM}{dt} = \alpha M$$

for some α. Of course, in the case of decay $\alpha < 0$. Solving this equation we again obtain

$$M = M_0 e^{\alpha t}. \tag{2}$$

In nuclear physics, an important concept concerning radioactive substances is that of the *half life* of a substance. That is the time which it takes a mass of radioactive substance to decay to half its mass.

Example 6.1. Suppose that we have a population of 1000 bacteria and let $P(t)$ be the population of bacteria after time t hours (thus, $P(0) = 1000$). Assuming the growth rate of the bacteria is proportional to the size of the population, and that the bacteria population reaches 2500 bacteria in 2 hours, find the number of bacteria in 10 hours.

We use the differential equation (1) and find that $P_0 = 1000$. We don't know α, so we determine it using the data. We have

$$\begin{aligned} P(2) &= 1000e^{2\alpha} \\ &= 2500. \end{aligned}$$

We therefore have that $\alpha = \frac{\ln 2.5}{2}$. Thus, we have

$$P(t) = 1000e^{\frac{\ln 2.5}{2}t}.$$

Substituting $t = 10$, we obtain that $P(10) = 1000 \cdot 2.5^5$.

Example 6.2. In this example, we discuss nuclear decay and half life. Suppose that we have a lump of radium and in 20 years this lump of radium lost 0.86% of its mass. Compute the half life of radium.

Let $M(t)$ be the mass of radium after t years, and M_0 its original mass. Our equation for the mass of radium has the form in (2), so we find $M(20) = (1 - .0086)M_0 = .9914M_0$. Hence,

$$.9914M_0 = M_0 e^{20\alpha},$$

implying that $\alpha = \ln(.9914)/20 \simeq -0.000432$. Therefore, to compute the half life of radium, we must find the time solution t_h to the equation

$$1/2M_0 = M_0 e^{-0.000432t_h}.$$

We see that $t_h = \ln(1/2)/(-0.000432) \simeq 1600$ years. Notice that throughout the computation, the value of M_0 was immaterial, which makes perfect sense (the half life does not depend on the initial mass).

Exercises 6.1.

1. Find the half life of a substance that loses a quarter of its mass in 10 years.

2. A bacterial population grows and assume that its rate of growth is proportional to the number of bacteria living. If the population increases by 50% in 1/2 an hour, how long will it take for the population to triple its original size? 5 times its original size?

6.2 Logistic growth

While Example 6.2 is a realistic model for nuclear decay, Example 6.1 is not as realistic as there is an underlying assumption that resources (food sources etc) are infinite. The following equation was first proposed as a model for population growth by Verhulst in 1838, in response to the exponential growth model. Let $P(t)$ be the size of a population at time t and consider the following differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) = rP - \frac{rP^2}{k}.$$

Here both r and k are positive constants. The constant r is the *growth proportionality constant* and k is called the *carrying capacity.* The second multiplier $(1 - \frac{P}{k})$ is often called the *competition term* and represents, for example, limited resources for which the population competes. Notice that as P grows to k, this term gets closer to 0. In this model, when P is small, $\dot{P}$ is approximately rP and so behaves like an exponential model. As time passes, the second multiplier goes from 1 to 0 as P grows and approaches k. Let's solve the equation and look at the behaviour of the system.

The equation is separable, so we can separate the variables.

$$\frac{dP}{rP(1 - \frac{P}{k})} = dt.$$

Using partial fractions on the left hand side we get

$$\begin{aligned}
&\left(\frac{1}{rP} + \frac{1/(kr)}{1 - \frac{P}{k}}\right) dP = dt \\
\Rightarrow\quad & \frac{1}{r}\ln P - \frac{1}{r}\ln\left(1 - \frac{P}{k}\right) = t + C \\
\Rightarrow\quad & \ln P - \ln\left(1 - \frac{P}{k}\right) = rt + D \\
\Rightarrow\quad & \frac{P}{1 - \frac{P}{k}} = Ae^{rt} \\
\Rightarrow\quad & P = Ae^{rt} - \frac{P}{k}Ae^{rt} \\
\Rightarrow\quad & P\left(k + Ae^{rt}\right) = kAe^{rt} \\
\Rightarrow\quad & P = \frac{kAe^{rt}}{k + Ae^{rt}}.
\end{aligned}$$

Setting $P(0) = P_0$ we find

$$\begin{aligned}
& P_0 = P(0) = \frac{kA}{k + A} \\
\Rightarrow\quad & kP_0 + AP_0 = kA \\
\Rightarrow\quad & A(k - P_0) = kP_0 \\
\Rightarrow\quad & A = \frac{kP_0}{k - P_0}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned} P(t) &= \frac{\dfrac{k^2 P_0}{k-P_0} e^{rt}}{k + \dfrac{kP_0}{k-P_0} e^{rt}} \\ &= \frac{kP_0 e^{rt}}{k - P_0 + P_0 e^{rt}} \\ &= \frac{kP_0 e^{rt}}{k + P_0\left(e^{rt}-1\right)}. \end{aligned} \tag{3}$$

Let's look at the behaviour of $P(t)$. When t is close to 0, the second term in the denominator is near 0, so $P(t) \simeq P_0 e^{rt}$. As $t \to \infty$, if $P_0 \neq 0$, we see

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{kP_0}{\frac{k}{e^{rt}} + P_0\left(1 - \frac{1}{e^{rt}}\right)} \\ &= \frac{kP_0}{0 + P_0(1-0)} \\ &= k. \end{aligned}$$

Thus, as t gets large, the population size P approaches k. Writing (3) as

$$P(t) = k\frac{P_0 e^{rt}}{k - P_0 + P_0 e^{rt}}, \tag{4}$$

we see that when $P_0 < k$, the denominator is bigger than the numerator, and therefore $P(t)$ is always smaller than k, thus P approaches k from below as $t \to \infty$. Similarly, when $P_0 > k$, we see that the denominator is smaller than the numerator and therefore P approaches k from above as $t \to \infty$.

