Solutions Chapter 4

Solutions to Exercises 4.1.

1. Differentiating $y^2 + xy = \cos(x + y) + x^2$ with respect to x (and remembering that $y = y(x)$ is a function of x) gives

$$2yy' + xy' + y = -\sin(x + y) \cdot (1 + y') + 2x.$$

Rearranging this we find

$$y' = \frac{-y - \sin(x + y) + 2x}{2y + x + \sin(x + y)}$$

as required.

2. Differentiating $y^2 + y = \ln x + \sin y$ with respect to x gives

$$2yy' + y' = \frac{1}{x} + \cos(y) \cdot y'.$$

Differentiating again gives

$$2yy'' + 2y'y' + y'' = \frac{-1}{x^2} + \cos(y) \cdot y'' - \sin(y)y'y',$$

which after rearranging becomes

$$y''(2y + 1 - \cos y) + (y')^2(2 + \sin y) = \frac{-1}{x^2}$$

as required.

Solutions to Exercises 4.2.

1. All four differential equations are separable.

   (a) Separating the variables gives $\int \frac{1}{y+1}\,\mathrm{d}y = \int 6x^2\,\mathrm{d}x$, and evaluating the integrals this becomes $\ln(y + 1) = 2x^3 + c$ where c is a constant. Hence the solution is
   $$y + 1 = A\exp(2x^3),$$
   where $A = e^c$ is a constant. (If you are careful with absolute values and the special case $y = -1$, you'll see that the constant A can be any real number.)

   (b) Separating the variables gives $\int \frac{1}{y}\,\mathrm{d}y = \int \frac{x}{x+1}\,\mathrm{d}x$, and evaluating the integrals this becomes $\ln y = x - \ln(x+1) + c$ where c is a constant. Hence the solution is
   $$y = A\frac{\exp(x)}{x + 1},$$
   where $A = e^c$ is a constant. (Again we were sloppy with absolute values and special cases.)

(c) Separating the variables we get $\int y^{3/2}\,\mathrm{d}y = \int x\sqrt{x^2+1}\,\mathrm{d}x$, and evaluating the integrals gives the solution

$$\frac{2}{5}y^{5/2} = \frac{1}{3}(x^2+1)^{3/2} + c,$$

where c is a constant.

(d) Separating the variables gives $\int e^{-y}\,\mathrm{d}y = \int xe^x\,\mathrm{d}x$, and evaluating the integrals this becomes $-e^{-y} = xe^x - e^x + c$ where c is a constant. Thus the solution is

$$y = -\ln(-xe^x + e^x - c).$$

2. This differential equation is separable. Separating the variables gives $\int 3y^2\,\mathrm{d}y = \int x\cos x\,\mathrm{d}x$, and evaluating the integrals this becomes $y^3 = x\sin x + \cos x + c$ where c is a constant. This is the general solution of the differential equation. When we substitute the boundary condition $x = \pi, y = 2$ into the general solution we find $2^3 = \pi\sin\pi + \cos\pi + c$, so $c = 9$. Thus the solution is

$$y = \sqrt[3]{x\sin(x) + \cos(x) + 9}.$$

3. This differential equation is homogeneous of degree 0. When we set $y = vx$, the equation becomes

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = 1 + \frac{vx}{x} + \left(\frac{vx}{x}\right)^2,$$

which can be simplified to

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^2.$$

Separating the variables gives

$$\int \frac{1}{1+v^2}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x,$$

hence

$$\arctan(v) = \ln x + c$$

where c is a constant. Thus $v = \tan(\ln x + c)$, and using that $y = vx$ we finally find

$$y = x\tan(\ln x + c).$$

Solutions to Exercises 4.3.

