4 Differential Equations

4.1 Terminology

A **differential equation** in y is an equation involving an unknown function y and its (partial) derivatives. When y is a function of one variable the differential equation is called an **ordinary differential equation** (ODE). When y is a function of at least two variables the differential equation is called a **partial differential equation** (PDE). The **order** of a differential equation is the highest order of derivative appearing in the equation.

Example 4.1. The following are all examples of differential equations. Equations (a)–(e) are ordinary differential equations; in all these equations the function is denoted by y and its variable by x. Equations (f)–(h) are partial differential equations. Equations (a), (b) and (f) have order one, (c), (d), (g) and (h) have order 2, and (e) has order three.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{e^x}{y+2}$

(c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4\sin x$

(d) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\omega^2 y$ (Here ω is a constant. This is the equation for **simple harmonic motion** which arises naturally in many problems in mechanics.)

(e) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^3y = 0$

(f) $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 3xy$ (Here z is a function of x and y.)

(g) $\dfrac{\partial^2 z}{\partial y \partial x} = x^2 + y^2$ (Here z is a function of x and y.)

(h) $\dfrac{\partial u}{\partial t} = \alpha\dfrac{\partial^2 u}{\partial x^2}$ (Here u is a function of t and x, and α is a constant. This PDE is called the **heat equation**.)

An ordinary differential equation of order n is called **linear** if it can be written in the form

$$a_n(x)\frac{\mathrm{d}^ny}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = b(x),$$

where $a_0(x), a_1(x), \ldots, a_{n-1}(x), a_n(x)$ and $b(x)$ are functions of x. Of the examples above, (a), (c), (d) and (e) are linear.

A **solution** of a differential equation is a function that satisfies the equation.

Example 4.2. It is easy to check that $y = x^2$ is a solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$. However, there are other solutions. For example, $y = x^2 + 5$, $y = x^2 - 7$ and $y = x^2 + 33.75$ are all solutions of this differential equation. In fact, $y = x^2 + c$ is a solution of this differential equation for any constant c. This last solution is called the **general solution** of the differential equation.

For most ordinary differential equations of order one, the general solution involves an arbitrary constant c. Given the value of y for a particular value of x we can determine the value of c. Such values of y and x are called **boundary conditions** for the differential equation.

Example 4.3. We want to find the solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ that satisfies the boundary condition $y = 7$ when $x = 3$. We know that the general solution is $y = x^2 + c$. Substituting $y = 7$ and $x = 3$ into the general solution gives $7 = 3^2 + c$ and hence $c = -2$. Thus the required solution is $y = x^2 - 2$.

For an ordinary differential equation of order k, the general solution normally involves k arbitrary constants. In this case, k independent boundary conditions are required to find a unique solution.

The solutions of the differential equation in Example 4.2 gave y **explicitly** in terms of x. However, solutions of differential equations often express y **implicitly** in terms of its variables, as the following example illustrates.

Example 4.4. The general solution of the ordinary differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x}{y+2}$ is

$$\frac{1}{2}y^2 + 2y = e^x + c, \tag{1}$$

where c is a constant. We can check this by differentiating expression (1) implicitly with respect to x to get

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = e^x.$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x}(y+2) = e^x,$$

and dividing by $y + 2$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x}{y+2},$$

as required.

Note that starting from the given solution we get

$$y^2 + 4y = 2e^x + d,$$

where $d = 2c$. Thus, completing the square of the left hand side of the above solution gives

$$(y+2)^2 - 4 = 2e^x + d,$$

and so

$$(y+2)^2 = 2e^x + f,$$

where $f = d + 4$. Taking square roots of both sides of this solution, and then subtracting 2 from both sides gives

$$y = \pm\sqrt{2e^x + f} - 2.$$

So in this case it is also possible to give the solution for y explicitly in terms x. However, this is not always possible (see the next example), and it is debatable whether it is worth the effort to find such a solution in this case.

Example 4.5. The general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y(x-1)}{x(y+1)}$ is

$$y + \ln y - x + \ln x = c.$$

This can easily be verified by differentiating the solution with respect to x and rearranging the result to get $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. However, it is not possible to rearrange this solution to express y explicitly in terms of x.

Exercises 4.1.

1. Show that $y^2 + xy = \cos(x+y) + x^2$ is a solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y - \sin(x+y) + 2x}{2y + x + \sin(x+y)}.$$

2. Show that $y^2 + y = \ln x + \sin y$ is a solution of

$$y''(2y + 1 - \cos y) + (y')^2(2 + \sin y) = \frac{-1}{x^2}.$$

4.2 ODEs of order one: Separation of variables

In this and the following two sections we consider the problem of finding the general solution of an ordinary differential equation of order one. This is not always possible, but there are many methods for finding the general solution for specific types of equations, and we consider some of these here. In this section we study variables separable equations and homogeneous equations.

Variables separable differential equations

Let M and N be functions of one variable, and consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{M(x)}{N(y)}.$$

Then

$$N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = M(x).$$

Integrating both sides with respect to x gives

$$\int N(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int M(x)\,\mathrm{d}x,$$

and so, by the chain rule,

$$\int N(y)\,\mathrm{d}y = \int M(x)\,\mathrm{d}x.$$

Thus, if we can evaluate the two integrals, we can find the general solution of the differential equation. Differential equations of this nature are called **variables separable**, and the method outlined for solving them is called the **separation of variables**.

Example 4.6. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay,$$

where a is a constant.

Separating the variables gives

$$\int \frac{\mathrm{d}y}{y} = \int a\,\mathrm{d}x.$$

Evaluating the two integrals gives

$$\ln|y| = ax + c,$$

where c is a constant, and so

$$|y| = e^{ax+c} = e^c e^{ax}.$$

Hence

$$y = Ae^{ax}$$

where $A = \pm e^c$ is a non-zero constant.

Note that in the first step we implicitly assumed $y \neq 0$ when we divided by y. In fact, the zero function (i.e. $y(x) = 0$ for all x) is also a solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = ay$. Since we can write the zero function as $y = 0 \cdot e^{ax}$, the general solution of the differential equation is $y = Ae^{ax}$ where $A \in \mathbb{R}$ is any constant.

In the following, we will usually be more sloppy than in the previous example and ignore absolute values, $\pm$, and special cases. This has the advantage that we can concentrate on the main ideas without being distracted by too many technical details. However the disadvantage of not being precise about all these details is that sometimes we will only find some but not all solutions of a differential equation.

Example 4.7. Consider the differential equation

$$9x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+1}{2y}.$$

Separating the variables gives

$$\int 2y\,\mathrm{d}y = \int \frac{x^2+1}{9x}\,\mathrm{d}x.$$

Thus

$$\int 2y\,\mathrm{d}y = \frac{1}{9}\int \left(x+\frac{1}{x}\right)\,\mathrm{d}x.$$

Evaluating the two integrals gives

$$y^2 = \frac{1}{9}\left(\frac{x^2}{2}+\ln x + c\right),$$

where c is a constant, and so

$$y = \frac{1}{3}\sqrt{\frac{x^2}{2}+\ln x + c}.$$

Example 4.8. Consider the differential equation

$$(3y+2)^3\cos^2 x\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

Separating the variables gives

$$\int (3y+2)^3\,\mathrm{d}y = \int \frac{\mathrm{d}x}{\cos^2 x}.$$

Thus

$$\int (3y+2)^3\,\mathrm{d}y = \int \sec^2 x\,\mathrm{d}x.$$

Evaluating the two integrals gives

$$\frac{\frac{1}{3}(3y+2)^4}{4} = \tan x + c,$$

where c is a constant, and so

$$(3y+2)^4 = 12\tan x + d,$$

where $d = 12c$.

