Solutions Chapter 5

Solutions to Exercises 5.1.

1. (a) Let $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$. Then $a_0 = 1$ because $y(0) = 1$. Furthermore

$$a_1 + 2a_2x + 3a_3x^2 + \cdots = y' = -y = -a_0 - a_1x - a_2x^2 - a_3x^3 - \dots,$$

so comparing coefficients gives

$$\begin{aligned} a_1 &= -a_0 & \Rightarrow \quad & a_1 = -1, \\ 2a_2 &= -a_1 & \Rightarrow \quad & a_2 = \tfrac{1}{2}, \\ 3a_3 &= -a_2 & \Rightarrow \quad & a_3 = -\tfrac{1}{2\cdot 3}, \end{aligned}$$

etc. Hence

$$\begin{aligned} y &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + - \cdots \\ &= e^{-x}. \end{aligned}$$

(b) Let $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$. Then

$$a_1 + 2a_2x + 3a_3x^2 + \cdots = y' = y - x = a_0 + (a_1 - 1)x + a_2x^2 + a_3x^3 + \dots,$$

so comparing coefficients gives

$$\begin{aligned} a_1 &= a_0 & & \\ 2a_2 &= a_1 - 1 & \Rightarrow \quad & a_2 = \tfrac{1}{2}(a_0 - 1) \\ 3a_3 &= a_2 & \Rightarrow \quad & a_3 = \tfrac{1}{2\cdot 3}(a_0 - 1) \end{aligned}$$

etc. Hence

$$\begin{aligned} y &= a_0 + a_0x + \frac{1}{2!}(a_0 - 1)x^2 + \frac{1}{3!}(a_0 - 1)x^3 + \dots \\ &= 1 + x + (a_0 - 1)\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots\right) \\ &= 1 + x + ce^x, \end{aligned}$$

where $c = a_0 - 1$ is a constant.

(c) Let $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$. Then $y''(x) = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \dots$. Hence

$$y'' + y = (2a_2 + a_0) + (3 \cdot 2a_3 + a_1)x + (4 \cdot 3a_4 + a_2)x^2 + \dots.$$

Using $y'' + y = 0$ we can therefore deduce

$$\begin{aligned}
2a_2 + a_0 = 0 \quad &\Rightarrow \quad a_2 = -\tfrac{1}{2!}a_0 \\
3 \cdot 2a_3 + a_1 = 0 \quad &\Rightarrow \quad a_3 = -\tfrac{1}{3!}a_1 \\
4 \cdot 3a_4 + a_2 = 0 \quad &\Rightarrow \quad a_4 = -\tfrac{1}{4\cdot 3}a_2 = \tfrac{1}{4!}a_0 \\
5 \cdot 4a_5 + a_3 = 0 \quad &\Rightarrow \quad a_5 = -\tfrac{1}{5\cdot 4}a_3 = \tfrac{1}{5!}a_1
\end{aligned}$$

etc. Hence

$$\begin{aligned}
y &= a_0 + a_1x - a_0\frac{1}{2!}x^2 - a_1\frac{1}{3!}x^3 + a_0\frac{1}{4!}x^4 + a_1\frac{1}{5!}x^5 - \dots \\
&= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right) \\
&= a_0 \cos x + a_1 \sin x,
\end{aligned}$$

where a_0 and a_1 are constants.

2. Suppose that $y(x) = a_0 + a_1x + a_2x^2 + \dots$. Then

$$a_1 + 2a_2x + 3a_3x^2 + \cdots = y' = 1/x^2,$$

hence

$$a_1x^2 + 2a_2x^3 + 3a_3x^4 + \cdots = 1.$$

However this is impossible because the constant coefficient on the left hand side is 0 and the constant coefficient on the right hand side is 1.

Solving the differential equation analytically gives $y = -1/x + c$, which is not defined at $x = 0$ and therefore can't be written as a series of the form $a_0 + a_1x + a_2x^2 + \dots$.

3. (a) From $y' = y$ we deduce $y'' = y'$, $y''' = y''$, etc. Hence $y'(0) = y(0) = 4$, $y''(0) = y'(0) = 4$, $y'''(0) = y''(0) = 4$, etc. Thus we get the Taylor series

$$\begin{aligned}
y &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots \\
&= 4\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\
&= 4e^x.
\end{aligned}$$

(b) From $x^2y' = 1$ we get $y' = x^{-2}$. Hence

$$y'' = -2x^{-3}, \quad y''' = 2 \cdot 3x^{-4}, \quad y^{(4)} = -2 \cdot 3 \cdot 4x^{-5}, \quad \dots,$$

and in general $y^{(n)} = (-1)^{n+1}n!x^{-(n+1)}$ for $n \geq 1$. Hence $y^{(n)}(1) =$

