Chapter 2

Generating Functions

Suppose we have a general counting problem which has some solution a_r for every nonnegative integer r. One way of keeping track of the solutions is to write them down in the form of a *generating function:* an infinite sum of the form

$$\sum_{r=0}^{\infty} a_r x^r$$

where the coefficient of x^r is the solution a_r. In fact, we have already seen an example of this:

Example 2.1. The number of ways of sampling r items from a set of size n if repetition is not allowed and order does not matter is $\binom{n}{r}$ when $0 \le r \le n$, and 0 for $r > n$. Thus the generating function for this counting problem is

$$\sum_{r=0}^{n} \binom{n}{r} x^r.$$

By the binomial theorem, we know we could also write this as $(1+x)^n$.

Generating functions of this form are sometimes referred to as *ordinary generating functions.* In this chapter we will see how they can be used to solve certain types of counting problems.

Example 2.2. Find the number of integer solutions of

$$X_1 + X_2 = r \text{ with } 0 \le X_1 \le 1 \text{ and } 1 \le X_2 \le 2$$

for $r \in \mathbb{Z}$.

Solution. The number of choices for X_1 and X_2 that make the sum $X_1 + X_2$ equal to r will be different for different values of r. Here we list all the possible combinations for X_1, X_2 and r.

X_1	X_2	r
0	1	1
0	2	2
1	1	2
1	2	3

Thus for $r = 1$ or $r = 3$ there is one solution, for $r = 2$ there are two solutions, and for all other values of r there are no solutions.

The number of possible values for X_1 and X_2 in this particular problem was small enough that we could easily write down all potential combinations of these values. This will not always be the case, however, so now we consider another approach to solving this problem. For each of the variables in our sum we construct a polynomial in x, in which the powers of x represent the possible values of that variable. So, for X_1 we have the polynomial $x^0 + x^1$ and for X_2 we have the polynomial $x^1 + x^2$. Consider what happens when we multiply these two polynomials:

$$(x^0 + x^1)(x^1 + x^2) = x^1 + x^2 + x^2 + x^3 = x^1 + 2x^2 + x^3.$$

The powers of x in the original two polynomials represented possible values of x_1 and x_2; each power of x in the new polynomial is the sum of an X_1 value and an X_2 value, that is they represent possible values of r. The coefficient of x^r (once we have collected like terms) is equal to the number of different combinations of X_1 and X_2 that sum to give r (observe that the coefficients match the numbers of possible solutions we computed earlier). We have thus found the generating function corresponding to this counting problem, and the number of solutions for any given value of r can be obtained by reading off the appropriate coefficient.

Example 2.3. Find the number of integer solutions of

$$X_1 + X_2 = r, \quad X_1, X_2 \geq 0.$$

Solution. At first glance this problem seems similar to the previous one, except that X_1 and X_2 can take on an infinite number of possible values, so we can no longer simply write them all down. Instead we take our motivation from the polynomial technique we have just seen, and consider the expression

$$(x^0 + x^1 + x^2 + \cdots)(x^0 + x^1 + x^2 + \cdots).$$

This is in fact the generating function for this problem; by determining the coefficient of x^r in this expression we can show that the number of solutions is $r + 1$.

We now revise how to manipulate expressions of this form.

2.1 Power Series

A formal power series is an infinite sum of the form

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$$

Here x is just an abstract variable, we never assign any kind of value to it. A power series can be thought of as being like a polynomial with an infinite number of terms. Note that when writing down a power series, we normally omit any terms for which $a_r = 0$, so a polynomial of degree t can be thought of as a power series with $a_r = 0$ for all $r > t$. Power series can be added and multiplied in a similar manner to polynomials, by "collecting like terms".