Note that, with a bit more effort, we can write the solution of this differential equation as $y = \frac{1}{3}\sqrt[4]{12\tan x + d} - \frac{2}{3}$, but it is a matter of mathematical taste as to whether this is better way of expressing the solution than the one given.

Homogeneous differential equations

Recall that a function f of two variables is homogeneous of degree n if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

A differential equation is called **homogeneous** if it is of the form

$$M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

where M and N are both homogeneous functions of the same degree n.

To solve a homogeneous differential equation we put $y = vx$, where v is a function of x. By the product rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v.$$

Also, since M is homogeneous of degree n,

$$\begin{aligned} M(x, y) &= M(x, vx) \\ &= M(x.1, x.v) \\ &= x^n M(1, v) \\ &= x^n \hat{M}(v), \end{aligned}$$

for some function $\hat{M}$ of one variable. Similarly $N(x, y) = x^n \hat{N}(v)$, for some function $\hat{N}$ of one variable.

So, with $y = vx$ the original differential equation becomes

$$x^n \hat{M}(v) + x^n \hat{N}(v)\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + v\right) = 0.$$

We can cancel the common factor of x^n to get

$$\hat{M}(v) + \hat{N}(v)\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + v\right) = 0.$$

This final differential equation is variables separable. Thus, using the method given earlier, we can find v, and consequently y.

Note When solving a homogeneous differential equation, it is not necessary to identify the functions $\hat{M}$ and $\hat{N}$. You just have to remember to eliminate y by putting $y = vx$. If the resulting differential equation is not variables separable then you have either made a mistake, or the original differential equation is not homogeneous.

Example 4.9. Consider the differential equation

$$x^3 + y^3 + 2xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

It is easy to verify that $M = x^3 + y^3$ and $N = 2xy^2$ are both homogeneous of degree 3. Putting $y = vx$ in the differential equation gives

$$x^3 + v^3x^3 + 2xv^2x^2 \left(x\frac{\mathrm{d}v}{\mathrm{d}x} + v \right) = 0.$$

Cancelling the common factor of x^3, and rearranging the resulting differential equation gives

$$2xv^2\frac{\mathrm{d}v}{\mathrm{d}x} = -(1 + 3v^3).$$

Separating the variables gives

$$\int \frac{2v^2}{1+3v^3}\,\mathrm{d}v = -\int \frac{\mathrm{d}x}{x}.$$

Evaluating the two integrals we get

$$\frac{2}{9}\ln(1+3v^3) = -\ln x + c,$$

where c is a constant. Since $y = vx$ then $v = \frac{y}{x}$, and so substituting for v in the above solution, we get the general solution of the original differential equation as follows:

$$\frac{2}{9}\ln\left(1 + \frac{3y^3}{x^3}\right) + \ln x = c.$$

Example 4.10. Consider the differential equation

$$y + x\sec\left(\frac{y}{x}\right) - x\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

It is easy to verify that $M = y + x\sec\left(\frac{y}{x}\right)$ and $N = -x$ are both homogeneous of degree 1. Putting $y = vx$ in the differential equation gives

$$vx + x\sec\left(\frac{vx}{x}\right) - x\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + v\right) = 0.$$

Cancelling the common factor of x, and rearranging the resulting differential equation gives

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \sec v.$$

Separating the variables gives

$$\int \frac{\mathrm{d}v}{\sec v} = \int \frac{\mathrm{d}x}{x},$$

and, since $\sec v = \frac{1}{\cos v}$, we get

$$\int \cos v\,\mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

Evaluating the two integrals, we get

$$\sin v = \ln x + c,$$

where c is a constant. Since $y = vx$ then $v = \frac{y}{x}$, and so substituting for v in the above solution, we get the general solution of the original differential equation as follows:

$$\sin\left(\frac{y}{x}\right) = \ln x + c.$$

Exercises 4.2.

1. Find the general solution of the following differential equations.

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6(y+1)x^2$

   (b) $(x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy$

   (c) $\dfrac{y^2}{x}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{(x^2+1)y}$

   (d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xe^{x+y}$

2. Find the solution of the differential equation

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\cos x}{3y^2},$$

   where $y = 2$ when $x = \pi$, expressing y explicitly in terms of x.

3. Find the general solution of the differential equation

   $$y' = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

4.3 ODEs of order one: Exact differential equations

In this section we study exact differential equations and integrating factors which allow us to reduce some non-exact differential equations to the exact case. In particular, we will solve linear differential equations of order one with the help of an integrating factor.

Exact differential equations

Let M and N be functions of two variables. The differential equation

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

When a differential equation of the given form is exact then the general solution has the form $f(x, y) = c$, for some constant c and some function f, of two variables, satisfying

$$\frac{\partial f}{\partial x} = M \text{ and } \frac{\partial f}{\partial y} = N.$$

To see this, we differentiate $f(x, y) = c$ with respect to x. This gives

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Since $\frac{\mathrm{d}x}{\mathrm{d}x} = 1$, $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ we get

$$M + N\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

as required.

Example 4.11. Consider the differential equation

$$2xy + (x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Here $M = 2xy$ and $N = x^2 + y^2$, and since $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$, the differential equation is exact. Thus we look for a function f of two variables such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. That is,

$$\frac{\partial f}{\partial x} = 2xy, \tag{2}$$

$$\frac{\partial f}{\partial y} = x^2 + y^2. \tag{3}$$

Integrating both sides of equation (2) with respect to x gives

$$f = x^2 y + g(y), \tag{4}$$

where g is a function of y. To find g we differentiate equation (4) with respect to y to get

$$\frac{\partial f}{\partial y} = x^2 + g'(y).$$

Comparing this expression for $\frac{\partial f}{\partial y}$ with the one in equation (3) gives

$$x^2 + g'(y) = x^2 + y^2.$$

Thus $g'(y) = y^2$ and so $g(y) = \frac{1}{3}y^3$. Hence $f(x, y) = x^2 y + \frac{1}{3}y^3$, and so the solution of the original differential equation is

$$x^2 y + \frac{1}{3}y^3 = c,$$

for some constant c.

Example 4.12. Consider the differential equation

$$1 + \ln x + 2x \ln y + \left(\frac{x^2}{y} - 2y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Here $M = 1+\ln x+2x\ln y$ and $N = \frac{x^2}{y} - 2y$, and since $\frac{\partial M}{\partial y} = \frac{2x}{y}$ and $\frac{\partial N}{\partial x} = \frac{2x}{y} = \frac{\partial M}{\partial y}$, the differential equation is exact. Thus we look for a function f of two variables such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. That is,

$$\frac{\partial f}{\partial x} = 1 + \ln x + 2x \ln y, \tag{5}$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} - 2y. \tag{6}$$

Integrating both sides of equation (5) with respect to x gives[1]

$$f = x \ln x + x^2 \ln y + g(y), \tag{7}$$

where g is a function of y. To find g we differentiate equation (7) with respect to y to get

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} + g'(y).$$

Comparing this expression for $\frac{\partial f}{\partial y}$ with the one in equation (6) gives

$$\frac{x^2}{y} + g'(y) = \frac{x^2}{y} - 2y.$$

Thus $g'(y) = -2y$ and so $g(y) = -y^2$. Hence $f(x,y) = x\ln x + x^2 \ln y - y^2$, and so the solution of the original differential equation is

$$x \ln x + x^2 \ln y - y^2 = c,$$

for some constant c.

Integrating factors

Let M and N be functions of two variables. In general the differential equation

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

will not be exact. However, suppose we can find a function μ of two variables such that the differential equation

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

[1] To evaluate $\int \ln x \, \mathrm{d}x$, use integration by parts with $u = \ln x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$.

is exact. Then we can solve the resulting differential equation using the method described, and hence find the general solution of the original differential equation. The function μ is called an **integrating factor** for the original differential equation.