1. All three differential equations are exact.

(a) Here $M = 3x^2y - 2xy^2$ and $N = x^3 - 2x^2y + 3\sqrt{y+1}$. A function f with $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ is $f(x,y) = x^3y - x^2y^2 + 2(y+1)^{3/2}$. Hence the general solution of the differential equation is

$$x^3y - x^2y^2 + 2(y+1)^{3/2} = c,$$

where c is a constant.

(b) Here $M = 2xy + 2y^3 + 4x$ and $N = x^2 + 6xy^2 + 3y^2$. A function f with $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ is $f(x, y) = x^2y + 2xy^3 + 2x^2 + y^3$. Hence the general solution of the differential equation is

$$x^2y + 2xy^3 + 2x^2 + y^3 = c,$$

where c is a constant.

(c) Here $M = xy^2 + 3x^2y + \sqrt{x}$ and $N = x^2y + x^3 + \cos y$. A function f with $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ is $f(x, y) = \frac{1}{2}x^2y^2 + x^3y + \frac{2}{3}x^{3/2} + \sin y$. Hence the solution of the differential equation is

$$\frac{1}{2}x^2y^2 + x^3y + \frac{2}{3}x^{3/2} + \sin y = c,$$

where c is a constant.

2. Here $M = e^{2x} + 4x^3 \ln y$ and $N = \frac{x^4}{y} + 15y^5$. Since $\frac{\partial M}{\partial y} = \frac{4x^3}{y} = \frac{\partial N}{\partial x}$, the differential equation is exact. A function f with $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ is $f(x, y) = \frac{1}{2}e^{2x} + x^4 \ln(y) + \frac{5}{2}y^6$. Hence the general solution of the differential equation is

$$\frac{1}{2}e^{2x} + x^4 \ln(y) + \frac{5}{2}y^6 = c,$$

where c is a constant. Substituting the boundary condition $x = 0$, $y = 1$ into the general solution gives $\frac{1}{2}e^{2\cdot 0} + 0^4 \ln(1) + \frac{5}{2} \cdot 1^6 = c$, so $c = 3$. Thus the solution of the differential equation with boundary condition is

$$\frac{1}{2}e^{2x} + x^4 \ln(y) + \frac{5}{2}y^6 = 3.$$

3. If we multiply both sides of the differential equation by x^r we get

$$x^r y^2 - x^{r+2} + x^{r+1}y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Then $M = x^r y^2 - x^{r+2}$ and $N = x^{r+1}y$, so $\frac{\partial M}{\partial y} = 2x^r y$ and $\frac{\partial N}{\partial x} = (r + 1)x^r y$. Thus this differential equation is exact if and only if $2x^r y = (r + 1)x^r y$. This implies that $2 = r + 1$ and therefore $r = 1$. Hence $\mu(x) = x$ is an integrating factor for the original differential equation.

Now solving the exact differential equation $xy^2 - x^3 + x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ gives the solution

$$\frac{1}{2}x^2y^2 - \frac{1}{4}x^4 = c,$$

where c is a constant. This is the general solution of the exact differential equation and hence also of the original differential equation.

4. All three differential equations are linear first order equations.

(a) Divide the equation by $x+1$ to get $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x+1}y = \frac{x^3}{x+1}$. The integrating factor is $\mu(x) = \exp\left(\int \frac{2}{x+1}\,\mathrm{d}x\right) = (x+1)^2$, and the general solution is

$$\begin{aligned} y &= \frac{1}{(x+1)^2}\int (x+1)^2 \frac{x^3}{x+1}\,\mathrm{d}x \\ &= \frac{1}{(x+1)^2}(x^5/5 + x^4/4 + C). \end{aligned}$$

(b) Divide the equation by 2 to get $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2x}y = \frac{7x^2+6}{2}$. The integrating factor is $\mu(x) = \exp\left(\int \frac{1}{2x}\,\mathrm{d}x\right) = \sqrt{x}$, and the general solution is

$$\begin{aligned} y &= \frac{1}{\sqrt{x}}\int \sqrt{x}\frac{7x^2+6}{2}\,\mathrm{d}x \\ &= x^3 + 2x + C/\sqrt{x}. \end{aligned}$$

