Chapter 3

Dynamical Systems

Contents

3.1 Terminology and Notation

Consider the economy of the UK. At any point in time there are various numbers that each quantify some aspect such as GDP, growth, inflation and unemployment. As time passes circumstances change and for various reasons the numbers we are interested in change (think of newspaper headlines along the lines of 'unemployment rises' or 'inflation eases off'). Each of these pieces of economic data will critically hinge on one another, and hence changes in one lead to changes in the others. We can use differential equations to capture these complicated relationships, enabling us to understand how and why complex changes occur. This is the picture you should have in your mind when thinking about dynamical systems.

Roughly speaking, a *dynamical system* consists of:

- A vector $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ whose components change with time. At a given time t_0 the vector $\mathbf{x}(t_0)$ is known as the *state* of the system.
- A set of rules for how the different co-ordinates depend on one another and the time t. This dictates how the system evolves.

As a dynamical system evolves through time from some initial state, the vector $\mathbf{x}(t)$ follows a *trajectory* (or *orbit*) through the *state space* (or *phase space*) of the system, which is generally all or a subset of $\mathbb{R}^n$. We will sometimes use $\dot{\mathbf{x}}$ to denote $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$, $\ddot{\mathbf{x}}$ to denote $\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$, etc.

Example 3.1. *Exponential growth and decay* are dynamical systems for modelling phenomena like the growth of populations with unlimited resources and the decay of radioactive material. The rule governing these dynamical systems is the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x,$$

where $\alpha \in \mathbb{R}$ models growth when $\alpha > 0$ and decay when $\alpha < 0$. The state space is the subset of $\mathbb{R}$ given by $x \geq x_0$ in the case of growth and $x \leq x_0$ in the case of decay, where x_0 is the value of x when $t = 0$.

Example 3.2. *Logistic population models* are models of the growth of a population that has access to only a limited pool of resources. In this case the rule governing the behaviour of the system is given by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP - \frac{rP^2}{k} = \frac{rP}{k}(k - P),$$

where $r, k \in \mathbb{R}$ are constants. In realistic scenarios we have $0 < P < k$ and so the phase space is the subset of $\mathbb{R}$ consisting of this interval.

Example 3.3. *Projectiles* model the flight of objects such as mortar shells and cricket balls. This time, since we are interested in both the height above the ground and how far horizontally the projectile reaches, the state space is a subset of $\mathbb{R}^2$. It turns out that the trajectory is a parabola whose precise shape depends on the initial conditions. Writing x for the horizontal distance traversed by the projectile and y for its distance above the ground, the equations governing this dynamical system are

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= 0\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -g,\end{aligned}$$

where g is the acceleration due to gravity (approximately $9.81\,\mathrm{m/s^2}$). The slightly more complicated example of a projectile that experiences air resistance is another example of a dynamical system.

Example 3.4. The *harmonic oscillator* models the motion of simple pendulums, the behaviour of certain electrical circuits, and springs obeying Hooke's law. In a model concerned only with the small displacement of a spring, the state space is the subset of $\mathbb{R}$ given by $|x| \leq A$, where A is the amplitude of the oscillations. The equation governing this motion is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x,$$

where ω gives the angular frequency of oscillation. The closely related examples of the damped harmonic oscillator and the forced harmonic oscillator are also examples of dynamical systems.

Usually the rules governing a dynamical system take the form of a differential equation or a set of differential equations, as in all the above examples. It is this case that we focus on in this course. Sometimes, however, it is more convenient to model a system in terms of discrete time intervals with the state at time t_n depending only on the state at time t_{n-1}. The evolution of such a system be described in terms of a difference equation. Much of the study of these discrete dynamical systems resembles what we do in the case of the continuous dynamical systems, but we will not consider them further.

A small word of warning: dynamical systems do not necessarily have unique solutions, though in most cases where the differential equation is motivated by some consideration of a real-world phenomenon problems of uniqueness do not arise.

Example 3.5. Consider the dynamical system defined by the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\sqrt{x}.$$

This system does not have a unique solution since $x(t) = 0$ for all t and $x(t) = t^2$ are both solutions of the above equation.

A dynamical system is typically represented as a system of first order ordinary differential equations of the form

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(x_1, x_2, \ldots, x_n) \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{\mathrm{d}x_n}{\mathrm{d}t} &= f_n(x_1, x_2, \ldots, x_n)
\end{aligned} \tag{3.1}$$

for some functions $f_1, f_2, \ldots, f_n$.

You might object to considering such a narrow form of equations. After all, even the phenomena you encountered in Calculus 2 were not all governed by equations of this form. The harmonic oscillator, for example, is governed by a second order differential equation. Surely ruling out such important examples is not a good idea? However, despite its appearance we have *not* ruled out the harmonic oscillator!

Example 3.6. Using the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ the equation for simple harmonic motion can be transformed into the form of the above system of differential equations. Simple harmonic motion is thus an example of a dynamical system of the form mentioned above corresponding to the system of first order equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\omega^2 x$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = y.$$

So here $x_1 \equiv y$ and $x_2 \equiv x$.

Exercise 3.7. Show that the equations of motion for a projectile (Example 3.3) can be expressed as a system of four first order linear differential equations.

In addition to *apparently* ruling out the harmonic oscillator, you may have the further objection that the above formulation does not include partial differential equations. This is indeed a valid criticism. Many important examples, from simple systems governed by straightforward rules such as the heat equation, to more complex systems, e.g. superconductive systems governed by the Ginzberg-Landau equations, are defined using partial differential equations. As we shall see, however, even when restricting ourselves to the domain of ordinary differential equations a wide range of systems can still be modelled, and a wide variety of dynamical phenomena still arises, including complex behaviour such as chaos. We must learn to walk before we can run, and working with partial differential equations is generally significantly harder than working with ordinary ones.

In the projectile example, we had a system in which the two variables x and y behaved completely independent of one another, since each variable did not feature in the equation governing the other. However, for the case of a simple harmonic oscillator Example 3.6 the pair of equations *do* very explicitly interact with one another.

Note that in Example 3.6, the variable x appears in the equation governing the evolution of y and vice versa. We say that these equations are *coupled*. It is this 'interactive nature' of coupled systems, with the behaviour of different variables reinforcing, resisting and feeding back into one another, that makes them both interesting and difficult to study. Most of the systems we study in this course will be coupled.

Observe that for the system (3.1), none of the functions $f_1, f_2, \dots, f_n$ depend on the time t. When our equations do not depend on time then the system is said to be *autonomous* or *time-invariant*. Such systems are typically easier to analyse.

One way to deal with a non-autonomous system is to convert it to an autonomous system by introducing a 'dud' variable $\theta = t$ with the uncoupled 'dud' equation $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$. This will give an autonomous system but with an extra variable.

Example 3.8. One model for a sort of electrical circuit that behave like a forced oscillators is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + x^3 = A\cos(t).$$

Express this equation as a set of coupled autonomous equations.

Solution. First re-express the equation as a set of coupled non-autonomous equations with the substitution $\frac{\mathrm{d}x}{\mathrm{d}t} = y$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = A\cos(t) - ky - x^3.$$

To make this system autonomous we include the equation $z = t$ so that the final set of equations is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = A\cos(z) - ky - x^3$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 1.$$

Exercise 3.9. Express the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t\frac{\mathrm{d}x}{\mathrm{d}t} + t^2 = 0$$

as a set of coupled autonomous equations.

3.2 Fixed Points

One useful and important feature of any system are its *fixed points* (also known as *equilibrium/steady/critical/stationary points/states/solutions*). As the name suggests, the system does not experience any change in this state. For a system to remain stationary at a particular point we require that $\dot{\mathbf{x}} = \mathbf{0}$, i.e. that $\frac{\mathrm{d}x_i}{\mathrm{d}t} = 0$ for $i = 1, 2, \ldots, n$. Consequently finding the fixed points of a system means solving the equations

$$f_1(x_1, x_2, \ldots, x_n) = f_2(x_1, x_2, \ldots, x_n) = \cdots = f_n(x_1, x_2, \ldots, x_n) = 0.$$

We begin by looking at some examples of simple linear systems.

Example 3.10. Find all the fixed points of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x + 4y.$$

Solution. This requires us to solve the equations

$$0 = x + 2y$$
$$0 = 3x + 4y.$$

The first equation tells us that $x = -2y$. Substituting this into the second equation gives $0 = 3(-2y) + 4y = -2y$ and so $y = 0$, implying $x = 0$. This system thus only has one fixed point at $(x, y) = (0, 0)$.

Example 3.11. Find all the fixed points of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + 4y.$$

Solution. This requires us to solve the equations

$$0 = x + 2y$$
$$0 = 2x + 4y.$$

Observing that the second equation is just the first one multiplied by 2 it follows that the set of fixed points is $\{(-2\lambda, \lambda) \mid \lambda \in \mathbb{R}\}$.

Exercise 3.12. Find all the fixed points of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x + 4y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x + 2y.$$

Exercise 3.13. Find all the fixed points of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x + 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6x + 3y.$$

The procedure is the same for non-linear systems, but often more difficult.

Example 3.14. Find the fixed points of the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = rP - \frac{rP^2}{k}$.
Solution. We need to solve the equation $0 = rP - \frac{rP^2}{k}$. Factorising shows that this amounts to solving $0 = \frac{rP}{k}(k - P)$, which implies that either $P = 0$ or $P = k$.

When interpreting this equation in terms of populations constrained by scarce resources these critical points correspond to there being nobody in the population at all using no resources whatsoever and the population having 'maxed out' using all the available resources.

