Calculus 2 - Assignment 1 solutions

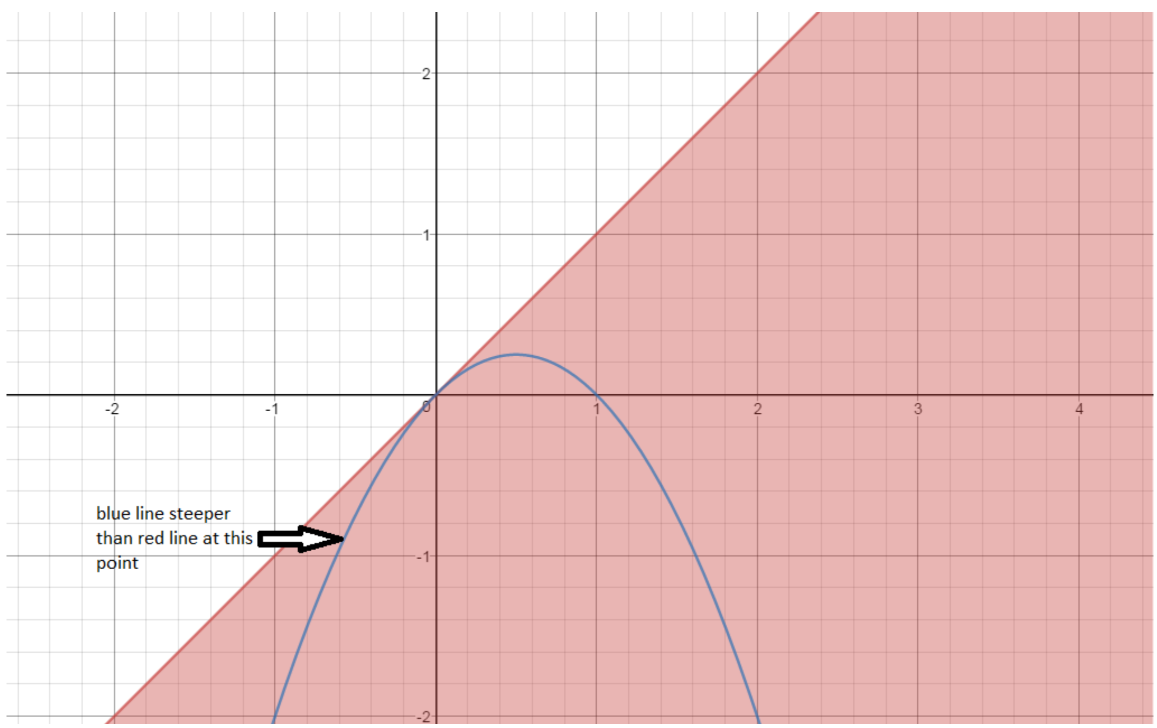
Question 1

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Suppose further that $f(0) = 0$. Decide if the statements below are true or false. If you think a statement is true then give a brief argument to support your conclusion, if false then give a counterexample. (Hint: pictures!)

(a) If $f(x) \leq x$ for all x, then $\frac{df}{dx} \leq 1$ for all x.

Solution

This statement is false. It is saying this: if my function never rises above the graph of $y = x$ and meets it at the origin then it can never be any *steeper* than the graph of $y = x$. Draw the graph of $y = x$. Almost any other line you draw that stays below or level with it and passes through the origin will be a counterexample. Having become convinced of this the task now is to find the equation such a function. Play around a bit and you'll come up with something like $f(x) = x - x^2$.



Feedback

This was the worst answered question on the assignment! Despite the hint to draw pictures, few people actually had pictures in the work they handed in. It's almost always a good idea to draw a picture when trying to get your head around a question, especially one involving graphs of functions and gradients.

A few of you tried to differentiate both sides of the expression $f(x) \leq x$ to get $\frac{df}{dx} \leq 1$.

This is *not* valid, differentiation does *not* respect inequalities. For example, $\sin x \leq 1$ for all x but differentiating both sides would give $\cos x \leq 0$ which clearly is not true for all x.

Other people gave an example where the statement is true and concluded that it is true in general - be very careful about making this kind of mistake. If I want to prove a statement is *false* then it is enough for me to give a single counterexample. But if I want to prove it is *true* then I have to give an argument that works *in all cases*, not just give some examples where it works.

