5 Method of series and numerical solutions to differential equations

Sometimes we are faced with an ODE that we cannot solve analytically. In fact, this can happen rather frequently, so we should develop some methods that are able to tackle some of these problems. For example, the simple looking differential equations

$$y' = x^2 + y^2 \qquad \text{and} \qquad xy'' + y' + xy = 0,$$

are impervious to the methods we have developed so far, despite their innocent looks.

Before we move on to some of these methods, let us make one comment on the existence of solutions to differential equations to justify some of these methods. Before, when we found analytical solutions, we were pretty confident that they were in fact solutions to the differential equations. When we are finding numerical solutions, what can justify this? The following is a deep theorem in the theory of differential equations, whose proof is beyond the scope of this module.

Theorem 5.1 (Existence and uniqueness of solutions). *Consider the first order differential equation $y' = M(x, y)$. Suppose that M and $\frac{\partial M}{\partial y}$ are continuous in some region R of the (x, y)-plane (of course, R can be the whole plane). Then there exists one and only one solution $y = g(x)$ which passes through any given point in R (specifying a point in R is the same as specifying an initial condition).*[1]

What this theorem tells us is that there is some solution out there. One way to think of this solution is that it is some curve in the (x, y)-plane, but doesn't necessarily have a "nice" functional description. However, we can estimate points on this curve using numerical methods.

We begin with the method of series for solving differential equations. This is not a numerical method, but has a close relationship with numerical methods.

5.1 Method of series

In the **method of series** for solving differential equations we assume that a solution exists that can be written as a suitable infinite series. We then use the differential equation to determine the coefficients of the series.

5.1.1 Basic method of series

Recall that the solution to the separable differential equation

$$y' = y \quad \text{with} \quad y(0) = 1 \tag{1}$$

[1]Higher derivative versions of this theorem exist.

is given by $y = e^x$ (see Example 4.6 in Chapter 4). We know that e^x has a series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ,$$

which is of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \tag{2}$$

This raises a question. Is it possible to assume a solution of the form (2) and obtain a solution to the differential equation (1)? Let's give it a try.

Suppose that $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is a solution to the differential equation (1). Then, differentiating $y(x)$ term by term[2], we have

$$\begin{aligned} a_1 + 2a_2 x + 3a_3 x^2 + \cdots &= y'(x) \\ &= y(x) \\ &= a_0 + a_1 x + a_2 x^2 + \cdots . \end{aligned}$$

The series $a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ is therefore equal to the series $a_0 + a_1 x + a_2 x^2 + \cdots$, and we equate coefficients of like terms. We get that

$$\begin{aligned} a_1 &= a_0 \\ a_2 &= a_1/2 = a_0/2 \\ a_3 &= a_2/3 = a_0/(2 \cdot 3) \end{aligned}$$

and so on. We therefore see that

$$a_n = a_0/n!.$$

Thus,

$$y(x) = a_0 \left(1 + x + \frac{x^2}{2!} + \cdots \right).$$

From the initial condition, we see that $a_0 = 1$ (we could have just used this information at the start). Hence $y(x) = 1 + x + \frac{x^2}{2!} + \cdots$ which is just e^x.

There are certain issues with this method, as there are with any computations involving infinite series. For example, does the solution even make sense? This question mainly pertains to whether or not the series obtained is valid (i.e. converges for some values x). For example, the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is valid, or converges, for any value x. In contrast, the series

$$1 + x + x^2 + x^3 + \cdots ,$$

which is the Taylor series of $\frac{1}{1-x}$, is only valid for $-1 < x < 1$. Another question is can we differentiate such a series term by term as we did in the above example?

[2] Is this allowed? See later discussion.

It turns out that if we have a series which is valid for a certain range of x, then within that range these operations and assumptions are just fine. We will, however, not bother with these subtleties right now, but will use this method nonetheless. Of course, if you find a series solution and it is the series of a known function (like e^x or $\sin x$), then you know that it converges for some x. Furthermore, plugging it into your original differential equation and showing that it satisfies the equation will show that it is a solution to your differential equation.

We mention one other subtle point. Sometimes we guess a series solution but the differential equation has no solution of that form. For example, the solution to the equation

$$y' = \frac{1}{x}$$

with $y(1) = 0$ is $y = \ln x$. Now $\ln x$ cannot be written as a power series about 0, so assuming a nice series solution like the one we have been assuming (or the Taylor series solution in the next section) won't work (give it a try if you don't believe me!). More complexity occurs if part of the solution to a differential equation has such a series expansion, but another part doesn't (say the solution to a second order differential equation is $e^x + \ln x$, and e^x and $\ln x$ are the two independent solutions). As mentioned, however, we will not deal with these subtleties or complex cases.