$(-1)^{n+1}n!$ for $n \geq 1$, and we obtain the Taylor series

$$\begin{aligned}
y(x) &= y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \cdots \\
&= 1 + (x-1) - (x-1)^2 + (x-1)^3 - \cdots \\
&= 1 + (x-1)\big(1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots\big) \\
&= 1 + (x-1)\big(1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots\big) \\
&= 1 + (x-1)\frac{1}{1-(1-x)} \\
&= 1 + \frac{x-1}{x} \\
&= 2 - 1/x.
\end{aligned}$$

Since the series $1+x+x^2+\cdots = \frac{1}{1-x}$ is valid for all $-1 < x < 1$, the series $1+(1-x)+(1-x)^2+\cdots = \frac{1}{1-(1-x)}$ is valid for all x with $-1 < 1-x < 1$, which implies that it is valid for $0 < x < 2$.

(c) From $y'' + y = \sin x$ we get

$$\begin{aligned}
y'' &= \sin x - y, & y''' &= \cos x - y', \\
y^{(4)} &= -\sin x - y'', & y^{(5)} &= -\cos x - y''', \\
y^{(6)} &= \sin x - y^{(4)}, & y^{(7)} &= \cos x - y^{(5)}, \quad \cdots
\end{aligned}$$

Hence

$$\begin{aligned}
y''(0) &= 0 - y(0) = 0, & y'''(0) &= 1 - y'(0) = 1, \\
y^{(4)}(0) &= 0 - y''(0) = 0, & y^{(5)}(0) &= -1 - y'''(0) = -2, \\
y^{(6)}(0) &= 0 - y^{(4)}(0) = 0, & y^{(7)}(0) &= 1 - y^{(5)}(0) = 3, \quad \cdots.
\end{aligned}$$

Thus we obtain the Taylor series

$$\begin{aligned}
y &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots \\
&= \frac{1}{3!}x^3 + \frac{-2}{5!}x^5 + \frac{3}{7!}x^7 + \cdots \\
&= \frac{x^3}{6} - \frac{x^5}{60} + \frac{x^7}{1680} - + \cdots.
\end{aligned}$$

The closed form of this series is hard to guess, but by solving the differential equation analytically we find

$$y = \frac{1}{2}(\sin x - x\cos x).$$

Solutions to Exercises 5.2.

1. (a) For all methods we have $x_0 = 0, x_1 = 0.1, x_2 = 0.2, \ldots$ and $y_0 = 0$.
   Euler's method: The iteration formula is $y_{i+1} = y_i + 0.1(2x_i + y_i)$.

i	x_i	y_i
0	0	0
1	0.1	0
2	0.2	0.02
3	0.3	0.062
4	0.4	0.1282
5	0.5	0.22102

Higher derivative Euler method: We have $y'' = 2 + y' = 2 + 2x + y$, hence the iteration formula is $y_{i+1} = y_i + 0.1(2x_i + y_i) + \frac{0.1^2}{2}(2 + 2x_i + y_i)$.

i	x_i	y_i
0	0	0
1	0.1	0.01
2	0.2	0.04205
3	0.3	0.09847
4	0.4	0.18180
5	0.5	0.29489

Corrected Euler method: The iteration is given by $k_1^i = 2x_i + y_i$, $k_2^i = 2(x_i + \frac{0.1}{2}) + (y_i + \frac{0.1}{2}k_1^i)$ and $y_{i+1} = y_i + 0.1k_2^i$.

i	x_i	y_i	k_1^i	k_2^i
0	0	0	0	0.1
1	0.1	0.01	0.21	0.3205
2	0.2	0.04205	0.44205	0.56415
3	0.3	0.09847	0.69847	0.83339
4	0.4	0.18180	0.98180	1.13089
5	0.5	0.29489		

Heun's method: The iteration is given by $k_1^i = 2x_i + y_i$, $k_2^i = 2(x_i + \frac{2\cdot 0.1}{3}) + (y_i + \frac{2\cdot 0.1}{3}k_1^i)$ and $y_{i+1} = y_i + \frac{0.1}{4}(k_1^i + 3k_2^i)$.

i	x_i	y_i	k_1^i	k_2^i
0	0	0	0	0.13333
1	0.1	0.01	0.21	0.35733
2	0.2	0.04205	0.44205	0.60485
3	0.3	0.09847	0.69847	0.87836
4	0.4	0.18180	0.98180	1.18059
5	0.5	0.29489		

(Why does Heun's method give the same result as the corrected Euler method in this case?)

Analytic solution: We can solve the differential equation analytically and find $y(x) = -2 - 2x + 2e^x$. Hence $y(0.5) = 0.29744$.

