Algebra 2 Assignment 2

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1. In order to show that $V_4 \trianglelefteq S_4$, we must show that for a set of V_4-coset representatives A, $xV_4 = V_4x \; \forall \; x \in A$. Because $|S_4 : v_4| = 6$ we know $|A| = 6$. Choosing the first member of A as 1, (now all we know about A is that $A = \{1, ...\}$) by the property of the identity element, we know that

$$1V_4 = V_41 = V_4 = \{1, (12)(34), (13)(24), (14)(23)\}.$$

Let B be the set of all $g \in S_4$ where g is a member of a V_4-coset we have already considered. Because the cosets "generated" by coset representatives partition S_4, we know that if we choose our next element x from $S_4 - B$, we will be fixing a new member in A.

The next member of A will be (12), we write

$$(12)V_4 = \{(12), (34), (1423), (1324)\}$$
$$V_4(12) = \{(12), (34), (1324), (1423)\} = (12)V_4.$$

We now know that $A = \{1, (12), ...\}$, and choose the next member as (13). We write

$$(13)V_4 = \{(13), (1432), (24), (1234)\}$$
$$V_4(13) = \{(13), (1234), (24), (1423)\} = (13)V_4.$$

Now $A = \{1, (12), (13), ...\}$, and consider (23). We write

$$(23)V_4 = \{(13), (1432), (24), (1234)\}$$
$$V_4(23) = \{(23), (1342), (1243), (14)\} = (23)V_4.$$

Now $A = \{1, (12), (13), (23), ...\}$, and consider (123). We write

$$(123)V_4 = \{(123), (243), (142), (134)\}$$
$$V_4(123) = \{(123), (134), (243), (142)\} = (123)V_4.$$

Now $A = \{1, (12), (13), (23), (123), ...\}$, and consider (234). We write

$$(234)V_4 = \{(234), (124), (132), (143)\}$$
$$V_4(234) = \{(234), (134), (143), (124)\} = (234)V_4.$$

Now $A = \{1, (12), (13), (23), (123), (234)\}$, and because we know $|A| = 6$, there are no more coset representatives to verify meet the condition $xV_4 = V_4x \ \forall \ x \in A$. As such, we have shown that $V_4 \trianglelefteq S_4$.

2. (a) The G-action on M_3 for $G = GL_3(\mathbb{R})$ is defined exactly as matrix multiplication. The identity of G is I_3. For M_3 to be a G-set, for $A, B \in G$ and for $C \in M_3$ it must hold that $(AB) \cdot C = A \cdot (B \cdot C)$ and $I_3C = C$. By the associativity of matrix multiplication the first is true, by the behaviour of the identity matrix under matrix multiplication the second is true. Therefore M_3 is a G-set and G acts on M_3.

   (b) Suppose $X_2 \in \text{orb}(X_1)$. Because orbits partition M_3,

$$\text{orb}(X_1) = \text{orb}(X_2)$$

and by the definition of orbit,

$$\{a \cdot X_1 : a \in G\} = \{b \cdot X_2 : b \in G\}.$$

Therefore there must be some particular $A, B \in G$ such that

$$A \cdot X_1 = B \cdot X_2.$$

Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}$. Now

$$\begin{aligned} A \cdot X_1 &= B \cdot X_2 \\ AX_1 &= BX_2 \\ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} &= \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 2a_1 & a_1 + 2a_3 & a_3 \\ 2a_4 & a_4 + 2a_6 & a_6 \\ 2a_7 & a_4 + 2a_9 & a_9 \end{pmatrix} &= \begin{pmatrix} 2b_1 & b_1 + b_3 & b_3 \\ 2b_4 & b_4 + b_6 & b_6 \\ 2b_7 & b_4 + b_9 & b_9 \end{pmatrix} \end{aligned}$$

Considering the relationship between coefficients of variables in each row is identical, we only need to consider $2a_1 = 2b_1$, $a_1 + 2a_3 = b_1 + b_3$ and $a_3 = b_3$. These equations only hold if $a_3 = b_3 = 0$ (which means $a_6 = b_6 = a_9 = b_9 = 0$) and the matrices A and B have a zero column, telling us $|A| = |B| = 0$ and hence $A, B \notin G$, $\text{orb}(X_1) \neq \text{orb}(X_2)$ and X_1 and X_2 must therefore be in different orbits.

