Discrete Assignment 1

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1. (a) Let $j = i - 1$

$$\begin{aligned}\sum_{i=1}^{1001} \binom{1000}{i-1} 2^i &= 2 \times \sum_{i=1}^{1001} \binom{1000}{i-1} 2^{i-1} \\ &= 2 \times \sum_{j=0}^{1000} \binom{1000}{j} 2^j\end{aligned}$$

Consider the binomial theorem when $x = 2$, $y = 1$, $n = 1000$

$$\begin{aligned}\sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} &= (x+y)^n \\ \sum_{r=0}^{1000} \binom{1000}{r} 2^r 1^{1000-r} &= (2+1)^{1000} \\ \sum_{r=0}^{1000} \binom{1000}{r} 2^r &= 3^{1000}\end{aligned}$$

Now let $r = j$

$$2 \times \sum_{j=0}^{1000} \binom{1000}{j} 2^j = 2 \times 3^{1000}$$

(b)

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{i} j &= \sum_{i=1}^{n} \frac{1}{2} i(i+1) \\
&= \frac{1}{2}\sum_{i=1}^{n}(i^2+i) \\
&= \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i \\
&= \frac{1}{12}n(n+1)(2n+1) + \frac{3}{12}n(n+1) \\
&= \frac{n(n+1)(2n+4)}{12} \\
&= \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}
\end{aligned}$$

2. (a) Assuming the order in which the films are played matters and the audience doesn't watch the same film twice in the same evening, this is an r-permutation problem. With a total of 33 films, the number of 3-permutations is

   i. ${}^{33}P_3 = 32736$.

   ii. Since there are 10 comedy films there are 10 ways of choosing the first film. Following this there are 32 remaining films which can be shown in ${}^{32}P_2$ ways. Hence $10 \times {}^{33}P_2 = 10560$.

   (b) Since the order doesn't matter and players cannot be picked twice, this is an r-combination problem. There are $\binom{30}{1}$ ways of choosing a captain, and $\binom{29}{7}$ ways of choosing the remaining players hence $\binom{30}{1} \times \binom{29}{7} = 46823400$.

   (c) Similarly, the number of ways of selecting 2 pairs of players from a team of 8 is $\binom{8}{2} \times \binom{6}{2} = 420$.

3. Let $X = \{n : 1 \leq n \leq 50\}$ and $|X| = c$.

   Let P be the statement $\exists\, x, y \in X$ st. $x - y \in 2\mathbb{N}$.

   The smallest value of c for which P is true is 2 (if we agree that $0 \notin \mathbb{N}$), so P is true for all $c > 1$.

   Let S_0, S_1, S_2 be sets. By the inclusion-exclusion theorem, the cardinality of their union $|S_0 \cup S_1 \cup S_2|$ is given by

$$\left|\bigcup_{i=0}^{2} S_i\right| = \sum_{i=0}^{2} |S_i| - \sum_{i,j=0}^{2} |S_i \cap S_j| + |S_0 \cap S_1 \cap S_2| \qquad (*)$$

   Now to address the problem. Let A, B and C be sets containing the numbers that Alice, Bob and Charlie (respectively) chose. We are told that $|A|, |B|, |C| = 15$, $|A \cap B| = 8$, $|A \cap C| = 6$, $|B \cap C| = 7$ and that $|A \cup B \cup C| = 29$. Now let $A = S_0$, $B = S_1$ and $B = S_2$ and substitute their values into $(*)$, giving

$$\begin{aligned} 29 &= (15 + 15 + 15) - (6 + 7 + 8) + |A \cap B \cap C| \\ |A \cap B \cap C| &= 5 \end{aligned}$$

Now let $A \cap B \cap C = X$, then $|X| = 5$ and $c = 5$. Since $c > 1$, P is true.

4. The generating function for the number of integer solutions of

$$X_1 + X_2 + X_3 + X_4 = r$$

where

$$\begin{aligned} X_1 &\leq 3 \\ X_2 &\leq 3 \\ X_3 &\leq 5 \\ 5 \leq X_4 &\leq 3 \end{aligned}$$

is

$$(1 + x + x^2 + x^3)^2(1 + x + \cdots + x^5)(x^5 + \cdots + x^{15})$$

Applying identities, factorising

$$\Big((1 - x^4)(1 + x + x^2 + \cdots)\Big)^2(1 - x^6)(1 + x + x^2 + \cdots)x^5(1 + x + \cdots + x^{10})$$

Expanding, collecting, applying identities

$$x^5(1 - x^4)^2(1 - x^6)(1 - x^{11})(1 + x + x^2 + \cdots)^4$$

Further application of identities

$$x^5\left(\sum_{i=0}^{2}(-1)^i\binom{2}{i}x^{4i}\right)(1 - x^6)(1 - x^{11})\left(\sum_{i=0}^{\infty}\binom{3+i}{i}x^i\right)$$

Expanding, distributing

$$(1 - 2x^4 + x^8)(x^{22} - x^{16} - x^{11} + x^5)\left(\sum_{i=0}^{\infty}\binom{3+i}{i}x^i\right)$$

To find the coefficient of x^{15}, we take all "paths" through the function where the sum of indices is 15 and sum them

$$\begin{array}{lcrcr}
\Big(1 & \times & -x^{11} & \times & \binom{7}{4}x^4\Big)+ \\
\Big(1 & \times & x^5 & \times & \binom{13}{10}x^{10}\Big)+ \\
\Big(-2x^4 & \times & -x^{11} & \times & \binom{3}{0}x^0\Big)+ \\
\Big(-2x^4 & \times & x^5 & \times & \binom{9}{6}x^6\Big)+ \\
\Big(x^8 & \times & x^5 & \times & \binom{5}{2}x^2\Big)
\end{array}$$

taking coefficients only

$$\begin{array}{lcrcr}
(1 & \times & -1 & \times & 35)+ \\
(1 & \times & 1 & \times & 286)+ \\
(-2 & \times & -1 & \times & 1)+ \\
(-2 & \times & 1 & \times & 84)+ \\
(1 & \times & 1 & \times & 10)
\end{array}$$

gives $-35 + 286 + 2 - 168 + 10 = 95$.