Example 3.15. The Rössler equations are used to model certain oscillating chemical reactions. Let x, y and z be the quantities of chemicals involved in a particular reaction. Then we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y - z$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + ay$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = b + zx - cz,$$

where a, b and c are parameters relating to the system being modelled. What are the fixed points of the Rössler equations?
Solution. We need to solve the equations $-y - z = x + ay = b + zx - cz = 0$. The first and second of these tell us respectively that $z = -y$ and $x = -ay$. Substituting these into the third equation leads us to the quadratic $0 = ay^2 + cy + b$, which can be solved in the usual way to

give $y = \frac{1}{2a}(-c \pm \sqrt{c^2 - 4ab})$. Substituting this back into the equations $z = -y$ and $x = -ay$ leads to the fixed points (assuming $c^2 - 4ab \neq 0$)

$$(x, y, z) = \left(\frac{c \mp \sqrt{c^2 - 4ab}}{2}, \frac{-c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \mp \sqrt{c^2 - 4ab}}{2a} \right).$$

Exercise 3.16. The following equations model the sequence of chemical reactions known as glycolysis used by cells to extract energy from sugar:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -x + ay + x^2 y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= b - ay - x^2 y, \end{aligned}$$

where a and b are positive constants and x and y are the concentrations of adenosine diphosphate and fructose-6-phosphate respectively. Find all the fixed points of these equations.

Exercise 3.17. The Maxwell-Bloch equations can be used to model the behaviour of lasers. Under certain conditions, they are given by

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \kappa(P - E) \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= \gamma_1(ED - P) \\ \frac{\mathrm{d}D}{\mathrm{d}t} &= \gamma_2(\lambda + 1 - D - \lambda EP), \end{aligned}$$

where E is the electric field, P is the polarisation of atoms and D is the population inversion. The parameters κ, γ_1 and γ_2 describe the decay rate of the laser cavity due to beam transmission, the atomic polarization and population inversion respectively, and λ is a pumping parameter. All parameters are positive constants. Find the fixed points of this dynamical system.

Exercise 3.18. The following set of equations arises in differential geometry:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(1 - x)(1 + x - y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(1 - y)(1 + y - x). \end{aligned}$$

What are the fixed points of the system?

If the above examples seemed difficult then there is worse to come: just like the linear case it is possible for a system of equations to have infinitely many fixed points!

Example 3.19. The Rikitake model of geomagnetic reversals uses the system of equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\nu x + zy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\nu y + (z - a)x \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 1 - xy, \end{aligned}$$

for some positive real constants ν and a. Show that for every real number k satisfying $\nu(k^2 - k^{-2}) = a$ there is a fixed point $(\pm k, \pm k^{-1}, \nu k^2)$.

Finally, we issue a small warning: it is possible that a system has no fixed points at all!

Example 3.20. The system $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + x^2$ has no fixed points, since the equation $1 + x^2 = 0$ has no real solutions (recall that we are taking our trajectories as being in $\mathbb{R}^n$).

3.2.1 Stability

Let $\mathbf{x}_*$ be a fixed point of a dynamical system, i.e., $\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(\mathbf{x}_*) = 0$ for $i = 1, \ldots, n$.

Definition 3.1. A fixed point $\mathbf{x}_*$ is said to be *Liapunov stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{x}(0) - \mathbf{x}_*\| < \delta \implies \|\mathbf{x}(t) - \mathbf{x}_*\| < \epsilon \; \forall t \geq 0$.[1]

Simply put, a fixed point is Liapunov stable whenever trajectories of the system that start near the fixed point always remain near the fixed point. A closely related concept is that of *attraction*, where trajectories are pulled towards a fixed point:

Definition 3.2. A fixed point $\mathbf{x}_*$ is said to be *(locally) attracting* if there exists $\delta > 0$ such that $\|\mathbf{x}(0) - \mathbf{x}_*\| < \delta \implies \mathbf{x}(t) \to \mathbf{x}_*$ as $t \to \infty$. If $\mathbf{x}_*$ is attracting for all $\mathbf{x}(0)$ then it is said to be *globally attracting*.

Note that Liapunov stability does not imply attractivity or vice versa. If a fixed point is Liapunov stable but not attractive then it is called *neutrally stable*; if $\mathbf{x}_*$ is both Liapunov stable and attractive then it is called *asymptotically stable*. A point which is neither Liapunov stable nor attracting is called *unstable*.

3.3 Phase Portraits

Often the most instructive way to analyse a dynamical system is to plot the solution $\mathbf{x}(t)$ graphically. Consider the pair of coupled first order autonomous differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f_1(x, y)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= f_2(x, y),\end{aligned}$$

with some functions f_1 and f_2. We can visualise the possible trajectories of the system by plotting a *phase portrait* that shows some representative trajectories starting from various different initial conditions $(x(0), y(0))$.[2] We usually use arrows to indicate what direction the forward flow of time is. Equilibrium states are shown as dots (some people we use solid dots to show Liapunov stable fixed points and open dots to show unstable fixed points). When plotting the phase portrait for higher dimensional systems it is common to focus on a particular two-dimensional 'slice' and plot the phase portrait in that plane.

Example 3.21. Consider the system of equations $\frac{\mathrm{d}x}{\mathrm{d}t} = y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = -x$, which govern the harmonic oscillator with $\omega = 1$ (see Example 3.4). It is easy to see that the only fixed point is $(x, y) = (0, 0)$. For this system there is a 'trick' to instantly find the shapes of the trajectories. Consider

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) &= 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= 2xy - 2xy\\ &= 0.\end{aligned}$$

[1] Recall that $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, the Euclidean norm or length of $\mathbf{x}$.

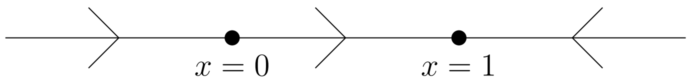
[2] For a two-dimensional dynamical system, the phase space is often referred to as the *phase plane*.

Hence trajectories $(x(t), y(t))$ must satisfy the equation $x^2 + y^2 = C$ for some constant C determined by the initial conditions. In other words, they are circles. The sense of rotation of clockwise or anticlockwise can be determined by considering the behaviour of the trajectories at some convenient point (e.g. when the trajectories crosses an axis).

Exercise 3.22. What is the phase portrait for the harmonic oscillator when $\omega \neq 1$?

Example 3.23. Consider the one-dimensional dynamical system defined by $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2(1-x)$. Find all the fixed points of this system and draw the corresponding phase portrait.
Solution. The system clearly only has fixed points at $x = 0$ and at $x = 1$, but the behaviour of the trajectories near each of these fixed points is very different. When $x < 0$, we have that $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$ and so the trajectory moves towards from the fixed point $x = 0$. When $0 < x < 1$, we have that $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$ and so trajectories still move upwards away from 0 and towards 1 (so it behaves a bit like logistic growth). When $x > 1$, we have that $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$ and so our trajectories approach the fixed point $x = 1$.

We thus have that the fixed point at $x = 1$ is asymptotically stable (take, for example, $\delta = \frac{1}{2}$ in the definitions above). The other fixed point, however, is *semi-stable* since the trajectories above it move away, but the trajectories below it are attracted to it. The one-dimensional phase space[3] can be visualised as follows:



Exercise 3.24. Consider the one-dimensional dynamical system defined by $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2(1-x^2)$. Find all the fixed points of this system and draw the corresponding phase portrait.

3.4 Linear Dynamical Systems

The pair of equations derived in Example 3.6 are an example of what we call a *linear dynamical system*. These are systems whose defining equations are of the form

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{\mathrm{d}x_n}{\mathrm{d}t} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n,
\end{aligned}$$

where the coefficients $a_{11}, a_{12}, \ldots a_{nn}$ are some real numbers. We concentrate on the autonomous case, so that a_{ij} are all independent of time. This is arguably the simplest, most natural and easiest to solve type of dynamical system.

Linear dynamical systems are often written using matrix notation. For instance, the equations above can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

[3]For a one-dimensional dynamical system, the phase space is sometimes referred to as the *phase line*.

or more compactly as $\dot{\mathbf{x}} = A\mathbf{x}$. An attractive feature of working with linear dynamical systems is that they can be explicitly solved.

Let us consider how to solve the two-dimensional linear dynamical system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \alpha x + \beta y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \gamma x + \delta y.\end{aligned}$$

1. Differentiate the first of the equations again, so that the second equations may be substituted into it (thereby giving us an equation to solve only involving the function $x(t)$ and its higher derivatives).

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \alpha\frac{\mathrm{d}x}{\mathrm{d}t} + \beta\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \alpha\frac{\mathrm{d}x}{\mathrm{d}t} + \beta(\gamma x + \delta y) \\ &= \alpha\frac{\mathrm{d}x}{\mathrm{d}t} + \beta\gamma x + \beta\delta\left(\frac{\frac{\mathrm{d}x}{\mathrm{d}t} - \alpha x}{\beta}\right) \\ &= \underbrace{(\alpha+\delta)}_{\operatorname{tr} A}\frac{\mathrm{d}x}{\mathrm{d}t} + \underbrace{(\beta\gamma - \alpha\delta)}_{-\det A}x\end{aligned}$$

   Note that we could in fact derive a single second order differential equation directly from the matrix invariants $\operatorname{tr} A$ and $\det A$.

2. Solve the above second order differential equation in the usual way. We first solve the auxiliary equation $m^2 - (\alpha+\delta)m - (\beta\gamma - \alpha\delta) = 0$. Observe that this is in fact the characteristic polynomial of A and hence the roots describe the eigenvalues of A, denoted λ_1 and λ_2. Then $x(t) = ae^{\lambda_1 t} + be^{\lambda_2 t}$ for some constants when $\lambda_1 \neq \lambda_2$ or $x(t) = (a+bt)e^{\lambda t}$ when $\lambda_1 = \lambda_2 = \lambda$, where a and b are constants determined by the initial conditions.

3. Having determined the function $x(t)$ we can substitute this back into the original equation for $\frac{\mathrm{d}x}{\mathrm{d}t}$ to give the solution $y = \frac{1}{\beta}\left(\frac{\mathrm{d}x}{\mathrm{d}t} - \alpha x\right)$. Note that $y(t)$ will also depend on the constants a and b. The initial conditions specified in the problem should then be used to give precise values for these.

Example 3.25. Solve the two-dimensional linear dynamical system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -3x + 24y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -2x + 11y,\end{aligned}$$

subject to the initial conditions that at time $t = 0$, $x = 7$ and $y = 2$.

