Calculus 3 Assignment 1

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1.

$$f(t) = \begin{cases} 5 & 0 \le t < 1 \\ t+4 & 1 \le t < 2 \\ 4t-2 & 2 \le t \end{cases}$$

(a) Where H is the unit step function

$$\begin{aligned} f(t) &= 5 - 5H(t-1) + (t+4)H(t-1) - (t+4)H(t-2) + (4t-2)H(t-2) \\ &= 5 + (t+4-5)H(t-1) + \Big((4t-2) - (t+4)\Big)H(t-2) \\ &= 5 + (t-1)H(t-1) + (3t-6)H(t-2) \\ &= 5 + H(t-1)(t-1) + 3H(t-2)(t-2) \end{aligned}$$

(b)

$$\begin{aligned} \mathcal{L}\Big(f(t)\Big) &= \mathcal{L}(5) + \mathcal{L}\Big(H(t-1)(t-1)\Big) + 3\mathcal{L}\Big(H(t-2)(t-2)\Big) \\ F(s) &= \frac{5}{s} + \frac{e^{-s}}{s} + \frac{3e^{-2s}}{s} \end{aligned}$$

2. (a)

$$\begin{aligned} \mathcal{L}(2t^4 + e^{-t}t^5) &= \mathcal{L}(2t^4) + \mathcal{L}(e^{-t}t^5) \\ &= 2\mathcal{L}(t^4) + \mathcal{L}(e^{-t}t^5) \\ &= \frac{2 \cdot 4!}{s^5} + \frac{5!}{s^{(5-1)+1}} \\ &= \frac{2 \cdot 4!}{s^5} + \frac{5!}{s^5} \end{aligned}$$

(b)

$$
\begin{aligned}
f(t) = \sin^2 t &= \frac{1}{2}\big(1-\cos(2t)\big)\\
&= \frac{1}{2}\Big(\mathcal{L}(1) - \mathcal{L}\big(\cos(2t)\big)\Big)\\
&= \frac{1}{2}\Big(\frac{1}{s} - \frac{s}{s^2+4}\Big)
\end{aligned}
$$

3.

$$
\begin{aligned}
F(s) &= \frac{s+2}{s^2+2s+5} = \frac{s+2}{(s+1)^2+2^2}\\
&= \frac{s}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\\
f(t) &= \frac{\sin(2t)}{2e^t} + \frac{\cos(2t)}{e^t}\\
&= e^{-t}\left(\frac{\sin(2t)}{2} + \cos(2t)\right)
\end{aligned}
$$

4. (a) Given $\mathcal{L}(y') = s\mathcal{L}(y) - y|_0$ we can write

$$
\begin{aligned}
\mathcal{L}(y'''') &= s\mathcal{L}(y''') - y'''|_0\\
&= s(s\mathcal{L}(y'') - y''|_0) - y'''|_0\\
&= s^2\mathcal{L}(y'') - sy''|_0 - y'''|_0\\
&= s^2(s\mathcal{L}(y') - y'|_0) - sy''|_0 - y'''|_0\\
&= s^3\mathcal{L}(y') - s^2y'|_0 - sy''|_0 - y'''|_0\\
&= s^3(s\mathcal{L}(y) - y|_0) - s^2y'|_0 - sy''|_0 - y'''|_0\\
&= s^4\mathcal{L}(y) - s^3y|_0 - s^2y'|_0 - sy''|_0 - y'''|_0.
\end{aligned}
$$

(b) Given initial conditions $y|_0 = 0$, $y'|_0 = 0$, $y''|_0 = 4$ and $y'''|_0 = 16$, we can write $\mathcal{L}(y'''') = \mathcal{L}(16y)$ and

$$
\begin{aligned}
16\mathcal{L}(y) &= s^4\mathcal{L}(y) - s^3y|_0 - s^2y'|_0 - sy''|_0 - y'''|_0\\
&= s^4\mathcal{L}(y) - s^3\cdot 0 - s^2\cdot 0 - s\cdot 4 - 16\\
&= s^4\mathcal{L}(y) - 4s - 16\\
\mathcal{L}(y)(s^4-16) &= 4s+16\\
\mathcal{L}(y) &= \frac{4s+16}{(s^2+4)(s^2-4)}.
\end{aligned}
$$

