# Discrete Mathematics 2016 Solutions
# Section A

1. (a)

$$\sum_{i=0}^{20} \binom{22}{i} = \sum_{i=0}^{22} \binom{22}{i} - \binom{22}{21} - \binom{22}{22},$$
$$= \mathbf{2^{22} - 23}.$$

[2]

(b)

$$\sum_{i=1}^{50} (i+1)(i-1) = \sum_{i=1}^{50} i^2 - 1,$$
$$= \frac{1}{6} 50(51)(101) - 50,$$
$$= \mathbf{42875}.$$

[3]

2. (a) $3^4 = \mathbf{81}$ [1]
   (b) Subtracting the number of words that contain no $X$s, we have $3^4 - 2^4 = \mathbf{65}$. [2]
   (c) There are $3^2$ words of the form $XY--$, $3^2$ words of the form $-XY-$ and $3^2$ words of the form $--XY$. There is one word that contains two copies of $XY$, namely $XYXY$. Hence the number of words that do not contain $XY$ is $81 - 27 + 1 = \mathbf{55}$. [2]

3. (a) $\binom{20+2}{2} = \mathbf{231}$ [1]
   (b) $\binom{14+2}{2} = \mathbf{120}$ [1]
   (c) Cases where the numbers of red and blue bowls are equal include 10 red and 10 blue, 9 red and 9 blue with 2 yellow, and so on, through to 0 red, 0 blue and 20 yellow, hence there are 11 such cases. Therefore there are $231 - 11 = 220$ cases with unequal numbers of red and blue bowls; in half of these there will be more red bowls, so the answer is **110**. [3]

4. (a) The first four terms of the sequence defined by this difference equation are $2, 1, -2, 1$. As it is a first-order difference equation, we conjecture that the solution repeats from this point onwards, so for $n \geq 1$, $u_n = 1$ when $n$ is odd, and $u_n = -2$ when $n$ is even. [2]

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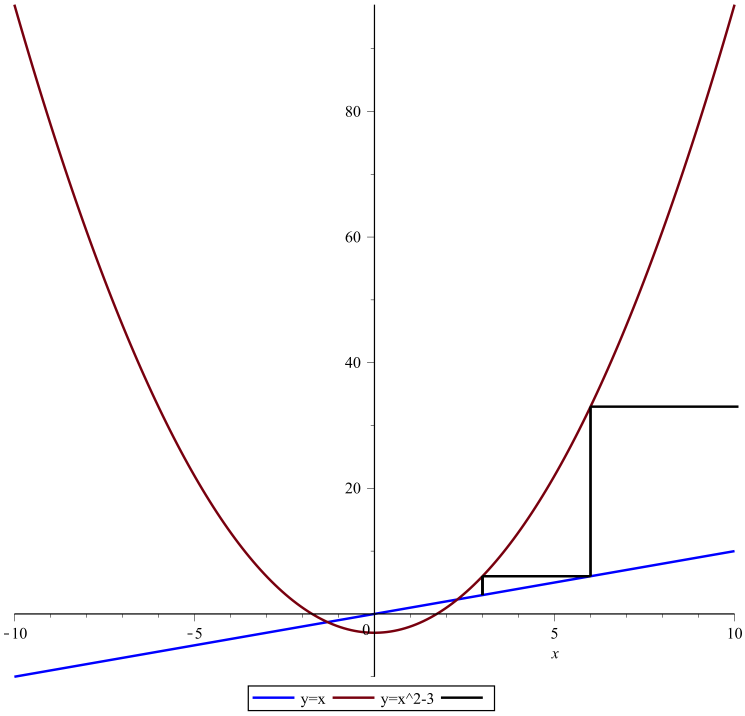


(b)

We have $u_n \to \infty$ as $n \to \infty$. [3]

5. (a) $[\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{3}, \mathbf{3}]$ [1]

