BIRKBECK
(University of London)

BSc EXAMINATION

SCHOOL OF BUSINESS, ECONOMICS AND INFORMATICS

Calculus 2: Multivariate Differential Equations-

SOLUTIONS

BUEM001S5

30 credits

Friday 5th June, 2015
10:00 a.m. - 13:00p.m.

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 marks each and questions in Section B are worth 20 marks each.
Candidates should attempt **all** *of the questions in Section A and* **two** *questions out of the four in Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorised calculators or has been approved by the chair of the Mathematics and Statistics Examination Sub-board.

Please turn over

Section A - with SOLUTIONS

1. (a) Explain why
$$\lim_{x\to 0}\frac{e^x+e^{-x}}{x^2}=\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{2}=1$$
is *not* correct.
Evaluate
$$\lim_{x\to 0}\frac{e^x+e^{-x}}{x^2}.$$
[2]

Because the first quotient does not have either of the inderterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x=0$ we cannot apply L'Hôpital's rule.
Evaluating the relevant limit we obtain
$$\lim_{x\to 0}(e^x+e^{-x})\frac{1}{x^2}=2(+\infty)=+\infty.$$

(b) Evaluate
$$\lim_{x\to\infty}\frac{\sqrt{9x^2+2}}{3-4x}.$$
[3]

Divide numerator and denominator with $x=\sqrt{x^2}$, then:
$$\lim_{x\to\infty}\frac{\sqrt{9x^2+2}}{3-4x}=\lim_{x\to\infty}\frac{\sqrt{9+\frac{2}{x^2}}}{\frac{3}{x}-4}=\frac{\lim_{x\to\infty}\sqrt{9+\frac{2}{x^2}}}{\lim_{x\to\infty}(\frac{3}{x}-4)}=\frac{\sqrt{9+0}}{0-4}=-\frac{3}{4}.$$

2. (a) Let $U\subseteq\mathbb{R}$ and let $f:U\to\mathbb{R}$ be a function.
   (i) Define what it means for f to be continuous at a point $a\in U$. [1]
   A function $f:U\to\mathbb{R}$ is continuous at a point $a\in U$ if $\lim_{x\to a}f(x)=f(a)$.
   (ii) State the definition of the derivative of f at $x\in U$. [1]
   The derivative of a function f at a point x is defined as
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$
   (if this limit exists).

   (b) Let $f:\{x\in\mathbb{R}:x\geq -1\}\to\mathbb{R}$ be the function $f(x)=\sqrt{x+1}$. Use the formal definition of the derivative to find $f'(x)$ for $x>-1$. If any limits appear as part of

Please turn over

your computation, you must evaluate them without using L'Hôpital's rule. [3]

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+1)+h}-\sqrt{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+1)+h}-\sqrt{x+1}}{h} \frac{\sqrt{(x+1)+h}+\sqrt{x+1}}{\sqrt{(x+1)+h}+\sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{1}{h} \frac{[(x+1)+h]-(x+1)}{\sqrt{(x+1)+h}+\sqrt{x+1}} \\
&= \frac{1}{2\sqrt{x+1}}, \text{ for } \quad x > -1.
\end{aligned}
$$

3. Evaluate the integral

$$\iint_D (x^3+4y)\, dxdy,$$

where D is the region in the $xy-$ plane bounded by the graphs of the equations $y = x^2$ and $y = 2x$. [5]

$$\iint_D (x^3+4y)\, dxdy = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^3+4y) dx\, dy = \int_0^4 (\frac{1}{4}y^2 + 4y^{3/2} - \frac{1}{64}y^4 - 2y^2) dy = \frac{32}{3}$$

4. Consider the differential equation

$$y' = 1 + x^2 + y.$$

Assume that the initial condition is $y(1) = 0$. Use the method of *Taylor series* about the point $x = 1$ to find the first five terms of the Taylor series of y about that point. [5]

The Taylor series of y about the point 1 has the form

$$y(x) = y(1) + y'(1)(x-1) + y''(1)\frac{(x-1)^2}{2!} + y'''(1)\frac{(x-1)^3}{3!} + y^{(4)}(1)\frac{(x-1)^4}{4!} + \dots.$$

