Calculus 2 **2013/14**

Multivariable Calculus and Differential Equations

Section A

1. (a) $\displaystyle\lim_{x\to\infty}\frac{x^2+2}{3x^2+4x}=\lim_{x\to\infty}\frac{1+\frac{2}{x^2}}{3+\frac{4}{x}}=\frac{\lim_{x\to\infty}\left(1+\frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(3+\frac{4}{x}\right)}=\frac{1+0}{3+0}=\frac{1}{3}$

   (b) Without using L'Hôpital's rule:
   $\displaystyle\lim_{h\to 0}\frac{e^{-2h}-1}{e^h-1}=\lim_{h\to 0}-e^{-2h}\frac{e^{2h}-1}{e^h-1}=\lim_{h\to 0}-e^{-2h}(e^h+1)=-1\cdot(1+1)=-2$
   Alternatively use L'Hôpital's rule.

2. (a) $\nabla f(x,y)=\begin{pmatrix} f_x(x,y)\\ f_y(x,y)\end{pmatrix}=\begin{pmatrix} 3x^2+y\\ x+2y\end{pmatrix}$

   (b) Note that $\mathbf{u}$ is a unit vector. We have $f_{\mathbf{u}}(1,3)=\nabla f(1,3)\cdot\mathbf{u}=\begin{pmatrix}6\\7\end{pmatrix}\cdot\begin{pmatrix}-3/5\\4/5\end{pmatrix}=-\frac{18}{5}+\frac{28}{5}=2.$

3. (a) By the chain rule we have $\frac{\partial g}{\partial x}(x,y)=f'(x^2-2y)\cdot 2x$ and $\frac{\partial g}{\partial y}(x,y)=f'(x^2-2y)\cdot(-2)$. Hence $g(2,1)=f(2)=1$, $\frac{\partial g}{\partial x}(2,1)=f'(2)\cdot 4=8$ and $\frac{\partial g}{\partial y}(2,1)=f'(2)\cdot(-2)=-4$.

   (b) We have $g(2+h,1+k)=g(2,1)+(hg_x(2,1)+kg_y(2,1))+\cdots=1+8h-4k+\ldots$. So the degree 1 Taylor approximation is $g(2+h,1+k)\approx 1+8h-4k$.

4.

$$\begin{aligned}\iint_D xy\,\mathrm{d}x\mathrm{d}y &= \int_0^2\int_0^{4-2x} xy\,\mathrm{d}y\mathrm{d}x\\ &= \int_0^2\left[\frac{1}{2}xy^2\right]_0^{4-2x}\mathrm{d}x\\ &= \int_0^2 (8x-8x^2+2x^3)\,\mathrm{d}x\\ &= \left[4x^2-\frac{8}{3}x^3+\frac{1}{2}x^4\right]_0^2=\frac{8}{3}.\end{aligned}$$

5. (a) After multiplying the equation with x we have (with the notation from the course) $M(x,y)=x+xy^2$ and $N(x,y)=x^2y$. Hence $\frac{\partial M}{\partial y}=2xy=\frac{\partial N}{\partial x}$, so the equation becomes exact, i.e. x is an integrating factor for the original equation.

   (b) We have to find a function $f(x,y)$ with $\frac{\partial f}{\partial x}=x+xy^2$ and $\frac{\partial f}{\partial y}=x^2y$. The first equation gives $f(x,y)=1/2x^2+1/2x^2y^2+g(y)$. Then the second equation gives $g'(y)=0$, so we can take $g(y)=0$. Then $f(x,y)=1/2x^2+1/2x^2y^2$, and the general solution is $1/2x^2+1/2x^2y^2=c$ where c is a constant.

6. With $z = x + y + 2$ we get

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + \frac{\mathrm{d}y}{\mathrm{d}x}\\ &= 1 + \frac{1}{z}\\ &= \frac{z+1}{z}.\end{aligned}$$

The differential equation $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z+1}{z}$ is variables separable, and separating the variables gives $\int \frac{z}{z+1}\,\mathrm{d}z = \int \mathrm{d}x$. After evaluating the integrals we find $z - \ln(z+1) = x + c$ where c is a constant. Hence the general solution of the differential equation is

$$x + y + 2 - \ln(x + y + 2 + 1) = x + c,$$

which can be simplified to

$$y + 2 - \ln(x + y + 3) = c.$$

7. (a) $\frac{\mathrm{d}M}{\mathrm{d}t} = \alpha M$

   (b) Separating the variables gives $\int \frac{1}{M}\,\mathrm{d}M = \int \alpha\,\mathrm{d}t$, so $\ln M = \alpha t + c$ and hence $M(t) = Ae^{\alpha t}$ for a constant $A = e^c$. From $M(0) = 2$ we get $A = 2$, so $M(t) = 2e^{\alpha t}$.

