Discrete Mathematics

Assignment 2 Solutions 2017

1. (a) Using the formula from the notes we have:

$$
\begin{aligned}
u_n &= \prod_{i=1}^{n} 16^{i^3} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} \prod_{j=i+1}^{i} 16^{j^3}, \\
&= 16^{1^3}16^{2^3}16^{3^3}\cdots 16^{n^3} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} \prod_{j=i+1}^{i} 16^{j^3}, \\
&= 16^{1^3+2^3+3^3+\cdots+n^3} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} \prod_{j=i+1}^{i} 16^{j^3}, \\
&= 16^{\frac{1}{4}n^2(n+1)^2} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} \prod_{j=i+1}^{i} 16^{j^3}, \\
&= 2^{n^2(n+1)^2} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} \prod_{j=i+1}^{i} 16^{j^3}, \\
&= 2^{n^2(n+1)^2} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} 16^{(i+1)^3}16^{(i+2)^3}16^{(i+3)^3}\cdots 16^{n^3}, \\
&= 2^{n^2(n+1)^2} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} 16^{(i+1)^3+(i+2)^3+(i+3)^3+\cdots+n^3}, \\
&= 2^{n^2(n+1)^2} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} 16^{\frac{1}{4}n^2(n+1)^2-\frac{1}{4}i^2(i+1)^2}, \\
&= 2^{n^2(n+1)^2} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} 16^{\frac{1}{4}n^2(n+1)^2}/16^{\frac{1}{4}i^2(i+1)^2}, \\
&= 2^{n^2(n+1)^2} + \sum_{i=1}^{n} 2^{i^2(i+1)^2} 2^{n^2(n+1)^2}/2^{i^2(i+1)^2}, \\
&= 2^{n^2(n+1)^2} + \sum_{i=1}^{n} 2^{n^2(n+1)^2}, \\
&= (1+n)2^{n^2(n+1)^2}.
\end{aligned}
$$

[3]

(b) We have $a_{n+2} - 4a_n = 10 \cdot 3^n$.
characteristic polynomial: $\lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$.

General solution to the homogeneous part: $G(n) = A2^n + B(-2)^n$.
Find a particular solution: Try $P(n) = M3^n$. This gives

$$M3^{n+2} - 4M3^n = 10 \cdot 3^n,$$

so after dividing through by 3^n we have:

$$9M - 4M = 10.$$

Hence $M = 2$ and $P(n) = 2 \cdot 3^n$.
General solution to the whole equation: $a_n = G(n) + P(n) = A2^n + B(-2)^n + 2 \cdot 3^n$.
Use the initial conditions to solve for A and B:

$$\begin{aligned} a_0 = A + B + 2 &= 9, \\ A + B &= 7. \\ a_1 = 2A - 2B + 6 &= 4, \\ 2A - 2B &= -2, \end{aligned}$$

whence we deduce $A = 3$ and $B = 4$.
Solution: $a_n = 3 \cdot 2^n + 4(-2)^n + 2 \cdot 3^n$ [3]

(c) We have $b_n - 6b_{n-1} + 5b_{n-2} = 120n - 33$.
characteristic polynomial: $\lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1)$.
General solution to the homogeneous part: $G(n) = A5^n + B$.
Find a particular solution: Try $P(n) = n(M_0 + M_1 n) = M_0 n + M_1 n^2$ (since 1 is a zero of the characteristic polynomial). This gives

$$M_0 n + M_1 n^2 - 6(M_0(n-1) + M_1(n-1)^2 + 5(M_0(n-2) + M_1(n-2)^2) = M_0 n + M_1 n^2 - 6$$

Comparing coefficients of n gives

$$\begin{aligned} M_0 - 6M_0 + 12M_1 + 5M_0 - 20M_1 &= 120, \\ -8M_1 &= 120, \\ M_1 &= -15. \end{aligned}$$

Comparing constant terms gives

$$\begin{aligned} 6M_0 - 6M_1 - 10M_0 + 20M_1 &= -33, \\ -4M_0 + 14M_1 &= -33, \\ M_0 &= -\frac{177}{4}. \end{aligned}$$

Hence we have $P(n) = -15n^2 - \frac{177}{4}n$.
General solution to the whole equation: $b_n = G(n) + P(n) = A5^n + B - 15n^2 -$

$\frac{177}{4}n$.