The initial condition is $y(1) = 0$, and the differential equation $y' = 1+x^2+y$ immediately gives

$$y'(1) = 1 + 1^2 + y(1) = 2.$$

By repeatedly differentiating the differential equation, we can find the higher derivatives:

$$
\begin{aligned}
y'' &= 2x + y' &\Rightarrow\quad & y''(1) = 2\cdot 1 + y'(1) = 4, \\
y''' &= 2 + y'' &\Rightarrow\quad & y'''(1) = 2 + y''(1) = 6, \\
y^{(4)} &= y''' &\Rightarrow\quad & y^{(4)}(1) = y'''(1) = 6.
\end{aligned}
$$

Please turn over

Hence the first five terms of the Taylor series of y about 1 are

$$0 + 2(x-1) + 4\frac{(x-1)^2}{2!} + 6\frac{(x-1)^3}{3!} + 6\frac{(x-1)^4}{4!}.$$

5. Solve the following differential equation using any appropriate method.

$$\frac{1}{x}\frac{dy}{dx} - y = e^{x^2/2}.$$

[5]

Rewrite as

$$\frac{dy}{dx} - xy = xe^{x^2/2}.$$

This is a linear equation of order one with $P(x) = -x$ and $Q(x) = xe^{x^2/2}$
With an integrating factor $\mu(x) = e^{\int P(x)dx} = e^{-x^2/2}$
the solution will be:

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx = e^{x^2/2}\int xdx = e^{x^2/2}\left(\frac{x^2}{2} + c\right).$$

6. Show that the differential equation

$$4x^3 - y^2 = 2xy\,\frac{dy}{dx}$$

is an exact differential equation. Solve it using the appropriate method subject to the condition that $y = 3$ when $x = 2$.
Express y explicitly as a function of x. [5]

First rewrite the equation as:

$$(4x^3 - y^2)dx + (-2xy)dy = 0$$

For $M(x,y) = 4x^3 - y^2$ and $N(x,y) = -2xy$ we have

$$\frac{\partial M}{\partial y} = -2y \text{ and } \frac{\partial N}{\partial x} = -2y$$

and since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this means that the equation is exact.
That means that there is a solution which has the general form $f(x,y) = c$ for some constant c and some function f of two variables satisfying

$$\frac{\partial f}{\partial x} = M = 4x^3 - y^2 \ (1) \quad \text{and} \quad \frac{\partial f}{\partial y} = -2xy. \ (2)$$

Please turn over

From (1) we will have

$$f(x,y) = \int M(x,y)dx = x^4 - y^2x + g(y) \quad (1')$$

where $g(y)$ is an arbitrary function.
By differentiating (1') with respect to y and equating with (2) we will have

$$\frac{\partial f}{\partial y} = -2yx + g'(y) = -2xy \quad \text{therefore} \quad g(y) = c_1$$

which means

$$f(x,y) = x^4 - xy^2 + c_1 = c \quad \text{or} \quad x^4 - xy^2 = C$$

which defines y implicitly in terms of x.
Using the information that $y = 3$ for $x = 2$ we will have that: $C = -2$.
The condition $y = 3$ implies that y is positive and therefore

$$y = \sqrt{x^3 + \frac{2}{x}}.$$

7. Radium decays exponentially and has a half-life of approximately 1600 years; that is, given any quantity, one-half of it will disintegrate in 1600 years.

   (a) Find a formula for the quantity $q(t)$ of radium (in mg) at time t (in years). It is known that at time $t = 0$ years the quantity of pure radium is $50\,mg$. [3]

   We have that $q(t) = q(0)e^{at}$ with $q(0) = 50$, hence

   $$q(t) = 50e^{at}$$

   Besides, $q(1,600) = 25$ or $50e^{1600t} = 25$ and therefore

   $$a = -\frac{\ln 2}{1600}$$

   which means

   $$q(t) = 50e^{-(\ln 2/1600)t}.$$

   (b) When will there be $20\,mg$ of radium left? [2]

   We need to solve the equation

   $$q(t) = 20 \quad \text{or} \quad 50e^{-(\ln 2/1600)t} = 20$$

   which gives us

   $$t = 1600(\ln(\frac{5}{2}))/\ln 2$$

   (approximately 2,115 years)