Solution. Differentiating the first equation and substituting gives

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -3\frac{\mathrm{d}x}{\mathrm{d}t} + 24\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= -3\frac{\mathrm{d}x}{\mathrm{d}t} + 24(-2x + 11y) \\ &= -3\frac{\mathrm{d}x}{\mathrm{d}t} - 48x + 11\left(\frac{\mathrm{d}x}{\mathrm{d}t} + 3x\right) \\ &= 8\frac{\mathrm{d}x}{\mathrm{d}t} - 15x,\end{aligned}$$

and so we are left with the task of solving the second order differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 8\frac{\mathrm{d}x}{\mathrm{d}t} + 15x = 0$. The corresponding auxiliary equation is $m^2 - 8m + 15 = (m-3)(m-5) = 0$ and hence our solution for x is of the form $x(t) = ae^{3t} + be^{5t}$ for some constants a and b. Differentiating this solution we obtain $\frac{\mathrm{d}x}{\mathrm{d}t} = 3ae^{3t} + 5be^{5t}$, and so for y we have

$$\begin{aligned}y(t) &= \frac{1}{24}\left(\frac{\mathrm{d}x}{\mathrm{d}t} + 3x\right) \\ &= \frac{1}{24}\left(6ae^{3t} + 8be^{5t}\right).\end{aligned}$$

Now using our initial conditions in our equations for $x(t)$ and $y(t)$ we see that $7 = a + b$ and $2 = \frac{1}{4}a + \frac{1}{3}b$. Solving these we find that $a = 4$ and $b = 3$, so $x(t) = 4e^{3t} + 3e^{5t}$ and $y(t) = e^{3t} + e^{5t}$.

Exercise 3.26. Apply the above method to solve the pair of equations arrived at in Example 3.6 for simple harmonic motion.

Exercise 3.27. Use the above method to solve the linear coupled differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x + 2y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2x + 4y,\end{aligned}$$

subject to the initial conditions that at time $t = 0$ we have $x = 2$ and $y = 4$.

Exercise 3.28. Use the above method to solve the linear coupled differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x - y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x + 3y,\end{aligned}$$

subject to the conditions $x(1) = -e^2$ and $y(1) = 0$.

3.4.1 Eigenvalues, Eigenvectors and Eigenspaces

Let us recall some basic concepts of linear algebra from the module Algebra 1.

Let A be an $n \times n$ matrix with entries in $\mathbb{R}$. A non-zero vector $\mathrm{x} \in \mathbb{C}^n$ is an *eigenvector* of A if there exists a complex number $\lambda \in \mathbb{C}$ such that $A\mathrm{x} = \lambda\mathrm{x}$. We call λ an *eigenvalue* of A. Recall that eigenvectors are only defined up to an arbitrary scalar multiple. Furthermore, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are both eigenvectors for the eigenvalue λ then $\mathbf{x}_1 + \mathbf{x}_2$ is also an eigenvector for λ (assuming $\mathbf{x}_1 + \mathbf{x}_2 \neq \mathbf{0}$). Denote by Λ the set of eigenvectors of λ; then $\Lambda \cup \{\mathbf{0}\}$ is the *eigenspace* of λ, denoted E_λ.

Recall that to find the eigenvalues of A we solve $A\mathbf{x} = \lambda\mathbf{x}$, or equivalently $(A - \lambda I)\mathbf{x} = \mathbf{0}$, where I is the identity matrix. In order to find the eigenvectors we solve $\det(A - \lambda I) = 0$. The resulting equation in λ is known as the characteristic polynomial and its roots are the eigenvalues of A. We find the eigenvectors corresponding to λ by looking for the non-trivial elements of the kernel of $A - \lambda I$.

If λ and μ are two distinct eigenvalues (i.e. $\lambda \neq \mu$) of the matrix A then E_λ and E_μ are two disjoint vector spaces, i.e. $E_\lambda \cap E_\mu = \{\mathbf{0}\}$. To see this consider $\mathbf{x} \in E_\lambda \cap E_\mu$. Then by definition of eigenvectors we have that $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{x} = \mu\mathbf{x}$. It follows that $\lambda\mathbf{x} = \mu\mathbf{x}$ and so $\mathbf{0} = \lambda\mathbf{x} - \mu\mathbf{x} = (\lambda - \mu)\mathbf{x}$; since $\lambda \neq \mu$ this can only hold if $\mathbf{x} = \mathbf{0}$.

Example 3.29. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

Solution. We have $\det(A - \lambda I) = (3-\lambda)(2-\lambda) - 2 \times 1 = (\lambda - 1)(\lambda - 4)$, and so it follows that the eigenvalues are $\lambda = 1$ and $\lambda = 4$. For the eigenvectors of the eigenvalue $\lambda = 1$ we need to solve

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0},$$

which amounts to solving the equation $2x + y = 0$. This has solution $x = 1$ and $y = -2$ (for example), and so an eigenvector for this eigenvalue is

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

A similar calculation shows that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector for the eigenvalue $\lambda = 4$.

Exercise 3.30. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$.

Exercise 3.31. Find the eigenvalues and eigenspaces of the matrix $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$.

3.4.2 Fixed Points of Two-Dimensional Systems

Remarkably, even in a linear system fixed points can exhibit a wide variety of behaviour. Suppose the equation $\dot{\mathbf{x}} = \mathbf{0}$ has solution $\mathbf{x}_*$. We have that $A\mathbf{x}_* = \mathbf{0}$, and so $\mathbf{x}_*$ lies in the kernel of A. Note that $\mathbf{x}_* = \mathbf{0}$ is always a solution. There exist more solutions if and only if $\det A = 0$.

In this section we will discuss the stability properties of fixed points in the two-dimensional linear dynamical system, $\mathbf{x}(t) = (x(t), y(t))$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Recall from our earlier discussion that solving a two-dimensional linear dynamical system involves solving the auxiliary equation for a second order differential equation. This gave us $m^2 - (\alpha + \delta)m - (\beta\gamma - \alpha\delta) = 0$, which is in fact the same as the characteristic equation for the 2×2 matrix A. In other words, we need to find the eigenvalues of A. We can think of the solutions of the system as eigenvectors in the phase plane and consider how the behaviour of the system at the origin (or indeed other fixed points) depends on the eigenvalues. As we shall see, there are numerous cases to consider and each behaves somewhat differently.

Let λ_1 and λ_2 be the eigenvalues of A corresponding to the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. The general solution of a trajectory typically (apart from some exceptions that we will discuss) takes the form

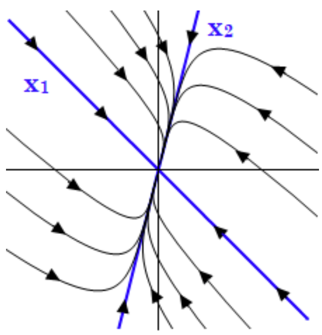
$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2, \tag{3.2}$$

where c_1 and c_2 are coefficients determined by the initial conditions. If $c_2 = 0$ then $\mathbf{x}(t)$ follows the direction $\mathbf{x}_1$. If $c_1 > 0$ ($c_1 < 0$) then the trajectory points in the direction of $\mathbf{x}_1$ ($-\mathbf{x}_1$). Similar considerations hold for $c_1 = 0$, in which case the trajectory follows the direction $\pm\mathbf{x}_2$.

For our phase portrait we are interested in the asymptotic behaviour of the trajectory, i.e. its behaviour at early times ($t \to -\infty$) and at late times ($t \to +\infty$). In practical terms this tells us the direction from which the trajectory came and the direction in which it is heading.

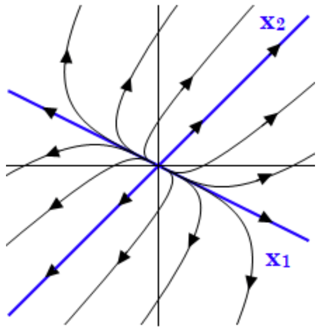
We can rewrite (3.2) as $\mathbf{x}(t) = e^{\lambda_1 t}(c_1\mathbf{x}_1 + c_2 e^{(\lambda_2-\lambda_1)t}\mathbf{x}_2) = e^{\lambda_2 t}(e^{(\lambda_1-\lambda_2)t}c_1\mathbf{x}_1 + c_2\mathbf{x}_2)$. Thus if $\lambda_1 < \lambda_2 \in \mathbb{R}$ and $c_1, c_2 \neq 0$ then at early times $\mathbf{x}_1$ dominates the trajectory while at late times $\mathbf{x}_2$ dominates the trajectory. Hence, for real eigenvalues, in the distant past the trajectory was parallel to $\mathbf{x}_1$, while at distant future times it will be parallel to $\mathbf{x}_2$. We will see that when the eigenvalues are complex the imaginary part of the eigenvalue induces rotation in the trajectory and so the above asymptotic analysis no longer quite works.

1. **Distinct real negative eigenvalues** ($\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2 < 0$).
   The exponential functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ both decay. As $t \to \infty$ the contributions of $\mathbf{x}_1$ and $\mathbf{x}_2$ diminish, so that the trajectory approaches the fixed point $\mathbf{x}_* = (0,0)$. Since $\lambda_1 < \lambda_2$, the contribution of $\mathbf{x}_1$ decays faster and hence the trajectory approaches the direction $c_2\mathbf{x}_2$. In contrast, as $t \to -\infty$ (early times) the first term dominates and the trajectory approaches the direction $c_1\mathbf{x}_1$. For any values of c_1 and c_2 the trajectories move towards the fixed point at the origin. Hence the fixed point $\mathbf{x}_*$ is asymptotically stable. This case is referred to as a *stable node* or *sink*.

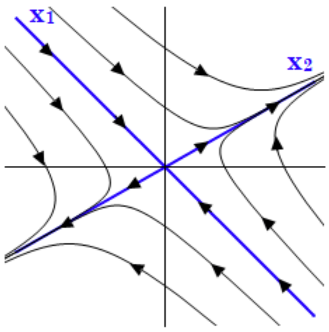


2. **Distinct real positive eigenvalues** ($\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 < \lambda_1 < \lambda_2$).
   Trajectories move out from the equilibrium point $(0,0)$. As $t \to \infty$ the term $c_2 e^{\lambda_2 t}\mathbf{x}_2$ dominates and so the direction of any trajectory approaches that of $\mathbf{x}_2$. As $t \to -\infty$, the term $c_1 e^{\lambda_1 t}\mathbf{x}_1$ dominates and so the direction of any trajectory approaches that of $\mathbf{x}_1$. In this case, the origin is unstable, and the fixed point is called an *unstable node* or

source. Note that this is simply a time-reversed version of the above case of distinct real negative eigenvalues.



