BIRKBECK

(University of London)

BSc EXAMINATION
SCHOOL OF BUSINESS, ECONOMICS AND INFORMATICS

Discrete Mathematics
BUEM002S5

30 credits

Tuesday, 26 May, 2015
Morning, 10:00 a.m. - 1:00 p.m.

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 *marks each and questions in Section B are worth* 20 *marks each.*
Candidates should attempt **all** *of the questions in Section A and* **two** *questions out of the four in Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorised calculators or has been approved by the chair of the Mathematics and Statistics Examination Sub-board.

Please turn over

Section A

1. Evaluate the following sums.

   (a) $\sum_{r=0}^{20} \frac{20!}{r!(20-r)!} 70^r 30^{20-r}(-1)^{20-r}$. [2]

   (b) $\sum_{r=1}^{20} (r + (r+2)^2)$. [3]

2. Seven cue balls are being shot towards six pockets numbered $1, 2, \ldots, 6$ on a billiards table. Find the number of different ways the seven balls can fall into the pockets in the following scenarios.

   (a) The balls are all coloured red and at most one ball can fit in each pocket. [1]

   (b) The balls are all coloured red and there is no limit to the number of balls in each pocket. [2]

   (c) The balls are numbered and there is no limit to the number of balls in a pocket. [2]

3. (a) Let A_3 and A_7 be the set of integers between 1 and 200 that are divisible by 3 and 7, respectively. Find $|A_3|$ and $|A_7|$. [2]

   (b) Use part (a) and inclusion-exclusion to find the number of integers n between 1 and 200 that are coprime to 21 (i.e. $\gcd(n, 21) = 1$). [3]

4. Consider the second order difference equation

$$u_n = au_{n-1} + bu_{n-2}.$$

   (a) Show that if the characteristic polynomial of the difference equation has a repeated root t, then $a = 2t$ and $b = -t^2$. [2]

   (b) Show that if the characteristic polynomial of the difference equation has a repeated root t, then $u_n = At^n$ and $u_n = Ant^n$ (where A is a constant) are both solutions to the difference equation. [3]

Please turn over

5. (a) State the Handshaking Lemma. [2]

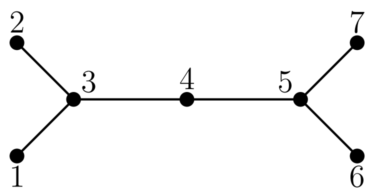
   (b) A graph G has the degree sequence $[1, 1, 2, 2, 3, 3]$. How many edges does G have? [1]

   (c) Draw two non-isomorphic simple graphs with the degree sequence $[1, 1, 2, 2, 3, 3]$. [2]

6. (a) State Cayley's formula for labelled trees. [1]

   (b) Determine whether the sequence $[6, 6, 6, 6]$ is a Prüfer sequence. Justify your answer. [1]

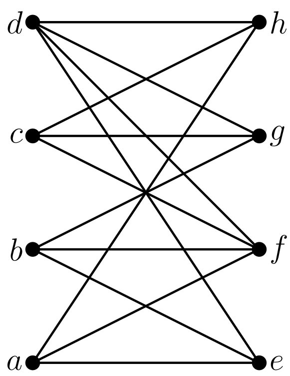
   (c) Find the Prüfer sequence corresponding to the following labelled tree:



[3]

7. (a) State Kuratowski's Theorem. [2]

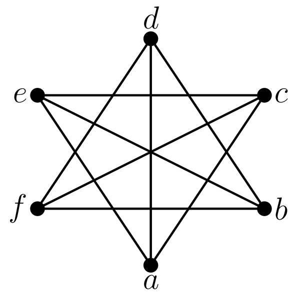
   (b) Determine whether the following graph is planar, giving full justification for your answer:



[3]

Please turn over

8. Let H be the following graph:



(a) State Dirac's theorem. [2]

(b) Determine whether H is Hamiltonian, giving full justification for your answer. [1]

(c) Determine whether H is Eulerian, giving full justification for your answer. [2]

Please turn over

Section B

9. (a) An owner of a clothing shop is restocking his scarf shelves. He stocks scarves of colours black, red, white and green, and needs to buy 40 scarves in total.

   The reds are super popular, so he wants at least 8 of those. The sales of whites and greens are a little unpredictable, so he wants at most 5 of each of those (he could take zero of each). Black is rather predictable in sales, so he is confident he needs between seven and twelve of those. We want to find the different ways the owner can stock his shelves. We solve this using the following steps.

   (i) Express the problem as an integer equation. [2]

   (ii) Find the generating series corresponding to the integer equation. [2]

   (iii) Use the generating series to solve the counting problem. [6]

   (b) Set $g(x) = \sum_{r=0}^{\infty} r^2 x^r$.

   (i) By applying $x\frac{d}{dx}$ to $\frac{1}{1-x}$, show that $g(x)$ can be expressed as $\frac{x(1+x)}{(1-x)^3}$. [3]

   (ii) Use your answer to part (i) to show that

$$\sum_{r=0}^{n} r^2 = \frac{(2r+1)(r+1)r}{6}.$$

[4]

   (c) Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be the generating series for Catalan numbers. It can be shown that $C(x)$ satisfies

$$C(x) = xC(x)^2 + 1.$$

   By taking the appropriate coefficient on both sides of the equation, show that the Catalan numbers satisfy the recurrence $c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$, with $c_0 = 1$. [3]

Please turn over

10. (a) It can be shown that the Catalan numbers c_n satisfy the recurrence relation

$$c_n = \frac{4n-2}{n+1}c_{n-1}, \text{ with } c_0 = 1.$$

Solve this difference equation to show that

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

(Hint: You may find the computation

$$(2n)! = \prod_{i=1}^{n}(2i) \cdot \prod_{i=1}^{n}(2i-1).$$

useful). [5]

(b) Solve the following difference equations.

