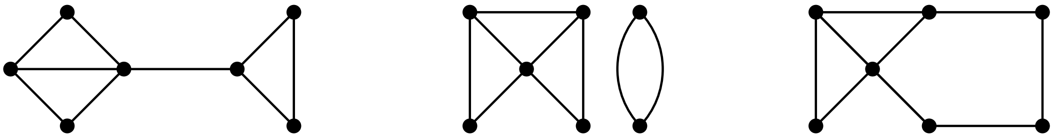
Discrete Mathematics Assignment 3 Solutions

1. (a) There are several vaild possibilities here, including the following:

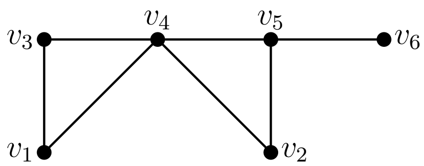


[1]

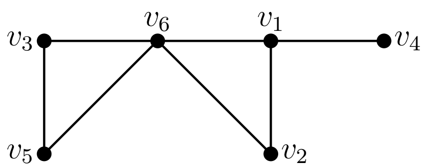
(b) The sum of the entries in G is 14. This tells us the sum of the degrees of the vertices in G, and hence is equal to twice the number of edges, by the Handshaking Lemma. Hence G has 7 edges. [1]

(c) i. The adjacency matrix of H has a zero in every position on the main diagonal, hence H has no loops. All the remaining entries are either zero or one, hence H has no multiple edges, and thus is simple. [1]

ii. G is the following graph:



H is the following graph, which is isomorphic to G:

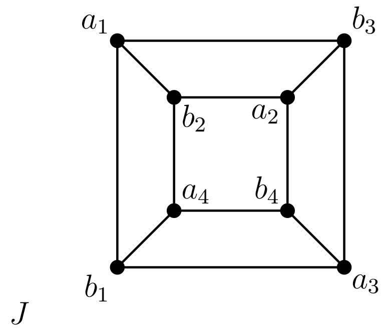


An isomorphism between G and H is given explicitly by $v_1 \mapsto v_5$, $v_2 \mapsto v_2$, $v_3 \mapsto v_3$, $v_4 \mapsto v_6$, $v_5 \mapsto v_1$, $v_6 \mapsto v_4$. [1]

2. (a) *e.g.* $K_{2,2}$ [1]

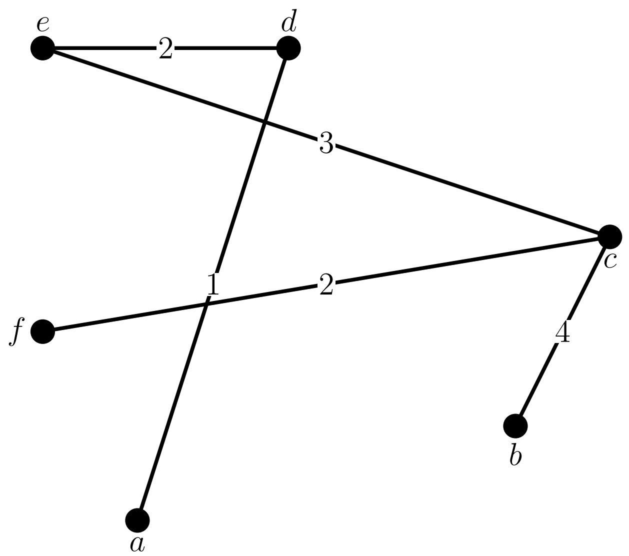
(b) *e.g.* C_3 [1]

(c) The graph J is planar, as can be shown by drawing it in the following way:



The graph K has 10 vertices and 20 edges. As it is bipartite, it has no triangles. Therefore the fact that $2 \times 10 - 4 < 20$ implies that it is nonplanar, by Corollary 4.10. [4]

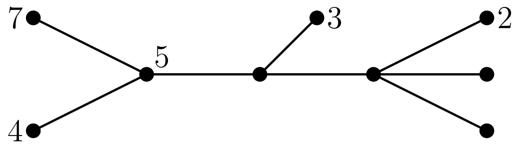
3. If we start with vertex a, then an appropriate order for choosing edges according to Prim's algorithm is ad (weight 1), de (weight 2), ec (weight 3), cf (weight 2), followed by cb (weight 4). The total weight of the minimum spanning tree is thus 12, and the resulting tree is shown below:



