7 Special functions

In this chapter we study several functions that are useful in calculus and other areas of mathematics.

7.1 Hyperbolic trigonometric functions

The functions we study in this section are similar to the standard trigonometric functions. They are known as the **hyperbolic trigonometric functions** or simply **hyperbolic functions**. We will refer to the trigonometric functions $\sin, \cos$ etc. as *circular trigonometric functions* to distinguish hyperbolic functions from the trigonometric functions we already know. Hyperbolic functions occur frequently in mathematics and science as well as having a number of interesting properties. So, even though they are simply sums of exponentials, they deserve their own name. Throughout this section, you should compare the various results and identities to their circular counterparts.

7.1.1 Definition of hyperbolic trigonometric functions

Definition 7.1. The following functions are defined for all $x \in \mathbb{R}$.

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2}, \\
\sinh x &= \frac{e^x - e^{-x}}{2}, \\
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}.
\end{aligned}$$

Notice that $\cosh x$ is symmetric about $x = 0$; that is, $\cosh(-x) = \cosh x$, so the cosh function is even. Using the trivial observation that $(e^x - 1)^2 \geq 0$ for all x, we have that

$$\begin{aligned}
(e^x - 1)^2 &\geq 0 \\
\Rightarrow \quad e^{2x} - 2e^x + 1 &\geq 0 \\
\Rightarrow \quad e^x - 2 + e^{-x} &\geq 0 \\
\Rightarrow \quad e^x + e^{-x} &\geq 2.
\end{aligned}$$

Thus $\cosh x \geq 1$ for all $x \in \mathbb{R}$, and $\cosh x = 1$ if and only if $x = 0$. Furthermore as x becomes large and positive, we see that $\cosh x \approx \frac{1}{2}e^x$. Since $\cosh x$ is even, it follows that $\cosh x \approx \frac{1}{2}e^{-x}$ as x becomes large and negative. From this we can deduce that the range of $\cosh x$ is all $y \geq 1$. The graph of $\cosh x$ is given in Figure 1.

The function $\sinh x$ is clearly an odd function, i.e. $\sinh(-x) = -\sinh x$. Furthermore, we use similar reasoning to the above to see that $\sinh x$ approaches $\frac{1}{2}e^x$ as x becomes large and positive, and $\sinh x$ approaches $-\frac{1}{2}e^{-x}$ as x becomes large and negative (since sinh is odd). Thus, we see that the range of $\sinh x$ is all of $\mathbb{R}$. The graph of $\sinh x$ is given in Figure 2.

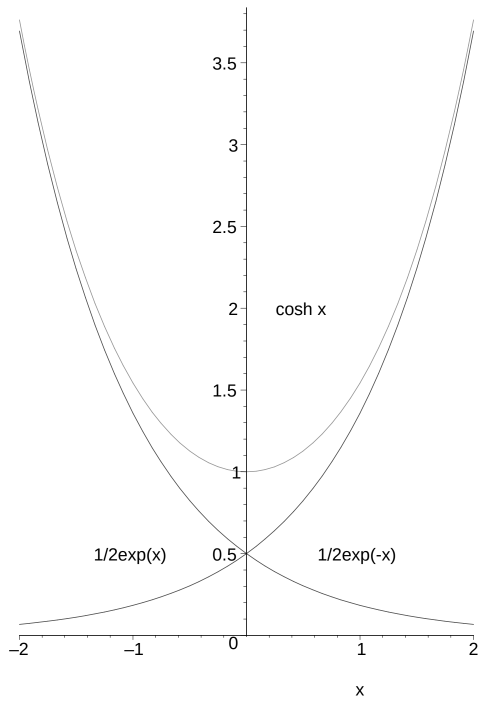

Figure 1: The graph of $\cosh x$ and $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$.

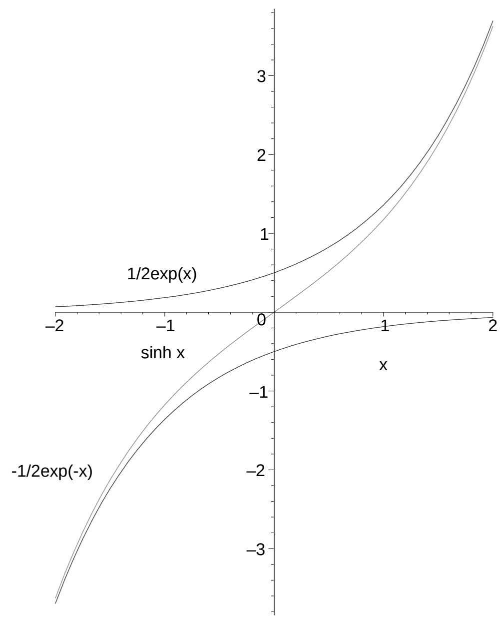

Figure 2: The graph of $\sinh x$, $\frac{1}{2}e^x$ and $-\frac{1}{2}e^{-x}$.

The properties of the graph of $\tanh x$ can be obtained in a similar way. Notice that for any x we have $|\cosh x| > |\sinh x|$ (because of the minus sign!), so $|\tanh x| < 1$. However, both approach $\frac{1}{2}e^x$ as x gets large and positive, so as x becomes large we see that $\sinh x/\cosh x$ approaches 1 (from below, since $\sinh x < \cosh x$). For large negative x, we saw that $\sinh x$ approaches $-\frac{1}{2}e^{-x}$ while $\cosh x$ approaches $\frac{1}{2}e^{-x}$, so the quotient $\sinh x/\cosh x$ approaches -1 (from above). Thus, we see the range for $\tanh x$ is all y such that $-1 < y < 1$. The graph of $\tanh x$ is given in Figure 3.