We further note two special initial conditions. If $P_0 = 0$, then $P(t) = 0$ for all t. This is not surprising; if there is no population to begin with, there never will be. Another interesting point is $P_0 = k$. We see from (4) that the equation for $P(t)$ simply becomes $P(t) = k$.

At both of these points ($P_0 = 0$ or k), the differential equation has a special form; $\frac{dP}{dt} = 0$ (not surprising since the solutions are constant for those initial conditions!). Such solutions are called *equilibrium solutions*. Notice, however, the behaviour near these two solutions is very different from each other. As noted, $P(t) \to k$ as $t \to \infty$ for any $P_0 \neq 0$. Thus, any small deviation from the initial condition $P_0 = 0$ (that is, if we had an initial condition P_0 near 0) changes the behaviour of the system dramatically. We therefore call the equilibrium solution $P(t) = 0$ for the initial condition $P_0 = 0$ *unstable*. However, the solution when $P_0 = k$ has the opposite behaviour; any small deviation from the initial condition $P_0 = k$, doesn't affect the long term behaviour of the system i.e. we would still have $P(t) \to k$ as $t \to \infty$. We therefore call the equilibrium solution $P(t) = k$ *stable*.

In Figures 1 and 2 we have a graph of logistic growth when $P_0 < k$ and $P_0 > k$.

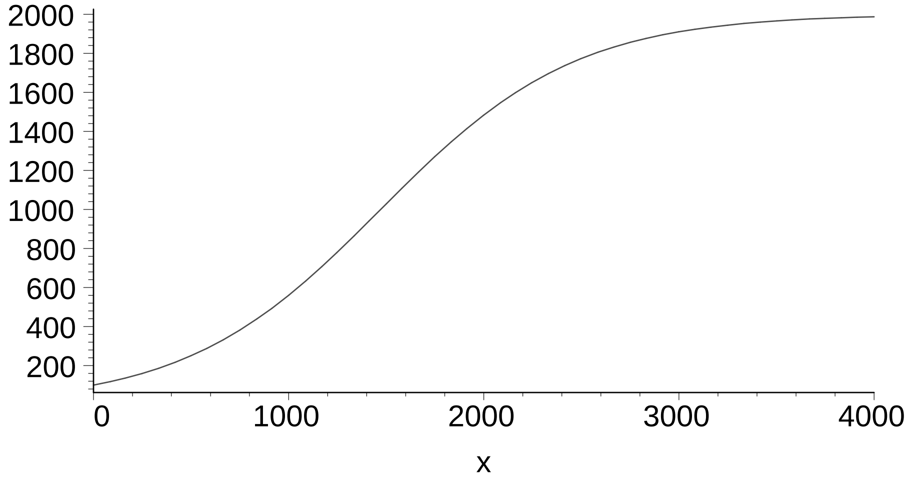

Figure 1: Logistic growth with $P_0 = 100, k = 2000$ and $r = .002$. Note, that the horizontal axis should be t and the vertical axis the size of the population P.

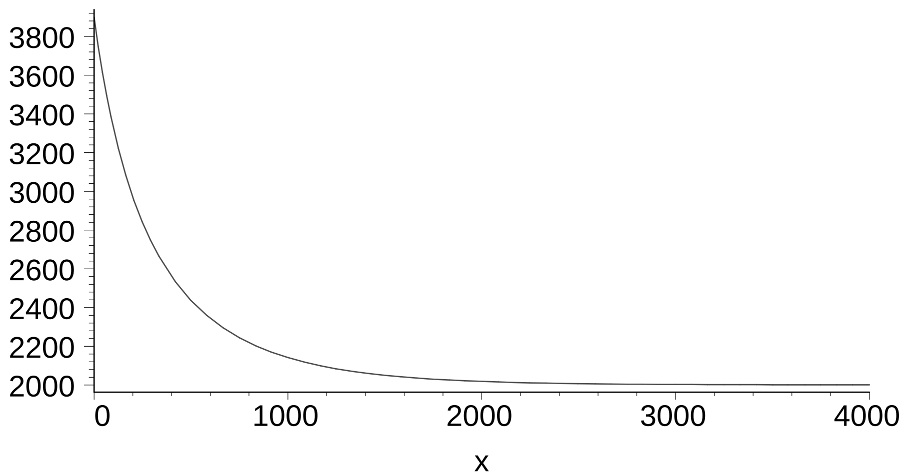

Figure 2: Logistic growth with $P_0 = 3900, k = 2000$ and $r = .002$. Note, that the horizontal axis should be t and the vertical axis the size of the population P.

Exercises 6.2.

1. You can see from the logistic growth graph in Figure 1, that if P_0 is small compared to k, then the graph begins by being concave up and then becomes concave down. Recall from Calculus 1 that this is the point of inflection. Find this point in terms of the population P.

2. Suppose that you have a population that grows according to the logistic growth model. In that model, the time t will be measured in hours. Suppose that the growth constant is $r = 1$, $P_0 = 100$ and the carrying capacity $k = 2000$. How long will it take the population to reach 90% of its maximum?

6.3 Applications to mechanics: projectiles

We now discuss applications from physics, namely mechanics. Basic to classical physics, are Newton's laws of motion. We review them here.

The following are Newton's laws of motion.

1. Objects in motion tend to stay in motion and objects at rest tend to stay at rest unless acted upon by an external force.

2. The time rate of change of momentum (which is mass $\times$ velocity) is proportional to the sum of the forces on that body.

3. Every action has an equal and opposite reaction.

We will often use x for horizontal position and y as vertical position. Here, velocity is the time rate of change of position and acceleration is the time rate of change of velocity. Again, we use the "dot" notation: the velocity v is $v = \dot{x} = \frac{dx}{dt}$, and the acceleration a is $a = \dot{v} = \ddot{x}$.

