Discrete Mathematics Solutions 2014
Section A

1. (a) $\sum_{r=0}^{3000} \frac{3000!}{r!(3000-r)!} = \sum_{r=0}^{3000} \binom{3000}{r} = \mathbf{2^{3000}}$ [2]

   (b) $\sum_{r=0}^{3000} 2^r + r^2 = \mathbf{2^{3001} - 1 + \frac{1}{6}(3000)(3001)(6001)}$ [3]

2. (a) $13^5 = \mathbf{371293}$ [1]

   (b) $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = \mathbf{154440}$ [1]

   (c) $\binom{5}{3}8^2 \cdot 5^3 + \binom{5}{4}8 \cdot 5^4 + 5^5 = \mathbf{108125}$ [3]

3. (a) $\binom{3+50}{3} = \mathbf{23426}$ [1]

   (b) After selecting 20 black and 15 blue markers we have 15 more markers to choose. $\binom{3+15}{3} = \mathbf{816}$ [2]

   (c) There are $\binom{39+3}{3} = 11480$ ways to choose combinations including at least 11 red markers. Therefore the number of selections with at most 10 red markers is $\binom{53}{3} - \binom{42}{3} = \mathbf{11946}$ [2]

4. (a) *characteristic polynomial:* $\lambda + \frac{2}{3}$
   general solution to homogeneous part: $G(n) = a\left(\frac{-2}{3}\right)^n$
   find particular solution: Try $P(n) = M$. We have

$$\begin{aligned} M &= -\frac{2}{3}M + 2, \\ M &= \frac{6}{5}. \end{aligned}$$

   general solution: $u_n = G(n) + P(n) = a\left(\frac{-2}{3}\right)^n + \frac{6}{5}$
   solve for a:

$$\begin{aligned} u_0 = a + \frac{6}{5} &= 20, \\ a &= \frac{94}{5} \end{aligned}$$

   solution: $u_n = \frac{94}{5}\left(\frac{-2}{3}\right)^n + \frac{6}{5}$ [3]

   (b) The sequence converges to $\frac{6}{5}$. [2]

Please turn over

5. (a) We have

$$\binom{2n}{n} - \binom{2}{n-1} = \frac{2n}{n!n!} - \frac{2n}{(n-1)!(n+1)!},$$
$$= \frac{2n(n+1)}{n!(n+1)!} - \frac{2n(n)}{n!(n+1)!},$$
$$= \frac{2n}{n!(n+1)!},$$
$$= \frac{1}{n+1}\binom{2n}{n} = C_n.$$

[2]

(b)

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$
$$= \frac{1 \times 2 \times 3 \times 4 \times \cdots \times (2n-2)(2n-1)(2n)}{(n+1)n!n!},$$
$$= \frac{1(2 \times 1)3(2 \times 2) \cdots \times (2 \times (n-1))(2n-1)(2 \times n)}{(n+1)!(1 \times 2 \times 3 \times \cdots \times (n-1) \times n)},$$
$$= \frac{(2 \times 1)(2 \times 3)(2 \times 5) \cdots (2 \times (2n-1))}{(n+1)!}.$$

[3]

6. (a) In any graph G, the sum of the degrees of the vertices of G is equal to twice the number of edges. [2]

(b) The Handshaking Lemma implies that the sum of the degrees of the vertices in a graph is even, yet we have $1+2+3+4+5=15$, which is odd. Hence $[1,2,3,4,5]$ is not the degree sequence of any graph. [1]

(c) The sum of the entries in column i is the degree of vertex i, hence if we sum the entries of the adjacency matrix we obtain the degree sum of the graph. In this case this sum is 10, hence G has **5** edges. [2]

Please turn over

7. (a) We can assign permanent labels to the vertices as follows:

$$\begin{aligned} P(a) &= 0, \\ P(b) &= 7, \\ P(c) &= 5, \\ P(d) &= 4, \\ P(e) &= 10, \\ P(g) &= 10, \\ P(f) &= 10, \\ P(h) &= 15, \\ P(j) &= 19, \\ P(i) &= 15, \\ P(k) &= 19, \\ P(l) &= 24, \\ P(m) &= 26. \end{aligned}$$

A critical path is a, d, g, f, h, j, l, m, corresponding to the sequence of tasks **D**, **E**, **F**, **H**, **I**, **M**. It has length **26**. [3]

(b) The earliest start time is **15**. The only path starting with J and ending at m has length 10. Hence the latest start time is $26 - 10 = \mathbf{16}$. [2]

8. (a) If v and w are two vertices of a connected graph G, then the maximum number of edge-disjoint vw-paths in G is equal to the size of the smallest vw-disconnecting set. [2]

(b) It is possible to find a set of three edge-disjoint ah-paths, for example $abfh$, $aceh$ and $adgh$. Similarly, it is also possible to find an ah-disconnecting set of size three (for example, all the edges incident with a). Hence by the edge form of Menger's theorem, this ah-disconnecting set is as small as possible. [3]

