# Calculus 3 Assignment 2

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1. Let $\frac{\mathrm{d}x}{\mathrm{d}t} = y$, now

$$\begin{aligned}(t^2+1)\frac{\mathrm{d}y}{\mathrm{d}t} - xty + x^2 &= x\cos t \\ (t^2+1)\frac{\mathrm{d}y}{\mathrm{d}t} &= x(\cos t + ty - x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x\left(\frac{\cos t + ty - x}{t^2+1}\right)\end{aligned}$$

let $z = t$. Now we can write the original differential equation as a system of first order autonomous differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x\left(\frac{\cos z + zy - x}{z^2+1}\right) \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 1.\end{aligned}$$

2. $\dot{x} = \alpha x^2 - x^4 = x^2(\alpha - x^2)$. The fixed points of this system are $x = 0$ and $x = \pm\sqrt{\alpha}$. To examine the behaviour of the sytem around these fixed points, we introduce an "amount of change" variable $\sqrt{\delta} \in \mathbb{R}$.

   In the case that $\alpha > 0$, when $x < -\sqrt{\alpha}$ we let $\sqrt{\delta} > 1$ and suppose that $x = \sqrt{\delta}(-\sqrt{\alpha}) = -\sqrt{\delta\alpha}$, now $x^2 = \delta\alpha > \alpha$, and the derivative of $x$, $\dot{x} = \delta\alpha(\alpha - \delta\alpha) < 0$. In the case $-\sqrt{\alpha} < x < 0$, let $0 < \sqrt{\delta} < 1$ and suppose $x = -\sqrt{\delta\alpha}$, now $x^2 = \delta\alpha < \alpha$ and $\dot{x} = \delta\alpha(\alpha - \delta\alpha) > 0$. In the case $0 < x < \sqrt{\alpha}$, let $0 < \sqrt{\delta} < 1$ and suppose $x = \sqrt{\delta\alpha}$, now $x^2 = \delta\alpha < \alpha$ and $\dot{x} = \delta\alpha(\alpha - \delta\alpha) > 0$. In the case $x > \sqrt{\alpha}$, let $\sqrt{\delta} > 1$ and suppose $x = \sqrt{\delta\alpha}$, now $x^2 = \delta\alpha > \alpha$ and $\dot{x} = \delta\alpha(\alpha - \delta\alpha) < 0$. We now have the necessary information to write our phase portrait for the system when $\alpha > 0$,

   It is clear that the fixed point $x = -\sqrt{\alpha}$ is unstable, $x = 0$ is semistable and $x = \sqrt{\alpha}$ is stable.

   In the case that $\alpha < 0$, since this dynamical system is one-dimensional, we know that $x \notin \mathbb{C}$ so the only fixed point is $x = 0$ and our dynamical system can be written $\dot{x} = -x^4$, but our phase portrait still provides a satisfactory illustration when the interval between $-\sqrt{\alpha}$ and $\sqrt{\alpha}$ is exactly zero. The fixed point $x = 0$ is semistable.

3. (a) $A = \begin{pmatrix} 3 & 0 \\ \beta & 3 \end{pmatrix}$, $\det(A - \lambda I) = (3 - \lambda)^2 = 0$, $\lambda = 3$ with algebraic multiplicity 2.

   (b) i. $\beta = 0$:

   $$(A - \lambda I)\mathbf{x} = \mathbf{0}$$
   $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   Any value of $x$ and $y$ will satisfy this equation, so the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Because $A$ is a $2 \times 2$ matrix and there are 2 independent eigenvectors, $A$ is not defective when $\beta = 0$.

ii. $\beta \neq 0$:

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$
$$\begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ \beta x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To satisfy this equation $x$ must be equal to zero, so the only eigenvector of this $A$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Becaues there is only one independent eigenvector, $A$ does not have a complete base of eigenvectors and is defective when $\beta \neq 0$.

(c) When $\beta = 0$ the fixed point $(0, 0)$ can be classified as an unstable star, and the phase portrait can be written

where eigenvectors are the axes of the graph.

When $\beta = 3$ the fixed point $(0, 0)$ can be classified as an unstable improper node and the phase portrait can be written

where the single eigenvector is the $y$-axis of the graph.