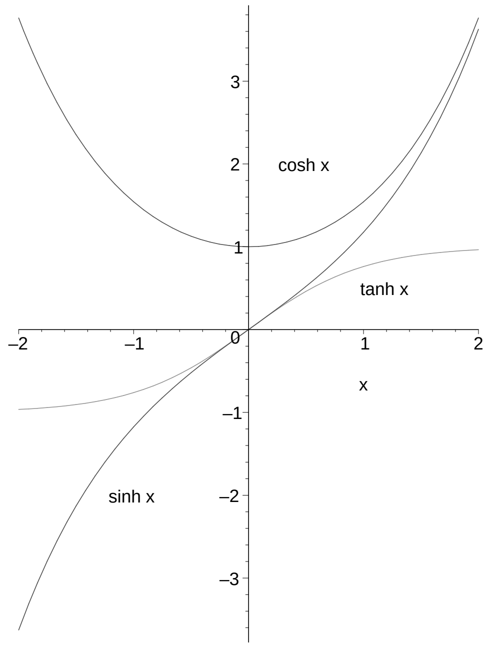

Figure 3: The graph of $\tanh x$, $\cosh x$ and $\sinh x$.

7.1.2 Identities

We have the following identities for the hyperbolic functions.

Proposition 7.2. *For all $x \in \mathbb{R}$, we have* $\cosh^2 x - \sinh^2 x = 1$.

Proposition 7.2 is the reason we call these *hyperbolic* functions. If we set $x = \cosh t$ and $y = \sinh t$ we get $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$, which is the equation of an hyperbola.

Proof. We have

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2 \\ &= \frac{1}{4}\big((e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\big) \\ &= \frac{1}{4}(4) = 1.\end{aligned}$$

□

We have the following identities, which are analogous to the angle sum identities for standard trigonometric functions.

Proposition 7.3. *We have*

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

and

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

One compares to the standard trigonometric identities:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

and

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Proof. For the first identity we have

$$\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \frac{1}{4}\big((e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})\big) \\
&= \frac{1}{4}\big(e^{x+y} + e^{x-y} + e^{y-x} + e^{-x-y} \\
&\qquad + e^{x+y} - e^{x-y} - e^{y-x} + e^{-x-y}\big) \\
&= \frac{1}{4}\big(2e^{x+y} + 2e^{-(x+y)}\big) \\
&= \cosh(x+y).
\end{aligned}$$

Similarly, for the second identity we have

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \frac{1}{4}\big((e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})\big) \\
&= \frac{1}{4}\big(e^{x+y} + e^{x-y} - e^{y-x} - e^{-x-y} \\
&\qquad + e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}\big) \\
&= \sinh(x+y).
\end{aligned}$$

□

Corollary 7.4. *We have the following "double angle" identities.*

$$\begin{aligned}
\cosh 2x &= \cosh^2 x + \sinh^2 x && (1) \\
&= 1 + 2\sinh^2 x && (2) \\
&= 2\cosh^2 x - 1 && (3)
\end{aligned}$$

and

$$\sinh 2x = 2 \sinh x \cosh x. \qquad (4)$$

Again we compare to the standard trigonometric double angle identities, where we have

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1\end{aligned}$$

and

$$\sin 2x = 2\sin x\cos x$$

Proof. Using the first part of Proposition 7.3 and setting $x = y$ we obtain (1), from which (2) and (3) easily following using Proposition 7.2. Using the second part of Proposition 7.3 and setting $x = y$ gives (4). □

7.1.3 Differentiation

We compute the derivatives of the hyperbolic functions from the definitions.

$$\begin{aligned}\frac{d}{dx}\cosh x &= \frac{d}{dx}\frac{1}{2}(e^x + e^{-x})\\ &= \frac{1}{2}(e^x - e^{-x})\\ &= \sinh x.\end{aligned}$$

Also,

$$\begin{aligned}\frac{d}{dx}\sinh x &= \frac{d}{dx}\frac{1}{2}(e^x - e^{-x})\\ &= \frac{1}{2}(e^x + e^{-x})\\ &= \cosh x.\end{aligned}$$

Finally, using the quotient rules for derivatives, we have

$$\begin{aligned}\frac{d}{dx}\tanh x &= \frac{d}{dx}\frac{\sinh x}{\cosh x}\\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}\\ &= 1 - \tanh^2 x.\end{aligned}$$

Using $\cosh^2 x - \sinh^2 x = 1$, the derivative of $\tanh x$ can also be expressed as

$$\begin{aligned}\frac{d}{dx}\tanh x &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}.\end{aligned}$$

The reader is urged to compare these derivatives with the derivatives for the circular trigonometric functions.

7.1.4 Relationship with standard trigonometric functions

We now show the relationship between the hyperbolic trigonometric functions and the standard trigonometric functions.

Recall that the Taylor series for $e^x, \cos x$ and $\sin x$ are

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots , \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \end{aligned}$$

We can use these series to prove Euler's formula:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

Now consider

$$\begin{aligned} \cosh i\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ &= \frac{\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)}{2} \\ &= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} \\ &= \cos\theta, \end{aligned} \tag{5}$$

where we have used the usual properties $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. In particular it follows that $\cosh i\theta \in \mathbb{R}$ when $\theta \in \mathbb{R}$. Similarly, we have

$$\begin{aligned} \sinh i\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2} \\ &= \frac{\cos\theta + i\sin\theta - \cos(-\theta) - i\sin(-\theta)}{2} \\ &= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2} \\ &= i\sin\theta. \end{aligned} \tag{6}$$

Therefore, $\sinh i\theta$ is purely imaginary when $\theta \in \mathbb{R}$.

More generally, we can evaluate cosh and sinh at any $z \in \mathbb{C}$. We see that if $z = x + iy \in \mathbb{C}$, then

$$\begin{aligned}\cosh z &= \cosh(x+iy)\\ &= \cosh x \cosh iy + \sinh x \sinh iy\\ &= \cosh x \cos y + i \sinh x \sin y. \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned}\sinh z &= \sinh(x+iy)\\ &= \sinh x \cosh iy + \cosh x \sinh iy\\ &= \sinh x \cos y + i \cosh x \sin y. \end{aligned} \tag{8}$$

Thus, we see that along the imaginary axis, both cosh and sinh are periodic. Also, we see that the sinh and cosh of a complex number is complex.

Using a combination of circular and hyperbolic trigonometric functions, we have evaluated cosh and sinh at any complex number. We will now do the same for cos and sin.