3. **Real eigenvalues of opposite signs** ($\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < 0 < \lambda_2$).
The general solution is again of the form (3.2). If $c_2 = 0$ and $c_1 \neq 0$, the trajectory is inwards along the direction $c_1\mathbf{x}_1$, whereas if $c_1 = 0$ and $c_2 \neq 0$, the trajectory is outwards along the direction of $c_2\mathbf{x}_2$. More generally, when $t \to -\infty$ the solution is dominated by $c_1 e^{\lambda_1 t}\mathbf{x}_1$ and when $t \to \infty$ the solution is dominated by $c_2 e^{\lambda_2 t}\mathbf{x}_2$. Hence trajectories follow $c_1\mathbf{x}_1$ inwards for early times, turn and follow $c_2\mathbf{x}_2$ outwards for late times. In this case, $\mathbf{x}_* = (0,0)$ is called a *saddle point*. This is an unstable fixed point.



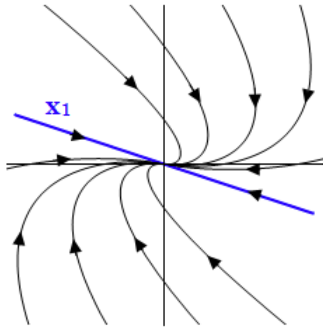
4. **Equal real negative eigenvalues** ($\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 = \lambda_2 = \lambda < 0$).
With a repeated eigenvalue λ, there are two options depending on the dimension of the eigenspace E_λ.

   (a) **Two-dimensional eigenspace**.
   If there are two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ corresponding to λ, then the general solution (3.2) can be written as $\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2)$. The trajectories are all aimed directly at the fixed point $\mathbf{x}_* = (0,0)$. In this case the fixed point at the origin is called a *stable star*. This is an asymptotically stable fixed point.

   (b) **One-dimensional eigenspace**.
   This is one of the exceptions to (3.2). Suppose that $\mathbf{x}_1$ spans the whole of E_λ. Then the general solution is of the form

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{x}_1 + c_2 e^{\lambda t}(t\mathbf{x}_1 + \mathbf{v}) = e^{\lambda t}(c_1\mathbf{x}_1 + c_2 t\mathbf{x}_1 + c_2\mathbf{v}),$$

   where, as usual, c_1 and c_2 are coefficients determined by the initial conditions. If A is the matrix corresponding to our original system of equations then $\mathbf{v}$ is a *generalised eigenvector* of A that satisfies $(A - \lambda I)\mathbf{v} = \mathbf{x}_1$. When $c_2 = 0$ there is a trajectory in the direction of $c_1\mathbf{x}_1$ that moves straight towards the fixed point at the origin. The general solution will be dominated by the term $c_1e^{\lambda t}t\mathbf{x}_1$ as $t \to \infty$ and as $t \to -\infty$! There are two possible 'S-shapes' of trajectory that are possible in this case, and the simplest way to gain additional information to work out which of the two is correct is to analyse the behaviour of the original equations at the axes (i.e. when $x = 0$, for example). The fixed point $(0, 0)$ is called a *stable improper node*; this is an asymptotically stable fixed point.



5. **Equal real positive eigenvalues** ($\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 = \lambda_2 = \lambda > 0$).
This case also separates into two different types of fixed point depending on the dimension of the eigenspace. Unsurprisingly, the phase portraits are very similar to the above case of equal negative eigenvalues, but now the fixed point $\mathbf{x}_* = (0, 0)$ is unstable. We call $\mathbf{x}_*$ an *unstable star* when E_λ is two-dimensional and an *unstable improper node* when E_λ is one-dimensional.

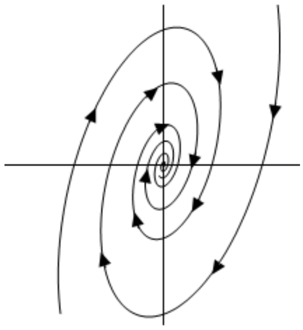
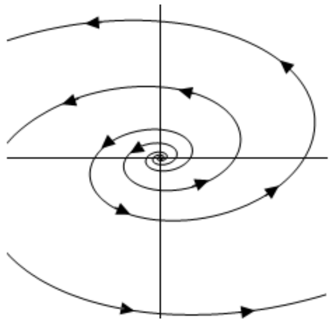
6. **Complex eigenvalues** ($\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 = \lambda_2^*, \mathrm{Im}(\lambda_1) \neq 0, \mathrm{Re}(\lambda_1) \neq 0$).
Recall that A is a matrix with real entries. So $A\mathbf{x}_1 = \lambda\mathbf{x}_1$ implies that $A\mathbf{x}_1^* = \lambda_1^*\mathbf{x}_1^*$, and hence eigenvalues and eigenvectors always come in complex conjugate pairs. We write $\mathbf{x}_1 = \mathbf{u} + i\mathbf{v}$ and $\lambda_1 = a + ib$, with $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. From (3.2) we have

$$\mathbf{x}(t) = c_1e^{(a+ib)t}(\mathbf{u} + i\mathbf{v}) + c_2e^{(a-ib)t}(\mathbf{u} - i\mathbf{v}),$$

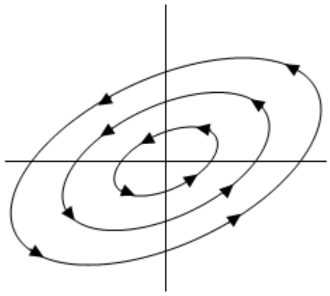
where c_1 and c_2 are constants determined by the initial conditions. Using $e^{\pm ibt} = \cos(bt) \pm i\sin(bt)$, we will obtain terms involving $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$. We can thus write

$$\mathbf{x}(t) = c_1'e^{at}\cos(bt)\mathbf{u}' + c_2'e^{at}\sin(bt)\mathbf{v}'$$

for some new real constants c_1', c_2' and vectors $\mathbf{u}', \mathbf{v}'$. Depending on the sign of a, this describes a trajectory that spirals into the fixed point $\mathbf{x}_* = \mathbf{0}$ (when $a < 0$) or away from the fixed point (when $a > 0$). To decide if the curve spirals clockwise or anticlockwise we go back to the original equation and consider whathappens when the trajectory crosses an axis. If the spiral moves towards the fixed point then we have a *stable spiral* (which is asymptotically stable); otherwise we have an *unstable spiral* (which is, unsurprisingly, unstable).

7. **Purely imaginary eigenvalues** ($\lambda_1 = -\lambda_2^*$ with $\mathrm{Re}(\lambda_1) = 0$).
   In this case the general solution is of the form $\mathbf{x}(t) = c_1 \cos(\lambda_1 t)\mathbf{u} + c_2 \sin(\lambda_1 t)\mathbf{v}$, where c_1 and c_2 are coefficients determined by the initial conditions. This describes a trajectory that orbits the origin in a circular or elliptical fashion. In this case $\mathbf{x}_*$ is called a *centre*, which is a neutrally stable fixed point.



8. **One eigenvalue vanishes and the other is negative** ($\lambda_1 < \lambda_2 = 0$).
   By definition of eigenvalues we must have $A\mathbf{x}_2 = \mathbf{0}$, and so all points on the line $\{s\mathbf{x}_2 \mid s \in \mathbb{R}\}$ are fixed points. These are called *non-isolated* fixed points. The general solution (3.2) now becomes $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2\mathbf{x}_2$. As t varies, the trajectories are half-lines in the direction of $c_1\mathbf{x}_1$. Our non-isolated fixed points are all neutrally stable.

9. **One eigenvalue vanishes and the other is positive** ($0 = \lambda_1 < \lambda_2$).
   The general solution here is the same as the previous case, but now the line $\{s\mathbf{x}_1 \mid s \in \mathbb{R}\}$ gives the non-isolated fixed points and the general solution has the form $\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$. As t varies, the trajectories are half-lines in the direction of $c_2\mathbf{x}_2$, but this time all trajectories move away from the line, and hence the fixed points are unstable.

10. **Both eigenvalues are zero** ($\lambda_1 = \lambda_2 = 0$).
    The fully degenerate case is not really of interest as solving the linear dynamical system becomes trivial, and the phase portrait is not especially enlightening (e.g. we have that every point in the phase space is a fixed point).

Example 3.32. Find and classify the fixed points of the linear dynamical system defined by the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x + 5y.$$

Hence sketch the corresponding phase portrait.
Solution. First we need to find the eigenvalues of the corresponding matrix. We find that

$$0 = \det \begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix} = (2-\lambda)(5-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1),$$

and so our eigenvalues are 1 and 6. We therefore have a pair of positive real eigenvalues, which gives us an unstable node.

For the phase portrait it is necessary to find the eigenvectors. To find the eigenvector for the eigenvalue $\lambda = 1$ we consider the equations

$$\begin{aligned} 0 &= x - 2y \\ 0 &= -2x + 4y, \end{aligned}$$

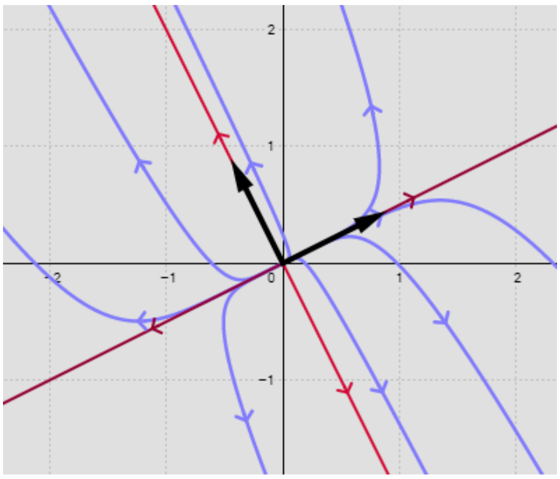
which are satisfied by the values $x = 2$ and $y = 1$, i.e. the corresponding eigenvector is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This dominates the behaviour at early times. Similar calculations show that the eigenvalue $\lambda = 6$ has the eigenvector

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

which dominates the late time behaviour. We thus have the following phase portrait.