Addition of power series

$$\sum_{r=0}^{\infty} a_r x^r + \sum_{r=0}^{\infty} b_r x^r = \sum_{r=0}^{\infty} (a_r + b_r) x^r.$$

Multiplication of power series

$$\left(\sum_{r=0}^{\infty} a_r x^r\right) \times \left(\sum_{r=0}^{\infty} b_r x^r\right) = \sum_{r=0}^{\infty} \left(\sum_{i=0}^{r} a_i b_{r-i}\right) x^r$$

We can also take the formal derivative of a power series, by differentiating each term.

(Formal) Differentiation of a power series

$$\frac{d}{dx}\left(\sum_{r=0}^{\infty} a_r x^r\right) = \sum_{r=1}^{\infty} r a_r x^{r-1}$$

Example 2.4. Compute the products of the following power series:

1. $(1 + x + 3x^2)$ and $(3x + x^3)$
2. x and $\sum_{r=0}^{\infty} x^r$
3. $1 - x$ and $\sum_{r=0}^{\infty} x^r$.

Solution.

1. $3x + 3x^2 + 10x^3 + x^4 + 3x^5$
2. $\sum_{r=0}^{\infty} x^{r+1} = \sum_{r=1}^{\infty} x^r$
3. Multiplying power series is like "expanding the brackets" when multiplying polynomials: we multiply every term in the first series by every term in the second series, and add up the results by collecting like terms. In this case, the first series only has two non-zero terms, so we simply have to add $1 \times \sum_{r=0}^{\infty} x^r$ to $-x \times \sum_{r=0}^{\infty} x^r = -\sum_{r=1}^{\infty} x^r$. This gives the (slightly surprising!) result of 1, since all higher powers of x are cancelled out. Therefore these two power series can be thought of as multiplicative inverses of each other, and we write
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
This result will be very useful later on, when we are manipulating power series in order to solve counting problems.

2.2 Finding the Generating Function of a Problem

Now that we have seen how to work with power series, the next step is to explore how to find the generating function corresponding to a given counting problem. We have already seen examples of how problems relating to counting the solutions of integer equations can be translated directly into generating functions. In general, our approach will be

1. express the problem in terms of an integer equation;

2. determine the generating function associated with that equation.

This second step can be performed in a straightforward way by using the following theorem.

Theorem 2.1. *The generating function for the number of integer solutions of*

$$X_1 + X_2 + \cdots + X_n = r$$

where, for $1 \leq i \leq n$, X_i *takes the values* $v_{i1}, v_{i2}, v_{i3}, \ldots$ *is*

$$(x^{v_{11}} + x^{v_{12}} + x^{v_{13}} + \cdots)(x^{v_{21}} + x^{v_{22}} + x^{v_{23}} + \cdots) \cdots (x^{v_{n1}} + x^{v_{n2}} + x^{v_{n3}} + \cdots).$$

The powers of x in the first bracket represent all the values that the variable X_1 can take, the powers of x in the second bracket represent the values X_2 can take, and so on.

Example 2.5. Find the generating function for the number of integer solutions of

$$X_1 + X_2 + X_3 + X_4 = r$$

with $0 \leq X_1 \leq 2$, $X_2 \in \{1, 3, 4\}$, $X_3 \geq 3$ and $X_4 \geq 0$.

Solution. The appropriate generating function is

$$(x^0 + x^1 + x^2)(x^1 + x^3 + x^4)(x^3 + x^4 + x^5 + \cdots)(x^0 + x^1 + x^2 + \cdots).$$

Example 2.6. Find the generating function for the number of integer solutions of

$$X_1 + 2X_2 + 4X_3 + 8X_4 = r$$

with $X_i \geq 0$ for $(1 \leq i \leq 4)$.

Solution. The given equation is equivalent to

$$Y_1 + Y_2 + Y_3 + Y_4 = r$$

where

$$\begin{aligned} &Y_1 \geq 0 \\ &Y_2 = 0, 2, 4, 6, \ldots \\ &Y_3 = 0, 4, 8, 12, \ldots \\ &Y_4 = 0, 8, 16, 24, \ldots \end{aligned}$$

Thus the required generating function is

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^4 + x^8 + x^{12} + \cdots)(1 + x^8 + x^{16} + x^{24} + \cdots).$$

Example 2.7. Find the generating function for the problem of distributing ten identical balls into four boxes such that each box contains an even number of balls.

