Calculus 2 - Assignment 2

Due 17^{th} January 2017

ANSWER ALL QUESTIONS

1. Let P be the polygon in the (x, y)-plane with vertices $(0,0),\ (1,2),\ (3,3)$ & $(2,1)$. We wish to evaluate the double intergral:

$$\iint_P e^x\, dydx,$$

over P and its interior. Suppose we make the change of variables: $x = \frac{2u+v}{3}$ and $y = \frac{u+2v}{3}$.

   (a) Describe geometrically the region P' in the (u, v)-plane to which P is mapped by this change of variables and find the limits of the double integral in terms of u and v. [2]

   (b) Compute the Jacobian for this change of variables [1]

   (c) Make the substitution and evaluate the integral. [2]

2. Recall that for real numbers $a > b > 0$, the equation of an ellipse, E, with semi-major axis a and semi-minor axis b is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

Consider now a rectangle, R, with sides aligned with the x- and y-axes and which is inscribed in the ellipse. Let $x > 0$ and $y > 0$ be the co-ordinates of the top, right-hand vertex of R.

   (a) Write down the co-ordinates of the other vertices of R and hence define the function $f(x, y)$ which gives the area of R in terms of x and y. [1]

   (b) Using the method of Lagrange multipliers, find the values of x and y that define the rectangle subscribed in E of greatest area. That is, maximise your function f subject to the constraint given in (1). [3]

   (c) Recall from lectures that we can find the area of E by evaluating the double integral $\iint_E 1\ dxdy$, where the double integral is evaluated over E and its interior. We will use this method to find the area of the ellipse.

      i. Show that the change of variables $u = \frac{x}{a},\ v = \frac{y}{b}$ transforms E into the unit disc. [1]

      ii. By changing to polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$ show that the area of $E = ab\pi$. [2]

      iii. Hence verify that the fraction of E occupied by the area of any inscribed rectangle is at most $\frac{2}{\pi}$. [1]

3. Let $f(x,y)$ and $g(x,y)$ be functions of two variables and suppose they both have stationary points at $x=a$ and $y=b$.

   (a) Show that the function $h(x,y)=f(x,y)+g(x,y)$ also has a stationary point at $x=a$ and $y=b$. [1]

   (b) Decide if the statements below are true or false. If true then give a brief argument to support your answer. If false then give a counterexample.

      i. If $f(a,b)$ and $g(a,b)$ are both local minima, then $h(a,b)$ is also a local minimum. [1]

      ii. If $f(a,b)$ is a minimum but $g(a,b)$ is a saddle point then $h(a,b)$ is a saddle point. [1]

4. Recall that a function of x can be defined *explicitly* via a formula $y=f(x)$ or *implicitly* via a relation $F(x,y)=0$. For example the ellipse from question (2) could be defined using either of these forms:

$$y=b\sqrt{1-\left(\frac{x}{a}\right)^2}. \tag{2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{3}$$

   Let $w=F(x,y)=0$ and (remembering that y is a function of x), follow these steps to develop a new technique for implicit differentiation:

   (a) Use the chain rule for partial derivatives to find an expression for $\frac{dw}{dx}$. [1]

   (b) Explain why $\frac{dw}{dx}$ must always equal zero and use this fact to rearrange your answer to part (a) to find $\frac{dy}{dx}$ in terms of F_x and F_y. [1]

   (c) Use your answer to part (b) to find $\frac{dy}{dx}$ for the ellipse defined implicitly in (3). [1]

   (d) Finally, differentiate equation (2) directly and verify that you get the same answer as you did in part (c). [1]