Solutions Chapter 3

Solutions to Exercises 3.1.

1. (a) $\frac{1}{2}x^6 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x + c$

   (b) $-2x^{-2} - \frac{5}{4}x^{-4} + c$

   (c) This integral can be evaluated using the substitution $u = x + 1$. Alternatively, we can first use polynomial division to get

   $$\frac{x^3 + 3x^2}{x + 1} = x^2 + 2x - 2 + \frac{2}{x + 1},$$

   and then integrate this to find $1/3x^3 + x^2 - 2x + 2\ln|x + 1| + c$.

   (d) $1/18(3x + 1)^6 + c$ (substitution $u = 3x + 1$)

   (e) $-1/3\cos(3x - 2) + c$ (substitution $u = 3x - 2$)

   (f) $1/9(x^2 + 7)^9 + c$ (substitution $u = x^2 + 7$)

   (g) $1/4\exp(2x^2) + c$ (substitution $u = 2x^2$)

   (h) $1/8\sin^8(x) + c$ (substitution $u = \sin(x)$)

   (i) $-\ln|\cos(x)| + c$ (substitution $u = \cos(x)$)

   (j) Note that $\frac{x+3}{x^2+4} = \frac{x}{x^2+4} + \frac{3}{x^2+4}$. Integrate separately

   $$\int \frac{x}{x^2 + 4}\,\mathrm{d}x = 1/2\ln(x^2 + 4) \qquad \text{(subst. } u = x^2 + 4\text{)},$$
   $$\int \frac{3}{x^2 + 4}\,\mathrm{d}x = \int \frac{3/4}{(x/2)^2 + 1}\,\mathrm{d}x = 3/2\arctan(x/2) \qquad \text{(subst. } u = x/2\text{)}.$$

   Add them together and throw in a constant c.

   (k) $x\sin(x) + \cos(x) + c$ (integration by parts with $u = x$, $v' = \cos x$)

   (l) $\frac{1}{11}x(x + 1)^{11} - \frac{1}{132}(x + 1)^{12} + c$ (integration by parts with $u = x$, $v' = (x + 1)^{10}$)

   (m) $-1/2x\cos(2x) + 1/4\sin(2x) + c$ (integration by parts with $u = x$, $v' = \sin(2x)$)

   (n) $-xe^{-x} - e^{-x} + c$ (integration by parts with $u = x$, $v' = e^{-x}$)

   (o) $x^2e^x - 2xe^x - 2e^x + c$ (use integration by parts twice: first time with $u = x^2$ and $v' = e^x$, second time with $u = x$ and $v' = e^x$)

   (p) We use the substitution $x = \tan u$. Then $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{\cos^2 u}$ and hence

   $$\int \arctan x\,\mathrm{d}x = \int \arctan(\tan u)\cdot\frac{1}{\cos^2 u}\,\mathrm{d}u = \int u\frac{1}{\cos^2 u}\,\mathrm{d}u.$$

   Integrating this by parts gives

   $$\int u\frac{1}{\cos^2 u}\,\mathrm{d}u = u\tan u - \int 1\cdot\tan u\,\mathrm{d}u = u\tan u + \ln|\cos u| + c.$$

Now note that $\frac{1}{\cos^2 u} = \frac{\sin^2 u + \cos^2 u}{\cos^2 u} = \tan^2 u + 1$ and hence $\cos u = 1/\sqrt{1+\tan^2 u}$. Thus

$$\int \arctan x \, \mathrm{d}x = u \tan u + \ln|\cos u| + c = x \arctan(x) + \ln\left(1/\sqrt{1+x^2}\right) + c.$$

(q) $x\ln(x) - x + c$ (integration by parts with $u = \ln x$ and $v' = 1$)

(r) $x \arccos(x) - \sqrt{1-x^2} + c$ (first substitution $x = \cos(u)$, then integration by parts)

2. (a) $\int \frac{\mathrm{d}x}{x \ln x} = \int \frac{1}{u} \, \mathrm{d}u = \ln|u| + c = \ln|\ln x| + c$

(b) $\int \frac{4x}{(3-2x)^2} \, \mathrm{d}x = \int \frac{u-3}{u^2} \, \mathrm{d}u = \ln|u| + \frac{3}{u} + c = \ln|3-2x| + \frac{3}{3-2x} + c$

(c) $\int \sec^5 x \tan x \, \mathrm{d}x = \int u^4 \, \mathrm{d}u = \frac{1}{5}u^5 + c = \frac{1}{5}\sec^5 x + c$

(d) $\int \frac{2e^{2x}}{1+e^{4x}} \, \mathrm{d}x = \int \frac{1}{1+u^2} \, \mathrm{d}u = \arctan(u) + c = \arctan(e^{2x}) + c$