Exercise 3.33. For each of the following linear dynamical systems, find and classify the fixed points and then sketch the phase portrait.

1. $\frac{\mathrm{d}x}{\mathrm{d}t} = 38x - 120y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 13x - 41y$.

2. $\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - y$.

3. $\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$.

3.5 Non-Linear Dynamical Systems

3.5.1 Locally Linear Systems

We have discussed the stability of fixed points in linear dynamical systems. What about the stability of fixed points in non-linear dynamical systems?

The answer is that we attempt to approximate a non-linear system by a linear system near its fixed points, using a result known as the *Hartman-Grobman Theorem*. Central to this theorem is the Jacobian matrix J. For a n-dimensional dynamical system with $\dot{x}_i = f_i(\mathbf{x})$, this is an $n \times n$ matrix with elements

$$J_{ij} = \frac{\partial f_i}{\partial x_j}.$$

Roughly speaking, the Hartman-Grobman Theorem states that if all the eigenvalues of the Jacobian matrix evaluated at a fixed point $\mathbf{x}_*$ have non-vanishing real parts[4] then the stability and topology of the fixed point is identical to that of the linearised system $\dot{\boldsymbol{\epsilon}} = J\boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} = \mathbf{x} - \mathbf{x}_*$.

We will not prove the Hartman-Grobman Theorem. Instead we take a look at a few examples and their linearisation from first principles. This should provide us with some intuition of how the theorem works. To analyse what happens near a fixed point $\mathbf{x}_*$, we substitute in $\mathbf{x} = \mathbf{x}_* + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ is treated as a small perturbation. Any terms higher-order than linear in $\boldsymbol{\epsilon}$ can be discarded. An analysis of the stability of the fixed solutions of these linear equations then provides a way of assessing the stability of the original non-linear fixed points.

Exercise 3.34. Consider the two-dimensional non-linear dynamical system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2y + xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x + y.\end{aligned}$$

Show that the fixed points are $(x_*, y_*) = (0, 0)$ and $(-2, 2)$.

To find the stability of these fixed points, we substitute in $x = x_* + \epsilon_x$ and $y = y_* + \epsilon_y$. The equations thus become

$$\begin{aligned}\frac{\mathrm{d}x_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= 2(y_* + \epsilon_y) + (x_* + \epsilon_x)(y_* + \epsilon_y) \\ \frac{\mathrm{d}y_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= (x_* + \epsilon_x) + (y_* + \epsilon_y).\end{aligned}$$

Note that, by definition of the fixed points, $2y_* + x_*y_* = x_* + y_* = 0$. Note also that x_* and y_* can be treated as constant and so their time derivatives vanish. Expanding the brackets and neglecting the second-order term then yields

$$\begin{aligned}\frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= y_*\epsilon_x + (2 + x_*)\epsilon_y \\ \frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= \epsilon_x + \epsilon_y.\end{aligned}$$

[4]Such fixed points are often known as *hyperbolic*, which is a misleading name but unfortunately quite standard.

To classify the behaviour of the fixed point we are thus led to considering the eigenvalues of the matrix

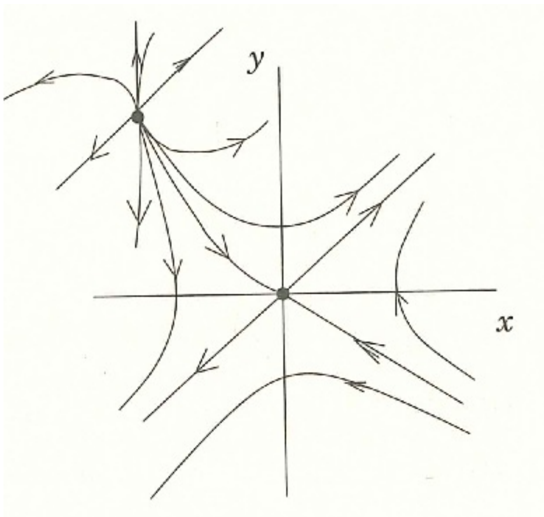
$$\begin{pmatrix} y_* & 2+x_* \\ 1 & 1 \end{pmatrix},$$

which you should verify is in fact the Jacobian matrix of the above system. At this stage it is easier to consider each fixed point separately, since they often exhibit very different behaviours.

Exercise 3.35. Show that the eigenvalues of the Jacobian at the fixed point $(x_*, y_*) = (0, 0)$ are 2 and -1 and find the corresponding eigenvectors.

Exercise 3.36. Show that the eigenvalues of the Jacobian at the fixed point $(x_*, y_*) = (-2, 2)$ are 2 and 1 and find the corresponding eigenvectors.

From the above we find that the fixed point $(0, 0)$ is a saddle point and the fixed point $(-2, 2)$ is an unstable node. Keeping in mind the eigenvectors found in the exercises we can sketch the following phase portrait for the system.



The general analysis for a locally linear dynamical system proceeds as follows:

1. Find the fixed points of the non-linear system.
2. At each fixed point evaluate the Jacobian matrix to linearise the system.
3. Find the eigenvalues and eigenvectors of the linearised system (i.e. those of the Jacobian matrix) and check that the real parts of the eigenvalues do not vanish.
4. Characterise the linearised fixed points.
5. Use the locally linear characterisation to sketch a phase portrait.

Exercise 3.37. Consider the two-dimensional non-linear dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - xy$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(x - 1).$$

Find the fixed points, linearise at each point and hence draw the corresponding phase portrait.

Exercise 3.38. Consider the two-dimensional non-linear dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x^3 - x.$$

Find the fixed points, linearise at each point and hence draw the corresponding phase portrait.

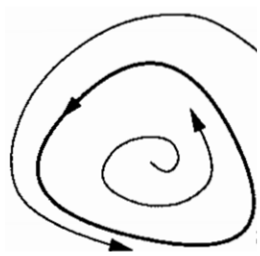
At this point we issue a small warning: linearisation does not always work! Not all fixed points are hyperbolic and hence the Hartman-Grobman Theorem cannot always be applied. Furthermore, the Hartman-Grobman Theorem cannot be used to analyse non-isolated fixed points. We will meet an example of non-hyperbolic fixed points in the next chapter when we meet the predator–prey equations; in these situations, where the Hartman-Grobman Theorem does not apply, there are more advanced methods available.

3.5.2 Periodic Orbits and Limit Cycles

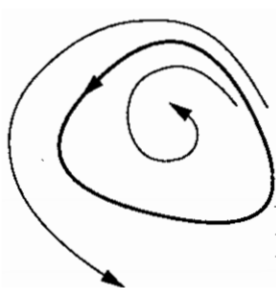
We have already seen in the example of the harmonic oscillator that trajectories can sometimes repeat themselves. If a trajectory $\mathbf{x}(t)$ has the property that for some k we have $\mathbf{x}(t+k) = \mathbf{x}(t)$ for all times t then we say that the dynamical system is *periodic* with period k. Such trajectories correspond to *closed orbits* in phase space.

For the case of the two-dimensional linear system $\dot{\mathbf{x}} = A\mathbf{x}$, we saw that solutions were periodic if and only if the eigenvalues of A were purely imaginary. This gave rise to elliptic orbits in the phase plane. For non-linear systems, another closely related type of trajectory can occur, known as a *limit cycle*. In contrast to periodic solutions for a linear system, this is an *isolated closed trajectory*, i.e. neighbouring trajectories are not closed but instead tend towards or away from the limit cycle. There are three varieties of limit cycle, classified by their orbital stability (as opposed to equilibrium stability):

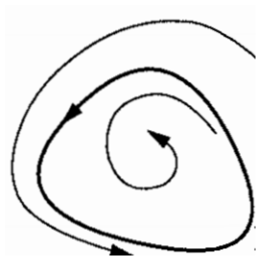
1. **Stable.** Every nearby trajectory will be attracted.



2. **Unstable.** Every nearby trajectory will be repelled.



3. **Semi-stable.** Nearby trajectories inside the cycle will behave one way while nearby trajectories outside the cycle will behave the opposite way.



Limit cycles are very important for physically realistic models of any system that exhibits some kind of self-sustained oscillation, e.g. the beating of a heart. If a system featuring a stable limit cycle is perturbed slightly from the regular periodic behaviour then it will naturally try to return. Although several techniques have been developed for analysing limit cycles, they are in general extremely difficult to handle.

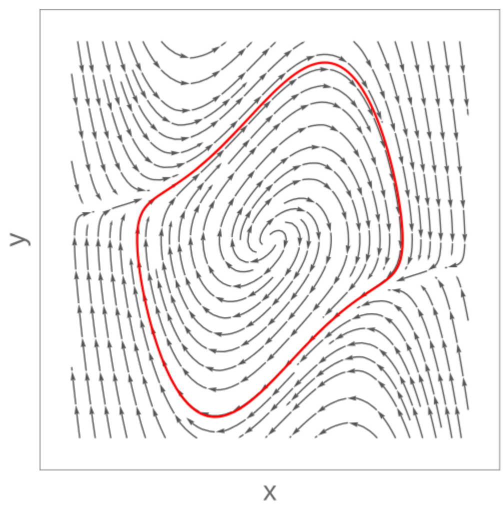
Example 3.39. The van der Pol equation was originally introduced to model a triode valve used in early radios, but it has since be used in biology to model action potentials of neurons and in seismology to model the behaviour of two plates at a geological fault.

The equation is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \mu(x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$

where x is the quantity oscillating (in an electrical context, this would be the current) and μ is a small positive constant. Note that the limiting case $\mu = 0$ describes a simple harmonic oscillator, and so we might expect some kind of periodicity from this system.

