BIRKBECK COLLEGE
(University of London)

BSc Examination
School of Business, Economics & Informatics

Calculus 2: Multivariable & Differential Equations BUEM001S5

Tuesday 21 May 2013
1430-1730

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 marks each and questions in Section B are worth 20 marks each. Candidates should attempt **all** *of the questions in Section A and* **two** *questions from Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorized calculators or has been approved by the chair of the Mathematics & Statistics Examination Sub-board.

Please turn over

Section A

1. (a) Evaluate

$$\lim_{x\to 2} \frac{x+2-\frac{8}{x}}{1-\frac{2}{x}}$$

without using L'Hôpital's rule. [2]

(b) Find the equation of the tangent plane to the surface $z = x^4 + 3x^3y - 2y^3 + y^2$ at $(1, -2)$. Express your answer in the form $ax + by + cz = d$, where a, b, c, d are numbers. [3]

2. Consider the integral $\iint_T x^3 \, \mathrm{d}x \, \mathrm{d}y$, where T is the triangle with vertices $(0, 0), (2, 2), (3, 0)$. Sketch the region of integration and evaluate the integral. [5]

3. Find the extreme values of $f(x, y) = x + y$ on the circle $x^2 + 6x + y^2 = 9$ using Lagrange multipliers. Are your answers local extrema or global? Justify your answer. [5]

4. Consider the following differential equation.

$$x^2 + y^2 \cos x + (y + 2y \sin x) \frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

where $y(0) = 1$.

(a) Show that the equation is exact. [1]

(b) Solve the equation. [4]

Please turn over

5. (a) Use the method of Taylor series to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y + 2,$$

where $y(1) = 2$. Find a closed form for your function. [3]

(b) Use Euler's method to estimate $y(1.2)$ for the above equation. Use a step size of $h = 0.1$. [2]

6. A box with mass 1kg is attached horizontally to a spring with restoring force given by $F_{\text{spring}} = -4x$. Here, x is the horizontal position of the box from equilibrium measured in metres. The box has an initial position of 0.5 metres and is released with zero velocity from this position. Neglect any drag forces.

(a) Using Newton's laws, find the differential equation governing the motion of the system. Clearly state the initial conditions. [1]

(b) Solve the differential equation. Does the mass pass through the equilibrium point? If yes, give a time when it does. Otherwise, explain why it doesn't. [4]

7. (a) Show that $y = \tanh(x)$ is a solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 1 = 0.$$

You may use any results from the module without proof as long as you clearly state them. [3]

(b) If $\sinh(x) = 12/5$, find all possible values of $\cosh(x)$ and $\tanh(x)$. [2]

8. (a) State the definitions of the Gamma and Beta functions. Make sure to state the domains of the functions. [2]

(b) Show that the Gamma function $\Gamma(x)$ satisfies $\Gamma(x) = (x-1)\Gamma(x-1)$ for $x > 1$. [3]

Please turn over

Section B

9. (a) Consider the two-variable function $f(x, y) = 3x^2y - 4x^3 + y^2 - 7y$.

   (i) Find f_x and f_y. [2]

   (ii) Find the three stationary points of f. All points will have either integer or rational coordinates. [5]

   (iii) Find f_{xx}, f_{yy} and f_{xy} and determine the nature of the stationary points. [4]

   (b) Suppose that $g(x, y) = x^2 + 2x + 6xy + y^2$. By finding the Taylor series of g about $(-2, 3)$, express g in terms of powers of $x + 2$ and $y - 3$. [6]

   (c) Suppose that $z(x, y)$ is a function of two variables. If $x = s + t$ and $y = s - t$, show that $z_s z_t = z_x^2 - z_y^2$. [3]

10. (a) State the definition of a derivative. Using the definition of a derivative, show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x.$$

   You may use without proof that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$. [3]

   (b) Consider the integral

$$\iint_R (x + y)^2 \, \mathrm{d}x\mathrm{d}y,$$

   where R is the half disc of radius 2 centred at $(0, 0)$ with $y \geq 0$.

   (i) Consider the change of variables $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that the Jacobean of this change of variables is r. [2]

   (ii) Sketch the region of integration and evalute the integral. [5]

   (c) Evaluate $\displaystyle\int_0^2 \int_{y^2}^4 \frac{y^3}{\sqrt{x^3 + 1}} \, \mathrm{d}x\mathrm{d}y$. [7]

   (d) Find the directional derivative of the function $f = x^4 + 3x^3y - 2y^3 + y^2$ at $(1, -2)$ in the direction $u = (1, 1)$. Notice, this is the same function as in Question 1. [3]

Please turn over

11. (a) Suppose that a population of bacteria grows according to the logistic growth model. That is,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{k}\right),$$

where $P := P(t)$ is the bacterial population as a function of time, r is the growth proportionality constant and k is the carrying capacity.

(i) Set $r = 10$ and $k = 5000$. Find the general solution to the differential equation. You may keep your answer in the form

$$\frac{P}{a - P} = Ae^{bt},$$

where a, b are numbers and A is the constant of integration. [4]

(ii) Find A if the initial population of bacteria is 4000. [1]

(iii) How long does it take the population to reach 90% of the maximum population? Assume time is measured in days. [3]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3e^{2x} + 2x + 3,$$

where $y(0) = 3$ and $y'(0) = 2$. [7]

(c) Consider the differential equation

$$ax^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

where a, b, c are real constants.

(i) By substituting $x = e^t$ and using the chain rule, show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$ [3]

(ii) Show that the above substitution turns the original differential equation into one with constant coefficients of the form

$$A\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + B\frac{\mathrm{d}y}{\mathrm{d}t} + Cy = 0,$$

for real constants A, B, C. You do not have to solve the equation. [2]

Please turn over

12. (a) Consider the differential equation

$$x^2y'' + 3xy' - 3y = x^2 - 4x + 2.$$

(i) Find the numbers k such that x^k is a solution to the homogeneous part of the above equation. [2]

(ii) By setting $y = vx$, change the above differential equation into

$$v'' + \frac{5}{x}v' = \frac{x^2 - 4x + 2}{x^3}. \qquad [3]$$

(iii) By solving the equation in part (ii), or otherwise, find the general solution to the original differential equation. [5]

(b) (i) State the definitions of $\cosh x$, $\sinh x$ and $\tanh x$. [1]

(ii) Show using only the definition of the sinh function that

$$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1}),$$

where $\operatorname{arcsinh}(x)$ is the inverse function of $\sinh(x)$. [4]

(c) (i) State the theorem expressing the Beta function in terms of the Gamma function. [1]

(ii) Evaluate

$$\int_0^\infty \frac{x}{1 + x^4}\,\mathrm{d}x. \qquad [4]$$