When needed, we use the metric or SI units (the are called SI units from the French *Système international d'unités*) of measurements. In these units, mass is given in kilograms kg, distance in metres m, time in seconds s. We will, however, try not to dwell on units. A common constant we will use is the acceleration of gravity, which is denoted by g and is $9.8\,\mathrm{m/s^2}$. It is often convenient to estimate g as $10\,\mathrm{m/s^2}$. This is the acceleration of gravity close to the surface of the earth. We will also use the unit the Newton N for force. $1\,\mathrm{N}$ is equivalent to $1\,\mathrm{kg}\cdot\mathrm{m/s^2}$. We note that position, velocity and acceleration are all vector quantities. We refer to the magnitude of velocity as speed.

The second law states that the sum of the forces is equal to $\dot{mv}$, and since mass is constant in Newton's mechanics, this simply becomes $m\dot{v} = ma = m\ddot{x}$ is the sum of the forces. Let's look at a simple example.

6.3.1 Projectiles without air resistance

Suppose that a ball is thrown/dropped straight up into the air. Let $y(t)$ be the height of the ball at time t in metres (so up is the positive y direction), with $y = 0$ being ground level. This is, of course, an arbitrary coordinate system. We could, for example, have a height of 10 metres as the point $y = 0$ and the positive y direction pointing down. Suppose that the ball is thrown straight up in the air with initial velocity v_0 and initial position y_0.

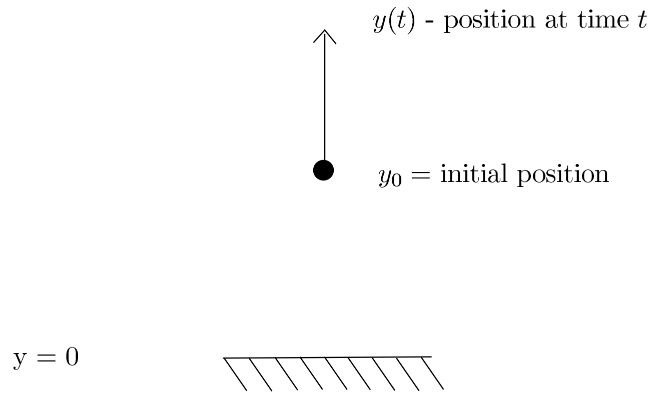

Figure 3: Schematic of the thrown ball.

The only force on the travelling ball (after it's initial throw) is gravity. The force of gravity is simply the mass of the ball times the acceleration of gravity; that is, F_{grav}, the force of gravity, has magnitude mg, and its sign will depend on whether the positive direction is up (minus sign) or down (plus sign). In this case, the positive y direction is up, so $F_{\text{grav}} = -mg$. Thus, using Newton's second law, we have the equation of motion

$$m\dot{v} = m\ddot{y} = -mg, \tag{5}$$

where m is the mass in kilograms and g is the acceleration of gravity. Notice the sign of the force; it is negative since the force is pointing in the negative y direction (down).

The equation in (5) is the simplest differential equation we can come across. Notice that there is no occurrence of y in the differential equation, so this equation becomes

$$\begin{aligned} & m\frac{dv}{dt} = -mg \\ \Rightarrow \quad & dv = -g\,dt \\ \Rightarrow \quad & v = -gt + v_0. \end{aligned} \tag{6}$$

To get the equation for position, we integrate again

$$\begin{aligned} & v = \dot{y} = -gt + v_0 \\ \Rightarrow \quad & \int dy = \int (-gt + v_0)\,dt \\ \Rightarrow \quad & y = -1/2gt^2 + v_0 t + y_0. \end{aligned} \tag{7}$$

Notice the independence of the position and velocity equations on mass. We can now do an example.

Example 6.3. Suppose that a ball is dropped from a height h metres. How long does it take to hit the ground?

We can use (7) to solve this problem. Since the ball is dropped (i.e. isn't given any initial velocity) from a height h, we see $v_0 = 0$ and $y_0 = h$. We want to know when the ball will reach the coordinate $y = 0$. Substituting these into (7) we get

$$\begin{aligned} 0 &= -1/2gt^2 + 0t + h \\ \Rightarrow \quad t &= \sqrt{\frac{2h}{g}}. \end{aligned}$$

(sanity check: time is positive!).

Example 6.4. Suppose the ball is thrown straight up with a velocity of $5\,\mathrm{m/s}$ from a height of $1\,\mathrm{m}$.[1] What is the maximum height of the ball?

At the maximum height of the ball, the velocity is 0. So, we want to find $y(t_{max_h})$ where $v(t_{max_h}) = 0$. Thus, we first need to find t_{max_h}. We use (6) to find this, so

$$\begin{aligned} 0 &= -gt_{max_h} + 5 \\ \Rightarrow t_{max_h} &= 5/g \simeq 1/2\,\mathrm{s}, \end{aligned}$$

where we have used the approximation that $g = 10\,\mathrm{m/s^2}$. Therefore, using t_{max_h} and (7) we get

$$\begin{aligned} y(t_{max_h}) &= -1/2gt_{max_h}^2 + v_0 t_{max_h} + y_0 \\ &= -5 \cdot (1/2)^2 + 5 \cdot 1/2 + 1 \\ &= -5/4 + 5/2 + 1 \\ &= 9/4 = 2.25. \end{aligned}$$

Thus, we get 2.25 metres as the answer.

6.3.2 Projectiles with air resistance and terminal velocity

The case in Section 6.3.1 contained a rather simple differential equation, but was useful in introducing notation to those unfamiliar with physics. We now talk about a more realistic scenario, that of projectiles moving in a fluid (say air) where there is a drag force, like wind resistance. Intuitively, we may think that magnitude of force due to wind resistance should depend on speed, and this turns out to be a good approximation. That is, the magnitude of force due to wind resistance can be

[1] Just as an aside, how fast is a $5\,\mathrm{m/s}$ ball? Apparently, the fastest recorded cricket pitch is whopping $160\,\mathrm{km/h} \simeq 45\,\mathrm{m/s}$, so not that fast! In comparison, a weakish gun apparently fires a bullet at around $400\,\mathrm{m/s}$.

taken as either proportional to speed or the square of the speed. Of course, any drag force will oppose the direction of motion; that is, it will act in a direction opposite to motion. Let's set up the differential equation in the case when the force due to resistance is proportional to speed.

