BIRKBECK

(University of London)

BSc EXAMINATION
SCHOOL OF BUSINESS, ECONOMICS AND INFORMATICS

# Calculus 2: Multivariable and Differential Equations BUEM001S5

## 30 credits

**27 May 2016**
**10:00-13:00**

*This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth* 5 *marks each and questions in Section B are worth* 20 *marks each.*
*Candidates should attempt* **all** *of the questions in Section A and* **two** *questions out of the four in Section B.*
*Candidates can use their own calculator, provided the model is on the circulated list of authorised calculators or has been approved by the chair of the Mathematics and Statistics Examination Sub-board.*

**Please turn over**

## Section A

1. (a) Using only the definition of a derivative, find the derivative of

$$f(x) = \sqrt{5x+1}, \qquad \text{for} \quad x > -\frac{1}{5}.$$

[3]

(b) Evaluate the right limit:

$$\lim_{x \to 0^+} \frac{\sqrt{x}}{\sin\left(\sqrt{x}\right)}.$$

[2]

2. Let $D$ be the triangle in the $(x, y)-$ plane bounded by $y = x$, $y = -x$ and $y = 2$. Sketch the region of integration and evaluate

$$\iint_D (x + y - 2xy)\, dxdy.$$

[5]

3. Let real-valued function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = e^x \sin y$.

(a) Find the second derivative of the function. [2]

(b) Write down the quadratic Taylor approximation at the point $(x, y) = (1, 0)$. [3]

4. Consider the differential equation

$$y' = x + 2y,$$

with initial condition $y(0) = 2$. Use the method of Taylor series about the point $x = 0$ to find the first five terms of the Taylor series of $y$. [5]

5. At time $t = 0$ a ball of mass 2 kilograms is dropped from the top of a 150 meter high building. Let $y(t)$ be the height in meters of the ball at time $t$, with $y = 0$ being ground level. The force of air resistance is four times the speed of the ball. Let the acceleration of gravity be $g = 9.8m/s^2$. What are the differential equation and initial condition(s) for $y(t)$? For full marks you must provide brief reasoning. You do not have to solve the differential equation. [5]

**Please turn over**

6. Consider the following differential equation

$$A\,x^2\frac{d^2y}{dx^2} + B\,x\frac{dy}{dx} + Cy = 0$$

for $A, B, C$ constants.

(a) By substituting $x = e^t$ and using the chain rule, show that:

$$\frac{dy}{dt} = x\frac{dy}{dx} \qquad \text{and} \qquad \frac{d^2y}{dt^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}.$$

[3]

(b) Show that the above variable substitution turns the original differential equation into one with constant coefficients. You do not have to solve the equation. [2]

7. The number of bacteria in a certain culture increases from 5,000 to 15,000 in 10 hours. Assuming that the rate of increase is proportional to the number of bacteria present, find a formula for the number of bacteria in the culture at any time t. When will the number be 50,000? [5]

8. (a) Show using only the definitions of $\cosh x$ and $\sinh x$ that

$$\cosh^2 x - \sinh^2 x = 1.$$

[2]

(b) Show that

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right) \qquad \text{for } -1 < x < 1.$$

[3]

**Please turn over**

## Section B

9. (a) Let $U \subseteq \mathbb{R}^2$ and consider $f : U \to \mathbb{R}$ be a function.
Define the following terms, for point $(a, b) \in U$:

   (i) stationary point of $f$,

   (ii) local minimum of $f$,

   (iii) global minimum of $f$, and

   (iv) boundary point of U. [4]

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x, y) = x^3 + x^2y - y^2.$$

   (i) Compute the partial derivatives $f_x, f_y, f_{xx}, f_{yy}$ and $f_{xy}$. [3]

   (ii) Find and classify the stationary points of $f$. If the Hessian gives you no information about a stationary point, you do not have to investigate the stationary point further. [7]

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = x^2 - 3y^2.$$

   (i) Let $P$ be the tangent plane of $f$ at the point $(2, 1, 1)$. Find the normal vector to the tangent plane at the point. [1]

   (ii) Find the Cartesian equation of $P$ to the graph of $f$ at the point $(2, 1, 1)$. [2]

   (iii) Find the directional derivative of $f$ at the point $(2, 1)$ in the direction $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. [2]

   (iv) Find the direction in which f is decreasing most rapidly as we move away from the point $(x, y) = (2, 1)$. [1]

**Please turn over**

10. (a) State the theorem of existence and uniqueness of solutions for a first order differential equation. [3]

(b) Consider the differential equation

$$\frac{dy}{dx} = x + y^2$$

with initial condition $y(1) = 0$. Using Euler's method with step length $h = 0.5$ estimate $y(2)$. [5]

(c) Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = x^2 - 4x + 2.$$

(i) Find the numbers $k$ such that $y = x^k$ is the solution to the homogeneous part of the given equation. [3]

(ii) By considering the substitution $y = vx$, change the given differential equation into

$$x^3 \frac{d^2v}{dx^2} + 5x^2\frac{dv}{dx} = x^2 - 4x + 2.$$

[3]

(iii) Solve the differential equation

$$x^3 \frac{d^2v}{dx^2} + 5x^2\frac{dv}{dx} = x^2 - 4x + 2,$$

and use your answer to find the general solution of the original equation. [6]

**Please turn over**

11. (a) Evaluate the integral

$$\int_0^4 \int_{\sqrt{y}}^2 y\cos(x^5)dxdy.$$

[5]

(b) Consider the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}} dydx.$$

(i) Assume the change of variables $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that the *Jacobian* of this change of variables is $r$. [2]

(ii) Use polar coordinates to evaluate the integral. [4]

(c) (i) State the definition of the Gamma function. State clearly the domain of the function. [1]

(ii) Show that the Gamma function $\Gamma(x)$ satisfies

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

for $x > 1$. [3]

(d) (i) Prove that: $\Gamma(x+y) > \Gamma(x)\Gamma(y)$. [3]

(ii) Give a simple counter-example to the statement:
"$\Gamma(x)$ is monotonically increasing for $x > 0$." [2]

**Please turn over**

12. (a) Let $U \subseteq \mathbb{R}$ and consider $f, g : U \to \mathbb{R}$ be real valued functions of one variable.

(i) State the definition of the derivative of $f$ at $x \in U$. [1]

(ii) Assume that $f', g'$ exist at $x$. Using only the definition of a derivative, show that:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

[4]

(b) (i) State L' Hôpital's rule. [1]

(ii) Find the value of the limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+1} - e^x}{x}.$$

[3]

(c) Consider the differential equation

$$y(9x+4y) + 6x(x+y)\frac{dy}{dx} = 0.$$

(i) Is the equation exact? Justify your answer. [2]

(ii) Show that $\mu(x, y) = xy$ is an integrating factor for this differential equation and find its general solution. [9]

**End of examination paper**