## SECTION A

(a)

**1** Using only the definition of a derivative, find the derivative of:

$$f(x) = \sqrt{5x+1} \quad , \text{ for } x > -\frac{1}{5}.$$

Solution

(1) Def.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0} \frac{\sqrt{5(x+h)+1} - \sqrt{5x+1}}{h}$$

(1) rationalise

$$= \lim_{h\to 0} \frac{\left(\sqrt{5(x+h)+1} - \sqrt{5x+1}\right)\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)}{h\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)}$$

$$= \lim_{h\to 0} \frac{5(\cancel{x}+h) + \cancel{1} - (\cancel{5x}+\cancel{1})}{h\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)}$$

(1) correct calculations

$$= \lim_{h\to 0} \frac{5\cancel{h}}{\cancel{h}\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)} = \lim_{h\to 0} \frac{5}{\sqrt{5(x+h)+1} + \sqrt{5x+1}}$$

$$= \frac{5}{2\sqrt{5x+1}}$$

END

[3 marks]

(b) Evaluate the right limit : $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sin(\sqrt{x})}$ [2 marks]

Solution

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sin(\sqrt{x})} \;\boxed{\underset{\text{De L'Hôpital}}{\overset{\frac{0}{0}}{=}}}\; \lim_{x\to 0^+} \frac{\frac{1}{\cancel{2\sqrt{x}}}}{\cos(\sqrt{x})\frac{1}{\cancel{2\sqrt{x}}}} = \lim_{x\to 0^+} \frac{1}{\cos(\sqrt{x})} = \underbrace{\frac{1}{1}}_{\text{0.5 correct answer}} = 1.$$

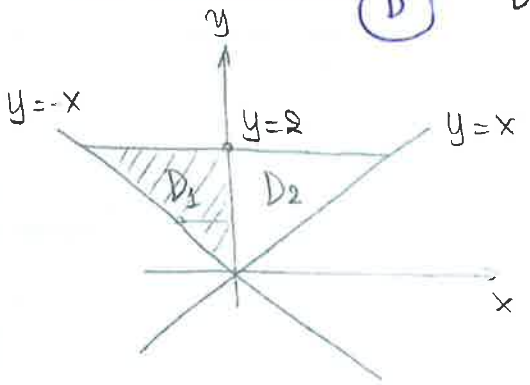
0.5 explanation

correct derivative (1)

END

(2) Let $D$ be the triangle in the $(x,y)$-plane bounded by $y=x$, $y=-x$ and $y=2$. Sketch the region of integration and $D$ Evaluate the integral

(1) $\iint_D (x+y-2xy)\,dx\,dy$ [5 marks]




(1) (1)

$$\boxed{\int_0^2} \boxed{\int_{-y}^{y}} (\dots)\,dx\,dy$$

$$\iint_D (x+y-2xy)\,dx\,dy = \int_0^2 \int_{-y}^{0} (\dots)\,dx\,dy + \int_0^2 \int_0^y (\quad)\,dx\,dy$$

$D_1$

$$= \int_0^2 \left[\frac{x^2}{2} + xy - x^2y\right]_{-y}^{0} dy + \int_0^2 \left[\frac{x^2}{2} + xy - x^2y\right]_0^y dy$$

$$= \int_0^2 -\left(\cancel{\frac{y^2}{2}} - y^2 - \cancel{y^3}\right) dy + \int_0^2 \left(\cancel{\frac{y^2}{2}} + y^2 - \cancel{y^3}\right) dy$$

(2)

$$= \int_0^2 2y^2\,dy = 2\left.\frac{y^3}{3}\right|_0^2 = 2 \times \frac{8}{3} = \frac{16}{3}$$

5 marks

END

[redacted]

Exercise seen in class

(3) Let real-valued function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by : $f(x,y) = e^x \sin y$

(i) Find the second derivative of the function [2]

