Calculus 2

Answers to the Exam Paper May 2013

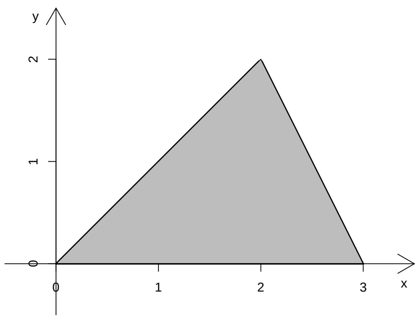
Note: The following are just the final answers for each question. To get full marks for any question, stating the final answer is not sufficient and intermediate steps and calculations are required as well.

Section A

1. (a) $\displaystyle\lim_{x\to 2}\frac{x+2-\frac{8}{x}}{1-\frac{2}{x}}=6$

   (b) $-14x-25y-z=21$

2. Sketch of T:



$$\iint_T x^3\,\mathrm{d}x\mathrm{d}y=\frac{39}{2}$$

3. maximum $f(0,3)=3$, minimum $f(-6,-3)=-9$

   The method of Lagrange multipliers finds local extrema of f on the set defined by our constraint $x^2+6x+y^2=9$. However as the set defined by our constraint is a circle (closed and bounded), we can deduce from the Extreme Value Theorem that the local extrema are in fact global extrema.

4. (a) With $M(x,y)=x^2+y^2\cos x$ and $N(x,y)=y+2y\sin x$ we have $\frac{\partial M}{\partial y}=2y\cos x=\frac{\partial N}{\partial x}$, hence the equation is exact.

   (b) $\frac{1}{3}x^3+\frac{1}{2}y^2+y^2\sin x=\frac{1}{2}$

5. (a) $y(x)=2+5(x-1)+6\frac{(x-1)^2}{2!}+6\frac{(x-1)^3}{3!}+\cdots=6e^{x-1}-x-3$

   (b) The iteration formula is $y_{i+1}=y_i+0.1(x_i+y_i+2)$. We find

$$y(1.2)\approx y_2=3.06.$$

6. (a) $\ddot{x}+4x=0$ with initial conditions $x(0)=0.5$, $\dot{x}(0)=0$

(b) The solution is $x(t) = \frac{1}{2}\cos(2t)$. The mass passes through the equilibrium point at the times $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$.

7. (a) We know that $\frac{\mathrm{d}}{\mathrm{d}x}\tanh(x) = 1 - \tanh^2(x)$. Hence for $y = \tanh(x)$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 1 = \frac{\mathrm{d}}{\mathrm{d}x}\tanh(x) + \tanh^2(x) - 1 = 0.$$

(b) If $\sinh(x) = \frac{12}{5}$ then $\cosh(x) = \frac{13}{5}$ and $\tanh(x) = \frac{12}{13}$.

8. (a) Gamma function: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ for $x > 0$

Beta function: $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$ for $x > 0$ and $y > 0$

(b) see the proof of Proposition 7.8 in the notes

Section B

9. (a) (i) $f_x = 6xy - 12x^2$, $f_y = 3x^2 + 2y - 7$

(ii) $\left(0, \frac{7}{2}\right)$, $(1, 2)$, $\left(-\frac{7}{3}, -\frac{14}{3}\right)$

(iii) $f_{xx} = 6y - 24x$, $f_{xy} = 6x$, $f_{yy} = 2$

$\left(0, \frac{7}{2}\right)$: local minimum

$(1, 2)$: saddle point

$\left(-\frac{7}{3}, -\frac{14}{3}\right)$: saddle point

(b)

$$\begin{aligned} g(x,y) &= -27 + \big(16(x+2) - 6(y-3)\big) \\ &\qquad + \frac{1}{2!}\big(2(x+2)^2 + 2\cdot 6(x+2)(y-3) + 2(y-3)^2\big) \\ &= -27 + 16(x+2) - 6(y-3) + (x+2)^2 + 6(x+2)(y-3) + (y-3)^2 \end{aligned}$$

(c) By the chain rule

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}, \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}. \end{aligned}$$

Hence

$$z_s z_t = (z_x + z_y)(z_x - z_y) = z_x^2 - z_y^2.$$

10. (a) The derivative of a function f at a point x is defined as

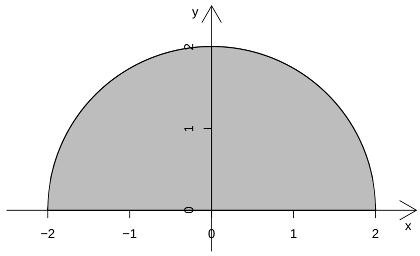
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For the function $\sin(x)$ we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \left(\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h} \right) \\
&= \sin(x) \lim_{h\to 0} \left(\frac{\cos(h)-1}{h} \right) + \cos(x) \lim_{h\to 0} \left(\frac{\sin(h)}{h} \right) \\
&= \sin(x)\cdot 0 + \cos(x) \cdot 1 \\
&= \cos(x).
\end{aligned}$$

(b) (i) The new variables (r, θ) are the polar coordinates of (x, y). See Section 3.7 of the notes for the computation of the Jacobean in this case.

(ii) Sketch of R:



$$\iint_R (x+y)^2 \,\mathrm{d}x\mathrm{d}y = 4\pi$$

(c) $\displaystyle\int_0^2 \int_{y^2}^4 \frac{y^3}{\sqrt{x^3+1}} \,\mathrm{d}x\mathrm{d}y = \frac{1}{6}\left(\sqrt{65} - 1\right)$

(d) $f_u(1, -2) = \frac{-39}{\sqrt{2}}$

11. (a) (i) $\dfrac{P}{5000 - P} = Ae^{10t}$

(ii) $A = 4$

(iii) It takes $\frac{1}{10}\ln\left(\frac{9}{4}\right) \approx 0.0811$ days (≈ 1.95 hours).

(b) $y(x) = 2e^{-x} + 2e^{2x} + xe^{2x} - x - 1$

(c) (i) See Example 4.28 in the notes.

(ii) From the previous part we have $x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}$ and $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}$. Substituting this into the differential equation gives

$$a\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right) + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0,$$

which can be rearranged as

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+(-a+b)\frac{\mathrm{d}y}{\mathrm{d}t}+cy=0.$$

This has the required form (with constant coefficients $A=a$, $B=-a+b$ and $C=c$).

12. (a) (i) $k=1$ and $k=-3$

(ii) If $y=xv$ then $y'=xv'+v$ and $y''=xv''+2v'$. Therefore the differential equation becomes

$$x^2(xv''+2v')+3x(xv'+v)-3xv=x^2-4x+2.$$

This simplifies to

$$x^3v''+5x^2v'=x^2-4x+2,$$

and dividing by x^3 gives the required form.

(iii) $y(x)=\frac{1}{5}x^2-x\ln(x)-\frac{2}{3}-\frac{c}{4}x^{-3}+dx$ where c and d are constants

(b) (i) $\cosh x=\dfrac{e^x+e^{-x}}{2}$, $\sinh x=\dfrac{e^x-e^{-x}}{2}$, $\tanh x=\dfrac{\sinh x}{\cosh x}$

(ii) See Section 7.2.1 in the notes.

(c) (i) $B(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

(ii) $\displaystyle\int_0^\infty\frac{x}{1+x^4}\,\mathrm{d}x=\frac{\pi}{4}$ (Note: This computation requires Proposition 7.18 in the notes which was not covered in the lectures this year.)