Please turn over

Section B

9. (a) (i)

$$\begin{aligned} C + O + L + S &= 30, \\ 10 \leq C &\leq 15, \\ 5 \leq O &\leq 10, \\ 5 \leq L &\leq 10, \\ 5 \leq S &\leq 10. \end{aligned}$$

[2]

(ii) $g(x) = (x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15})(x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})^3$. [2]

(iii) We have

$$\begin{aligned} g(x) &= (x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15})(x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})^3, \\ &= x^{25}(1 + x + x^2 + x^3 + x^4 + x^5)^4, \\ &= x^{25}(1 - x^6)^4(1 - x)^{-4}, \\ &= x^{25}(1 - 4x^6 + 6x^{12} - 4x^{18} + x^{24}) \sum_{r=0}^{\infty} \binom{r+3}{r} x^r. \end{aligned}$$

The coefficient of x^{30} in this expression is $\binom{8}{5} = \mathbf{56}$. [6]

(b) (i) This sequence satisfies a homogeneous linear difference equation of order k if and only if p is a polynomial of degree less than k, and q is a polynomial of degree k whose constant term is nonzero. [2]

(ii) We have

$$\begin{aligned} g(x) &= \frac{1}{1 + 4x^2}, \\ &= 1 - 4x^2 + 16x^4 - 64x^6 + ... \end{aligned}$$

Hence the first four terms of the sequence are $\mathbf{1, 0, -4, 0}$. [2]

(c) (i) $\mathbf{2(1 - x)^{-1}}$ [2]

(ii) $\mathbf{(1 - 2x)^{-1}}$ [2]

(iii) $\mathbf{2\frac{x}{(1-x)^2}}$ [2]

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10. (a) (i) *characteristic polynomial:* $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$,
general solution to homogeneous part: $G(n) = a3^n + b$,
find a particular solution: Try $P(n) = cn^2 + dn$. We have

$$cn^2 + dn = 4c(n-1)^2 + 4d(n-1) - 3c(n-2)^2 - 3d(n-2) - 20n + 26,$$

$$\begin{aligned}\text{coefficient of } n\text{: } d &= -8c + 4d + 12c - 3d - 20,\\ c &= 5,\\ \text{constants: } 0 &= 4c - 4d - 12c + 6d + 26,\\ 40 - 26 &= 2d,\\ d &= 7.\end{aligned}$$

Hence $P(n) = 5n^2 + 7n$ is a particular solution.
general solution: $u_n = a3^n + b + 5n^2 + 7n$,
solve for a, b:

$$\begin{aligned}u_0 &= a + b = 10,\\ u_1 &= 3a + b + 12 = 24.\end{aligned}$$

This gives $a = 1$, $b = 9$.
solution: $\mathbf{u_n = 3^n + 5n^2 + 7n + 9}$ [5]

(ii) *characteristic polynomial:* $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2$,
general solution to the homogeneous part: $G(n) = a7^n + bn7^n$,
find a particular solution: Try $P(n) = C3^n$. We have

$$\begin{aligned}C3^n &= -C \cdot 14 \cdot 3^{n-1} - 49C3^{n-2} + \frac{500}{9}3^n,\\ 9C &= -42C - 49C + 500,\\ 100C &= 500,\\ C &= 5,\end{aligned}$$

hence $P(n) = 5 \cdot 3^n$ is a particular solution.
general solution: $u_n = G(n) + P(n) = a7^n + bn7^n + 5 \cdot 3^n$.
solve for a, b:

$$\begin{aligned}u_0 &= a + 5 = 7,\\ u_1 &= 7a + 7b + 15 = -20.\end{aligned}$$

This gives $a = 2$, $b = -7$.
solution: $\mathbf{u_n = 2 \cdot 7^n - n7^{n+1} + 5 \cdot 3^n}$.

[5]

(b) Applying the suggested substitution we have

$$\begin{aligned}2^{b_n} &= \frac{(2^{b_{n-1}})^5}{(2^{b_{n-2}})^6},\\ &= 2^{5b_{n-1} - 6b_{n-2}},\end{aligned}$$

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whence

$$b_n = 5b_{n-1} - 6b_{n-2}.$$

The characteristic equation for this homogeneous difference equation is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. Hence the general solution is $b_n = c_1 2^n + c_2 3^n$. For initial conditions, we have $2^{b_0} = \frac{1}{4}$, so $b_0 = -2$, and $2^{b_1} = 2$, hence $b_1 = 1$. Thus $c_1 + c_2 = -2$, and $2c_2 + 3c_1 = 1$, which gives $c_1 = -7$, $c_2 = 5$. Hence we have the solution $b_n = -7(2^n) + 5(3^n)$, which gives $\mathbf{a_n = 2^{-7(2^n)+5(3^n)}}$. [6]

(c) (i) The homogeneous part of (1) is

$$u_n = Au_{n-1} + Bu_{n-2}. \tag{1}$$

[1]

(ii) If $P(n)$ is a particular solution of (1), and $G(n)$ is a solution to the homogeneous part of (1) then we have

$$\begin{aligned} & A\left(P(n-1) + G(n-1)\right) + B\left(P(n-2) + G(n-2)\right) + f(n) \\ &= \left(AP(n-1) + BP(n-2) + f(n)\right) + \left(AG(n-1) + BG(n-2)\right), \\ &= P(n) + G(n), \end{aligned}$$

hence $P(n) + G(n)$ is a solution of (1). [3]