Sol $f''(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{pmatrix}$

since

$f_x = e^x \sin y$

$f_y = e^x \cos y$

$f_{xx} = e^x \sin y$

$f_{xy} = e^x \cos y$

$f_{yy} = -e^x \sin y$

(ii) Write down the quadratic Taylor approximation at the point $(x,y) = (1,0)$

Sol $f(x,y) = \cancel{f(1,0)}^{0} + f'(1,0) \begin{pmatrix} x-1 \\ y-0 \end{pmatrix} + \frac{1}{2}(x-1, y) f''(1,0) \begin{pmatrix} x-1 \\ y \end{pmatrix}$

$= (0 \;\; e) \begin{pmatrix} x-1 \\ y \end{pmatrix} + \frac{1}{2}(x-1 \;\; y) \underset{2\times 2}{\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} x-1 \\ y \end{pmatrix}}$

$= ey + \frac{1}{2}(x-1 \;\; y) \begin{pmatrix} ey \\ e(x-1) \end{pmatrix}$

$= ey + \frac{1}{2}[ey(x-1) + ey(x-1)]$

$= ey + ey(x-1)$ [3]

$= ey(\cancel{1} + x - \cancel{1})$

$= eyx$

(4.) Consider the differential equation

$$y' = x + 2y,$$

with initial condition $y(0) = 2$.

ok

Use the method of Taylor series (~~about the point $x=0$~~) to find the first five terms of the Taylor series of $y$ about that point. [5]

Solution

The Taylor series of $y$ about the point 0 has the form

(1) $$y(x) = y(0) + y'(0)(x-0) + \frac{y''(0)}{2!}(x-0)^2 + \frac{y'''}{3!}(x-0)^3 + \frac{y^{(4)}}{4!}(x-0)^4$$

The initial condition $y(0)=2$, and the differential equation

$$y' = x + 2y$$

(1) give that:

for $x=0$ : $y'(0) = 2y(0) = 4$

By repeatedly differentiating the differential equation, we can find the higher derivatives:

no need

$$y'' = 2y' \left[= \cancel{2(x+2y)} = 2x + 4y\right] \Rightarrow y''(0) = 2y'(0) = 8$$

$$y''' = 2y'' \Rightarrow y''' = 2 \times 8 = 16$$

$$y^{(4)} = 2y''' \Rightarrow y^{(4)} = 2 \times 16 = 32$$

Therefore the first five terms of the Taylor series of $y$ about 1 are:

$$2 + 4x + \frac{8}{2!}x^2 + \frac{16}{3!}x^3 + \frac{32}{4!}x^4.$$

END

(5.) At time $t=0$ a ball of mass $2kg$ is dropped from the top of a $150\,m$ high building.
Let $y(t)$ be the height of the ball at time $t$, with $y=0$ being ground level.
The force of air resistance is four times the speed of the ball. Let the accelaration of gravity to be:

$$g = 9.8\, m/s^2.$$

What are the differential equation and initial condition(s) for $y(t)$?

For full marks you must provide brief reasoning
[5 marks] You do not have to solve the differential equation.

<u>Answer</u>

1 Due to Newton's second law: $ma = F$
$m = 2kgr$
$a = \ddot{y}$ the accelaration
$F$: sum of forces

We have 2 forces:

1 - gravity: $F_{grav} = - mg$

negative direction in our coordinate system since $y=0$ is ground level

1 - air resistance: $F_{drag} = - 4\dot{y}$

minus sign comes from the fact that the drag force is opposed to the direction of motion

Putting everything together:

$$ma = -mg - 4\dot{y} \Rightarrow 2\ddot{y} + 4\dot{y} + 2g = 0$$

1 The initial conditions are $y(0) = 150$ – initially the ball is on the top of the building

1 $\dot{y}(0) = 0$ – the ball is dropped with no initial velocity

END

Bookwork

(6.) Consider the following differential equation

$$Ax^2 \frac{d^2y}{dx^2} + Bx\frac{dy}{dx} + Cy = 0 \quad (*)$$

for A, B, C : constants.

