Discrete Mathematics Examination 2015
Section A

1. (a) This sum is simply $(x+y)^n$ by the binomial theorem with $x = 70$, $y = -30$, and $n = 20$. Thus, the sum is 40^{20}. [2]

   (b) We have

$$\sum_{r=1}^{20} \left(r + (r+2)^2\right) = \sum_{r=1}^{20} r + \sum_{r=1}^{20} (r+2)^2.$$

   The first sum is known to be $20(21)/2 = 210$. The second sum becomes (through our equation for the sum of r^2)

$$\begin{aligned} \sum_{r=0}^{22} r^2 - 2^2 - 1^2 &= \frac{22(23)(45)}{6} - 4 - 1 \\ &= 11 \cdot 23 \cdot 15 - 5 \\ &= 3790. \end{aligned}$$

   Thus, the final answer is $3970 + 210 = 4000$. [3]

2. (a) We don't need formulae for this. The answer is clearly 0. [1]

   (b) We have indistinguishable balls with non-exclusive occupancy. Using the formula from the notes, the answer is $\binom{6-1+7}{7} = 792$. [2]

   (c) We have distinguishable balls with non-exclusive occupancy. So, from the notes, the answer is 6^7. [2]

3. (a) We have

$$|A_3| = |\{3, 6, \ldots, 198\}| = |\{3 \cdot 1, 3 \cdot 2, \ldots, 3 \cdot 66\}| = 66,$$

   and

$$|A_7| = |\{7, 14, \ldots, 198\}| = |\{7 \cdot 1, 7 \cdot 2, \ldots, 7 \cdot 28\}| = 28.$$

   Thus, $|A_3| = 66$ and $|A_7| = 28$. [2]

   (b) The set of elements coprime to 21 are those elements precisely not in $A_3 \cup A_7$, and therefore we must compute $200 - |A_3 \cup A_7|$. Using inclusion-exclusion, we get

$$|A_3 \cup A_7| = |A_3| + |A_7| - |A_3 \cap A_7|.$$

   We must compute the last quantity in the previous equation. So,

$$|A_3 \cap A_7| = |\{21, 42, \ldots, 189\}| = |\{21 \cdot 1, 21 \cdot 2, \ldots, 21 \cdot 9\}| = 9.$$

   Thus, $200 - |A_3 \cup A_7| = 200 - 66 - 28 + 9 = 115$. [3]

Please turn over

4. If the characteristic polynomial has a repeated root t, then the characteristic polynomial must be $(\lambda - t)^2 = \lambda^2 - 2t\lambda + t^2$. Thus, we must have $a = 2t$ and $b = -t^2$. Therefore, if $u_n = At^n$, we have

$$\begin{aligned} 2tu_{n-1} - t^2 u_{n-2} &= 2tAt^{n-1} - t^2 At^{n-2} \\ &= 2At^n - At^n \\ &= At^n = u_n. \end{aligned}$$

Thus, $u_n = At^n$ is a solution. Now set $u_n = Ant^n$. We have

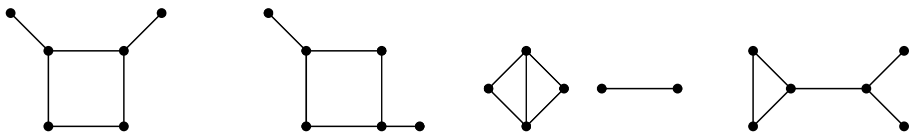
$$\begin{aligned} 2tu_{n-1} - t^2 u_{n-2} &= 2tA(n-1)t^{n-1} - t^2 A(n-2)t^{n-2} \\ &= 2Ant^n - 2At^n - Ant^n + 2At^n \\ &= Ant^n = u_n. \end{aligned}$$

Thus, $u_n = Ant^n$ is also a solution. [5]

5. (a) The sum of the degrees of the vertices in a graph G is equal to twice the number of edges in G. [2]

   (b) $1 + 1 + 2 + 2 + 3 + 3 = 12$ so G has **6** edges. [1]

   (c) Possibilities include:



(one mark each, for a total of up to two marks.) [2]

6. (a) There are n^{n-2} distinct labelled trees on n vertices. [1]

   (b) A Prüfer sequence of length $n - 2$ consists of $n - 2$ integers that each lie between 1 and n inclusive. In this case we have $n = 6$, so this sequence does indeed satisfy these conditions and hence is a Prüfer sequence. [1]

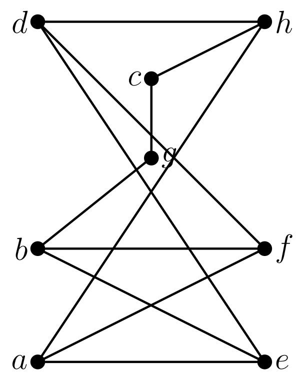
   (c) $[3, 3, 4, 5, 5]$ [3]

Please turn over

7. (a) A graph is planar if and only if it does not contain a subgraph that is isomorphic to a subdivision of either K_5 or $K_{3,3}$. [2]

(b) Being bipartite, this graph is triangle-free. Therefore we can apply the result from the notes that states that all triangle-free connected planar simple graphs with $n \geq 3$ vertices and m edges satisfy $m \leq 2n - 4$. This is a triangle-free connected simple graph with $m = 13$ and $n = 8$, so we deduce it is not planar.

Alternatively, we observe that the following subgraph is a subdivision of $K_{3,3}$:



[3]

8. (a) Let G be a simple connected graph with $n \geq 3$ vertices. If $d(v) \geq n/2$ for all vertices v, then G is Hamiltonian. [2]

(b) The graph H has 6 vertices, and is regular of degree 3. Hence it is Hamiltonian, by Dirac's theorem. [1]

(c) H is not Eulerian, as it contains vertices of odd degree. [2]

Please turn over

Section B

9. (a) (i) The problem can be expressed as the integer equation

$$X_1 + X_2 + X_3 + X_4 = 40,$$

with $X_1 \geq 8$, $0 \leq X_2, X_3 \leq 5$ and $7 \leq X_4 \leq 12$. [2]

(ii) From the notes, the generating series is

$$(x^8 + x^9 + \cdots)(1 + x + x^2 + x^3 + x^4)^2(x^7 + x^8 + x^9 + x^{10} + x^{12}).$$

The compact form will come next. [2]

(iii) The generating series can be expressed as

$$\begin{aligned} x^8 \frac{1}{1-x} \frac{(1-x^6)^2}{(1-x)^2} x^7 \frac{1-x^6}{1-x} &= x^{15} \frac{(1-x^6)^3}{(1-x)^4} \\ &= x^{15}(1 - 3x^6 + 3x^{12} - x^{18}) \sum_{r \geq 0} \binom{4-1+r}{r} x^r. \end{aligned}$$

We want the coefficient of x^{40} in the above generating series, which is the coefficient of x^{25} of the series after the x^{15}. Therefore, the coefficient of the remaining is

$$\binom{28}{25} - 3\binom{22}{19} + 3\binom{16}{13} - \binom{10}{7}.$$

This answer is sufficient. Simplifying, you get 216. [6]

(b) (i) We see that

$$\begin{aligned} \sum_{r \geq 0} r x^r &= x \frac{d}{dx} \frac{1}{1-x} \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

We do so again to obtain

$$\begin{aligned} g(x) &= x \frac{d}{dx} \frac{x}{(1-x)^2} \\ &= \frac{x(1+x)}{(1-x)^3}. \end{aligned}$$

[3]

Please turn over

(ii) To answer our question, we want to the coefficient of x^n in $\frac{g(x)}{1-x} = \frac{x(1+x)}{(1-x)^4}$. So, we find

$$\begin{aligned}
x(1+x)\left(\sum_{r=0}^{\infty}\binom{3+r}{r}x^r\right) &= \sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+1} + \sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+2} \\
&= \sum_{r=1}^{\infty}\binom{2+r}{r-1}x^r + \sum_{r=2}^{\infty}\binom{1+r}{r-2}x^r \\
&= x + \sum_{r=2}^{\infty}\left[\binom{2+r}{r-1} + \binom{1+r}{r-2}\right]x^r \\
&= x + \sum_{r=2}^{\infty}\left[\frac{(r+2)(r+1)r}{6} + \frac{(r+1)r(r-1)}{6}\right]x^r \\
&= x + \sum_{r=2}^{\infty}\frac{(2r+1)(r+1)r}{6}x^r.
\end{aligned}$$

