BIRKBECK COLLEGE
(University of London)

BSc Examination
School of Business, Economics & Informatics

Calculus 2: Multivariable & Differential Equations BUEM001S5

Wednesday 11 June 2014
10:00am–1:00pm

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 marks each and questions in Section B are worth 20 marks each. Candidates should attempt **all** *of the questions in Section A and* **two** *questions from Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorized calculators or has been approved by the chair of the Mathematics & Statistics Examination Sub-board.

Please turn over

Section A

1. (a) Without using L'Hôpital's rule, evaluate $\lim_{x\to\infty} \dfrac{x^2+2}{3x^2+4x}$. [2]

   (b) Evaluate by any method $\lim_{h\to 0} \dfrac{e^{-2h}-1}{e^h-1}$. If you are using L'Hôpital's rule, you must show that its assumptions are satisfied. [3]

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^3 + xy + y^2$.

   (a) Find the gradient $\nabla f(x,y)$. [2]

   (b) Compute the directional derivative $f_{\mathbf{u}}(1,3)$ where $\mathbf{u} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$. [3]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(2) = 1$ and $f'(2) = 2$. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = f(x^2 - 2y)$.

   (a) Compute $g(2,1)$, $\dfrac{\partial g}{\partial x}(2,1)$ and $\dfrac{\partial g}{\partial y}(2,1)$. [3]

   (b) Find the Taylor approximation of degree 1 for g centred at $(2,1)$. [2]

4. Evaluate the integral

$$\iint_D xy \,\mathrm{d}x\mathrm{d}y$$

where D is the triangle in the (x,y)-plane which is bounded by the x-axis, the y-axis and the line $y = 4 - 2x$. [5]

Please turn over

5. Consider the differential equation

$$1 + y^2 + xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

   (a) Show that $\mu(x, y) = x$ is an integrating factor for this differential equation. [2]

   (b) Find its general solution. [3]

6. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x + y + 2}.$$

   [Hint: Use the substitution $z = x + y + 2$.] [5]

7. Suppose we have a sample of a radioactive substance. Let $M(t)$ be the mass of this substance (in grams) at time t (in days). At time $t = 0$ days we have $M(0) = 2$ grams of the substance.

   (a) We know from nuclear physics that the rate of change in mass is proportional to the present mass, with an unknown proportionality constant α. Express this statement as a differential equation for M. [1]

   (b) Find the solution of this differential equation with our initial condition $M(0) = 2$. [3]

   (c) At time $t = 100$ days only $M(100) = 0.5$ grams of the substance are left. Compute α. [1]

Please turn over

8. Recall that the Gamma function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ for $x > 0$.

   (a) Show that $\Gamma(x) = (x-1)\Gamma(x-1)$ for $x > 1$. [3]

   (b) Compute $\Gamma\left(\frac{5}{2}\right)$. In your computation you may use without proof any result from the module. [2]

Please turn over

Section B

9. (a) Let $U \subseteq \mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be a function.

   (i) Define the terms "stationary point of f", "local maximum of f" and "global maximum of f". [3]

   (ii) Let $(a, b) \in \mathbb{R}^2$. What does it mean to say that (a, b) is a boundary point of U? [1]

   (b) From now on let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x, y) = 2x^2 + xy + 2y^2 + 6x.$$

   (i) Compute the partial derivatives f_x, f_y, f_{xx}, f_{yy} and f_{xy}. [2]

   (ii) Find and classify the stationary points of f. [5]

   (iii) Use the method of Lagrange multipliers to find the extrema of f on the circle $x^2 + y^2 = 8$. [7]

   (iv) What are the global maximum and minimum values of f on the disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 8\}$? Justify your answer. [2]

Please turn over

10. (a) (i) Let $U \subseteq \mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function. Define what it means for f to be continuous at a point $a \in U$. [1]

(ii) State the definition of a derivative. Using this definition, show that

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x.$$

You may use without proof that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. [3]

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = xye^{-(x^2+y^2)}.$$

(i) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. [2]

(ii) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 2)$. Express your answer in the form $z = ax + by + c$ where $a, b, c \in \mathbb{R}$. [4]

(iii) Use the tangent plane to estimate the value $f(0.9, 2.1)$. [2]

(c) Evaluate the integral

$$\iint_D xye^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y$$

where D is the region $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 \leq 2\}$. [8]

Please turn over

11. (a) Let $P, Q, R \in \mathbb{R}$ and consider the differential equation

$$P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = 0.$$

(i) Show that if t is a constant such that $y(x) = e^{tx}$ is a solution of this differential equation, then $Pt^2 + Qt + R = 0$. [4]

(ii) In the case where the equation $Pt^2 + Qt + R = 0$ has two distinct real roots α and β, what is the general solution of the differential equation? You do not need to justify your answer. [1]

(b) Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 0.$$

(i) Find the general solution of this differential equation. [4]

(ii) Find the solution satisfying the initial conditions $y(0) = 2$ and $y'(0) = 5$, and compute $\lim_{x\to\infty} y(x)$ for this solution. [5]

(c) Now consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 39\sin(2x).$$

Find the general solution of this differential equation. [6]

Please turn over

12. (a) State the definition of $\sinh x$ and $\cosh x$, and compute the derivatives $\frac{\mathrm{d}}{\mathrm{d}x}\sinh x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\cosh x$. [2]

(b) Let $y(x)$ be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sinh x$$

with initial condition $y(0) = 1$. We want to estimate the value $y(1)$.

(i) Use Euler's method with step length $h = 0.5$ to estimate $y(1)$. [4]

(ii) Use the method of Taylor series about the point $x = 0$ to find the first four terms of the Taylor series of y. Use this to estimate $y(1)$. [5]

(iii) Finally solve the differential equation analytically and use the result to compute $y(1)$. [6]

(c) Show using only the definition of the sinh function that

$$\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{1 + x^2}\right),$$

where $\operatorname{arcsinh}(x)$ is the inverse function of $\sinh(x)$. [3]