Unfortunately, in general, it is difficult to find an integrating factor for most differential equations. However, there are certain classes of differential equations for which there is a method for finding one. In the next subsection we consider one important class of differential equations that can be solved using an integrating factor.

Linear differential equations

A ordinary differential equation of order one is **linear** if it can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

where P and Q are functions of one variable.

Given a linear differential equation of this form, let

$$\mu(x) = e^{\int P(x)\,\mathrm{d}x}.$$

We claim that μ is an integrating factor for the linear differential equation.

First note that[2]

$$\mu'(x) = P(x)e^{\int P(x)\,\mathrm{d}x} = P(x)\mu(x).$$

Now rearrange the linear differential equation in the form

$$P(x)y - Q(x) + \frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and then multiply both sides by $\mu(x)$ to get

$$\mu(x)(P(x)y - Q(x)) + \mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{8}$$

We now need to show that this differential equation is exact. Let $M = \mu(x)(P(x)y - Q(x))$ and $N = \mu(x)$. Then

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \mu(x)P(x), \\
\frac{\partial N}{\partial x} &= \mu'(x) \\
&= \mu(x)P(x).
\end{aligned}$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and so the differential equation (8) is exact. Thus $\mu(x)$ is an integrating factor for the linear differential equation.

[2] To differentiate $\mu(x)$, we use the chain rule and the fact that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int P(x)\,\mathrm{d}x\right) = P(x)$.

To solve the exact differential equation (8) we have to look for a function f of two variables such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$, that is

$$\frac{\partial f}{\partial x} = \mu(x)(P(x)y - Q(x)), \tag{9}$$

$$\frac{\partial f}{\partial y} = \mu(x). \tag{10}$$

Integrating (10) with respect to y gives

$$f = \mu(x)y + g(x) \tag{11}$$

where g is a function of x. Differentiating (11) with respect to x gives

$$\frac{\partial f}{\partial x} = \mu'(x)y + g'(x) = P(x)\mu(x)y + g'(x).$$

Comparing this expression for $\frac{\partial f}{\partial x}$ with the one in equation (9) gives

$$P(x)\mu(x)y + g'(x) = \mu(x)(P(x)y - Q(x)).$$

Thus $g'(x) = -\mu(x)Q(x)$ and so $g(x) = -\int \mu(x)Q(x)\,\mathrm{d}x$. Hence we have $f(x,y) = \mu(x)y - \int \mu(x)Q(x)\,\mathrm{d}x$, and so the solution of the original differential equation is

$$\mu(x)y - \int \mu(x)Q(x)\,\mathrm{d}x = c,$$

where c is a constant. Solving this for y and including the constant c in the indefinite integral gives

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)\,\mathrm{d}x. \tag{12}$$

So to solve the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

first find the integrating factor $\mu(x) = e^{\int P(x)\,\mathrm{d}x}$, and then use identity (12) to find y.

Example 4.13. Consider the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ay = Ae^{bx},$$

where A, a and b are constants.

Here $P(x) = a$ and $Q(x) = Ae^{bx}$. Let

$$\mu(x) = e^{\int a\,\mathrm{d}x} = e^{ax}.$$

Then

$$\begin{aligned} y &= \frac{1}{\mu(x)} \int \mu(x) Q(x) \, \mathrm{d}x \\ &= e^{-ax} \int (e^{ax})(Ae^{bx}) \, \mathrm{d}x \\ &= e^{-ax} \int Ae^{(a+b)x} \, \mathrm{d}x \\ &= e^{-ax} \left(\frac{Ae^{(a+b)x}}{a+b} + c \right) \\ &= \frac{A}{a+b} e^{bx} + ce^{-ax}. \end{aligned}$$

So the general solution of the differential equation is

$$y = \frac{A}{a+b} e^{bx} + ce^{-ax},$$

where c is a constant.

Example 4.14. Consider the linear differential equation

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = x(x^2+1).$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{(x^2+1)} y = x.$$

Thus the differential equation is linear with $P(x) = \frac{x}{(x^2+1)}$ and $Q(x) = x$. Let

$$\begin{aligned} \mu(x) &= e^{\int \frac{x}{(x^2+1)} \, \mathrm{d}x} \\ &= e^{\frac{1}{2} \int \frac{2x}{(x^2+1)} \, \mathrm{d}x} \\ &= e^{\frac{1}{2} \ln(x^2+1)} \\ &= \left(e^{\ln(x^2+1)} \right)^{\frac{1}{2}} \\ &= (x^2+1)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{aligned} y &= \frac{1}{\mu(x)} \int \mu(x) Q(x) \, \mathrm{d}x \\ &= (x^2+1)^{-\frac{1}{2}} \int \left((x^2+1)^{\frac{1}{2}} \right) x \, \mathrm{d}x \\ &= (x^2+1)^{-\frac{1}{2}} \left(\frac{1}{3} (x^2+1)^{\frac{3}{2}} + c \right) \\ &= \frac{1}{3}(x^2+1) + c(x^2+1)^{-\frac{1}{2}} \\ &= \frac{1}{3}(x^2+1) + \frac{c}{\sqrt{x^2+1}}. \end{aligned}$$

So the general solution of the differential equation is

$$y = \frac{1}{3}(x^2+1) + \frac{c}{\sqrt{x^2+1}},$$

where c is a constant.

Exercises 4.3.

1. Find the general solution of the following differential equations.

   (a) $\left(x^3 - 2x^2y + 3\sqrt{y+1}\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2 - 3x^2y$

   (b) $2xy + 2y^3 + 4x + (x^2 + 6xy^2 + 3y^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

   (c) $xy^2 + 3x^2y + \sqrt{x} + (x^2y + x^3 + \cos y)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

2. Find the solution of the differential equation

$$\left(\frac{x^4}{y} + 15y^5\right)\frac{\mathrm{d}y}{\mathrm{d}x} + e^{2x} + 4x^3\ln y = 0,$$

   where $y = 1$ when $x = 0$.

3. Show that the differential equation

$$(y^2 - x^2) + (xy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

   has an integrating factor of the form $\mu(x) = x^r$, for some real number r. Hence find the general solution of this differential equation.

4. Find the general solution of the following differential equations.

   (a) $(x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3$

   (b) $2\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{x} = 7x^2 + 6$

   (c) $(x^2+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4xy = 16x$

4.4 ODEs of order one: Change of variables

When solving a homogeneous differential equation we changed the variables with the substitution $y = vx$. Often other first order differential equations can be solved by changing the variables x or y.

Obvious change of variables

For some differential equations it is clear what the change of variables should be, as illustrated in the following example.

Example 4.15. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x + 4y + 3)^2.$$

Here we put $z = x + 4y + 3$. Then

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + 4\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= 1 + 4(x + 4y + 3)^2 \\ &= 1 + 4z^2,\end{aligned}$$

and so we get the variables separable equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + 4z^2.$$

Separating the variables gives

$$\int \frac{\mathrm{d}z}{1 + 4z^2} = \int \mathrm{d}x,$$

and evaluating the two integrals we get

$$\frac{1}{2}\arctan(2z) = x + c,$$

for some constant c.

Substituting for z gives

$$\frac{1}{2}\arctan(2(x + 4y + 3)) = x + c,$$

which is the general solution of the original equation. The general solution can be tidied up (with a little effort) to get

$$y = \frac{1}{8}(\tan(2x + d) - 2x - 6),$$

where $d = 2c$.

In general, it is not always obvious what the change of variables should be. However, there are some classes of differential equations that can be solved with a standard change of variable. We discuss such a class in the next subsection.