Please turn over

8. Recall that the hyperbolic function $\tanh x$ is defined, for all $x \in \mathbb{R}$, by

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

(a) State the domain and codomain for the *inverse* hyperbolic function $\operatorname{arctanh} x$. [1]

$$\operatorname{arctanh} x : (-1, 1) \to \mathbb{R}$$

(b) Deduce the explicit formula for the *inverse* hyperbolic function $\operatorname{arctanh} x$. [4]

$$\operatorname{arctanh} x = \frac{1}{2} \ln \left(\frac{x+1}{1-x}\right)$$

The process to deduce the formula is provided in the course notes
Let $x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$. Setting $z = e^{2y}$ we find

$$\begin{aligned}
& x = \frac{z-1}{z+1} \\
\Rightarrow \quad & zx + x = z - 1 \\
\Rightarrow \quad & z(x-1) = -x - 1 \\
\Rightarrow \quad & z = \frac{x+1}{1-x} \\
\Rightarrow \quad & 2y = \ln\left(\frac{x+1}{1-x}\right).
\end{aligned}$$

Hence

$$\operatorname{arctanh} x = \frac{1}{2} \ln\left(\frac{x+1}{1-x}\right),$$

where $-1 < x < 1$.

Please turn over

Section B- with SOLUTIONS

9. (a) Let $U \subseteq \mathbb{R}^2$ and consider $f : U \to \mathbb{R}$ be a function.
Define the following terms, for point $(a, b) \in U$:
   (i) stationary point of f,
   (ii) local minimum of f, and
   (iii) global minimum of f [3]

   The definitions are given in the coursenotes (pages 31-32,40)
   (i) $(a, b) \in U$ is called a stationary point of f if the tangent plane at (a, b) is horizontal; that is the tangent plane exists and is parallel to the $(x, y)-$ plane
   (ii) $f(x, y)$ has a local minimum at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) in some small enough disk around (a, b)
   (iii) the point $(a, b) \in U$ is a global minimum if $f(a, b) \leq f(x, y)$ for all $(x, y) \in U$

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^2 + 6xy + 6y^2 + 7.$$

   (i) Compute the partial derivatives f_x, f_y, f_{xx}, f_{yy} and f_{xy}. [3]

$$\begin{aligned} f_x &= 2x + 6y, \\ f_y &= 6x + 12y, \\ f_{xx} &= 2, \\ f_{yy} &= 12, \\ f_{xy} &= 6. \end{aligned}$$

   (ii) Find and classify the stationary points of f. [5]
   To find the stationary points we have to solve

$$\begin{aligned} f_x = 0 \quad &\text{or} \quad 2x + 6y = 0. \\ f_y = 0 \quad &\text{or} \quad 6x + 12y = 0. \end{aligned}$$

   which gradually gives us that the only stationary point of f is $(x, y) = (0, 0)$.
   In order to classify this stationary point we calculate the second derivative of f at the point $(0, 0)$.
   We have $f"(x) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 6 & 12 \end{pmatrix}$ for all $x \in \mathbb{R}^2$.
   The Hessian matrix has determinant $-12 < 0$ and diagonal elements > 0 (at the point $(0, 0)$) so the stationary point $(0, 0)$ is a saddle point.

   (iii) Investigate if this function has any global extrema and, if yes, identify them. [2]

   It follows that f (being differentiable in $\mathbb{R}^2$) cannot have any global extremum because there is no other candidate for a global extremum.

Please turn over

(c) Consider now the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^2 \cos y + y^2.$$

(i) Find the *quadratic* Taylor approximation to f at the point $(0, 0)$. Note that $(0, 0)$ is a stationary point of the function f. [5]

The second order Taylor approximation to f at the point (x_0, y_0) is

$$\begin{aligned} g(x,y) \cong f(x_0,y_0) \quad &+ \quad \left(f_x(x_0,y_0),\ f_y(x_0,y_0)\right) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \\ &+ \quad \frac{1}{2}(x - x_0,\ y - y_0) \begin{pmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \end{aligned}$$

where the Hessian matrix is $\begin{pmatrix} 2\cos y & -2x \sin y \\ -2x \sin y & -x^2 \cos y + 2 \end{pmatrix}$
Since $(0, 0)$ is a stationary point this means $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$ and therefore

$$\begin{aligned} g(x,y) \quad &\cong \quad f(0,0) + \frac{1}{2}(x,\ y) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \quad x^2 + y^2 \end{aligned}$$

(ii) Using the (c)(i) classify the point $(0, 0)$ as a local minimum, local maximum or saddle point. [2]

Since $g(x, y) > g(0, 0) = 0$ for every $(x, y) \neq (0, 0)$ this means that $(0, 0)$ is a local minimum.