Note that if $z = x + iy \in \mathbb{C}$, then $iz = ix + i^2 y = -y + ix$. Now using (5) and (7) we get

$$\begin{aligned}\cos z &= \cosh iz\\ &= \cosh(-y+ix)\\ &= \cosh(-y)\cos x + i\sinh(-y)\sin x\\ &= \cos x \cosh y - i \sin x \sinh y. \end{aligned} \tag{9}$$

Similarly, using (6) and (8) we get

$$\begin{aligned}\sin z &= \tfrac{1}{i}\sinh(iz)\\ &= \tfrac{1}{i}\sinh(-y+ix)\\ &= \tfrac{1}{i}\big(\sinh(-y)\cos x + i\cosh(-y)\sin x\big)\\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned} \tag{10}$$

Exercises 7.1.

1. Prove
$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$
2. Prove for any real n that
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx.$$
3. If $\cosh x = 5/3$, find $\sinh x$ and $\tanh x$.
4. Consider the differential equation $y'' = m^2 y$.
   (a) Show that the solution is $A \sinh mx + B \cosh mx$. Does this agree with our work in differential equations?
   (b) Find the solution to $y'' = 9y$, where $y(0) = -4$ and $y'(0) = 6$.

7.2 Inverses of hyperbolic trigonometric functions

We now study the inverses of the hyperbolic trigonometric functions cosh, sinh and tanh.

7.2.1 Formulas for the inverse hyperbolic functions

We follow the convention used for standard trigonometric functions that the inverse function of a hyperbolic function has the same name as the original function with "arc" prepended to the name. Hence, the inverse functions are named arccosh, arcsinh and arctanh.

As noted in the definition of the hyperbolic functions, their domains are all of $\mathbb{R}$. In Section 7.1, we saw that the ranges of the hyperbolic functions are as follows:

$$\begin{aligned} \cosh x &: \text{Range is } [1,\infty) \\ \sinh x &: \text{Range is } \mathbb{R} \\ \tanh x &: \text{Range is } (-1,1). \end{aligned}$$

We therefore see that domains and codomains for the inverse hyperbolic functions are

$$\begin{aligned} \operatorname{arccosh} x &: [1,\infty) \longrightarrow \mathbb{R} \\ \operatorname{arcsinh} x &: \mathbb{R} \longrightarrow \mathbb{R} \\ \operatorname{arctanh} x &: (-1,1) \longrightarrow \mathbb{R}. \end{aligned}$$

Notice that of the three functions $\sinh, \cosh$ and $\tanh$, only cosh is not one-to-one. Thus, we pick the upper branch (i.e. the part $y \geq 0$) for the range of the arccosh function. This is analogous to the way we pick the upper branch for the inverse of the function $y = x^2$. Namely, we pick the upper branch for $y = \sqrt{x}$.

Let us now deduce explicit formulas for the inverse functions.

Consider $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$. We solve this for y and hence obtain a formula for arcsinh x. Set $z = e^y$. Then we see that

$$\begin{aligned} & 2x = z - \frac{1}{z} \\ \Rightarrow\quad & z^2 - 2xz - 1 = 0 \\ \Rightarrow\quad & z = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}. \end{aligned}$$

Since $z = e^y$, which is always greater than 0, we take the positive root. Thus, $e^y = z = x + \sqrt{x^2+1}$, which implies that $y = \ln(x + \sqrt{x^2+1})$. We conclude that $\operatorname{arcsinh} x = \ln(x + \sqrt{x^2+1})$. Note that this expression makes sense for all real x, further justifying the domain for arcsinh x.

Next consider $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$. Again, let $z = e^y$. Then $2x = z + \frac{1}{z}$, so

$$\begin{aligned} & 2x = z + \frac{1}{z} \\ \Rightarrow \quad & z^2 - 2xz + 1 = 0 \\ \Rightarrow \quad & z = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}. \end{aligned} \tag{11}$$

Notice that when $x = \cosh y$ then $y \to \infty$ (if it is positive) as $x \to \infty$, which implies that $z \to \infty$ as $x \to \infty$. So, we take the positive root in (11), for otherwise $z \to 0$ as $x \to \infty$. Therefore, we have $z = x + \sqrt{x^2 - 1}$ which implies that $y = \ln(x + \sqrt{x^2 - 1})$. We conclude that $\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1})$. Note that this expression only makes sense for $x \geq 1$ (otherwise the radical in the ln will be less than 0), justifying the domain for $\operatorname{arccosh} x$.

As noted above, cosh is the only hyperbolic function that is not one-to-one. Thus, while arccosh is given above, when solving the equation $\cosh x = y$ for a particular y, you will get two solutions for x; namely, $x = \pm \ln(y + \sqrt{y^2 - 1})$ (again, in analogy with $y = x^2$ and $x = \pm\sqrt{y}$).

Finally, we compute $y = \operatorname{arctanh} x$. Then $x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$. Setting $z = e^{2y}$ we find

$$\begin{aligned} & x = \frac{z-1}{z+1} \\ \Rightarrow \quad & zx + x = z - 1 \\ \Rightarrow \quad & z(x-1) = -x - 1 \\ \Rightarrow \quad & z = \frac{x+1}{1-x} \\ \Rightarrow \quad & 2y = \ln\left(\frac{x+1}{1-x}\right). \end{aligned}$$

Hence

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right),$$

where $-1 < x < 1$.

7.2.2 Differentiation

One can compute the derivatives of the inverses of the hyperbolic functions directly from the relations found in Section 7.2.1, however, we choose to use implicit differentiation here.

Consider $y = \operatorname{arcsinh} x$. Then $x = \sinh y$, so differentiating both sides with respect to x we get

$$\begin{aligned} 1 &= \cosh y \cdot \frac{dy}{dx} \\ &= \sqrt{1+\sinh^2 y} \cdot \frac{dy}{dx} \\ &= \sqrt{1+x^2} \cdot \frac{dy}{dx}. \end{aligned}$$

Thus, we see that

$$\frac{d}{dx}\text{arcsinh } x = \frac{1}{\sqrt{1+x^2}}.$$

Similarly, let $y = \text{arccosh } x$. Then $x = \cosh y$, so differentiating both sides with respect to x we get

$$\begin{aligned} 1 &= \sinh y \cdot \frac{dy}{dx} \\ &= \sqrt{\cosh^2 y - 1} \cdot \frac{dy}{dx} \\ &= \sqrt{x^2-1} \cdot \frac{dy}{dx}. \end{aligned}$$