Bookwork (a) By substituting $x = e^t$ and using the chain rule, show that:

$$\frac{dy}{dt} = x\frac{dy}{dx} \quad (1) \quad \text{and} \quad \frac{d^2y}{dt^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} \quad (2)$$

[3 marks]

Proof

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t = x\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dt}$$

$$= \left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right)e^t = \left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right)x$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

(b) Show that the above variable substitution turns the original differential equation into one with constant coefficients.
You do not have to solve the equation [2 marks]

Sol : Because of (1) and (2)

$$\left(A\frac{d^2y}{dt^2} - Ax\frac{dy}{dx}\right) + B\frac{dy}{dt} + Cy = 0$$

$$A\frac{d^2y}{dt^2} - A\frac{dy}{dt} + B\frac{dy}{dt} + Cy = 0$$

$$A\frac{d^2y}{dt^2} + (B-A)\frac{dy}{dt} + Cy = 0$$

END

Swokowski (p.341 / Question 19)

The number of bacteria in a certain culture increases from 5,000 to 15,000 in 10 hours. Assuming that the rate of increase is proportional to the number of bacteria present, find a formula for the number of bacteria in the culture at any time t. ~~Estimate the number at the end of 20 hours.~~ no need for this one When will the number be 50,000?

Answer

If $N(t)$ number of bacteria at time t then:

(2) $\boxed{\frac{dN}{dt} = cN}$ 0.5 $\Rightarrow \underbrace{\frac{1}{N} dN = c\,dt \Rightarrow \ln N = ct}_{\text{(1) for extra explanation}} \Rightarrow \boxed{N(t) = Ae^{ct}}$ 0.5

Using the provided information:

(1) $N(0) = 5{,}000 = Ae^{c \cdot 0} \Rightarrow A = 5{,}000$ (1)

(1) $N(10) = 15{,}000 = Ae^{c10} \overset{(1)}{\Rightarrow} e^{10c} = \frac{15{,}000}{5{,}000} = 3,$

$\Rightarrow 10c = \ln(3)$

$\Rightarrow c = \frac{1}{10}\ln(3)$

Hence: $\boxed{N(t) = 5{,}000\, e^{\frac{1}{10}(\ln 3)t}}$

The number will be 50,000 when:

$$50{,}\cancel{000} = 5{,}\cancel{000}\, e^{\frac{1}{10}(\ln 3)t}$$

$$\Rightarrow 10 = e^{\frac{1}{10}(\ln 3)t}$$

(1) $\Rightarrow \ln 10 = \frac{1}{10}\ln 3\, t \Rightarrow t = \frac{10 \ln 10}{\ln 3} \cong 20.959$ hours

END

Book work

8 (a) Show using only the definitions of $\cosh x$ and $\sinh x$ that
$$\cosh^2 x - \sinh^2 x = 1 \qquad [2]$$

(b) Show that
$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right) \quad \text{for } -1<x<1 \qquad [3]$$

Proof

(a) For $\cosh x = \dfrac{e^x+e^{-x}}{2}$

$\sinh x = \dfrac{e^x-e^{-x}}{2}$

We have $\cosh^2 x - \sinh^2 x = \frac{1}{4}(e^x+e^{-x})^2 - \frac{1}{4}(e^x-e^{-x})^2$

$$= \frac{1}{4}\left((\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}) - (\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}\right)$$

$$= \frac{1}{4}4 = 1$$

(b) let $x = \tanh y = \dfrac{e^{2y}-1}{e^{2y}+1}$

substitute $z = e^{2y}$ $\Rightarrow$ $x = \dfrac{z-1}{z+1}$

$\Rightarrow zx + x = z - 1$

$\Rightarrow z(x-1) = -x-1$

$\Rightarrow z = \dfrac{x+1}{1-x}$

$\Rightarrow 2y = \ln\left(\dfrac{x+1}{1-x}\right)$

$\Rightarrow y = \dfrac{1}{2}\ln\left(\dfrac{x+1}{1-x}\right)$

$\Rightarrow \operatorname{arctan} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{1-x}\right)$

with $-1<x<1$

## SECTION B

9. (a) Let $U \subseteq \mathbb{R}^2$ and $f: U \to \mathbb{R}$ be a function

(i) Define the terms: "stationary point of $f$", "local maximum of $f$", "global minimum of $f$" and

(ii) boundary point $(a,b)$ of $U$. [4].