   (c) From $M(100) = 0.5$ we get $2e^{100\alpha} = 0.5$, so $\alpha = \ln(0.25)/100 \approx -0.0139$.

8. (a) Use integration by parts with $u = t^{x-1}$ and $v' = e^{-t}$. Then

$$\begin{aligned}\Gamma(x) &= \left[-t^{x-1}e^{-t}\right]_0^\infty + \int_0^\infty (x-1)t^{x-2}e^{-t}\,\mathrm{d}t\\ &= (x-1)\int_0^\infty (x-1)t^{(x-1)-1}e^{-t}\,\mathrm{d}t\\ &= (x-1)\Gamma(x-1).\end{aligned}$$

   (b) From the given formula with $x = \frac{1}{2}$ we get $\Gamma(\frac{1}{2})\Gamma(1-\frac{1}{2}) = \frac{\pi}{\sin(\frac{\pi}{2})} = \pi$, so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hence $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$.

Section B

9. (a) (i) stationary point of f: A stationary point of f is a point $(a, b) \in U$ such that the tangent plane to f at (a, b) exists and is horizontal.
   local maximum of f: A local maximum of f is a point $(a, b) \in U$ such that $f(a, b) \geq f(x, y)$ for all $(x, y) \in U$ that are in a sufficiently small disk around (a, b).
   global maximum of f: A global maximum of f is a point $(a, b) \in U$ such that $f(a, b) \geq f(x, y)$ for all $(x, y) \in U$.

   (ii) A boundary point of U is a point $(a, b) \in \mathbb{R}^2$ such that every disk around (a, b) contains points from U and from $\mathbb{R}^2 \setminus U$.

   (b) (i) $f_x = 4x + y + 6$, $f_y = x + 4y$, $f_{xx} = 4$, $f_{xy} = 1$, $f_{yy} = 4$

(ii) To find the stationary points we have to solve $f_x = 0$ and $f_y = 0$. This gives the stationary point $(-\frac{8}{5}, \frac{2}{5})$. The Hessian matrix is $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ which has determinant > 0 and diagonal elements > 0, so the stationary point is a local minimum.

(iii) The Lagrange function is

$$L(x, y, \lambda) = 2x^2 + xy + 2y^2 + 6x - \lambda(x^2 + y^2 - 8).$$

We have to solve $L_x = L_y = L_\lambda = 0$, i.e.

$$\begin{aligned} 4x + y + 6 - 2\lambda x &= 0 \\ x + 4y - 2\lambda y &= 0 \\ x^2 + y^2 - 8 &= 0. \end{aligned}$$

The first two equations give $y^2 + 6y - x^2 = 0$. Substituting $x^2 = 8 - y^2$ into this equation gives the quadratic equation $y^2 + 3y - 4 = 0$ with solutions $y = 1$ and $y = -4$. Clearly $y = -4$ is impossible because of $x^2 + y^2 - 8 = 0$. So $y = 1$ and it follows that $x = \pm\sqrt{7}$. So the two critical points are $(\sqrt{7}, 1)$ and $(-\sqrt{7}, 1)$. Since

$$\begin{aligned} f(\sqrt{7}, 1) &= 16 + 7\sqrt{7} \approx 34.5 \\ f(-\sqrt{7}, 1) &= 16 - 7\sqrt{7} \approx -2.5, \end{aligned}$$

it follows that the first point is the maximum and the second is the minimum on the circle.

(iv) Note that the stationary point is inside the disk and we have $f(-\frac{8}{5}, \frac{2}{5}) = \frac{-24}{5} \approx -4.8$. The global extrema are either local extrema inside the disk or extrema on the boundary. Hence the global minimum value is -4.8 and the global maximum value is $16 + 7\sqrt{7}$.