Solution. The corresponding integer equation is

$$X_1 + X_2 + X_3 + X_4 = r,$$

where $X_i = 0, 2, 4, 6, \ldots$ for $(1 \leq i \leq 4)$. (Note that while the question asks for the number of ways of distributing ten balls, the generating function will in fact describe the solution in the case of r balls, for all positive r.) The generating function for this problem is therefore

$$(1 + x^2 + x^4 + x^6 + \cdots)^4.$$

Example 2.8. Find the generating function for the number of 6-multisets of $\{A, B, C, D, E\}$ containing at most two As, at least one B and any number of Cs, Ds and Es.

Solution. If we let the number of As, Bs, Cs, Ds and Es be denoted by X_A, X_B, X_C, X_D and X_E, then we get the integer equation

$$X_A + X_B + X_C + X_D + X_E = r,$$

where $X_A \in \{0, 1, 2\}$, $X_B \geq 1$ and $X_C, X_D, X_E \geq 0$. Thus the required generating function is

$$(1 + x + x^2)(x + x^2 + x^3 + \cdots)(1 + x + x^2 + \cdots)^3.$$

Example 2.9. Find the generating function for the number of ways to throw n distinct dice and obtain a sum of r.

Solution. This problem is equivalent to the number of integer solutions of

$$X_1 + X_2 + \cdots + X_n = r,$$

where $1 \leq X_i \leq 6$ $(1 \leq i \leq n)$. Thus the generating function is

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

Exercise 2.10. Find the generating function that describes the number of ways a head-teacher can distribute 24 laptops to four class teachers so that each teacher receives at least three, but not more than eight laptops.

Exercise 2.11. A unit of 20 special forces is assembled for operations in a foreign country. Three nations will have soldiers in the unit. There must be at least 11 U.S. soldiers, at least one, but no more than three Australian soldiers, and the remainder must be British soldiers. Express the number of ways of doing this in terms of an integer equation, and write down the corresponding generating function.

1. $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^n$
2. $(1-x^m)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{mr}$
3. $\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$
4. $(1+x+x^2+\cdots)^n = \sum_{r=0}^{\infty} \binom{n-1+r}{r} x^r$
5. $(1+x+\cdots+x^{m-1}) = (1-x^m)(1+x+x^2+\cdots)$

Figure 2.1: Helpful identities for simplifying generating functions

2.3 Finding the Coefficients of a Generating Function

Now that we know how to write down an expression for the generating function associated with a counting problem, we need to be able to simplify the generating function so that we can determine the coefficients and thus solve the problem. Figure 2.1 contains a list of identities that are useful for this purpose. The first three follow from results we have seen already; it would be a useful exercise for you to try and see if you can show why the remaining two results are true.

Example 2.12. Find the coefficient of x^{15} in $(x^2+x^3+x^4+\cdots)^3$.

Solution.

$$\begin{aligned}(x^2+x^3+x^4+\cdots)^3 &= x^6(1+x+x^2+\cdots)^3 \\ &= x^6\left(\binom{2}{0}+\binom{3}{1}x+\cdots+\binom{2+r}{r}x^r\right) \text{ (Identity 4)}\end{aligned}$$

Thus the coefficient of x^{15} is $\binom{11}{9} = 55$.

Example 2.13. Find the coefficient of x^{10} in $(1+x+x^2+x^3)^6$.