Let's suppose the scenario is that we have some object falling and the force of air resistance is proportional to speed. Since we will be thinking of a falling object, we use the coordinate system that $y = 0$ is the height from which the object was dropped and the positive y direction is down. We could, as before, use a different coordinate system, but we use this system in this case as it is convenient. Using Newton's second law we see that there are two forces on the object, gravity and air resistance. Since we are assuming the drag force is proportional to velocity, we see that if F_{drag} is the drag force that $F_{\text{drag}} = -\beta v$, for some constant $\beta > 0$ (notice the sign on the drag force). In SI units, β has units kg/s, making βv measured in N (remember, βv is force, so should be measured in Newtons!). The constant β is ordinarily dependent on the fluid (in this case air) and on the shape and volume of the object.

Thus, the equation of motion for our projectile is

$$ma = m\dot{v} = mg + F_{\text{drag}} = mg - \beta v.$$

Again, notice the signs on the forces; gravity is in the positive y direction (down) and so has a positive sign and the air resistance is against the direction of velocity, hence the negative sign. This equation is separable, so we get

$$\begin{aligned}
& m\dot{v} = mg - \beta v \\
\Rightarrow \quad & \int \frac{dv}{g - \frac{\beta}{m} v} = \int dt \\
\Rightarrow \quad & -\frac{m}{\beta} \ln\left(g - \frac{\beta}{m} v\right) = t + c \\
\Rightarrow \quad & \ln\left(g - \frac{\beta}{m} v\right) = -\frac{\beta}{m} t + D \\
\Rightarrow \quad & g - \frac{\beta}{m} v = A e^{-\frac{\beta}{m} t} \\
\Rightarrow \quad & v = \frac{mg}{\beta} - B e^{-\frac{\beta}{m} t} \qquad (8)
\end{aligned}$$

Now, let us analyse the behaviour of this system. Suppose that the object was dropped, so that $v(0) = 0$, so that we get

$$\begin{aligned}
0 &= \frac{mg}{\beta} - B \\
\Rightarrow \quad B &= \frac{mg}{\beta}.
\end{aligned}$$

Thus, we have

$$v = \frac{mg}{\beta}\left(1 - e^{-\frac{\beta}{m} t}\right). \qquad (9)$$

When $t \to \infty$, we see that $v \to \frac{mg}{\beta}$. Thus, we see that the velocity starts with a value of 0 and increases in the down direction until the maximum velocity mg/β has been reached. This is known as the *terminal velocity* of an object. We can of course find an equation for the position of the object after time t by simply integrating (9).

Example 6.5. Notice that a dropped object never actually reaches terminal velocity (because the exponential term in (9) is never 0, it just approaches it). Let's determine when a dropped object reaches 90% of terminal velocity. The terminal velocity is $\frac{mg}{\beta}$ so we want the time t_{90} such that

$$.9\frac{mg}{\beta} = \frac{mg}{\beta}\left(1 - e^{-\frac{\beta}{m}t_{90}}\right).$$

Therefore,

$$\begin{aligned}
\Rightarrow \quad & e^{-\frac{\beta}{m}t_{90}} = 0.1 \\
\Rightarrow \quad & -\frac{\beta}{m}t_{90} = \ln 0.1 \\
\Rightarrow \quad & t_{90} = -\frac{m}{\beta}\ln 0.1.
\end{aligned}$$

Notice that t_{90} is positive because $\ln 0.1 < 0$.

Exercises 6.3.

1. Find the equation for the position of a mass m which is dropped (i.e. $v(0) = 0$) from some height and has a drag force equivalent to $-\beta v$.

2. A person of mass 50 kg with a parachute is falling and opens the chute when travelling at 5 m/s, and then floats down to safety because of air resistance. Take time $t = 0$ to be the point when the parachute is opened. Suppose that the air resistance produces a force of magnitude 200 N when the velocity of the person is 5 m/s. Assume that the force of air resistance is proportional to the velocity (as in the notes).

   (a) Find the constant β, the coefficient of v for air resistance.

   (b) Write down the differential equation of motion and solve it. Take care to state the initial conditions (note the initial velocity is not 0). Find both the velocity and position as a function of time. Use the coordinate system that $y = 0$ when the chute is opened.

   (c) Find the terminal velocity.

6.4 The harmonic oscillator

The harmonic oscillator is a classical example from physics of a second order differential equation with constant coefficients. It turns out that there are other systems

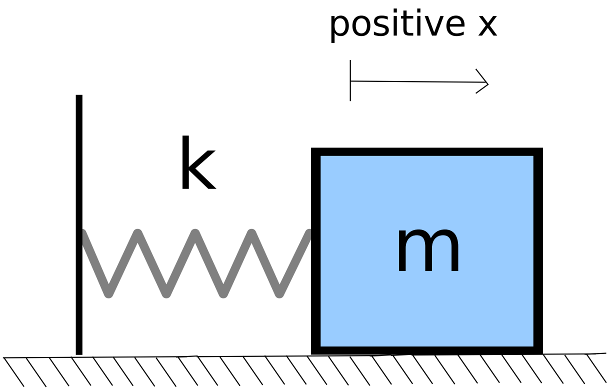

Figure 4: The simple harmonic oscillator. When the spring is not stretched it is at equilibrium. The distance x is measured from equilibrium and the positive x direction is usually to the right.

that are governed by these types of equations, namely in electrical circuits, but this type of system lends itself to being very instructive as we can use physical intuition to help us gain understanding into the various results we obtain. Let us begin with the undamped motion of a harmonic oscillator.