10. (a) (i) A function $f : U \to \mathbb{R}$ is continuous at a point $a \in U$ if $\lim_{x\to a} f(x) = f(a)$.

(ii) The derivative of a function f is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

For the function $f(x) = e^x$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} e^x &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^x. \end{aligned}$$

(b) (i) $\frac{\partial f}{\partial x} = (y - 2x^2y)e^{-(x^2+y^2)}$, $\frac{\partial f}{\partial y} = (x - 2y^2x)e^{-(x^2+y^2)}$

(ii) We have $f(1,2) = 2e^{-5}$, $f_x(1,2) = -2e^{-5}$, $f_y(1,2) = -7e^{-5}$. The tangent plane has equation

$$\begin{pmatrix} x-1 \\ y-2 \\ z-f(1,2) \end{pmatrix} \cdot \begin{pmatrix} f_x(1,2) \\ f_y(1,2) \\ -1 \end{pmatrix} = 0,$$

so in this case we obtain

$$z = -2e^{-5}x - 7e^{-5}y + 18e^{-5}.$$

(iii) $f(0.9, 2.1) \approx (-2 \cdot 0.9 - 7 \cdot 2.1 + 18)e^{-5} = \frac{3}{2}e^{-5} \approx 0.0101.$

(c) Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we get:

$$\begin{aligned} \iint_D xye^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y &= \int_0^{\pi/2}\int_0^{\sqrt{2}} r\cos\theta \cdot r\sin\theta \cdot e^{-r^2} \cdot r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\pi/2} \cos\theta\sin\theta\,\mathrm{d}\theta \cdot \int_0^{\sqrt{2}} r^3e^{-r^2}\,\mathrm{d}r \end{aligned}$$

Substituting $u = \sin\theta$ in the first integral gives

$$\int_0^{\pi/2} \cos\theta\sin\theta\,\mathrm{d}\theta = \int_0^1 u\,\mathrm{d}u = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}.$$

Substituting $u = -r^2$ in the second integral gives

$$\int_0^{\sqrt{2}} r^3e^{-r^2}\,\mathrm{d}r = \frac{1}{2}\int_0^{-2} ue^u\,\mathrm{d}u = \frac{1}{2}\left[ue^u - e^u\right]_0^{-2} = \frac{1}{2}\left(-2e^{-2} - e^{-2} + 1\right).$$

Hence

$$\iint_D xye^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \frac{1}{4}\left(-3e^{-2} + 1\right).$$

11. (a) (i) For $y(x) = e^{tx}$ we have $\frac{\mathrm{d}y}{\mathrm{d}x} = te^{tx}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = t^2e^{tx}$. Thus if $y(x)$ is a solution of the differential equation then

$$Pt^2e^{tx} + Qte^{tx} + Re^{tx} = 0.$$

We can divide by e^{tx} because $e^{tx} \neq 0$, and obtain $Pt^2 + Qt + R = 0$ as required.

(ii) In this case the general solution is $y(x) = Ae^{\alpha x} + Be^{\beta x}$ where A and B are constants.

(b) (i) The auxiliary equation is $t^2 + 6t + 25 = 0$. This has solutions $t = -3 \pm 4i$. Hence the general solution of the differential equation is

$$y(x) = e^{-3x}(A\cos(4x) + B\sin(4x))$$

where A and B are constants.

(ii) We have

$$\begin{aligned}y'(x) &= -3e^{-3x}(A\cos(4x) + B\sin(4x)) + e^{-3x}(-4A\sin(4x) + 4B\cos(4x)) \\ &= e^{-3x}\big((-3A+4B)\cos(4x) + (-4A-3B)\sin(4x)\big).\end{aligned}$$

Hence $y(0) = 2$ and $y'(0) = 5$ imply that $A = 2$ and $-3A + 4B = 5$, so $B = \frac{11}{4}$. Therefore the required solution is

$$y(x) = e^{-3x}(2\cos(4x) + \tfrac{11}{4}\sin(4x)).$$

Since $\lim_{x\to\infty} e^{-3x} = 0$ and $2\cos(4x) + \frac{11}{4}\sin(4x)$ is bounded, it follows that $\lim_{x\to\infty} y(x) = 0$.

