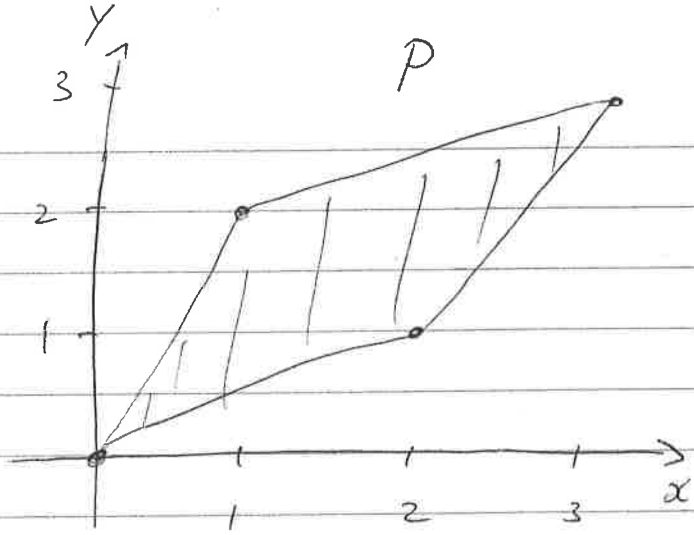
Calculus 2

Assignment 2

Solutions

1\.
(a)



$$x = \frac{2}{3}u + \frac{v}{3}$$

$$y = \frac{1}{3}u + \frac{2}{3}v$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/3 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

in matrix form: $\underline{x} = A\underline{u}$

so $\underline{u} = A^{-1}\underline{x} = \frac{1}{|A|}\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$= \frac{1}{\frac{4}{9} - \frac{1}{9}}\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

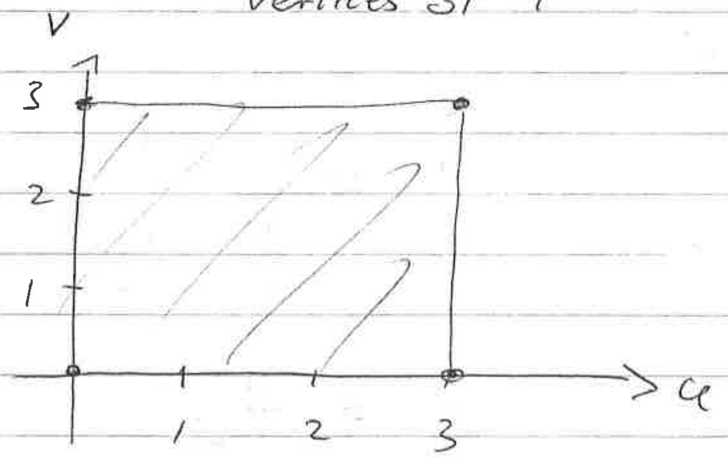
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow u = 2x - y$
$v = -x + 2y$

(or solve simultaneously to get this)

So to find the vertices of P' we can apply A^{-1} to the vertices of P.

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 2 & 3 & 1 \end{pmatrix}}_{\text{vertices of } P} = \underbrace{\begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 3 & 3 & 0 \end{pmatrix}}_{\text{vertices of } P'}$$



$$\int_0^3 \int_0^3 (\qquad) \, du \, dv$$

So P' is the square in the uv-plane shown here.

1 (b) To calculate the Jacobian we evaluate

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det \begin{pmatrix} 2/3 & \frac{1}{3} \\ \frac{1}{3} & 2/3 \end{pmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

$$1(c)\quad \iint_P e^{x}\,dy\,dx = \iint_{P'} e^{\frac{2u+v}{3}} \cdot \frac{1}{3} \cdot du\,dv$$

$$= \frac{1}{3}\int_0^3\int_0^3 e^{\frac{2u}{3}} e^{\frac{v}{3}}\,du\,dv$$

$$= \frac{1}{3}\int_0^3 \frac{3}{2}\left[e^{\frac{2u}{3}}\right]_0^3 e^{\frac{v}{3}}\,dv$$