(e) $\int 5\sin^7 x \, \mathrm{d}x = \int -5(1-u^2)^3 \, \mathrm{d}u = -5u + 5u^3 - 3u^5 + \frac{5}{7}u^7 + c$
$= -5\cos(x) + 5\cos^3(x) - 3\cos^5(x) + \frac{5}{7}\cos^7(x) + c$

(f) $\int 14x^2\sqrt{1+x} \, \mathrm{d}x = \int 28u^2(u^2-1)^2 \, \mathrm{d}u = 4u^7 - \frac{56}{5}u^5 + \frac{28}{3}u^3 + c$
$= 4(x+1)^{7/2} - \frac{56}{5}(x+1)^{5/2} + \frac{28}{3}(x+1)^{3/2} + c$

(g) $\int \sqrt{1-x^2} \, \mathrm{d}x = \int \cos^2 u \, \mathrm{d}u = \frac{1}{2}u + \frac{1}{2}\sin u \cos u + c$
$= \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + c$
(The integral of $\cos^2 u$ in this computation can be found using integration by parts:
$\int \cos^2 u \, \mathrm{d}u = \int \cos u \cos u \, \mathrm{d}u = \cos u \sin u - \int(-\sin u)\sin u \, \mathrm{d}u$
$= \cos u \sin u + \int(1-\cos^2 u) \, \mathrm{d}u = \cos u \sin u + u - \int \cos^2 u \, \mathrm{d}u$,
hence $2\int \cos^2 u \, \mathrm{d}u = u + \sin u \cos u + c$.)

Solutions to Exercises 3.2.

1. (a) $\int_0^\infty e^{-5x} \, \mathrm{d}x = \lim_{T\to\infty} \left[-\frac{1}{5}e^{-5ex}\right]_0^T = \frac{1}{5}$

(b) undefined, because $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{T\to\infty} [2\sqrt{x}]_1^T$ and the limit $\lim_{T\to\infty} \sqrt{T}$ does not exist

(c) $\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \lim_{T\to\infty} [\arctan x]_0^T = \pi/2$

(d) $\int_0^\infty e^{-x}\cos x \, \mathrm{d}x = \lim_{T\to\infty} \left[\frac{1}{2}e^{-x}(\sin x - \cos x)\right]_0^T = \frac{1}{2}$

2. If $s \neq -1$ then

$$\lim_{T\to\infty} \int_1^T x^s \, dx = \lim_{T\to\infty} \frac{T^{s+1}}{s+1} - \frac{1}{s+1}$$

which converges if and only if $s < -1$. If $s = -1$ then we end up with $\lim_{T\to\infty} \ln(T)$, which diverges. Thus, the integral exists for $s < -1$ and doesn't exist for $s \geq -1$.

3. We define $\displaystyle\int_a^b f(x)\ \mathrm{d}x = \lim_{T\to a}\int_T^b f(x)\ \mathrm{d}x$, where in the limit T approaches a from the right.
   (a) $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{T\to 0}\left[2x^{1/2}\right]_T^1 = 2$
   (b) $\int_1^{10} \frac{\mathrm{d}x}{(x-1)^{2/3}} = \lim_{T\to 1}\left[3(x-1)^{1/3}\right]_T^{10} = 3\cdot 9^{1/3}$

Solutions to Exercises 3.3.

1. $\int_1^3\int_0^2 x^2y\ \mathrm{d}x\,\mathrm{d}y = \int_1^3 \frac{8}{3}y\ \mathrm{d}y = \frac{32}{3}$
2. $\int_0^{\pi/4}\int_0^{\pi/4}\sin(x+y)\ \mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/4}(-\cos(\pi/4+y)+\cos(y))\ \mathrm{d}y$
   $= -\sin(\pi/2)+2\sin(\pi/4) = -1+\sqrt{2}$

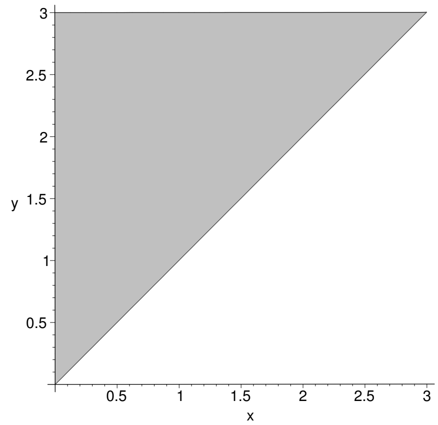
Solutions to Exercises 3.4.