Example 5.2. Find the solution to $y' = xy$, where $y(0) = 1$ using the series method. If the solution has a closed form, find it.

We assume a solution of the form $y(x) = a_0 + a_1x + a_2x^2 + \cdots$. Since $y(0) = 1$, we see that $a_0 = 1$. Furthermore, using the differential equation we see that

$$\begin{aligned}
a_1 + 2a_2x + 3a_3x^2 + \cdots &= y' \\
&= xy \\
&= x(a_0 + a_1x + a_2x^2 + \cdots) \\
&= a_0x + a_1x^2 + a_2x^3 + \cdots .
\end{aligned}$$

Thus, the first equation is equal to the last and equating coefficients of like terms we have

$$\begin{aligned}
a_1 &= 0 \\
2a_2 &= a_0 = 1 \quad \Rightarrow \quad a_2 = 1/2 \\
3a_3 &= a_1 = 0 \quad \Rightarrow \quad a_3 = 0 \\
4a_4 &= a_2 = 1/2 \;\Rightarrow \; a_4 = 1/8.
\end{aligned}$$

It is easy to see that $a_i = 0$ when i is odd. When i is even, we see that

$$\begin{aligned}
ia_i &= a_{i-2} \\
\Rightarrow \quad \frac{a_i}{a_{i-2}} &= \frac{1}{i}.
\end{aligned}$$

Setting $i = 2n$ for even i we see that

$$\begin{aligned}
a_{2n} &= \frac{a_{2n}}{a_{2n-2}} \cdot \frac{a_{2n-2}}{a_{2n-4}} \cdots \frac{a_2}{a_0} \\
&= \frac{1}{2n} \cdot \frac{1}{2n-2} \cdot \frac{1}{2n-4} \cdots \frac{1}{2} \\
&= \frac{1}{2^n n(n-1)\cdots 1} \\
&= \frac{1}{2^n n!}.
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
y(x) &= 1 + \frac{x^2}{2^1 1!} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \cdots \\
&= 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \cdots.
\end{aligned}$$

This is the series for $e^{x^2/2}$.

Of course, the equation in Example 5.2 is easily solvable since it is separable. The power of this technique is that it can be used to solve equations like $y' = x^2 + y^2$.

5.1.2 Method of Taylor series

In the last section we assumed a solution of the form $y = a_0 + a_1 x + a_2 x^2 + \cdots$ and often obtained a Taylor series. We can go right ahead and assume that the solution has a Taylor series form

$$y(x) = y(a) + y'(a)(x-a) + y''(a)\frac{(x-a)^2}{2!} + \cdots.$$

There are a couple of advantages to this method over the method of (plain) power series. First, you can avoid a lot of unnecessary equating of coefficients. Second, you can tailor the series (pun intended!) to fit your initial conditions. For example, if your initial condition is given as $y(1) = c$, it is convenient to use a Taylor series about $a = 1$.

Example 5.3. Solve the equation

$$y' = x + y + 1,$$

where $y(0) = 1$. We see that we can easily compute a series about $x = 0$. We have

$$y'' = 1 + y', \qquad y''' = y'', \qquad y'''' = y''' \tag{3}$$

and so on. Hence $y'(0) = 0 + y(0) + 1 = 2$, $y''(0) = 1 + y'(0) = 3$, $y'''(0) = y''(0) = 3$, etc. We therefore see that

$$\begin{aligned}
y(x) &= 1 + 2x + 3\frac{x^2}{2!} + 3\frac{x^3}{3!} + \cdots \\
&= 3(e^x - x - 1) + 2x + 1 \\
&= 3e^x - x - 2.
\end{aligned}$$

Notice that was relatively painless.

Example 5.4. Solve the equation

$$y' = x + y + 1,$$

where $y(1) = 1$. In this case, it may be more appropriate to use the Taylor series about $x = 1$. We use the computations (3) and get that $y'(1) = 3$, $y''(1) = 4$, $y'''(1) = 4$ and so on. Therefore we have

$$\begin{aligned} y(x) &= 1 + 3(x-1) + 4\frac{(x-1)^2}{2!} + 4\frac{(x-1)^3}{3!} + \cdots \\ &= 4(e^{x-1} - (x-1) - 1) + 1 + 3(x-1) \\ &= 4e^{x-1} - x - 2. \end{aligned}$$

Exercises 5.1.

