BIRKBECK
(University of London)

BSc EXAMINATION
SCHOOL OF BUSINESS, ECONOMICS AND INFORMATICS

Discrete Mathematics
BUEM002S5

30 credits

Wednesday 4 June 2014
Morning, 10:00 a.m. - 1:00 p.m.

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 *marks each and questions in Section B are worth* 20 *marks each.*
Candidates should attempt **all** *of the questions in Section A and* **two** *questions out of the four in Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorised calculators or has been approved by the chair of the Mathematics and Statistics Examination Sub-board.

Please turn over

Section A

1. Evaluate the following sums:

   (a) $\sum_{r=0}^{3000} \frac{3000!}{r!(3000-r)!}$ [2]

   (b) $\sum_{r=0}^{3000} 2^r + r^2$ [3]

2. A toy piano has eight white keys and five black keys. A cat jumps on the piano and plays five notes. Assuming that the notes are played one at a time, how many different patterns of notes can the cat play if:

   (a) there are no further restrictions on the notes the cat plays? [1]

   (b) the cat does not repeat any notes? [1]

   (c) the cat plays at least three black notes? [3]

3. An administrator is ordering fifty new whiteboard markers for his department. The markers are available in black, blue, green, or red. How many different selections of markers can he order if:

   (a) there are no further restrictions? [1]

   (b) the department requires at least twenty black markers and at least fifteen blue markers? [2]

   (c) the department wants to order at most ten red markers? [2]

4. A sequence $(u_i)_{i=0}^{\infty}$ is described by the difference equation

$$u_n = -\frac{2}{3}u_{n-1} + 2$$

   for $n \geq 1$, with initial condition $u_0 = 20$.

   (a) Solve this difference equation to find an explicit expression for the n^{th} term of the sequence. [3]

   (b) Describe the behaviour of the sequence as $n \to \infty$. [2]

Please turn over

5. The formula for the n^{th} Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

   (a) Show that C_n can be written in the form $\binom{2n}{n} - \binom{2n}{n-1}$. [2]

   (b) Show that

$$C_n = \frac{(2 \times 1)(2 \times 3)(2 \times 5)\cdots(2 \times (2n-1))}{(n+1)!}.$$

[3]

6. (a) State the Handshaking Lemma. [2]

   (b) Use the Handshaking Lemma to show that there is no graph with the degree sequence $[1, 2, 3, 4, 5]$. [2]

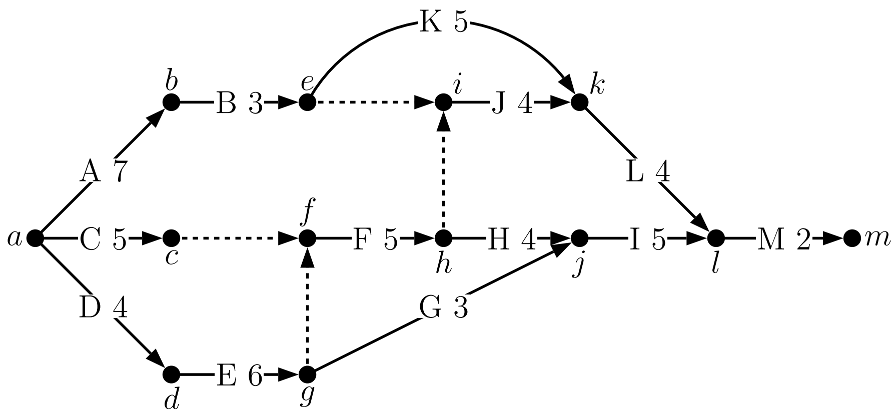
   (c) How many edges are there in the graph G described by the following adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}?$$

[1]

Please turn over

7. The following activity network depicts the tasks A, B, C, D, E, F, G, H, I, J, K, L, M required to complete a project, together with the time required to complete each task, as well as indicating the relationships between tasks that must completed before other tasks can begin:

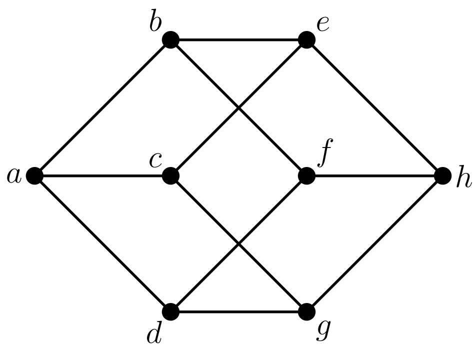


(a) Use the Longest Path Algorithm to find a critical path for this activity network. [3]

(b) Find the earliest and latest possible start time for task J if the project is to be completed as quickly as possible. [2]

8. (a) State the edge form of Menger's theorem. [2]

(b) Use Menger's theorem to show that the smallest ah-disconnecting set in the following graph has size three.



[3]

Please turn over

Section B

9. (a) A shop owner is purchasing cans of soft drink to fill his vending machine. He wishes to purchase at least ten and at most fifteen cans of cola flavour, and at least five but at most ten cans each of orange fizz flavour, lemonade flavour, and sarsaparilla flavour (these are the only available flavours). Thirty cans in total are required to fill the vending machine. How many different choices of flavours satisfy the shop owner's requirements?

