Calculus 2, Assignment 4

BM Corser

1. (a) In this context, k cannot be 0, since $k = 0$ implies there is no relationship between $\frac{\mathrm{d}T}{\mathrm{d}t}$ and $T_s - T$. This gives us $|k| > 0$. However, I don't see a "colloquial" reason for $k > 0$, since either or both of T_s and T can be negative. I can see it's in some sense meaningless to take a negative factor, since if a is a factor of b then $-a$ is also a factor of b.

   ...

   Also, having found T, it is clear that (for the model to be sane) as $t \to \infty$, $T \to T_s$ and so we can write

$$\text{when } T_s \geq T, T_s - T \geq 0, \frac{\mathrm{d}T}{\mathrm{d}t} \geq 0 \text{ therefore } k > 0$$

   and

$$\text{when } T_s \leq T, T_s - T \leq 0, \frac{\mathrm{d}T}{\mathrm{d}t} \leq 0 \text{ therefore } k > 0.$$

   (b) $\frac{\mathrm{d}T}{\mathrm{d}t} = k(T_s - T)$ is a first order variables separable ordinary differential equation and as such

$$\begin{aligned}
\int \frac{1}{T_s - T}\mathrm{d}T &= k \int \mathrm{d}t \\
-\ln(T_s - T) &= k(t + c) \text{ where } c \text{ is of the form } a + \tfrac{b}{k} \\
T_s - T &= e^{-k(t+c)} \\
T &= T_s - e^{-k(t+c)}.
\end{aligned}$$

   If $T = T_0$ when $t = 0$

$$\begin{aligned}
T_0 &= T_s - e^{-kc} \\
e^{-kc} &= T_s - T_0 \\
c &= -\frac{\ln(T_s - T_0)}{k}
\end{aligned}$$

so

$$
\begin{aligned}
T &= T_s - e^{-k\left(t - \frac{\ln(T_s - T_0)}{k}\right)} \\
&= T_s - e^{\ln(T_s - T_0) - kt} \\
&= T_s - \frac{e^{\ln(T_s - T_0)}}{e^{kt}} \\
&= T_s - \frac{T_s - T_0}{e^{kt}}.
\end{aligned}
$$

(c) We are given $T(0) = 37$ and $T_s = 24$.

i. Let the amount of time between death and discovery be A, now $T(A) = 34$, $T(A + 30) = 32$ and

$$
\begin{aligned}
e^{kA} &= \frac{13}{10} \\
kA &= \ln\left(\tfrac{13}{10}\right), \\
e^{k(A+30)} &= \frac{13}{8} \\
kA + k30 &= \ln\left(\tfrac{13}{8}\right).
\end{aligned}
$$

Substituting our value for kA into the second equation

$$
\begin{aligned}
k &= \frac{\ln\left(\tfrac{13}{8}\right) - \ln\left(\tfrac{13}{10}\right)}{30} \\
&= \frac{\ln\left(\tfrac{5}{4}\right)}{30}.
\end{aligned}
$$

ii. With a value for k,

$$
T = T_s - (T_s - T_0) \cdot \exp\left(-\frac{A \ln\left(\tfrac{5}{4}\right)}{30}\right)
$$

and

$$
\begin{aligned}
34 &= 24 - (-13) \cdot \exp\left(-\frac{A \ln\left(\tfrac{5}{4}\right)}{30}\right) \\
A &= -\frac{30 \ln\left(\tfrac{10}{13}\right)}{\ln\left(\tfrac{5}{4}\right)} \\
&\approx 35.2729347...,
\end{aligned}
$$

which tells us the time of death was about 35 minutes before high noon.

iii. I was returning some video tapes.

iv. Since the model we have considers surrounding temperature to be constant, I have found in the past that the turning the thermostat up or down before leaving the scene of the crime works nicely. What? Sorry, I have to return some video tapes.