1. Find the solution by method of series. Find a closed form if you can. Of course, if initial conditions aren't given, find the general solution.
   (a) $y' = -y$, $y(0) = 1$
   (b) $y' = y - x$
   (c) $y'' + y = 0$. This is second order, but the method is the same.
2. Try a series solution of the form $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ to the differential equation $y' = 1/x^2$ (this won't work). Solve the equation analytically to see why a series solution didn't work.
3. Find the solution using the method of Taylor series about the given point a. Find a closed form for the series if possible.
   (a) $y' = y$, $y(0) = 4$, $a = 0$
   (b) $x^2 y' = 1$, $y(1) = 1$, $a = 1$
   (c) $y'' + y = \sin x$, $y(0) = y'(0) = 0$, $a = 0$. Verify the solution analytically.

5.2 Numerical methods

Here we consider numerical methods for finding the approximate solution of the ordinary first order differential equation

$$\frac{dy}{dx} = M(x, y), \tag{4}$$

with initial condition $y(x_0) = y_0$.

In particular, for a fixed $h > 0$ (called the step length) we want to estimate the value of y for $x_1, x_2, x_3, \ldots$ where $x_{i+1} = x_i + h$. So, $x_1 = x_0 + h, x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 3h$ and in general $x_i = x_0 + ih$. We denote by y_i, y_i', y_i'' the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the value x_i.

To be clear, in principle, there is some function y out there by Theorem 5.1 that we may not be able to find analytically. However, we can find the behaviour of y by finding a suitable number of pairs (x_i, y_i) which approximate y.

5.2.1 Euler's method

Recall that

$$\frac{dy}{dx} = \lim_{k\to 0} \frac{y(x+k) - y(x)}{k}.$$

Putting $x = x_i$ gives

$$\frac{dy}{dx} = \lim_{k\to 0} \frac{y(x_i+k) - y(x_i)}{k}.$$

For small h, however, we have the following approximation

$$\lim_{k\to 0} \frac{y(x_i+k) - y(x_i)}{k} \simeq \frac{y(x_i+h) - y(x_i)}{h} = \frac{y_{i+1} - y_i}{h}.$$

Therefore, we see that for small h

$$\frac{dy}{dx} \simeq \frac{y_{i+1} - y_i}{h}.$$

Of course, the smaller h is, the more accurate the previous equation is. Now, at x_i, we have $\frac{dy}{dx}|_{x=x_i} = M(x_i, y_i)$ and it follows that

$$M(x_i, y_i) \simeq \frac{y_{i+1} - y_i}{h},$$

from which we obtain the following relation

$$y_{i+1} \simeq y_i + hM(x_i, y_i). \tag{5}$$

The relation in (5) is central to **Euler's method**, and is probably the simplest method for finding the approximate solutions of a first order differential equation.

Example 5.5. Consider

$$\frac{dy}{dx} = 2xy, \tag{6}$$

with $y(0) = 0.5$. We will estimate y at $x = 0.3$ using a step length of i) 0.1 and ii) 0.05.

In the first case, we set $h = 0.1$, so that $y_{i+1} \simeq y_i + 0.2x_iy_i$. Table 1 gives the values of x_i and y_i. The first set of values (i.e. the first line in the table) is given by the initial condition, so we know that is exact, and this will make the starting point for our iteration. Hence, at $x = 0.3$ we see that $y \simeq 0.5304$.

i	x_i	y_i
0	0	0.5
1	0.1	0.5
2	0.2	0.51
3	0.3	0.5304

Table 1: The values of x_i and y_i for step size $h = 0.1$.

i	x_i	y_i
0	0	0.5
1	0.05	0.5
2	0.1	0.5025
3	0.15	0.507525
4	0.2	0.515138
5	0.25	0.525441
6	0.3	0.538577

Table 2: The values of x_i and y_i for step size $h = 0.05$.

Next, setting $h = 0.05$ we have $y_{i+1} \simeq y_i + 0.1x_iy_i$. Table 2 gives the values of x_i and y_i. So, at $x = 0.3$ we find $y \simeq 0.538577$.

Incidentally, the differential equation in (6) is easy to solve analytically using the techniques of Chapter 4. It is separable and the general solution is $y = A\exp(x^2)$. We use the initial condition $x = 0$ and $y = 0.5$ to find that $A = 1/2$. At $x = 0.3$, we have $y = \frac{1}{2}\exp(0.3^2) = 0.547087142$.

