Solutions to Chapter 1 Exercises

1.1 1. $\frac{1-(-\gamma)^{n+1}}{1+\gamma}$

2. $\frac{1}{12}n(n+1)(3n^2+11n-14)$

1.11

What are we counting?	Order?	Repetition?
How many seven digit telephone numbers are there?	yes	yes
In how many ways can a football coach distribute one of eight shirts numbered 7 to 14 inclusive to each of his five forwards?	yes	no
How many ways are there to choose 12 identical cups available in white, green or blue?	no	yes
How many ways are there of selecting a subcommittee of 5 people from a committee of 16?	no	no
I wish to place a bet on which horses will come first, second and third in a race with 10 horses. How many ways can I select the horses for my bet?	yes	no
A bakery makes cupcakes with 5 smarties on top. If smarties come in eight colours, how many different ways are there to pick the smarties that go on a cake?	no	yes
A library has 20 DVDs available for borrowing. Fred wishes to borrow 4 DVDs to watch on the weekend; in how many ways can he make his selection?	no	no
The conductor of an orchestra is selecting the program of music for a concert. How many ways are there for her to do this if the orchestra knows how to play six pieces of music and she wants them to play 3 pieces in the concert?	yes	no
The boardgame Mastermind requires one player to fill four hidden holes with coloured pegs; the other player then has to guess which colour peg is in each hole. There is a large supply of pegs in six different colours –how many ways are there to guess which colours have been chosen?	yes	yes
If there are 20 people in a swim squad, how many ways are there of choosing 4 swimmers for a medley relay? (The medley relay involves backstroke, breaststroke, butterfly and freestyle swimming, with a different swimmer performing each stroke.)	yes	no

1.16

1. 1000000, 151200, 114265
2. 35, 25
3. 30240, 252
4. 2850120
5. 7
6. 458752
7. 16
8. $\binom{d+n-1}{d}$, $\binom{d-4+n}{d-3}$

1.21

1. 10
2. 120
3. set $x = 1$, $y = 1$
4. The coefficient of x^n on the RHS of the equation is $\binom{2n}{n}$.
   The coefficient of x^n on the LHS of the equation is
   $\sum_{r=0}^{n}$ [coefficient of x^r in $(1+x)^n$] [coefficient of x^{n-r} in $(1+x)^n$].
   Alternatively, count the number of ways of choosing n elements from a set of size $2n$. This equals the sum from $r = 0$ to n of the number of ways of choosing r items from the first half of the set times the number of ways of choosing $n - r$ items from the second half of the set.

1.23 3^n

1.26 3

1.27 There are $\binom{n}{2}$ pairs of marks, and the number of distinct integers between 1 and x is x. If the largest distance between marks is less than $\frac{n(n-1)}{2}$, then by the pigeonhole principle two of the differences measured must coincide.

1.28 There are 48 elements in the first set and 5 in the second. As $\lceil \frac{48}{5} \rceil = 10$ there are 10 elements from the first set that get mapped to the same element of the second set by the pigeon hole principle.

1.29

1. 286
2. 54
3. yes

1.33 12

1.34

1.

$$\begin{aligned} 0.2 + 0.04 + 0.008 + 0.0016 + \cdots + 0.0000001024 &= \sum_{i=1}^{10} 0.2^i \\ &= \frac{0.2(1-0.2^{10})}{1-0.2} \\ &= \frac{1-0.2^{10}}{4} \\ &= \mathbf{0.2499999744} \end{aligned}$$

2. *either:*

$$\begin{aligned}
\sum_{r=1}^{n}(r+2)^3 &= \sum_{r=3}^{n+2} r^3 \\
&= \left(\sum_{r=1}^{n+2} r^3\right) - 2^3 - 1^3 \\
&= \frac{1}{4}(n+2)^2(n+3)^2 - 9 \\
&= \frac{1}{4}(n^4 + 10n^3 + 37n^2 + 60n) \\
&= \mathbf{\frac{1}{4}n(n+5)(n^2+5n+12)}
\end{aligned}$$

or:

$$\begin{aligned}
\sum_{r=1}^{n}(r+2)^3 &= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 12\sum_{r=1}^{n} r + 8\sum_{r=1}^{n} 1 \\
&= \frac{1}{4}n^2(n+1)^2 + n(n+1)(2n+1) + 6n(n+1) + 8n \\
&= \frac{1}{4}(n^4 + 2n^3 + n^2 + 8n^3 + 4n^2 + 8n^2 + 4n + 24n^2 + 24n + 32n) \\
&= \frac{1}{4}(n^4 + 10n^3 + 37n^2 + 60n) \\
&= \mathbf{\frac{1}{4}n(n+5)(n^2+5n+12)}
\end{aligned}$$

1.35 1. $26^5 =$ **11881376**

2. There are $\binom{5}{3}$ ways of choosing the positions of the js, and 25^2 ways of choosing the remaining letters. Hence the number of such words is $\binom{5}{3}25^2 =$ **6250**.

3. A five-letter word has four consecutive vowels if either the first four letters are vowels and the last letter is a consonant, or the first letter is a consonant and the remaining four letters are vowels, or all five letters are vowels. There are $5^4 \times 21$ words of the first type, 21×5^4 words of the second type, and 5^5 words of the third type. Hence the number of words without four consecutive vowels is $26^5 - 42 \times 5^4 - 5^5 =$ **11852001**. (The answer is 11855126 if you understand the question to imply that words with five vowels are permitted.)

1.36 There are 251 integers between 0 and 250. Of these, $1 + \lfloor\sqrt{250}\rfloor = 16$ are squares, $1 + \lfloor 250^{\frac{1}{3}}\rfloor = 7$ are cubes, and $1 + \lfloor\frac{250}{6}\rfloor = 42$ are divisible by 6. The number of them that are both squares and cubes is $1 + \lfloor 250^{\frac{1}{6}}\rfloor = 3$. The number of them that are squares that are divisible by 6 is $1 + \lfloor\frac{\sqrt{250}}{6}\rfloor = 3$. The number of them that are cubes that are divisible by 6 is $1 + \lfloor\frac{250^{\frac{1}{3}}}{6}\rfloor = 2$. The number that are squares, cubes, and divisible by six is 1.

By the inclusion-exclusion principle, the number of integers between 0 and 250 that are either squares or cubes or divisible by 6 is

$$16 + 7 + 42 - 3 - 3 - 2 + 1 = 58.$$

Thus the number of integers that are neither squares, cubes, nor divisible by 6 is $251 - 58 =$ **193**.

1.37 1. $\binom{21+4-1}{21} = \mathbf{2024}$

2. If the teacher is to buy at least six basketballs, then after purchasing six basketballs there are $\binom{15+4-1}{15} = 816$ ways of choosing the 15 remaining balls. Thus the number of ways of purchasing 21 balls including at most five basketballs is $2024 - 816 = \mathbf{1208}$.

   Alternatively, adding up the number of ways making the purchase if she buys 0, 1, 2, 3, 4 or 5 soccer balls we obtain

$$\binom{21+3-1}{21} + \binom{20+3-1}{20} + \binom{19+3-1}{19} + \binom{18+3-1}{18} + \binom{17+3-1}{17} + \binom{16+3-1}{16} = \mathbf{1208}.$$

3. 208 (This questions is more easily handled using techniques from chapter 2.)

4. **Yes**, since there are 21 balls and 4 sports, and $\left\lfloor \frac{21}{4} \right\rfloor = 6$, so by the extended pigeonhole principle there must be some sport for which there are at least 6 balls.