Thus, the coefficient of x^n gives us the correct formula (notice, this is true for $n = 0$ and $n = 1$ as well). [4]

(c) We take the coefficient of x^n of both sides. First notice that the coefficient of x^0 of the LHS is c_0 and the coefficient of x^0 for the RHS is 1. That takes care of the base case.

Now take the coefficient of a general term x^{n+1}. On the LHS that's c_{n+1}. On the RHS, that becomes the coefficient of x^n in $C(x)^2$. We find that

$$C(x)^2 = \left(\sum_{i\geq 0}c_i x^i\right)\left(\sum_{j\geq 0}c_j x^j\right) = \sum_{i\geq 0}\sum_{j\geq 0}c_i c_j x^{i+j}.$$

Substituting $n = i + j$, we see that $n \geq 0$ and $j = n - i$ and $0 \leq i \leq n$. Thus, the previous equation becomes

$$\sum_{n\geq 0}\sum_{i\geq 0}^{n}c_i c_{n-i}x^n,$$

which has $\sum_{i\geq 0}^{n} c_i c_{n-i}$ as the coefficient of x^n, giving the result. [3]

Please turn over

10. (a) Using the formula in the notes, if a difference equation has the form $u_n = f(n)u_{n-1}$, with initial condition $u_0 = U$, then its solution is

$$u_n = U\prod_{i=1}^{n} f(n).$$

Thus, the solution is

$$\begin{aligned}
c_n &= \prod_{i=1}^{n} \frac{4i-2}{i+1} \\
&= \frac{2^n \prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{n} i+1} \\
&= \frac{2^n n! \prod_{i=1}^{n}(2i-1)}{n!(n+1)!} \\
&= \frac{\prod_{i=1}^{n}(2i) \prod_{i=1}^{n}(2i-1)}{n!(n+1)!} \\
&= \frac{(2n)!}{n!(n+1)!} \\
&= \frac{1}{n+1}\frac{(2n)!}{n!n!} \\
&= \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

[5]

(b) (i) This is a linear second order DE with constant coefficients with characteristic polynomial $(\lambda-2)(\lambda-3)$. Thus, the solution to the homogeneous equation is

$$A2^n + B3^n.$$

For the particular solution $P(n)$ we try $Cn3^n + D4^n$, because 3^n already occurs in the solution to the homogeneous part. Thus,

$$\begin{aligned}
Cn3^n + D4^n &= u_n \\
&= 5u_{n-1} - 6u_{n-2} + 3\cdot 2^n - 4^n \\
&= 5C(n-1)3^{n-1} + 5D4^{n-1} - 6C(n-2)3^{n-2} - 6D4^{n-2} + 2\cdot 3^n - 4^n \\
&= 3Cn3^{n-1} - C3^{n-1} + 5D4^{n-1} - 6D4^{n-2} + 2\cdot 3^n - 4^n \\
&= Cn3^n - C3^{n-1} + 14D4^{n-2} + 2\cdot 3^n - 4^n.
\end{aligned}$$

Thus, $-C3^{n-1} + 2\cdot 3^n = 0$ implying that $(-C+6)\cdot 3^{n-1} = 0$. Therefore, $C = 6$. Also, we have $14D4^{n-2} - 4^n = D4^n$, implying that $2D4^{n-2} = -4^n = 16\cdot 4^{n-2}$. Thus, $D = -8$. Therefore, our solution without accounting for initial conditions is $A2^n + B3^n + 6n3^n - 8\cdot 4^n$. With $u_0 = u_1 = 1$, we get the linear system

$$A + B = 9, 2A + 3B = 15.$$

Please turn over

Solving, we get $A = 12$ and $B = -3$. Thus, the solution is $u_n = 3 \cdot 2^{n+2} - 3^{n+1} + 2n3^{n+1} - 2 \cdot 4^{n+1}$. [5]