Finally here's an intuitive way to see why we should suspect the statement is not true. Suppose we thought that if $f(x) \leq g(x)\ \forall x$ then this must mean $f'(x) \leq g'(x)\ \forall x$. (This was the implicit assumption some of you were making by differentiating both sides).

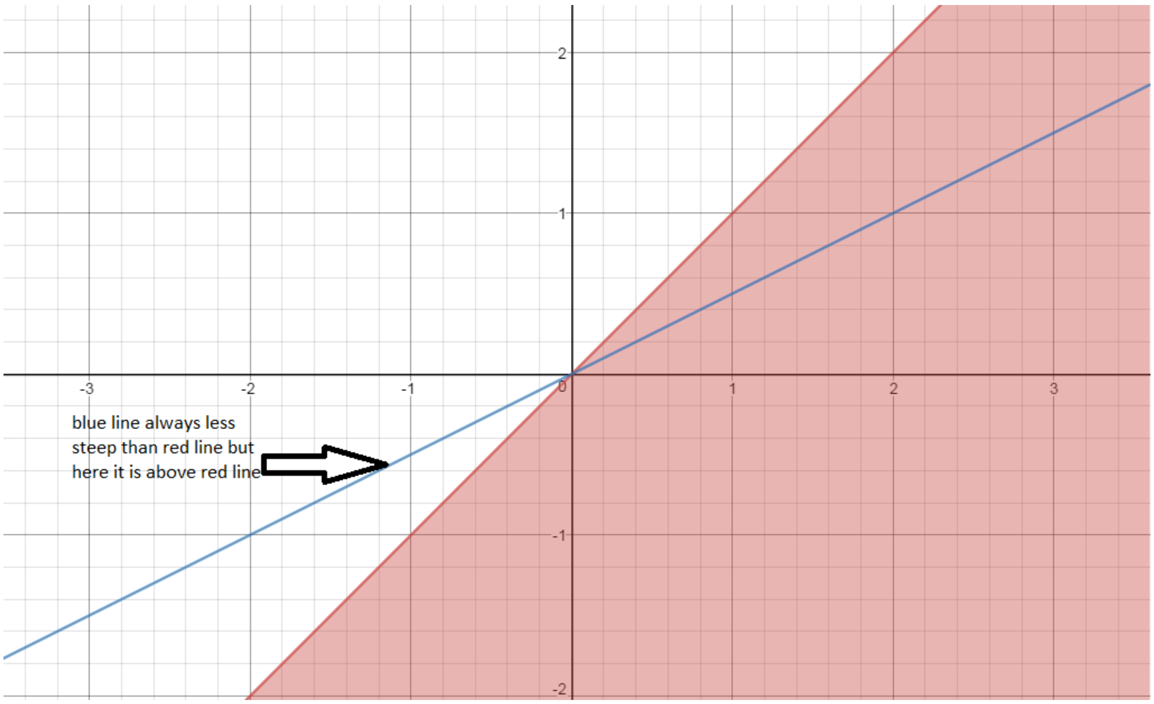
Imagine that $f(x)$ is the distance along the road of a car and $g(x)$ is this distance along the road of another car and when $x = 0$ they are neck and neck (I guess x is time). Then the statement $f(x) \leq g(x)$ is saying that $f()$ never gets ahead of $g()$. Does that mean that $g()$ is *at least as fast* as $f()$?

No: $f()$ could have been faster in the past then caught up with $g()$ (when $x = 0$), then could slow down again and fall further behind, then speed up to double the speed of $g()$ and catch up again, then stop for a bit, then speed up . . .

(b) If $\frac{df}{dx} \leq 1$ for all x, then $f(x) \leq x$ for all x.

Solution

This statement is also false. It is saying: if my function is never steeper than the graph of $y = x$ and meets that line at the origin then it can never be above that line. Again, messing around with drawing graphs you'll quickly find something like $f(x) = \frac{x}{2}$. This function has gradient of $\frac{1}{2}$, so always $\frac{df}{dx} \leq 1$, and goes through the origin but for negative x it is above the line $y = x$. One of you had $f(x) = \sin x$ as a counter example which is very slick.



