The Diophantine Equation

$$y(y+1) = x(x+1)(x+2) \tag{1}$$

In this section we use the arithmetic of the cubic field

$$K = \mathbb{Q}(\theta), \theta - 4\theta + 2 = 0, \tag{2}$$

if we set : $X = 2x+2$, $Y = 2y+1$ we will simplify $X = 2x+2$ and $Y = 2y+2$ to find x , y after that we will substitute in equation (1)we will get:

$$2Y^2 = X^3 - 4X + 2 \tag{3}$$

Clearly any solution of equation (3) must have X even and Y odd . We will show that the only solution of equation (3) are
$(X,Y) = (-2,\pm 1), (0,\pm 1), (2,\pm 1), (4,\pm 5), (12,\pm 29)$.
We let $\theta, \theta', \theta'' \in \mathbb{C}$ be the three roots of $x^3 - 4x + 2 = 0$ so that $x^3 - 4x + 2 = (x-\theta)(x-\theta')(x-\theta'')$. We will expand the right hand side and take common factors of x and x^2. After that we will equate the coefficients of x, x^2 and for the constant term. These equalities can be written $\theta + \theta' + \theta'' = 0$ for x^2, $\theta\theta' + \theta'\theta'' + \theta''\theta = -4$ for x and $\theta\theta'\theta'' = -2$ for the constant term.
the following solution in the source ().

We will take another example from the source (An introduction to diophantine equation).

$$6x + 10y - 15z = 1 \tag{4}$$

we have $y = 1(mod 3)$, hence $y = 1 + 3s$, s$\in \mathbb{Z}$. We will substitute y in equation (4) the equation becomes $6x - 15z = -9 - 30s$, or equivalently ,

$$2x - 5z = -3 - 10s \tag{5}$$

Because $z = 1(mod 2)$, $z = 1 + 2t$, where $t \in \mathbb{Z}$,we will substitute z in equation (5) the equation becomes $x = 1 - 5s + 5t$.Hence the solution are :
$(x,y,z) = (1 - 5s + 5t, 1 + 3s, 1 + 2t)$, $s,t \in \mathbb{Z}$.

Prove that equation

$$x^3 - x^2 + 8 = y^2$$

is not solvable in integer to solve this equation for x odd , we will write the equation as
$(x + 2)(x^2 - 2x + 4) = x^2 + y^2$. It is clear that $gcd(x, y) = 1$. the greatest common divisor (gcd) of two or more integers, which are not all zero, is the largest positive integer that divides each of the integers. If $x = 4k + 1$, then $x + 2 = 4k + 3$ has a prime divisor of this form that divides $x^2 + y^2$, impossible .If $x = 4k+3$, then $x^2 - 2x+4$ is of the form $4m+3$, and by the same argument , we again get a contradiction.
For $x = 2u$, the equation becomes

$$2u^3 - u^2 + 2 = z^2.$$

If u is odd, then the left hand side is congruent to 3 (mod 4), and so it cannot be a perfect square . If u is even , then the left hand side is congruent to 2(mod4) and again cannot be a perfect square.

We will take another example from source (Algebraic number theory and fermat's last theorem).

$$x^2 + 7 = 2^n$$

to solve this equation we work in $\mathbb{Q}(\sqrt{-7})$ whose ring of integers has unique factorization.
unique factorization domain means (UFD) is an integral domain (a non-zero commutative ring in which the product of non-zero elements is non-zero) in which every non-zero non-unit element can be written as a product of prime elements (or irreducible elements), uniquely up to order and units, analogous to the fundamental theorem of arithmetic for the integers. For x is odd and we will suppose x is positive. Assume first that n is even we have factorization of integers:

$$(2^{n/2} + x)(2^{n/2} - x) = 7$$

so that $2^{n/2} + x = 7$, $2n/2 - xz = 1$,

so

$$2^{1+n/2} = 8$$

and n=4, x=3. Now let n be odd, and assume $n > 3$.
We have to use (Dedekinds Theorem) to factorization into prime

$$2 = (1 + \sqrt{-7}/2)(1 - \sqrt{-7}/2)$$

. Now let x is odd, $x = 2k+1$, so $x^2+7 = 4k^2+4k+8$ is divisible by 4.Putting $m = n - 2$, we can rewite the equation to be solved as

$$\frac{x^2 + 7}{4} = 2^m$$

so that

$$(\frac{x+\sqrt{-7}}{2})(\frac{x-\sqrt{-7}}{2}) = (\frac{1+\sqrt{-7}}{2})^m(\frac{1-\sqrt{-7}}{2})^m$$

where the right hand side is a prime factorization. Neither $(1+\sqrt{-7})/2$ nor $(1-\sqrt{-7})/2$ is a common factor of the terms on the left because such a factor would divide their difference, $\sqrt{-7}$, which is seen to be impossible by taking norms. Comparing the two factorizations, since the only units in the integers of $\mathbb{Q}(\sqrt{-7})$ are $+1$, we must have

$$\frac{x+\sqrt{-7}}{2} = +(\frac{1+\sqrt{-7}}{2})^m$$

for which we derive

$$+\sqrt{-7} = (\frac{1+\sqrt{-7}}{2})^m - (\frac{1-\sqrt{-7}}{2})^m.$$

we claim that the positive sign cannot occur. For, putting $(\frac{1+\sqrt{-7}}{2})^m = a$, $(\frac{1-\sqrt{-7}}{2}) = b$ we have

$$a^m - b^m = a - b.$$

Then $a^2 \equiv (1-b)^2 \equiv 1 \pmod{b^2}$
since ab=2, and so
$a^m \equiv a(a^2)^{\frac{m-1}{2}} \equiv a \pmod{b^2}$
where$a \equiv a - b \pmod{b^2}$, a contradiction. The only solution of the equation $x^2 + 7 = 2^n$ in integers x,n are:
x=1 3 5 11 181
n= 3 4 5 7 15
We can find the rest of solution in the source (Algebraic number theory and fermat's last theorem).