(d) Because the system has equal eigenvalues, $\lambda_1 = \lambda_2 = \lambda$, the general form

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 \\ &= e^{\lambda t}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2).\end{aligned}$$

Substituting values for the initial conditions of $\mathbf{x}(t)$ when $t = 0$, $\lambda = 3$ and eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ we can write

$$\begin{aligned}\begin{pmatrix} 2 \\ -1 \end{pmatrix} &= e^{3\cdot 0}\left(c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\end{aligned}$$

therefore $c_1 = 2$, $c_2 = -1$ and $\mathbf{x}(t) = e^{3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

4. The dynamical system in the question can be written

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -4\sin x - 3y.\end{aligned}$$

The fixed points of this system are those that satisfy $\dot{x} = \dot{y} = 0$. We can write $y = 0$ and see that $-4\sin x = 0$. The latter is true when $x$ takes values in $\{n\pi : n \in \mathbb{Z}\}$. Therefore the fixed points of this system are $\{(n\pi, 0) : n \in \mathbb{Z}\}$. Computing the Jacobian of the system gives

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4\cos x & -3 \end{pmatrix}.$$

Given that values of $x$ are $n\pi$, we can see that there are two possibilities for the bottom left entry $-4\cos x$, namely, 4 when $n$ is odd and $-4$ when $n$ is even.

Evaluating $J$ at the fixed point $(n\pi, 0)$ when $n$ is odd, gives the Jacobian matrix $J_\pi = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$.

That

$$\det(J_\pi - \lambda I) = -\lambda(-3 - \lambda) - 4 = (\lambda + 4)(\lambda - 1)$$

tells us that the eigenvalues for $J_\pi$ are $\lambda = -4$ and $\lambda = 1$. Since these eigenvalues both have a nonzero real part, we know that the fixed points $\{(n\pi, 0) : n \text{ is odd}\}$ are hyperbolic and we can apply Hartman-Grobman.

For $\lambda = -4$,

$$
\begin{aligned}
(J_\pi - \lambda I)\mathbf{x} &= \mathbf{0} \\
\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

and the eigenvectors associated with $\lambda = 4$ are of the form $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

For $\lambda = 1$,

$$
\begin{aligned}
(J_\pi - \lambda I)\mathbf{x} &= \mathbf{0} \\
\begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

and the eigenvectors associated with $\lambda = 1$ are of the form $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The general form of the linearised system in the neighbourhood of fixed points $\{(n\pi, 0) : n \text{ is odd}\}$ can therefore be written

$$
\begin{aligned}
\mathbf{x}(t) &= Ae^{\lambda_1 t}\mathbf{x}_1 + Be^{\lambda_2 t}\mathbf{x}_2 \\
&= Ae^{-4t}\begin{pmatrix} 4 \\ -1 \end{pmatrix} + Be^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}
$$

Because eigenvalues are real and with opposite signs we can say that, locally, these fixed points are saddle points with $\mathbf{x}_1$ pushing trajectories inwards and $\mathbf{x}_2$ pushing trajectories outwards. Sending $t \to -\infty$ tells us that the $\mathbf{x}_1$ dominates early times and $t \to \infty$ tells us $\mathbf{x}_2$ dominates late times.

Evaluating $J$ at the fixed point $(n\pi, 0)$ when $n$ is even, gives the Jacobian matrix $J_0 = \begin{pmatrix} 0 & 1 \\ -4 & -3 \end{pmatrix}$.

$$
\det(J_0 - \lambda I) = -\lambda(-3 - \lambda) + 4
$$

doesn't factorise, which tells us $\lambda \in \mathbb{C}$. Applying the quadratic equation, $\lambda = \frac{-3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} = \frac{1}{2}(-3 \pm i\sqrt{7})$. These complex conjugate values for $\lambda$ both have a nonzero real part, so this fixed point is also hyperbolic and we can apply Hartman-Grobman.

We don't need to caculate the eigenvectors in this case because we know these fixed points behave like a spirals locally. Because $\mathrm{Re}(\lambda) < 0$ we also know that these spirals are stable. As for the winding direction, because $\dot{x} = y$ and $\dot{y} = -3y$ when $x = 0$ these spirals wind in a clockwise direction.

One last thing to consider before writing the phase portrait of this system is the interaction between the two classes of fixed points with respect to a given trajectory. We have establised that for those fixed points with $n$-odd, $\mathbf{x}_1$ dominates early times and $\mathbf{x}_2$ dominates late times. For fixed points with $n$-even, the idea of early/late times doesn't really make sense, since the conjugate eigenvalues mean the eigenvectors "share" dominance over time, but always in a spiral inwards. In that sense, fixed points with $n$-even have a constant amount of "spiralling dominance" over the trajectory. To get an idea of what happens when all the fixed points have most similar amount of influence on the trajectory, we consider what happens at *mid times* with, let's say, $t = 0$.

At mid times, then, the coefficients of the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are 1. Between these two (very roughly, not considering $c_1$ or $c_2$) the vector with the larger magnitude will be more influential, that is $\mathbf{x}_1$. However, the spiral fixed point has exactly the same amount of "spiralling dominance" as it did when $t \to \pm\infty$, so there is more "sum dominance" over a trajectory that would cause it to behave as if it were being attracted to a fixed point.

This is an attempt to explain why trajectories don't follow $\mathbf{x}_2$ vectors as $t \to \infty$. A more informal argument would be that this dynamical system describes a pendulum, and pendulums come to rest.

The phase portrait is written overleaf.