1. $\iint_D (x+y)\ \mathrm{d}x\,\mathrm{d}y = \int_0^1\int_y^{1+2y}(x+y)\ \mathrm{d}x\,\mathrm{d}y = \int_0^1\left(\frac{1}{2}+3y+\frac{5}{2}y^2\right)\ \mathrm{d}y = \frac{17}{6}$
2. $\iint_D (x-y)^2\ \mathrm{d}x\,\mathrm{d}y = \int_1^9\int_{\sqrt{y}}^3 (x-y)^2\ \mathrm{d}x\,\mathrm{d}y = \int_1^9\left(9-9y+4y^2-\frac{1}{3}y^{3/2}-y^{5/2}\right)\ \mathrm{d}y = \frac{904}{35}$
3. If you plot the points you get a square rotated by 45 degrees. So, I split the integral into two. The first has limits $0\le y\le 1$ and $1-y\le x\le 1+y$ and the second integral is $1\le y\le 2$ and $y-1\le x\le 3-y$.
$$\begin{aligned}\iint_D xy\ \mathrm{d}x\,\mathrm{d}y &= \int_0^1\int_{1-y}^{1+y} xy\ \mathrm{d}x\,\mathrm{d}y + \int_1^2\int_{y-1}^{3-y} xy\ \mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1 2y^2\ \mathrm{d}y + \int_1^2 (4y-2y^2)\ \mathrm{d}y\\ &= 2\end{aligned}$$
4. Another split. First region is $0\le y\le 1$ and $0\le x\le 2$ and the second is $1\le y\le 2$ and $0\le x\le 3-y$.
$$\begin{aligned}\iint_P xy\ \mathrm{d}x\,\mathrm{d}y &= \int_0^1\int_0^2 xy\ \mathrm{d}x\,\mathrm{d}y + \int_1^2\int_0^{3-y} xy\ \mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1 2y\ \mathrm{d}y + \int_1^2\left(\frac{9}{2}y-3y^2+\frac{1}{2}y^3\right)\ \mathrm{d}y\\ &= \frac{21}{8}\end{aligned}$$
5. The integral $\iint_D \mathrm{d}x\,\mathrm{d}y$ is the volume under the surface $z=1$ and above the region D. This volume is the area of D times the height 1, i.e. it's just the same as the area of the region D. Limits of integration: $-1\le y\le 2$ and $y^2\le x\le y+2$.
$$\iint_D \mathrm{d}x\,\mathrm{d}y = \int_{-1}^2\int_{y^2}^{y+2} 1\ \mathrm{d}x\,\mathrm{d}y = \int_{-1}^2 (y+2-y^2)\,\mathrm{d}y = \frac{9}{2}$$

Solutions to Exercises 3.5.

1. (a) (i) $\displaystyle\int_0^3 \int_0^y (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^3 \frac{4}{3}y^3\,\mathrm{d}y = 27$

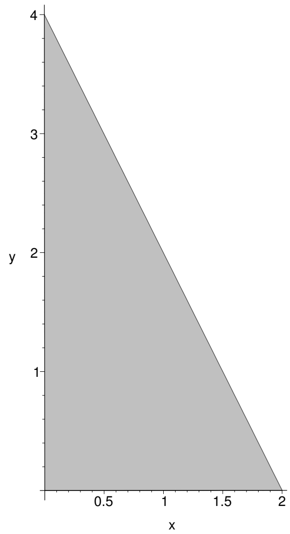
(ii)



(iii) new limits: $0 \le x \le 3$ and $x \le y \le 3$

(b) (i) $\displaystyle\int_0^2 \int_0^{4-2x} xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 (8x - 8x^2 + 2x^3)\,\mathrm{d}x = \frac{8}{3}$

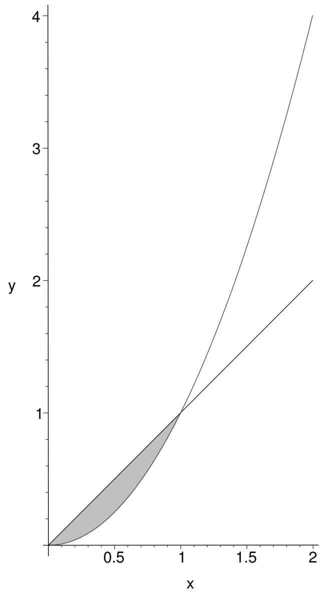
(ii)



(iii) new limits: $0 \le y \le 4$ and $0 \le x \le (4-y)/2$

(c) (i) $\displaystyle\int_0^1 \int_y^{\sqrt{y}} y^2 x\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\frac{1}{2}y^3 - \frac{1}{2}y^4\right)\mathrm{d}y = \frac{1}{40}$

(ii)



(iii) new limits: $0 \le x \le 1$ and $x^2 \le y \le x$

2. (a) It is difficult to integrate in the current order. Switching the order gives the new limits $0 \le x \le 2$ and $0 \le y \le x/2$.

$$\begin{aligned}\int_0^1 \int_{2y}^2 6y\sqrt{1+x^3}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \int_0^{x/2} 6y\sqrt{1+x^3}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^2 \frac{3}{4}x^2\sqrt{1+x^3}\,\mathrm{d}x \\ &= \frac{13}{3}\end{aligned}$$