Solution.

$$\begin{aligned}(1+x+x^2+x^3)^6 &= (1-x^4)^6(1+x+x^2+\cdots)^6 \text{ (Identity 5)} \\ &= \left(\binom{6}{0}-\binom{6}{1}x^4+\binom{6}{2}x^8-\cdots+\binom{6}{6}x^{24}\right) \\ &\qquad \times (1+x+x^2+\cdots)^6 \text{ (Identity 2)} \\ &= \left(\binom{6}{0}-\binom{6}{1}x^4+\binom{6}{2}x^8-\cdots+\binom{6}{6}x^{24}\right) \\ &\qquad \times \left(\binom{5}{0}+\binom{6}{1}x+\binom{7}{2}x^2+\cdots\right) \text{ (Identity 4)}\end{aligned}$$

Thus the coefficient of x^{10} is

$$\binom{6}{0}\binom{15}{10}-\binom{6}{1}\binom{11}{6}+\binom{6}{2}\binom{7}{2} = 546.$$

Example 2.14. A crisp manufacturer produces selection packs of bags of crisps in four flavours: *Sea Salt*, *Cheesy Chive*, *Italian Herb* and *Hot Spice*. In each selection pack there is at least one bag of each flavour, at most four bags of *Italian Herb* and at most five of *Hot Spice*. How many different possible selection packs are there if each pack contains (1) 6 bags of crisps? (2) 12 bags of crisps? (3) 24 bags of crisps?

Solution Let X_s, X_C, X_I and X_H denote the number of bags of *Sea Salt*, *Cheesy Chive*, *Italian Herb* and *Hot Spice* in each selection pack. Then the problem is equivalent to finding the number of solutions of the integer equation

$$X_S + X_C + X_I + X_H = r,$$

where $X_S \geq 1$, $X_C \geq 1$, $1 \leq X_I \leq 4$ and $1 \leq X_H \leq 5$. The generating function for this problem is

$$\begin{aligned}
&(x + x^2 + x^3 + \cdots)^2(x + x^2 + x^3 + x^4)(x + x^2 + x^3 + x^4 + x^5) \\
&= x^4(1 + x + x^2 + \cdots)^2(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4) \\
&= x^4(1 + x + x^2 + \cdots)^2(1 - x^4)(1 + x + x^2 + \cdots)(1 - x^5)(1 + x + x^2 + \cdots) \text{ (Identity 5)} \\
&= x^4(1 - x^4 - x^5 + x^9)(1 + x + x^2 + \cdots)^4 \\
&= x^4(1 - x^4 - x^5 + x^9)\left(\binom{3}{0} + \binom{4}{1}x + \binom{5}{2}x^2 + \cdots + \binom{3+r}{r}x^r + \cdots\right) \text{ (Identity 4)}
\end{aligned}$$

1. The number of different selection packs with 6 bags of crisps is just the coefficient of x^6 in the generating function. Thus there are

$$\binom{5}{2} = 10$$

different selection packs in this case.

2. The number of different selection packs with 12 bags of crisps is just the coefficient of x^{12} in the generating function. Thus there are

$$\binom{11}{8} - \binom{7}{4} - \binom{6}{3} = 110$$

different selection packs in this case.

3. The number of different selection packs with 24 bags of crisps is just the coefficient of x^{24} in the generating function. Thus there are

$$\binom{23}{20} - \binom{19}{16} - \binom{18}{15} + \binom{14}{11} = 350$$

different selection packs in this case.

Exercise 2.15.

1. Express the following problem in terms of an integer equation, and hence determine its generating function: how many ways are there of making r pence in change using only 1p, 2p and 5p coins?

2. What is the coefficient of x^9 in the following generating function?

$$(x + x^2 + x^3 + ...)(x^3 + x^4 + x^5 + x^6)$$

3. In how many ways can I fill my fruit bowl with 15 pieces of fruit if the only fruit available are apples, oranges and bananas, and I must use at most four bananas and at least three apples?

4. In how many ways can I order 10 pizzas for a party if the available kinds are hawaiian, vegetarian, or pepperoni and I want to have between 1 and 5 pizzas of each kind?