(b) We have

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 2 \\ 1 & 0 & 2 & 2 & 0 \\ 1 & 0 & 2 & 3 & 1 \\ 1 & 2 & 0 & 1 & 3 \end{pmatrix},$$

hence the answer is **1** (in either direction). [2]

(c) The graphs $G$ and $H$ have the same degree sequence, but are not isomorphic to each other, as the vertices of degree 3 are adjacent in $G$ but not in $H$. [2]

6. (a) **3** (in general the edge-connectivity of $K_n$ is $n - 1$). [1]

(b) 3 (by convention.) [1]

(c) Examples include $K_{3,3}$ or the cube graph. For full marks justifcation is required, for example an observation that there are three edge-disjoint paths between every pair of vertices in the graph. [3]

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7. (a) Using the longest path algorithm from the notes, we assign labels to the vertices.

$$\begin{aligned} P(b) &= 5, \\ P(c) &= 2, \\ P(d) &= 6, \\ P(e) &= 8, \\ P(f) &= 10, \\ P(g) &= 15, \\ P(h) &= 15, \\ P(j) &= 17, \\ P(k) &= 21. \end{aligned}$$

The critical path is **aCdFfHhJk**, and it has length 21. [3]

(b) The earliest start time for activity $G$ is given by the label of $e$, the vertex at the start of the arc corresponding to $G$, and hence is **8**. There is only one path to $k$ that starts with arc $G$, and it has length 11. Hence the latest start time for $G$ is $21 - 11 = \mathbf{10}$. [2]

8. (a) A perfect matching in a graph $G$ is a set of edges of $G$ with the property that each vertex of $G$ is incident with precisely one edge in the set. [2]

(b) The first graph has an odd number of vertices, and hence cannot have a perfect matching. The second graph cannot have a perfect matching, as the fact that vertices $f$ and $c$ both have degree 1 implies that edges $ef$ and $ec$ would have to be included in such a matching; this is not possible as then $e$ would be incident with more than one edge. [3]

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# Section B

9. (a) (i)

$$\begin{aligned} A + C + P &= 30, \\ 5 \le A &\le 10, \\ 10 &\le C, \\ P &\le 7. \end{aligned}$$

[2]

(ii) $g(x) = (x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})(x^{10} + x^{11} + \cdots)(x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)$ [2]

(iii)

$$\begin{aligned} g(x) &= (x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})(x^{10} + x^{11} + \cdots)(x^0 + x^1 + x^2 + \cdots + x^7), \\ &= x^{15}(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + \cdots)(x^0 + x^1 + x^2 + \cdots + x^7), \\ &= x^{15}(1 - x^6)(1 - x^8)(1 + x + x^2 + \cdots)^3, \\ &= x^{15}(1 - x^6 - x^8 + x^{14}) \sum_{r=0}^{\infty} \binom{r+2}{r} x^r. \end{aligned}$$

The coefficient of $x^{30}$ is $\binom{17}{15} - \binom{11}{9} - \binom{9}{7} + \binom{3}{1} = \mathbf{48}$. [5]

(b) Let $g(x)$ be the generating function associated with the sequence $1, 1, 1, \ldots$. Then $g(x) = 1 + x + x^2 + \cdots$. We have

$$\begin{aligned} (1 - x)g(x) &= 1 + x + x^2 + x^3 + \cdots \\ &\quad - x - x^2 - x^3 - \cdots \\ &= 1, \end{aligned}$$

hence it is the case that $g(x) = (1 - x)^{-1}$. [3]

(c) We have

$$\begin{aligned} \frac{1}{(1 - 5x + 6x^2)} &= \frac{1}{(1 - 3x)(1 - 2x)}, \\ &= \frac{3}{1 - 3x} - \frac{2}{1 - 2x}. \end{aligned}$$