Please turn over

10. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = 5y^2 - x^2$.

(a) (i) Find the gradient vector of f and evaluate it at the point $(x, y) = (1, 1)$. [2]

We have

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x \\ 10y \end{pmatrix}$$

and therefore

$$\nabla f(1, 1) = \begin{pmatrix} -2 \\ 10 \end{pmatrix}.$$

(ii) Find the directional derivative of the function f at $(1, 1)$ in the direction of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. [3]

If $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ then

$$f_{\mathbf{u}}(1, 1) = \nabla f(1, 1)\hat{\mathbf{u}} = \frac{\mathbf{1}}{\sqrt{\mathbf{5}}} \begin{pmatrix} -2 \\ 10 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

where $\hat{\mathbf{u}}$ is the unit vector $\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(iii) Write down the equation of the tangent plane to the surface $z = f(x, y) = 5y^2 - x^2$ at the point $(2, 0, f(2, 0))$. [3]

At the point $(x, y) = (2, 0)$ we have $z = -4$ and $f_x(2, 0) = -4 \quad f_y(2, 0) = 0$ and the equation of the tangent plane will be:

$$f(2, 0) - z + f_x(2, 0)(x - 2) + f_y(2, 0)(y - 0) = 0$$

or

$$4x + z = 4.$$

(b) Consider again the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 5y^2 - x^2$.

(i) Show that f is a homogeneous function of degree 2. [2]

For $f(x, y) = 5y^2 - x^2$ we have:

$$f(\lambda x, \lambda y) = 5(\lambda y)^2 - (\lambda x)^2 = \lambda^2(5y^2 - x^2) = \lambda^2 f(x, y)$$

Therefore, f is truly homogeneous of degree 2.

Please turn over

(ii) Verify *Euler's Theorem* on homogeneous functions by evaluating $x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y}$. [2]

For the given function f we have that $f_x = -2x$ and $f_y = 10y$, therefore:

$$xf_x + yf_y = -2x^2 + 10y^2 = 2f$$

as required.

(c) Consider the differential equation

$$5xy\frac{dy}{dx} = x^2 - 5y^2.$$

Our aim is to find the general solution of the differential equation.

(i) By setting $y = vx$, change the above differential equation into

$$5xv\frac{dv}{dx} = 1 - 10v^2.$$

Explain with a sentence why one would set $y = vx$ in the first instance. [3]

We may rewrite the given equation as

$$(5y^2 - x^2) + 5xy\frac{dy}{dx} = 0$$

We can observe that $f(x,y) = 5y^2 - x^2$ is homogeneous of degree 2 (shown in the previous question) and it is also easy to show that $g(x,y) = xy$ is also homogeneous of degree 2.
Hence we are dealing with a homogeneous differential equation and we may proceed accordingly by letting

$$y = vx; \qquad \Big(\frac{dy}{dx} = v + x\frac{dv}{dx}\Big).$$

We will therefore have:

$$\begin{aligned} 5x(vx)\Big(v + x\frac{dv}{dx}\Big) &= x^2 - 5(vx)^2 \quad \text{or} \\ 5v^2 + 5vx\frac{dv}{dx} &= 1 - 5v^2 \quad \text{or} \\ 5vx\frac{dv}{dx} &= 1 - 10v^2. \end{aligned}$$

Please turn over

(ii) By solving the equation in (i), find the general solution to the original differential equation. [5]

$$\frac{5v}{1-10v^2}dv = \frac{1}{x}dx \quad \text{is a separable equation, hence}$$
$$\int \frac{5v}{1-10v^2}dv = \int \frac{1}{x}dx \quad \text{which will give us}$$
$$-\frac{1}{4}\ln|1-10v^2| = \ln|x| + \ln c \text{ or}$$
$$\ln\left|1-10\frac{y^2}{x^2}\right|^{-1/4} = \ln|cx|$$

and the solution will be

$$\left|1-10\frac{y^2}{x^2}\right|^{-1/4} = cx$$

for c arbitrary constant.

