Algebra 2 Assignment 4

Due on or before Friday 27^{th} April 2018

Answer all questions. A total mark out of 20 will be given. This assignment is worth 5% of the marks for the module. Note: college regulations mean that, unless there are mitigating circumstances, work submitted late (up to 14 days, so up to Friday 11^{th} May) will have the mark capped at 40% (i.e. 8/20), and work submitted after 14 days late will score 0. Marked work will be available for collection on May 14^{th}.

1. Recall from lectures that the set $\mathbb{R}[x]$ of polynomials in x with real coefficients is a real vector space. Define

$$\begin{aligned}
U &= \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\};\\
V &= \{a_0 + a_1x + a_2x^2 + a_3x^3 \in U : a_0 + a_1 + a_2 + a_3 = 0\};\\
W &= \{a_0 + a_1x + a_2x^2 + a_3x^3 \in U : a_0 + a_1 + a_2 + a_3 = 1\}.
\end{aligned}$$

   (a) Show that U and V are subspaces of $\mathbb{R}[x]$, but that W is not a subspace of $\mathbb{R}[x]$. [5]

   (b) Let $B = \{x - 1, x^2 - 1, x^3 - 1\}$. Show that B is a basis for V. [2]

   (c) The map $\theta : V \to U$ is defined as follows. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be an element of V. Define

$$\theta(f(x)) = (a_0 + a_1)x + (a_1 + a_2)x^2 + (a_2 + a_3)x^3.$$

      i. Show that θ is a linear transformation. [2]

      ii. Find $\ker\theta$ and the nullity of θ. Hence find the rank of θ. [3]

      iii. It can be shown that the set $T = \{1, x, x^2, x^3\}$ is a basis for U (you do **not** have to do this). Find the matrix M for θ with respect to the bases B for V and T for U. [2]

2. Let $T : \mathbb{Z}_3^5 \to \mathbb{Z}_3^3$ denote the linear transformation satisfying

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_4 + x_5, x_2 + 2x_3 + x_4 + 2x_5, x_1 + x_2 + 2x_3 + 2x_4).$$

   (You do **not** need to show that T is a linear transformation.)

   (a) Let M denote the matrix of T with respect to the bases B_1 and B_2, where

$$B_1 = \{(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)\}$$

   and

$$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}.$$

   Write down M. [2]

   (b) Let R denote the subspace of $\mathbb{Z}_3^5$ spanned by the rows of M. Find bases for R and for $\ker T$. [2]

   (c) Determine a basis for the vector space $R + \ker T$ and **hence** determine the dimension of $R \cap \ker T$. [2]

Coursework should be neatly written or typed in black or blue ink on A4 paper. Please staple the sheets of paper together. **Please do not submit your work in a plastic wallet / folder that is closed on three sides** — those that are closed on two sides are fine. You should submit your work by placing it, with a signed cover sheet, inside the Assignment Box, which is opposite the lifts on the 7^{th} floor. Full coursework regulations are given in the programme handbook which can be downloaded from the Moodle page for this module.