2.4 Application of Generating Functions to Sequences

We have seen how generating functions can be used to solve counting problems, but this is not their only application within discrete mathematics. In this section we will explore how they can be exploited in the study of sequences.

If we are given a sequence $(a_r)_{r=0}^{\infty}$, we can associate it with the generating function

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$$

Example 2.16. What is the generating function of the sequence $1, 1, 1, \ldots$?

Solution. By the above definition, the generating function is given by

$$\sum_{r=0}^{\infty} x^r = (1 - x)^{-1}.$$

Example 2.17. What is the generating function of the sequence $(a_r)_{r=0}^{\infty}$, where $a_r = a^r$?

Solution. If we replace x by ax in the previous example, we see that the required generating function is

$$\sum_{r=0}^{\infty} a^r x^r = (1 - ax)^{-1}$$

A useful way of finding the generating function of a sequence is to try and construct it from other sequences whose generating functions we already know. Figure 2.2 gives a (by no means exhaustive) list of rules that can be used in this process.

Example 2.18. Find the generating function for the sequence $(a_r)_{r=0}^{\infty}$ with $a_r = 3r + 2$.

Solution. Let $g(x) = (1 - x)^{-1} = 1 + x + x^2 + \cdots$, which is the generating function of the sequence $(b_r)_{r=0}^{\infty}$ with $b_r = 1$. Using Rule 1, we have that

$$g_1(x) = xg'(x) = x(1 - x)^{-2}$$

is the generating function for the sequence $(c_r)_{r=0}^{\infty}$ with $c_r = r$.

Suppose $g(x)$ and $h(x)$ are the generating functions for the sequences $(a_r)_{r=0}^{\infty}$ and $(b_r)_{r=0}^{\infty}$ respectively. Then we have the following results:

1. The generating function for the sequence $(ra_r)_{r=0}^{\infty}$ is $xg'(x)$;
2. The generating function for the sequence
$$(a_0 + a_1 + a_2 + \cdots + a_r)_{r=0}^{\infty} = \left(\sum_{i=0}^{r} a_i\right)_{r=0}^{\infty}$$
is $\frac{g(x)}{1-x}$;
3. For constants C_1 and C_2, the generating function for the sequence $(C_1 a_r + C_2 b_r)_{r=0}^{\infty}$ is $C_1 g(x) + C_2 h(x)$;

Figure 2.2: Rules for transforming the generating functions of sequences

Thus, using Rule 3, the generating function for $(a_r)_{r=0}^{\infty}$ with $a_r = 3r + 2$ is

$$\begin{aligned}
3g_1(x) + 2g(x) &= \frac{3x}{(1-x)^2} + \frac{2}{(1-x)} \\
&= \frac{2+x}{(1-x)^2}.
\end{aligned}$$

Example 2.19. The sequence $(a_r)_{r=0}^{\infty}$ has generating function $f(x) = e^{2x} - e^x$.

1. What are the first five terms of the sequence?
2. What is the generating function for the sequence $(ra_r)_{r=0}^{\infty}$?
3. What are the first five terms of the sequence $(ra_r)_{r=0}^{\infty}$?

Solution.

1.
$$\begin{aligned}
f(x) &= e^{2x} - e^x \\
&= \left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots\right] - \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right] \\
&= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right) \\
&= 0 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{5}{8}x^4 + \cdots
\end{aligned}$$
So the first five terms are $0, 1, \frac{3}{2}, \frac{7}{6}, \frac{5}{8}$.
2. The generating function for $(ra_r)_{r=0}^{\infty}$, using Rule 1, is $xf'(x) = x\left[2e^{2x} - e^x\right]$.