   (i) Express this problem as an integer equation. [2]

   (ii) Write the generating function corresponding to this integer equation. [2]

   (iii) Use the generating function to find the solution to this counting problem. [6]

   (b) (i) A sequence $(u_r)_{r=0}^{\infty}$ has a generating function of the form $\frac{p(x)}{q(x)}$ where p and q are polynomials. State necessary and sufficient conditions on p and q for this sequence to satisfy a homogeneous linear difference equation of order k. [2]

   (ii) Find the first four terms of the sequence corresponding to the generating function

$$g(x) = \frac{1}{1+4x^2}.$$

[2]

   (c) Find the generating function for each of the following sequences:

   (i) $2, 2, 2, 2, 2, 2, \ldots$ [2]

   (ii) $0, 2, 4, 8, 16, 32, \ldots$ [2]

   (iii) $0, 2, 4, 6, 8, 10, 12, \ldots$ [2]

Please turn over

10. (a) Solve the following difference equations:

(i)

$$u_n = 4u_{n-1} - 3u_{n-2} - 20n + 26, \qquad u_0 = 10, \quad u_1 = 24.$$

[5]

(ii)

$$u_n = -14u_{n-1} - 49u_{n-2} + \frac{500}{9}3^n, \qquad u_0 = 7, \quad u_1 = -20.$$

[5]

(b) Use the substitution $a_n = 2^{b_n}$ to solve the following difference equation:

$$a_n = \frac{(a_{n-1})^5}{(a_{n-2})^6}, \qquad a_0 = \frac{1}{4}, \quad a_1 = 2.$$

[6]

(c) Consider the inhomogeneous linear difference equation

$$u_n = Au_{n-1} + Bu_{n-2} + f(n). \tag{1}$$

(i) Write down the homogeneous part of (1). [1]

(ii) Show that if $P(n)$ is a particular solution to (1), and $G(n)$ is a solution to the homogeneous part of (1), then $P(n) + G(n)$ is also a solution to (1). [3]

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11. (a) (i) State Hall's conditions for a bipartite graph to have a complete matching. [3]

(ii) Use the pigeonhole principle to prove that Hall's conditions are necessary. (That is, show that a bipartite graph that does not satisfy Hall's conditions cannot have a complete matching.) [2]

(b) Five men and five women rank each other in order of preference as follows:

W_1: M_1, M_2, M_3, M_4, M_5 $\quad$ M_1: W_3, W_2, W_5, W_4, W_1
W_2: M_2, M_3, M_4, M_5, M_1 $\quad$ M_2: W_4, W_1, W_3, W_5, W_2
W_3: M_1, M_3, M_5, M_4, M_2 $\quad$ M_3: W_4, W_2, W_1, W_3, W_5
W_4: M_3, M_2, M_4, M_1, M_5 $\quad$ M_4: W_3, W_4, W_5, W_1, W_2
W_5: M_4, M_5, M_2, M_1, M_3 $\quad$ M_5: W_5, W_2, W_4, W_1, W_3

(i) State the definition of a male-optimal stable matching. [2]

(ii) Find a male-optimal stable matching. [5]

(iii) Prove that any male-optimal stable matching is female pessimal. [4]

(iv) For each of the following matchings, show whether or not they are stable. [4]

A. $\{M_1, W_1\}, \{M_2, W_2\}, \{M_3, W_3\}, \{M_4, W_4\}, \{M_5, W_5\}$

B. $\{M_1, W_3\}, \{M_2, W_1\}, \{M_3, W_4\}, \{M_4, W_2\}, \{M_5, W_5\}$

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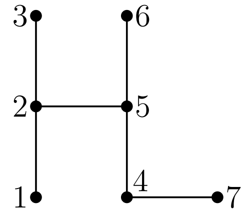
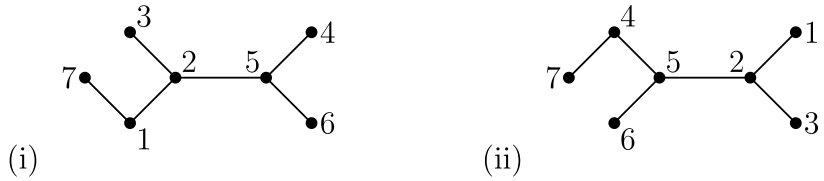
12. (a) Let G be the following labelled tree:



For each of the following labelled trees, state whether they are isomorphic to G as a labelled tree, giving evidence for your answer:



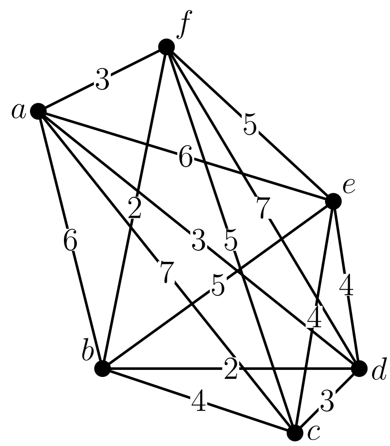
[4]

(b) Let H be the complete graph with five vertices, K_5. Any spanning subtree of H corresponds to a unique labelled tree with five vertices. How many spanning subtrees does H have? [3]

(c) (i) State the travelling salesman problem for a weighted complete graph on n vertices. [2]

(ii) Suppose you possess a machine that outputs a solution to any given instance of the travelling salesman problem. Describe how this machine can be used to determine whether a graph on n vertices possesses a Hamiltonian cycle. [3]

(d) Let B be the following weighted graph:



(i) Use Kruskal's algorithm to find a minimum spanning tree for B. [3]

(ii) Starting with vertex d, use an algorithm from the course to find an upper bound for the length of a solution to the travelling salesman problem for B. [2]

(iii) Starting with vertex d, use an algorithm from the course to find a lower bound for the length of a solution to the travelling salesman problem for B. [3]