Thus, we see that

$$\frac{d}{dx}\text{arccosh } x = \frac{1}{\sqrt{x^2-1}}.$$

Similarly, let $y = \text{arctanh } x$. Then $x = \tanh y$, so differentiating both sides with respect to x we get

$$\begin{aligned} 1 &= (1-\tanh^2 y) \cdot \frac{dy}{dx} \\ &= (1-x^2) \cdot \frac{dy}{dx}. \end{aligned}$$

Thus, we see that

$$\frac{d}{dx}\text{arctanh } x = \frac{1}{1-x^2}.$$

Of course, from the above it follows, by the chain rule that

$$\begin{aligned} \frac{d}{dx}\text{arcsinh } ax &= \frac{a}{\sqrt{1+a^2x^2}}, \\ \frac{d}{dx}\text{arccosh } ax &= \frac{a}{\sqrt{a^2x^2-1}}, \\ \frac{d}{dx}\text{arctanh } ax &= \frac{a}{1-a^2x^2}. \end{aligned}$$

7.2.3 Application to integration

From Section 7.2.2 we have the following integral formulas.

$$\int \frac{1}{\sqrt{x^2 - r^2}}\, dx = \operatorname{arccosh}\left(\frac{x}{r}\right) + c \tag{12}$$

$$\int \frac{1}{\sqrt{x^2 + r^2}}\, dx = \operatorname{arcsinh}\left(\frac{x}{r}\right) + c \tag{13}$$

$$\int \frac{1}{r^2 - x^2}\, dx = \frac{1}{r}\operatorname{arctanh}\left(\frac{x}{r}\right) + c \tag{14}$$

Notice that (14) can be evaluated using partial fractions.

Example 7.5. We evaluate the following integrals.

1.
$$\begin{aligned}\int \frac{1}{\sqrt{5 + 9x^2}} dx &= \frac{1}{3}\int \frac{1}{\sqrt{5/9 + x^2}} dx\\ &= \frac{1}{3}\operatorname{arcsinh}\left(\frac{3x}{\sqrt{5}}\right) + c.\end{aligned}$$

2.
$$\begin{aligned}\int \frac{1}{\sqrt{4x^2 + 12x - 7}} dx &= \frac{1}{2}\int \frac{1}{\sqrt{x^2 + 3x - 7/4}} dx\\ &= \frac{1}{2}\int \frac{1}{\sqrt{(x + 3/2)^2 - 4}} dx\\ &= \frac{1}{2}\operatorname{arccosh}\left(\frac{x + 3/2}{2}\right) + c\\ &= \frac{1}{2}\operatorname{arccosh}\left(\frac{2x + 3}{4}\right) + c.\end{aligned}$$

3.
$$\begin{aligned}\int \frac{1}{8 - 6x - x^2} dx &= \int \frac{1}{17 - (x + 3)^2} dx\\ &= \frac{1}{\sqrt{17}}\operatorname{arctanh}\left(\frac{x + 3}{\sqrt{17}}\right) + c.\end{aligned}$$

Notice that we could have used partial fractions at the second stage of the above derivation. We could have done

$$\int \frac{1}{17 - (x + 3)^2} dx = \int \frac{1}{(\sqrt{17} - x - 3)(\sqrt{17} + x + 3)} dx,$$

but this is rather messy. The hyperbolic functions are a much neater way to do this.

7.2.4 Inverse hyperbolic functions for complex numbers

Combining the results on trigonometric functions of complex numbers in Section 7.1.4 with the formulas for the inverse hyperbolic functions in Section 7.2.1 allows us to compute the inverse of trigonometric functions at complex numbers.

Example 7.6. Find all $z \in \mathbb{C}$ such that $\cosh z = i$.

Let $z = x + iy$. Then from (7) we see that $\cosh x \cos y + i \sinh x \sin y = \cosh z = i$. Clearly, then $\cosh x \cos y = 0$ and $\sinh x \sin y = 1$. Since $\cosh x$ is never 0, we must have $\cos y = 0$, implying that $y = k\frac{\pi}{2}$, where k is any odd integer, i.e. k is of the form $2m + 1$ with $m \in \mathbb{Z}$.

If $k = 4m + 1$, $m \in \mathbb{Z}$, then $\sin y = \sin(k\pi/2) = 1$. It follows that $\sinh x = 1$. Clearly, only one x satisfies $\sinh x = 1$, namely $x = \operatorname{arcsinh} 1 = \ln(1 + \sqrt{2})$.

If $k = 4m+3$, $m \in \mathbb{Z}$, then $\sin y = \sin(k\pi/2) = -1$. It follows that $\sinh x = -1$ and that $x = \operatorname{arcsinh}(-1) = \ln(\sqrt{2} - 1)$.

Thus, the solutions to $\cosh z = i$ are $z = \ln(1 + \sqrt{2}) + ik\frac{\pi}{2}$, for $k = 4m + 1$ and $m \in \mathbb{Z}$, and $z = \ln(\sqrt{2} - 1) + ik\frac{\pi}{2}$ for $k = 4m + 3$ and $m \in \mathbb{Z}$.

If we like, we can verify these solutions. For example for the second solution we have

$$\begin{aligned}
\cosh z &= \cosh\left(\ln(\sqrt{2}-1) + i(4m+3)\tfrac{\pi}{2}\right) \\
&= \cosh\left(\ln(\sqrt{2}-1)\right)\cos\left((4m+3)\tfrac{\pi}{2}\right) + i\sinh\left(\ln(\sqrt{2}-1)\right)\sin\left((4m+3)\tfrac{\pi}{2}\right) \\
&= 0 + i\cdot\frac{1}{2}\left(\sqrt{2}-1-\frac{1}{\sqrt{2}-1}\right)\cdot(-1) \\
&= -\frac{i}{2}\left(\sqrt{2}-1-\frac{\sqrt{2}+1}{2-1}\right) \\
&= i.
\end{aligned}$$

Example 7.7. Find all $z \in \mathbb{C}$ such that $\cos z = 5$.

Let $z = x + iy$. Using (9), we get

$$\cos x \cosh y - i \sin x \sinh y = \cos z = 5.$$

Therefore, we must have $\cos x \cosh y = 5$ and $\sin x \sinh y = 0$. It follows that either $\sin x = 0$ or $\sinh y = 0$.

If $\sinh y = 0$, then $y = 0$, which gives $\cos x \cdot 1 = 5$, which is impossible for $x \in \mathbb{R}$.