A class of ODEs solvable by change of variables

Let $a, b, c, d, e, f \in \mathbb{R}$ with at least one of a and b nonzero and at least one of d and e nonzero. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax + by + c}{dx + ey + f}.$$

The differential equations in this class can be solved using a change of variables, depending on whether the lines $ax + by + c = 0$ and $dx + ey + f = 0$ are parallel.

Suppose first that $ax + by + c = 0$ and $dx + ey + f = 0$ are not parallel. Then let (α, β) be the point where these two lines intersect, and put

$$\begin{aligned} x &= X + \alpha, \\ y &= Y + \beta. \end{aligned}$$

We claim that $\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}y}{\mathrm{d}x}$. To see this, suppose that $y = g(x)$ for some function g. Then $\frac{\mathrm{d}y}{\mathrm{d}x} = g'(x)$. Also $Y + \beta = g(X + \alpha)$ and so, by the chain rule,

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = g'(X + \alpha) = g'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Thus $\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}y}{\mathrm{d}x}$, as claimed.

Since (α, β) is on the line $ax + by + c = 0$, it follows that $a\alpha + b\beta + c = 0$. Thus

$$\begin{aligned} ax + by + c &= a(X + \alpha) + b(Y + \beta) + c \\ &= aX + bY + a\alpha + b\beta + c \\ &= aX + bY. \end{aligned}$$

Similarly

$$dx + ey + f = dX + eY.$$

Using the above, after the change of variables, the differential equation becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{aX + bY}{dX + eY}.$$

This differential equation is homogeneous, and so can be solved using the method given earlier. Thus we can find the general solution of the original equation.

Suppose now that $ax + by + c = 0$ and $dx + ey + f = 0$ are parallel. Then there are $\lambda, g \in \mathbb{R}$ such that

$$dx + ey + f = \lambda(ax + by + g).$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax + by + c}{\lambda(ax + by + g)}.$$

Put $z = ax + by$. Then

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= a + b\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= a + b\frac{ax+by+c}{\lambda(ax+by+g)}\\ &= a + b\frac{z+c}{\lambda(z+g)}.\end{aligned}$$

Thus we obtain the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a + \frac{b(z+c)}{\lambda(z+g)}$$

which variables separable, and so can be solved using the method given earlier. Thus we can find the general solution of the original equation.

Example 4.16. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-6}{x-y-2}.$$

The lines $x+y-6=0$ and $x-y-2=0$ are not parallel[3] and intersect at the point $(4,2)$. Putting $x = X+4$ and $y = Y+2$ into the differential equation gives

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{(X+4)+(Y+2)-6}{(X+4)-(Y+2)-2} = \frac{X+Y}{X-Y}.$$

This differential equation is homogeneous, so putting $Y = XV$ we get $\frac{\mathrm{d}Y}{\mathrm{d}X} = X\frac{\mathrm{d}V}{\mathrm{d}X} + V$, and the differential equation becomes

$$X\frac{\mathrm{d}V}{\mathrm{d}X} + V = \frac{X+XV}{X-XV} = \frac{1+V}{1-V}.$$

Thus

$$X\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{1+V}{1-V} - V = \frac{1+V-V(1-V)}{1-V} = \frac{1+V^2}{1-V}.$$

The last differential equation is variables separable and so

$$\int \frac{1-V}{1+V^2}\,\mathrm{d}V = \int \frac{\mathrm{d}X}{X}.$$

Hence

$$\int \left(\frac{1}{1+V^2} - \frac{V}{1+V^2}\right)\mathrm{d}V = \int \frac{\mathrm{d}X}{X}.$$

[3]To find the point where the lines intersect solve the equations

$$\begin{aligned} x+y &= 6,\\ x-y &= 2.\end{aligned}$$

Evaluating the two integrals gives

$$\arctan V - \frac{1}{2}\ln(1+V^2) = \ln X + c,$$

where c is a constant. Multiplying throughout by 2 and rearranging the solution gives

$$2\arctan V - (\ln(1+V^2) + 2\ln X) = d,$$

where $d = 2c$. Now $2\ln X = \ln X^2$, and so

$$\ln(1+V^2) + 2\ln X = \ln(1+V^2) + \ln X^2 = \ln(X^2 + X^2V^2).$$

Thus

$$2\arctan V - \ln(X^2 + X^2V^2) = d.$$

Since $V = \frac{Y}{X}$ the solution becomes

$$2\arctan\left(\frac{Y}{X}\right) - \ln(X^2+Y^2) = d.$$

Finally, putting $X = x-4$ and $Y = y-2$ we get

$$2\arctan\left(\frac{y-2}{x-4}\right) - \ln((x-4)^2 + (y-2)^2) = d,$$

which is the general solution of the original differential equation.

Example 4.17. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y+5}{2x+4y-3}.$$

The lines $x+2y+5=0$ and $2x+4y-3=0$ are parallel so we put $z = x+2y$. Then

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= 1 + 2\frac{x+2y+5}{2x+4y-3}\\ &= 1 + 2\frac{z+5}{2z-3}\\ &= \frac{4z+7}{2z-3}.\end{aligned}$$

Hence

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{4z+7}{2z-3}.$$

This differential equation is variables separable, and separating the variables gives

$$\int \frac{2z-3}{4z+7}\,\mathrm{d}z = \int \mathrm{d}x. \tag{13}$$

Now

$$\begin{aligned}\int \frac{2z-3}{4z+7}\,\mathrm{d}z &= \frac{1}{2}\int \frac{4z-6}{4z+7}\,\mathrm{d}z\\ &= \frac{1}{2}\int \frac{4z+7-13}{4z+7}\,\mathrm{d}z\\ &= \frac{1}{2}\int \left(1-\frac{13}{4z+7}\right)\mathrm{d}z\\ &= \frac{1}{2}\left(z-\frac{13}{4}\ln(4z+7)\right).\end{aligned}$$

Thus evaluating the two integrals in equality (13) gives

$$\frac{1}{2}\left(z-\frac{13}{4}\ln(4z+7)\right) = x+c,$$

where c is a constant. Multiplying both sides of the last equality by 8 we get

$$4z-13\ln(4z+7) = 8x+d,$$

where $d = 8c$. Substituting $z = x+2y$ in the last expression and simplifying the result gives

$$8y-4x-13\ln(4x+8y+7) = d,$$

which is the general solution of the original differential equation.

Exercises 4.4.

1. Find the general solution to the following differential equations.

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+y}{x-y+2}$

   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+y}{3x+3y+1}$

2. Find the solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+y}{2x+2y+1}$ which satisfies the boundary condition $y(0) = 0$.

3. Find the general solution using the given substitution.

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2}{x+2y-3}$ using the substitution $v = x+2y-3$

   (b) $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 = \sin x$ using the substitution $u = y^2$

   (c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{x} = x^3y^6$ using the substitution $u = y^{-5}$

4.5 ODEs of order two: linear with constant coefficients

To find the general solution of a differential equation of order two is more difficult than for order one. In this and the next section we only consider linear differential equations, that is, differential equations of the form

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)y = S(x),$$

where P, Q, R and S are functions of one variable.

The homogeneous case with constant coefficients

We start with the simplest case where P, Q and R are constant functions and S is identically zero. That is, the differential equation has the form

$$P\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = 0, \tag{14}$$

where $P, Q, R \in \mathbb{R}$. Such a differential equation is called **homogeneous** with **constant coefficients**.

