Calculus 2 - Assignment 1

Due 2^{nd} December

ANSWER ALL QUESTIONS

Questions 1, 2 & 3 are meant to get you thinking about the concepts of limits, continuity and derivatives. I am not looking for very formal proofs in your answers to these questions. Don't worry about $\epsilon - \delta$ type arguments (unless you want to) but you do need to make your argument clear and convince me you know what you're talking about.

The remaining questions are more along of the lines of practising the various kinds of calculations we have covered in the lectures.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Suppose further that $f(0) = 0$. Decide if the statements below are true or false. If you think a statement is true then give a brief argument to support your conclusion, if false then give a counterexample. (Hint: pictures!)

   (a) If $f(x) \leq x$ for all x, then $\frac{df}{dx} \leq 1$ for all x. [1]

   (b) If $\frac{df}{dx} \leq 1$ for all x, then $f(x) \leq x$ for all x. [1]

2. Give an example of two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, where $f(g(x)) \neq g(f(x))$ and for all x,
$$\frac{d}{dx}(f(g(x)) = \frac{d}{dx}(g(f(x)).$$
   Justify your answers. [2]

3. Suppose we are told that $f : U \to \mathbb{R}$ is a continuous function and that $\lim_{x\to 0} \frac{f(x)}{x} = 7$. Deduce that $f(0) = 0$ and that $f'(0) = 7$. Give an example of a function f that satisfies these conditions other than $f(x) = 7x$. [2]

4. Define the function $f : U \to \mathbb{R}$ by,
$$f(x) = \frac{x + a}{bx + 1},$$
   where $a > 0,\ b > 0$ are real numbers. $U \subset \mathbb{R}$ is the domain of definition of f.

   (a) Find values of a and b so that, [2]
$$\lim_{x\to\infty} f(x) = 3 \text{ and } \lim_{x\to 0} f(x) = 9.$$

   (b) For your values of a and b, what is the domain of definition of f? [1]

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by,

$$f(x, y) = x^2 - 2x + y^2 - 4y + 5.$$

(a) Draw a contour-plot for the values $f(x, y) = a$ where $a \in \{0, 1, 2, 3, 4\}$. Explain why it does not make sense to plot contours for $a < 0$. [2]

(b) Find the equation of the tangent plane to the point $P = (0, 2, f(0, 2))$. [2]

(c) Find the gradient ∇f and verify that it is perpendicular to the contour line at the point $(0, 2)$. [2]

6. Let Π be the plane $2x + y + 3z = 6$. By considering a suitable double integral over a corresponding region of the (x, y)-plane, find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$ and Π. Your answer should include a check by doing the double integral again in the other order. [5]