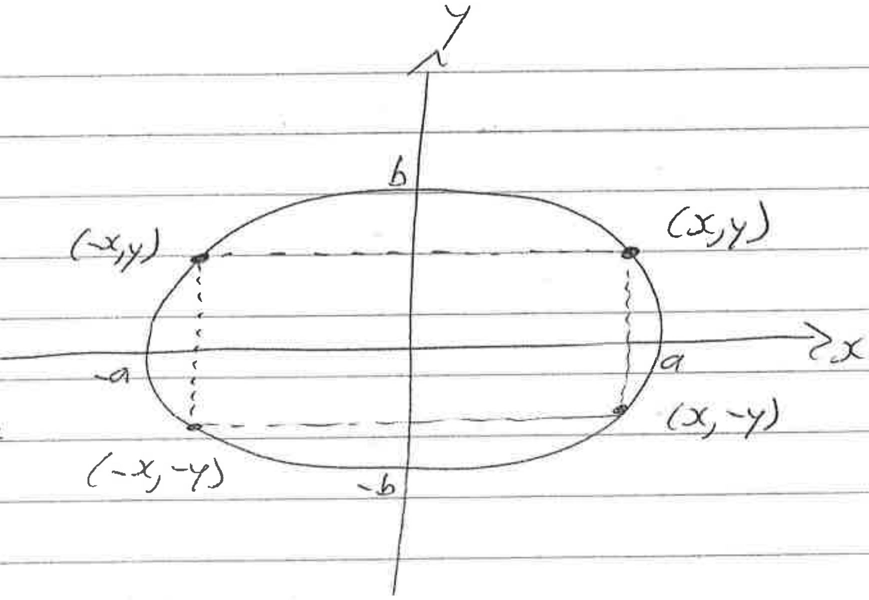
$$= \frac{1}{3}\cdot\frac{3}{2}\left(e^{2}-1\right)\int_0^3 e^{\frac{v}{3}}\,dv$$

$$= \frac{1}{2}\left(e^{2}-1\right)\left[3e^{\frac{v}{3}}\right]_0^3$$

$$= \frac{1}{2}\left(e^{2}-1\right)\cdot 3\left(e^{1}-1\right)$$

$$= \frac{3}{2}\left(e^{2}-1\right)\left(e-1\right).$$

2(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Vertices of R are
(x,y), $(x,-y)$
$(-x,y)$, $(-x,-y)$.

area rectangle $= f(x,y) =$ width $\times$ height

$$= 2x - 2y$$

$$= 4xy.$$

(b) maximise $f(x,y) = 4xy$ subject to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = g(x,y)$$

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

$$= 4xy - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$L_x = 4y - \frac{2\lambda x}{a^2} = 0 \quad \text{——— } ①$$

$$L_y = 4x - \frac{2\lambda y}{b^2} = 0 \quad \text{——— } ②$$

Rearrange ① and ② and divide to get:

$$\frac{y^2}{x^2} = \frac{b^2}{a^2} \Rightarrow \frac{y}{x} = \frac{b}{a}$$

$$\Rightarrow y = \frac{b}{a}x$$ (since x, y, a, b all > 0 no issue with -ve roots)

now subst. ~~$y = \frac{b}{a}x$ into the constraint~~

$\frac{y^2}{b^2} = \frac{x^2}{a^2}$ into ellipse

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \Rightarrow 2x^2 = a^2$$

$$\Rightarrow x = \frac{a}{\sqrt{2}}$$

similarly $y = \frac{b}{\sqrt{2}}$

So point of interest is $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$, call it P.

Need to check P is a max not min.

At P, $f\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) = 4\frac{ab}{\sqrt{2}\sqrt{2}} = 2ab > 0$

but $f(a, 0) = 0$ $\therefore$ P is a max and max area is $2ab$.