Thus the first 5 terms are

$$\begin{aligned} 3^1 - 2^1 &= \mathbf{1}, \\ 3^2 - 2^2 &= \mathbf{5}, \\ 3^3 - 2^3 &= \mathbf{19}, \\ 3^4 - 2^4 &= \mathbf{65}, \\ 3^5 - 2^5 &= \mathbf{211}. \end{aligned}$$

[2]

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(d) Let $g(x) = (1-x)^{-1}$. Then the generating function of the sequence $0, 1, 2, 3, \ldots$ is given by $xg'(x) = \frac{x}{(1-x)^2}$. The $n^{\text{th}}$ triangle number is the sum of the first $n$ terms of this sequence, hence the generating function for the sequence of triangle numbers is $\mathbf{\frac{x}{(1-x)^3}}$. [2]

(e) (i) We have established that the generating function of the sequence $0, 1, 2, 3, \ldots$ is $\frac{x}{(1-x)^2}$. It follows that the generating function of this sequence is $\frac{x^2}{(1-x^2)^2}$. [2]

(ii) This is the Fibonacci sequence, which satisfies the linear difference equation $u_n - u_{n-1} - u_{n-2} = 0$, with $u_0 = 0$ and $u_1 = 1$. Let $g(x)$ be the generating function for this sequence. Then we have

$$\begin{array}{rclllll} g(x) & = & x & +x^2 & +2x^3 & +\cdots \\ -xg(x) & = & & -x & -x^2 & -\cdots \\ -x^2g(x) & = & & & -x^2 & -\cdots, \end{array}$$

so $g(x) = \frac{x}{1-x-x^2}$. [2]

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10. (a) Solve the following difference equations:

(i) *Find the characteristic polynomial:* $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$.
*General solution to homogeneous part:* $G(n) = A5^n + B(-2)^n$.
*Find a particular solution:* Try $P(n) = Cn + D$. Then

$$\begin{aligned} Cn + d &= 3C(n-1) + 3D + 10C)n - 2) + 10D - 36n + 9, \\ C &= 3C + 10C - 36, \\ C &= 3. \\ D &= -3C + 3D - 20C + 10D + 9, \\ 12D &= 60, \\ D &= 5. \end{aligned}$$

*General solution:* $u_n = G(n) + P(n) = A5^n + B(-2)^n + 3n + 5$.
*Apply initial conditions:*

$$\begin{aligned} u_0 = A + B + 5 &= 11, \\ A + B &= 6. \\ u_1 = 5A - 2B + 8 &= 10, \\ 5A - 2B &= 2, \\ A &= 2, \\ B &= 4. \end{aligned}$$

*Solution:* $\mathbf{u_n = 2 \cdot 5^n + 4(-2)^n + 3n + 5}$. [4]

(ii) *Find the characteristic polynomial:* $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$.
*General solution to homogeneous part:* $G(n) = An + B$.
*Find a particular solution:* Try $P(n) = Cn^2$. Then

$$\begin{aligned} Cn^{"} &= 2C(n-1)^2 - C(n-2)^2 + 8, \\ \text{comparing constants: } 0 &= 2C - 4C + 8, \\ C &= 4. \end{aligned}$$

*General solution:* $u_n = 4n^2 + An + B$.
*Apply initial conditions:*

$$\begin{aligned} u_0 &= B = 6, \\ u_1 &= 4 + A + 6 = 13, \\ A &= 3. \end{aligned}$$

*Solution:* $\mathbf{u_n = 4n^2 + 3n + 6}$. [4]

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(iii) We have

$$\begin{aligned} u_n &= \prod_{i=1}^{n} \frac{i+2}{i} + \sum_{i=1}^{n}(i+1) \prod_{j=i+1}^{n} \frac{j+1}{j}, \\ &= n+1+\sum_{i=1}^{n}(i+1)\frac{n+1}{i+1}, \\ &= n+1+n(n+1), \\ &= (n+1)^2. \end{aligned}$$

[4]

