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Probability and Statistics

Solutions 3

1. For $X \sim B(10, 0.3)$,

(i) $\Pr(X \le 4) = F_4 = 0.8497$.

(ii) $\Pr(X > 4) = 1 - F_4 = 1 - 0.8497 = 0.1503$.

(iii) $\Pr(X = 4) = F_4 - F_3 = 0.8497 - 0.6496 = 0.2001$.

(iv) $E(X) = np = 10 \times 0.3 = 3$.

(v) $\text{var}(X) = npq = 10 \times 0.3 \times 0.7 = 2.1$.

2. (i) $B(n, 1/5)$.

(ii) $(4/5)^n$.

(iii) We require

$$1 - \left(\frac{4}{5}\right)^n \ge \frac{19}{20} \quad \text{i.e.} \quad \left(\frac{4}{5}\right)^n \le \frac{1}{20} \quad \text{i.e.} \quad \left(\frac{5}{4}\right)^n \ge 20.$$

Thus we require

$$n \ge \frac{\ln(20)}{\ln(1.25)} = \frac{2.9957}{0.2231} = 13.43.$$

The smallest value of n that satisfies the required condition is 14.

(iv) Using the $B(14, 0.2)$ distribution for the number X of left-handed people in the sample, we have

$$\Pr(X \ge 1) = 1 - F_0 = 1 - 0.0440 = 0.9560.$$

3. (i) Using the Poisson distribution with parameter 2.5,

$$p_3 = F_3 - F_2 = 0.7576 - 0.5438 = 0.2138.$$

(ii)

$$1 - p_0 = 1 - F_0 = 1 - 0.0821 = 0.9179.$$

(iii) Using the Poisson distribution with parameter 10,

$$1 - F_5 = 1 - 0.0671 = 0.9329.$$

4. (i) $X \sim B(300, 0.02)$.

(ii) $E(X) = 300 \times 0.02 = 6$, $\text{var}(X) = 300 \times 0.02 \times 0.98 = 5.88$.

(iii) The Poisson distribution with parameter 6.

(iv) Comparing the $B(300, 0.02)$ distribution with the Poisson distribution with mean 6,

r	F_r binomial	F_r Poisson
0	0.0023	0.0025
1	0.0166	0.0174
2	0.0602	0.0620
3	0.1485	0.1512
4	0.2824	0.2851
5	0.4441	0.4457
6	0.6063	0.6063
7	0.7454	0.7440
8	0.8493	0.8472
9	0.9182	0.9161
10	0.9590	0.9574

The values for the binomial distribution are not available from tables. `R` (or Excel) may be used. The required values of the cumulative distribution functions are obtained in the `R` output below.

```r
r <- 0:10
BinomialFr <-  pbinom(r, 300, 0.02)
PoissonFr <-  ppois(r, 6)
cbind(r, BinomialFr, PoissonFr)

##        r  BinomialFr   PoissonFr
##  [1,]  0 0.002332506 0.002478752
##  [2,]  1 0.016613153 0.017351265
##  [3,]  2 0.060183698 0.061968804
##  [4,]  3 0.148510382 0.151203883
##  [5,]  4 0.282352346 0.285056500
##  [6,]  5 0.444055291 0.445679641
##  [7,]  6 0.606308246 0.606302782
##  [8,]  7 0.745382207 0.743979760
##  [9,]  8 0.849332897 0.847237494
## [10,]  9 0.918161926 0.916075983
## [11,] 10 0.959037941 0.957379076
```

(v)

$$\begin{aligned} \Pr(X \geq 5) &= 1 - F_4 \\ \text{(a):} &= 1 - 0.2824 = 0.7176 \quad \text{(exact binomial)} \\ \text{(b):} &= 1 - 0.2851 = 0.7149 \quad \text{(approximating Poisson)} \end{aligned}$$

5. Let X denote the number of bacteria in a sample.

(i) X has the Poisson distribution with mean $500/1000 = 0.5$.

(ii) $\Pr(X \geq 1) = 1 - F_0 = 1 - 0.6065 = 0.3935$.

(iii) Now X has the Poisson distribution with mean 5.
$\Pr(X \leq 3) = F_3 = 0.2650$.

6. (i) Clearly $p_r > 0$ $(r = 0, 1, 2, \ldots)$. We need to check that $\sum_{r=0}^{\infty} p_r = 1$. Recall the formula for the sum of a geometric series: for $|x| < 1$,

$$\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}.$$

It follows that

$$\sum_{r=0}^{\infty} p_r = \sum_{r=0}^{\infty} q^r p = \frac{p}{1-q} = \frac{p}{p} = 1.$$

(ii)

$$G(t) = \sum_{r=0}^{\infty} p_r t^r = \sum_{r=0}^{\infty} p\, q^r t^r = p\, \sum_{r=0}^{\infty} (qt)^r = \frac{p}{1-qt},$$

provided that t satisfies $|qt| < 1$, i.e., $|t| < 1/q$.

(iii) Differentiating the p.g.f. twice,

$$G'(t) = \frac{pq}{(1-qt)^2} \quad \text{and} \quad G''(t) = \frac{2pq^2}{(1-qt)^3}.$$

Hence

$$\mu = G'(1) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

and

$$\begin{aligned}
\sigma^2 &= G''(1) + \mu - \mu^2 \\
&= \frac{2pq^2}{(1-q)^3} + \frac{q}{p} - \left(\frac{q}{p}\right)^2 \\
&= \frac{2pq^2}{p^3} + \frac{q}{p} - \frac{q^2}{p^2} \\
&= \frac{q^2}{p^2} + \frac{q}{p} = \frac{q^2 + qp}{p^2} = \frac{q(q+p)}{p^2} = \frac{q}{p^2}.
\end{aligned}$$