We introduce the variable $y = \dot{x}$ to give a system of coupled non-linear first order differential equations. As shown in the phase portrait below, the system features a stable limit cycle ($\mu = 1$ pictured):



3.5.3 Chaos Theory

In 1963, Lorenz obtained a dynamical system of three equations for a heavily simplified model of convection in the atmosphere:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y-x)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= rx - y - xz\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz,\end{aligned} \tag{3.3}$$

where $\sigma, r, b > 0$ are parameters related to fluid dynamics. Lorenz found that these equations had some extremely bizarre properties that seemed to defy everything known about dynamical systems at the time. What Lorenz had stumbled across was a phenomenon we today call *chaos*, which is now known to pervade almost every complex system we attempt to model, including weather, electrical circuits, chemical reactions, population dynamics, inflation, commodity prices, neurons, electrical activity in the heart, eye movements of schizophrenics, plate tectonics, geomagnetism and even dripping taps!

Chaotic systems are still very much an area of active research, and many questions remain unanswered. As such, our treatment of the topic will not be very deep, but we will look at some of the key properties of the Lorenz equations (3.3).

There is no universally accepted definition of chaos, but most would agree on the following working definition: chaos is *aperiodic long-term behaviour* in a *deterministic* system that exhibits *sensitive dependence on initial conditions*. Let us look at the three key parts of this definition.

1. *Aperiodic long-term behaviour* means that there are trajectories that do not tend towards fixed points or periodic orbits as $t \to \infty$. We also exclude the case that the trajectory merely shoots off to infinity.
2. *Deterministic* means that the irregular behaviour stems from the non-linearity of the system itself, rather than from any uncertainty in the evolution of a trajectory.
3. *Sensitive dependence on initial conditions* means that trajectories that start off close to each other will eventually separate from one another at an exponential rate (to use the terminology we shall shortly be introducing, the system has positive *Liapunov exponent*). This is sometimes known as the *butterfly effect*.

Let us now examine some elementary properties of the Lorenz equations.

- **Non-linearity.** The equations are not linear since they contain two non-linear terms, namely the terms xy and xz. It is known that linear systems can never behave chaotically.

- **Three variables.** One reason for chaos only being discovered relatively recently is that for chaotic behaviour we require a dynamical system with at least three variables. This is a consequence of the *Poincaré-Bendixson Theorem*, which heavily constrains the topology of portraits in the phase plane.

- **Fixed points.** As with any other dynamical system, determining the fixed points is an essential part of the analysis.

  Exercise 3.40. Show that the only fixed points of the Lorenz equations in the case $r > 1$ are $(0, 0, 0)$ and $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$.

Sticking to Lorenz's original notation, we will label these latter two fixed points C^+ and C^-. For realistic values of b and r we will always have $b > 0$ and $r \neq 1$.

- **Stability at the origin.** Consider the stability of the fixed point $(x_*, y_*, z_*) = (0, 0, 0)$.

  Exercise 3.41. Show that the linearisation of the Lorenz equations at the origin is

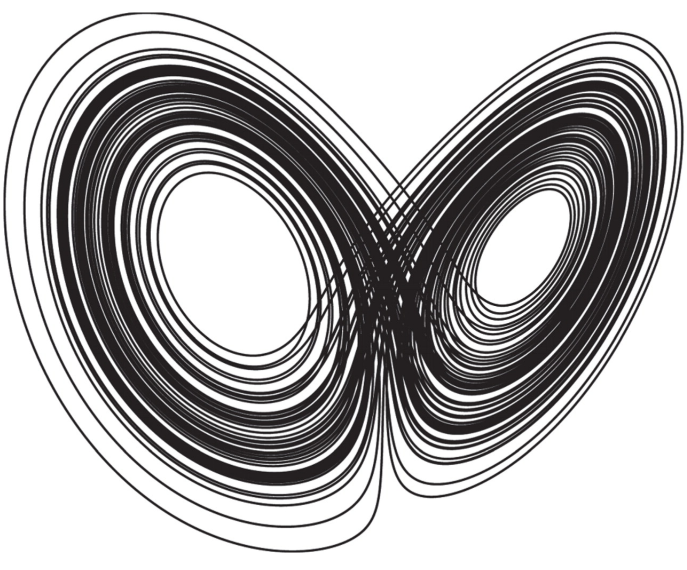
$$
\begin{aligned}
\frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= \sigma(\epsilon_y - \epsilon_x) \\
\frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= r\epsilon_x - \epsilon_y \\
\frac{\mathrm{d}\epsilon_z}{\mathrm{d}t} &= -b\epsilon_z.
\end{aligned}
$$

  Note that z has decoupled from the other two variables. Focussing on the xy plane, the first two equations boil down to studying the two-dimensional system

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}\begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix}.
$$

  Computing the eigenvalues of the above 2×2 matrix, we find that for $0 < r < 1$, we have two negative eigenvalues, and for $r > 1$, we have one negative and one positive eigenvalue. Applying the Hartman-Grobman theorem, we therefore see that for the Lorenz system the origin is a stable node for $0 < r < 1$ and a saddle point for $r > 1$.

- **Stability at $C^{\pm}$.** Calculations similar to those used to find the stability of the origin (but involving the solution of a cubic) show that C^+ and C^- are stable when $1 < r < \rho = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ (assuming that $\sigma - b - 1 > 0$). However, when $r > \rho$, C^+ and C^- are kinds of unstable limit cycle, and hence all three fixed points of the system are unstable. Trajectories are therefore repelled from all three, but it can be shown that no trajectory will shoot off to infinity. As they are also aperiodic, any trajectory must have a very strange behaviour indeed. Trajectories that start near either of the limit cycles will move erratically between the two 'orbiting' one fixed point for a while, but never repeating, before jumping over to the other side seemingly at random. The resulting portrait in phase space describes a *strange attractor*:



- **Sensitive dependence on initial conditions.** Suppose we have a trajectory $\mathbf{x}(t)$ and consider what happens in a nearby trajectory $\mathbf{x}(t) + \boldsymbol{\delta}(t)$. Numerical experiments show that even if $\|\boldsymbol{\delta}(0)\|$ is extremely small then in the long run this 'error term' behaves like $\|\boldsymbol{\delta}(t)\| \sim \|\boldsymbol{\delta}(0)\| e^{\lambda t}$. The value of λ is the *Liapunov exponent*. For the original values that Lorenz himself used, namely $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$, we have $\lambda \approx 0.9$.

The consequence of a positive Liapunov exponent is that accurate long-term prediction becomes impossible. Even if we know the initial conditions very precisely, we find that in the long run the small difference errors grow so large that they overwhelm the whole system. The point beyond which we cannot predict the behaviour of the dynamical system accurately is known as the *time horizon* of the system. Let a denote our tolerance, i.e. if $\|\boldsymbol{\delta}(t)\| \geq a$ then we no longer consider our prediction to be acceptable. The value of the time horizon associated with this is $\frac{1}{\lambda} \ln\left(\frac{a}{\|\boldsymbol{\delta}(\mathbf{0})\|}\right)$. Owing to the logarithmic dependence, we can essentially never give accurate predictions beyond a few multiples of $1/\lambda$. Lorenz suggested that this is why weather prediction is such a difficult problem (although improvements in modelling, denser observation and the use of supercomputer have increased accuracies considerably since 1963).

Solutions to Exercises

3.7 We first need some variable to represent $\frac{dx}{dt}$. We thus set $w = \frac{dx}{dt}$ and substituting this into the first of the equations we have $\frac{dw}{dt} = 0$. Similarly we let $z = \frac{dy}{dt}$ and substituting this into the second of the original equations we have that $\frac{dz}{dt} = -g$. We thus have the following set of coupled equations:

$$\frac{dw}{dt} = 0$$
$$\frac{dx}{dt} = w$$
$$\frac{dy}{dt} = z$$
$$\frac{dz}{dt} = -g.$$

3.9 We substitute $y = \frac{dx}{dt}$. In addition to this compensating for the presence of time in the equation we need to add the substitution $z = t$ and the equation $\frac{dz}{dt} = 1$ to the set of equations we thus have the following set of coupled autonomous differential equations.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = zy - z^2$$
$$\frac{dz}{dt} = 1.$$

3.12 We need to solve the equations

$$5x + 4y = 3x + 2y = 0.$$

The first equation tells us that $x = -\frac{4}{5}y$ and substituting this into the second equation we see that $0 = -\frac{4}{5}y + 2y = \frac{6}{5}y$ and so $y = x = 0$ is the only fixed point.

3.13 We need to solve the equations

$$4x + 2y = 6x + 3y = 0.$$

Observing that the second equation is just the first one multiplied by $\frac{3}{2}$ it follows that the set of fixed points consists of the points $\{(\lambda, -2\lambda) \mid \lambda \in \mathbb{R}\}$.

3.16 We need to solve the equations

$$-x + ay + x^2y = b - ay - x^2y = 0.$$

Adding these two equations together we obtain the equation $0 = b - x$ so we must have that $x = b$ at any fixed points. Substituting this into the first equation we are now faced with the task of solving the equation

$$-b + ay + b^2y = 0$$

and rearranging this tells us that $y = \frac{b}{a+b^2}$. It follows that our only fixed point is at

$$(x_*, y_*) = (b, \frac{b}{a + b^2}).$$

3.17 We need to solve the equations

$$\kappa(P - E) = \gamma_1(ED - P) = \gamma_2(\lambda + 1 - D - \lambda EP) = 0.$$

For $\kappa > 0$ the first equation implies that $P = E$. Substituting this into the second and third equations and supposing that $\gamma_1, \gamma_2 > 0$ we are now faced with the problems of solving the equations

$$ED - E = \lambda + 1 - D - \lambda E^2 = 0.$$

Factorising the first of these tells us that $E(D - 1) = 0$ and so either $E = 0$ or $D = 1$. In the first case, substituting this into the third equation tells us that $\lambda + 1 - D = 0$ and so $D = -\lambda - 1$. In the second case, substituting into the third equation tells us that $\lambda - \lambda E^2 = 0$ that is $E^2 = 1$ and so $E = \pm 1$. Substituting these back into the original equations tells us that our only fixed points are at

$$(D_*, E_*, P_*) = (\lambda + 1, 0, 0), (1, -1, -1) \text{ or } (1, 1, 1).$$

3.18 We need to solve the equations

$$x(1 - x)(1 + x - y) = y(1 - y)(1 + y - x) = 0.$$

The first of these tells us that either $x = 0, 1 - x = 0$ or $1 + x - y = 0$. In the first case the second equation tells us that $y(1 - y)(1 + y) = 0$ and so $y = 0, 1$ or -1. In the second case we have that $x = 1$ and thus the second equation tells us that $y^2(1 - y) = 0$ and so $y = 0$ or 1. In the final case we have that $x = y - 1$ and substituting this into the second equation we have that $2y(1 - y) = 0$ and so $y = 0$ or 1 giving us $x = -1$ or 0 respectively. We thus have the fixed points

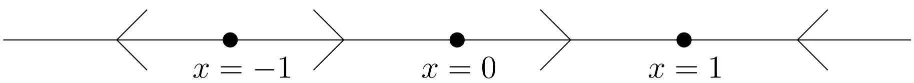
$$(x, y) = (0, -1), (0, 0), (0, 1), (1, 0), (1, 1) \text{ and } (-1, 0).$$

3.22 If $\omega \neq 1$ then we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega^2 x^2 + y^2) = 2\omega^2 x \frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2\omega^2 xy - 2\omega^2 xy = 0.$$

The trajectories thus satisfy $\omega^2 x^2 + y^2 = C$ for some constant C determined by the initial conditions. The trajectories are therefore now ellipses rather than circles.