In looking for the general solution of equation (14) we note first that if y is a solution of the differential equation then so is Ay, for a constant A. Also, since the differential equation is of order two, we expect the general solution to involve two arbitrary constants. Finally, in Example 4.6 we found that the general solution of a homogeneous linear differential equation of order one was an exponential function. Inspired by this, we look for solutions of equation (14) having the form $y = e^{tx}$, for some constant t. For this y, $\frac{\mathrm{d}y}{\mathrm{d}x} = te^{tx}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = t^2e^{tx}$. Substituting for y, $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in equation (14) gives

$$Pt^2e^{tx} + Qte^{tx} + Re^{tx} = 0.$$

Since $e^{tx} \neq 0$, we can cancel this factor from the previous equation to get

$$Pt^2 + Qt + R = 0.$$

This is a quadratic equation in t, and is called the **auxiliary equation** of the differential equation given in (14).

Let α and β be the roots of the auxiliary equation. Then the general solution of (14) depends on the nature of α and β. There are three cases to consider.

Case 1: $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$. This is the easiest case; since we know that $e^{\alpha x}$ and $e^{\beta x}$ are distinct solutions of equation (14), the general solution is

$$y = Ae^{\alpha x} + Be^{\beta x},$$

where A and B are (real) constants.

Case 2: $\alpha, \beta \in \mathbb{R}$ with $\alpha = \beta$. In this case, since $\alpha = \beta$, $e^{\alpha x} = e^{\beta x}$ and so we only have one solution of equation (14). However, it is straightforward to check that $xe^{\alpha x}$ is also a solution of equation (14), and so the general solution is

$$y = (A + Bx)e^{\alpha x},$$

where A and B are (real) constants.

Case 3: $\alpha, \beta \notin \mathbb{R}$. We first recall that it is not possible to have one of α or β in $\mathbb{R}$ and the other not in $\mathbb{R}$, since the non-real roots of a polynomial equation with coefficients in $\mathbb{R}$ occur in conjugate pairs. Thus, in this case, $\alpha = a + bi$ and $\beta = a - bi$ where $a, b \in \mathbb{R}$ with $b \neq 0$ and $i = \sqrt{-1}$. Doing the same as in case 1 we could write the general solution of equation (14) in the form

$$y = \hat{A}e^{(a+bi)x} + \hat{B}e^{(a-bi)x},$$

where $\hat{A}$ and $\hat{B}$ are complex constants. However, it is unsatisfactory to have the solution of a differential equation involving a real function expressed in terms of complex functions, so we aim to find a better way of expressing the solution. Now[4]

$$e^{(a+bi)x} = e^{ax+bxi} = e^{ax}e^{bxi} = e^{ax}(\cos(bx) + i\sin(bx)).$$

Similarly,

$$e^{(a-bi)x} = e^{ax-bxi} = e^{ax}e^{-bxi} = e^{ax}(\cos(bx) - i\sin(bx)).$$

Using these equivalent expressions for $e^{(a+bi)x}$ and $e^{(a-bi)x}$ we get

$$\begin{aligned} \hat{A}e^{(a+bi)x} + \hat{B}e^{(a-bi)x} &= \hat{A}e^{ax}(\cos(bx) + i\sin(bx)) + \\ &\quad \hat{B}e^{ax}(\cos(bx) - i\sin(bx)) \\ &= e^{ax}((\hat{A} + \hat{B})\cos(bx) + i(\hat{A} - \hat{B})\sin(bx)) \\ &= e^{ax}(A\cos(bx) + B\sin(bx)), \end{aligned}$$

where $A = \hat{A} + \hat{B}$ and $B = i(\hat{A} - \hat{B})$.[5]

In summary, we have the following method for finding the general solution of equation (14). First find the solutions α and β of the auxiliary equation

$$Pt^2 + Qt + R = 0.$$

[4]Here we use the standard rules for exponentials together with the result that

$$e^{\theta i} = \cos\theta + i\sin\theta,$$

for all $\theta \in \mathbb{R}$.

[5]Note, since $A, B \in \mathbb{R}$, it follows that $\hat{B}$ is the complex conjugate of $\hat{A}$.

Then the following table gives the general solution of equation (14), where A and B are constants.

Nature of the roots of the auxiliary equation	General solution of equation (14)
$\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$	$y = Ae^{\alpha x} + Be^{\beta x}$
$\alpha, \beta \in \mathbb{R}$, $\alpha = \beta$	$y = (A + Bx)e^{\alpha x}$
$\alpha = a + bi$, $\beta = a - bi$, $a, b \in \mathbb{R}$, $b \neq 0$	$y = e^{ax}(A\cos(bx) + B\sin(bx))$

Example 4.18. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$$

The auxiliary equation of this differential equation is

$$t^2 - 5t + 6 = 0.$$

Thus $(t-2)(t-3) = 0$, and so $t = 2, 3$. Hence the general solution of the differential equation is

$$y = Ae^{2x} + Be^{3x},$$

where A and B are constants.

Example 4.19. Consider the differential equation

$$4\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

The auxiliary equation of this differential equation is

$$4t^2 - 4t + 1 = 0.$$

Thus $(2t-1)^2 = 0$, and so $t = \frac{1}{2}$ (twice). Hence the general solution of the differential equation is

$$y = (A + Bx)e^{\frac{1}{2}x},$$

where A and B are constants.

Example 4.20. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 10\frac{\mathrm{d}y}{\mathrm{d}x} + 29y = 0.$$

The auxiliary equation of this differential equation is

$$t^2 - 10t + 29 = 0.$$

Thus

$$\begin{aligned} t &= \frac{10 \pm \sqrt{100 - 4 \times 29}}{2} \\ &= \frac{10 \pm \sqrt{-16}}{2} \\ &= 5 \pm 2i. \end{aligned}$$

Hence the general solution of the differential equation is

$$y = e^{5x}(A\cos(2x) + B\sin(2x)),$$

where A and B are constants.

The nonhomogeneous case with constant coefficients

We now consider the case where P, Q and R are constant functions and S is a function of x. That is, the differential equation has the form

$$P\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = S(x), \tag{15}$$

where $P, Q, R \in \mathbb{R}$. Such a differential equation is called **nonhomogeneous** with **constant coefficients**.

To find the general solution of equation (15) we have to do two things. First we find the general solution $C(x)$ of the **homogeneous part**

$$P\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = 0.$$

This is called the **complementary function** of equation (15). Next we find any solution $p(x)$ of equation (15). This is called a **particular integral** of the differential equation. Then the general solution of equation (15) is

$$y = C(x) + p(x).$$

Finally, if there are any boundary conditions we use them to determine the arbitrary constants in the general solution.

Finding a particular integral

In the above method, we know how to find the complementary function. However, finding a particular integral is very hard for most functions $S(x)$, but there are some families of functions for which it is possible to find a particular integral. Here we give three such families of functions.

Family 1: $S(x) = Me^{\gamma x}$, where $M, \gamma \in \mathbb{R}$. In this case we look for a particular integral of the form
$$p(x) = Gx^j e^{\gamma x},$$
where j is the number of times γ is a root of the auxiliary equation, and G is a constant.

Family 2: $S(x) = M\cos(cx) + N\sin(cx)$, where $M, N, c \in \mathbb{R}$. In this case we look for a particular integral of the form
$$p(x) = G\cos(cx) + H\sin(cx),$$
where G and H are constants, unless the roots of the auxiliary equation are $\pm ci$, in which case we look for a particular integral of the form
$$p(x) = x(G\cos(cx) + H\sin(cx)).$$

Family 3: $S(x) = M_k x^k + M_{k-1}x^{k-1} + \cdots + M_1 x + M_0$, where $M_k, M_{k-1}, \ldots, M_1, M_0 \in \mathbb{R}$. In this case we look for a particular integral of the form
$$p(x) = x^j(m_k x^k + m_{k-1}x^{k-1} + \cdots + m_1 x + m_0),$$
where j is the number of times 0 is a root of the auxiliary equation, and $m_k, m_{k-1}, \ldots, m_1, m_0$ are constants.

