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Probability and Statistics

Solutions Assignment 1

1. A sample of 20 high-powered copper coils yields the following tensile strength values (in units of kilogram per square centimetre):

280, 155, 329, 140, 307, 116, 202, 262, 130, 131,
187, 187, 292, 83, 207, 197, 134, 294, 163, 217

Solution:

(i) $n = 20$

$$\bar{x} = \frac{4013}{20} = 200.65$$

(ii)

$$s^2 = \frac{903279 - 20 \times 200.65^2}{19} = \frac{903279 - 805208.4}{19} = \frac{98070.55}{19} = 5161.608$$

(iii)

```r
strength <- c(280, 155, 329, 140, 307, 116, 202, 262,
              130, 131, 187, 187, 292,  83, 207, 197,
              134, 294, 163, 217)
xbar <- mean(strength)
xbar

## [1] 200.65

s2 <- var(strength)
s2

## [1] 5161.608
```

2. Given two events A and B for which $P(A) = 0.3$, $P(B) = 0.15$ and $P(A|B) = 0.67$. Find:

(i) $P(B^C)$

(ii) $P(A \cap B)$

(iii) $P(A \cap B^C)$

(iv) $P(A|B^C)$

(v) Are the events A and B independent? Justify your answer.

(vi) Are the events A and B mutually exclusive? Justify your answer.

Solution:

(i)

$$\begin{aligned}\Pr(B^C) &= 1 - \Pr(B) \\ &= 1 - 0.15 \\ &= 0.85\end{aligned}$$

(ii)

$$\begin{aligned}P(A \cap B) &= \Pr(A|B)\Pr(B) \\ &= 0.67 \times 0.15 \\ &= 0.1005\end{aligned}$$

(iii)

$$\begin{aligned}\Pr(A \cap B^c) &= \Pr(A) - \Pr(A \cap B) \\ &= 0.3 - 0.1005 \\ &= 0.1995\end{aligned}$$

(iv)

$$\begin{aligned}P(A|B^c) &= \frac{\Pr(A \cap B^c)}{\Pr(B^c)} \\ &= \frac{0.1995}{0.85} \\ &= 0.2347\end{aligned}$$

(v) $\Pr(A|B) \neq \Pr(A) \implies A$ and B are not mutually independent.
[Or, $\Pr(A \cap B) \neq \Pr(A)\Pr(B) \implies A$ and B are not mutually independent,
Or, $\Pr(B|A) \neq \Pr(B) \implies A$ and B are not mutually independent]

(vi) $\Pr(A \cap B) \neq 0 \implies A$ and B are not mutually exclusive.
[Or, $\Pr(A|B) \neq 0 \implies A$ and B are not mutually exclusive.]

3. The probability that a dog is affected by a certain rare disease is 0.002. We have collected a random sample of 3000 dogs. We are interested in the probability that there is at most one dog affected by that disease.

(i) Calculate the exact probability. Give also the `R` commands.

(ii) Calculate the probability by using the Poisson approximation. Give also the `R` commands.

(iii) Calculate the probability by using the Normal approximation. Give also the `R` commands.

(iv) Suppose there is a blood test to detect this disease. This test is very accurate: in fact, the probability of the test resulting positive given that the dog has the disease is 0.99, and the probability of the test resulting negative given that the dog does not have the disease is 0.95.

Find the probability that the test results positive given that the dog does not have the disease.

Solution:

(i)

$$
\begin{aligned}
\Pr(X \leq 1) &= \Pr(X = 0) + \Pr(X = 1) \\
&= \binom{3000}{0}(0.002)^0(1-0.002)^{3000} + \binom{3000}{1}(0.002)^1(1-0.002)^{2999} \\
&= 1 \times 1 \times 0.002463904 + 3000 \times 0.002 \times 0.002468842 \\
&= 0.002463904 + 0.01481305 \\
&= 0.01727696 \\
&\approx 0.0173
\end{aligned}
$$

```
n <- 3000
p <- 0.002

pbinom(1, n, p)

## [1] 0.01727696
```

(ii) $\mu = n \times p = 3000 \times 0.002 = 6$ $Y \sim Poisson(6)$ use Table 2.

$$
\begin{aligned}
\Pr(X \leq 1) &\approx \Pr(Y \leq 1) \\
&= 0.0174
\end{aligned}
$$

```
ppois(1, lambda = n * p)

## [1] 0.01735127
```

(iii) $\mu = n \times p = 3000 \times 0.002 = 6$, $\sigma^2 = n \times p \times (1-p) = 3000 \times 0.002 \times 0.998 = 5.988$, $W \sim N(6, 5.988)$ use Table 4.

$$
\begin{aligned}
\Pr(X \leq 1) &\approx \Pr(W \leq 1) \\
&= \Pr\left(Z \leq \frac{1-6}{\sqrt{5.988}}\right) \qquad [1] \\
&= \Pr\left(Z \leq \frac{-5}{2.447039}\right) \\
&= \Pr(Z \leq -2.043286) \\
&= 1 - \Pr(Z \leq 2.043286) \\
&\approx 1 - 0.97932 \\
&= 0.02068
\end{aligned}
$$

```
pnorm(1, mean = n * p, sd = sqrt(n * p * (1 - p)))

## [1] 0.02051208
```

(iv) Let's indicate $\Pr(D) = 0.002$ the probability that a dog has the disease, and $\Pr(D^c) = 0.998$ is the probability that a dog does not have the disease.

Let's indicate $\Pr(T|D) = 0.99$ the probability that the test reports correctly that the dog has the disease.

Let's indicate $\Pr(T^c|D^c) = 0.95$ the probability that the test reports correctly that the dog does not have the disease.

Let's indicate $\Pr(T|D^c)$ the probability that the test results come back positive given that the dog does not have the disease.

$$\Pr(T|D^c) = 1 - \Pr(T^c|D^c) = 1 - 0.95 = 0.05$$

4. A salesman assumes that the demand for a certain product is distributed according to a Poisson distribution with mean of two items per day.

> How many items does he have to buy in order to have 95% probability of not having a demand higher than the supply for 7 days?

Solution:

X is the number of items sold in 7 days, and follows a Poisson with mean $= 2 \times 7 = 14$.

We have to find the smallest k such that $\Pr(X \leq k) \geq 0.95$. Using the tables $k = 20$ $\Pr(X \leq 20) = 0.9521$.

The salesman have to buy at least 20 items.

5. In a science test the marks were normally distributed with mean 68 and standard deviation 10.

(i) What is the probability that a student scored 75 or more?

(ii) What is the probability that a student scored between 70 and 75?

Solution:

(i)

$$\begin{aligned}
\Pr(X \geq 75) &= 1 - \Pr(X < 75) \\
&= 1 - \Pr(Z < \frac{75 - 68}{10}) \\
&= 1 - \Pr(Z < \frac{7}{10}) \\
&= 1 - \Pr(Z < 0.7) \\
&= 1 - 0.7580 \\
&= 0.242
\end{aligned}$$

(ii)

$$\begin{aligned}
\Pr(70 \leq X \leq 75) &= \Pr(X \leq 75) - \Pr(X \leq 70) \\
&= \Pr(Z \leq \frac{75 - 68}{10}) - \Pr(Z \leq \frac{70 - 68}{10}) \\
&= \Pr(Z \leq \frac{7}{10}) - \Pr(Z \leq \frac{2}{10}) \\
&= \Pr(Z < 0.7) - \Pr(Z < 0.2) \\
&= 0.7580 - 0.5793 \\
&= 0.1787
\end{aligned}$$