(b) (i) This is a second order linear homogeneous difference equation with constant coefficients, hence its characteristic polynomial is the quadratic polynomial in which the coefficient of $\lambda^i$ matches the coefficient of term $u_{n-2+i}$ in the equation, namely $\lambda^2 - A\lambda + B$. [1]

(ii) We have that $\omega^n$ is a solution of (1) if and only if $\omega^n = A\omega^{n-1} + B\omega^{n-2}$. This occurs if and only if $\omega^{n-2}(\omega^2 - A\omega - B) = 0$, that is, if and only if $\omega = 0$ or $\omega$ is a zero of the characteristic polynomial $\lambda^2 - a\lambda + B$. [2]

(c) (i) $C_n = \frac{1}{n+1}\binom{2n}{n}$. [2]

(ii) We have

$$\begin{aligned} C_n &= \frac{1}{n+1}\binom{2n}{n}, \\ &= \frac{(2n)!}{(n+1)n!n!}, \\ &= \frac{\prod_{i=1}^{2n} i}{(n+1)\left(\prod_{i=1}^{n} i\right)\left(\prod_{i=1}^{n} i\right)}, \\ &= \frac{\prod_{i=n+1}^{2n} i}{(n+1)\left(\prod_{i=1}^{n} i\right)}, \\ &= \frac{1}{n+1}\prod_{k=1}^{n}\frac{n+k}{k}, \\ &= \prod_{k=2}^{n}\frac{n+k}{k}, \end{aligned}$$

as required. [3]

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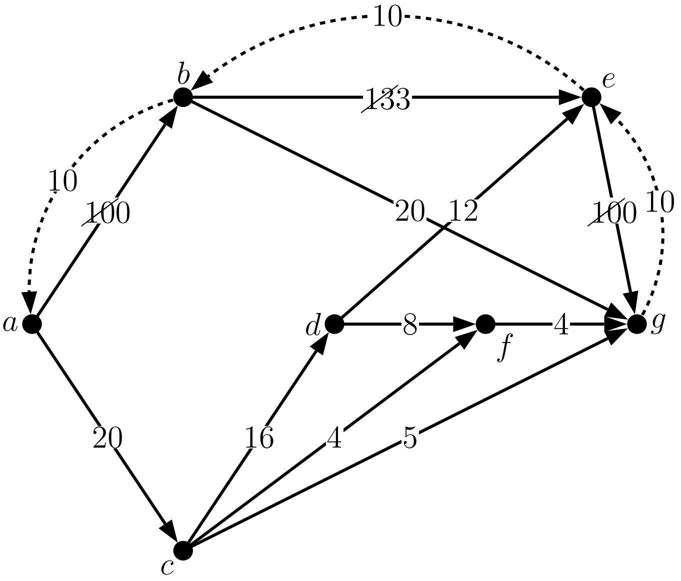
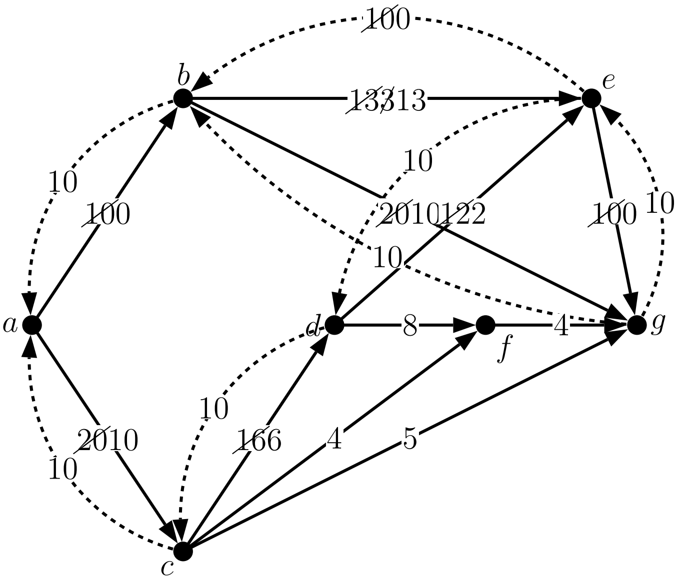
11. (a) (i) The size of a minimum cut in a network is equal to the value of a maximum flow. [2]