Observe that, if 0 is a root of the auxiliary equation then the differential equation
$$P\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = S(x)$$
must have $R = 0$. That is the differential equation is
$$P\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} = S(x).$$
In this case, it is easier to change the variable by putting $u = \frac{\mathrm{d}y}{\mathrm{d}x}$ (so $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$) to get
$$P\frac{\mathrm{d}u}{\mathrm{d}x} + Qu = S(x),$$
and then solve the resulting first order differential equation. If we do this, then whenever $S(x)$ is a polynomial of degree k the particular integral is also a polynomial of degree k.

In general, we note that for all three families of functions considered, the particular integral has the same form as $S(x)$, except when the roots of the auxiliary equation take certain values. In the latter case we multiply the form of the particular integral by a suitable power of x; this is required because, in this case, $S(x)$ is part of the complementary function and so cannot be a particular integral of the nonhomogeneous differential equation.

Finally, we observe that if $S(x)$ is a sum of functions from the three families considered then the particular integral will be a sum of the corresponding particular integrals.

Example 4.21. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 20e^{-3x},$$

with boundary conditions $y = 5$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$.

From Example 4.18, the auxiliary equation of the homogeneous part of this differential equation is $t^2 - 5t + 6 = 0$, with roots $t = 2, 3$. Hence the complementary function of the differential equation is

$$C = Ae^{2x} + Be^{3x},$$

where A and B are constants.

Here $S = 20e^{-3x}$, so we look for a particular integral of the form $p = Ge^{-3x}$. Then $\frac{\mathrm{d}p}{\mathrm{d}x} = -3Ge^{-3x}$ and $\frac{\mathrm{d}^2p}{\mathrm{d}x^2} = 9Ge^{-3x}$. Now, since p is a particular integral of the differential equation it must satisfy the differential equation. So putting $y = p$ in the differential equation gives

$$9Ge^{-3x} - 5(-3Ge^{-3x}) + 6Ge^{-3x} = 20e^{-3x}.$$

Cancelling the common factor of e^{-3x} from this expression, and simplifying the result gives $30G = 20$, and so $G = \frac{2}{3}$. Thus the particular integral is

$$p = \frac{2}{3}e^{-3x}.$$

From the above, the general solution of the differential equation is

$$y = C + p = Ae^{2x} + Be^{3x} + \frac{2}{3}e^{-3x}.$$

We now use the boundary conditions to determine A and B. Differentiating the general solution with respect to x gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2Ae^{2x} + 3Be^{3x} - 2e^{-3x}.$$

Now, when $x = 0$ then $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$. Thus, putting $x = 0$ in the general solution and its derivative, and then comparing them with the corresponding boundary condition gives

$$\begin{aligned} A + B + \frac{2}{3} &= 5, \\ 2A + 3B - 2 &= 3. \end{aligned}$$

Simplifying these equations we get

$$\begin{aligned} 3A + 3B &= 13, & (16)\\ 2A + 3B &= 5. & (17) \end{aligned}$$

Subtracting equation (17) from equation (16) gives $A = 8$; substituting $A = 8$ in equation (17) gives $16 + 3B = 5$, which gives $B = -\frac{11}{3}$. Thus the solution of the differential equation with the given boundary conditions is

$$y = 8e^{2x} - \frac{11}{3}e^{3x} + \frac{2}{3}e^{-3x}.$$

Example 4.22. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 3e^{2x}.$$

As in the previous example, the roots of the auxiliary equation are $t = 2, 3$ and the complementary function of the differential equation is

$$C = Ae^{2x} + Be^{3x},$$

where A and B are constants.

Here $S = 3e^{2x}$. Since 2 is a single root of the auxiliary equation, we look for a particular integral of the form $p = Gxe^{2x}$. Then

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x} &= Ge^{2x} + 2Gxe^{2x}\\ &= G(1+2x)e^{2x},\\ \frac{\mathrm{d}^2p}{\mathrm{d}x^2} &= 2Ge^{2x} + 2G(1+2x)e^{2x}\\ &= 4G(1+x)e^{2x}. \end{aligned}$$

So putting $y = p$ in the differential equation gives

$$4G(1+x)e^{2x} - 5G(1+2x)e^{2x} + 6Gxe^{2x} = 3e^{2x}.$$

Cancelling the common factor of e^{2x} from this expression gives

$$G(4 + 4x - 5 - 10x + 6x) = 3.$$

Thus $-G = 3$, [6] and so $G = -3$. Thus the particular integral is

$$p = -3xe^{2x}.$$

From the above, the general solution of the differential equation is

$$y = C + p = Ae^{2x} + Be^{3x} - 3xe^{2x}.$$

[6]Note that all of the x terms cancel. If this does not happen then we have done something wrong.

Example 4.23. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 10\frac{\mathrm{d}y}{\mathrm{d}x} + 29y = 29x^2 + 38x - 18.$$

From Example 4.20, the auxiliary equation of the homogeneous part of this differential equation is $t^2 - 10t + 29 = 0$, with roots $t = 5 \pm 2i$. Hence the complementary function of the differential equation is

$$C = e^{5x}(A\cos(2x) + B\sin(2x)),$$

where A and B are constants.

Here $S = 29x^2 + 38x - 18$, so we look for a particular integral of the form $p = ax^2 + bx + c$. Then $\frac{\mathrm{d}p}{\mathrm{d}x} = 2ax + b$ and $\frac{\mathrm{d}^2p}{\mathrm{d}x^2} = 2a$. So putting $y = p$ in the differential equation gives

$$2a - 10(2ax + b) + 29(ax^2 + bx + c) = 29x^2 + 38x - 18.$$

Comparing the coefficients of x^2 and x, and the constant term, respectively, for both sides of the previous expression gives

$$\begin{aligned} 29a &= 29, \\ -20a + 29b &= 38, \\ 2a - 10b + 29c &= -18. \end{aligned}$$

Solving the system of linear equations gives $a = 1$, $b = 2$ and $c = 0$. Thus the particular integral is

$$p = x^2 + 2x.$$

From the above, the general solution of the differential equation is

$$y = C + p = e^{5x}(A\cos(2x) + B\sin(2x)) + x^2 + 2x.$$

Example 4.24. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 4\sin x + 3\cos x.$$

The auxiliary equation of the homogeneous part of this differential equation is $t^2 + 1 = 0$, and so $t = \pm i$. Hence the complementary function of the differential equation is

$$C = A\cos x + B\sin x,$$

where A and B are constants.

Here $S = 4\sin x + 3\cos x$. Since $\pm i$ are roots of the auxiliary equation, we look for a particular integral of the form $p = x(G\sin x + H\cos x)$. Then

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x} &= G\sin x + H\cos x + x(G\cos x - H\sin x), \\ \frac{\mathrm{d}^2p}{\mathrm{d}x^2} &= G\cos x - H\sin x + G\cos x - H\sin x + x(-G\sin x - H\cos x) \\ &= 2(G\cos x - H\sin x) - x(G\sin x + H\cos x). \end{aligned}$$

Putting $y = p$ in the differential equation gives

$2(G\cos x - H\sin x) - x(G\sin x + H\cos x) + x(G\sin x + H\cos x) = 4\sin x + 3\cos x.$

Thus $2(G\cos x - H\sin x) = 4\sin x + 3\cos x$. Thus $G = \frac{3}{2}$ and $H = -2$, and so the particular integral is

$$p = x\left(\frac{3}{2}\sin x - 2\cos x\right).$$

From the above, the general solution of the differential equation is

$$y = C + p = A\cos x + B\sin x + x\left(\frac{3}{2}\sin x - 2\cos x\right).$$

Example 4.25. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = 2e^x + 3x + 4.$$

The auxiliary equation of the homogeneous part of this differential equation is $t^2 + t + 1 = 0$, and so

$$\begin{aligned} t &= \frac{-1 \pm \sqrt{1-4}}{2} \\ &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \end{aligned}$$

Hence the complementary function of the differential equation is

$$C = e^{-\frac{1}{2}x}\left(A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right),$$

where A and B are constants.