If $\sin x = 0$, then $x = k\pi$ for $k \in \mathbb{Z}$. In this case, $\cos x = (-1)^k$. Therefore $(-1)^k \cosh y = 5$. When k is odd, we have no solution since $\cosh y \geq 1$. Therefore k is even and we have to solve for $\cosh y = 5$. There are two such y satisfying this: $y = \pm\operatorname{arccosh} 5 = \pm\ln(5 + 2\sqrt{6})$. Thus, our solutions are $z = 2m\pi \pm i\ln(5 + 2\sqrt{6})$ with $m \in \mathbb{Z}$.

Exercises 7.2.

1. Find $\dfrac{d}{dx}\operatorname{arctanh}(\sin x)$.

2. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^4-1}}\,dx$

   (b) $\displaystyle\int \frac{\sqrt{x^2-19}+x^2-19}{(x^2-19)^{3/2}}\,dx$

3. Find all complex numbers z such that:

   (a) $\sinh z = i$

   (b) $\sin z = 5$

   (c) $\sin z = 1$

7.3 The Gamma function

The Gamma and Beta functions occur frequently in statistics, probability, number theory, calculus and combinatorics. In this section we study the Gamma function which was discovered by Euler when looking for a function that generalised the factorial function. In the next section we will then discuss the Beta function, along with links between the two functions. We will further give a number of applications of both functions to calculus.

7.3.1 Definition and properties

For $x \in \mathbb{R}^+$, the Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Though it may seem strange to define a function as an integral, many important mathematical objects are defined in this way (in particular, Laplace and Fourier transforms). We will discuss various properties of the Gamma function and give some applications in calculus. We first begin with an important basic property of this function.

Proposition 7.8. *For $x > 1$, $\Gamma(x) = (x-1)\Gamma(x-1)$.*

Proof. We use integration by parts. Set $u = t^{x-1}$ and $\frac{dv}{dt} = e^{-t}$. Then $\frac{du}{dt} = (x-1)t^{x-2}$ and $v = -e^{-t}$. Then,

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}dt \\ &= \big[-t^{x-1}e^{-t}\big]_0^\infty + \int_0^\infty (x-1)t^{x-2}e^{-t}dt \\ &= 0 + (x-1)\int_0^\infty t^{(x-1)-1}e^{-t}dt \\ &= (x-1)\Gamma(x-1).\end{aligned}$$

□

Note, from Proposition 7.8, we need only evaluate Γ for any real number $0 < x \leq 1$ in order to evaluate for any $x \in \mathbb{R}^+$.

Proposition 7.9. *If k is a positive integer, then $\Gamma(k) = (k-1)!$.*

In light of Proposition 7.9, we see that the Gamma function is in fact a generalisation of the factorial function. Gauss, apparently, proposed defining the Gamma function as $\Pi(x) = x\Gamma(x)$ so that $\Pi(k) = k!$ for integer k. However, it didn't catch on, so we have this slightly irritating shift.

Proof. We prove this by induction. For $k = 1$, we have

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}dt \\ &= [-e^{-t}]_0^\infty \\ &= 0 - (-1) = 1.\end{aligned}$$

Therefore, $\Gamma(1) = 1 = 0!$.

Now suppose that $k > 1$ and that this proposition holds for $k-1$. We then have

$$\begin{aligned}\Gamma(k) &= (k-1)\Gamma(k-1) && \text{(by Proposition 7.8)} \\ &= (k-1)(k-2)! && \text{(by induction)} \\ &= (k-1)!,\end{aligned}$$

completing the proof. □

Another important property of the Gamma function is as follows.

Theorem 7.10. *For any $0 < x < 1$, we have $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$.*

Notice Theorem 7.10 clearly does not hold for any integer x, as $\sin(\pi x) = 0$ for those x, and we have not defined $\Gamma(x)$ for negative x, hence the restriction to our range. We omit the proof of Theorem 7.10 as it is somewhat technical. We now show how to use the previous results to evaluate the Gamma function for certain values.

Example 7.11. For example, we can use Theorem 7.10 to obtain $\Gamma(1/6)\Gamma(5/6) = \frac{\pi}{\sin(\pi/6)} = \frac{\pi}{1/2} = 2\pi$.

Example 7.12. We can use Theorem 7.10 to evaluate $\Gamma(1/2)$. It follows from Theorem 7.10 that we have $\Gamma(1/2)\Gamma(1/2) = \frac{\pi}{\sin(\pi/2)} = \pi$. Thus $\Gamma(1/2)^2 = \pi$, implying $\Gamma(1/2) = \sqrt{\pi}$.

Example 7.13. We evaluate $\Gamma(11/2)$ by repeatedly using Proposition 7.8.

$$\begin{aligned}
\Gamma(11/2) &= \frac{9}{2} \cdot \Gamma\left(\frac{9}{2}\right) \\
&= \frac{9}{2} \cdot \frac{7}{2} \cdot \Gamma\left(\frac{7}{2}\right) \\
&= \frac{9}{2} \cdot \frac{7}{2} \cdots \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \\
&= \frac{9 \cdot 7 \cdots 1}{2^5} \cdot \Gamma\left(\frac{1}{2}\right) \\
&= \frac{9 \cdot 7 \cdots 1}{2^5} \cdot \sqrt{\pi}.
\end{aligned}$$

For any odd positive integer k define $M(k) = k(k-2)(k-4)\cdots 1$. Then, generalising the above computation we get

$$\Gamma\left(\frac{2m+1}{2}\right) = \frac{M(2m-1)}{2^m}\sqrt{\pi}.$$

7.3.2 Application to integration

We now give a number of examples of using the Gamma function to evaluate integrals. In each case, we evaluate the integral by changing it into an integral that looks like a Gamma function. These types of integrals occur frequently in statistics and probability, since we often want to integrate a function against a measure given by an exponential function. We don't say what this is, but it amounts to evaluating integrals whereby you have a polynomial times an exponential in the integrand.