(c) We look for a particular integral of the form $p(x) = a\cos(2x) + b\sin(2x)$. Then $p'(x) = -2a\sin(2x) + 2b\cos(2x)$ and $p''(x) = -4a\cos(2x) - 4b\sin(2x)$. Substituting this into the differential equation gives

$$\begin{aligned}(-4a\cos(2x) - 4b\sin(2x)) + 6(-2a\sin(2x) + 2b\cos(2x)) + 25(a\cos(2x) + b\sin(2x)) \\ = 39\sin(2x).\end{aligned}$$

Hence

$$\begin{aligned}21a + 12b &= 0 \\ -12a + 21b &= 39.\end{aligned}$$

This has the solution $a = -\frac{4}{5}$, $b = \frac{7}{5}$. Hence a particular integral is $p(x) = -\frac{4}{5}\cos(2x) + \frac{7}{5}\sin(2x)$. The general solution is the complementary function plus the particular integral, so

$$y(x) = e^{-3x}(A\cos(4x) + B\sin(4x)) - \frac{4}{5}\cos(2x) + \frac{7}{5}\sin(2x).$$

12. (a) $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$,
$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \frac{e^x - (-1)e^{-x}}{2} = \cosh x$, $\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \frac{e^x + (-1)e^{-x}}{2} = \sinh x$

(b) (i) We have $x_{i+1} = x_i + 0.5$ and $y_{i+1} = y_i + 0.5(y_i + \sinh x_i) = 1.5y_i + 0.5\sinh x_i$ with $x_0 = 0$, $y_0 = 1$.

i	x_i	y_i
0	0	1
1	0.5	1.5
2	1	2.5105

Hence $y(1) \approx 2.5105$.

(ii) We assume the solution has the form

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \dots.$$

Now

$$\begin{aligned} & & y(0) &= 1 \\ y' &= y + \sinh x &\Rightarrow\quad y'(0) &= y(0) + \sinh 0 = 1 \\ y'' &= y' + \cosh x &\Rightarrow\quad y''(0) &= y'(0) + \cosh 0 = 2 \\ y^{(3)} &= y'' + \sinh x &\Rightarrow\quad y^{(3)}(0) &= y''(0) + \sinh 0 = 2\end{aligned}$$

Hence

$$y(x) \approx 1 + x + x^2 + \frac{1}{3}x^3,$$

so $y(1) \approx 3.3333$.

(iii) The equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y = \sinh x$ is linear. The integrating factor is $\mu(x) = \exp(\int -1\,\mathrm{d}x) = e^{-x}$. Hence the general solution is

$$\begin{aligned} y &= \frac{1}{e^{-x}} \int e^{-x} \sinh x \,\mathrm{d}x \\ &= e^x \int e^{-x} \frac{e^x - e^{-x}}{2} \,\mathrm{d}x \\ &= \frac{e^x}{2} \int (1 - e^{-2x}) \,\mathrm{d}x \\ &= \frac{e^x}{2} \left(x + \frac{e^{-2x}}{2} + c \right) \\ &= \frac{xe^x}{2} + \frac{e^{-x}}{4} + \frac{ce^x}{2} \end{aligned}$$

where c is a constant. The initial condition $y(0) = 1$ gives $0 + \frac{1}{4} + \frac{c}{2} = 1$, so $c = \frac{3}{2}$. Thus the required solution is

$$y = \frac{xe^x}{2} + \frac{e^{-x}}{4} + \frac{3e^x}{4}.$$

We obtain

$$y(1) = \frac{e}{2} + \frac{e^{-1}}{4} + \frac{3e}{4} = \frac{5e}{4} + \frac{1}{4e} \approx 3.4898.$$

(c) If $x = \sinh y = \frac{e^y - e^{-y}}{2}$ then $y = \operatorname{arcsinh}(x)$. Set $z = e^y$. Then $2x = z - \frac{1}{z}$ and hence $z^2 - 2xz - 1 = 0$. It follows that $z = x \pm \sqrt{x^2+1}$, and since $z = e^y > 0$ we must have $z = x + \sqrt{x^2+1}$. Now $e^y = x + \sqrt{x^2+1}$ implies $y = \ln\left(x + \sqrt{x^2+1}\right)$, as required.