Calculus 2 Assignment 1

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1. (a) P' is a square $(0,0),(0,3),(3,3),(3,0)$ and so the limits of both the inner and outer integrals are 0 and 3.

   (b) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{2}{3}$, $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{1}{3}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

   (c)

$$\begin{aligned} \iint_\Delta f(u,v)\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}u\mathrm{d}v &= \frac{1}{3}\cdot\int_0^3\int_0^3 e^{\frac{2u+v}{3}}\mathrm{d}u\mathrm{d}v \\ &= \tfrac{3}{2}(e^3 - e^2 - e + 1) \\ &\approx 16.46729849369695841750672208398107137907551776145351486798\ldots \end{aligned}$$

2. (a) The other co-ordinates of R are $(x,-y)$, $(-x,-y)$, $(-x,y)$ and the area of R, $A_R = 4xy$.

   (b) Let $f(x,y) = A_R$, $g(x,y) = c_1x^2 + c_2y^2 - 1$, $c_1 = a^{-2}$, $c_2 = b^{-2}$. We will maximise f subject to the constraint $g(x,y) = 0$. Now let

$$\begin{aligned} L(x,y,\lambda) &= f(x,y) - \lambda g(x,y) \\ &= 4xy - \lambda(c_1x^2 + c_2y^2 - 1). \end{aligned}$$

   Then

$$\begin{aligned} L_x &= 4y - 2\lambda c_1 x \\ L_y &= 4x - 2\lambda c_2 y \\ L_\lambda &= -(c_1x^2 + c_2y^2 - 1). \end{aligned}$$

Let $L_x = L_y = 0$, then

$$\begin{aligned} 4y - 2\lambda c_1 x &= 0 \\ 4c_2 y^2 - 2\lambda c_1 c_2 xy &= \end{aligned}$$

$$\begin{aligned} 4x - 2\lambda c_2 y &= 0 \\ 4c_1 x^2 - 2\lambda c_1 c_2 xy &= \end{aligned}$$

and

$$\begin{aligned} (4c_2 y^2 - 2\lambda c_1 c_2 xy) - (4c_1 x^2 - 2\lambda c_1 c_2 xy) &= 0 \\ c_2 y^2 &= c_1 x^2. \end{aligned}$$

Since $c_1 x^2 + c_2 y^2 - 1$, $2c_1 x^2 = 1$ then

$$x = \pm\sqrt{\tfrac{1}{2c_1}} = \pm\frac{a}{\sqrt{2}},$$

$$y = \pm\sqrt{\tfrac{1}{2c_2}} = \pm\frac{b}{\sqrt{2}}$$

and the maximum is therefore $f\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) = 2ab$.

(c) i. With the change of variables $u = \frac{x}{a}$, $v = \frac{y}{b}$ the inequality $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ becomes $u^2 + v^2 \leq 1$. The set of points satisfying this inequality such that $(u, v) \in \mathbb{R}^2$ describe the unit disc. Let's call this set of points D. Under the change of variables, $x = ua$, $y = vb$, so $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = ab$ and the area of E can then be written

$$A_E = A_D = ab \iint_D 1 \mathrm{d}u\mathrm{d}v$$

ii. Let's do another change of variables! In particular, $r\cos\theta = u$ and $r\sin\theta = v$. Let $\Delta = D$, then the area of Δ can be written as the integral

$$\begin{aligned} ab\iint_{\Delta} r\mathrm{d}r\mathrm{d}\theta &= ab\int_0^{2\pi}\int_0^1 r\mathrm{d}r\mathrm{d}\theta \\ &= ab\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 \mathrm{d}\theta \\ &= ab\int_0^{2\pi}\frac{1}{2}\mathrm{d}\theta \\ &= ab\left[\frac{\theta}{2}\right]_0^{2\pi} \\ A_E = A_D = A_\Delta &= \pi ab \end{aligned}$$

iii. $\frac{A_R}{A_E} = \frac{2ab}{\pi ab} = \frac{2}{\pi}$

3. (a) Because

$$h_x(a,b) = f_x(a,b) + g_x(a,b) = f_y(a,b) + g_y(a,b) = h_y(a,b) = 0,$$

$h(a,b)$ is a stationary point.

(b) i. If $f(a,b)$ and $g(a,b)$ are at their (locally) lowest then their sum $h(a,b)$ will also be at its lowest. The statement is true.

ii. The statement is false, a counterexample is $f(x,y) = x^2 - \frac{y^2}{10}$, $g(x,y) = x^2 + y^2$, $(a,b) = (0,0)$. Here $f(a,b)$ is a saddle point, $g(a,b)$ is a local minimum. However, $h(a,b)$ is a not a saddle point, but instead is a local minimum.

4. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, with $g(u(x,y), v(x,y))$ where

$$u(x,y) = \tfrac{x}{a} \text{ and } v(x,y) = \tfrac{y}{b}.$$

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, with $f(u,v) = u^2 + v^2 - 1$. Then $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$, where

$$F(x,y) = f(g(x,y)) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0.$$

Now let $w = F$.

(a)

$$\begin{aligned} \frac{\partial w}{\partial x} = F_x = \frac{\partial F}{\partial x} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} \\ &= 2u\cdot\frac{1}{a} + 2v\cdot 0 \\ &= \frac{2x}{a^2} \end{aligned}$$

and

$$\frac{\partial w}{\partial y} = F_y = \frac{2y}{b^2}$$

Here's where I started getting confused by the notation used in the question. Assuming that $\frac{dw}{dx}$ in the question means $\frac{\mathrm{d}w}{\mathrm{d}x}$ and not $\frac{\partial w}{\partial x}$ (it reads as something in between), then Wolfram Alpha defines $\frac{\mathrm{d}}{\mathrm{d}x}f(x,y) = f_x + f_y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$. Applying this to our function and its partial derivatives gives

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}x} &= F_x + F_y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\end{aligned}$$

(b) Since $\frac{\mathrm{d}w}{\mathrm{d}x}$ describes the gradient in w along the x-axis, and the function w describes a "flat" ellipse on the x-y axis – meaning $w = 0$ for all (x, y), and hence the gradient of w in the x-axis will always be 0, ie. $\frac{\mathrm{d}w}{\mathrm{d}x} = 0$. Then we can write

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}x} = F_x + F_y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{F_x}{F_y}\end{aligned}$$

(c) Substituting F_x and F_y in terms of x gives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{2x}{a^2}}{\frac{2y}{b^2}} \\ &= -\frac{b^2 x}{a^2 y} \\ &= -\frac{bx}{a^2\sqrt{1-\left(\frac{x}{a}\right)^2}}\end{aligned}$$

(d) Let $u(x) = 1 - \left(\frac{x}{a}\right)^2$, $f = b\sqrt{u}$. Now

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= u'(x)f'(u) \\
&= \frac{2x}{a^2} \cdot \frac{b}{2\sqrt{1 - \left(\frac{x}{a}\right)^2}} \\
&= -\frac{bx}{a^2\sqrt{1 - \left(\frac{x}{a}\right)^2}}
\end{aligned}$$