# Calculus 3 Assignment 2

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1. (a) $P_0$ is even, $P_1$ is odd, $P_2$ is even, $P_3$ is odd.

   (b) Since for any odd function $f(x)$ we know that $\int_{-M}^{M} f(x)\mathrm{d}x = 0$, all we need to demonstrate is any pairwise product $P_i P_j$ is odd. The product of two odd functions is odd and the product of an odd and an even function is also odd (ie. $\langle P_1, P_i\rangle = \langle P_3, P_i\rangle = 0$), so it only remains to prove $\langle P_0, P_2\rangle = 0$, that is

$$\begin{aligned}
\int_{-1}^{1} 1 \cdot \frac{1}{2}(3x^2 - 1)\mathrm{d}x &= 0 \\
\frac{1}{2}\left(\int_{-1}^{1} 3x^2\mathrm{d}x - \int_{-1}^{1} 1\mathrm{d}x\right) &= 0 \\
\frac{1}{2}\left([x^3]_{-1}^{1} - [x]_{-1}^{1}\right) &= 0 \\
\frac{1}{2}\left((1 - -1) - (1 - -1)\right) &= 0 \\
\frac{1}{2}\left(2 - 2\right) &= 0 \\
\frac{1}{2}\left(0\right) &= 0 \\
0 &= 0
\end{aligned}$$

   QED

2. (a) The definition of the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(\tfrac{\pi n x}{L}) + b_n \sin(\tfrac{\pi n x}{L})\big)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\tfrac{\pi n x}{L})\mathrm{d}x \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\tfrac{\pi n x}{L})\mathrm{d}x
\end{aligned}$$

and here $f(x) = e^x$, $L = \pi$ so

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \mathrm{d}x \\
&= \frac{1}{2\pi}(e^{\pi} + e^{-\pi}) \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left[e^x\left(\frac{\cos(nx) + n\sin(nx)}{n^2+1}\right)\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}(e^{\pi} - e^{-\pi})\frac{(-1)^n}{n^2+1} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left[e^x\left(\frac{\sin(nx) - n\cos(nx)}{n^2+1}\right)\right]_{-\pi}^{\pi} \\
&= -n\frac{1}{\pi}(e^{\pi} - e^{-\pi})\frac{(-1)^n}{n^2+1}
\end{aligned}$$

and we can write the Fourier series as

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}(e^{\pi} + e^{-\pi}) + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}(e^{\pi} - e^{-\pi})\frac{(-1)^n}{n^2+1}\cos(nx) - n\frac{1}{\pi}(e^{\pi} - e^{-\pi})\frac{(-1)^n}{n^2+1}\sin(nx)\right) \\
&= \frac{1}{2\pi}(e^{\pi} + e^{-\pi}) + \frac{1}{\pi}(e^{\pi} - e^{-\pi})\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\big(\cos(nx) - n\sin(nx)\big)
\end{aligned}$$

and because $\sinh(\pi) = \frac{1}{2}(e^{\pi} + e^{-\pi})$ we can finally write

$$f(x) = \frac{\sinh(\pi)}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\big(\cos(nx) - n\sin(nx)\big)\right)$$

(b) Because $f(x)$ is continuous at $x = 0$, by Theorem 2.3, the Fourier series at $x = 0$ converges to $f(0) = e^0 = 1$ and we can write

$$\begin{aligned}
1 &= f(0) \\
&= \frac{\sinh(\pi)}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\right) \\
\frac{1}{\sinh(\pi)} &= \frac{1}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\right) \\
\operatorname{cosech}(\pi) &= \\
\pi\operatorname{cosech}(\pi) &= 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}.
\end{aligned}$$

Because $f(x)$ has a discontinuity at $x = \pi$, the Fourier series at $x = \pi$ will converge to

$$\tfrac{1}{2}\left(\lim_{x\to\pi^+} f(x) + \lim_{x\to\pi} f(x)\right) = \tfrac{1}{2}(e^{\pi} + e^{-\pi}) = \cosh(\pi)$$

now

$$\begin{aligned}
\cosh(\pi) &= f(\pi) \\
&= \frac{\sinh(\pi)}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right) \\
\frac{\cosh(\pi)}{\sinh(\pi)} &= \frac{1}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right) \\
\coth(\pi) &= \\
\pi\coth(\pi) &= 1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2+1}
\end{aligned}$$

3. (a) The steady state solution $g(x)$ must be a linear function that satisfies the boundary conditions $g(0) = 0$ and $g(400) = 200$. $g(x) = \frac{1}{2}x$ is such a function.

(b) $h(x) = u_0(x) - g(x) = \frac{3}{2}x - \frac{x^2}{400} - \frac{1}{2}x = x - \frac{x^2}{400}$. If $h(x)$ is even about $x = 200$ then $h(200 + x) = h(200 + x)$ and

$$\begin{aligned}
(200+x) - \frac{(200+x)^2}{400} &= (200-x) - \frac{(200-x)^2}{400} \\
200 + x - \frac{200^2 + 400x + x^2}{400} &= 200 - x - \frac{200^2 - 400x + x^2}{400} \\
400x - 200^2 + 400x + x^2 &= -400x - 200^2 - 400x + x^2 \\
x^2 - 200 &= x^2 - 200
\end{aligned}$$

(c)

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\tfrac{\pi n x}{L}\right) e^{\lambda_n^2 t} + \frac{x}{2}$$

where $L = 400$, $\lambda_n = \alpha\frac{\pi n}{L}$, $\alpha^2 = 1$,

$$\begin{aligned}
B_n &= \frac{2}{400}\int_0^{400} (x - \tfrac{x^2}{400}) \sin\left(\tfrac{\pi n x}{400}\right) \mathrm{d}x \\
&= \frac{1}{200}\left(\Big[ -(x - \tfrac{x^2}{400}) \cos\left(\tfrac{\pi n x}{400}\right)\left(\tfrac{400}{\pi n}\right)\Big]_0^{400} + \int_0^{400} (1 - \tfrac{x}{200}) \cos\left(\tfrac{\pi n x}{400}\right)\left(\tfrac{400}{\pi n}\right) \mathrm{d}x\right) \\
&= \frac{2}{\pi n}\left(\int_0^{400} (1 - \tfrac{x}{200}) \cos\left(\tfrac{\pi n x}{400}\right) \mathrm{d}x\right) \\
&= \frac{2}{\pi n}\left(\Big[(1 - \tfrac{x}{200})\left(\tfrac{400}{\pi n}\right) \sin\left(\tfrac{\pi n x}{400}\right)\Big]_0^{400} + \tfrac{1}{200}\left(\tfrac{400}{\pi n}\right)\int_0^{400} \sin\left(\tfrac{\pi n x}{400}\right) \mathrm{d}x\right) \\
&= \frac{4}{(\pi n)^2}\left(\int_0^{400} \sin\left(\tfrac{\pi n x}{400}\right) \mathrm{d}x\right) \\
&= \frac{4}{(\pi n)^2}\Big[ -\left(\tfrac{400}{\pi n}\right) \cos\left(\tfrac{\pi n x}{400}\right)\Big]_0^{400} \\
&= -\frac{1600}{(\pi n)^3}\left(\cos(\pi n) - 1\right).
\end{aligned}$$

We can now write

$$u(x,t) = -1600 \sum_{n=1}^{\infty} \frac{1}{(\pi n)^3}\big((-1)^n - 1\big) sin\big(\tfrac{\pi n x}{400}\big) e^{\frac{(\pi n)^2}{400^2}t} + \frac{x}{2}$$