(ii) Same format, with characteristic polynomial given by $(\lambda - 1)^2$. Thus, there is a repeated root 1. So, the solution to the homogeneous equation is $A + Bn$. For our particular solution, we are told to try $P(n) = Cn^2$, since 1 is a repeated root of the characteristic equation. Doing so, we get

$$\begin{aligned} Cn^2 &= 2C(n-1)^2 - C(n-2)^2 + 1 \\ &= 2Cn^2 - 4Cn + 2C - Cn^2 + 4Cn - 4C + 1 \\ &= Cn^2 - 2C + 1. \end{aligned}$$

Thus, $-2C + 1 = 0$, implying that $C = 1/2$. Thus, the general solution is $A + Bn + \frac{1}{2}n^2$. Using the initial conditions, we get $u_0 = 1 = A$ and $u_1 = 1 = 1 + B + \frac{1}{2}$. Thus, $B = -\frac{1}{2}$. Thus, our solution is $u_n = 1 - \frac{1}{2}n + \frac{1}{2}n^2$. [5]

(c) Let $U(x)$ be the generating series $\sum_{n\geq 0} u_n x^n$. Then, using the difference equation, we get that

$$\begin{aligned} \sum_{n\geq 1} u_n x^n &= \sum_{n\geq 1} a u_{n-1} x^n + b \sum_{n\geq 1} x^n \\ &= axU(x) + b\frac{x}{1-x}. \end{aligned}$$

The LHS is $U(x) - U$. Thus, we find that

$$U(x) = U\frac{1}{1-ax} + b\frac{x}{(1-x)(1-ax)}. \tag{1}$$

The coefficient of x^n on the LHS gives u_n. Thus, we find that the coefficient of the RHS. The coefficient of x^n of the first term of the RHS is clearly Ua^n. For the second term, we use partial fractions:

$$\begin{aligned} &\frac{bx}{(1-x)(1-ax)} = \frac{A}{1-x} + \frac{B}{1-ax} \\ &\Rightarrow A(1-ax) + B(1-x) = bx \\ &\Rightarrow (A+B)1 + (-Aa - B)x = bx. \end{aligned}$$

This gives the linear system

$$A + B = 0, -Aa - B = b.$$

Solving, we get $A = \frac{b}{1-a}$ and $B = \frac{-b}{1-a}$. So, the second term in (1) becomes

$$b\frac{x}{(1-x)(1-ax)} = \frac{b}{1-a}\frac{1}{1-x} - \frac{b}{1-a}\frac{1}{1-ax},$$

and the coefficient of x^n is

$$\frac{b}{1-a} - \frac{b}{1-a}a^n = \frac{b(1-a^n)}{1-a}.$$

Adding Ua^n to the previous equation (i.e. combining the coefficients of x^n of both terms in (1)) gives the answer. [5]

Please turn over

11. (a) (i) The connectivity of a connected graph (other than a complete graph) is the minimum number of vertices that must be removed in order for the resulting graph to be disconnected. Define the connectivity of K_n to be $n-1$ for $n \geq 2$. [2]

(ii) *e.g.* $\{fd, fg, bg\}$ [1]

(iii) *e.g.* $\{ef, eb, af, ab\}$ [1]

(iv) 3 [1]

(b) (i) If v and w are two vertices of a connected graph G, then the maximum number of edge-disjoint vw-paths in G is equal to the size k of the smallest vw-disconnecting set. [2]

(ii) A suitable set of paths is $\{vhw, vgejw, vdbfw, vaecw\}$. There are four edge-disjoint vw-paths in this set, and the set $\{vg, vh, vd, va\}$ is a vw-disconnecting set of size 4, hence by Menger's theorem this is a largest possible set of edge-disjoint vw-paths. [3]