(i) $u_n = 5u_{n-1} - 6u_{n-2} + 2 \cdot 3^n - 4^n$, with $u_0 = u_1 = 1$. [5]

(ii) $u_n = 2u_{n-1} - u_{n-2} + 1$, with $u_0 = u_1 = 1$. [5]

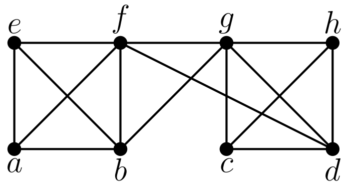
(c) Consider the first order difference equation $u_n = au_{n-1} + b$, with $u_0 = U$. Show using generating functions that the solution to this difference equation is

$$u_n = a^n U + b\frac{a^n - 1}{a - 1}.$$

[5]

Please turn over

11. (a) Let J be the following graph:



(i) State the definition of the connectivity of a graph. [2]

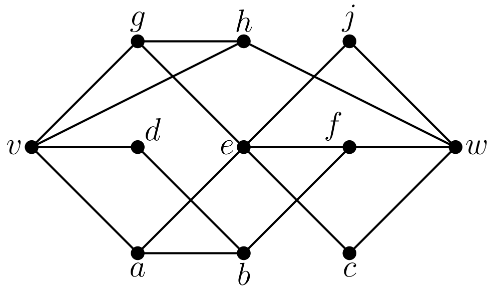
(ii) Find a cutset in J of size 3. [1]

(iii) Find a cutset in J of size 4. [1]

(iv) What is the edge connectivity of J? [1]

(b) (i) State the edge form of Menger's theorem. [2]

(ii) Find a set of edge-disjoint vw-paths that is as large as possible in the following graph and demonstrate that it is not possible to find a larger set of edge-disjoint vw-paths:



[3]

(c) (i) State the Kőnig-Egerváry Theorem. [2]

(ii) Determine the size of the largest independent set of zero entries in the following matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

[2]

(iii) The following table lists the prices that six contractors C_1, C_2, C_3, C_4, C_5 and C_6 charge to perform six tasks T_1, T_2, T_3, T_4, T_5 and T_6. Find an assignment of contractors to tasks that minimises the total cost.

	T_1	T_2	T_3	T_4	T_5	T_6
C_1	7	6	5	4	3	6
C_2	4	6	6	6	8	5
C_3	7	7	6	7	9	6
C_4	5	8	5	8	5	6
C_5	8	7	4	6	4	8
C_6	6	6	5	4	7	5

[6]

Please turn over

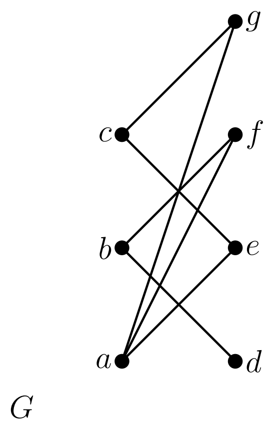
12. (a) A pairing of a finite set S is a partition of the elements of S into disjoint pairs of elements. Prove that if S has size $2n$ then the number of possible pairings of S is given by the following expression:

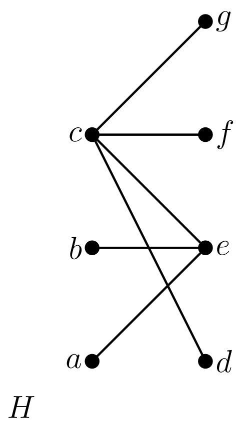
$$\frac{(2n)!}{(2!)^n n!}.$$

[4]

(b) (i) State the definition of a complete matching of a bipartite graph. [2]

(ii) For each of the two following graphs G and H, determine (with justification) whether they possess a complete matching:

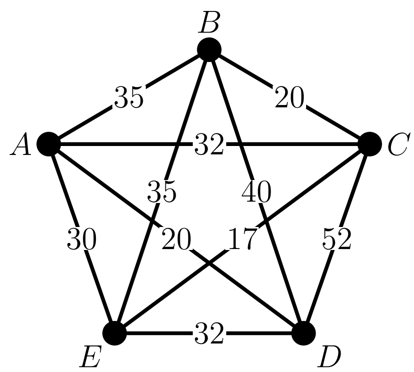




[4]

(c) (i) State the Travelling Salesman Problem. [2]

(ii) Use Kruskal's algorithm to find a minimum spanning tree in the following graph:



[4]

Please turn over

(iii) Hence determine a lower bound for the value of the solution to the Travelling Salesman Problem in the case where the salesman is required to visit cities A, B, C, D, E and F with the distances between cities given in the following table:

	B	C	D	E	F
A	35	32	20	30	41
B	–	20	40	35	27
C	–	–	52	17	34
D	–	–	–	32	29
E	–	–	–	–	27

[2]

(iv) Use the Upper Bound Algorithm to find an upper bound for the value of the solution to the instance of the Travel Salesman Problem given in (iii). [2]