Please turn over

11. (a) (i) Let $G = (V_1, V_2, E)$ be a bipartite graph. Then G has a complete matching if and only if for each subset $S \subseteq V_1$, the set $N(S) \subseteq V_2$ of vertices of V_2 that are adjacent to vertices in S satisfies $|N(S)| \geq |S|$. [3]

(ii) Let $G = (V_1, V_2, E)$ be a bipartite graph that does not satisfy Hall's conditions. Then there exists a set $S \subseteq V_1$ of n vertices (say) whose set $N(S) \subseteq V_2$ of neighbours has size $m < n$. By the pigeonhole principle, if we are joining each of the n vertices in S to a vertex in $N(S)$, then one of the vertices in $N(S)$ is joined to more than one vertex in S, since $\lceil \frac{n}{m} \rceil > 1$. Hence G contains no complete matching. [2]

(b) (i) We use the Gale-Shapley algorithm:

Round 1	$M_1 \to W_3$,
	$M_2 \to W_4$,
	$M_3 \to W_4$,
	$M_4 \to W_3$,
	$M_5 \to W_5$,
engagements:	$(M_1, W_3), (M_3, W_4), (M_5, W_5)$
Round 2	$M_2 \to W_1$,
	$M_4 \to W_4$,
engagements:	$(M_1, W_3), (M_2, W_1), (M_3, W_4), (M_5, W_5)$
Round 3	$M_4 \to W_5$,
engagements:	$(M_1, W_3), (M_2, W_1), (M_3, W_4), (M_4, W_5)$
Round 4	$M_5 \to W_2$,
engagements:	$(\mathbf{M_1}, \mathbf{W_3}), (\mathbf{M_2}, \mathbf{W_1}), (\mathbf{M_3}, \mathbf{W_4}), (\mathbf{M_4}, \mathbf{W_5}), (\mathbf{M_5}, \mathbf{W_2})$

[7]

(ii) Suppose $\mathcal{M}_1$ is a male optimal stable matching that is not female pessimal. Then for some woman W_1 who is married to a man M_1 in $\mathcal{M}_1$, there is a matching $\mathcal{M}_2$ in which she marries a man M_2 whom she likes less than M_1. Let W_2 be the wife of M_1 in $\mathcal{M}_2$. Then $\{(M_1, W_2), (M_2, W_1)\}$ is an unstable pair, as woman W_1 prefers M_1 to M_2, and M_1 prefers W_1 to W_2 (as she is his optimal partner). This contradicts the assumption that $\mathcal{M}_2$ is stable. [4]

(iii) Neither are stable.

A. $\{M_1, W_1\}, \{M_3, W_3\}$ is an unstable pair, as M_1 and W_3 rank each other above their current partners. [1]

B. $\{M_4, W_2\}, \{M_5, W_5\}$ is an unstable pair, as M_4 and W_5 rank each other above their current partners. [3]

Please turn over

12. (a) The graph (i) is not isomorphic as a labelled tree, as vertex 4 is not adjacent to vertex 7. [2]

The graph (ii) is isomorphic to G as a labelled tree (in both cases 2 is adjacent to 1, 3 and 5, we have 5 is adjacent to 4 and 6 as well as 2, 4 is adjacent to 5 and 7, and 1, 3, 6 and 7 are leaves). [2]

(b) By Cayley's theorem, H has $5^{5-2} = \mathbf{125}$ spanning subtrees. [3]

(c) (i) One suitable choice of edges is $\mathbf{bf}, \mathbf{bd}, \mathbf{af}, \mathbf{dc}, \mathbf{de}$. [3]

(ii) Travelling salesman problem: given a weighted complete graph G on n vertices, find a Hamiltonian cycle of G of minimum total weight. [2]

(iii) Given a graph G on n vertices, assign the weight 0 to each edge. Now add extra edges to G to obtain a complete graph, assigning the weight 1 to all the new edges. Use this weighted complete graph as an input to the machine. If the Hamilton cycle output by the machine has weight 0 then each of the edges used in this cycle were present in G, and hence G itself contains a Hamilton cycle. If the output of the machine has weight ≥ 1 then it is not possible to construct a Hamiltonian cycle using only edges of G, and hence G is not Hamiltonian. [3]

(iv) The edge db is the edge of smallest weight incident with d. Place the vertices d and b into a cycle. The next smallest edge incident to one of either d or b is bf; place f in the cycle following b. A next smallest edge adjacent to one vertex in the cycle is fa; place a in the cycle following f. Continuing in a similar manner we can select the edges dc and ce, resulting in the cycle a, d, c, e, b, f of total weight 20. [2]

(v) Smallest edges incident with d are db and de of weights 2 and 3. A minimum spanning tree for the induced subgraph on vertices a, b, c, e, f has weight 13. Hence a lower bound for the TSP is $2 + 3 + 13 = 18$. [3]