2. (a) i. Since f is even, $g(f(-x)) = g(f(x))$ and so $g \circ f$ is even. Similarly, $f(g(-x)) = f(-g(x)) = f(g(x))$ so $f \circ g$ is also even.

ii. $f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x)$ so $f(x)g(x)$ is odd. Similarly, $\frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -\frac{f(x)}{g(x)}$ and so $\frac{f(x)}{g(x)}$ is also odd.

iii. The definition of the chain rule is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$. Since $f(x) = f(-x)$, if we let $y = f$ and $u = -x$ then

$$\begin{aligned}
\frac{\mathrm{d}f(-x)}{\mathrm{d}x} &= \frac{\mathrm{d}f(u)}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}, \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= -1, \\
\frac{\mathrm{d}f(-x)}{\mathrm{d}x} &= \frac{\mathrm{d}f(u)}{\mathrm{d}u} \cdot (-1) \\
&= -\frac{\mathrm{d}f(x)}{\mathrm{d}x}.
\end{aligned}$$

Similarly, since $g(-x) = -g(x)$,

$$\begin{aligned}
\frac{\mathrm{d}g(-x)}{\mathrm{d}x} &= \frac{\mathrm{d}g(u)}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}, \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= -1, \\
\frac{\mathrm{d}g(-x)}{\mathrm{d}x} &= \frac{\mathrm{d}g(u)}{\mathrm{d}u} \cdot (-1) \\
&= \frac{\mathrm{d}g(-x)}{\mathrm{d}x} \cdot (-1) \\
&= \frac{\mathrm{d}(-g(x))}{\mathrm{d}x} \cdot (-1) \\
&= (-1) \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x} \cdot (-1) \\
&= \frac{\mathrm{d}g(x)}{\mathrm{d}x}
\end{aligned}$$

(b) i. $h_e(-x) = \frac{1}{2}(h(-x)+h(x)) = h_e(x)$ and h_e is even, also $h_o(-x) = \frac{1}{2}(h(-x) - h(x)) = -h_o(x)$ and h_o is odd.

ii. $f(x) = x^2 + x$, f is their even nor odd. Let $h_e = x^2$ and $h_o = x$, h_e is even, h_o is odd, $f(x) = h_e(x) + h_o(x)$.

3. (a)

$$\begin{aligned}
\ddot{x} &= t^2 + bt + c \\
\dot{x} &= \frac{t^3}{3} + b\frac{t^2}{2} + ct \\
x &= \frac{t^4}{12} + b\frac{t^3}{6} + ct^2
\end{aligned}$$

(b) Since we are told that the particle is "at rest" when $t = 0$, we know $c = ct = ct^2 = 0$ (this is also why I have ignored other constants of integration above). Now let $b = -\frac{1}{2}$ and consider our equation for displacement at $t = 1$.

$$\begin{aligned}
x(1) &= \frac{1}{12} + (-\frac{1}{2}) \cdot \frac{1}{6} + 0 \\
&= \frac{1}{12} - \frac{1}{12} \\
&= 0
\end{aligned}$$

(c) As $t \to \infty$ all of $\ddot{x}$, $\dot{x}$ and x will also go to infinity; the particle will have infinite acceleration, achieve infinite speed and travel infinitely far.

4.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3xy \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y \\
&= 9x^2y + 3y \\
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 3\frac{\mathrm{d}y}{\mathrm{d}x} + 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 6\frac{\mathrm{d}y}{\mathrm{d}x} + 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
&= 18x^2y + 27x^3y + 9xy
\end{aligned}$$

and

$$y_{i+1} = y_i + h\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{h^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{h^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3}.$$

Writing a Python program that expresses the above (and is is general over derivative terms, see the `terms` variable) might look like the following

```
1  from math import factorial
2  
3  def d1(x, y):  # first derivative
4      return 3 * x * y
5  
6  def d2(x, y):  # second derivative
7      return 3 * x * d1(x, y) + 3 * y
8  
9  def d3(x, y):  # third derivative
10     return 6 * d1(x, y) + 3 * x * d2(x, y)
11 
12 terms = [d1, d2, d3]
13 initial_condition = 0.5
14 step = 0.1
15 
16 def x(i):
17     return step * i
18 
19 def h(i):
20     return pow(step, i) / factorial(i)
21 
22 def euler(i, n, d):
23     return h(n) * d(x(i), y(i))
24 
25 def y(i_1):  # i_1 means i + 1
26     if i_1 == 0:
27         return initial_condition
28     i = i_1 - 1
29     return y(i) + sum([
30         euler(i, n + 1, d) for n, d in enumerate(terms)
31     ])
32 
33 tab_fmt = "{0} & {1} & {2:.6f} \\\\"
34 for i in range(0, 5):
35     print(tab_fmt.format(i, x(i), y(i)))
```

Lines 33-35 format output suitable for LaTeX as follows

i	x_i	y_i
0	0.0	0.500000
1	0.1	0.507500
2	0.2	0.530797
3	0.3	0.572059
4	0.4	0.635283