Here $S = 2e^x + 3x + 4$. This is the sum of functions from two of the families considered. Hence we look for a particular integral of the form $p = Ge^x + Hx + K$. Then

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x} &= Ge^x + H, \\ \frac{\mathrm{d}^2p}{\mathrm{d}x^2} &= Ge^x. \end{aligned}$$

Putting $y = p$ in the differential equation gives

$$Ge^x + (Ge^x + H) + (Ge^x + Hx + K) = 2e^x + 3x + 4.$$

Thus $3Ge^x + Hx + H + K = 2e^x + 3x + 4$. Thus $G = \frac{2}{3}$, $H = 3$ and $H + K = 4$, so $K = 1$. Thus the particular integral is

$$p = \frac{2}{3}e^x + 3x + 1.$$

From the above, the general solution of the differential equation is

$$y = C + p = e^{-\frac{1}{2}x}\left(A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right) + \frac{2}{3}e^{x} + 3x + 1.$$

Exercises 4.5.

1. Find the general solution of the following differential equations.

   (a) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} - 8y = 0$

   (b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\dfrac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0$

   (c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0$

2. Find a particular integral of the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 8y = F(x),$$
when $F(x)$ is (a) $12e^{-2x}$; (b) $10e^{2x}$; (c) $16x - 12$; (d) $30\cos 2x$.

3. Find a particular integral of the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = G(x),$$
when $G(x)$ is (a) $6e^{x}$; (b) $10e^{-3x}$; (c) $54x^2$; (d) $3e^{-x} + 10\sin x$.

4. Find the solution of the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 15y = 14e^{2x}$$
with initial conditions $y(0) = 1$ and $y'(0) = 0$.

5. Find the solution of the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 12e^{-2x},$$
where $y = 5$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3$ when $x = 0$.

4.6 ODEs of order two: general linear equations

In this section we continue our study of order two linear differential equations by considering the general case, that is, differential equations of the form

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)y = S(x), \tag{18}$$

where P, Q, R and S are functions of one variable.

Finding the general solution from a solution of the homogeneous part

The **homogeneous part** of equation (18) is the differential equation

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+Q(x)\frac{\mathrm{d}y}{\mathrm{d}x}+R(x)y=0.$$

As in the case with constant coefficients, finding the general solution of equation (18) depends upon being able to find a solution of the homogeneous part. To see this, suppose $y=g$ is a solution of the homogeneous part, for some function g of x. Then

$$P(x)\frac{\mathrm{d}^2g}{\mathrm{d}x^2}+Q(x)\frac{\mathrm{d}g}{\mathrm{d}x}+R(x)g=0.$$

Now we change the variable in equation (18) by putting $y=gv$ for some function v of x. Then, using the product rule, we get

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= g\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}g}{\mathrm{d}x},\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= g\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+2\frac{\mathrm{d}g}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}^2g}{\mathrm{d}x^2}.\end{aligned}$$

Thus, with the change of variable, equation (18) becomes

$$P(x)\left(g\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+2\frac{\mathrm{d}g}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}^2g}{\mathrm{d}x^2}\right)+Q(x)\left(g\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}g}{\mathrm{d}x}\right)+R(x)gv=S(x),$$

which can be rewritten as

$$\begin{aligned}&v\left(P(x)\tfrac{\mathrm{d}^2g}{\mathrm{d}x^2}+Q(x)\tfrac{\mathrm{d}g}{\mathrm{d}x}+R(x)g\right)+\\ &P(x)g\tfrac{\mathrm{d}^2v}{\mathrm{d}x^2}+\left(2P(x)\tfrac{\mathrm{d}g}{\mathrm{d}x}+Q(x)g\right)\tfrac{\mathrm{d}v}{\mathrm{d}x}=S(x).\end{aligned}\tag{19}$$

Now, since g is a solution of the homogeneous part of equation (18), the first term of equation (19) is zero. Thus equation (19) becomes

$$P(x)g\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+\left(2P(x)\frac{\mathrm{d}g}{\mathrm{d}x}+Q(x)g\right)\frac{\mathrm{d}v}{\mathrm{d}x}=S(x).\tag{20}$$

Now we put $w=\frac{\mathrm{d}v}{\mathrm{d}x}$ in equation (20) to get

$$P(x)g\frac{\mathrm{d}w}{\mathrm{d}x}+\left(2P(x)\frac{\mathrm{d}g}{\mathrm{d}x}+Q(x)g\right)w=S(x),$$

which is a first order differential equation for w. Solving this differential equation gives w. We can then find the general solution of equation (18) by integrating w with respect to x to get v and finally getting $y=vg$.

Note You do not have to remember the first order differential equation for w in terms of P, Q, S and g to solve equation (18); just remember the change of variable $y = gv$, and derive the first order differential equation from scratch.

Example 4.26. Consider the differential equation

$$x^2(x+1)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x(3x-2)\frac{\mathrm{d}y}{\mathrm{d}x} - (3x-2)y = 10x^3.$$

It is easy to verify that $y = x$ is a solution of the homogeneous part of this equation. So putting $y = vx$ we get

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x\frac{\mathrm{d}v}{\mathrm{d}x} + v, \\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= x\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x}. \end{aligned}$$

Thus the differential equation becomes

$$x^2(x+1)\left(x\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x}\right) + x(3x-2)\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + v\right) - (3x-2)vx = 10x^3.$$

Collecting terms in the previous equation involving $\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$ we get

$$x^3(x+1)\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \left(2x^2(x+1) + x^2(3x-2)\right)\frac{\mathrm{d}v}{\mathrm{d}x} = 10x^3,$$

and simplifying gives

$$x^3(x+1)\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 5x^3\frac{\mathrm{d}v}{\mathrm{d}x} = 10x^3.$$

Dividing the resulting equation throughout by $x^3(x+1)$, and then putting $w = \frac{\mathrm{d}v}{\mathrm{d}x}$ we get

$$\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{5w}{x+1} = \frac{10}{x+1},$$

which is a first order linear differential equation. To solve this differential equation we first find the integrating factor

$$\begin{aligned} \mu(x) &= e^{\int \frac{5}{x+1}\,\mathrm{d}x} \\ &= e^{5\ln(x+1)} \\ &= e^{\ln(x+1)^5} \\ &= (x+1)^5. \end{aligned}$$

Then

$$\begin{aligned} w &= \frac{1}{\mu(x)} \int \mu(x) \frac{10}{x+1}\, \mathrm{d}x \\ &= \frac{1}{(x+1)^5} \int (x+1)^5 \frac{10}{x+1}\, \mathrm{d}x \\ &= \frac{1}{(x+1)^5} \int 10(x+1)^4\, \mathrm{d}x \\ &= \frac{1}{(x+1)^5} \left(2(x+1)^5 + C\right) \\ &= 2 + \frac{C}{(x+1)^5}, \end{aligned}$$

where C is a constant.

Now, $\frac{\mathrm{d}v}{\mathrm{d}x} = w$, and so

$$\begin{aligned} v &= \int w\, \mathrm{d}x \\ &= \int \left(2 + \frac{C}{(x+1)^5}\right) \mathrm{d}x \\ &= 2x + \frac{D}{(x+1)^4} + E, \end{aligned}$$

where $D = -\frac{C}{4}$ and E is a constant.

