Discrete Mathematics

Assignment 1 Solutions 2016

1. Find the value of the following sums:

   (a) Let $k = i - 1$. We observe that when $i = 1$ we have $k = 0$, and when $i = 1001$ we have $k = 1000$. Thus we have

$$\begin{aligned}
\sum_{i=1}^{1001} \binom{1000}{i-1} 2^i &= \sum_{k=0}^{1000} \binom{1000}{k} 2^{k+1}, \\
&= 2\sum_{k=0}^{1000} \binom{1000}{k} 2^k, \\
&= 2(1+2)^{1000} \qquad \text{(by the binomial theorem)} \\
&= 2 \cdot 3^{1000}.
\end{aligned}$$

[2]

   (b)

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{i} j &= \sum_{i=1}^{n} \frac{i(i+1)}{2}, \\
&= \frac{1}{2} \sum_{i=1}^{n} (j^2 + j), \\
&= \frac{1}{2} \left(\frac{1}{6} n(n+1)(2n+1) + \frac{1}{2} n(n+1) \right), \\
&= \frac{1}{2} \left(\frac{1}{6} n(n+1)(2n+4) \right), \\
&= \frac{1}{6} n(n+1)(n+2).
\end{aligned}$$

[3]

2. (a) It is reasonable to suppose that the youth club will not show the same film twice, so repetition is not allowed. Furthermore, order matters (in the sense that we could, for instance, tell the difference between the programme "4:00 Rocky, 7:00 Home Alone, 10:00 Batman Begins" and the programme "4:00 Batman Begins, 7:00 Rocky, 10:00 Home Alone").

      i. We are selecting three films (so $r = 3$) from a set of $10 + 15 + 8 = 33$ possible films (so $n = 33$). Hence the number of ways of doing this is $33!/((33-3)!) = 33 \cdot 32 \cdot 31 = 32736$. [1]

ii. There are 10 possible choices for the first film, 32 possible choices for the second film and 31 choices for the last film, hence the total number of ways of choosing the films is $10 \cdot 32 \cdot 31 = 9920$. [2]

(b) There are 30 ways to choose the captain. We then must choose 7 more players from the remaining 29 club members. The order does not matter, and repetition is not allowed, hence there are $\binom{29}{7}$ ways to do this. Hence the total number of ways to choose the team is $30\binom{29}{7} = 46823400$. [1]

(c) There are $\binom{8}{4}$ ways to choose the four team members who will play singles matches. We then have to count the number of possible ways of arranging the remaining four players into two pairs. Call the players A, B, C and D. If A is paired with B, then C must play with D. If A plays with C then B plays with D, and if A plays with D then B and C play together. These are the only possible options. Hence the total number of ways of selecting the pairs to play in the doubles matches is $3\binom{8}{4} = 210$. [2]

3. A logical first step to trying to answer this question is to determine whether there are in fact any numbers that have been written down by Alice and by Bob and by Charlie. Let X be the quantity of such numbers (note that $X = 0$ is a potential option here.) By the inclusion-exclusion theorem we have

$$29 = 15 + 15 + 15 - 6 - 7 - 8 + X,$$

so

$$\begin{aligned} X &= 29 - 15 - 15 - 15 + 6 + 7 + 8, \\ &= 5. \end{aligned}$$

Thus there are five numbers that all three people have written down. Each of these numbers is either even, or odd. We note that if we subtract an even number from an even number we obtain an even number, and if we subtract an odd number from an odd number we also obtain an even number. Thus to obtain x and y for which $x - y$ is even it suffices for x and y to be both even, or both odd. Now $\lceil \frac{5}{2} \rceil = 3$, so by the Pigeonhole Principle the set of numbers written down by all three people contains at least three numbers that are either all even, or all odd. Hence it is indeed possible to take two of them x and y for which $x - y$ is even. [4]

4. Let M_1 denote the number of hampers on the first motorbike, M_2 denote the number of hampers on the second motorbike, C denote the number of hampers in the car and V denote the number of hampers in the van. Then we have

$$M_1 + M_2 + C + V = 15,$$

with

$$\begin{aligned} 0 \leq M_1 &\leq 3, \\ 0 \leq M_2 &\leq 3, \\ 0 \leq C &\leq 5, \\ 5 \leq V &\leq 15. \end{aligned}$$

The generating function for this problem is

$$
\begin{aligned}
&(1+x+x^2+x^3)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4+x^5)(x^5+x^6+x^7+\cdots+x^{15}),\\
&= x^5(1+x+x^2+x^3)^2(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+\cdots+x^{10}),\\
&= x^5(1-x^4)^2(1+x+x^2+\cdots)^2(1-x^6)(1+x+x^2+\cdots)(1-x^{11})(1+x+x^2+\cdots),\\
&= x^5(1-x^4)^2(1-x^6)(1-x^{11})(1+x+x^2+\cdots)^4,\\
&= x^5(1-2x^4+x^8)(1-x^6)(1-x^{11})(1+x+x^2+\cdots)^4,\\
&= x^5(1-2x^4-x^6+x^8+2x^{10}-x^{14})(1-x^{11})(1+x+x^2+\cdots)^4,\\
&= x^5(1-2x^4-x^6+x^8+2x^{10}-x^{11}-x^{14}+2x^{15}+x^{17}-x^{19}-2x^{21}+x^{35})(1+x+x^2+\cdots)^4,\\
&= x^5(1-2x^4-x^6+x^8+2x^{10}-x^{11}-x^{14}+2x^{15}+x^{17}-x^{19}-2x^{21}+x^{35})\sum_{r=0}^{\infty}\binom{4-1+r}{r}x^r.
\end{aligned}
$$

Hence the number of ways of distributing the hampers is $\binom{13}{10}-2\binom{9}{6}-\binom{7}{4}+\binom{5}{2}+2\binom{3}{0}=95.$ [5]