5.2.2 Higher derivative Euler methods

The following method is similar to Euler's method, except we use not only the first derivative in our approximation, but we use higher derivatives. Again consider the differential equation (4). Consider now the Taylor series of y centred at x_i. Then, truncating the series after the $(n+1)^{\text{th}}$ term, we obtain the approximation

$$y(x_i + h) \simeq y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \cdots + \frac{h^n}{n!}y^{(n)}(x_i).$$

Thus, we have

$$y_{i+1} \simeq y_i + hy'_i + \frac{h^2}{2!}y''_i + \cdots + \frac{h^n}{n!}y^{(n)}_i. \tag{7}$$

Taking only the first two terms gives $y_{i+1} \simeq y_i + hy'_i = y_i + hM(x_i, y_i)$, so we get the equation in Euler's method. Taking more terms of the series in (7) produces, in general, better approximations. However, this is only practical if M is a relatively simple function. As usual, we have to trade some work for better approximations.

Example 5.6. Consider again the differential equation $\frac{dy}{dx} = 2xy$, with $y(0) = 0.5$. We will estimate y at $x = 0.3$ using a step length of $h = 0.1$ by taking i) the first three terms of the Taylor series; and ii) the first four terms of the Taylor series.

Note first that $\frac{dy}{dx} = 2xy$, so we must use the product rule to find the higher derivatives. Thus,

$$\frac{d^2y}{dx^2} = 2y + 2x\frac{dy}{dx} = 2y + 2x(2xy) = 2y(1 + 2x^2),$$
$$\frac{d^3y}{dx^3} = 2\frac{dy}{dx}(1 + 2x^2) + 2y(4x) = 4xy(1 + 2x^2) + 8xy = 4xy(3 + 2x^2).$$

For the first three terms of the Taylor series, we get

$$\begin{aligned} y_{i+1} &\simeq y_i + hy_i' + \frac{h^2}{2!}y_i'' \\ &= y_i + 0.2x_iy_i + 0.01y_i(1 + 2x_i^2), \end{aligned} \tag{8}$$

where the second equation follows from $h = 0.1$. For the first four terms we get

$$\begin{aligned} y_{i+1} &\simeq y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' \\ &= y_i + 0.2x_iy_i + 0.01y_i(1 + 2x_i^2) + \frac{0.002}{3}x_iy_i(3 + 2x_i^2). \end{aligned} \tag{9}$$

Table 3 gives the values of x_i and y_i in these two cases.

i	x_i	y_i (for 3 terms)	y_i (for 4 terms)
0	0	0.5	0.5
1	0.1	0.505	0.505
2	0.2	0.520251	0.520353
3	0.3	0.546680	0.547000

Table 3: Using more terms of the Taylor series. The third column gives the value of y_i using three terms of the Taylor series, as in (8), and the fourth column gives the value of y_i using four terms, as in (9).

As mentioned above, we can solve the differential equation analytically and find that $y = 0.547087142$ when $x = 0.3$. As we can see, the values in Table 3 are reasonably accurate.

5.2.3 Runge-Kutta methods

Euler's method is known as a *single step method*; that is, y_{i+1} is estimated using x_i, x_{i+1} and y_i only. The Runge-Kutta methods are also single step methods but are the most general and the most accurate. They allow for the use of a large number of varying parameters that can help increase accuracy. The derivation of why these methods work is not difficult, but is lengthy and very technical, so we omit it. We do, however, give the basic details and an example for the 2-stage methods, and make some comments about the general R-stage methods in Remark 5.8.

The 2-stage Runge-Kutta methods for solving $\frac{dy}{dx} = M(x,y)$ with $y(x_0) = y_0$, with step length h, are given by

$$\begin{aligned} k_1^i &= M(x_i, y_i) \\ k_2^i &= M(x_i + ah, y_i + ahk_1^i) \\ y_{i+1} &= y_i + h\left(\left(1 - \tfrac{1}{2a}\right)k_1^i + \tfrac{1}{2a}k_2^i\right), \end{aligned}$$

for some $a \in \mathbb{R}$ with $0 < a < 1$. So, any choice of $a \in \mathbb{R}$ yields an iterative method for solving $\frac{dy}{dx} = M(x,y)$ with $y(x_0) = y_0$. This is one of the reasons why the Runge-Kutta methods are so powerful. Numerically, it is easy to tweak the parameter a and get possibly more accurate results.