6.4.1 Undamped harmonic motion

We study the very simple system of a mass attached to a spring with one end fixed to an immovable object (a wall say). We refer the student to the diagram in Figure 4. We start with the simplest case where the only horizontal force on the mass is the spring; so, friction is negligible and there is no drag force. For a spring, there is a position known as *the equilibrium position*; in this position the spring does not exert any force on the mass. When the spring is stretched it exerts a *linear restoring force*; that is, the force of the spring tends to pull (or push) the mass towards the equilibrium position and the force is proportional to the distance the mass is displaced. This is known as *Hooke's law*. Referring to Figure 4, when the spring is stretched in the positive x direction, the force is in the minus x direction and when the spring is compressed in the minus x direction (to the left), the force is in the positive x direction. Thus, we see that if the force due to the spring is labelled F_{spring}, then $F_{\text{spring}} = -kx$ (notice that the force is always in the correct direction in this form). Here, x is the position from equilibrium (so $x = 0$ is the equilibrium position) and k is known as the *spring constant*, which depends on the stiffness of the spring. Since x is measured in metres, k is usually measured in N/m. We can therefore conclude the equation of motion is

$$ma = F_{\text{spring}} = -kx.$$

Hence, we obtain the differential equation

$$m\ddot{x} + kx = 0.$$

This is a second order differential equation with constant coefficients. We know how to solve it. The auxiliary equation is $mr^2 + k = 0$, which implies $r = \pm i\sqrt{\frac{k}{m}}$. Set $\omega_0 = \sqrt{\frac{k}{m}}$. The parameter ω_0 is known as the *fundamental* or *natural frequency.* Thus, the general solution to the differential equation is from Chapter 4

$$x(t) = B\sin\omega_0 t + C\cos\omega_0 t, \tag{10}$$

where B and C are constants. We can write (10) in the form

$$x(t) = A\sin(\omega_0 t + \phi) \tag{11}$$

(there is nothing special about sin, we can also write it as $A\cos(\omega_0 t + \phi)$). To see this, we use the double angle identity and expand

$$\begin{aligned} A\sin(\omega_0 t + \phi) &= A\sin\omega_0 t\cos\phi + A\cos\omega_0 t\sin\phi \\ &= B\sin\omega_0 t + C\cos\omega_0 t, \end{aligned}$$

where $B = A\cos\phi$ and $C = A\sin\phi$. Thus, we see that in the form (11), the A and ϕ are the two arbitrary constant. We call ϕ the *fundamental displacement.* Notice further that the constant A is the amplitude of the motion; that is, it is the maximum displacement from equilibrium.

Thus, we see that the behaviour of the system is purely sinusoidal, which makes sense intuitively; if there is no drag force, the system will simply oscillate forever back and forth. We can now see how initial conditions affect our system.

Example 6.6. Suppose that $x(0) = 0$ and $x'(0) = 0$, and let's take (11) as the equation of motion for the position x. We therefore have the following system of equations:

$$0 = x(0) = A\sin\phi$$

and

$$0 = x'(0) = A\omega_0\cos\phi.$$

Since no angle ϕ makes both $\cos\phi$ and $\sin\phi$ equal 0, we must have $A = 0$. Therefore, $x(t) = 0$ is the equation of the position of the mass at time t.

This is intuitively the case. If the initial position is at equilibrium, the spring is neither stretched nor compressed, and there is no initial velocity, then the system will never move.

Example 6.7. Suppose that a mass m attached to a spring is stretched $d > 0$ metres from equilibrium and released. Assuming simple harmonic motion with no damping, find the equation of motion for x.

First, let us discuss the intuition behind the system. Since this is simple harmonic motion with no damping, we assume (11) as the equation for the displacement x. Since this is purely harmonic sinusoidal motion, we would expect that the maximum distance from equilibrium is d and this is reached at every half period. This process should repeat indefinitely. Now, let's see if this intuition can be extracted from the solution to the differential equation.

If the mass is pulled to some distance d (metres) and released at $t = 0$, we see that $x(0) = d$ and $x'(0) = 0$. Then these two conditions imply

$$d = A\sin(\omega_0 0 + \phi) = A\sin\phi \tag{12}$$
$$0 = A\omega_0\cos(\omega_0 0 + \phi) = A\omega_0\cos\phi. \tag{13}$$

Since $d > 0$, from (12), $A \neq 0$, so $\cos\phi = 0$ from (13), which implies that $\phi = \pi/2$ (we can choose ϕ from 0 to 2π). Then, from (12), we see that $d = A\sin(\omega_0 0 + \pi/2) = A$. Thus, $A = d$. We therefore have that the equation for x is

$$x(t) = d\sin(\omega_0 t + \pi/2).$$

Thus, we see that the amplitude is d and this motion occurs forever.

We know from elementary trigonometry that the period of the motion for (11) is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k/m}}.$$

The period is usually measured in seconds (consistent with our other units!).

6.4.2 Simple harmonic motion with damping

In the last section we saw an idealised version of the simple harmonic oscillator as there is no damping force. Intuitively, we know that the system pictured in Figure 4 should not simply oscillate forever, but it should slowly or quickly come to a stop. We now consider the more realistic case where there is a damping force. We assume this force is proportional to velocity (similar to the case of the projectile in a fluid).

Suppose that there is damping force equal to βv in the system in Figure 4 for some $\beta > 0$. This force will of course oppose motion. Thus, if F_{drag} is the the drag force, we see that $F_{\text{drag}} = -\beta v$. So, we obtain the following equation for the motion of the system:

$$ma = F_{\text{drag}} + F_{\text{spring}} = -\beta v - kx.$$

Since $a = \ddot{x}$ and $v = \dot{x}$, we therefore find that

$$m\ddot{x} + \beta\dot{x} + kx = 0.$$

This is a linear differential order two equation with constant coefficients. The auxiliary equation is

$$mr^2 + \beta r + k = 0,$$

from which we obtain

$$\begin{aligned} r &= \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} \\ &= \frac{-\beta}{2m} \pm \sqrt{\frac{\beta^2}{4m^2} - \frac{k}{m}} \\ &= \frac{-\beta}{2m} \pm \sqrt{\frac{\beta^2}{4m^2} - \omega_0^2}. \end{aligned} \tag{14}$$

It will be useful to set

$$\tau = \sqrt{\frac{\beta^2}{4m^2} - \omega_0^2},$$

where we are again setting $\omega_0 = \sqrt{k/m}$ (as in the last section). There are three cases for τ: τ is real and non-zero, τ is zero and τ is imaginary. Note that from (14), these three scenarios correspond to $\beta^2 > 4mk$, $\beta^2 = 4mk$ and $\beta^2 < 4mk$, respectively. We can interpret these three scenarios physically as the damping force is strong, medium, or weak, respectively, relative to the product mk. Let's look at each case separately.