C, Bookwork

(i) $(a,b) \in U$ is called a stationary point of $f$ if the tangent plane at $(a,b)$ is horizontal.

(ii) $(a,b) \in U$ is called a local minimum if $f(x,y) \geq f(a,b)$ for all $(x,y)$ in some small enough disk around $(a,b)$

(iii) $(a,b) \in U$ is a global minimum of if $f(a,b) \leq f(x,y)$ for all $(x,y) \in U$.

(iv) $(a,b) \in U$ is a boundary point of $U$ if every disk around the point, no matter how small, contains both points in $U$ and outside $U$.

## SECTION B

**9** (a) Let $U \subseteq \mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a function

(i) Define the terms: "stationary point of $f$", "local maximum of $f$", "global minimum of $f$"
and

(ii) boundary point of $U$. [4]
(a,b)

↳ Bookwork

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = x^3 + x^2y - y^2$

(i) Write down the partial derivatives $f_x, f_y, f_{xx}, f_{yy}$ and $f_{xy}$ [3]

$f_x = 3x^2 + 2xy$ , $f_{xx} = 6x + 2y$

$f_y = x^2 - 2y$ $\quad f_{yy} = -2$

$f_{xy} = 2x = f_{yx}$

(ii) Find and classify the stationary points of $f$. If the Hessian gives you no information about a stationary point, you do not have to investigate that point further [7]

Sol: $\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right| \Rightarrow \begin{array}{ll} x(3x + 2y) = 0 & (1) \\ x^2 - 2y = 0 & (2) \end{array}$ $\Rightarrow x = 0$ or $3x + 2y = 0$

$y = 0$

in (2):

$x^2 + 3x = 0$

$x(x+3) = 0$

$x = 0 \quad x = -3$

Hence, the stationary points will be:

$(0,0)$ , $(-3, \frac{9}{2})$

$\Delta = \left.\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\right|_{(0,0)}$ : no information

$\Delta = \left.\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\right|_{(-3, \frac{9}{2})} = \begin{pmatrix} -9 & -6 \\ -6 & -2 \end{pmatrix} \Rightarrow \Delta = -18 < 0$

$\Rightarrow (-3, \frac{9}{2})$ : saddle pt

(c) Let $f:\mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = x^2 - 3y^2$

(i) normal vector to the tangent plane, P, to the graph of f at point $(2,1,1)$

Solution $\vec{n} = \begin{pmatrix} 4 \\ -6 \\ -1 \end{pmatrix}$ for $f_x = 2x$, $f_y = -6y$ (1)

(ii) cartesian equation of the tangent plane P to the graph f at the point $(2,1,1)$

$$\Rightarrow \quad z - f(2,1) = f_x(2,1)(x-2) + f_y(2,1)(y-1) \qquad (2)$$

$$z - 1 = 4(x-2) - 6(y-1)$$

(iii) directional derivative of f at the point $(2,1)$ in the direction $\binom{3}{4}$

Sol: $f_u = \frac{1}{5}\nabla f(2,1)\binom{3}{4} = \frac{1}{5}\binom{4}{-6}\binom{3}{4} = -\frac{12}{5}$ (2)

(iv) Find the direction in which f is decreasing most rapidly as we move away from the point $(2,1)$

Ans: $-\nabla f(2,1) = \binom{-4}{6}$ (1)

END

10 (b) Consider the differential equation.

$$\frac{dy}{dx} = x + y^2$$

with initial condition $y(1)=0$.
Using Euler's method with step length $h=0.5$, estimate $y(2)$.