1.38 1.

$$\begin{aligned}\sum_{r=1}^{50} 2^{-r} &= \frac{2^{-1}(1-2^{-50})}{1-2^{-1}} \quad \text{(sum of a geometric series)}\\ &= 1 - 2^{-50}\\ &\approx 1.\end{aligned}$$

2.

$$\begin{aligned}\sum_{r=1}^{50} \binom{r}{2} &= \sum_{r=1}^{50} \frac{r(r-1)}{2}\\ &= \frac{1}{2}\sum_{r=1}^{50} r^2 - \frac{1}{2}\sum_{r=1}^{50} r\\ &= \frac{1}{2}\left(\frac{1}{6}50 \times 51 \times 101 - \frac{1}{2}50 \times 51\right)\\ &= 20825.\end{aligned}$$

3.

$$\begin{aligned}\sum_{r=1}^{50} \binom{n}{r} 2^{-r} &= \sum_{r=0}^{50} \binom{n}{r} 2^{-r} - \binom{50}{0} 2^0\\ &= (1+2^{-1})^{50} - 1\\ &= \left(\frac{3}{2}\right)^{50} - 1\\ &\approx 5.38 \times 10^8.\end{aligned}$$

1.39 1.

$$\binom{40}{10} = 847660528$$

2. Number of committees with no women: $\binom{23}{10} = 11244066$
   Number of committees with 1 woman: $17 \times \binom{23}{9} = 13892230$
   $\therefore$ Number of committees with at least two women$=\binom{40}{10} - \binom{23}{10} - 17 \times \binom{23}{9} =$ 832624232.

3. There are 30 employees who are not on the committee; if the committee members stand in a line, then the number of ways of finding partners for them are the number of ways of picking 10 of the 30 to stand in a line (so the order matters here, and there is no repetition). Therefore the number of outcomes is $\frac{30!}{20!} \approx 1.09 \times 10^{14}$.

1.40

1. The difference between two integers is divisible by 3 precisely when they are in the same residue class modulo 3. There are 3 such residue classes, hence by the pigeonhole principle, there must be some residue class that contains at least two of the four integers. Hence we can select two integers from this class, and their difference is divisible by 3.

2. We can split the numbers from 1 to 99 into the sets $S_1 = \{n \in \mathbb{Z} | 1 \leq n \leq 33\}$, $S_2 = \{n \in \mathbb{Z} | 34 \leq n \leq 66\}$, $S_3 = \{n \in \mathbb{Z} | 67 \leq n \leq 99\}$. As there are 3 such sets, one of them must contain at least two numbers by the pigeonhole principle. The difference between two numbers in any one of these sets is at most 32.

1.41

1. This is equivalent to a distribution problem of placing 15 identical balls into 3 boxes with nonexclusive occupancy, hence the answer is $\binom{15+2}{15} = 136$.

2. The possibilities are: Terry and Chris each deliver 0 tables, each deliver 1 table, each deliver 2 tables, each deliver 3 tables,each deliver 4 tables, each deliver 5 tables, each deliver 6 tables, or each deliver 7 tables. Hence there are 8 possibilities.

3. 36 (This question is more easily handled using techniques from Chapter 2.)

Solutions to Chapter 2 Exercises

2.10 $(x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$

2.11 $X_1 + X_2 + X_3 = r$,
$11 \leq X_1$,
$1 \leq X_2 \leq 3$,
$0 \leq X_3$,
$(x^{11} + x^{12} + \cdots)(x + x^2 + x^3)(1 + x + x^2 + \cdots)$

2.15.1 $X_1 + X_2 + X_3 = r$
$X_1 \geq 0$,
$X_2 \in \{0, 2, 4, \ldots\}$,
$X_3 \in \{0, 5, 10, \ldots\}$
$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)$

2.15.2 4

2.15.3 55

2.15.4 18

2.21.1 $\frac{x(1+x)}{(1-x)^3} + \frac{x}{(1-x)^2} + \frac{1}{(1-x)}$

2.21.2 $\frac{-x}{(1+x)^2}$

Solutions to Chapter 3 Exercises

3.11.1 $u_n = 11 \times 2^n - 7 \times 3^n$

3.11.2 $u_n = 2 \times 2^n \cos\left(\frac{n\pi}{3}\right) - \sqrt{3} \times 2^n \sin\left(\frac{n\pi}{3}\right)$

3.11.3 $u_n = 3^{n-1}(6 - 4n)$

3.20.1 $u_n = (2 + n)(n!)^2$

3.20.2 $(6 + \sqrt{3})2^{n-2} \cos\left(\frac{n\pi}{6}\right) + (5 - 4\sqrt{3})2^{n-2} \sin\left(\frac{n\pi}{6}\right) + (4\sqrt{3} - 8)^{-1}2^n$

