Calculus 2 assignment 4

Solutions

1(a) We have $\frac{dT}{dt} = k(T_s - T)$.

If $T_s > T$, then $T_s - T > 0$ and in this case we expect the object to be warming up to match its surroundings, ie $\frac{dT}{dt} > 0$

$\therefore k > 0$.

Conversely, if $T_s < T$ then the object should be cooling down, $\frac{dT}{dt} < 0$ and again $k > 0$.

(b) $$\frac{dT}{dt} = k(T_s - T)$$

$$\int \frac{dT}{T_s - T}\, dT = \int k\, dt$$

$$-\ln(T_s - T) = kt + C \quad , \quad C \text{ const}$$

$$\Rightarrow \quad \ln(T_s - T) = -kt + D \quad , \quad D \text{ "}$$

$$T_s - T = Fe^{-kt} \quad , \quad F \text{ "}$$

$$\Rightarrow \quad T = T_s - Fe^{-kt}$$

when $t = 0$, $T = T_0$, so

$$T_0 = T_s - F \Rightarrow F = \cancel{\quad} = T_s - T_0$$

$$\therefore \quad T = T_s - (T_s - T_0)e^{-kt} \qquad (*)$$

(c) (i) At noon: $t = A$, $T_s = 24$, $T = 34$, $T_0 = 37$, subst into $(*)$:

$$34 = 24 - (24 - 37)e^{-kA}$$

$$\frac{34 - 24}{24 - 37} = -e^{-kA}$$

$$\frac{10}{-13} = -e^{-kA} \Rightarrow -kA = \ln\left(\frac{10}{13}\right) \qquad (**)$$

(ii) At 12:30, $t = A + 30$, $T_s = 24$, $T = \cancel{34}\ 32$, $T_0 = 37$

subst into $(*)$:

$$32 = 24 - (24 - 37)e^{-k(A+30)}$$

$$\frac{32 - 24}{24 - 37} = -e^{-kA - 30k}$$

$$\frac{8}{13} = e^{-kA - 30k} \Rightarrow -kA - 30k = \ln \frac{8}{13}$$

From $(**)$, $-kA = \ln\left(\frac{10}{13}\right)$, subst. this to get

$$\ln\left(\frac{10}{13}\right) - 30k = \ln\left(\frac{8}{13}\right)$$

$$\Rightarrow \quad -30k = \ln\left(\frac{8}{\cancel{13}} \cdot \frac{\cancel{13}}{10}\right)$$

$$\Rightarrow \quad k = \frac{1}{30} \ln\left(\frac{5}{4}\right)$$

Now we have k, we can subst. this value into $(**)$:

$$-kA = \ln\left(\frac{10}{13}\right)$$

$$\Rightarrow \quad A = -\frac{1}{k} \ln\left(\frac{10}{13}\right)$$

$$= \frac{-30 \ln\left(\frac{10}{13}\right)}{\ln\left(\frac{5}{4}\right)} = 35.27293479 \text{ mins}$$

$$\approx 35 \text{ mins}$$

$\therefore$ time of death is approx 11:25 am

(iii) Murder the victim in a room which is at 37°C.

2(a) Let $f(x)$ be even and $g(x)$ be odd.

(i) Write $h(x) = g(f(x))$. Then we have:

$$h(-x) = g(f(-x)) = g(f(x)) = \cancel{g}\, h(x)$$

since f is even

$\therefore$ h is even

Now write $h(x) = f(g(x))$, then:

$$h(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = h(x)$$

since g is odd   since f is even

$\therefore$ h is again even.

Statement is TRUE.

(ii) Write $h(x) = f(x)g(x)$, then:

$$h(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x)$$

$$= -h(x)$$

since f is even & g is odd

$\therefore$ h is odd

When $h(x) = \dfrac{f(x)}{g(x)}$ proof is similar.

Statement is TRUE.

