Solutions Chapter 6

Solutions to Exercises 6.1.

1. The mass at time time is given by $M(t) = M_0 e^{\alpha t}$ where M_0 is the initial mass. We measure the time t in years. The statement that the substance looses a quarter of its mass in 10 years becomes
$$M(10) = M_0 - \tfrac{1}{4}M_0.$$
Hence $M_0 e^{10\alpha} = \frac{3}{4}M_0$ which implies $\alpha = \frac{1}{10}\ln\left(\frac{3}{4}\right)$. To find the half life t_h we have to solve
$$M(t_h) = \tfrac{1}{2}M_0,$$
i.e.
$$M_0 e^{\frac{1}{10}\ln\left(\frac{3}{4}\right)t_h} = \tfrac{1}{2}M_0.$$
Hence $\frac{1}{10}\ln\left(\frac{3}{4}\right)t_h = \ln\left(\frac{1}{2}\right)$, which gives $t_h \approx 24.1$ years.

2. We are assuming exponential growth, so the size of the population at time t (measured in hours) is given by $P(t) = P_0 e^{\alpha t}$. If the population increases by 50% in 1/2 an hour, then $P(1/2) = 1.5P_0$. This implies $\alpha = 2\ln(1.5)$. The population will triple its original size when $P(t) = 3P_0$, i.e. $t = \ln(3)/\alpha \approx 1.35$ hours. The population will grow to 5 times its original size when $P(t) = 5P_0$, i.e. $t = \ln(5)/\alpha \approx 1.98$ hours.

Solutions to Exercises 6.2.

1. To find the point(s) of inflection of the function $P(t)$ we have to solve $P''(t) = 0$. We have a formula for $P(t)$, but computing the second derivative from that is a bit messy. Instead we will use the differential equation
$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{k}\right) = rP - \frac{r}{k}P^2.$$
Differentiating both sides with respect to t gives
$$\begin{aligned}\frac{\mathrm{d}^2P}{\mathrm{d}t^2} &= rP' - \frac{r}{k}2PP' \\ &= r\left(1 - \frac{2P}{k}\right)P' \\ &= r\left(1 - \frac{2P}{k}\right)rP\left(1 - \frac{P}{k}\right).\end{aligned}$$
Hence $P''(t) = 0$ implies
$$1 - \frac{2P}{k} = 0 \quad \text{or} \quad P = 0 \quad \text{or} \quad 1 - \frac{P}{k} = 0,$$

i.e.

$$P = \frac{k}{2} \quad \text{or} \quad P = 0 \quad \text{or} \quad P = k.$$

If we assume $0 < P_0 < k$ then $0 < P(t) < k$ for all t, so the second and third cases are impossible. Therefore we find one point of inflection at $P = k/2$.

If we like we can also find the time t for which we have $P(t) = k/2$. Solving $\frac{kP_0 e^{rt}}{k+P_0(e^{rt}-1)} = k/2$ for t gives $t = \frac{1}{r}\ln\left(\frac{k-P_0}{P_0}\right)$.

2. With $r = 1$, $P_0 = 100$ and $k = 2000$ we have

$$P(t) = \frac{200,000e^t}{2000 + 100(e^t - 1)} = \frac{200,000e^t}{1900 + 100e^t}.$$

The maximum of the population is $k = 2000$, so we have to find the time t for which $P(t) = 0.9 \times 2000 = 1800$. Solving

$$\frac{200,000e^t}{1900 + 100e^t} = 1800$$

gives $t = \ln(171) \approx 5.14$ hours.

Solutions to Exercises 6.3.

1. In the notes it was shown that in this case $v(t) = \frac{mg}{\beta}\left(1 - e^{-\frac{\beta}{m}t}\right)$. Since $v(t) = \dot{y}(t)$, we just have to integrate the expression for $v(t)$ to get $y(t)$. In this way we get

$$y(t) = \frac{mg}{\beta}t + \frac{m^2 g}{\beta^2}e^{-\frac{\beta}{m}t} + c,$$

where c is a constant. If the initial height is $y(0) = y_0$, then $\frac{m^2 g}{\beta^2} + c = y_0$, i.e. $c = y_0 - \frac{m^2 g}{\beta^2}$.

