Calculus 2, Assignment 4

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1. (a) In this context, k cannot be 0, since $k = 0$ implies there is no relationship between $\frac{\mathrm{d}T}{\mathrm{d}t}$ and $T_s - T$. This gives us $|k| > 0$. However, I don't see a colloquial reason for $k > 0$, since either or both of T_s and T can be negative. I can see it's in some sense meaningless to take a negative factor, since if a is a factor of b then $-a$ is also a factor of b.

   ...

   After having found a value for T, on the other hand, I can see that a negative value of k would lead to a situation where instead of T approaching T_s as t approaches infinity, T would also go to minus infinity (which breaks the model of reality).

   (b) $\frac{\mathrm{d}T}{\mathrm{d}t} = k(T_s - T)$ is a first order variables separable ordinary differential equation and as such we can write

$$\begin{aligned} \int \frac{1}{T_s - T} \mathrm{d}T &= k \int \mathrm{d}t \\ k(t + c) &= -\ln(T_s - T) \\ e^{-k(t+c)} &= T_s - T \\ T &= T_s - e^{-k(t+c)}. \end{aligned}$$

   If $T = T_0$ when $t = 0$ we can write

$$\begin{aligned} T_0 &= T_s - e^{-kc} \\ e^{-kc} &= T_s - T_0 \\ c &= -\frac{\ln(T_s - T_0)}{k} \end{aligned}$$

and

$$
\begin{aligned}
T &= T_s - e^{-k\left(t - \frac{\ln(T_s - T_0)}{k}\right)} \\
&= T_s - e^{\ln(T_s - T_0) - kt} \\
&= T_s - \frac{e^{\ln(T_s - T_0)}}{e^{kt}} \\
&= T_s - \frac{T_s - T_0}{e^{kt}}.
\end{aligned}
$$

(c) We are given $T(0) = 37$ and $T_s = 24$.

i. Let the amount of time between death and discovery be A, now $T(A) = 34$, $T(A+30) = 32$ and

$$
\begin{aligned}
e^{kA} &= \frac{13}{10} \\
kA &= \ln\left(\tfrac{13}{10}\right), \\
e^{k(A+30)} &= \frac{13}{8} \\
kA + k30 &= \ln\left(\tfrac{13}{8}\right).
\end{aligned}
$$

Substituting our value for kA into the second equation

$$
\begin{aligned}
k &= \frac{\ln\left(\tfrac{13}{8}\right) - \ln\left(\tfrac{13}{10}\right)}{30} \\
&= \frac{\ln\left(\tfrac{5}{4}\right)}{30}.
\end{aligned}
$$

ii. With a value for k, we can write

$$
T = T_s - (T_s - T_0) \cdot \exp\left(-\frac{A \ln\left(\tfrac{5}{4}\right)}{30}\right)
$$

and

$$
\begin{aligned}
34 &= 24 - (-13) \cdot \exp\left(-\frac{A \ln\left(\tfrac{5}{4}\right)}{30}\right) \\
\frac{10}{13} &= \exp\left(-\frac{A \ln\left(\tfrac{5}{4}\right)}{30}\right) \\
A &= -\frac{30 \ln\left(\tfrac{10}{13}\right)}{\ln\left(\tfrac{5}{4}\right)} \\
&\approx 35.2729347...,
\end{aligned}
$$

which tells us the time of death was about 35 minutes before high noon.