Example 7.14. Evaluate the integral

$$\int_0^\infty x^6 e^{-5x} dx.$$

We make the substitution $t = 5x$. Then $\frac{dt}{dx} = 5$. Hence,

$$\begin{aligned}
\int_0^\infty x^6 e^{-5x} dx &= \int_0^\infty x^6 e^{-5x} \frac{dx}{dt}\, dt \\
&= \int_0^\infty \left(\frac{t}{5}\right)^6 e^{-t} \frac{1}{5}\, dt \\
&= \frac{1}{5^7} \int_0^\infty t^6 e^{-t}\, dt \\
&= \frac{1}{5^7} \Gamma(7) \\
&= \frac{1}{5^7} 6! \qquad \text{(by Proposition 7.9)} \\
&= \frac{144}{15625}.
\end{aligned}$$

Example 7.15. Evaluate the integral

$$\int_0^\infty x^3 e^{-\frac{1}{2}x^2} dx.$$

Set $t = \frac{1}{2}x^2$, so $\frac{dt}{dx} = x$. Therefore, we have

$$\begin{aligned}
\int_0^\infty x^3 e^{-\frac{1}{2}x^2} dx &= \int_0^\infty x^3 e^{-\frac{1}{2}x^2} \frac{dx}{dt}\, dt \\
&= \int_0^\infty x^2 e^{-\frac{1}{2}x^2}\, dt \\
&= \int_0^\infty 2te^{-t}\, dt \\
&= 2\Gamma(2) \\
&= 2.
\end{aligned}$$

Example 7.16. Evaluate the integral

$$\int_0^\infty x^2 e^{-\frac{1}{4}x^2}\, dx.$$

Make the substitution $t = \frac{1}{4}x^2$, so that $\frac{dt}{dx} = \frac{1}{2}x$. Thus, we have

$$\begin{aligned}
\int_0^\infty x^2 e^{-\frac{1}{4}x^2} dx &= \int_0^\infty x^2 e^{-\frac{1}{4}x^2} \frac{dx}{dt}\, dt \\
&= 2\int_0^\infty x e^{-\frac{1}{4}x^2}\, dt \\
&= 2\int_0^\infty 2t^{\frac{1}{2}} e^{-t}\, dt \\
&= 4\int_0^\infty t^{\frac{1}{2}} e^{-t}\, dt \\
&= 4\Gamma\left(\frac{3}{2}\right) \\
&= 4 \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \\
&= 2\sqrt{\pi}.
\end{aligned}$$

Exercises 7.3.

1. Evaluate the following integrals.
   (a) $\int_0^\infty x^3 e^{-x}\, dx$.
   (b) $\int_0^\infty x^6 e^{-2x}\, dx$.
2. Evaluate $\int_0^\infty x^m e^{ax^n}\, dx$ where m, n and a are positive integers.
3. Evaluate $\int_0^a \frac{1}{\sqrt{\ln\left(\frac{a}{x}\right)}}\, dx$.
4. Evaluate $\int_0^\infty \sqrt{y} e^{-y^3}\, dy$.
5. Prove that $\int_0^1 x^m (\ln x)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$.

7.4 The Beta function

We now introduce the Beta function and study its relation to the Gamma function and applications.

7.4.1 Definition of the Beta function

The Beta is a function of two real variables.

For any $x, y \in \mathbb{R}^+$, define

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

There are several equivalent forms for the Beta function, and we give some now.

Proposition 7.17. *We have*

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}(u)\cos^{2y-1}(u)\, du.$$

Proof. Set $t = \sin^2 u$ in the definition of the Beta function. Then, $\frac{dt}{du} = 2\sin u \cos u$. Also, we see that $u = 0$ when $t = 0$ and $u = \pi/2$ when $t = 1$. Therefore, we have

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \\ &= \int_0^{\pi/2} (\sin^2 u)^{x-1}(\cos^2 u)^{y-1} 2\sin u \cos u\, du \\ &= \int_0^{\pi/2} \sin^{2x-2}(u)\cos^{2y-2}(u) 2\sin u\cos u\, du \\ &= 2\int_0^{\pi/2} \sin^{2x-1}(u)\cos^{2y-1}(u)\, du. \end{aligned}$$

□

Proposition 7.18. *We have*

$$B(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du.$$

Proof. Set $t = \frac{u}{u+1}$ in the definition of the Beta function. Then $\frac{dt}{du} = \frac{1}{(u+1)^2}$. Also, $u = 0$ when $t = 0$ and notice that as $u \to \infty$, we have $t \to 1$. Thus, we have

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \\ &= \int_0^\infty \left(\frac{u}{u+1}\right)^{x-1}\left(1-\frac{u}{u+1}\right)^{y-1}\frac{dt}{du}\, du \\ &= \int_0^\infty \left(\frac{u}{u+1}\right)^{x-1}\left(1-\frac{u}{u+1}\right)^{y-1}\frac{1}{(u+1)^2}\, du \\ &= \int_0^\infty \left(\frac{u}{u+1}\right)^{x-1}\left(\frac{1}{u+1}\right)^{y-1}\frac{1}{(u+1)^2}\, du \\ &= \int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}}\, du. \end{aligned}$$

□

7.4.2 The link between the Gamma and Beta functions

Here is the link between the two functions.

Theorem 7.19. *The Gamma and Beta function are linked via the following equation.*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The following corollary easily follows.

Corollary 7.20. *For any $x, y \in \mathbb{R}^+$, we have*

$$B(x, y) = B(y, x).$$

Proof of Theorem 7.19. First note if we put $t = a^2, (a > 0)$ in the integral for $\Gamma(x)$ then $\frac{dt}{da} = 2a$. Also, when $t = 0, a = 0$ and as $t \to \infty, a \to \infty$. Therefore, we have

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty t^{x-1} e^{-t}\, dt \\
&= \int_0^\infty a^{2(x-1)} e^{-a^2} 2a\, da \\
&= \int_0^\infty 2a^{2x-1} e^{-a^2}\, da.
\end{aligned}$$

Thus, if we take the product of $\Gamma(x)\Gamma(y)$ using the substitution above we have

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \int_0^\infty t^{x-1} e^{-t}\, dt \int_0^\infty u^{y-1} e^{-u}\, du \\
&= \int_0^\infty 2a^{2x-1} e^{-a^2}\, da \int_0^\infty 2b^{2y-1} e^{-b^2}\, db \\
&= \int_0^\infty \int_0^\infty 4a^{2x-1} b^{2y-1} e^{-a^2-b^2}\, da\, db. \qquad (15)
\end{aligned}$$