For $a = 1/2$, we get the *corrected Euler method.* When $a = 1/2$ then $1 - \frac{1}{2a} = 0$ and $\frac{1}{2a} = 1$, therefore

$$\begin{aligned} k_1^i &= M(x_i, y_i) \\ k_2^i &= M(x_i + 1/2h, y_i + 1/2hk_1^i) \\ y_{i+1} &= y_i + hk_2^i. \end{aligned}$$

For $a = 2/3$, we get *Heun's method.* When $a = 2/3$ then $1 - \frac{1}{2a} = 1/4$ and $\frac{1}{2a} = 3/4$, therefore

$$\begin{aligned} k_1^i &= M(x_i, y_i) \\ k_2^i &= M(x_i + 2/3h, y_i + 2/3hk_1^i) \\ y_{i+1} &= y_i + \frac{1}{4}h(k_1^i + 3k_2^i). \end{aligned}$$

Example 5.7. Using a step length of $h = 0.1$ find the approximate value of y when $x = 0.5$ for the differential equation

$$\frac{dy}{dx} = 3x^2y, \text{ with } y = 1 \text{ when } x = 0,$$

using i) the corrected Euler method and ii) Heun's method.

In the case of the corrected Euler method we have $y_{i+1} = y_i + hk_2^i = y_i + 0.1k_2^i$, where

$$\begin{aligned} k_1^i &= M(x_i, y_i) = 3x_i^2 y_i, \\ k_2^i &= M(x_i + 0.05, y_i + 0.05k_1^i) = 3(x_i + 0.05)^2(y_i + 0.05k_1^i). \end{aligned}$$

If we want to express y_{i+1} purely in terms of x_i and y_i, we can substitute the expression for k_1^i into the formula for k_2^i and then substitute k_2^i into $y_{i+1} = y_i + 0.1k_2^i$ and obtain

$$y_{i+1} = y_i + 0.3(x_i + 0.05)^2(y_i + 0.15x_i^2 y_i).$$

i	x_i	y_i	k_1^i	k_2^i
0	0	1	0	0.0075
1	0.1	1.00075	0.030023	0.067652
2	0.2	1.007515	0.120902	0.190043
3	0.3	1.026519	0.277160	0.382339
4	0.4	1.064753	0.511082	0.662362
5	0.5	1.130989		

Table 4: Table of the values for the corrected Euler method.

In Table 4, we see the values of x_i and y_i for this case. So, $y(0.5) \simeq 1.130989$.

For Heun's method we have $y_{i+1} = y_i + \frac{1}{4}h(k_1^i + 3k_2^i) = y_i + 0.025(k_1^i + 3k_2^i)$, where

$$\begin{aligned} k_1^i &= M(x_i, y_i) = 3x_i^2 y_i, \\ k_2^i &= M\left(x_i + \frac{2}{30}, y_i + \frac{2}{30}k_1^i\right) \\ &= 3\left(x_i + \frac{1}{15}\right)^2\left(y_i + \frac{1}{15}k_1^i\right). \end{aligned}$$

As explicit formula for y_{i+1} we therefore obtain

$$y_{i+1} = y_i + 0.025\left(3x_i^2 y_i + 9\left(x_i + \frac{1}{15}\right)^2\left(y_i + \frac{1}{5}x_i^2 y_i\right)\right).$$

In Table 5, we see the values of x_i and y_i for this case. So, $y(0.5) \simeq 1.13256$.

i	x_i	y_i	k_1^i	k_2^i
0	0	1	0	0.013333
1	0.1	1.001	0.03003	0.083584
2	0.2	1.008020	0.120962	0.216765
3	0.3	1.027301	0.277371	0.421803
4	0.4	1.065870	0.511618	0.718652
5	0.5	1.132560		

Table 5: Table of values for Heun's method.

Again, we can solve the equation analytically, and get $y = A\exp(x^3)$ and $A = 1$ from the initial condition. Using a calculator, we see that $y(0.5) = 1.133148453$.

Remark 5.8. In general, the R-stage Runge-Kutta methods for solving $\frac{dy}{dx} = M(x, y)$ with $y(x_0) = y_0$, with step length h are given by

$$\begin{aligned} k_1^i &= M(x_i, y_i) \\ k_s^i &= M\left(x_i + ha_s, y_i + h\sum_{j=1}^{s-1} b_{s,j}k_j^i\right), s = 2, 3, \ldots, R \\ y_{i+1} &= y_i + h\sum_{j=1}^{R} c_j k_j^i \end{aligned}$$

where the a_s (for $s = 2, 3 \ldots, R$), $b_{s,j}$ (for $s = 2, 3, \ldots, R$ and $j = 1, 2, \ldots, s-1$), and c_j (for $j = 1, 2, \ldots, R$) are constants. By choosing suitable constants it is possible to reduce the error of the approximation.