Now,

$$\frac{4s+16}{(s^2+4)(s^2-4)} \equiv \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2-4}$$

and

$$\begin{aligned}4s+16 &= (As+B)(s^2-4)+(Cs+D)(s^2+4)\\ &= As^3-4As+Bs^2-4B+Cs^3+4Cs+Ds^2+4D.\end{aligned}$$

Equating coefficients for s^3 gives

$$A+C=0,$$

for s^2

$$B+D=0,$$

for s

$$\begin{aligned}4 &= 4C-4A\\ C-A &= 1,\end{aligned}$$

and comparing constant terms

$$\begin{aligned}16 &= 4D-4B\\ D-B &= 4.\end{aligned}$$

For these equations to hold, it must be true that $A=-\frac{1}{2}$, $B=-2$, $C=\frac{1}{2}$ and $D=2$. Now let $y=f(t)$,

$$\begin{aligned}\mathcal{L}\big(f(t)\big) &= \frac{As+B}{s^2+4}+\frac{Cs+D}{s^2-4}\\ &= \frac{-\frac{s}{2}-2}{s^2+4}+\frac{\frac{s}{2}+2}{s^2-4}\\ &= \frac{1}{2}\cdot\left(\frac{-s-4}{s^2+4}+\frac{s+4}{s^2-4}\right)\\ 2\mathcal{L}\big(f(t)\big) &= -\frac{s}{s^2+2^2}-2\cdot\left(\frac{2}{s^2+2^2}\right)+\frac{s}{s^2-2^2}+2\cdot\left(\frac{2}{s^2-2^2}\right)\\ 2f(t) &= -\cos(2t)-2\sin(2t)+\cosh(2t)+2\sinh(2t)\\ f(t) &= \frac{1}{2}\Big(\cosh(2t)+2\sinh(2t)-\cos(2t)-2\sin(2t)\Big).\end{aligned}$$

5. Let $g(t)=e^{-t}$. Now we can write $\mathcal{L}(g(t))=G(s)=\frac{1}{s+1}$ and wrt the question

$$f(t)+\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau = 1. \tag{1}$$

By the definition of convolution, $\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau = \mathcal{L}^{-1}\big(F(s)G(s)\big)$. Now when we consider the Laplace transform of equation (1), we can write

$$\begin{aligned}
\mathcal{L}\big(f(t)\big) + \mathcal{L}\left(\mathcal{L}^{-1}\big(F(s)G(s)\big)\right) &= \mathcal{L}(1) \\
F(s) + F(s)G(s) &= \frac{1}{s} \\
F(s)\left(1 + \frac{1}{s+1}\right) &= \\
F(s)\left(\frac{s+1}{s+1} + \frac{1}{s+1}\right) &= \\
F(s)\left(\frac{s+2}{s+1}\right) &= \\
F(s) &= \frac{s+1}{s(s+2)}.
\end{aligned}$$

Taking the inverse Laplace transform of the equation gives

$$\begin{aligned}
\mathcal{L}^{-1}\big(F(s)\big) &= \mathcal{L}^{-1}\left(\frac{s+1}{s(s+2)}\right) \\
f(t) &= \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s+2}\right) \\
&= e^{-2t} + (1 * e^{-2t})(t) \\
&= e^{-2t} + (e^{-2t} * 1)(t) \\
&= e^{-2t} + \int_0^t \big(e^{-2\tau}\cdot 1\big)\,\mathrm{d}\tau \\
&= e^{-2t} + \left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\
&= e^{-2t} + \Big((-\frac{1}{2}e^{-2t}) - (-\frac{1}{2}e^{-2\cdot 0})\Big) \\
&= e^{-2t} + (1 - \frac{1}{2}e^{-2t}) \\
&= \frac{1}{2}e^{-2t} + 1.
\end{aligned}$$