Finally, $y = vx$, and so the general solution of the original differential equation is

$$y = 2x^2 + \frac{Dx}{(x+1)^4} + Ex.$$

Euler equations

In general, it is difficult to find a solution of the homogeneous part of an order two linear differential equation. However, there are some families of equations where this is possible. Here we consider one such family, the **Euler equations**. These are differential equations of the form

$$ax^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx \frac{\mathrm{d}y}{\mathrm{d}x} + cy = S(x),$$

where a, b and c are constants, and S is a function of one variable.

There are two possible ways to look for the general solution of an Euler equation.

- Look for a solution of the homogeneous part of the equation of the form $y = x^k$. If a suitable value of k is found then apply the method described above to find the general solution of the nonhomogeneous equation. (This method does not always work as the values of k found could be non-real.)

- Change the variable by putting $x = e^t$. This reduces the differential equation to one with constant coefficients. (This method always works, but is probably more difficult.)

We now give some examples to illustrate the two methods.

Example 4.27. Consider the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7x\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 16x^4.$$

We look for a solution of

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7x\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 0$$

of the form $y = x^k$. Then $\frac{\mathrm{d}y}{\mathrm{d}x} = kx^{k-1}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = k(k-1)x^{k-2}$. Substituting for y, $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in the homogeneous part gives

$$x^2k(k-1)x^{k-2} - 7xkx^{k-1} + 16x^k = 0,$$

and so

$$k(k-1)x^k - 7kx^k + 16x^k = 0.$$

The last equation is valid for all x and so we can cancel the x^k term. Thus

$$\begin{aligned} k(k-1) - 7k + 16 = 0 \quad &\Leftrightarrow \quad k^2 - 8k + 16 = 0 \\ &\Leftrightarrow \quad (k-4)^2 = 0 \\ &\Leftrightarrow \quad k = 4. \end{aligned}$$

So $y = x^4$ is a solution of the homogeneous part of the differential equation. We now change the variable in the differential equation by putting $y = vx^4$. Then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x^4\frac{\mathrm{d}v}{\mathrm{d}x} + 4x^3v \\ &= x^3\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + 4v\right), \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= x^3\left(x\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \frac{\mathrm{d}v}{\mathrm{d}x} + 4\frac{\mathrm{d}v}{\mathrm{d}x}\right) + 3x^2\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + 4v\right) \\ &= x^2\left(x^2\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 8x\frac{\mathrm{d}v}{\mathrm{d}x} + 12v\right). \end{aligned}$$

Substituting the expressions found for y, $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in the differential equation we get

$$x^4\left(x^2\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 8x\frac{\mathrm{d}v}{\mathrm{d}x} + 12v\right) - 7x^4\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + 4v\right) + 16vx^4 = 16x^4.$$

Cancelling the common factor of x^4 in the previous equation and simplifying the result gives

$$x^2\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\frac{\mathrm{d}v}{\mathrm{d}x} = 16.$$

Putting $w = \frac{\mathrm{d}v}{\mathrm{d}x}$ in this equation, and dividing throughout by x^2 we get

$$\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{w}{x} = \frac{16}{x^2},$$

which is a first order linear differential equation. To solve this differential equation we first find the integrating factor

$$\begin{aligned}\mu(x) &= e^{\int \frac{1}{x}\,\mathrm{d}x}\\ &= e^{\ln x}\\ &= x.\end{aligned}$$

Then

$$\begin{aligned}w &= \frac{1}{\mu(x)}\int \mu(x)\frac{16}{x^2}\,\mathrm{d}x\\ &= \frac{1}{x}\int x\frac{16}{x^2}\,\mathrm{d}x\\ &= \frac{1}{x}\int \frac{16}{x}\,\mathrm{d}x\\ &= \frac{1}{x}\left(16\ln x + C\right)\\ &= \frac{16\ln x}{x} + \frac{C}{x}\end{aligned}$$

where C is a constant.

Now, $\frac{\mathrm{d}v}{\mathrm{d}x} = w$, and so

$$\begin{aligned}v &= \int w\,\mathrm{d}x\\ &= \int \left(\frac{16\ln x}{x} + \frac{C}{x}\right)\mathrm{d}x\\ &= 8(\ln x)^2 + C\ln x + D,\end{aligned}$$

where D is a constant.[7]

Finally, $y = vx^4$, and so the general solution of the original differential equation is

$$y = \big(8(\ln x)^2 + C\ln x + D\big)x^4.$$

Example 4.28. Consider again the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7x\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 16x^4.$$

[7] To evaluate $\int \frac{16\ln x}{x}\,\mathrm{d}x$, put $u = \ln x$.

We now consider the second way of solving this equation using the change of variable $x = e^t$. Then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\mathrm{d}y}{\mathrm{d}x}e^t \\ &= x\frac{\mathrm{d}y}{\mathrm{d}x}, \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \left(\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\cdot e^t \\ &= \left(\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\cdot x \\ &= x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}. \end{aligned}$$

Using the final expressions for $\frac{\mathrm{d}y}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$ we get

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}, \\ x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}. \end{aligned}$$

Replacing the terms $x\frac{\mathrm{d}y}{\mathrm{d}x}$ and $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ of the original equation with the above equivalent terms gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} - 7\frac{\mathrm{d}y}{\mathrm{d}t} + 16y = 16x^4.$$

Simplifying the previous equation, and noting that $x^4 = (e^t)^4 = e^{4t}$, we get

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 8\frac{\mathrm{d}y}{\mathrm{d}t} + 16y = 16e^{4t}.$$

This is a linear differential equation with constant coefficients, so we can solve it using the method given earlier. The auxiliary equation of this equation is $s^2 - 8s + 16 = 0$. Thus $(s-4)^2 = 0$, and so $s = 4$ (twice). Hence the complementary function of the differential equation is

$$C(t) = (At + B)e^{4t},$$

where A and B are constants.

Since 4 is a double root of the auxiliary equation, we look for a particular integral of the form $p = Gt^2e^{4t}$. Then

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= 2Gte^{4t} + 4Gt^2e^{4t}\\ &= 2Gt(1+2t)e^{4t},\\ \frac{\mathrm{d}^2p}{\mathrm{d}t^2} &= 2G\left((1+4t)e^{4t} + 4t(1+2t)e^{4t}\right)\\ &= 2G(1+8t+8t^2)e^{4t}.\end{aligned}$$

So putting $y = p$ in the differential equation gives

$$2G(1+8t+8t^2)e^{4t} - 16Gt(1+2t)e^{4t} + 16Gt^2e^{4t} = 16e^{4t}.$$

Cancelling the common factor of e^{4t} from this expression gives

$$G(2+16t+16t^2-16t-32t^2+16t^2) = 16.$$

Thus $2G = 16$, and so $G = 8$. Thus the particular integral is

$$p = 8t^2e^{4t}.$$

From the above, the general solution of the differential equation is

$$\begin{aligned}y &= C + p\\ &= (At+B)e^{4t} + 8t^2e^{4t}\\ &= (At+B+8t^2)e^{4t}.\end{aligned}$$

Now $e^t = x$, and so $e^{4t} = x^4$ and $t = \ln x$. Thus

$$y = \left(A\ln x + B + 8(\ln x)^2\right)x^4.$$

Exercises 4.6.

1. Find the general solutions of the following differential equations.

   (a) $x^2y'' - 2y = x$

   (b) $x^2y'' - xy' + 2y = \ln x$

   (c) $x^2y'' + 5xy' + 4y = x^2 + 16(\ln x)^2$

2. Find the general solution of $x^3y'' + 3x^2y' = 1 + x$. (Hint: substitution)