Discrete Mathematics Assignment 4 Solutions

1. (a) 3 [1]

   (b) 2 [1]

   (c) The set $\{ec, fc, fd\}$ is an ea-disconnecting set. The set $\{eca, efcda, eghfdba\}$ contains three edge-disjoint ea-paths, hence by Menger's theorem we conclude that the smallest ea-disconnecting set has size 3. [2]

   (d) The set $\{c, f\}$ is a be-separating set of size 2. The set $\{bce, bdfe\}$ contains two vertex-disjoint be-paths, hence by Menger's theorem the smallest be-separating set has size 2. [2]

2. (a)

a	b	c	d	e	f	g	h	j	z
0									
	9	5	6	**3**					
	9	5	**4**					7	
	9	**5**				13	5	7	
	9				13	12	**5**	7	
	9				13	8		**7**	9
	9				13	**8**			9
	9				13				**9**

   The shortest path is a, e, d, h, z, and its total length is 9. [2]

   (b) $P(a) = 0$
   $P(e) = 3$
   $P(d) = 6$
   $P(c) = 11$
   $P(b) = 15$
   $P(j) = 10$
   $P(h) = 13$
   $P(g) = 18$
   $P(f) = 23$
   $P(z) = 28$
   The longest path is a, d, c, g, f, z, and its total length is 26. [2]

3. Round 1:
   $E_1 \to C_5$
   $E_2 \to C_5$
   $E_3 \to C_2$
   $E_4 \to C_3$
   $E_5 \to C_3$
   $(E_1, C_5), (E_3, C_2), (E_4, C_3)$
   Round 2:
   $E_2 \to C_4$
   $E_5 \to C_1$

$(E_1, C_5), (E_3, C_2), (E_4, C_3), (E_2, C_4), (E_5, C_1)$
The required matching is $(E_1, C_5), (E_3, C_2), (E_4, C_3), (E_2, C_4), (E_5, C_1)$. [4]

4. (a) Since we wish to maximise the total suitability, we begin by replacing each entry of the matrix with nine minus that entry.

$$\begin{bmatrix} 3 & 1 & 7 & 5 & 9 & 5 \\ 2 & 0 & 8 & 6 & 4 & 6 \\ 9 & 9 & 3 & 4 & 5 & 7 \\ 9 & 7 & 5 & 4 & 3 & 5 \\ 2 & 4 & 4 & 3 & 3 & 3 \\ 5 & 5 & 5 & 2 & 4 & 3 \end{bmatrix}$$

Preprocess the rows:

$$\begin{bmatrix} 2 & 0 & 6 & 4 & 8 & 4 \\ 2 & 0 & 8 & 6 & 4 & 6 \\ 6 & 6 & 0 & 1 & 2 & 4 \\ 6 & 4 & 2 & 1 & 0 & 2 \\ 0 & 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 0 & 2 & 1 \end{bmatrix}$$

Preprocess the columnns:

$$\begin{bmatrix} 2 & 0 & 6 & 4 & 8 & 3 \\ 2 & 0 & 8 & 6 & 4 & 5 \\ 6 & 6 & 0 & 1 & 2 & 3 \\ 6 & 4 & 2 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 3 & 3 & 3 & 0 & 2 & 0 \end{bmatrix}$$

Crossing out columns 2 and 3, and rows 4,5,6 and applying the Hungarian algorithm yields the following matrix:

$$\begin{bmatrix} 1 & 0 & 6 & 3 & 7 & 2 \\ 1 & 0 & 8 & 5 & 3 & 4 \\ 5 & 6 & 0 & 0 & 1 & 2 \\ 5 & 5 & 3 & 0 & 0 & 1 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 3 & 4 & 4 & 0 & 2 & 0 \end{bmatrix}$$

Crossing out column 2 and rows 3,4,5,6 and applying the Hungarian algorithm yields the following matrix:

$$\begin{bmatrix} \mathbf{0} & 0 & 5 & 2 & 6 & 1 \\ 0 & \mathbf{0} & 7 & 4 & 2 & 3 \\ 5 & 7 & \mathbf{0} & 0 & 1 & 2 \\ 5 & 5 & 3 & 1 & \mathbf{0} & 1 \\ 0 & 4 & 3 & 1 & 1 & \mathbf{0} \\ 3 & 5 & 4 & \mathbf{0} & 2 & 0 \end{bmatrix}$$

The highlighted entries correspond to the assignment Alice-hosing, Barbara-scrubbing, Curt-window washing, Danielle-polishing, Elmer-waxing, Fred-vacuuming, with a total suitability of 40. [4]

(b) The assignment given as answer to 4(a) has smallest suitability value of 6. To test whether this is best possible we create a 0-1 matrix whose entry is 0 whenever the entry in the given table is greater than six:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

As this array has no independent set of zero entries of size six, we conclude that the assignment given as answer to 4(a) is best possible. [2]