Calculus 2 Assignment 3

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1. The variables separable, exact, homogeneous, first order ordinary differential equation

$$x^2 + y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{1}$$

can be solved by

(a) separation of variables; if we rewrite (1) as

$$y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = -x^2$$

and integrate both sides with respect to x

$$\begin{aligned} \int \left(y^2 \frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}x &= -\int x^2 \mathrm{d}x \\ \int y^2 \mathrm{d}y &= -\int x^2 \mathrm{d}x \\ \frac{y^3}{3} &= c_1 - \frac{x^3}{3} \end{aligned}$$

then rearrange to make y the subject

$$\begin{aligned} y^3 &= c_2 - x^3 \\ y &= \sqrt[3]{c_2 - x^3} \end{aligned}$$

is a general solution to (1), where $c_2 = 3c_1$.

(b) observing that if we consider (1) to be of the form

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

where $M = x^2$ and $N = y^2$ then $\frac{\partial M}{\partial x} = \frac{\partial M}{\partial y} = 0$ and as such (1) is exact. We can then say that (1) has a general solution of the form $f(x, y) = c$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. Integrating M with respect to x gives

$$f = \frac{x^3}{3} + g(y) \tag{2}$$

and integrating (2) with respect to y gives

$$\frac{\partial f}{\partial y} = g'(y).$$

We can therefore deduce that

$$g'(y) = N = y^2$$

and

$$g(y) = \int N \mathrm{d}y = \frac{y^3}{3}$$

so

$$f(x, y) = \frac{x^3 + y^3}{3}.$$

$$\begin{aligned}
\frac{x^3 + y^3}{3} &= c_1 \\
y^3 &= c_2 - x^3 \\
y &= \sqrt[3]{c_2 - x^3}
\end{aligned}$$

is therefore a general solution to (1), where $c_2 = 3c_1$.

(c) using the substitution $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$. Now

$$\begin{aligned}
x^2 + (vx)^2(x\frac{\mathrm{d}v}{\mathrm{d}x} + v) &= 0 \\
x^2 + v^2x^3\frac{\mathrm{d}v}{\mathrm{d}x} + x^2v^3 &= 0 \\
1 + v^2x\frac{\mathrm{d}v}{\mathrm{d}x} + v^3 &= 0
\end{aligned}$$

which is variables separable, so we can rearrange and integrate with respect to x as follows

$$
\begin{aligned}
1 + v^2 x \frac{\mathrm{d}v}{\mathrm{d}x} + v^3 &= 0 \\
\frac{1}{v^2} + x\frac{\mathrm{d}v}{\mathrm{d}x} + v &= 0 \\
\frac{1+v^3}{v^2} &= -x\frac{\mathrm{d}v}{\mathrm{d}x} \\
\frac{v^2}{v^3+1}\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{x} \\
\int \frac{v^2}{v^3+1}\mathrm{d}v &= -\int \frac{\mathrm{d}x}{x} \\
\frac{1}{3}\ln(v^3+1) &= -\ln x + c_1 \\
\ln(v^3+1) &= -3\ln x + c_1 \\
&= \ln x^{-3} + c_1 \\
\exp(\ln(v^3+1)) &= \exp(\ln x^{-3} + c_1) \\
v^3 + 1 &= x^{-3}c_2 \\
\frac{y^3}{x^3} + 1 &= \\
y^3 + x^3 &= c_2 \\
y &= \sqrt[3]{c_2 - x^3}.
\end{aligned}
$$

Then $y = \sqrt[3]{c_2 - x^3}$ where $c_2 = e^{c_1}$.

2. The non-linear, non-exact, first order differential equation

$$
y^2 + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{3}
$$

can be solved by

(a) separation of variables; if we rewrite (3) as

$$
\begin{aligned}
x^2\frac{\mathrm{d}y}{\mathrm{d}x} &= -y^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -y^2x^{-2} \\
y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} &= -x^{-2}
\end{aligned}
$$

and integrate both sides with respect to x

$$\begin{aligned}
\int\left(y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x &= -\int x^{-2}\mathrm{d}x \\
\int y^{-2}\mathrm{d}y &= -\int x^{-2}\mathrm{d}x \\
x^{-1}+y^{-1} &= c
\end{aligned}$$

(b) using the substitution $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$, giving

$$\begin{aligned}
v^2 + x\frac{\mathrm{d}v}{\mathrm{d}x} + v &= 0 \\
x\frac{\mathrm{d}v}{\mathrm{d}x} &= -(v^2+v) \\
\frac{1}{v^2v}\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{x}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{v^2v} &= \frac{1+v-v}{v(v+1)} \\
&= \frac{1+v-v}{v(v+1)} \\
&= \frac{1+v}{v(v+1)} - \frac{v}{v(v+1)} \\
&= \frac{1}{v} - \frac{1}{v+1}
\end{aligned}$$

it follows that

$$\begin{aligned}
\int\frac{1}{v}\mathrm{d}v - \frac{1}{v+1}\mathrm{d}v &= -\int\frac{\mathrm{d}x}{x} \\
\ln v - \ln(v+1) &= -\ln x + c_1 \\
\ln\left(\frac{v}{v+1}\right) + \ln x &= c_1 \\
x\cdot\frac{v}{v+1} &= c_2 \\
\frac{yx}{y+x} &= c_2
\end{aligned}$$

and

$$\begin{aligned}
\frac{y+x}{yx} &= c_3 \\
\frac{y}{yx} + \frac{x}{yx} &= c_3 \\
\frac{1}{x} + \frac{1}{y} &= c_3 \\
x^{-1} + y^{-1} &= c_3.
\end{aligned}$$