Case 1: τ is real and non-zero. In this case, we can further show that both solutions are negative, since we clearly have

$$\begin{aligned} & \frac{\beta}{2m} > \sqrt{\left(\frac{\beta}{2m}\right)^2 - \omega_0^2} \\ \Rightarrow \quad & 0 > \frac{-\beta}{2m} + \sqrt{\frac{\beta^2}{4m^2} - \omega_0^2}. \end{aligned}$$

Thus, we see that $\frac{-\beta}{2m} \pm \tau$ is always negative. From Chapter 4.5, we know the solution to this system is

$$x(t) = Ae^{\left(-\frac{\beta}{2m}+\tau\right)t} + Be^{\left(-\frac{\beta}{2m}-\tau\right)t}. \tag{15}$$

That is, we see the motion of the system exponentially decays and the mass comes to a rest at $x = 0$ after a long time. Notice that there are no oscillations in this case. This makes sense: a very strong damping force brings the system to a halt very quickly. This type of system is called *over damped*.

Case 2: $\tau = 0$. In this case we have only one solution to our auxiliary equation and it is $\frac{-\beta}{2m}$. Hence, from Chapter 4, the solution to our differential equation is

$$x(t) = Ae^{\frac{-\beta}{2m}t} + Bte^{\frac{-\beta}{2m}t}. \tag{16}$$

Again, we see that our system has no oscillations but comes to a halt very quickly. This type of system is called *critically damped*, as any reduction in β will produce oscillations, as we shall see in the next case.

Case 3: τ is not real. Hence, $\tau = i\omega_d$, ω_d is real. In this case, we see that the solution to our auxiliary equation is $-\frac{\beta}{2m} \pm i\omega_d$. Thus, from Chapter 4, the solution is

$$x(t) = Ae^{-\frac{\beta}{2m}t}\sin\omega_d t + Be^{-\frac{\beta}{2m}t}\cos\omega_d t.$$

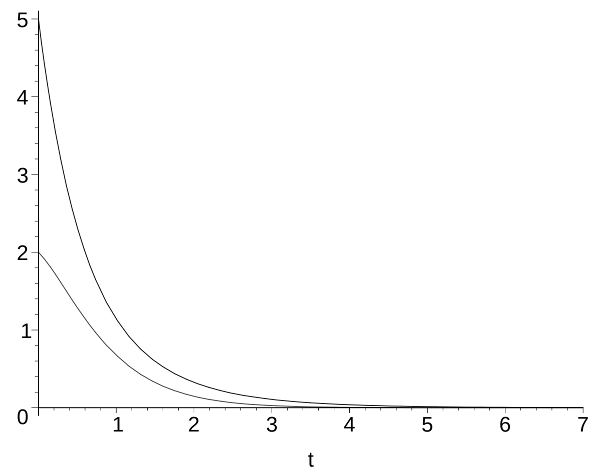

Figure 5: A critically damped system and an overdamped system. In this graph, the critically damped function is $x(t) = 2e^{-2t} + 3te^{-2t}$ and the over damped system is $x(t) = 2e^{-t} + 3e^{-2t}$, and the overdamped system is above the critically damped. Thus, the curve which returns to equilibrium faster is the critically damped one. In the graph, the vertical axis is the position x.

Using the same algebraic trick as in Section 6.4 in (11), we can rewrite the previous equation as

$$x(t) = Ae^{-\frac{\beta}{2m}t}\sin(\omega_d t + \phi),$$

where A and ϕ are two arbitrary constants which depend on the initial conditions. We call ω_d the *modified* or *quasi frequency*. We can set $\mathcal{A}(t) = Ae^{-\frac{\beta}{2m}t}$, and think of this as the amplitude term. Since $\mathcal{A}(t)$ depends on time, we refer to it as the *time varying amplitude*. Thus, we see that in this case we do get oscillations, but they die out slowly as time progresses because the amplitude term $\mathcal{A}(t) = Ae^{-\frac{\beta}{2m}t}$ diminishes as time passes. We call this type of system *under damped*.

The difference between an over damped and critically damped system is that generally critically damped systems return to equilibrium faster. Critically damped systems are often found in doors of office buildings. See Figure 5 for a comparison. In Figure 6, we have a classic picture of the under damped case.

Let us look at some specific examples and see how initial conditions affect the system.

Example 6.8. Suppose that we have a critically damped system and suppose that the initial conditions are $x(0) = A_{max}$ and $x'(0) = v_0$. Notice that physically this corresponds to taking the mass, pulling it to a position A_{max} and releasing it giving it some initial velocity. To make our lives a little simpler, let's assume that $m = 1, \beta = 2$ and $k = 1$. This is, however, not necessary, but will just make our algebraic lives easier. The important thing is that the system is critically damped and that $x(0) = A_{max}$ and $x'(0) = v_0$. Then the equation of motion for the mass is

$$\ddot{x} + 2\dot{x} + 1x = 0,$$

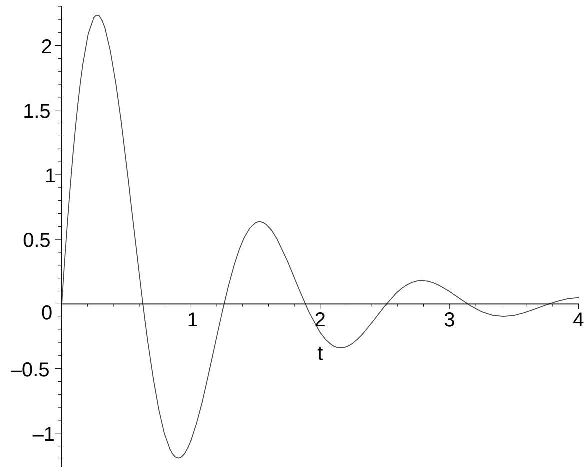

Figure 6: An under damped oscillator. The equation in the graph above is $x(t) = 3e^{-t}\sin(5t)$. In the graph, the vertical axis is x, the position of the mass.