(b) Again, it is worth switching on account of the integration. The new limits are $1/2 \le x \le 1$ and $1 \le y \le 1/x$.

$$\begin{aligned}\int_1^2 \int_{1/2}^{1/y} x^3\cos(x^2y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{1/2}^1 \int_1^{1/x} x^3\cos(x^2y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{1/2}^1 (x\sin(x) - x\sin(x^2))\,\mathrm{d}x \\ &= -\frac{1}{2}\cos(1/4) - \sin(1/2) + \frac{1}{2}\cos(1/2) - \frac{1}{2}\cos(1) + \sin(1)\end{aligned}$$

3. If you draw the region, you will see that the region can be expressed more compactly as $0 \le x \le 4$ and $1/2x \le y \le x$.

$$\int_0^4 \int_{x/2}^x \sin\left(\frac{\pi y}{x}\right)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4 \frac{x}{\pi}\,\mathrm{d}x = \frac{8}{\pi}$$

Solutions to Exercises 3.6.

1. $\displaystyle\int_1^2 \int_{y^2}^{\infty} \frac{1}{x^2y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 \frac{1}{y^4}\,\mathrm{d}y = \frac{7}{24}$

2. $\displaystyle\int_0^{\infty} \int_0^{\sqrt{y}} xe^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\infty} \frac{1}{2}ye^{-y^2}\,\mathrm{d}y = \frac{1}{4}$

Solutions to Exercises 3.7.

1. Set $u = x + y$ and $v = y - x^2$. Then the region D in the (x, y)-plane becomes the rectangle Δ in the (u, v)-plane given by $4 \le u \le 6$ and $0 \le v \le 2$. The Jacobian is $\frac{\partial(x,y)}{\partial(u,v)} = 1/(1 + 2x)$, which is always positive in the region of interest. Thus, the integral becomes
$$\begin{aligned}\iint_D (x+y)(1+2x)\ \mathrm{d}x\ \mathrm{d}y &= \iint_\Delta (x+y)(1+2x)\frac{\partial(x,y)}{\partial(u,v)}\ \mathrm{d}u\ \mathrm{d}v \\ &= \iint_\Delta u\ \mathrm{d}u\ \mathrm{d}v \\ &= \int_0^2\int_4^6 u\ \mathrm{d}u\ \mathrm{d}v\end{aligned}$$
which is now easy to integrate. You should get 20.

2. Set $u = x^2y$ and $v = y/\sqrt{x^3}$. Then the region D in the (x, y)-plane becomes the rectangle Δ in the (u, v)-plane given by $1 \le u \le 3$ and $1 \le v \le \sqrt{5}$. The Jacobian is $\frac{\partial(x,y)}{\partial(u,v)} = \frac{2}{7}\frac{\sqrt{x}}{y}$, which is always positive in the region of interest. Thus
$$\begin{aligned}\iint_D \frac{1}{xy}\ \mathrm{d}x\ \mathrm{d}y &= \iint_\Delta \frac{1}{xy}\frac{\partial(x,y)}{\partial(u,v)}\ \mathrm{d}u\ \mathrm{d}v \\ &= \int_1^{\sqrt{5}}\int_1^3 \frac{2}{7}\frac{1}{uv}\ \mathrm{d}u\ \mathrm{d}v,\end{aligned}$$
which can now be easily integrated to give $2/7\ln(\sqrt{5})\ln(3)$.

3. Use polar coordinates. The integral becomes
$$\int_0^{|a|}\int_0^{\pi} (r\cos(\theta))^2\sqrt{r^2}\cdot r\ \mathrm{d}\theta\,\mathrm{d}r = \int_0^{|a|}\int_0^{\pi} r^4\cos^2(\theta)\ \mathrm{d}\theta\,\mathrm{d}r = \frac{\pi}{10}|a|^5.$$

4. Use polar coordinates. We have that $1 \le r^2 \le 4$, so $1 \le r \le 2$. The integral becomes
$$\int_1^2\int_0^{\pi/2} r\cos(\theta)r\sin(\theta)\sqrt{r^2}\cdot r\ \mathrm{d}\theta\,\mathrm{d}r = \int_1^2\int_0^{\pi/2} r^4\sin(\theta)\cos(\theta)\ \mathrm{d}\theta\,\mathrm{d}r = \frac{31}{10}.$$

5. The Jacobian of the change of variables is $\frac{\partial(x,y)}{\partial(r,\theta)} = r$, and the integral becomes
$$\int_0^{2\pi}\int_0^1 (1 + r\cos(\theta))(1 + r\sin(\theta))r\ \mathrm{d}r\ \mathrm{d}\theta = \pi.$$