Solution

Use the Euler iteration formula

$$y_{i+1} = y_i + h\,m(x_i, y_i)$$

where $h = 0.5$

$$m(x_i, y_i) = x_i + y_i^2$$

(2)

$$y_{i+1} = y_i + 0.5\left(x_i + y_i^2\right), \quad x_{i+1} = x_i + h$$

(3)

| $i$ | $x_i$ | $y_i$ |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 1.5 | 0.5 |
| 2 | 2 | $0.5 + 0.5(1.5 + 0.5^2)$ |

$0 + 0.5(1.0 + 0^2) = 0.5$

$x_i + 0.5$, $1.5$

$\cong 1.375$

[5 marks]

(a) State (without proof) the theorem of existence and uniqueness of solutions for a first order differential equation. (Bookwork) [3 marks]

Statement

Consider the 1st order differential equation: $y' = m(x, y)$.

1 Suppose that $m$ and $\frac{\partial m}{\partial y}$ are continuous in some region $R$ of the $(x,y)$-plane.

1 Then there exists one and only one solution $y = g(x)$ which

1 passes through any given point in $R$.

8 marks

(c) Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = x^2 - 4x + 2 \quad (*)$$

(i) ~~[illegible]~~ Find the numbers $k$ such that $y = x^k$ is the solution to the homogeneous part of the given equation

$$y = x^k$$

$$\frac{dy}{dx} = kx^{k-1}$$

1

$$\frac{d^2y}{dx^2} = k(k-1)x^{k-2}$$

$$x^2 k(k-1)x^{k-2} + 3x k x^{k-1} - 3x^k = 0$$

$$\Rightarrow (k-1)k + 3k - 3 = 0 \quad 1$$

$$\Rightarrow k^2 - k + 3k - 3 = 0$$

$$k^2 + 2k - 3 = 0$$

$$\Delta = \beta^2 - 4\alpha\gamma$$

$$= 4 - 4(-3) = 4 + 12 = 16$$

$$k_{1,2} = \frac{-\beta \pm \sqrt{\Delta}}{2\alpha} = \frac{-2 \pm 4}{2} = -3,\ 1$$

[3 marks]

1

(ii) By considering the substitution $y = vx$, change the given differential equation into:

$$x^3 \frac{d^2v}{dx^2} + \frac{5x^2}{\cancel{4}}\frac{dv}{dx} = (x^2 - 4x + 2)$$

Since $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ (1)

[3 marks]

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx} \quad (1)$$

or $$\frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Hence (from (*)) $\Rightarrow$ $$\left(2x^2\frac{dv}{dx} + x^3\frac{d^2v}{dx^2}\right) + \cancel{3xv} + 3x^2\frac{dv}{dx} - \cancel{3vx} = x^2 - 4x + 2$$

or $$x^3\frac{d^2v}{dx^2} + 5x^2\frac{dv}{dx} = x^2 - 4x + 2 \quad (1)$$

Question 10 (continue)

(iii) Solve the differential equation $x^3 \frac{d^2v}{dx^2} + 5x^2 \frac{dv}{dx} = x^2 - 4x + 2$

and therefore find the general solution of the original equation

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = x^2 - 4x + 2 \qquad [6]$$

(1) Let $w = \frac{dv}{dx}$ then $\frac{dw}{dx} = \frac{d^2v}{dx^2}$ and we may obtain

$x^3 \frac{dw}{dx} + 5x^2 w = x^2 - 4x + 2$ which is linear equation with respect to $w$

(1) $\Rightarrow \frac{dw}{dx} + \underbrace{\frac{5}{x}}_{P(x)} w = \underbrace{\frac{1}{x^3}(x^2 - 4x + 2)}_{Q(x)}$

with solution $w = \frac{1}{\mu(x)} \int \mu(x) Q(x)\, dx = \frac{1}{x^5} \int \frac{x^5}{x^3}(x^2 - 4x + 2)\, dx$

$= \frac{1}{x^5} \int x^2(x^2 - 4x + 2)\, dx = \frac{1}{x^5}\left(\frac{x^5}{5} - \frac{4x^4}{4} + 2\frac{x^3}{3} + c\right.$