Please turn over

11. (a) If D is the region bounded by $y = x^2$, $x = 3$ and $y = 0$, evaluate

$$\iint_D e^{x^3}\, dxdy.$$

[4]

Since it is not possible to integrate at first $\int e^{x^3} dx$ we change the order of integration and we will have

$$\begin{aligned}\iint_D e^{x^3}\, dxdy &= \int_0^3 \int_0^{x^2} e^{x^3}\, dydx = \int_0^3 ye^{x^3}\Big|_0^{x^2} dx \\ &= \int_0^3 x^2 e^{x^3} dx = \frac{1}{3}e^{x^3}\Big|_0^3 = \frac{1}{3}(e^{27} - 1).\end{aligned}$$

(b) Consider the integral

$$\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} (x^2 + y^2)^{3/2}\, dydx.$$

(i) Assume the change of variables $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that the *Jacobean* of this change of variables is r. [2]

The proof is given in the course notes: Chapter 3.7, page 78.
If we change the polar coordinates then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

(ii) Use polar coordinates to evaluate the integral. [4]

Replacing $(x^2 + y^2)$ in the integrand by r^2, $dydx$ by $rdrd\theta$ and changing the limits, we have:

$$\begin{aligned}\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} (x^2 + y^2)^{3/2}\, dydx &= \int_0^{\pi} \int_0^{a} r^3 r dr d\theta \\ &= \int_0^{\pi} \frac{r^5}{5}\Big|_0^a d\theta = \frac{a^5}{5}\theta\Big|_0^{\pi} = \frac{\pi a^5}{5}.\end{aligned}$$

(c) (i) State the definition of the Gamma function. State clearly the domain of the function. [1]

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad \text{for} \quad x > 0.$$

Please turn over

(ii) Show that the Gamma function $\Gamma(x)$ satisfies

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

for $x > 1$. [3]

Proof of Proposition 7.8 in the notes.
We use integration by parts. Set $u = t^{x-1}$ and $\frac{dv}{dt} = e^{-t}$. Then $\frac{du}{dt} = (x-1)t^{x-2}$ and $v = -e^{-t}$. Then,

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}dt \\
&= \big[-t^{x-1}e^{-t}\big]_0^\infty + \int_0^\infty (x-1)t^{x-2}e^{-t}dt \\
&= 0 + (x-1)\int_0^\infty t^{(x-1)-1}e^{-t}dt \\
&= (x-1)\Gamma(x-1).
\end{aligned}$$

(d) (i) Evaluate

$$\int_0^\infty x^4 e^{-3x} dx.$$

[3]

Set $t = 3x$ then $\frac{dt}{dx} = 3$, then

$$\begin{aligned}
\int_0^\infty x^4 e^{-3x} dx &= \int_0^\infty (\frac{t}{3})^4 e^{-t}(\frac{1}{3})dt \\
&= \frac{1}{3^5}\int_0^\infty t^4 e^{-t}dt = \frac{1}{3^5}\Gamma(5) = \frac{1}{3^5}4!\,.
\end{aligned}$$

(ii) Prove that

$$2\int_0^2 x\sqrt[3]{1-\big(\frac{x}{2}\big)^3}\,dx = \frac{8}{3}B(\frac{2}{3},\frac{4}{3}).$$

[3]

Let $t = (\frac{x}{2})^3$ then $dt = \frac{1}{8}3x^2dx$ and the requested integral becomes

$$2\int_0^1 x\sqrt[3]{1-t}\,\big(\frac{8}{3}\frac{1}{x^2}\big)dt = 2\int_0^1 \frac{8}{3}\big(\frac{1}{2}t^{-1/3}\big)(1-t)^{1/3}dt = \frac{8}{3}B\big(\frac{2}{3},\frac{4}{3}\big).$$

Please turn over

12. (a) Suppose that a population of bacteria grows according to the logistic growth model. That is

$$\frac{dP}{dt} = rP(1 - \frac{P}{k}),$$

where $P := P(t)$ is the bacteria population as a function of time t, r is the growth proportionality constant and k is the carrying capacity.

