Partial Fractions Decomposition

Dr. Philippe B. Laval
Kennesaw State University

August 26, 2008

Abstract

This handout describes partial fractions decomposition and how it can be used when integrating rational functions.

1 Partial Fraction Decomposition

1.1 Introduction

This handout describes a method to rewrite a fraction we do not know how to integrate into simpler (partial) fractions we know how to integrate. This method works for rational functions, that is functions which can be written as the quotient of two polynomial functions.

For the remaining part of this document, we will assume that we have a rational function $\dfrac{p(x)}{q(x)}$ in which degree of $p(x) <$ degree of $q(x)$. If this is not the case, we can always perform long division. For example, if we were given the fraction $\dfrac{x^3}{x^2-1}$. We would perform long division to obtain

$$\begin{aligned} x^3 &= x\left(x^2-1\right)+x \\ \frac{x^3}{x^2-1} &= x+\frac{x}{x^2-1} \end{aligned}$$

We then would apply partial fraction decomposition to $\dfrac{x}{x^2-1}$.

In this class, we will use partial fraction decomposition as an integration technique. The ultimate goal is to decompose a fraction so we can integrate it. In this document, we will focus on the decomposition. Keep in mind why we are doing this decomposition.

How we perform partial fraction decomposition depends of the denominator of the fraction. We consider several cases.

1.2 Case 1: q(x) is a product of distinct linear factors.

Let us assume that $q(x)$ is a product of n distinct linear factors that is

$$q(x) = (a_1 x + b_1)(a_2 x + b_2) \ldots (a_n x + b_n)$$

Then,

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \ldots + \frac{A_n}{a_n x + b_n}$$

Finding the decomposition amounts to finding the coefficients A_1, ..., A_n. This can be done two different ways. We illustrate this in the examples below.

Remark 1 *$p(x)$ does not play a role in the way the decomposition is written.*

Example 2 *Find the decomposition for* $\dfrac{x}{x^2 + 2x - 3}$

We begin by factoring the denominator. We obtain $\dfrac{x}{(x+3)(x-1)}$. *According to the decomposition we wrote above, we have:*

$$\frac{x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

We need to find A and B. Multiplying each side by the denominator of the fraction on the left and simplifying, we obtain:

$$\begin{aligned} x &= A(x-1) + B(x+3) \\ &= Ax - A + Bx + 3B \\ &= (A+B)x - A + 3B \end{aligned}$$

Two polynomials are equal if their corresponding coefficients are equal. This gives us the following system:

$$\begin{cases} A + B = 1 \\ -A + 3B = 0 \end{cases}$$

The solution of this system is: $\begin{cases} A = \dfrac{3}{4} \\ B = \dfrac{1}{4} \end{cases}$ *Thus, we have*

$$\frac{x}{x^2 + 2x - 3} = \frac{1}{4}\left(\frac{3}{x+3} + \frac{1}{x-1}\right)$$

There is an easier way which works in this case. We show it on the next example.

Example 3 *Find the decomposition for* $\dfrac{x^2 - x}{(x-1)(x+5)(x-3)}$

The denominator is already factored. The decomposition is:

$$\frac{x^2 - x}{(x-1)(x+5)(x-3)} = \frac{A}{x-1} + \frac{B}{x+5} + \frac{C}{x-3}$$

We need to find A, B, and C. We begin the same way, we multiply each side by the denominator of the fraction on the left, and simplify. We obtain:

$$x^2 - x = A(x+5)(x-3) + B(x-1)(x-3) + C(x-1)(x+5)$$

Then we notice that since the above equality is true for every x, it will be true for specific values of x. We select values for x which will make all but one of the coefficients go away. We will then be able to solve for that coefficient. More precisely,

- *When $x = 1$, we obtain:*

$$\begin{aligned} 0 &= A(6)(-2) \\ A &= 0 \end{aligned}$$

- *When $x = -5$, we obtain*

$$\begin{aligned} 30 &= B(-6)(-8) \\ B &= \frac{30}{48} \\ B &= \frac{5}{8} \end{aligned}$$

- *When $x = 3$, we obtain*

$$\begin{aligned} 6 &= C(2)(8) \\ C &= \frac{3}{8} \end{aligned}$$

Therefore, the decomposition is:

$$\frac{x^2 - x}{(x-1)(x+5)(x-3)} = \frac{1}{8}\left(\frac{5}{x+5} + \frac{3}{x-3}\right)$$

1.3 Case 2: q(x) is a product of linear factors, some being repeated

The factors which are not repeated will be decomposed as above. Suppose that $q(x)$ also contains $(ax+b)^m$ that is $ax+b$ is repeated m times. The decomposition for this factor will be

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_m}{(ax+b)^m}$$

Example 4 *Find a decomposition for* $\dfrac{x^2-2}{(x-2)(x+1)^3}$

The decomposition is:

$$\frac{x^2-2}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

Then, we proceed as before. We multiply each side by the denominator of the fraction on the left and simplify. We obtain:

$$x^2 - 2 = A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2)$$

We need to find A, B, C, and D. We can use either of the methods described in the first case.

