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Probability and Statistics

Solutions 2

1. Using the distributive law,

$$A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c) = (A \cup B) \cap S = A \cup B.$$

Since $A \subseteq B$, $A \cup B = B$. Hence

$$B = A \cup (B \cap A^c). \tag{1}$$

Since $A \cap (B \cap A^c) \subseteq A \cap A^c = \emptyset$, the events in the union of the right hand side of Equation (1) are mutually exclusive. It follows from Axiom 3 that

$$\Pr(B) = \Pr(A) + \Pr(B \cap A^c) \geq \Pr(A).$$

2. (i) Using Theorem 5 of Section 2,

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = \frac{1}{4} + \frac{1}{3} - \frac{5}{12} = \frac{1}{6}.$$

(ii) If A and B were mutually exclusive we would have $\Pr(A \cap B) = 0$. So in this case A and B are <u>not</u> mutually exclusive.

(iii)

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/6}{1/4} = \frac{2}{3}.$$

(iv) If A and B were independent we would have $\Pr(B|A) = \Pr(B)$, which is clearly not the case. Alternatively, using the formal definition of independence, we may check that

$$\Pr(A)\Pr(B) = \frac{1}{4} \times \frac{1}{3} \neq \frac{1}{6} = \Pr(A \cap B),$$

from which it follows that A and B are <u>not</u> independent.

(v) Using Lemma 4 of Section 2 with the roles of A and B interchanged,

$$\Pr(A^c \cap B) = \Pr(B) - \Pr(A \cap B) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}.$$

3. (i) Using Lemma 4 of Section 2,

$$\Pr(A \cap B) = \Pr(A) - \Pr(A \cap B^c) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

(ii)

$$\Pr(B) = \Pr(A^c \cap B) + \Pr(A \cap B) = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}.$$

(iii)

$$\Pr(A)\Pr(B) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} = \Pr(A \cap B),$$

from which it follows that A and B are independent.

4. If A and B are independent then $\Pr(A \cap B) = \Pr(A)\Pr(B)$. It follows that

$$\begin{aligned} \Pr(A \cap B^c) &= \Pr(A) - \Pr(A \cap B) \\ &= \Pr(A) - \Pr(A)\Pr(B) \\ &= \Pr(A)[1 - \Pr(B)] \\ &= \Pr(A)\Pr(B^c). \end{aligned}$$

Thus A and B^c are also independent.

5. (i) Using Theorem 5 of Section 2 and the distributive law,

$$\begin{aligned} \Pr((A \cup B) \cup C) &= \Pr(A \cup B) + \Pr(C) - \Pr((A \cup B) \cap C) \\ &= \Pr(A) + \Pr(B) - \Pr(A \cap B) + \Pr(C) \\ &\quad - \Pr((A \cap C) \cup (B \cap C)) \\ &= \Pr(A) + \Pr(B) - \Pr(A \cap B) + \Pr(C) \\ &\quad - \Pr(A \cap C) - \Pr(B \cap C) + \Pr((A \cap C) \cap (B \cap C)) \\ &= \Pr(A) + \Pr(B) + \Pr(C) - \Pr(B \cap C) - \Pr(A \cap C) \\ &\quad - \Pr(A \cap B) + \Pr(A \cap B \cap C). \end{aligned}$$

(ii) Using the definition of conditional probability,

$$\Pr(C)\Pr(B|C)\Pr(A|B\cap C) = \Pr(C)\frac{\Pr(B \cap C)}{\Pr(C)}\frac{\Pr(A \cap B \cap C)}{Pr(B \cap C)} = \Pr(A\cap B\cap C).$$

6. Using the Law of Total Probability,

$$\begin{aligned} \Pr(E) &= \Pr(B_1)\Pr(E|B_1) + \Pr(B_2)\Pr(E|B_2) + \Pr(B_3)\Pr(E|B_3) \\ &= (0.7)(0.1) + (0.2)(0.2) + (0.1)(0.9) \\ &= 0.07 + 0.04 + 0.09 = 0.2\ . \end{aligned}$$

Using Bayes' Theorem, the posterior probabilities are given by

$$\Pr(B_j|E) = \frac{\Pr(B_j)\Pr(E|B_j)}{\Pr(B_1)\Pr(E|B_1) + \Pr(B_2)\Pr(E|B_2) + \Pr(B_3)\Pr(E|B_3)}.$$

Specifically,

$$\begin{aligned} \Pr(B_1|E) &= \frac{0.07}{0.2} = 0.35, \\ \Pr(B_2|E) &= \frac{0.04}{0.2} = 0.20, \\ \Pr(B_3|E) &= \frac{0.09}{0.2} = 0.45. \end{aligned}$$