3.

$$f_1(x) \cdot g_1(y) + f_2(x) \cdot g_2(y) \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{4}$$

(a) The equation (4) is of the general form

$$M_0(x,y) + N_0(x,y) \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Let $\mu = \frac{1}{f_2(x) g_1(y)}$. Multiplying both sides of (4) by μ gives

$$\begin{aligned}
M_0(x,y) + N_0(x,y) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\mu M_0(x,y) + \mu N_0(x,y) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{f_1(x)}{f_2(x)} + \frac{g_2(y)}{g_1(y)} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

which is of the form

$$M_1(x) + N_1(y) \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and as such $\frac{\partial N_1}{\partial x} = \frac{\partial M_1}{\partial y} = 0$ and the ODE is exact. This demonstrates that μ is an integrating factor.

(b) Now, let $\mu = \frac{1}{x^2 y^2}$ and multiply (3) by μ, resulting in

$$\frac{1}{x^2} + \frac{1}{y^2} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{5}$$

Now (5) is of the form

$$M(x) + N(y) \frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

which means (as shown above) that it is exact. In this form we can solve by finding a function $f(x, y) = c$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. Now

$$\begin{aligned}\int \frac{\partial f}{\partial x}\mathrm{d}x &= \int x^{-2}\mathrm{d}x \\ f &= -x^{-1} + g(y) \\ \frac{\partial f}{\partial y} &= g'(y) = y^{-2}\end{aligned}$$

then $g(y) = \int y^{-2}\mathrm{d}y = -y^{-1}$ and $f = -x^{-1} - y^{-1} = c$.

4. Let $t = x^{-1}$ and $y = f(t)$. Now, by the chain rule

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}t}{\mathrm{d}x} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (-x^{-2}) \cdot \frac{\mathrm{d}y}{\mathrm{d}t}\end{aligned}$$

and by the product rule and chain rule

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2t}{\mathrm{d}x^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}t}{\mathrm{d}x} \cdot \left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}\right) \\ &= 2x^{-3} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + (-x^{-2}) \cdot \left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \cdot (-x^{-2})\right) \\ &= 2x^{-3} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + x^{-4} \cdot \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\end{aligned}$$

then

$$x^4 \cdot \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

and

$$2x^3 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = -2x \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$$

so, under the substitution $x = t^{-1}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4y = 4$. This is a linear, non-homogeneous, second order, ordinary differential equation with constant coefficients. Its homogeneous part has characteristic polynomial $w^2 - 4 = 0$ which has roots $\alpha = -2$ and $\beta = 2$, so with $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq \beta$, the general solution to the homogeneous part of the ODE is $y = Ae^{-2t} + Be^{2t}$ with $A, B \in \mathbb{R}$. A particular solution to the original ODE can be found by assuming that $p = n$ with $n \in \mathbb{R}$ and considering $\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} - 4p = 4$, hence a particular solution is $p = -1$ and the general solution is therefore $y = Ae^{-2x^{-1}} + Be^{2x^{-1}} - 1$.

5. The equation we are trying to find has the form

$$P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = S(x) \tag{6}$$

where the P,Q and R terms are the homogeneous part yielding the complimentary function $C(x)$ and $S(x)$ is the non-homogeneous part yielding the particular integral $p(x)$. The general solution to (6) given in the question can be written in the form $y = C(x) + p(x)$. Let's consider the complimentary function first. The form of the general solution suggests that $C(x) = e^{2x}(A\cos(3x)+B\sin(3x))$. We will proceed under this assumption since it also gives us $p(x) = 4x^2 + x - 1$. The form of $C(x)$ tells us that the auxiliary function $Pt^2 + Qt + R = 0$ has roots $2 \pm 3i$. If a polynomial has roots $\pm a$ then $(x \pm a)$ are factors of that polynomial. As such

$$\begin{aligned} 0 &= (x - 2 - 3i)(x - 2 + 3i) \\ &= x^2 - 4x + 13 \end{aligned}$$

and we infer $P = 1$, $Q = -4$, $R = 13$. Now we will consider the particular integral $p(x)$ and its relationship to $S(x)$. Because p is a solution to (6), $\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} - 4\frac{\mathrm{d}p}{\mathrm{d}x} + 13p = S(x)$. Now

$$\begin{aligned} p &= 4x^2 + x - 1 \\ \frac{\mathrm{d}p}{\mathrm{d}x} &= 8x + 1 \\ \frac{\mathrm{d}^2 p}{\mathrm{d}x^2} &= 8 \end{aligned}$$

and

$$\begin{aligned} S(x) &= 8 - 4(8x + 1) + 13(4x^2 + x - 1) \\ &= 52x^2 - 19x - 9 \end{aligned}$$

so

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 52x^2 - 19x - 9$$