and the solution is

$$x(t) = Ae^{-t} + Bte^{-t} \tag{17}$$

(you can either solve this directly or use (16) above to obtain this solution). Using the initial conditions we have

$$\begin{aligned} A_{max} &= x(0) = A \\ v_0 &= x'(0) = -A + B. \end{aligned}$$

Solving this linear system we obtain $A = A_{max}$ and $B = A_{max} + v_0$. Thus, our solution is

$$x(t) = A_{max}e^{-t} + (A_{max} + v_0)te^{-t}.$$

Does this system ever pass through the equilibrium point? For this to happen we would need

$$\begin{aligned} & 0 = A_{max}e^{-t} + (A_{max} + v_0)te^{-t} \\ \Rightarrow \quad & -A_{max}e^{-t} = (A_{max} + v_0)te^{-t} \\ \Rightarrow \quad & t = \frac{-A_{max}}{A_{max} + v_0} \end{aligned} \tag{18}$$

Notice that physically, we want solutions such that $t \geq 0$. If we simply release the mass with no initial velocity (i.e. $v_0 = 0$), we won't have any positive solution. In fact, we only have a positive solution if $v_0 < -A_{max}$ (notice the sign on v_0). See Figure 7. If the reader wishes, you can abandon the parameters m, β and k given and do the computation for arbitrary critically damped parameters. The result should be the same. Namely, that a critically damped system returns to equilibrium at most once. Furthermore, we can make a similar computation in the over damped case; that is, the system will only return to equilibrium at most once.

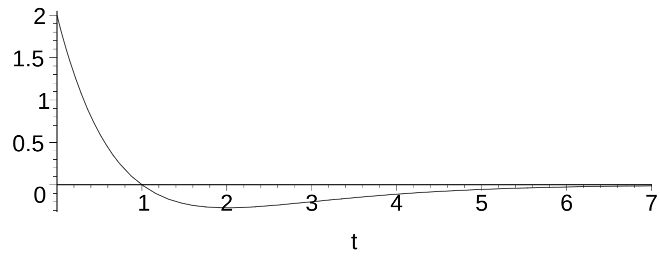

Figure 7: This is a critically damped oscillator that passes through the equilibrium position. Here the equation of motion is $x(t) = 2e^t - 2te^t$.

6.4.3 Harmonic motion with a periodic forcing function

Until now, we have considered unforced harmonic oscillators. That is, there is no external force applied. It is common to have an external force which is periodic with respect to time (think of a child on a swing being pushed).

Let $F(t)$ be the external force on a damped oscillating mass. Then, the differential equation governing the system is

$$m\ddot{x} = -\beta\dot{x} - kx + F(t)$$

where β and k are our usual physical constants associated with drag and springs. We have learned to solve such equations under certain conditions.

Example 6.9. Suppose that we have a 1 kg mass attached to a spring. The damping force is equivalent to 8 times the instantaneous velocity. Suppose that the spring stretches 1/4 of a metre when 16 N of force is applied to the mass. There is a periodic forcing function of $32\cos 8t$. Suppose that mass is pulled to 1/4 metre and then released with no initial velocity. Find the motion of the system.

First of all, we need to find the spring constant k. Since the spring stretches 1/4 of a metre under 16 N of force, we see that $k(1/4) = 16$ implying that $k = 64$. Thus, the differential equation governing the system is

$$\ddot{x} + 8\dot{x} + 64x = 32\cos 8t.$$

From Chapter 4, we need both the complementary function and a particular solution to solve this equation. We see that the auxiliary equation is $r^2 + 8r + 64 = 0$, which has roots $r = -4 + i4\sqrt{3}$. Thus, the complementary solution is

$$Ae^{-4t}\sin(4\sqrt{3}t + \phi).$$

Let's try $p = a\sin 8t + b\cos 8t$ for the particular integral. We find that

$$\begin{aligned}\dot{p} &= 8a\cos 8t - 8b\sin 8t\\ \ddot{p} &= -64a\sin 8t - 64b\cos 8t.\end{aligned}$$

So, we see that

$$\begin{aligned}
&(-64a\sin 8t - 64b\cos 8t) + 8(8a\cos 8t - 8b\sin 8t)\\
&\qquad + 64(a\sin 8t + b\cos 8t) = 32\cos 8t\\
\Rightarrow\quad &64a\cos 8t - 64b\sin 8t = 32\cos 8t.
\end{aligned}$$

Thus $a = 1/2$ and $b = 0$. The general solution of the differential equation is therefore

$$x(t) = Ae^{-4t}\sin(4\sqrt{3}t + \phi) + \frac{1}{2}\sin 8t,$$

where A and ϕ are arbitrary constants.