(ii) The size of any cut in a network is given by a sum of weights of arcs in the network, so if the weights are integers, so too is this sum. In particular, the size of a minimum cut must be an integer, and therefore so too must the value of a maximum flow, by the maximum flow/minimum cut theorem. [2]

(iii) Let $\phi$ be the zero flow. Add 10 to $\phi$ along all arcs in the path $abeg$.




Add 10 to $\phi$ along all arcs in the path $acdebg$.




Add 4 to $\phi$ along all arcs in the path $acdfg$.

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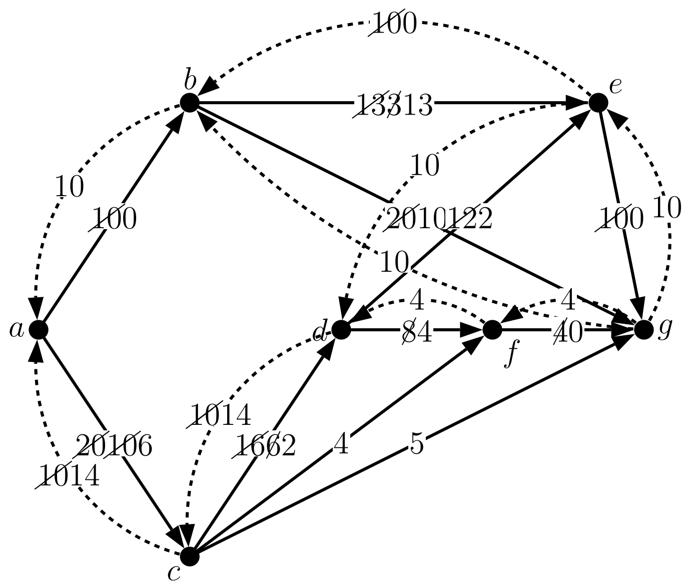
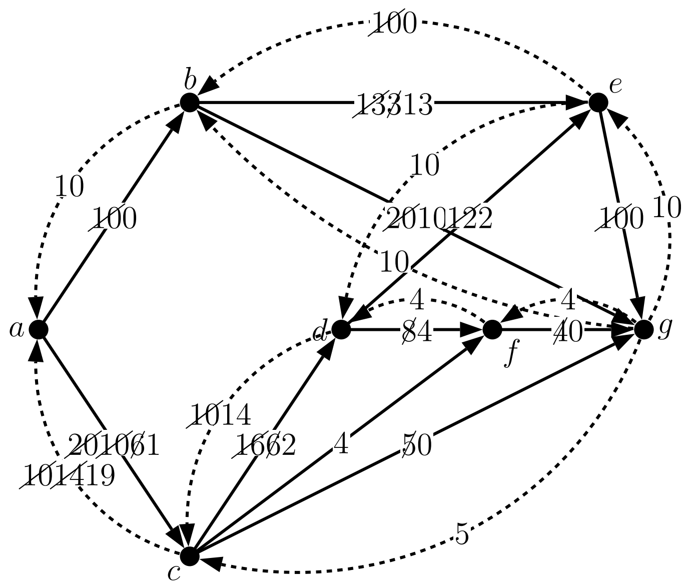


Add 5 to $\phi$ on all arcs in the path $acg$



Reversing the direction of the dashed arcs yields $\phi$, which is now a maximum flow with value 29. [6]

(b) (i) A convex polyhedron with $v$ vertices, $e$ edges and $f$ faces satisfies $v+f=e+2$. [2]