(c) (i) Let M be a square matrix. The size of the largest independent set of entries that are equal to zero is equal to the smallest number of rows and/or columns that between them contain all zero entries of the matrix. [2]

(ii) All the zeros in the matrix are contained in the first two rows and/or first three and last column. The highlighted entries below form a set of 5 independent zero entries, hence the answer is 5 by the Kőnig-Egerváray Theorem.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & \mathbf{0} & 0 \\ 0 & 1 & 0 & 1 & 1 & \mathbf{0} \\ 1 & 1 & \mathbf{0} & 1 & 1 & 0 \\ 0 & \mathbf{0} & 1 & 1 & 1 & 1 \\ \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

[2]

(iii) First we process the rows:

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 & 3 \\ 0 & 2 & 2 & 2 & 4 & 1 \\ 1 & 1 & 0 & 1 & 2 & 0 \\ 0 & 3 & 0 & 3 & 0 & 1 \\ 4 & 3 & 0 & 2 & 0 & 4 \\ 2 & 2 & 1 & 0 & 3 & 1 \end{bmatrix}$$

Then we process the columns:

$$\begin{bmatrix} 4 & 2 & 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 2 & 4 & 1 \\ 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 3 & 0 & 1 \\ 4 & 2 & 0 & 2 & 0 & 4 \\ 2 & 0 & 1 & 0 & 3 & 1 \end{bmatrix}$$

Please turn over

Now we continue with the Hungarian algorithm. Covering columns 1, 3, 5 and rows 3, 6 leads to the following array:

$$\begin{bmatrix} 4 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 4 & 0 \\ 2 & 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 & 3 \\ 3 & 2 & 2 & 0 & 4 & 1 \end{bmatrix}.$$

There are sets of six independent zero entries, these give optimal assigments. For example: $C_1 - T_5, C_2 - T_1, C_3 - T_2, C_4 - T_6, C_5 - T_3, C_6 - T_4$. [6]

Please turn over

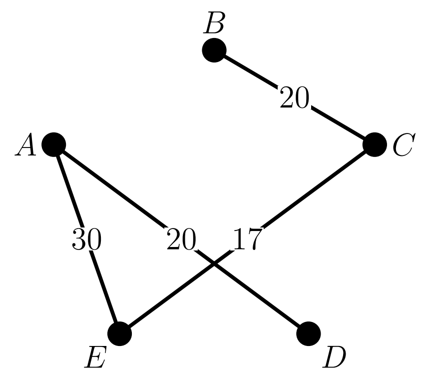
12. (a) There are $(2n)!$ ways of writing the elements in a list, and then pairing the first two, second two and so on. However there are then $n!$ ways to change the order of the resulting pairs, and $(2!)^n$ ways to reorder elements within the pairs, thus giving the desired total. [4]

(b) (i) A complete matching from V_1 to V_2 in a bipartite graph $G(V_1, V_2)$ is a one-to-one correspondence between the vertices in V_1 and a subset of the vertices in V_2 with the property that corresponding vertices are joined by an edge of G. [2]

(ii) The graph G has a complete matching $\{cg, bf, ae\}$. The graph H has no perfect matching as e is the only neighbour of a and the only neighbour of b, violating Hall's conditions. [4]

(c) (i) The Travelling Salesman Problem is the problem of finding a Hamilton cycle of minimum total weight in a weighted complete graph. [2]

(ii) One possible order in which edges are chosen according to Kruskal's algorithm is EC, BC, AD, AE, leaving the following spanning tree, with total weight 87.



[4]

(iii) Starting at F, we observe the two lowest weight edges adjacent to F both have weights 27. Removing F, the resulting graph is the one depicted in part (ii), hence the weight of a minimum spanning tree for this graph is 87. This gives a total of $87 + 27 + 27 = 141$. [2]

(iv) Starting at A, we create a cycle containing vertices A and D. We then place E clockwise from A in the cycle, C clockwise from E, B clockwise from C and F clockwise from B. This gives the cycle $AECBDA$, of total weight 143. This is an upper bound for the TSP on this graph.

[2]