Feedback

More people came up with a valid counterexample for this part than part (a). Again, some of you took $\frac{df}{dx} \leq 1$ and integrated both sides to get $f(x) \leq x + c$ then used the conditions to get $c = 0$.

Again, this is not valid though at first sight it does seem reasonable. It *is* true if we have a definite integral:

$$\int_a^b \frac{df}{dx} dx \leq \int_a^b 1 dx.$$

(Thinking of the area under a graph which never gets higher than $y = 1$ helps here, draw a sketch).

Going back to the two cars: if $g()$ is always at least as fast as $f()$ and they are neck and neck at time $= 0$, does that mean that $g()$ was always in front? No: $g()$ could have started behind $f()$ then overtaken it at time$= 0$. What is true is that once $g()$ gets ahead then since it's never slower than $f()$ it never falls behind again.

Question 2

2. Give an example of two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, where $f(g(x)) \neq g(f(x))$ and for all x,

$$\frac{d}{dx}(f(g(x)) = \frac{d}{dx}(g(f(x)).$$

Solution

On the face of it this seems rather a mysterious question. What is it really saying? Well $f(g(x)) \neq g(f(x))$ so these are different functions. But $\frac{d}{dx}(f(g(x)) = \frac{d}{dx}(g(f(x))$ so their derivatives are the same. Well then $f(g(x))$ and $g(f(x))$ must differ by a constant. Does that insight help? Not really...

Hm, time to start messing around. Guess that $f(x) = x$ and $g(x) = 2x$. Then $f(g(x)) = f(2x) = 2x$ and $g(f(x)) = g(x) = 2x$. Oh no, they're the same.

What about $f(x) = x^2$ and $g(x) = x + 1$? Then $f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$ whereas $g(f(x)) = g(x^2) = x^2 + 1$. They're different, great. What about the derivatives? $(x^2 + 2x + 1)' = 2x + 2$ but $(x^2 + 1)' = 2x$. Oh no, they're different.

Back to the drawing board, what about $f(x) = 2x$ and $g(x) = 2x + 1$? Then $f(g(x)) = f(2x+1) = 2(2x+1) = 4x + 2$. Now $g(f(x)) = g(2x) = 4x + 1$.

OK, so they are different, do they have the same derivatives? Yes! They both have gradient 4.

Now you're done and you can move on to the next question. Or maybe by now you're into this so you try to come up with the simplest possible answer. What's the simplest possible derivative? How about a constant function with zero gradient.

Say $f(x) = 1$ and $g(x) = 0$. Then $f(g(x)) = 1$ and $g(f(x)) = 0$ so they are different. And since they're both horizontal they have zero gradients. Job done.

Feedback

This was the best answered question on the assignment. Maybe it was too easy. One of you gave a solution $f(x) = ax + b$ and $g(x) = px + q$ and did the substitution that way keeping $f()$ and $g()$ as general linear functions. This allows us to come up with as many solutions as we please by playing with the controls $a,\ b,\ p,$ and q.

Can anyone come up with an answer where f or g (or both) are non-linear?

Question 3

3. Suppose we are told that $f : U \to \mathbb{R}$ is a continuous function and that $\lim_{x\to 0} \frac{f(x)}{x} = 7$. Deduce that $f(0) = 0$ and that $f'(0) = 7$. Give an example of a function f that satisfies these conditions other than $f(x) = 7x$.

Solution

We have $\lim_{x\to 0} x = 0$ and since f is continuous $\lim_{x\to 0} f(x) = f(0)$. So if we look at the quotient of the limits we would get,

$$\frac{f(0)}{0}.$$

This doesn't look good. But we are told that the limit of this quotient is 7. If $f(0)$ is anything other than zero then the limit of the quotient is either $\infty, -\infty$ or does not exist. Therefore we must have that $f(0) = 0$.

This means we are in the "$\frac{0}{0}$" scenario when we would want to apply l'Hopital's rule:

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim \frac{(f(x))'}{(x)'} \lim_{x\to 0} \frac{f'(x)}{1} = 7.$$

Well, then, $\lim_{x\to 0} f'(x) = f'(0) = 7$ is the only option.