(c) Divide the equation by x^2+1 to get $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4x}{x^2+1}y = \frac{16x}{x^2+1}$. The integrating factor is $\mu(x) = \exp\left(\int \frac{4x}{x^2+1}\,\mathrm{d}x\right) = (x^2+1)^2$, and the general solution is

$$\begin{aligned} y &= \frac{1}{(x^2+1)^2}\int (x^2+1)^2 \frac{16x}{x^2+1}\,\mathrm{d}x \\ &= \frac{1}{(x^2+1)^2}(4x^4 + 8x^2 + C). \end{aligned}$$

Solutions to Exercises 4.4.

1. (a) This is the type of equation found in Section 4.4 of the notes. Here the relevant lines are not parallel and intersect in the point $(-1,1)$. With the change of variables $x = X-1$ and $y = Y+1$ the differential equation becomes $\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{X+Y}{X-Y}$. To solve this homogeneous differential equation we set $Y = XV$ and obtain the separable differential equation $X\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{1+V^2}{1-V^2}$. Solving this by separation of variables gives

$$\arctan(V) - \frac{1}{2}\ln(1+V^2) = \ln(X) + C$$

where C is a constant. Now substituting back $V = Y/X$ and then $X = x+1$ and $Y = y-1$ gives the general solution of the original differential equation

$$\arctan\left(\frac{y-1}{x+1}\right) - \frac{1}{2}\ln\left(1 + \frac{(y-1)^2}{(x+1)^2}\right) = \ln(x+1) + C.$$

Note that this solution can be simplified as

$$2\arctan\left(\frac{y-1}{x+1}\right) = \ln\left((x+1)^2 + (y-1)^2\right) + c,$$

where $c = 2C$.

(b) Here the relevant lines are parallel. Put $z = x + y$. Then the differential equation becomes $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{4z+1}{3z+1}$ which is separable. Solving this by separating the variables gives $\frac{3}{4}z + \frac{1}{16}\ln(4z+1) = x + C$ where C is a constant. Substituting $z = x+y$ gives the general solution of the original differential equation

$$\frac{3}{4}(x+y) + \frac{1}{16}\ln(4(x+y)+1) = x + C.$$

This solution can be simplified as

$$12y + \ln(4x+4y+1) = 4x + c$$

where $c = 16C$.

2. This is the type where the relevant lines are parallel. With $z = x + y$ the differential equation becomes $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{3z+1}{2z+1}$ which is separable. Solving this by separation of variables gives $\frac{2}{3}z + \frac{1}{9}\ln(3z+1) = x + C$ where C is a constant. Substituting $z = x + y$ gives the general solution of the original differential equation

$$\frac{2}{3}(x+y) + \frac{1}{9}\ln(3(x+y)+1) = x + C.$$

This solution can be simplified as

$$6y + \ln(3x+3y+1) = 3x + c$$

where $c = 9C$. The boundary condition $y(0) = 0$ implies that $c = 0$. Thus the solution of the differential equation with boundary condition is

$$6y + \ln(3x+3y+1) = 3x.$$

3. (a) After making the substitution, we get the separable equation $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v+4}{v}$. Solving this by separation of variables gives $v - 4\ln(v+4) = x + C$. Hence the solution of the original differential equation is

$$x + 2y - 3 - 4\ln(x+2y+1) = x + C.$$

(This solution can be simplified to $2y - 3 - 4\ln(x+2y+1) = C$.)

(b) After making the substitution, we get the linear equation $\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{x}u = \frac{\sin x}{x}$. Solving this gives $u = \frac{1}{x}(-\cos(x) + c)$. Hence the solution of the original differential equation is

$$y^2 = -\frac{\cos x}{x} + \frac{c}{x}.$$

(c) With the given substitution and some algebraic manipulations the differential equation becomes $\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{5}{x}u = -5x^3$, which is linear. Solving this gives $u = x^4(5 + cx)$. Hence the solution of the original differential equation is

$$y^{-5} = x^4(5 + cx).$$

Solutions to Exercises 4.5.