3.20.3 $1 + (4/3)n - (1/3)n^3$

3.21.1 $u_n = (3/4)3^n + (1/4)(-1)^n$

3.21.2 $u_n = (3/5)3^n + (2/5)(-2)^n$

3.24

$$\begin{aligned} q(x) &= 1 + a_1x + a_2x^2 + \cdots + a_kx^k \\ r(x) &= x^k + a_1x^{k-1} + \cdots + a_k \end{aligned}$$

so

$$\begin{aligned} r(1/x) &= x^{-k} + a_1x^{1-k} + \cdots + a_k \\ \therefore x^k r(1/x) &= 1 + a_1x + a_2x^2 + \cdots + a_kx^k \\ &= q(x) \end{aligned}$$

3.25 see answers to 3.11

3.28

$$\begin{aligned} \binom{2n}{n}/(n+1) &= \frac{(2n)!}{n!n!(n+1)} \\ &= \frac{2n(2n-1)(2n-2)(2n-3)\cdots 3 \cdot 2 \cdot 1}{(n+1)!n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1} \\ &= \frac{2n(2n-1)2(n-1)(2n-3)\cdots 3 \cdot 2 \cdot 1}{(n+1)!n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1} \\ &= \frac{2(2n-1)2(2n-3)2(2n-5)\cdots 2 \cdot 3 \cdot 2 \cdot 1}{(n+1)!} \\ &= \frac{2 \cdot 6 \cdot 10 \cdots 2(2n-1)}{(n+1)!} \end{aligned}$$

3.29

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{n!n!} \\ &= \frac{(n+1)(2n)!}{n!(n+1)!} - \frac{n(2n)!}{n!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!} \\ &= \frac{(n+1)(2n)!}{n!n!} \\ &= (n+1)\binom{2n}{n} \\ &= C_n\end{aligned}$$

3.32.1 the compound interest is a better choice

3.32.2 12 years, 11 years 7 months

3.32.3 358.22

3.33 If $p_{t-1} = \frac{d-b}{c+a}$ then

$$\begin{aligned}p_t &= \frac{a}{c}\frac{d-b}{c+a} + \frac{d-b}{c} \\ &= \frac{a(d-b)}{c(c+a)} + \frac{(c+a)(d-b)}{c(c+a)} \\ &= \frac{c(d-b)}{c(c+a)} \\ &= \frac{d-b}{c+a} \\ &= p_{t-1}.\end{aligned}$$

3.34 1) arithmetic progression 2) grows exponentially 3) stays constant 4) decays to 0 5) not particularly!

Solutions to Chapter 4 Exercises

4.10 e.g. G has no loop but H does, H has no multiple edges but G does, H has two vertices of degree 4 but G has one, G has two vertices of degree three, but H has none, etc.

4.12 H and K

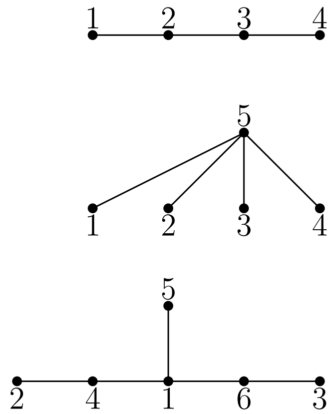
Solutions to Chapter 5 Exercises

5.7 weights of trees are 720, 145

5.9 weights of trees are 720, 145

5.11 £48

5.15



Solutions to Chapter 6 Exercises

6.1 no

6.2 the rightmost 2

6.3 yes. no.