Use the initial conditions to solve for A and B:

$$\begin{aligned} b_0 = A + B &= 9, \\ b_1 = 5A + B - \frac{177}{4} - 15 &= 30, \\ 4A - \frac{177}{4} - 15 &= 21, \\ 4A &= \frac{177}{4} + 36, \\ A &= \frac{177}{16} + 9, \\ B &= \frac{-177}{16}. \end{aligned}$$

Solution: $b_n = \frac{321}{16} \cdot 5^n - \frac{177}{16} - 15n^2 - \frac{177}{4}n.$ [3]

2. Let Y_n be the number of plants after n years have passed. The biologist starts with three plants, so $Y_0 = 3$. As they are initially one-year-old plants, at the end of the first year they will all reproduce, giving 6 new plants each, so $Y_1 = 21$. In general the plants existing at the end of year n will be those that were there at the end of year $n-1$ (of which there are Y_{n-1}) together with any new ones (of which there will be $6Y_{n-2}$, since the plants of age 2 or more years that reproduce at this point must be those that were in the greenhouse at the end of year $n-2$.) Hence the difference equation describing the number of plants is

$$Y_n = Y_{n-1} + 6Y_{n-2}, \qquad Y_0 = 3,\ Y_1 = 21.$$

Let $g(x)$ be the generating function describing the number of plants at each time period. Then we have

$$\begin{array}{rclcccccccc} g(x) & = & Y_0 & + & Y_1x & + & Y_2x^2 & + & Y_4x^3 & + & \ldots, \\ -xg(x) & = & & - & Y_0x & - & Y_1x^2 & - & Y_2x^3 & - & \ldots, \\ -6x^2g(x) & = & & & & - & 6Y_0x^2 & - & 6y_1x^3 & - & \ldots, \end{array}$$

So we have:

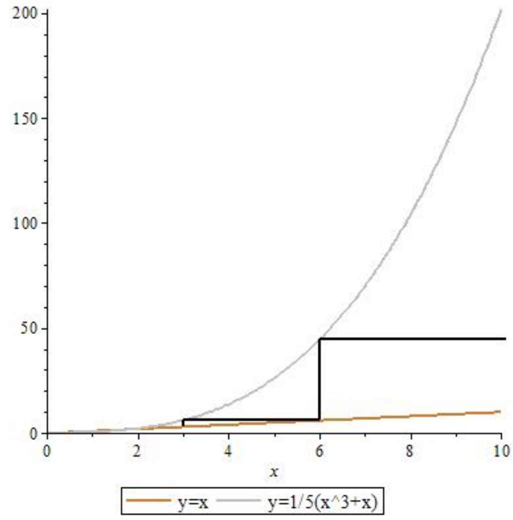
$$\begin{aligned} (1 - x - 6x^2)g(x) &= Y_0 + (Y_1 - Y_0)x \\ g(x) &= \frac{3 + 18x}{1 - x - 6x^2}, \\ &= \frac{3 + 18x}{(1 - 3x)(1 + 2x)}, \\ &= \frac{27/5}{1 - 3x} + \frac{-12/5}{1 + 2x}, \\ &= \frac{27}{5}\sum_{r=0}^{\infty} 3^r x^r - \frac{12}{5}\sum_{r=0}^{\infty}(-2)^r x^r, \end{aligned}$$

from which we deduce $Y_n = \frac{27}{5}3^n - \frac{12}{5}(-2)^n$.

[4]

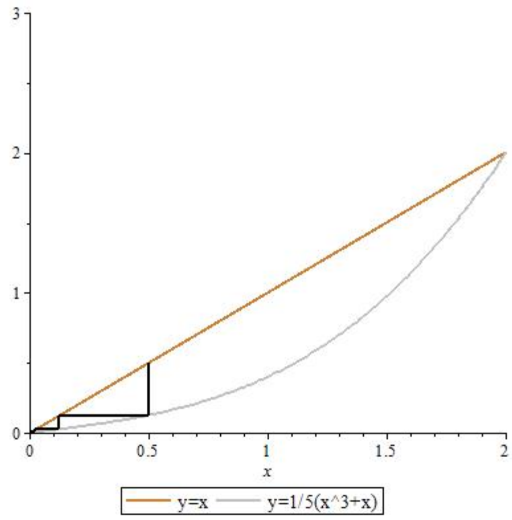
3. (a) When $u_0 = 0$ we have $u_1 = \frac{1}{5}(0^3+0) = 0$, and hence $u_n = 0$ for all n. Similarly, when $u_0 = 2$, we have $u_1 = \frac{1}{5}(8+2) = 2$, and thus $u_n = 2$ for all n. Thus in each of these cases the terms of the sequence are constant. [1]

(b) The presence of the $\frac{1}{5}u_{n-1}^3$ term in the difference equation suggests the sequence will go to infinity as $n \to \infty$, which agrees with the cobweb diagram below:



[3]

(c) In this case the sequence converges to 0 as $n \to \infty$, as illustrated by the following cobweb diagram:



[3]