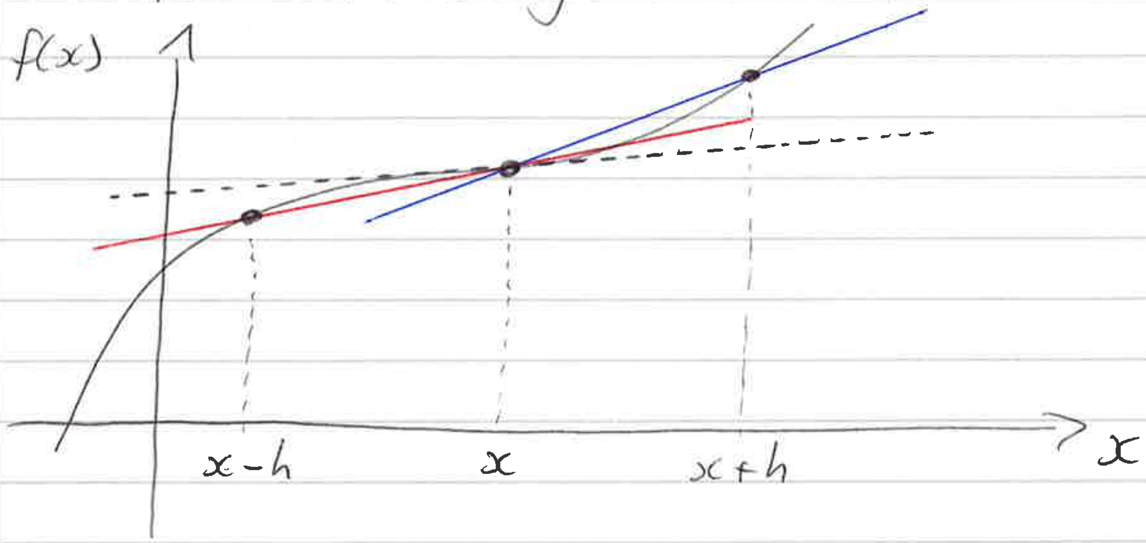
(iii) Recall the definition of $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{—— } ①$$

But in fact this is exactly the same as

$$② \text{ —— } \lim_{h \to 0} \frac{f(x) - f(x-h)}{h},$$ as you can

see from the diagram below.



As $h \to 0$, you see that the slope of the blue line = limit ① and that of the red line = limit ② both $\to$ slope of the black line, which is $f'(x)$.

So then for $f(x)$ even we have:

$$f'(-x) = \lim_{h \to 0} \frac{f(-x) - f(-x-h)}{h} \quad (\text{using } 2)$$

$$= \lim_{h \to 0} \frac{f(x) - f(x+h)}{h}, \quad \text{since } f \text{ is even}$$

$$= \lim_{h \to 0} \frac{-(f(x+h) - f(x))}{h}$$

$$= -\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= -f'(x) \quad (\text{using } 1)$$

ie $f'(-x) = -f'(x)$ so $f'(x)$ is odd.

Now consider $g(x)$ which is odd.

Then $g'(-x) = \lim_{h \to 0} \frac{g(-x) - f(-x-h)}{h}$ (using 2)

$$\cancel{= \lim_{h \to 0} \frac{-g(x) + f(x+h)}{h}}$$ sorry!

$$= \lim_{h \to 0} \frac{-g(x) - -g(x+h)}{h} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$= g'(x)$.

$\therefore\ g'(-x) = g'(x)$ so $g'(x)$ is even.

statement is TRUE.

(b)(i) $h_e(x)$ is even:

$h_e(-x) = \dfrac{h(-x) + h(x)}{2} = h_e(x) \ \therefore$ even

$h_o(x)$ is odd:

$h_o(-x) = \dfrac{h(-x) - h(--x)}{2} = \dfrac{h(-x) - h(x)}{2}$

$= -\dfrac{(h(x) - h(-x))}{2} = -h_o(x) \ \therefore$ odd.

Note that

$h_e(x) + h_o(x) = \dfrac{h(x) + h(-x)}{2} + \dfrac{h(x) - h(-x)}{2}$

$= \dfrac{2h(x)}{2} = h(x)$.

So for any function $h(x)$ we have an even part and an odd part which sum to $h(x)$.

(ii) Lots of possible examples.

e.g. $h(x) = \dfrac{1}{x-1}$

$h(2) = 1$, $h(-2) = -\dfrac{1}{3}$ so $h(x)$ is neither odd nor even.