Now put $a = r\sin\theta$ and $b = r\cos\theta$, i.e. change to polar coordinates. From Chapter 3 in the notes, we see that the integral (15) becomes

$$\begin{aligned}
&= \int_0^{\pi/2} \int_0^\infty 4(r\sin\theta)^{2x-1} (r\cos\theta)^{2y-1} e^{-r^2} r\, dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^\infty 4r^{2x+2y-1} e^{-r^2} \sin^{2x-1}(\theta) \cos^{2y-1}(\theta)\, dr\, d\theta \\
&= \int_0^{\pi/2} 2\sin^{2x-1}(\theta) \cos^{2y-1}(\theta)\, d\theta \int_0^\infty 2r^{2(x+y)-1} e^{-r^2}\, dr. \qquad (16)
\end{aligned}$$

Notice the change in limits. We refer the reader to Chapter 3 for an explanation of the limits of integration for polar coordinates. Notice that in the original integral, we are integrating over all points in the first quadrant. Notice the first integral in (16) is $B(x, y)$ from Proposition 7.17. Put $v = r^2$ in the second integral. Thus,

$\frac{dv}{dr} = 2r$. Therefore, we have

$$\begin{aligned}\Gamma(x)\Gamma(y) &= B(x,y)\int_0^\infty 2r^{2(x+y)-1}e^{-v}\,\frac{dr}{dv}\,dv\\ &= B(x,y)\int_0^\infty 2r^{2(x+y)-1}e^{-v}\,\frac{dv}{2r}\\ &= B(x,y)\int_0^\infty r^{2(x+y-1)}e^{-v}\,dv\\ &= B(x,y)\int_0^\infty v^{x+y-1}e^{-v}\,dv\\ &= B(x,y)\Gamma(x+y).\end{aligned}$$

Therefore, we have

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \qquad \square$$

7.4.3 Evaluating integrals with the Beta function

Using Theorem 7.19, we can use the Beta function to evaluate integrals. Again, these type of integrals appear in probability and statistics.

Example 7.21. Evaluate the integral

$$\int_0^1 x^2\sqrt{1-x^2}\,dx.$$

First, put $t = x^2$, so that $\frac{dt}{dx} = 2x$. Hence,

$$\begin{aligned}\int_0^1 x^2\sqrt{1-x^2}dx &= \int_0^1 t\sqrt{1-t}\,\frac{dx}{dt}\,dt\\ &= \int_0^1 t\sqrt{1-t}\,\frac{dt}{2x}\\ &= \frac{1}{2}\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt \qquad \left(\text{since } \frac{1}{x} = \frac{1}{t^{\frac{1}{2}}}\right)\\ &= \frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right)\\ &= \frac{1}{2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}\\ &= \frac{1}{2}\frac{\left(\frac{1}{2}\Gamma(\frac{1}{2})\right)^2}{2!}\\ &= \frac{\pi}{16}.\end{aligned}$$

Example 7.22. Evaluate the integral

$$\int_0^2 (2x - x^2)^{\frac{3}{2}}\ dx.$$

We put $x = 2t$, so that $\frac{dx}{dt} = 2$. Therefore, we have

$$\begin{aligned}
\int_0^2 (2x - x^2)^{\frac{3}{2}}\ dx &= \int_0^1 (4t)^{\frac{3}{2}}(1-t)^{\frac{3}{2}}\ 2\ dt \\
&= \int_0^1 16t^{\frac{3}{2}}(1-t)^{\frac{3}{2}} dt \\
&= 16B\left(\frac{5}{2}, \frac{5}{2}\right) \\
&= 16\frac{\left(\Gamma\left(\frac{5}{2}\right)\right)^2}{\Gamma(5)} \\
&= 16\frac{(\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi})^2}{4!} \\
&= \frac{9\pi}{4!} \\
&= \frac{3}{8}\pi
\end{aligned}$$

Note, the purpose of the substitution is to get a factor of $(1-t)$ into the integral.

Example 7.23. Evaluate the integral

$$\int_0^\infty \frac{\sqrt{x}}{16 + x^2}\ dx.$$

We are going to use the Beta formula given in Proposition 7.18. In that case, we want to make a substitution so that we have a $1 + u$ in the denominator. So, let's try $x^2 = 16u$. Then $x = 4u^{1/2}$, $\sqrt{x} = 2u^{1/4}$ and $\frac{dx}{du} = \frac{8}{x}$. Thus, we have

$$\begin{aligned}
\int_0^\infty \frac{\sqrt{x}}{16 + x^2} dx &= \int_0^\infty \frac{\sqrt{x}}{16 + 16u}\frac{dx}{du} du \\
&= \frac{1}{16}\int_0^\infty \frac{2u^{1/4}}{1+u}\frac{8}{x}\ du \\
&= \frac{1}{16}\int_0^\infty \frac{2u^{1/4}}{1+u}\frac{2}{u^{1/2}}\ du \\
&= \frac{1}{4}\int_0^\infty \frac{u^{-1/4}}{1+u}\ du.
\end{aligned}$$

This is a Beta function in the form of Proposition 7.18. Here, $x - 1 = -1/4$, which implies $x = 3/4$, and $x + y = 1$, which implies $y = 1/4$. Therefore, continuing with the computation, we have

$$
\begin{aligned}
&= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{4}\right) \\
&= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} \\
&= \frac{1}{4} \frac{\pi/\sin(\pi/4)}{1} \\
&= \frac{1}{4}\sqrt{2}\pi.
\end{aligned}
$$

Example 7.24. Evaluate the integral

$$\int_0^{\pi/2} \sin^4 u \cos^3 u \, du.$$

This integral is already in the form of Proposition 7.17, so we go right ahead and use that formula. In the form of Proposition 7.17, we have $2x - 1 = 4$, implying that $x = 5/2$ and we have $2y - 1 = 3$, implying that $y = 2$. Hence, we see that

$$
\begin{aligned}
\int_0^{\pi/2} \sin^4 u \cos^3 u du &= \frac{1}{2} B\left(\frac{5}{2}, 2\right) \\
&= \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{\Gamma\left(\frac{9}{2}\right)} \\
&= \frac{1}{2}\left(\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot 1}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}\right) \\
&= \frac{1}{2}\left(\frac{1}{\frac{7}{2}\cdot\frac{5}{2}}\right) \\
&= \frac{2}{35}.
\end{aligned}
$$

