Games, Choice and Optimisation Assignment 1

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1. (a) Let the variable x_1 represent the number 2-storey executive homes, x_2 represent the number of 3-storey blocks, x_3 represent the number of 1-storey bungalows and x_4 represent the number of 2-storey social housing. The constraints of the linear programme representing the development problem can be written, for

$$
\begin{aligned}
\text{time in weeks } & 3x_1 + 2x_2 + x_3 + x_4 \leq 140, \\
\text{units of land } & 4x_1 + 3x_2 + x_3 + \tfrac{3}{2}x_4 \leq 600, \\
\text{storey limit } & \frac{1}{4}\left(2x_1 + 3x_2 + x_3 + 2x_4\right) \leq \frac{9}{5}, \\
& 8x_1 + 12x_2 + x_3 + 4x_4 \leq \frac{36}{5}, \\
\text{social housing } & \frac{1}{4}\left(x_1 + x_2 + x_3 + x_4\right) \leq x_4, \\
& x_1 + x_2 + x_3 - 3x_4 \leq 0.
\end{aligned}
$$

Since the objective in the development problem is to maximise profit, the objective function is, where coefficients represent units of 1000 pounds, $70x_1 + 30x_2 + 25x_3 + 5x_4$.
In standard form, then, our linear programme is written

$$
\begin{array}{ll}
\text{maximise} & 70x_1 + 30x_2 + 25x_3 + 5x_4, \\
\text{subject to} & 3x_1 + 2x_2 + x_3 + x_4 \leq 140, \\
& 4x_1 + 3x_2 + x_3 + \tfrac{3}{2}x_4 \leq 600, \\
& 8x_1 + 12x_2 + x_3 + 4x_4 \leq \dfrac{36}{5}, \\
& x_1 + x_2 + x_3 - 3x_4 \leq 0, \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{array}
$$

(b)

$$
\begin{array}{llrrr}
\text{maximise} & & -2x_1+ & (\bar{x}-\hat{x})+ & 3x_3 \\
\text{subject to} & & x_1+ & 2(\bar{x}-\hat{x})+ & 3x_3 \leq 25 \\
& & -2x_1+ & 3(\bar{x}-\hat{x})+ & x_3 \leq 17 \\
& x_1, x_2, x_3, \bar{x}, \hat{x} \geq 0. & & &
\end{array}
$$

(c) i. One slack variable is introduced per constraint, so in this case, we introduce $x_4, x_5, x_6 \geq 0$.
Because x_1, x_2 and x_3 appear in the objective function, x_1, x_2 and x_3 will be made nonbasic.
Since x_3 has the greatest coefficient in z and as such will contribute most to the maximisation of that function, x_3 will be our pivot variable.

$$
\begin{aligned}
x_4 &= 136 - x_1 + 6x_2 - 4x_3 \\
x_5 &= 44 - 2x_1 - 3x_2 - 8x_3 \\
x_6 &= 56 - 4x_1 + 2x_2 - 4x_3
\end{aligned}
$$

Since $x_1 = x_2 = x_3 = 0$, $x_4 = 136$, $x_5 = 44$ and $x_6 = 56$.
We increase the pivot variable x_3 and write it in terms of our basic variables x_4, x_5, x_6 and using the fact that $x_1 = x_2 = 0$ have

$$x_4 = 136 - 4x_3 \geq 0 \text{ and } x_3 \leq 34,$$

and

$$x_5 = 44 + 8x_3 \geq 0 \text{ and } x_3 \geq -\frac{11}{2},$$

also

$$x_6 = 56 - 4x_3 \geq 0 \text{ and } x_3 \leq 14.$$

Here the most restrictive value for nonbasic x_3 comes from the equation for basic x_6, so we set $x_3 = 14$ and $x_6 = 0$, making x_4 basic and x_6 nonbasic.

ii. Now we use the equation for now-nonbasic x_6 to write basic x_3 in terms of nonbasic variables

$$\begin{aligned} x_6 &= 56 - 4x_1 + 2x_2 - 4x_3 \\ x_3 &= \frac{1}{4}(56 - 4x_1 + 2x_2 - x_6) \\ &= 14 - x_1 + \frac{1}{2}x_2 - \frac{1}{4}x_6 \end{aligned}$$

and substitute this into our equations for basic x_4 and x_5 and for z

$$\begin{aligned} x_4 &= 136 - x_1 + 6x_2 - (56 - 4x_1 + 2x_2 - x_6) \\ &= 80 - 5x_1 + 4x_2 + x_6, \\ x_5 &= 44 - 2x_1 - 3x_2 - 2(56 - 4x_1 + 2x_2 - x_6) \\ &= -68 + 6x_1 - 5x_2 + 2x_6, \\ z &= 3x_1 - 7x_2 + 10(14 - x_1 + \frac{1}{2}x_2 - \frac{1}{4}x_6) \\ &= 140 - 7x_1 - 13x_2 - \frac{5}{2}x_6 \end{aligned}$$

At this stage the basic feasible solution is $x_1 = x_2 = 0$ and $x_3 = 14$ with value 140.

