Calculus 3 Assignment 4

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1. (a) To show that $\bar{\beta}$ is independent of time, we must show $\frac{\mathrm{d}\bar{\beta}}{\mathrm{d}t} = 0$. First note that, since $x = x_1 + x_2$,

$$\begin{aligned}\dot{x} = \dot{x}_1 + \dot{x}_2 &= -\alpha y \frac{x_1}{x} - \alpha y \frac{x_2}{x} \\ &= -\alpha y \frac{1}{x}(x_1 + x_2) \\ &= -\alpha y\end{aligned}$$

also that, by the chain rule

$$\frac{\mathrm{d}(x^{-1})}{\mathrm{d}t} = \dot{x}\left(-\frac{1}{x^2}\right) = -\frac{\dot{x}}{x^2} = \frac{\alpha y}{x^2}$$

and that

$$\begin{aligned}\frac{\mathrm{d}(\beta_1 x_1 + \beta_2 x_2)}{\mathrm{d}t} &= \beta_1 \dot{x}_1 + \beta_2 \dot{x}_2 \\ &= -\frac{\alpha y}{x}(\beta_1 x_1 + \beta_2 x_2).\end{aligned}$$

By the product rule, then,

$$\begin{aligned}\frac{\mathrm{d}\bar{\beta}}{\mathrm{d}t} &= \frac{\mathrm{d}(x^{-1})}{\mathrm{d}t}(\beta_1 x_1 + \beta_2 x_2) + (x^{-1})\frac{\mathrm{d}(\beta_1 x_1 + \beta_2 x_2)}{\mathrm{d}t} \\ &= \frac{\alpha y}{x^2}(\beta_1 x_1 + \beta_2 x_2) + (x^{-1})(-\frac{\alpha y}{x}(\beta_1 x_1 + \beta_2 x_2) \\ &= \frac{\alpha y}{x^2}(\beta_1 x_1 + \beta_2 x_2) - \frac{\alpha y}{x^2}(\beta_1 x_1 + \beta_2 x_2) \\ &= 0\end{aligned}$$

and $\bar{\beta}$ is independent of time. Now notice that $\bar{\beta} = -\frac{\dot{y}}{x}$, $\bar{\beta}x = -\dot{y}$ and $\dot{y} = -\bar{\beta}x$. We have shown that $\bar{\beta}$ is constant and $\dot{x} = -\alpha y$, $\dot{y} = -\bar{\beta}x$, so by Lanchester's square law $\frac{\mathrm{d}(\alpha y^2 - \bar{\beta}x^2)}{\mathrm{d}t}$ is also constant with respect to time (ie. constant throughout the battle). We can interpret $\bar{\beta}$ as the "aggregate effectiveness" of a soldier in army X.

(b) Since $\bar{\beta}$ is constant, we may evaluate it for any value of t to obtain its quantity. Let $t = 0$, now $\bar{\beta} = \frac{4\times100+1\times500}{100+500} = \frac{3}{2}$. Since $c = \alpha y^2 - \bar{\beta} x^2$ is constant,we may obtain a its value in a similar fashion, namely $c = (2 \times 1000^2) - (\frac{3}{2} \times 600^2) = 1460000$. Because $c > 0$, we know Y is the victor, with $y = \sqrt{\frac{c}{\alpha}} \approx 854$ archers remaining (plus one poor guy missing $\frac{3}{5}$ths of his body) when $x = 0$.

2. (a) As above, to show N is constant, we must show its derivative is zero with respect to time.

$$\begin{aligned} N &= S + I \\ &= I + S \\ \dot{N} &= \dot{I} + \dot{S} \\ &= \beta SI - \beta SI \\ &= 0 \end{aligned}$$

(b) ...

(c) By the definition of $\dot{I}$ and N, we may write

$$\begin{aligned}\dot{I} &= -\beta SI \\ &= -\beta I(N-I) \\ \frac{\dot{I}}{I(N-I)} &= -\beta.\end{aligned}$$

Since we know $\frac{1}{I(N-I)} \equiv \frac{X}{I} + \frac{Y}{N-I}$, we may write

$$\begin{aligned}1 &= X(N-I) + YI \\ &= I(X-Y) + YN\end{aligned}$$

and infer by equating coefficients that $Y = \frac{1}{N}$ and $X = Y$. We may now write

$$\begin{aligned}\frac{\dot{I}}{I(N-I)} = \frac{1}{N}\left(\frac{\dot{I}}{I} + \frac{\dot{I}}{N-I}\right) &= -\beta \\ \frac{\dot{I}}{I} + \frac{\dot{I}}{N-I} &= -\beta N \\ \int \frac{\dot{I}}{I}\mathrm{d}t + \int \frac{\dot{I}}{N-I}\mathrm{d}t &= -\beta N \int 1\mathrm{d}t.\end{aligned}$$

By substitution and laws of logarithms, let $A = e^c$ and write

$$\begin{aligned}\ln(I) + \ln(N-I) &= -\beta Nt + c \\ \ln(I(N-I)) &= \\ I(N-I) &= Ae^{-\beta Nt}.\end{aligned}$$

We can now see $SI = Ae^{-\tau}$. It is given that when $t = 0$, $S = I$, so we know $A = I^2$. We can now write

$$\begin{aligned}SI &= I^2 e^{-\tau} \\ e^{\tau} &= \frac{I}{S}.\end{aligned}$$

Finally, returning to $N = S + I$ and dividing by I, we can write

$$\begin{aligned} \frac{N}{I} &= \frac{S}{I} + 1 \\ &= e^{-\tau} + 1 \\ N &= I(e^{-\tau} + 1) \\ I &= \frac{N}{e^{-\tau} + 1} \end{aligned}$$

and, dividing by S

$$\begin{aligned} \frac{N}{S} &= 1 + \frac{I}{S} \\ &= 1 + e^{\tau} \\ N &= S(1 + e^{\tau}) \\ S &= \frac{N}{1 + e^{\tau}} \end{aligned}$$

3. There general form of the Lotka-Volterra equations is $\dot{x} = \alpha x - \beta xy$, $\dot{y} = \gamma xy - \delta y$. In this case it is given that $\alpha = 2, \beta = \frac{1}{2}, \gamma = 1$ and $\delta = \frac{1}{2}$.

   (a) Because we know the non-trivial fixed points of these equations are $(x_*, y_*) = \left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)$, we know there is a fixed point at $x = \frac{\delta}{\gamma} = 2$ and $y = \frac{\alpha}{\beta} = 4$, namely when there are 2000 rabbits and 4000 foxes.

   (b) The Jacobian of the given system is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 2 - \frac{1}{2}x & -\frac{1}{2}x \\ \frac{1}{2}y & x - \frac{1}{2} \end{pmatrix} \end{aligned}$$

   which, when we evalulate it at $x = y = 0$, gives $J_0 = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$.

   When $x = 10$ and $y = 10$, $\dot{x} = -30$ and $\dot{x} = 95$. Since we know $(2, 4)$ is a fixed point, we know a prey population can support a predator population double its size, so it makes sense that having predator and prey populations equal will lead to predator population growth due to abundance of delicious rabbits. It also makes sense (in this situation) that prey population should be decreasing, because predators do not hunt according to the reproduction coefficients of their prey (ie. $\beta y > \alpha$).