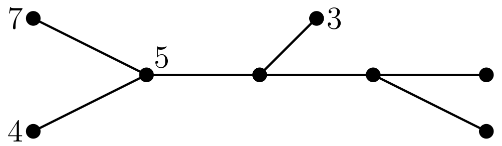
[3]

4. (a) The smallest label on any leaf is 1, and this leaf is connected to vertex 10,

hence the sequence begins [10,. Removing that leaf gives the following tree:

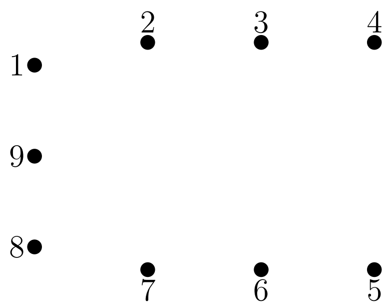


Now the smallest leaf label is 2. That leaf is connected to vertex 6, hence we extend the sequence to [10, 6 and remove leaf 2, leaving the following tree:

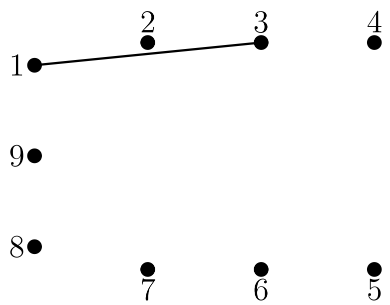


At this point the smallest leaft label is 3, and leaf 3 is adjacent to vertex 10, hence the sequence becomes [10, 6, 10. Proceeding in this manner gives rise to the sequence [10, 6, 10, 5, 5, 10, 6, 6]. [2]

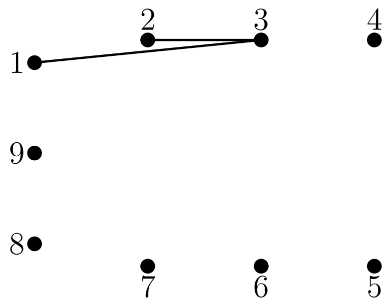
(b) This sequence has length seven, hence the resulting tree wil have nine vertices. We construct the list (1, 2, 3, 4, 5, 6, 7, 8, 9) and write down the vertices:



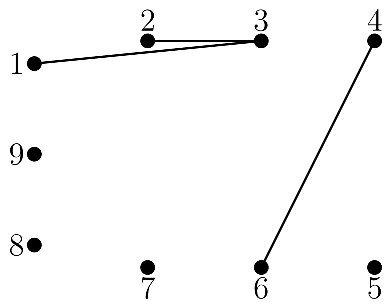
The smallest entry that appears in the list but not in the sequence is 1, and the first element of the sequence is 3, hence we join vertices 1 and 3:



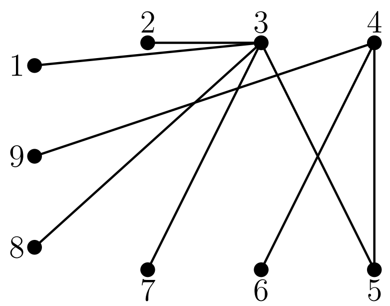
Now the list becomes $(2, 3, 4, 5, 6, 7, 8, 9)$ and the sequence becomes $[3, 4, 3, 3, 5, 4]$. The smallest remaining list entry that does not appear in the sequence is 2, and the current first element of the list is 3, hence we join vertices 2 and 3:



Now the list is $(3, 4, 5, 6, 7, 8, 9)$ and the sequence is $[4, 3, 3, 5, 4]$. The smallest list element that does not appear in the sequence is 6, and the first list element is 4. Hence we join vertices 6 and 4:



Continuing in this manner gives rise to the following tree:



[2]

(c) Cayley's formula tells us there are $8^6 = 2^{18}$ such labelled trees. [1]

(d) The tree has eight vertices, hence the labels we have to assign are $1, 2, 3, 4, 5, 6, 7, 8$. There are eight ways to choose the label for the degree 4 vertex, which leaves seven possible choices for the label on the degree 3 vertex, then six possible choices for the degree 2 vertex. At this point we can choose a label for the leaf adjacent to the degree two vertex (five options remain) and then a label for the leaf adjacent to the degree 3 vertex (four options remain). There are now three remaining vertices and three remaining labels. However any two ways of assigning these labels will result in isomorphic labelled trees (these remaining vertices are all leaves attached to a single vertex, and thus the order in which we draw them on the page makes no difference to the resulting graph). Therefore the overall number of possibilities is $8 \times 7 \times 6 \times 5 \times 4 = 6720$. [2]