3. **Either** from 2. the generating function is

$$\begin{aligned}
&x\left[2\left(1+2x+2x^2+\frac{4}{3}x^3+\cdots\right)-\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots\right)\right]\\
&=x\left[1+3x+\frac{7}{2}x^2+\frac{5}{2}x^3+\cdots\right]\\
&=0+x+3x^2+\frac{7}{2}x^3+\frac{5}{2}x^4+\cdots
\end{aligned}$$

so the first five terms are $0, 1, 3, \frac{7}{2}, \frac{5}{2}$.

or $(a_r)_{r=0}^{\infty}$ is the sequence $0, 1, \frac{3}{2}, \frac{7}{6}, \frac{5}{8}, \ldots$ by 1., so$(ra_r)_{r=0}^{\infty}$ is the sequence

$$0\times 0, 1\times 1, 2\times\frac{3}{2}, 3\times\frac{7}{6}, 4\times\frac{5}{8}, \ldots,$$

i.e. $0, 1, 3, \frac{7}{2}, \frac{5}{2}, \ldots$

Example 2.20. Use generating functions to find an expression for the sum of the first n squares.
Solution Let $g(x)$ be the generating function for the sequence of squares $g(x)=\sum_{r=1}^{\infty} r^2x^r$.

We know that $(1-x)^{-1}=\sum_{r=0}^{\infty} x^r$. In order to find an expression for $g(x)$, we apply Rule 1 twice to this sequence: first we obtain that

$$\begin{aligned}
\sum_{r=0}^{\infty} rx^r &= x\frac{d}{dx}\frac{1}{(1-x)}\\
&=\frac{x}{(1-x)^2}.
\end{aligned}$$

Applying Rule 1 a second time yields

$$\begin{aligned}
g(x) &= \sum_{r=0}^{\infty} r^2x^r\\
&= x\frac{d}{dx}\frac{x}{(1-x)^2}\\
&= \frac{x(1+x)}{(1-x)^3}.
\end{aligned}$$

By Rule 2, the generating function for the sequence $\left(\sum_{i=0}^{r} i^2\right)_{r=0}^{\infty}$ whose r^{th} term is the sum of the first r squares is given by $g(x)(1-x)^{-1}=x(1+x)(1-x)^{-4}$.

In order to solve the problem we now need to find an expression for the coefficient of x^n in this generating function. By Identity 4, we observe that this function is equal to

$$\begin{aligned}
x(1+x)\left(\sum_{r=0}^{\infty}\binom{3+r}{r}x^r\right) &= \sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+1}+\sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+2} \\
&= \sum_{r=1}^{\infty}\binom{2+r}{r-1}x^r+\sum_{r=2}^{\infty}\binom{1+r}{r-2}x^r \\
&= x+\sum_{r=2}^{\infty}\left[\binom{2+r}{r-1}+\binom{1+r}{r-2}\right]x^r \\
&= x+\sum_{r=2}^{\infty}\left[\frac{(r+2)(r+1)r}{6}+\frac{(r+1)r(r-1)}{6}\right]x^r \\
&= x+\sum_{r=2}^{\infty}\frac{(2r+1)(r+1)r}{6}x^r.
\end{aligned}$$

We see that the expression of the coefficient of x^n in this generating function agrees with the expression obtained in Section 1.1 for the sum of the first n squares. This approach can be used to find a similar expression for $\sum_{r=1}^{n} r^k$ for $k = 3, 4, \ldots,$ through k applications of Rule 2.

Exercise 2.21.

1. Find the generating function for the sequence $(a_r)_{r=0}^{\infty}$ given by $a_r = r^2 + r + 1$.
2. Find the generating function for the sequence $-1, 2, -3, 4, -5, 6 \ldots$
   (Hint: can you write down an expression for a general term of this sequence? Compare with Example 2.18)

Learning Outcomes

After completing this chapter and the related problems you should be able to:

- be able to find the generating function of an integer equation;
- know how to model certain counting problems with integer equations and find the corresponding generating function for the problem;
- be able to apply the identities given in order to find the coefficient of a specified power of x in a generating function;
- understand the relationship between generating functions and sequences;
- be able to use the rules given to find the generating functions of particular sequences;
- be able to determine the sequence represented by a given generating function.