To find these constants we use the initial conditions. We see that

$$\begin{aligned}
1/4 &= x(0)\\
&= A\sin\phi,
\end{aligned}$$

which implies that $A = \frac{1}{4\sin\phi}$. Also, we have

$$\begin{aligned}
0 &= x'(0)\\
&= -4Ae^{-4t}\sin(4\sqrt{3}t+\phi) + 4\sqrt{3}Ae^{-4t}\cos(4\sqrt{3}t+\phi) + 4\cos 8t|_{t=0}\\
&= -4A\sin\phi + 4\sqrt{3}A\cos\phi + 4.
\end{aligned}$$

Substituting $A = \frac{1}{4\sin\phi}$ into the last equation we get,

$$\tan\phi = -\frac{\sqrt{3}}{3}.$$

Pick ϕ to be the smallest angle that does this, so $\phi = 5\pi/6$. You work out, if you like, that $\sin\phi = 1/2$ and $\cos\phi = -\frac{\sqrt{3}}{2}$. Thus, our solution is

$$x(t) = \frac{1}{2}e^{-4t}\sin\left(4\sqrt{3}t + \frac{5\pi}{6}\right) + \frac{1}{2}\sin 8t.$$

We call $\frac{1}{2}e^{-4t}\sin\left(4\sqrt{3}t + \frac{5\pi}{6}\right)$ the *transient solution* because it dies out with time and $\frac{1}{2}\sin 8t$ the *steady state solution* since it persists as time goes on. That is, after some time we have

$$x(t) \approx \frac{1}{2}\sin 8t.$$

Exercises 6.4.

1. A 1 kg mass is attached to a spring with spring constant $k = 16$. Neglect any drag forces. Suppose that the weight is pulled 10 cm to the right and released at $t = 0$.

   (a) Set up the differential equation and state the initial conditions.

(b) Find the equation for the velocity v and the position x as a function of time.

(c) Find the maximum distance from the equilibrium point reached by the mass and period of motion.

2. Redo Question 1 parts b and c if the initial conditions are that mass starts at the equilibrium position and has a initial velocity of $-0.3\,\mathrm{m/s}$.

3. A 2 kg weight is attached to a spring. If you pull the mass with a force of 2 N, the mass is displaced by 1/4 of a metre. Assume a drag force equivalent to $-8v$, where v is the instantaneous velocity. Suppose that the system is set into motion at $t = 0$ by pulling the mass a distance of 0.1 m in the positive x-directions and given a speed in the opposite direction of $0.3\,\mathrm{m/s}$.

   (a) Find the spring constant k.

   (b) Find the equation for the position x. Is the system over, under or critically damped?

   (c) Does the mass ever pass through the equilibrium point?

4. Suppose that a damped oscillator has damping force equal 10 times its velocity. When the mass is pulled with a force of 2.5 N the spring stretches 0.1 m. Suppose that the system is set into motion from equilibrium with an initial velocity of $0.3\,\mathrm{m/s}$. An external force $F(t) = 5\sin t$ is applied to the system. The mass is 5 kg.

   (a) Find the value of the spring constant k.

   (b) Find the differential equation that governs the system, including the initial conditions.

   (c) Solve the differential equation.

   (d) Is the system over damped, under damped or critically damped? If is is under or over damped, find out if the mass ever passes equilibrium. If it is under damped, state the transient part of the solution and the steady state part of the solution.

5. Suppose that in the previous question we neglect damping and there is no external force. Find the maximum distance from equilibrium reached by the mass.

6.5 An application from chemistry

In this application we are going to see how to model the change in concentration of a substance in water when a solution of fixed concentration is being added.

Consider the following scenario. A tank has 40 L of water with 3 kg of salt forming a brine. The concentration of salt can be measured in units kg/L. A brine of concentration 0.25 kg/L enters the tank at 8 L/min (min = minute). The mixture is continuously stirred and leaves the tank at the same rate. Hence, we do not need to account for any concentration gradient in the tank (i.e. all the water in the tank is of the same concentration) and the amount of water in the tank is constant.

Let us determine the following.

1. The salt at time t.
2. How much salt is present in 10 minutes.
3. The long term concentration of salt.

For the first question, let $A(t)$ be the amount of salt at time t minutes. Clearly,

$$\frac{dA}{dt} = \text{ rate of amount gained} - \text{rate of amount lost}$$

Since brine is entering with concentration 0.25 kg/L, and entering at a rate of 8 L/min, we see that the rate the amount of salt entering the tank is

$$8\,\mathrm{L/min} \times 0.25\,\mathrm{kg/L} = 2\,\mathrm{kg/min}.$$

The rate at which salt is leaving the tank is

$$A\,\mathrm{kg}/40\,\mathrm{L} \times 8\,\mathrm{L/min} = \frac{A}{5}\,\mathrm{kg/min}.$$

Thus, we see that

$$\frac{dA}{dt} = 2 - \frac{A}{5}.$$

We therefore have a separable differential equation. The initial condition is $A(0) = 3$. Solving we get

$$\begin{aligned} \frac{dA}{10 - A} &= \frac{1}{5}dt \\ \Rightarrow \quad -\ln(10 - A) &= \frac{t}{5} + c \\ \Rightarrow \quad \ln(10 - A) &= -\frac{t}{5} - c \\ \Rightarrow \quad 10 - A &= Be^{-\frac{t}{5}} \\ \Rightarrow \quad A &= 10 - Be^{-\frac{t}{5}}, \end{aligned}$$

where B is the arbitrary constant of integration. Using the initial condition we have $B = 7$. Thus, we have the equation

$$A(t) = 10 - 7e^{-\frac{t}{5}}, \tag{19}$$

expressing the amount of salt in the tank as a function of time.

For the second part of the question, we set $t = 10$. We obtain $A(10) = 10 - 7e^{-2} = 9.05$.

For the final part of the question, let's first think about this physically. It makes sense that after a long time, the concentration of the water in the tank should reach the concentration of the water entering the tank. Since there is 40 L in the tank and the concentration of the water entering the tank is 0.25 kg/L, we should eventually have 10 kg of salt in the tank. Taking the limit as $t \to \infty$ in (19) gives this.

Exercises 6.5.

1. Suppose we have a melting snowball whose volume decreases at a rate proportional to its surface area. If the snowball had radius 10 cm at time $t = 0$ and has shrunk to radius 5 cm at time $t = 1$ hour, then when will it have melted completely? [We assume that the snowball remains spherical while melting. The volume and surface area of a sphere with radius r are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ respectively.]