(ii) By the Handshaking Lemma it has $3(20)/2 = 30$ edges. Euler's polyhedral formula then implies $20+f = 30+2$, hence it divides the plane into 12 distinct regions. [3]

(iii) The complete bipartite graph $K_{m,n}$ is planar whenever $m$ or $n$ (or both) are less than 3. [2]

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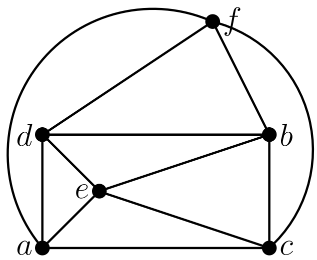
(iv) The graph is planar and hence does not contain a subgraph isomorphic to a subdivision of $K_{3,3}$:




[3]

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12. (a) (i) Let $G$ be a connected simple graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for any non-adjacent pair of vertices $u$ and $v$, then $G$ is Hamiltonian. [2]

(ii) e.g. $C_{500}$ [2]

(iii) e.g. $K_{3,3}$ [2]

(iv) There are $n!$ ways to order the vertices. For each such choice there are $n$ vertices from which the cycle could start, and two orders in which you could write the vertices of the cycle. Hence there are $(n-1)!/2$ distinct Hamiltonian cycles in $K_n$. [2]

(b) Let $M$ be a square matrix. The size of the largest independent set of entries that are equal to zero is equal to the smallest number of rows and/or columns that between them contain all zero entries of the matrix. [2]

(c) A manager wishes to have five tasks completed, Task $A$, Task $B$, Task $C$, Task $D$ and Task $E$. He approaches five contractors, Contractor 1, Contractor 2, Contractor 3, Contractor 4 and Contractor 5. The amounts they quote in pounds to perform each task are given in the following table:

| | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|
| 1 | 4 | 7 | 5 | 6 | 9 |
| 2 | 6 | 8 | 10 | 8 | 9 |
| 3 | 7 | 7 | 9 | 7 | 7 |
| 4 | 6 | 9 | 5 | 10 | 6 |
| 5 | 8 | 9 | 6 | 9 | 8 |

.

(i) We use the Hungarian algorithm. After preprocessing the rows we have the matrix

$$\begin{pmatrix} 0 & 3 & 1 & 2 & 5 \\ 0 & 2 & 4 & 2 & 3 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 4 & 0 & 5 & 1 \\ 2 & 3 & 0 & 3 & 2 \end{pmatrix}.$$

Covering up columns 1 and 3 and row 3 covers all the zeros. The smallest entry in the rest of the matrix is 1, so we update it accordingly:

$$\begin{pmatrix} 0 & 2 & 1 & 1 & 4 \\ 0 & 1 & 4 & 1 & 2 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 3 & 0 & 4 & 0 \\ 2 & 2 & 0 & 2 & 1 \end{pmatrix}.$$

This does not have a set of five independent zero entries, so we cover column 1

**Please turn over**

and rows 3,4,5 and update the matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 1 \\ 2 & 0 & 3 & 0 & 0 \\ 2 & 3 & 0 & 4 & 0 \\ 3 & 2 & 0 & 2 & 1 \end{pmatrix}.$$

This matrix does have a set of five independent zero entries. An appropriate assignment is 1-A, 2-B, 3-D, 4-E, 5-C, for a total cost of **31**. [5]

(ii) In order to maximise the total, we could begin by replacing each entry $x$ with $10 - x$ then performing the Hungarian Algorithm as before. [1]

(iii) We would add an extra row to the matrix for Contractor 6, then to make it a square matrix we would add a dummy task in column 6, with (for example) costs of 0 for each entry. We would then apply the Hungarian algorithm to this new matrix, and whichever Contractor was assigned task 6 would not be hired. [2]

(iv) The largest entry in column 1 is 8, so the assignment 1-E, 2-D, 3-C, 4-B, 5-A is best possible here. [2]

**End of examination paper**