3.24 The fixed points are clearly at $x = -1, 0$ or 1. For$x < -1$ we have that $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$ and so the trajectory moves away from the fixed point $x = -1$. For $-1 < x < 1$ we have that $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$ whenever $x \neq 0$. The trajectories therefore move away from $x = -1$ and towards $x = 0$ when $-1 < x < 0$ and away from 0 towards $x = 1$ when $0 < x < 1$. For we again have that $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$ and so the trajectory moves towards the fixed point $x = 1$. It follows that the fixed point $x = 1$ is attracting and thus stable, but all the other fixed points are unstable. A diagram of the phase space is as follows.



3.26 We are aiming to solve the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = -\omega^2 x.$$

Differentiating the original equation gives

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}y}{\mathrm{d}t} = -\omega^2 x.$$

Unsurprisingly we have arrived at the original equation and will thus arrive at the original solution.

3.27 Differentiating the first equation gives us

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \frac{\mathrm{d}x}{\mathrm{d}t} + 4x + 8y \\ &= \frac{\mathrm{d}x}{\mathrm{d}t} + 4x + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 4x \\ &= 5\frac{\mathrm{d}x}{\mathrm{d}t}.\end{aligned}$$

Solving the auxiliary equation $m^2 - 5m = 0$ tells us that $m = 0$ or 5. The general solution is therefore of the form

$$x(t) = A + Be^{5t}$$

since the solutions to the auxiliary equation are distinct and real. Differentiating $x(t)$ gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5Be^{5t}$$

and so we have

$$y(t) = \frac{1}{2}(4Be^{5t} - A).$$

The boundary conditions now tell us that to find the values of A and B we need to solve the equations.

$$2 = A + B \text{ and } 8 = -A + 4B.$$

Solving these in the usual way we find that $A = 0$ and $B = 2$ hence

$$x(t) = 2e^{5t} \text{ and } y(t) = 4e^{5t}.$$

3.28 Differentiating the first equation gives us

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \frac{\mathrm{d}x}{\mathrm{d}t} - x - 3y \\ &= 4\frac{\mathrm{d}x}{\mathrm{d}t} - 4x.\end{aligned}$$

Solving the auxiliary equation $m^2 - 4m + 4 = (m-2)^2 = 0$ tells us that $m = 2$. The general solution is therefore of the form

$$x(t) = (A + Bt)e^{2t}$$

since the solutions to the auxiliary equation are real but repeated. Differentiating $x(t)$ gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (2A + B + 2Bt)e^{2t}$$

and so we have

$$y(t) = (A + Bt)e^{2t} - (2A + B + 2Bt)e^{2t}$$

which simplifies to

$$y(t) = (-A - B - Bt)e^{2t}$$

The boundary conditions now tell us that to find the values of A and B we need to solve the equations.

$$-e^2 = (A + B)e^2 \text{ and } 0 = (-A - 2B)e^2$$

Solving these in the usual way we find that $A = -2$ and $B = 1$ and so

$$x(t) = (-2 + t)e^{2t} \text{ and } y(t) = (-3 - t)e^{2t}.$$

3.30 We have that

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix}$$

and so $\det(A - \lambda I) = (-1-\lambda)(4-\lambda) - 3 \times 2 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$. Our eigenvalues are hence $\lambda = -2$ and $\lambda = 5$. For the eigenvectors of the eigenvalue $\lambda = -2$ we need to solve

$$\mathbf{0} = (A + 2I)\mathbf{x} = \begin{pmatrix} 4+2 & 2 \\ 3 & -1+2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

which amounts to solving the equations

$$6x + 2y = 0 \text{ and } 3x + y = 0.$$

This is easily seen to have the solutions $x = -1$ and $y = 3$ (for example) and so an eigenvector for this eigenvalue is

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Similarly for the eigenvectors of the eigenvalue $\lambda = 5$ we need to solve

$$\mathbf{0} = (A - 5I)\mathbf{x} = \begin{pmatrix} 4-5 & 2 \\ 3 & -1-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

which amounts to solving the equations

$$-x + 2y = 0 \text{ and } 3x - 6y = 0.$$

This is easily seen to have the solutions $x = 2$ and $y = -1$ (for example) and so an eigenvector for this eigenvalue is

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

3.31 First we proceed in the usual way to find the eigenvalues

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 2 & -2 \\ 1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 5\lambda \\ 1 & 1 \end{vmatrix}$$
$$\begin{aligned} &= (4-\lambda)((5-\lambda)(2-\lambda)+2) - (2(2-\lambda)+2) - (2-(5-\lambda)) \\ &= (4-\lambda)(\lambda^2-7\lambda+12) + 2\lambda - 6 + 3 - \lambda \\ &= (-\lambda^3+11\lambda^2-40\lambda+48) + \lambda - 3 \\ &= -\lambda^3 + 11\lambda - 39\lambda + 45, \end{aligned}$$

and so we are confronted with the task of solving the equation $0 = \lambda^3 - 11\lambda + 39\lambda - 45$. By trial and error we notice that $\lambda = 3$ is a root of this polynomial, and so proceeding as we would in the module Algebra 1 we see that

$$0 = (\lambda-3)(\lambda^2-8\lambda+15) = (\lambda-3)^2(\lambda-5)$$

and so the eigenvalues of the above matrix are $\lambda = 5$ and $\lambda = 3$ (repeated).

To find the eigenspaces we need to find eigenvectors. For the eigenvalue 5, as before, this entails finding solutions to the equations

$$-x+y-z = 2x-2z = x+y-3z = 0.$$

Solving these in the usual way we find that the only solutions are scaled multiples of $x = 1$, $y = 2$ and $z = 1$ and so the eigenspace for this eigenvalue is the span of the vector

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

The real interest in this exercise comes from considering the other eigenvalue $\lambda = 3$. This time we are confronted with the task of solving the equations

$$x+y-z = 2x+2y-2z = x+y-z = 0.$$

this time we find that $x = 1$, $y = -1$ and $z = 0$ are all solutions. However, this time we find that $x = 1$, $y = 0$ and $z = 1$ is also a solution. In particular there is a solution that is not just a scaled version of the first solution. In terms of eigenspaces we have that the eigenspace for the eigenvalue $\lambda = 3$ is the span of the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

3.33

1. To find the fixed point(s) we need to solve the equations

$$38x - 120y = 13x - 41y = 0.$$

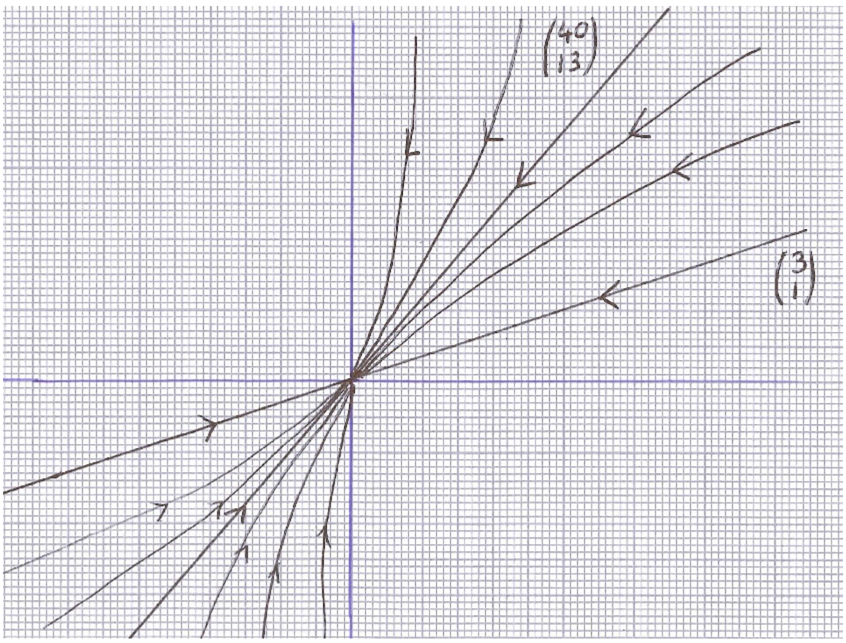
Substituting $x = \frac{60}{19}y$ into the second equation tells us that $-\frac{389}{19}y = 0$ and so $x = y = 0$ is the only fixed point. To determine the nature of this fixed point we need find the eigenvalues of the corresponding matrix. We find that

$$0 = \begin{vmatrix} 38-\lambda & -120 \\ 13 & -41-\lambda \end{vmatrix} = (41+\lambda)(\lambda-38)+1560 = \lambda^2+3\lambda+2 = (\lambda+2)(\lambda+1)$$

and so $\lambda = -1$ and -2. Since these are distinct real eigenvalues that are both negative it follows that our fixed point is a stable node. To draw the phase diagram we need to find the eigenvectors. In the usual way we find that the eigenvectors are

$$\begin{pmatrix} 40 \\ 13 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

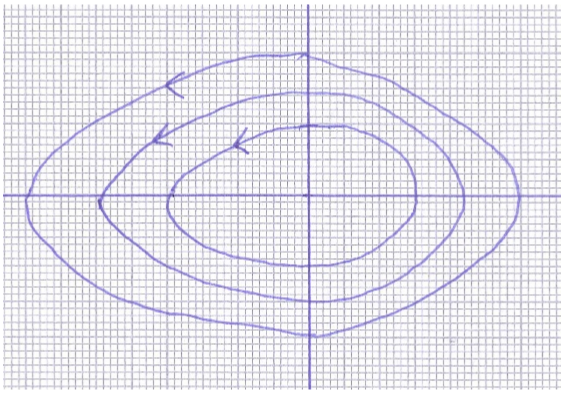
for $\lambda = -1$ and $\lambda = -2$ respectively. We thus have the following phase portrait.