(i) Suppose that $r = 20$ and $k = 1000$. Find the general solution to the differential equation. You may keep your answer in the form

$$\frac{P}{a - P} = Ae^{bt},$$

where a, b are numbers and A is the constant of integration. [4]

Here

$$\frac{dP}{dt} = 20P(1 - \frac{P}{1000})$$

which is a separable equation

$$\frac{dP}{20P\big(1 - P/1000\big)} = dt$$

or

$$(\frac{1}{20P} + \frac{1/(20 \times 1000)}{1 - P/1000})dP = dt$$

which gives

$$\frac{P}{1 - P/1000} = Be^{20t}$$

or

$$\frac{P}{1000 - P} = Ae^{20t}$$

for A, B arbitrary constants.

(ii) Find A if the initial population of bacteria is 500. [1]

Since $P(0) = 500$ from $\frac{P}{1000-P} = Ae^{20t}$ and for $t = 0$ we will obtain

$$A = 1.$$

(iii) How long will it take the population to reach 80% of its maximum size? Assume that the time is measured in days. [3]

For $A = 1$ we have that $\frac{P}{1000-P} = e^{20t}$ while the maximum of the population is $k = 1000$. With this information we just need to find the time t for which

Please turn over

$P(t) = 0.8 \times 1000 = 800$.
Solving

$$\frac{800}{1000 - 800} = e^{20t}$$

for t it is found that

$$t = \frac{1}{20} \ln 4$$

i.e. approximately 0.069 days (or approximately 1.66 hours).

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = e^{3x} + e^{2x}.$$

[7]

The relevant auxiliary equation is : $t^2 + 2t - 8 = 0$ with roots $t = 2,\ t = -4$.
Therefore the general solution $C(x)$ of the homogeneous differential equation will be

$$C(x) = Ae^{2x} + Be^{-4x}$$

with A, B arbitrary constants.
For the particular integral and since 2 is a (single) root of the auxiliary equation we look for a particular integral of the form

$$p(x) = Gxe^{2x} + He^{3x}$$

for G, H arbitrary constants.
In fact

$$p(x) = \frac{1}{6}xe^{2x} + \frac{1}{7}e^{3x}$$

since

$$\begin{aligned} p'(x) &= Ge^{2x} + 2Gxe^{2x} + 3He^{3x} \\ p''(x) &= 4Ge^{2x} + 4Gxe^{2x} + 9He^{3x} \end{aligned}$$

and therefore due to:

$$p''(x) + 2p'(x) - 8p(x) = e^{3x} + e^{2x}$$

or

$$6Ge^{2x} + 7He^{3x} = e^{2x} + e^{3x}$$

we obtain

$$H = \frac{1}{7}, \qquad \text{and} \qquad G = \frac{1}{6}.$$

In conclusion, the required general solution will be

$$y(x) = C(x) + p(x) = Ae^{2x} + Be^{-4x} + \frac{1}{6}xe^{2x} + \frac{1}{7}e^{3x}$$

for A, B arbitrary constants.

Please turn over

(c) Consider the differential equation

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-8y=0.$$

(i) By substituting $x=e^t$ and using the chain rule, show that

$$\frac{dy}{dt}=x\frac{dy}{dx}\quad\text{and}\quad\frac{d^2y}{dt^2}=x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}.$$

[3]

The proof is given in the Example 4.28 in the course notes, page 115.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\
&= \frac{\mathrm{d}y}{\mathrm{d}x}e^t\\
&= x\frac{\mathrm{d}y}{\mathrm{d}x},\\
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\frac{\mathrm{d}x}{\mathrm{d}t}\\
&= \left(\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\cdot e^t\\
&= \left(\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\cdot x\\
&= x\frac{\mathrm{d}y}{\mathrm{d}x}+x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.
\end{aligned}$$

(ii) Show that the above substitution (in (i)) turns the original differential equation into one with constant coefficients of the form

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}-8y=0.$$

You do not have to solve the equation. [2]

Replacing the terms $x\frac{dy}{dx}$ and $x^2\frac{d^2y}{dx^2}$ of the original equation with the above equivalent terms gives:

$$\begin{aligned}
\left(\frac{d^2y}{dt^2}-x\frac{dy}{dx}\right)+3\frac{dy}{dt}-8y=0 &\quad\text{or}\\
\frac{d^2y}{dt^2}+(3-1)\frac{dy}{dt}-8y=0 &\quad\text{or}\\
\frac{d^2y}{dt^2}+2\frac{dy}{dt}-8y=0.&
\end{aligned}$$

Please turn over