2. (a) The drag force is given by $F_{\text{drag}} = -\beta v$. At velocity $v = 5\,\text{m/s}$ we have the drag force $F_{\text{drag}} = -200\,\text{N}$ (negative because it's opposed to the direction of motion), so $\beta = -F_{\text{drag}}/v = 200/5 = 40$.

   (b) This question is the same as in the notes in Section 6.3.2, except that now the initial velocity is $5\,\text{m/s}$. So we have the differential equation

$$m\dot{v} = mg - \beta v$$

with solution

$$v(t) = \frac{mg}{\beta} - Be^{-\frac{\beta}{m}t}.$$

Here, $\beta = 40$, $m = 50$ and $g = 9.8$, so $\frac{mg}{\beta} = 12.25$ and $\beta/m = 0.8$. Hence we have

$$v(t) = 12.25 - Be^{-0.8t}.$$

The initial condition $v(0) = 5$ implies that $B = 7.25$. Thus,

$$v(t) = 12.25 - 7.25e^{-0.8t}.$$

For the position we integrate v to get

$$y(t) = 12.25t + 7.25/0.8e^{-0.8t} + c.$$

The initial condition $y(0) = 0$ implies that $c = -9.0625$. Thus,

$$y(t) = 12.25t + 9.0625e^{-0.8t} - 9.0625.$$

(c) The terminal velocity is $\lim_{t\to\infty} v(t) = 12.25\,\mathrm{m/s}$.

Solutions to Exercises 6.4.

1. (a) The differential equation for the position x is $m\ddot{x}+kx = 0$, so $\ddot{x}+16x = 0$. The initial conditions are $x(0) = 0.1$ (because the weight is pulled 0.1 m to right at time $t = 0$) and $\dot{x}(0) = 0$ (because the weight is released without any initial velocity).

   (b) The fundamental frequency is $\omega_0 = \sqrt{\frac{k}{m}} = 4$, so the solution is $x(t) = A\sin(4t + \phi)$ for some constants A and ϕ. Since $\dot{x}(t) = 4A\cos(4t + \phi)$, the initial conditions $x(0) = 0.1$ and $\dot{x}(0) = 0$ give $A\sin(\phi) = 0.1$ and $4A\cos(\phi) = 0$. This implies that $\phi = \pi/2$ and $A = 0.1$. Therefore,

   $$x(t) = 0.1\sin(4t + \pi/2)$$

   and

   $$v(t) = \dot{x}(t) = 0.4\cos(4t + \pi/2).$$

   (c) The maximum distance from the equilibrium point is $0.1\,\mathrm{m} = 10\,\mathrm{cm}$ and the period of motion is $2\pi/4 = \pi/2$ seconds.

2. We have the same differential equation as in Question 1, but now the initial conditions are $x(0) = 0$ and $\dot{x}(0) = -0.3\,\mathrm{m/s}$.

   (b) Again, $x(t) = A\sin(4t + \phi)$. However, now the initial conditions give $\phi = 0$ and $A = -0.3/4 = -0.075$. Hence

   $$x(t) = -0.075\sin(4t)$$

   and

   $$v(t) = \dot{x}(t) = -0.3\cos(4t).$$

   (c) The period is the same as in Question 1 ($\pi/2$ seconds). The maximum distance is $0.075\,\mathrm{m} = 7.5\,\mathrm{cm}$.

3. (a) We have $F_{\text{spring}} = -kx$. We know that the spring is stretched by 0.25 m when a force of 2 N is applied, hence $k = 2/0.25 = 8$. (Note that this is the correct sign: if we pull the mass with a force of 2 N, then the spring pulls the mass with the same force into the opposite direction, i.e. $F_{\text{spring}} = -2\,\text{N}$.)