An example of $f(x)$? A classic example where l'Hopital is used is

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Multiplying both sides of this by 7 gives,

$$7 \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{7 \sin x}{x} = 7.$$

Feedback

This question was not answered too badly, but again a lot of people just showed it to be true for an example and not in general.

I think a handful of you did not read the question carefully: the phrase "Deduce that $f(0) = 0$ and that $f'(0) = 7$" is an instruction for *you* to desmonstrate these facts but some of you didn't try to do this.

It's worth pointing out that if I am telling you something but don't expect you to prove it in an assignment or an exam question then I would not use the word "deduce". And in an exam I would have to be even more specific saying something like "...which means that $f(0) = 0$ and that $f'(0) = 7$ (you are not expected to show this)".

Finally, although I wasn't expecting a very formal solution and would have been happy if you had come up with my answer above a couple of you gave a very nice, short, clear proof so here it is:

We are told that

$$\lim_{x\to 0} \frac{f(x)}{x} = 7.$$

Multiplying both sides by $\lim_{x\to 0} x$ gives,

$$\begin{aligned}
\lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{f(x)}{x} &= 7 \lim_{x\to 0} x \\
\lim_{x\to 0} x \cdot \frac{f(x)}{x} &= 7 \lim_{x\to 0} x, \text{ (product of limits is limit of product)} \\
\lim_{x\to 0} f(x) &= 0 \\
f(0) &= 0. \text{ (since } f(x) \text{ is continuous)}
\end{aligned}$$

And the rest follows via l'Hopital as above.

Question 4

4. Define the function $f : U \to \mathbb{R}$ by,

$$f(x) = \frac{x + a}{bx + 1},$$

where $a > 0,\ b > 0$ are real numbers. $U \subset \mathbb{R}$ is the domain of definition of f.

(a) Find values of a and b so that,$\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to 0} f(x) = 9$.

(b) For your values of a and b, what is the domain of definition of f?

Solution

For part (a) we have:

$$\lim_{x\to\infty}\frac{x+a}{bx+1}=\lim_{x\to\infty}\frac{(x+a)}{(bx+1)}\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{x}{x}+\frac{a}{x}}{\frac{bx}{x}+\frac{1}{x}}=\frac{1+0}{b+0}=\frac{1}{b}=3\Rightarrow b=\frac{1}{3}.$$

And:

$$\lim_{x\to 0}\frac{x+a}{bx+1}=\lim_{x\to\infty}\frac{0+a}{0+1}=9\Rightarrow a=9.$$

For part (b), we see that $f(x)=\frac{x+9}{\frac{1}{3}x+1}$ is defined except when the denominator is zero:

$$\frac{1}{3}x+1=0\Rightarrow x=-3.$$

Therefore the domain of definition is $\mathbb{R}\backslash\{-3\}$.

Feedback

This question was generally well done. However quite a few people had something like this:

$$\lim_{x\to\infty}\frac{x+9}{\frac{1}{3}x+1}=\frac{\infty}{\frac{1}{3}\infty}=3.$$

Now the intuition behind writing this is correct - in comparison to infinity the 9 and the 1 don't count. However this is *not* the way to write it and in fact that collection of symbols is literally meaningless mathematically.

In part (b) some of you had $\{x\in\mathbb{R}: 3\le x\le 9\}$ for the domain of f. I'm not sure what you were thinking here...

Question 5

5. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=x^2-2x+y^2-4y+5$.

(a) Draw a contour-plot for the values $f(x,y)=a$ where $a\in\{0,1,2,3,4\}$. Explain why it does not make sense to plot contours for $a<0$.