2. The only fixed point of this system is where $x = y = 0$. To determine the nature of this fixed point we need find the eigenvalues of the corresponding matrix. We find that

$$0 = \begin{vmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1)+4 = \lambda^2+3$$

and so the eigenvalues are $\lambda = \pm i\sqrt{3}$. Since these are conjugate pure imaginaries we must have a periodic solution that orbits the fixed point. To draw the phase portrait we note that when $x = 0$ (i.e. where the trajectories meet the y-axis) we have that $\frac{\mathrm{d}x}{\mathrm{d}t} = -2y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = -y$ and so when y is positive the value of x is decreasing more rapidly than the value of y. The phase diagram is thus as follows.



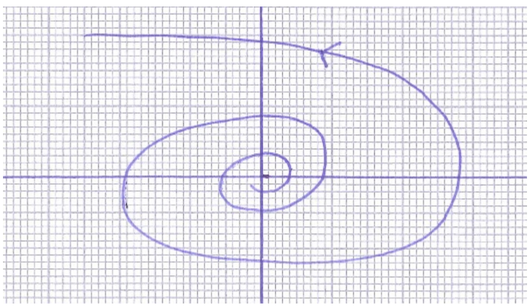
3. To find the fixed point(s) we need to solve the equations

$$x - 2y = 2x + y = 0.$$

Substituting $x = 2y$ into the second equation tells us that $5y = 0$ and so $x = y = 0$ is the only fixed point.
To determine the nature of this fixed point we need find the eigenvalues of the corresponding matrix. We find that

$$0 = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) + 4 = \lambda^2 - 2\lambda + 5$$

and so $\lambda = 1 \pm 2i$. Since these are complex conjugate eigenvalues with a non-zero imaginary part and a positive real part our fixed point is an unstable spiral point.
For a clearer picture of what the trajectories look like we again look at the original equation. When $x = 0$ (i.e. when the trajectories on the phase portrait cross the y-axis) we find that $\frac{\mathrm{d}x}{\mathrm{d}t} = -2y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = y$. Consequently when y is positive the value of y will increase and the value of x will decrease. Our phase portrait will therefore be as follows.



3.34 This entails solving the equations

$$2x + xy = x + y = 0.$$

The second equation tells us that $-x = y$, and substituting this back into the original equation tells us that $0 = 2x - x^2 = x(2-x)$, and so either $x = 0$ or $x = 2$. In the first case $y = 0$ and in the second case $y = -2$. It follows that the only fixed points are $(x_*, y_*) = (0, 0)$ and $(2, -2)$.

3.35 We need to find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

To find the eigenvalues we see that

$$0 = \begin{vmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = \lambda(\lambda-1) - 2 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

and so the eigenvalues are 2 and -1. In the usual way we find that the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

for the eigenvalues 2 and -1 respectively.

3.36 We need to find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

To find the eigenvalues we see that

$$0 = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)$$

and so the eigenvalues are 2 and 1. In the usual way we find that the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for the eigenvalues 2 and 1 respectively.

3.37 To find the fixed points we need to solve the equations

$$1 - xy = y(x-1) = 0.$$

The second of these equations tells us that either $y = 0$ or $x = 1$. If $y = 0$ then the first equation tells us that $1 = 0$ which is clearly absurd so this cannot happen. If $x = 1$ then the first equation immediately tells us that $y = 1$. It follows that the only fixed point is

$$(x_*, y_*) = (1, 1).$$

To linearise these equations we substitute $1 + \epsilon_x$ and $1 + \epsilon_y$. The above equations thus become

$$\begin{aligned} \frac{\mathrm{d}x_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= 1 - (1+\epsilon_x)(1+\epsilon_y) \\ \frac{\mathrm{d}y_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= (1+\epsilon_y)\epsilon_x, \end{aligned}$$

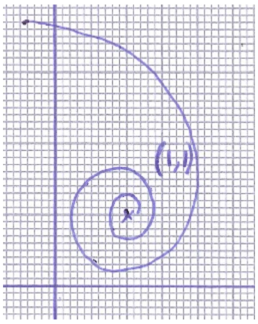
which after the usual manipulation becomes

$$\begin{aligned} \frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= -\epsilon_x - \epsilon_y \\ \frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= \epsilon_x, \end{aligned}$$

and so we need to consider the matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

We find that the eigenvalues of the above matrix are $\frac{-1\pm\sqrt{-3}}{2}$ which are complex conjugate eigenvalues with negative real part and so we have a stable spiral point. To decide if the trajectory is clockwise or anticlockwise we see how the trajectories behave as they cross the y-axis, that is, when $x = 0$. In this case we see that $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = -y$ and so when y is positive the value of x is increasing and the value of y is decreasing and so we are spiralling towards the fixed point clockwise. We thus have the following phase portrait.



3.38 For the fixed points the first equation immediately tells us that we must have $y = 0$ at each of them and the second immediately tells us that we must have $x = -1, 0$ or 1, that is, the fixed points of this system are $(x_*, y_*) = (-1, 0), (0, 0)$ and $(1, 0)$.

To linearise these equations we substitute $x_* + \epsilon_x$ and $y_* + \epsilon_y$. The above equations thus become

$$\begin{aligned} \frac{\mathrm{d}x_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= y_* + \epsilon_y \\ \frac{\mathrm{d}y_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= (x_* + \epsilon_x)^3 - (x_* + \epsilon_x), \end{aligned}$$

which after the usual manipulation becomes

$$\begin{aligned} \frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= \epsilon_y \\ \frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= 3x_*^2\epsilon_x - \epsilon_x, \end{aligned}$$

and so we need to consider the matrix

$$\begin{pmatrix} 0 & 1 \\ 3x_*^2 - 1 & 0 \end{pmatrix}.$$

At the fixed points $(-1, 0)$ and $(1, 0)$ the eigenvalues and eigenvectors are the same. More specifically using the usual method we see that in these cases the eigenvalues are $\pm\sqrt{2}$ which are real of distinct sign and so these fixed points are saddle points. For the phase portrait we therefore need to find the eigenvectors and by the usual approach these prove to be

$$\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

for the eigenvalues of $-\sqrt{2}$ and $\sqrt{2}$ respectively.

At the other fixed point $(0,0)$ we that using the usual method the eigenvalues are $\pm i$ and are therefore purely imaginary making the fixed point a centre. Again, to decide which way round the trajectories move around this fixed point we again look at the original equations when $x = 0$. At this point, when $y > 0$ we immediately see that $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$ and the value of x is increasing, that is, the trajectories move clockwise around the fixed point. We thus have the following phase portrait.

3.40 We need to solve the equations

$$\sigma(y - x) = rx - y - xz = xy - bz = 0.$$

The first of these equations tells us that $x = y$. Substituting this into the other two equations tells us that

$$(r-1)x - xz = x^2 - bz = 0.$$

The second of these equations tells us that $z = \frac{1}{b}x^2$ and substituting this into the equation $(r-1)x - xz = 0$ tells us that

$$0 = (r-1)x - \frac{1}{b}x^3 = x(r - 1 - \frac{1}{b}x^2)$$

and so either $x = 0$ or $0 = r - 1 - \frac{1}{b}x^2$. If $x = 0$ then earlier equations tells us that $y = z = 0$, that is $(0,0,0)$ is a fixed point. If $0 = r - 1 - \frac{1}{b}x^2$ then $x^2 = b(r-1)$ and so $x = \pm\sqrt{b(r-1)}$ since $r > 1$ this is a real number. Since $y = x$ we must also have $y = \pm\sqrt{b(r-1)}$. Since $z = \frac{1}{b}x^2$ we must have $z = r - 1$. It follows that the only fixed points must be

$$(x_*, y_*, z_*) = (0,0,0) \text{ and } (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1).$$

3.41 Substituting $x_* + \epsilon_x$, $y_* + \epsilon_y$ and $z_* + \epsilon_z$ into these equations we have

$$\begin{aligned}
\frac{\mathrm{d}x_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= \sigma((y_* + \epsilon_y) - (x_* + \epsilon_x)) \\
\frac{\mathrm{d}y_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= r(x_* + \epsilon_x) - (y_* + \epsilon_y) - (x_* + \epsilon_x)(z_* + \epsilon_z) \\
\frac{\mathrm{d}z_*}{\mathrm{d}t} + \frac{\mathrm{d}\epsilon_z}{\mathrm{d}t} &= (x_* + \epsilon_x)(y_* + \epsilon_y) - b(z_* + \epsilon_z),
\end{aligned}$$

which after the usual manipulation becomes

$$\begin{aligned}
\frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} &= \sigma(\epsilon_y - \epsilon_x) \\
\frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} &= r\epsilon_x - \epsilon_y - z_*\epsilon_x - x_*\epsilon_z \\
\frac{\mathrm{d}\epsilon_z}{\mathrm{d}t} &= y_*\epsilon_x + x_*\epsilon_y - b\epsilon_z.
\end{aligned}$$

At the fixed point $(x_*, y_*, z_*) = (0, 0, 0)$ the above equations become

$$\frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} = \sigma(\epsilon_y - \epsilon_x)$$
$$\frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} = r\epsilon_x - \epsilon_y$$
$$\frac{\mathrm{d}\epsilon_z}{\mathrm{d}t} = -b\epsilon_z.$$