Example 7.25. Evaluate the integral

$$\int_0^{\pi/2} \cos^8 u \, du.$$

Again, this is already in the form of Proposition 7.17. In this case, $2x - 1 = 0$, implying that $x = 1/2$, and $2y-1=8$, implying that $y = 9/2$. Thus,

$$\begin{aligned}\int_0^{\pi/2} \cos^8 u\, du &= \frac{1}{2}B\left(\frac{1}{2},\frac{9}{2}\right)\\ &= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma(5)}\\ &= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4!}\\ &= \frac{35\cdot 3\pi}{2^5 4!}\\ &= \frac{35}{256}\pi.\end{aligned}$$

Example 7.26. Evaluate the integral

$$\int_0^{\pi} \sin^3 x \cos^4 x\, dx.$$

Notice here that the limit of integration is 0 to π and not 0 to $\pi/2$. The integral can be written as

$$\int_0^{\pi} \sin^3 x\cos^4 x\, dx = \int_0^{\pi/2} \sin^3 x\cos^4 x\, dx + \int_{\pi/2}^{\pi} \sin^3 x\cos^4 x\, dx.$$

The first integral is just the Beta function in the form of Proposition 7.17, with $2x-1=3$, so $x=2$, and $2y-1=4$, so $y=5/2$. We therefore have $\frac{1}{2}B\left(2,\frac{5}{2}\right)$ for the first integral.

For the second integral, we make the substitution $u = x-\pi/2$. Thus, $\frac{du}{dx}=1$ and when $x=\pi/2, u=0$ and when $x=\pi, u=\pi/2$. From the standard trigonometric sum identities we have

$$\sin(u+\pi/2) = \sin u\cos\pi/2 + \cos u\sin\pi/2 = \cos u$$

and

$$\cos(u+\pi/2) = \cos u\cos\pi/2 - \sin u\sin\pi/2 = -\sin u.$$

Thus, we have

$$\begin{aligned}\int_0^{\pi} \sin^3 x\cos^4 dx &= \tfrac{1}{2}B\left(2,\tfrac{5}{2}\right) + \int_0^{\pi/2}\cos^3 u\sin^4 u\, du\\ &= \tfrac{1}{2}B\left(2,\tfrac{5}{2}\right) + \tfrac{1}{2}B\left(\tfrac{5}{2},2\right)\\ &= B\left(2,\tfrac{5}{2}\right) \qquad \text{(from Corollary 7.20, } B(x,y)=B(y,x))\\ &= \frac{\Gamma(2)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}\\ &= \frac{1\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}\\ &= \frac{4}{35}.\end{aligned}$$

Note, that you can save yourself some computation by noticing that

$$\frac{\Gamma(2)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = \frac{\Gamma(2)\cdot\Gamma\left(\frac{5}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)} = \frac{4}{35},$$

which is quicker!

Example 7.27. Evaluate the integral

$$\int_0^{2\pi} \sin^2 x \cos^4 x \, dx.$$

Again, we break this up into integrals from 0 to $\pi/2$, from $\pi/2$ to π, from π to $3\pi/2$, and from $3\pi/2$ to 2π. We therefore have

$$\int_0^{2\pi} \sin^2 x \cos^4 x \, dx = \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx + \int_{\pi/2}^{\pi} \sin^2 x \cos^4 x \, dx + \int_{\pi}^{3\pi/2} \sin^2 x \cos^4 x \, dx + \int_{3\pi/2}^{2\pi} \sin^2 x \cos^4 x \, dx. \tag{17}$$

The first integral on the RHS of (17) is already a Beta function. We make the following substitutions: $x = u$ (i.e. not a real substitution), $x = u + \pi/2$, $x = u + \pi$, and $x = u + 3\pi/2$ into the first, second, third and fourth integral, respectively. Using the trigonometric sum formulae, the effects of the three change of variables can be seen in the following table.

x	$\sin x$	$\cos x$
$u + \pi/2$	$\cos u$	$-\sin u$
$u + \pi$	$-\sin u$	$-\cos u$
$u + 3\pi/2$	$-\cos u$	$\sin u$

Notice that these variable changes make all the limits of integration 0 to $\pi/2$ in all the integrals in the RHS of (17). Thus, the RHS of (17) becomes

$$\int_0^{\pi/2} \sin^2(u)\cos^4(u)\, du + \int_0^{\pi/2} \cos^2(u)\sin^4(u)\, du + \int_0^{\pi/2} \sin^2(u)\cos^4(u)\, du + \int_0^{\pi/2} \cos^2(u)\sin^4(u)\, du \tag{18}$$

Setting $2x - 1 = 2$ and $2y - 1 = 4$ implies that $x = 3/2$ and $y = 5/2$ (these will be the arguments of the Beta function). Thus, (18) becomes

$$\begin{aligned}
\frac{1}{2}B\left(\frac{3}{2},\frac{5}{2}\right) &+ \frac{1}{2}B\left(\frac{5}{2},\frac{3}{2}\right) \\
&\qquad + \frac{1}{2}B\left(\frac{3}{2},\frac{5}{2}\right) + \frac{1}{2}B\left(\frac{5}{2},\frac{3}{2}\right) \\
&= 2B\left(\frac{3}{2},\frac{5}{2}\right) \qquad \text{(by Corollary 7.20)} \\
&= \frac{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)} \\
&= \frac{2\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} \\
&= \frac{\frac{3}{4}\cdot\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{3!} \\
&= \frac{1}{8}\pi \qquad \left(\text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right).
\end{aligned}$$

Exercises 7.4.

1. Evaluate the following integrals.
   (a) $\int_0^1 x^4(1-x)^3\,dx$.
   (b) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$.
2. Evaluate the following integrals.
   (a) $\int_0^{\pi/2} \sin^4 x\cos^5 x\,dx$.
   (b) $\int_0^{\pi} \cos^4 x\,dx$.
   (c) $\int_0^{2\pi} \sin^8 x\,dx$.
3. Prove Theorem 7.10 given that $\int_0^\infty \frac{x^{p-1}}{x+1}\,dx = \frac{\pi}{\sin p\pi}$.