Solution

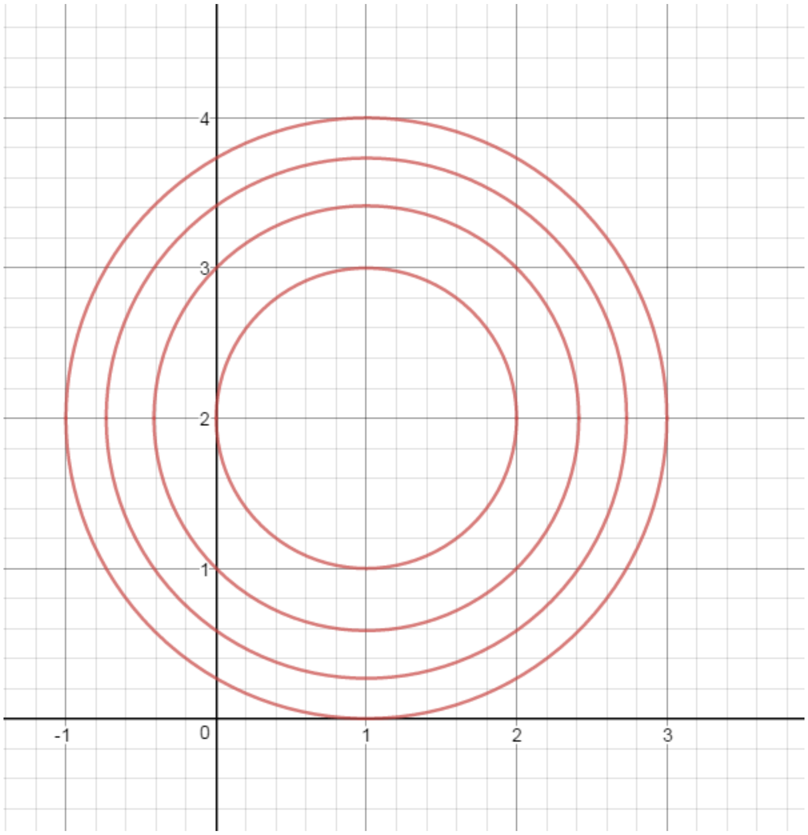
One of the techniques in our maths toolbox should be completing the square:

$$f(x,y)=x^2-2x+y^2-4y+5=(x-1)^2+(y-2)^2.$$

Then the contours $f(x,y)=$ constant are these curves:

$$(x-1)^2+(y-2)^2=a,$$

which are circles centre $(1,2)$ radius $\sqrt{a}$.



Feedback

Well done on the whole. Lot's of evidence of people playing with desmos which is great but of course you won't have access to it in the exam. If you need to draw a diagram but, like me, you can't draw then the way to avoid losing marks is to describe the curve in words: "these are circles centre (1,2) radius $\sqrt{a}$" because then whatever your diagram looks like I know you know the maths.

Several of you lost a mark because you had the radii as a not $\sqrt{a}$.

(b) Find the equation of the tangent plane to the point $P = (0, 2, f(0, 2))$.

Solution

From the notes we have the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is given by:

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) - z = 0.$$

Then we have $(a, b) = (0, 2)$, $f(0, 2) = 1$, $f_x(x, y) = 2(x - 1) \Rightarrow f_x(0, 2) = -2$, $f_y(x, y) = 2(y - 2) \Rightarrow f_y(0, 2) = 0$, Plugging in all the ingredients into the expression above gives us:

$$-2(x - 0) + 0(y - 2) + 1 - z = 0, \text{ or,}$$
$$2x + z - 1 = 0$$

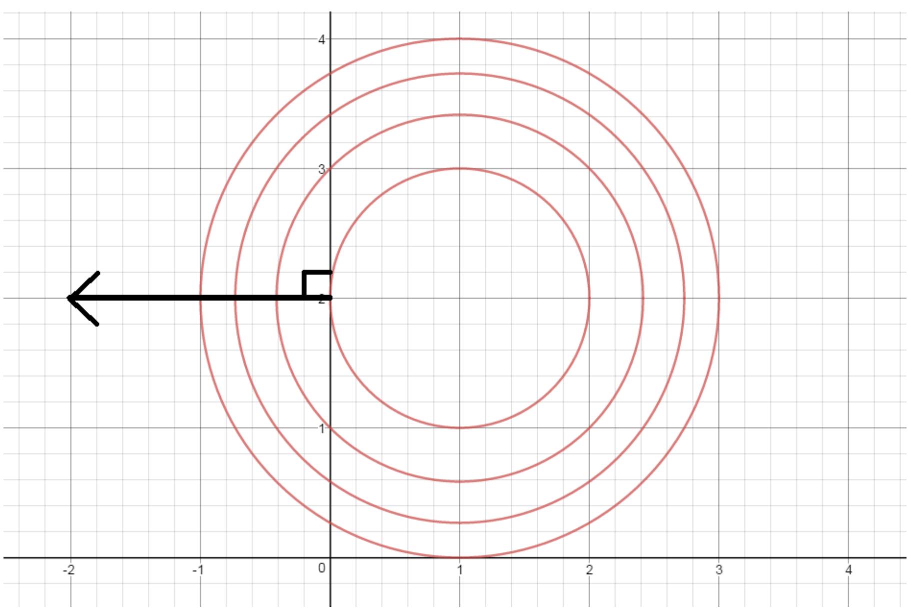
Feedback

Very well done. A few people went through the whole derivation of the tangent plane formula using the dot product but it would have been fine just to quote the formula on page 23 and fill in the ingredients to get the answer.