- *If $x = 2$, we get*

$$\begin{aligned} 2 &= 27A \\ A &= \frac{2}{27} \end{aligned}$$

- *If $x = -1$, we get*

$$\begin{aligned} -1 &= -3D \\ D &= \frac{1}{3} \end{aligned}$$

  We still have to find B and C. For this, we choose two more values for x and write the corresponding system. We now know A and D, so we can use the value we found for them.

- *If $x = 1$, we get*

$$\begin{aligned} -1 &= 8A - 4B - 2C - D \\ &= \frac{16}{27} - 4B - 2C - \frac{1}{3} \\ 4B + 2C &= \frac{16}{27} - \frac{9}{27} + \frac{27}{27} \\ 4B + 2C &= \frac{34}{27} \end{aligned}$$

- *If $x = 0$, we get*

$$\begin{aligned} -2 &= A - 2B - 2C - 2D \\ &= \frac{2}{27} - 2B - 2C - \frac{2}{3} \\ 2B + 2C &= \frac{2}{27} - \frac{18}{27} + \frac{54}{27} \\ 2B + 2C &= \frac{38}{27} \end{aligned}$$

- *Thus, we need to solve*

$$\left\{ \begin{aligned} 4B + 2C &= \frac{34}{27} \\ 2B + 2C &= \frac{38}{27} \end{aligned} \right.$$

The solution is : $\left\{B=-\frac{2}{27}, C=\frac{7}{9}\right\}$. *Putting all this together, we get*

$$\frac{x^2-2}{(x-2)(x+1)^3}=\frac{2}{27(x-2)}-\frac{2}{27(x+1)}+\frac{7}{9(x+1)^2}+\frac{1}{3(x+1)^3}$$

1.4 Case 3: q(x) is a product of distinct irreducible quadratic factors.

Recall that a term is called irreducible if it cannot be factored any further. Thus x^2+x+1 is irreducible, so is x^2+1. Be careful, $(x+1)^2$ is not considered a quadratic term. You must think of it as a linear term appearing twice. We can generalize what we did in the previous two cases as follows. Instead of thinking of linear factors, think that when we write the decomposition, the degree of the term in the numerator is 1 less than the degree of the term in the denominator. When we had linear factors in the denominator, it meant that we had to have terms of degree 0 in the numerator, that is we had constant terms. If the denominator contains irreducible quadratic factors, then the numerator will contain linear terms. We look at an example to see how the decomposition is written.

Example 5 *Decompose* $\dfrac{x}{(x^2+1)(x^2+2)}$
If we apply a method similar to that of case 1, the decomposition will contain two fractions, one for each irreducible factor. However, since the degree of the denominator is now 2, the degree of the numerator will be 1, that is we will have linear terms. Recall that a linear term is of the form $ax+b$. *Thus,*

$$\frac{x}{(x^2+1)(x^2+2)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}$$

To solve, we proceed as above. First, we multiply each side by the denominator of the fraction on the left, and simplify. We obtain:

$$x=(Ax+B)\left(x^2+2\right)+(Cx+D)\left(x^2+1\right)$$

We then pick 4 different values for x *to get a system of 4 equations, which we solve. The answer is:*

$$\frac{x}{(x^2+1)(x^2+2)}=\frac{x}{x^2+1}-\frac{x}{x^2+2}$$

1.5 Case 4: q(x) is a product of irreducible quadratic factors, some being repeated

This is similar to case 2, with linear terms in the numerator and quadratic terms in the denominator.

Example 6 *Write the decomposition for* $\dfrac{2x-1}{\left(x^2+x+1\right)^3}$

$$\frac{2x-1}{\left(x^2+x+1\right)^3}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{\left(x^2+x+1\right)^2}+\frac{Ex+F}{\left(x^2+x+1\right)^3}$$

We would find the coefficients as above.

1.6 General Case: q(x) is a mixture of the above

Example 7 *Decompose* $\dfrac{2x-1}{(x-1)^2\left(x^2+x+1\right)^3}$

$$\begin{aligned}\frac{2x-1}{(x-1)^2\left(x^2+x+1\right)^3}&=\frac{A}{x-1}+\frac{B}{(x-1)^2}\\&\quad+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{\left(x^2+x+1\right)^2}+\frac{Gx+H}{\left(x^2+x+1\right)^3}\end{aligned}$$

The answer is

$$\begin{aligned}\frac{2x-1}{(x-1)^2\left(x^2+x+1\right)^3}&=\frac{1}{27\,(x-1)^2}-\frac{1}{27\,(x-1)}\\&\quad+\frac{1}{27}\frac{x+1}{x^2+x+1}-\frac{1}{9\left(x^2+x+1\right)^2}-\frac{1}{3}\frac{x+3}{\left(x^2+x+1\right)^3}\end{aligned}$$

1.7 Application

The idea behind this decomposition is that once the fraction is decomposed, we can integrate it.