(1) $\boxed{w = \frac{1}{5} - \frac{1}{x} + \frac{2}{3}\frac{1}{x^2} + \frac{c}{x^5}}$

(1) where $\mu(x) = e^{\int P(x)dx} = e^{\int \frac{5}{x}dx} = e^{5\ln x} = x^5$

Therefore $w = \frac{dv}{dx} \Rightarrow v = \int w\, dx$

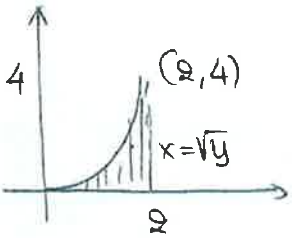
(1) $\Rightarrow v = \frac{1}{5}x - \ln x - \frac{2}{3}\frac{1}{x} - 4\frac{c}{x^4} + k$ $c, k$: constan

(1) Finally: $y = vx = \frac{1}{5} - x\ln x - \frac{2}{3} - \frac{4c}{x^3} + kx$, $c, k$: constant

END

(11) (a) Evaluate the integral: $\int_0^4 \int_{\sqrt{y}}^2 y\cos(x^5)\,dx\,dy$ [5]

Sol: (Need to change the order)




$$\int_0^4 \int_{\sqrt{y}}^2 y\cos(x^5)\,dx\,dy = \int_0^2 \int_0^{x^2} y\cos(x^5)\,dy\,dx$$

$$= \int_0^2 \left[ \frac{y^2}{2} \Big|_0^{x^2} \right] \cos(x^5)\,dx$$

$$= \int_0^2 \frac{x^4}{2} \cos(x^5)\,dx$$

$$= \left[ \frac{\sin(x^5)}{10} \right]_0^2 = \frac{1}{10}\sin(32) \cong 0.055$$

(b) Consider the integral: $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\,dy\,dx$

(i) Assume the change of variables $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that the Jacobian of this change of variables is $r$.
(Bookwork, p. 78/notes) [2]

(ii) Use polar coordinates to evaluate the integral [4]

Sol $$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^{\pi/2} \int_0^1 r e^r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left( [re^r]_0^1 - \int_0^1 e^r\,dr \right) d\theta$$

$$= \int_0^{\pi/2} (e - (e-1))\,d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2}$$

(c) (i) State the definition of the Gamma function. State clearly the domain of the function [1]

(ii) Show that $\Gamma(x) = (x-1)\Gamma(x-1)$ , $x>1$ [3]

Bookwork

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \ , \ x>0$$

(d) (i) Prove that $\Gamma(x+y) > \Gamma(x)\Gamma(y)$ [3]

<u>Proof</u>: $\Gamma(x+y) > \Gamma(x)\Gamma(y) \Rightarrow 1 > \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ , $x,y>1$

$\Rightarrow 1 > B(x,y)$ , $x,y>1$

$\Rightarrow 1 > \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ , $x,y$

The last statement is true because for $0 \le t \le 1$ , $x-1>0$ and $y-1>0$

$\Rightarrow 0 \le t^{x-1}(1-t)^{y-1} < 1$

(ii) Give a simple counter-example to the statement "$\Gamma(x)$ is monotonically increasing for $x>0$" [2]

<u>Example</u>: We have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1) = 0! = 1$

Hence for $\frac{1}{2} < 1$

$$\Gamma\left(\tfrac{1}{2}\right) > \Gamma(1)$$

Bookwork

11 (b) (i)

<u>Proof</u> $$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

(c) (ii) $\Gamma(x) = (x-1)\Gamma(x-1)$ for $x > 1$

<u>Proof</u>: We use integration by parts.
Set $u = t^{x-1}$ and $\frac{dv}{dt} = e^{-t}$

Then $\frac{du}{dt} = (x-1)t^{x-2}$ and $v = -e^{-t}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$$

$$= \left[-t^{x-1}e^{-t}\right]_0^\infty + \int_0^\infty (x-1)t^{x-2}e^{-t}\,dt$$

$$= 0 + (x-1)\int_0^\infty t^{(x-1)-1} e^{-t}\,dt$$

$$= (x-1)\Gamma(x-1)$$

(12.) (a) Let $U \subseteq \mathbb{R}$ and $f, g : U \to \mathbb{R}$ be real valued functions of one variable.