(c) Find the gradient ∇f and verify that it is perpendicular to the contour line at the point $(0, 2)$.

Solution

$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2x-2 \\ 2y-4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$. This is a horizontal vector and so is clearly in the direction of a radius from the centre of the circle $f(x, y) = 1$ to the point $(0, 2)$ on the circle and hence must meet the circle at right angles.



Feedback

This question was not well done at all. I'm not really sure what the problem was here other than it was all covered in that grueling lecture where we went through gradients and tangent planes. I'll let you all look at my solution and if it's not clear then maybe we will need to come back to this in the revision lectures.

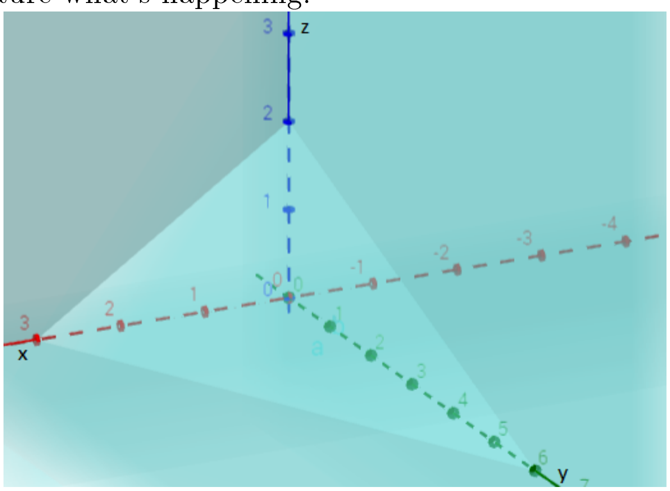
Question 5

5. Let Π be the plane $2x+y+3z=6$. By considering a suitable double integral over a corresponding region of the (x, y)-plane, find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$ and Π. Your answer should include a check by doing the double integral again in the other order.

Solution

This is an absolutely typical kind of exam question for a course like calculus 2. I'm sorry to say that the way I phrased the question here was incomplete. I should have said "...find the volume of the tetrahedron bounded by the planes, $x = 0,\ y = 0,\ z = 0$ and Π". A tetrahedron has four faces so I should have given you four planes to bound it. However everyone seems to have understood what I meant.

This question is easy once you've figured out what you have to do. Use diagrams to help you picture what's happening.



If you think of a carboard box where one of the corners is the origin, the plane Π comes in at some angle and cuts away that corner. The question is asking us to find the volume of that piece which is a tetrahedron (i.e. a four sided solid object).

The plane Π meets the (x, y)-plane when $z = 0$ so we have $2x + y = 6$ and rearranging this we get the line of intersection of these two planes is given by $y = -2x + 6$. If we think of the base of the tetrahedron as lying in the (x, y)-plane then it is the region bounded by the lines $x = 0,\ y = 0\ \&\ y = -2x + 6$. This will be our region for the double integral.

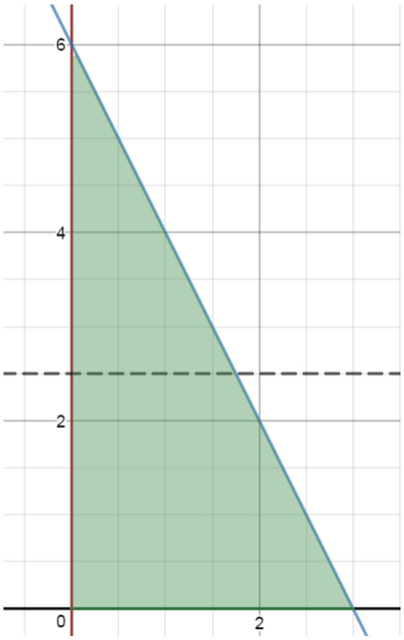
What function are we going to integrate? Well it must be the height of the tetraheron which is just the height of Π over the corresponding point in the (x, y)-plane. The eqation of Π is