(b) For all methods we have $x_0 = 1, x_1 = 1.1, x_2 = 1.2, \ldots$ and $y_0 = 0$.

Euler's method: The iteration formula is $y_{i+1} = y_i + 0.1(x_i^2 - y_i)$.

i	x_i	y_i
0	1	0
1	1.1	0.1
2	1.2	0.211
3	1.3	0.3339
4	1.4	0.46951
5	1.5	0.61856
6	1.6	0.78170

Higher derivative Euler method: We have $y'' = 2x - y' = 2x - (x^2 - y) = 2x - x^2 + y$, hence the iteration formula is

$$y_{i+1} = y_i + 0.1(x_i^2 - y_i) + \frac{0.1^2}{2}(2x_i - x_i^2 + y_i).$$

i	x_i	y_i
0	1	0
1	1.1	0.105
2	1.2	0.22098
3	1.3	0.34878
4	1.4	0.48920
5	1.5	0.64292
6	1.6	0.81060

Corrected Euler method: The iteration is given by $k_1^i = x_i^2 - y_i$, $k_2^i = (x_i + \frac{0.1}{2})^2 - (y_i + \frac{0.1}{2}k_1^i)$ and $y_{i+1} = y_i + 0.1k_2^i$.

i	x_i	y_i	k_1^i	k_2^i
0	1	0	1	1.0525
1	1.1	0.10525	1.10475	1.16201
2	1.2	0.22145	1.21855	1.28012
3	1.3	0.34946	1.34054	1.40601
4	1.4	0.49006	1.46994	1.53894
5	1.5	0.64396	1.60604	1.67824
6	1.6	0.81178		

Heun's method: The iteration is given by $k_1^i = x_i^2 - y_i$, $k_2^i = (x_i + \frac{2\cdot 0.1}{3})^2 - (y_i + \frac{2\cdot 0.1}{3}k_1^i)$ and $y_{i+1} = y_i + \frac{0.1}{4}(k_1^i + 3k_2^i)$.

i	x_i	y_i	k_1^i	k_2^i
0	1	0	1	1.07111
1	1.1	0.10533	1.10467	1.18213
2	1.2	0.22161	1.21839	1.30161
3	1.3	0.34969	1.34031	1.42873
4	1.4	0.49035	1.46965	1.56278
5	1.5	0.64430	1.60570	1.70310
6	1.6	0.81218		

Analytic solution: We can solve the differential equation analytically and find $y(x) = 2 - 2x + x^2 - e^{-x+1}$. Hence $y(1.6) = 0.81119$.

2. Assume that $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$. The initial condition $y(0) = 0$ gives $a_0 = 0$. From $y' = 2x + y$ we obtain

$$a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots = a_0+(2+a_1)x+a_2x^2+a_3x^3+a_4x^4+\ldots.$$

Equating coefficients gives

$$\begin{aligned} a_1 &= a_0 = 0 \\ 2a_2 &= 2 + a_1 = 2 &\Rightarrow\quad & a_2 = 1 \\ 3a_3 &= a_2 = 1 &\Rightarrow\quad & a_3 = \tfrac{1}{3} \\ 4a_4 &= a_3 = \tfrac{1}{3} &\Rightarrow\quad & a_4 = \tfrac{1}{3\cdot4} \\ 5a_5 &= a_4 = \tfrac{1}{3\cdot4} &\Rightarrow\quad & a_5 = \tfrac{1}{3\cdot4\cdot5}. \end{aligned}$$

Hence the series solution is

$$y(x) = 0 + 0x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\cdot4} + \frac{x^5}{3\cdot4\cdot5} + \cdots.$$

Using the first 6 terms (i.e. up to the term x^5) we get $y(0.5) \simeq 0.29739583$

3. We determine the Taylor series about the point $a = 1$, i.e.

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \ldots.$$

We have $y(1) = 0$ from the initial condition. Furthermore

$$\begin{aligned} y' &= x^2 - y &\Rightarrow\quad & y'(1) = 1 \\ y'' &= 2x - y' &\Rightarrow\quad & y''(1) = 1 \\ y''' &= 2 - y'' &\Rightarrow\quad & y'''(1) = 1 \\ y^{(4)} &= -y''' &\Rightarrow\quad & y^{(4)}(1) = -1 \\ y^{(5)} &= -y^{(4)} &\Rightarrow\quad & y^{(5)}(1) = 1. \end{aligned}$$

Hence the Taylor series is

$$y(x) = 0 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} - \frac{(x-1)^4}{4!} + \frac{(x-1)^5}{5!} - + \ldots.$$

Using the first six terms of the Taylor series gives the estimate $y(1.6) = 0.811248$.