(c) i. Using the same reasoning as above, for some $A, B \in G$, that

$$AX_1 = BX_3$$
$$\begin{pmatrix} 2a_1 & a_1 + 2a_3 & a_3 \\ 2a_4 & a_4 + 2a_6 & a_6 \\ 2a_7 & a_4 + 2a_9 & a_9 \end{pmatrix} = \begin{pmatrix} 2b_2 - 2b_3 & b_1 + b_2 - b_3 & b_1 \\ 2b_5 - 2b_6 & b_4 + b_5 - b_6 & b_4 \\ 2b_8 - 2b_9 & b_7 + b_8 - b_9 & b_7 \end{pmatrix}$$

tells us $a_1 = b_2 - b_3$, $a_1 + 2a_3 = b_1 + b_2 - b_3$ and $a_3 = b_1$. These equations only hold if $b_1 = 0$, which means that $A, B \notin G$, $\mathrm{orb}(X_1) \neq \mathrm{orb}(X_3)$ and X_1 and X_3 must therefore be in different orbits.

ii. Again, for some $A, B \in G$, setting

$$AX_2 = BX_3$$
$$\begin{pmatrix} 2a_1 & a_1 + a_3 & a_3 \\ 2a_4 & a_4 + a_6 & a_6 \\ 2a_7 & a_4 + a_9 & a_9 \end{pmatrix} = \begin{pmatrix} 2b_2 - 2b_3 & b_1 + b_2 - b_3 & b_1 \\ 2b_5 - 2b_6 & b_4 + b_5 - b_6 & b_4 \\ 2b_8 - 2b_9 & b_7 + b_8 - b_9 & b_7 \end{pmatrix}$$

tells us $a_1 = b_2 - b_3$, $a_1 + a_3 = b_1 + b_2 - b_3$ and $a_3 = b_1$. These equations are consistent without requiring some any variable to be zero, consequently there exist matrices A and B in G. As such $\mathrm{orb}(X_3) = \mathrm{orb}(X_2)$ and X_2 and X_3 are in the same orbit.

3. (a) Let $(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) \in \mathrm{Fix}(\alpha)$ where a_i are the sides of the hexagon and b_i are the diagonals. Then

$$\begin{aligned}(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) &= \alpha \cdot (a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) \\ &= (a_2, a_3, a_4, a_5, a_6, a_1, b_2, b_3, b_1).\end{aligned}$$

(Note a_i and b_i are rotated idependently)

Hence $a_1 = a_2 = ... = a_3$ and $b_1 = b_2 = b_3$ and $|\mathrm{Fix}(\alpha)| = 3^2 = 9$.

Let $(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) \in \mathrm{Fix}(\alpha^2)$. Then

$$\begin{aligned}(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) &= \alpha^2 \cdot (a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) \\ &= (a_3, a_4, a_5, a_6, a_1, a_2, b_3, b_1, b_2).\end{aligned}$$

Hence $a_1 = a_3 = a_5$, $a_2 = a_4 = a_6$ and $b_1 = b_2 = b_3$ and $|\mathrm{Fix}(\alpha^2)| = 3^3 = 27$.

Let $(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) \in \text{Fix}(\alpha^3)$. Then

$$\begin{aligned}(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) &= \alpha^3 \cdot (a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3) \\ &= (a_4, a_5, a_6, a_1, a_2, a_3, b_1, b_2, b_3).\end{aligned}$$

Hence $a_1 = a_4$, $a_2 = a_5$, $a_3 = a_6 = a_6$ and there are no restrictions on b_i and $|\text{Fix}(\alpha^3)| = 3^6 = 729$.

Now note that $|\text{Fix}(\alpha^4)| = |\text{Fix}(\alpha^2)| = 27$, that $|\text{Fix}(\alpha^{-1})| = |\text{Fix}(\alpha)| = 9$ and that $|\text{Fix}(1)| = |\text{Fix}(X)| = 19683$.

(b)

$$\begin{aligned}\#\text{colourings} &= \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| \\ &= \frac{1}{6}(19683 + 9 + 27 + 729 + 27 + 9 \\ &= 3414.\end{aligned}$$