BIRKBECK COLLEGE
(University of London)

BSc Examination
School of Business, Economics & Informatics

Calculus 2: Multivariable & Differential Equations
BUEM001S5

Friday 1 June 2012

1430-1730

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 marks each and questions in Section B are worth 20 marks each.
Candidates should attempt **all** *of the questions in Section A and* **two** *questions from Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorized calculators or has been approved by the chair of the Mathematics & Statistics Examination Sub-board.

Please turn over

Section A

1. (a) Using only the definition of a derivative, find the derivative of $f(x) = x^2 + 3x$. [2]

   (b) Let $f(x, y) = x^2 + 3xy + x^3y^2$. Find the equation of the tangent plane to $z = f(x, y)$ at $(-1, 2)$. Write your answer in the form $z = ax + by + c$, for constants a, b and c. [3]

2. Let D be the set of points (x, y) in $\mathbb{R}^2$ such that $x, y \geq 0$ and $1 \leq x^2 + y^2 \leq 2$. Sketch the region D and evaluate

$$\iint_D x^2 \, \mathrm{d}x\mathrm{d}y.$$ [5]

3. Find the extreme values of $f(x, y) = x + y$ subject to $x^2 + y^2 = 1$ using Lagrange multipliers. Are your answers local extrema or global? Justify your answer. [5]

4. Consider the following differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5x \frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 0.$$

   (a) By substituting $x = e^t$ and using the chain rule, show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x \frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = x \frac{\mathrm{d}y}{\mathrm{d}x} + x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$ [3]

   (b) Show that the above variable substitution turns the original differential equation into one with constant coefficients. You do not have to solve the equation. [2]

Please turn over

5. Solve
$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 5x,$$
where $y(1) = 1$. [5]

6. (a) Use the method of Taylor series to solve the differential equation $y' = x + y$, where $y(1) = 1$. Find at least four terms of the series solution. [2]

   (b) Recall that the higher derivative Euler method is a generalisation of Euler's method. Use the higher derivative Euler method with 3 terms (i.e. one more than the regular Euler method) to estimate $y(1.2)$ for y in part a). Use a step size of $h = 0.1$. [3]

7. A box with mass $m = 1kg$ is attached horizontally to a spring with restoring force $F_{\text{spring}} = -4x$ and is subject to a drag force $F_{\text{drag}} = -4v$. Here, x is the horizontal position of the box from equilibrium measured in metres, and v is the velocity measured in metres per second. The mass has an initial position of 1 metre and its initial velocity is 0 metres per second.

   (a) Using Newton's laws, find the differential equation governing the motion of the system. Clearly state the initial conditions. [1]

   (b) Solve the differential equation and classify the system as over damped, under damped or critically damped. [3]

   (c) Does the mass pass through the equilibrium point? Explain your answer. [1]

8. (a) Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(2x) = 2\cosh^2(x) + 2\sinh^2(x).$$
[2]

   (b) Recall that $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ for $x > 0$. Show that if $x \geq 1$ that $\Gamma(x) = (x-1)\Gamma(x-1)$ by using integration by parts. [3]

Please turn over

Section B

9. (a) Consider $z = x^3 + y^3 - 3xy^2 + 5 + \frac{3}{2}y^2$.

   (i) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. [1]

   (ii) Find the stationary points of z. [5]

   (iii) Find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y^2}$. [2]

   (iv) Determine the nature of the stationary points of z. If the Hessian gives you no information about a stationary point, you do not have to investigate that stationary point further. [4]

   (b) Let $g(x, y) = f(u)$ where $u = 2x^3 + 3y^2$. Show that
   $$y\frac{\partial g}{\partial x} = x^2\frac{\partial g}{\partial y}.$$ [3]

   (c) Find the degree 2 Taylor approximation of $f(x, y) = ye^{xy}$ about the point $(1, 1)$ (recall, degree 2 Taylor approximations involve terms up to and including quadratic terms). Use your answer to estimate $f(1.1, 0.9)$. You may leave exponentials in your answer (i.e. you do not need to convert e to a numerical value). [5]

Please turn over

10. (a) Use L'Hôpital's rule to find

$$\lim_{h\to 0} \frac{e^h - \cosh h}{h}.$$

[3]

(b) Consider the integral

$$\iint_D 5x^2y + 2 \, \mathrm{d}x\mathrm{d}y,$$

where D is the region bounded by $y = |x|$ and $y = 3$. Sketch D and evaluate the integral. [6]

(c) Evaluate the integral

$$\iint_R y^3 \, \mathrm{d}x\mathrm{d}y,$$

where R is the region bounded $xy = 1, xy = 3, y = x^2$ and $y = 2x^2$. [7]

(d) Solve

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + y + e^x = 0,$$

where $y(0) = 1$. [4]

Please turn over

11. (a) A mass of m kilograms is launched from Earth. Since the distance from the mass to the earth is large, the force of gravity is not constant and is given by $F_{\text{grav}}(y) = -\frac{mgR^2}{(R+y)^2}$, where y is the height of the rocket from the earth's surface and R is the radius of the earth (both measured in metres), and g is the usual gravitational constant.

(i) Use Newton's laws to explain why the equation governing the motion of the rocket is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{mgR^2}{(R+y)^2}.$$ [1]

(ii) Using the chain rule on $\frac{\mathrm{d}v}{\mathrm{d}t}$, show that the above equation becomes

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = -\frac{gR^2}{(R+y)^2}.$$ [2]

(iii) Find the general solution to the equation in part ii). [3]

(b) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x+y-2}{x+y+1}.$$

(i) Find the intersection of the two lines $4x+y=2$ and $x+y=-1$. [1]

(ii) By making an appropriate substitution of the form $X = x-\alpha$ and $Y = y-\beta$ for constants α and β, and noting that $\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}y}{\mathrm{d}x}$, show that the above equation becomes

$$(X+Y)\frac{\mathrm{d}Y}{\mathrm{d}X} - 4X - Y = 0.$$ [3]

(iii) Show that $M(X,Y) = X+Y$ and $N(X,Y) = -4X-Y$ are both homogeneous of degree k for some k. [2]

(iv) By making an appropriate substitution, or otherwise, find the general solution of the equation in part ii). [6]

(v) Using part iv) or otherwise, find the general solution to the original differential equation. Express your final answer in a form that contains no logs or exponentials. [2]

Please turn over

12. (a) (i) Show that

$$\int_{\sqrt{3}}^{3} \frac{1}{\sqrt{3+x^2}} \mathrm{d}x = \ln\left(\frac{\sqrt{3}+2}{1+\sqrt{2}}\right)$$

[5]

(ii) Find all complex numbers z such that $\cos z = 2$. [5]

(b) Recall that $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

(i) By making an appropriate substitution, show that

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}(\theta)\cos^{2y-1}(\theta)\,\mathrm{d}\theta.$$

[4]

(ii) Evaluate

$$\int_0^{\pi} \cos^6(\theta)\,\mathrm{d}\theta.$$

[6]