$$2x + y + 3z = 6 \Rightarrow z = -\frac{2}{3}x - \frac{1}{3}y + 2.$$

Therefore the volume is given by the double integral

$$\iint_R -\frac{2}{3}x - \frac{1}{3}y + 2\ dxdy.$$

Finally we use our standard procedure to work out the limits:



$$
\begin{aligned}
\int_{y=0}^{y=6}\int_{x=0}^{3-\frac{y}{2}} -\frac{2}{3}x - \frac{1}{3}y + 2 \; dxdy &= -\frac{1}{3}\int_{y=0}^{y=6}\int_{x=0}^{3-\frac{y}{2}} 2x + y - 6 \; dxdy \\
&= -\frac{1}{3}\int_{y=0}^{y=6}\left[\frac{2}{2}x^2 + yx - 6x\right]_0^{3-\frac{y}{2}} \; dy \\
&= -\frac{1}{3}\int_{y=0}^{y=6} (3-\frac{y}{2})^2 + y(3-\frac{y}{2}) - 6(3-\frac{y}{2}) \; dy \\
&= -\frac{1}{3}\int_{y=0}^{y=6} 9 - 3y + \frac{y^2}{4} + 3y - \frac{y^2}{2} - 18 + 3y \; dy \\
&= -\frac{1}{3}\int_{y=0}^{y=6} -9 + 3y - \frac{y^2}{4} \; dy \\
&= -\frac{1}{3}\left[-9y + \frac{3}{2}y^2 - \frac{1}{12}y^3\right]_0^6 \\
&= -\frac{1}{3}\left[-54 + 54 - 18\right] \\
&= 6.
\end{aligned}
$$

And doing the double integral in the other order:

$$
\begin{aligned}
\int_{x=0}^{x=3}\int_{y=0}^{y=-2x+6} -\frac{2}{3}x - \frac{1}{3}y + 2\ dydx &= -\frac{1}{3}\int_{x=0}^{x=3}\int_{y=0}^{y=-2x+6} 2x + y - 6\ dydx \\
&= -\frac{1}{3}\int_{x=0}^{x=3}\left[2xy + \frac{1}{2}y^2 - 6y\right]_0^{-2x+6}\ dx \\
&= -\frac{1}{3}\int_{x=0}^{x=3} 2x(-2x+6) + \frac{1}{2}(-2x+6)^2 - 6(-2x+6)\ dx \\
&= -\frac{1}{3}\int_{x=0}^{x=3} -4x^2 + 12x + 2x^2 - 12x + 18 + 12x - 36\ dx \\
&= -\frac{1}{3}\int_{x=0}^{x=3} -2x^2 + 12x - 18\ dx \\
&= -\frac{1}{3}\left[-\frac{2}{3}x^3 + 6x^2 - 18x\right]_0^3 \\
&= -\frac{1}{3}\left[-18 + 54 - 54\right] \\
&= 6.
\end{aligned}
$$

Feedback

On the whole very well done, lots of you got full marks for this question. One problem I noticed from some of you was the first step in the integration.

You went from this:

$$\int_{y=0}^{y=6}\int_{x=0}^{3-\frac{y}{2}} -\frac{2}{3}x - \frac{1}{3}y + 2\ dxdy$$

To this:

$$\int_{y=0}^{y=6}\left[-\frac{2}{6}x^2 - \frac{1}{3}y + 2x\right]_0^{3-\frac{y}{2}}\ dy.$$

In other words you did not integrate the $-\frac{1}{3}y$ with respect to x to get $-\frac{1}{3}yx$. This is a common error in multivariable calculus - forgetting to consider the variable not currently being integrated or differentiated as a constant. For some reason we are tempted to ignore it.

A number of people didn't bother to do the integration in the other order and so lost a couple of marks.

For some the integration went disastrously wrong - you have my sympathies here. Double integrals with fractions and minus signs are conceptually easy but often go wrong for me. That's why I do them with Mathematica.