And $h(x) = \dfrac{1}{x-1} = \underbrace{\dfrac{1}{x^2-1}}_{h_e(x)} + \underbrace{\dfrac{x}{x^2-1}}_{h_o(x)}$

3. $\ddot{x} = t^2 + 6t + c, \; b, c \in \mathbb{R}.$

Recap: $\ddot{x}$ = acceleration = $\frac{dv}{dt}$

$\dot{x}$ = velocity = v = $\frac{dx}{dt}$

x = displacement

(a) $$\frac{dv}{dt} = t^2 + bt + c$$

$$\Rightarrow \quad v = \int t^2 + bt + c \, dt$$

$$= \frac{1}{3}t^3 + \frac{b}{2}t^2 + ct + \text{const}$$

"released from rest at $t=0$" means $v=0$ when $t=0$ so const $=0$

$$\therefore \quad v = \frac{1}{3}t^3 + \frac{b}{2}t^2 + ct$$

so $$\frac{dx}{dt} = \frac{1}{3}t^3 + \frac{b}{2}t^2 + ct$$

$$\Rightarrow \quad x = \int \frac{1}{3}t^3 + \frac{b}{2}t^2 + ct \, dt$$

$$= \frac{1}{12}t^4 + \frac{b}{6}t^3 + \frac{c}{2}t^2 + C, \quad C \text{ a const.}$$

$t=0 \Rightarrow x = C$, so C is starting position.

$$\therefore \quad x = \frac{1}{12}t^4 + \frac{b}{6}t^3 + \frac{c}{2}t^2 + C.$$

(b) For the particle to return to its starting position means $x = C$ again:

$$C = \frac{1}{12}t^4 + \frac{b}{6}t^3 + \frac{c}{2}t^2 + C$$

$$\Rightarrow \quad \frac{1}{12}t^4 + \frac{b}{6}t^3 + \frac{c}{2}t^2 = 0$$

$$\Rightarrow \quad t^4 + 2bt^3 + 6ct^2 = 0$$

So we need to choose b and c so that this polynomial has a positive root.

Make life easy for yourself – choose $b = 0$.

$$\therefore \quad t^4 + 6ct^2 = 0$$

$$\Rightarrow \quad t^2(t^2 + 6c) = 0$$

$$\Rightarrow \quad t^2 = 0 \quad \text{or} \quad t^2 = -6c$$

$t^2 = 0$ corresponds to start time $\quad \therefore \quad t = \sqrt{-6c}$

Obvious choice is $c = -6$
so $t = 6$.

Check, so when $b=0$ and $c=-6$

$$x(t) = \frac{1}{12}t^4 - 3t^2 + C$$

$$x(6) = \frac{1}{12}(6^4) - 3(6^2) + C = C$$

which is what we wanted.

(c) Finally it is clear to see that the t^4 term dominates and so $x(t) \to +\infty$ as $t \to \infty$.

For $\dot{x}(t)$, t^3 term dominates and $\dot{x}(t) \to +\infty$ as $t \to \infty$

and for $\ddot{x}(t)$, t^2 term dominates so $\ddot{x}(t) \to +\infty$ as $t \to \infty$

4 We have $\frac{dy}{dx} = y' = 3xy$.

First we need to make y'' and y'''.

$$y'' = \frac{d}{dx}(y') = 3y + 3xy' = 3y + 3x \cdot 3xy$$
$$= 3y(1+3x^2)$$

$$y''' = \frac{d}{dx}(y'') = 3y'(1+3x^2) + 3y(6x)$$

$$= 3.3xy(1+3x^2) + 3y(6x)$$
$$= 3y(3x + 9x^3 + 6x)$$
$$= 27xy(1+x^2)$$

Then Euler's method with higher derivatives is:

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i'''$$

So $h = \frac{1}{10}$ and use derivatives above to get

$$y_{i+1} = y_i + \frac{3x_i y_i}{10} + \frac{3y_i(1+3x_i^2)}{200} + \frac{27x_i y_i(1+x_i^2)}{6000}$$

So this is the equation to input into excel.

Then we can make this table:

i	x_i	y_i
0	0	0.5
1	0.1	0.5075
2	0.2	0.530796534
3	0.3	0.572058533