Example 8 *Find* $\displaystyle\int\frac{x}{x^2+2x-3}dx$

We found earlier that $\dfrac{x}{x^2+2x-3}=\dfrac{1}{4}\left(\dfrac{3}{x+3}+\dfrac{1}{x-1}\right)$. *Therefore,*

$$\begin{aligned}\int\frac{x}{x^2+2x-3}dx&=\frac{1}{4}\left(3\int\frac{dx}{x+3}+\int\frac{dx}{x-1}\right)\\&=\frac{3}{4}\ln|x+3|+\frac{1}{4}\ln|x-1|\end{aligned}$$

Using substitution

Example 9 *Find* $\displaystyle\int\frac{x}{\left(x^2+1\right)\left(x^2+2\right)}dx$

We saw earlier as an example that the partial fraction decomposition of $\dfrac{x}{\left(x^2+1\right)\left(x^2+2\right)}$ *was*

$$\frac{x}{\left(x^2+1\right)\left(x^2+2\right)}=\frac{x}{x^2+1}-\frac{x}{x^2+2}$$

Therefore

$$\int \frac{x}{(x^2+1)(x^2+2)}dx = \int \frac{x}{x^2+1}dx - \int \frac{x}{x^2+2}dx$$

We can do the first integral by substitution. If we let $u = x^2+1$*, then* $du = 2xdx$ *and therefore*

$$\begin{aligned}\int \frac{x}{x^2+1}dx &= \frac{1}{2}\int \frac{du}{u} \\ &= \frac{1}{2}\ln|u| \\ &= \frac{1}{2}\ln\left(x^2+1\right)\end{aligned}$$

The second integral is done in a similar way. We obtain

$$\int \frac{x}{x^2+2}dx = \frac{1}{2}\ln\left(x^2+2\right)$$

It follows that

$$\begin{aligned}\int \frac{x}{(x^2+1)(x^2+2)}dx &= \frac{1}{2}\ln\left(x^2+1\right) - \frac{1}{2}\ln\left(x^2+2\right) \\ &= \frac{1}{2}\ln\left(\frac{x^2+1}{x^2+2}\right) \\ &= \ln\sqrt{\frac{x^2+1}{x^2+2}} + C\end{aligned}$$

Example 10 *Find* $\int \frac{x^3}{x^2-1}dx$

The function we are integrating is a rational function. However, the degree of the numerator is greater than or equal to the degree of the denominator. So, the first step is to perform long division. We did this at the beginning of this document and found that

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

It follows that

$$\int \frac{x^3}{x^2-1}dx = \int xdx + \int \frac{x}{x^2-1}dx \tag{1}$$

We can do the first integral. The second, is the integral of a rational function. To be able to evaluate it, we first decompose $\frac{x}{x^2-1}$ *into partial fractions.*

$$\begin{aligned}\frac{x}{x^2-1} &= \frac{x}{(x-1)(x+1)} \\ &= \frac{A}{x-1} + \frac{B}{x+1}\end{aligned}$$

We need to find A and B. We do it using the techniques described above. First, we multiply each side by the denominator of the fraction on the left to obtain

$$x = A(x+1) + B(x-1)$$

When $x = 1$, we get $1 = 2A$ or $A = \frac{1}{2}$. When $x = -1$, we get $-1 = -2B$ or $B = \frac{1}{2}$. Therefore,

$$\frac{x}{x^2-1} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

If we replace what we just found in equation 1, we obtain

$$\int \frac{x^3}{x^2-1}dx = \int x dx + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

These are integrals we can handle.

$$\begin{aligned} \int \frac{x^3}{x^2-1}dx &= \frac{x^2}{2} + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| \\ &= \frac{x^2}{2} + \frac{1}{2}\ln\left|x^2-1\right| \end{aligned}$$

1.8 Problems

Do the problems below:

1. Find $\int_4^2 \frac{4x-1}{(x-1)(x+2)}dx$
2. Find $\int \frac{x^2}{x+1}dx$
3. Find $\int \frac{x^2+1}{x^2-x}dx$
4. Find $\int \frac{dx}{x^2-a^2}$ where $a \neq 0$
5. Find $\int_0^1 \frac{2x+3}{(x+1)^2}dx$
6. Find $\int \frac{dx}{(x-1)^2(x+4)}$
7. Do # 15, 16, 17, 19, 21, 25, 27 on page 405