   (b) The differential equation for the position x is

$$2\ddot{x} + 8\dot{x} + 8x = 0.$$

   The auxiliary equation $2r^2 + 8r + 8 = 0$ has the double root -2, so the general solution is

$$x(t) = Ae^{-2t} + Bte^{-2t}.$$

   The initial conditions $x(0) = 0.1$ and $\dot{x}(0) = -0.3$ imply that $A = 0.1$ and $B = -0.1$, so the equation for the position x is

$$x(t) = 0.1e^{-2t} - 0.1te^{-2t}.$$

   Since the auxiliary equation has only one root (with multiplicity two), the system is critically damped.

   (c) Setting $x(t) = 0$ and solving for t we get $t = 1\,\text{s}$.

4. (a) The spring is stretched $x = 0.1\,\text{m}$ when pulled with $F = 2.5\,\text{N}$. Since $F_{\text{spring}} = -kx$, it follows that $k = 2.5/0.1 = 25$.

   (b) The differential equation is $5\ddot{x} + 10\dot{x} + 25x = 5\sin t$. Simplifying, we get $\ddot{x} + 2\dot{x} + 5x = \sin t$. The initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0.3$.

   (c) The auxiliary equation is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. Hence the complementary solution is $Ae^{-t}\sin(2t + \phi)$, where A and ϕ are constants.

   Next we are looking for a particular solution of the nonhomogeneous equation of the form $p(t) = a\sin t + b\cos t$. We find $p(t) = 1/5\sin t - 1/10\cos t$. Hence the general solution of the differential equation is

$$x(t) = Ae^{-t}\sin(2t + \phi) + 1/5\sin t - 1/10\cos t.$$

   The initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.3$ imply that

$$\begin{aligned} A\cos\phi - 1/10 &= 0, \\ -A\sin\phi + 2A\cos\phi + 1/5 &= 0.3. \end{aligned}$$

   From this we deduce $A = \sqrt{2}/10$ and $\phi = \pi/4$. Thus,

$$x(t) = \sqrt{2}/10e^{-t}\sin(2t + \pi/4) + 1/5\sin t - 1/10\cos t.$$

   (d) The system is under damped. The transient solution is $\sqrt{2}/10e^{-t}\sin(2t + \pi/4)$ and the steady state solution is $1/5\sin t - 1/10\cos t$.

5. If there is no damping and no external force, the differential equation is $5\ddot{x} + 25x = 0$. The auxiliary equation is $5r^2 + 25 = 0$, so $r = \pm i\omega_o$ with $\omega_0 = \sqrt{5}$. Thus, the general solution is

$$x(t) = A\sin(\sqrt{5}t + \phi)$$

where A and ϕ are constants. The initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.3$ imply that $\phi = 0$ and $A = \frac{3}{10\sqrt{5}}$. Hence, the answer is $\frac{3}{10\sqrt{5}}$ (metres).

Solutions to Exercises 6.5.

1. Let $r(t)$ be the radius of the snowball (in cm) at time t (in hours). We know that $r(0) = 10$ and $r(1) = 5$. The volume and surface area at time t are given by $V(t) = \frac{4}{3}\pi r(t)^3$ and $S(t) = 4\pi r(t)^2$ respectively. The statement that the rate of change of volume is proportional to the surface area can be expressed as the equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \alpha S \tag{1}$$

for some constant α. Now using the chain rule we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}.$$

Hence (1) becomes

$$4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha 4\pi r^2,$$

which can be simplified to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha.$$

Therefore

$$r(t) = \alpha t + c$$

for some constant c. The initial condition $r(0) = 10$ gives $c = 10$, and from $r(1) = 5$ we then deduce $\alpha = -5$. Thus

$$r(t) = -5t + 10.$$

The snowball will have melted completely when $r(t) = 0$ which happens at time $t = 2$ hours.