2(c) (i) $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ subst. into

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{to get} \quad \underbrace{u^2 + v^2 = 1}_{\text{unit disc, call it } D}$$

(ii) Jacobian : $\begin{vmatrix} \dfrac{\partial x}{\partial u} = a, & \dfrac{\partial x}{\partial v} = 0 \\ \dfrac{\partial y}{\partial u} = 0, & \dfrac{\partial y}{\partial v} = b \end{vmatrix} = ab.$

$$\therefore \iint_E 1 \, dx\,dy = \iint_D 1.ab \, du\,dv = ab \iint_D 1 \, du\,dv$$

$$= ab \int_0^{2\pi}\int_0^1 1 \, r \, dr \, d\theta = ab \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$= ab \int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{ab}{2}\left[\theta\right]_0^{2\pi} = \frac{2\pi ab}{2} - 0$$

$$= \pi ab.$$

(iii) Fraction of E occupied by inscribed rectangle is

$$\frac{f(x,y)}{\pi ab} \leq \frac{\text{area max rectangle}}{\pi ab} = \frac{2ab}{\pi ab}$$

$$= \frac{2}{\pi}$$

3 (a) $h(x,y)$ is stationary at point (a,b) if the tangent plane is horizontal at that point.

ie $h_x(a,b) = h_y(a,b) = 0$.

Then since $h(x,y) = f(x,y) + g(x,y)$

$h_x(a,b) = f_x(a,b) + g_x(a,b) = 0 + 0 = 0$

$h_y(a,b) = f_y(a,b) + g_y(a,b) = 0 + 0 = 0$

$\therefore$ $h(x,y)$ is stationary at (a,b).

(b) (i) $f(a,b)$ & $g(a,b)$ min $\Rightarrow$ $h(a,b)$ min.

This is <u>true</u>.

Write $f(a,b) = M$ and $g(a,b) = N$, some $M, N \in \mathbb{R}$.

Then $f(a,b)$ min $\Rightarrow$ $f(x,y) \geq M$ in some disc centre (a,b) radius $d_1 > 0$

$g(a,b)$ min $\Rightarrow$ $g(x,y) \geq N$ in some disc centre (a,b) radius $d_2 > 0$.

Then let d be whichever of d_1 and d_2 is smaller.

$\therefore$ $h(x,y) = \underbrace{f(x,y)}_{\geq M} + \underbrace{g(x,y)}_{\geq N} \geq M + N$ in the disc centre (a,b) radius d.

(ii) $f(a,b)$ min & $g(a,b)$ saddle $\Rightarrow$ $h(a,b)$ saddle

This is <u>false</u>.

<u>Counterexample</u>:

$f(x,y) = x^2 + y^2$, clearly has min. at $(0,0)$

$g(x,y) = 2xy$, has saddle at $(0,0)$

$h(x,y) = x^2 + y^2 + 2xy$

$= (x+y)^2$, has <u>min</u> at $(0,0)$ not saddle.

4 (a) $w = F(x,y) = 0$

$$\frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx}$$ by chain rule for partial derivatives.

now $\frac{dx}{dx} = 1$ so this becomes:

$$\frac{dw}{dx} = F_x + F_y \frac{dy}{dx}$$

(b) since $w = 0$, the value of w is constant and so does not vary with x. ie $\frac{dw}{dx} = 0$.

Then, $\frac{dw}{dx} = F_x + F_y \frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

(c) $w = F(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$

$$F_x = \frac{2x}{a^2}, \quad F_y = \frac{2y}{b^2}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{a^2} \cdot \frac{b^2}{2y} = -\frac{b^2 x}{a^2 y}.$$

(d) $y = b\left[1-\left(\frac{x}{a}\right)^2\right]^{\frac{1}{2}} = b\left[u\right]^{\frac{1}{2}}, \quad u = 1-\frac{x^2}{a^2}$

By chain rule:

$$\frac{du}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{du} = \frac{1}{2}bu^{-\frac{1}{2}}$$

$$= \frac{1}{2}bu^{-\frac{1}{2}}\cdot\frac{-2x}{a^2}$$

$$= -\frac{b}{a^2}\cdot\frac{x}{\sqrt{1-x^2/a^2}}$$

$$= -\frac{b^2}{a^2}\frac{x}{y}, \quad \text{since } \sqrt{1-\frac{x^2}{a^2}} = \frac{y}{b}.$$