For the 2-stage Runge-Kutta methods (so $R = 2$) we have

$$\begin{aligned} k_1^i &= M(x_i, y_i) \\ k_2^i &= M(x_i + ha_2, y_i + hb_{2,1}k_1^i) \\ y_{i+1} &= y_i + h(c_1k_1^i + c_2k_2^i). \end{aligned}$$

In this case, it can be shown that we get the smallest error by choosing the constants as

$$a_2 = a, b_{2,1} = a, c_1 = 1 - \frac{1}{2a} \text{ and } c_2 = \frac{1}{2a}$$

for any $a \in \mathbb{R}$ with $0 < a < 1$. Thus we obtain the 2-stage Runge-Kutta methods as described earlier.

5.2.4 The method of series as a numerical method

Some may have noticed that the method of series can be seen as a numerical method, as we know that a truncated series approximates a function. The only technicality is that the series should be valid for those values of the variables being used. We don't have the technical tools to deal with this; we will nonetheless forge ahead anyway.

Example 5.9. Suppose that $y' = x + y$ and $y(0) = 1$. Determine $y(1)$.

Assume a standard series solution of the form

$$y = a_0 + a_1x + a_2x^2 + \cdots .$$

We can reduce our work by using $y(0) = 1$ from the beginning. So,

$$y = 1 + a_1x + a_2x^2 + \cdots .$$

From the differential equation, we obtain

$$\begin{aligned} a_1 + 2a_2x + 3a_3x^2 + \cdots &= y' \\ &= x + y \\ &= x + 1 + a_1x + a_2x^2 + \cdots \\ &= 1 + (a_1 + 1)x + a_2x^2 + \cdots . \end{aligned}$$

We can now equate coefficients to obtain

$$a_1 = 1, \qquad 2a_2 = a_1 + 1 = 2 \Rightarrow a_2 = 1, \qquad 3a_3 = a_2 = 1 \Rightarrow a_3 = 1/3,$$
$$a_4 = \frac{1}{3 \cdot 4}, \qquad a_5 = \frac{1}{3 \cdot 4 \cdot 5},$$

and so on. We therefore find that

$$y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3 \cdot 4} + \frac{x^5}{3 \cdot 4 \cdot 5} + \cdots .$$

It can be shown that this series is valid for all x. To find $y(1)$ we can take the first 6 terms of the series above (the ones shown) and obtain that $y(1) \simeq 1+1+1+0.333+0.083+0.017 = 3.433$. Of course, taking more terms of the series gives increased accuracy.

We can of course use the Taylor series method as well.

Example 5.10. Find $y(1.5)$, where $y' = x + y$ and $y(1) = 2$. We use the Taylor series method and assume a solution of the form

$$y(x) = y(1) + y'(1)(x-1) + y'(1)\frac{(x-1)^2}{2!} + \cdots .$$

We find that $y'' = 1 + y'$, $y''' = y''$ and so on. Thus, $y'(1) = 1 + 2 = 3$, $y''(1) = 1 + 3 = 4$, $y'''(1) = 4$ and so on. We therefore find that

$$y(x) = 2 + 3(x-1) + 2(x-1)^2 + \frac{2}{3}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots .$$

It can be show that this series is valid for all x (it is not too hard to find its closed form). Using the first 5 terms we get

$$y(1.5) \simeq 2 + 3(0.5) + 2(0.5)^2 + 2/3(0.5)^3 + 1/6(0.5)^4 = 4.094.$$

Exercises 5.2.

1. Solve numerically using i) the Euler method ii) the higher derivative Euler method with 3 terms iii) the corrected Euler method and iv) Heun's method. The step size h is given.
   (a) $y' = 2x + y, y(0) = 0, h = 0.1$. Estimate $y(0.5)$.
   (b) $y' = x^2 - y, y(1) = 0, h = 0.1$. Estimate $y(1.6)$.
2. Estimate the value of the function in 1.(a) by using the method of series with 6 terms.
3. Estimate the value of the function in 1.(b) by using the method of Taylor series with 6 terms.