Bookwork (i) State the definition of the derivative of $f$ at $x \in U$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \text{ if the limit exists.}$$ [1]

(ii) Assume that $f', g'$ exist at $x$.
Using only the definition of a derivative, show that
$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

1 $$(f(x)g(x))' = \lim_{h \to 0} \frac{fg(x+h) - fg(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

1 $$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left\{ f(x+h)\frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} g(x) \right\}$$

1 $$= \lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x)$$

1 $$= f(x)g'(x) + f'(x)g(x)$$

since the relevant limits exist [4]

(b) (i) State the L'Hôpital's rule [1]

$\Rightarrow$ If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(ii) Find the value of the limit:

$$\lim_{x\to 0} \frac{\sqrt{x^2+1} - e^x}{x}$$

Solution

We have $\lim_{x\to 0} \frac{\sqrt{x^2+1} - e^x}{x}$ $\boxed{= \frac{0}{0}}$ 1 $\lim_{x\to 0} \frac{\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} - e^x}{1}$ 1

therefore use L'Hôpital

$$= \lim_{x\to 0} \frac{x - e^x\sqrt{x^2+1}}{\sqrt{x^2+1}} = \frac{0 - e^0\sqrt{0+1}}{\sqrt{0+1}} = -1$$

[3]

(c) Consider the differential equation:

$$y(9x+4y) + 6x(x+y)\frac{dy}{dx} = 0$$

(i) Is the equation exact? Justify your answer. [2]

Following the notation used in class

$M(x,y) = 9xy + 4y^2$

$N(x,y) = 6x^2 + 6xy$

Hence $\frac{\partial M}{\partial y} = 9x + 8y$

$\frac{\partial N}{\partial x} = 12x + 6y$

$\Bigg\}$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ Hence the equation is not exact

Question 12 (continue)

(ii) Show that $\mu(x,y) = xy$ is an integrating factor for this differential equation. and find its general solution

+ 9

$\mu(x,y) = xy$ will be an integrating factor if by multiplying the
1 equation with it then the new equation becomes exact

We have

$$\underbrace{\overbrace{xy^2(9x+4y)}^{9x^2y^2+4xy^3}}_{M(x,y)} + \underbrace{\overbrace{6x^2y(x+y)}^{6x^3y+6x^2y^2}}_{N(x,y)}\frac{dy}{dx} = 0$$

This time
2
$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 18x^2y + 12xy^2\\ \frac{\partial N}{\partial x} &= 18x^2y + 12xy^2\end{aligned}\right\}$$ equal and therefore the assumption is correct

1 [ The general solution of the equation will be a function $f$ such that $f(x,y) = C$ with
1
$$\left[\begin{aligned}\frac{\partial f}{\partial x} &= M(x,y) = 9x^2y^2 + 4xy^3\\ \text{and } \frac{\partial f}{\partial y} &= N(x,y) = 6x^3y + 6x^2y^2\end{aligned}\right.$$

In fact

1 $$f(x,y) = \int (9x^2y^2 + 4xy^3)\,dx = 9\frac{x^3}{3}y^2 + 4\frac{x^2}{2}y^3 + g(y)$$

$$= 3x^3y^2 + 2x^2y^3 + g(y)$$

1 and
$$\left.\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= 6x^3y + 6x^2y^2 + g'(y)\\ &= 6x^3y + 6x^2y^2\end{aligned}\right] \Rightarrow g'(y) = 0$$

or $g(y) = \text{constant}$ ] 1

Finally the general solution of the equation will be
1 $f(x,y) = 3x^3y^2 + 2x^2y^3 = \text{constant}$

END