6.5.1 K_n is Eulerian when n is odd

6.5.2 K_{nm} is Eulerian if n and m are both even.

6.5.3 only the octahedron

6.8 e.g. 2,2,0,0,3,3,1,1,4,4,3,2,1,0,4

6.9 see `http://en.wikipedia.org/wiki/Icosian_game` for an example of a solution

6.12.1 all of them with $n \geq 3$

6.12.2 K_{mn} is Hamiltonian iff $m = n$.

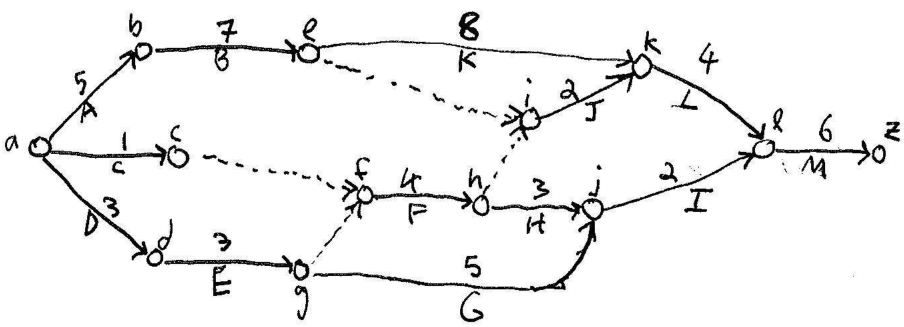
6.18, 6.19 These are provided as examples of the TSP, you're not expected to solve them!

6.23 For the 6.18 graph, starting at a lower bound =618, upper bound=923. For the 6.19 graph, starting at H lower bound=750, upper bound=950.

Solutions to Chapter 7 Exercises

7.2 a, e, h, z length= 7

7.5 a, c, d, j, h, l, z length= 68



7.7 1.

2. A, B, K, L, M length= 30

3.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	5	0	0	3	6	6	10	13	12	12	20	24

4.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	5	13	8	11	14	17	19	22	18	12	20	24

5.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	0	13	8	8	8	11	9	9	6	0	0	0

7.11 1. The set of all edges incident with a given vertex is a cutset of size $n-1$, hence the edge-connectivity of K_n is at most $n-1$.
2. The edge-connectivity is no greater than the smallest degree of a vertex in the graph.

7.20 1. 3
2. 2,2
3. for example, vertices $\{x, y, z\}$ edges $\{xy, xy, xy, yz, yz, yz\}$

7.30 value= 25

Solutions to Chapter 8 Exercises

8.10 1. 2, no 4, yes
2. Go round the cycle, including every second edge in the matching. A Hamiltonian graph has a cycle that includes every vertex; go round the cycle, including every second edge in the matching.

8.13 for example: $\{E_1, J_2\}, \{E_2, J_3\}, \{E_3, J_4\}$

8.14 yes

8.19 $\{M_2, W_1\}, \{M_3, W_2\}, \{M_1, W_3\}$

8.20 1. Each man is matched with his top ranking woman and thus will not swap.
2. $\{M_1, W_1\}, \{M_2, W_2\}, \{M_3, W_3\}$ unstable pair: $\{M_2, W_2\}$ and $\{M_3, W_3\}$ (for example); $\{M_3, W_1\}, \{M_2, W_2\}, \{M_1, W_3\}$ unstable pair: $\{M_2, W_2\}$ and $\{M_1, W_3\}$ (for example); $\{M_2, W_1\}, \{M_3, W_2\}, \{M_1, W_3\}$ unstable pair: $\{M_2, W_1\}, \{M_1, W_3\}$ (for example)

8.21 1. $\{M_1, W_1\}, \{M_2, W_2\}, \{M_3, W_3\}, \{M_4, W_4\}$
2. $\{M_1, W_3\}, \{M_2, W_2\}, \{M_3, W_1\}, \{M_4, W_4\}$

8.27
$$\begin{array}{ccccc} 1 & 0 & 2 & 2 & 4 \\ 1 & 0 & 0 & 3 & \mathbf{0} \\ \mathbf{0} & 1 & 2 & 4 & 5 \\ 2 & 5 & 0 & \mathbf{0} & 2 \\ 3 & \mathbf{0} & 2 & 3 & 1 \end{array}$$
zeros can be covered by rows 2,3,4 and column 2

8.29 $(W_1, j_5), (W_2, j_1), (W_3, j_4), (W_4, j_2), (W_5, j_3)$

8.31 $(W_1, j_4), (W_2, j_3), (W_3, j_1)$

8.33 smallest value= 3