2. (a) We introduct two slack variables, one for each constraint

$$\begin{aligned} x_4 &= 21 - 2x_1 - 3x_2 + 3x_3, \\ x_5 &= 72 - 4x_1 - 9x_2 + 4x_3. \end{aligned}$$

Setting $x_2 = x_3 = 0$ as our nonbasic variables and choosing x_1 as our pivot variable, we write

$$\begin{aligned} x_4 &= 21 - 2x_1 \geq 0 \text{ and } x_1 \leq \frac{21}{2} \\ x_5 &= 72 - 4x_1 \geq 0 \text{ and } x_1 \leq 18. \end{aligned}$$

Since the inequality arising from x_4 is most restrictive, we set $x_1 = \frac{21}{2}$ and $x_4 = 0$ and write

$$x_1 = \frac{21}{2} - \frac{3}{2}x_2 + \frac{3}{2}x_3 - \frac{1}{2}x_4$$

and

$$x_5 = 30 - 3x_2 - 2x_3 + 2x_4$$

and

$$\begin{aligned} z &= 5\left(\frac{21}{2} - \frac{3}{2}x_2 + \frac{3}{2}x_3 - \frac{1}{2}x_4\right) + 4x_2 - 7x_3 \\ &= \frac{105}{2} - \frac{15}{2}x_2 + \frac{15}{2}x_3 - \frac{5}{2}x_4 + 4x_2 - 7x_3 \\ &= \frac{105}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \end{aligned}$$

Since the only variable with a positive coefficient is x_3, we choose it as our pivot variable and write it in terms of our basic variables x_1 and x_5

$$\begin{aligned} x_1 &= \frac{21}{2} + \frac{3}{2}x_3 \geq 0 \text{ and } x_3 \geq -7 \\ x_5 &= 30 - 2x_3 \geq 0 \text{ and } x_3 \leq 15. \end{aligned}$$

Of these inequalities, the one arising from x_5 is most restrictive, so we set $x_3 = 15$ and $x_5 = 0$ and write

$$\begin{aligned} x_5 &= 30 - 3x_2 - 2x_3 + 2x_4, \\ x_3 &= \frac{1}{2}(30 - 3x_2 + 2x_4 - x_5) \\ &= 15 - \frac{3}{2}x_2 + x_4 - \frac{1}{2}x_5 \end{aligned}$$

and

$$\begin{aligned} x_1 &= \frac{21}{2} - \frac{3}{2}x_2 + \frac{3}{2}\left(15 - \frac{3}{2}x_2 + x_4 - \frac{1}{2}x_5\right) - \frac{1}{2}x_4 \\ &= \frac{21}{2} - \frac{3}{2}x_2 + \left(\frac{45}{2} - \frac{9}{4}x_2 + \frac{3}{2}x_4 - \frac{3}{4}x_5\right) - \frac{1}{2}x_4 \\ &= 33 - \frac{15}{4}x_2 + x_4 - \frac{3}{4}x_5 \end{aligned}$$

now

$$\begin{aligned} z &= \frac{105}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \\ &= \frac{105}{2} - \frac{7}{2}x_2 + \frac{1}{2}\left(15 - \frac{3}{2}x_2 + x_4 - \frac{1}{2}x_5\right) - \frac{5}{2}x_4 \\ &= \frac{120}{2} - \frac{12}{4}x_2 - 2x_4 - \frac{1}{2}x_5 \end{aligned}$$

and all coefficients in the objective function of the linear programme are negative and the basic feasible solution $x_1 = 33$, $x_3 = 15$ with value 60 is an optimal solution of the linear programme.

(b) Because x_3 has a coefficient of zero in the objective function, there are alternative solutions of $\mathcal{L}$, for example

x_1	x_2	x_3	x_4	x_5	x_6	
$\frac{1}{2}$	$-\frac{3}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	18
-2	2	0	1	$\frac{1}{2}$	3	25
0	$\frac{5}{2}$	0	0	$\frac{9}{2}$	$\frac{3}{2}$	427

(c) i.

$$\begin{array}{ll} \text{maximise} & -x_0 \\ \text{subject to} & -x_0 + 5x_1 - 4x_2 - 6x_3 - 2x_4 \leq -68, \\ & -x_0 + 3x_1 + x_2 - 2x_3 - 4x_4 \leq -32, \\ & x_0, x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

ii. The auxilliary linear programme for $\mathcal{L}$ is

x_0	x_1	x_2	x_3	x_4	x_5	x_6	
-1	5	-4	-6	-2	1	0	-68
-1	3	1	-2	-4	0	1	-32
1	0	0	0	0	0	0	0

Pivot on row 2 and column 1, eros $r_1 \to r_1 - r_2$, $r_3 \to r_3 + r_1$, $r_2 \to -r_2$ give tableau

x_0	x_1	x_2	x_3	x_4	x_5	x_6	
0	2	-5	-4	2	1	-1	-36
1	-3	-1	2	4	0	-1	32
0	3	1	-2	-4	0	1	-32

Pivot on row 2 and column 4, eros $r_3 \to r_3 + r_2$, $r_1 \to r_1 + \frac{1}{2}r_2$, $r_2 \to \frac{1}{4}r_2$ give tableau

x_0	x_1	x_2	x_3	x_4	x_5	x_6	
$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{9}{2}$	-5	0	1	$-\frac{1}{2}$	-52
$\frac{1}{4}$	$-\frac{3}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	1	0	$-\frac{1}{4}$	8
1	0	0	0	0	0	0	0

That the object function here has value zero tells us that the linear programme $\mathcal{L}$ is feasible.