1. All three differential equations are order two linear homogeneous with constant coefficients.

   (a) The auxiliary equation is $t^2+2t-8=0$ which has roots -4 and 2. Hence the general solution of the differential equation is
   $$y(x) = Ae^{-4x} + Be^{2x},$$
   where A, B are constants.

   (b) The auxiliary equation is $t^2+6t+9=0$ which has the root -3 with multiplicity two. Hence the general solution of the differential equation is
   $$y(x) = Ae^{-3x} + Bxe^{-3x},$$
   where A, B constants.

   (c) The auxiliary equation is $t^2+2t+5=0$ which has roots $t=-1\pm 2i$. Hence the general solution of the differential equation is
   $$y(x) = Ae^{-x}\cos 2x + Be^{-x}\sin 2x,$$
   where A, B are constants.

2. The auxiliary equation of the homogeneous part is $t^2+2t-8=0$ which has roots -4 and 2 (see Exercise 1.(a)).

   (a) Looking for a particular integral of the form Ge^{-2x} gives the solution $-\frac{3}{2}e^{-2x}$.

   (b) Note that 2 is a (single) root of the auxiliary equation. Looking for a particular integral of the form Gxe^{2x} gives the solution $\frac{5}{3}xe^{2x}$.

   (c) Looking for a particular integral of the form $ax+b$ gives the solution $-2x+1$.

   (d) Looking for a particular integral of the form $G\cos 2x + H\sin 2x$ gives the solution $-\frac{9}{4}\cos 2x + \frac{3}{4}\sin 2x$.

3. The auxiliary equation of the homogeneous part is $t^2+6t+9=0$ which has the double root -3. (see Exercise 1.(b)).

   (a) Looking for a particular integral of the form Ae^x gives the solution $\frac{3}{8}e^x$.

   (b) Note that -3 is a double root of the auxiliary equation. Looking for a particular integral of the form Ax^2e^{-3x} gives the solution $5x^2e^{-3x}$.

   (c) Looking for a particular integral of the form ax^2+bx+c gives the solution $6x^2-8x+4$.

   (d) Looking for a particular integral of the form $Ae^{-x}+B\cos x+C\sin x$ gives the solution $\frac{3}{4}e^{-x}-\frac{3}{5}\cos x+\frac{4}{5}\sin x$.

4. The auxiliary equation of the homogeneous part is $t^2 + 2t - 15 = 0$ which has roots -5 and 3. Hence the complementary function is $C = Ae^{-5x} + Be^{3x}$, where A and B are constants. Looking for a particular integral of the form Ge^{2x} gives the solution $p = -2e^{2x}$. Thus the general solution of the differential equation is
$$y = C + p = Ae^{-5x} + Be^{3x} - 2e^{2x}.$$
From the initial conditions $y(0) = 1$ and $y'(0) = 0$ we deduce that $A = \frac{5}{8}$ and $B = \frac{19}{8}$. Hence the final solution is
$$y = \frac{5}{8}e^{-5x} + \frac{19}{8}e^{3x} - 2e^{2x}.$$

5. The auxiliary equation of the homogeneous part is $t^2 + 7t + 10 = 0$ which has roots -5 and -2. Hence the complementary function is $C = Ae^{-5x} + Be^{-2x}$, where A and B are constants. Since -2 is a (single) root of the auxiliary equation, we have to look for a particular integral of the form Gxe^{-2x}; we find $p = 4xe^{-2x}$. Thus the general solution of the differential equation is
$$y = C + p = Ae^{-5x} + Be^{-2x} + 4xe^{-2x}.$$
From the initial conditions $y(0) = 5$ and $y'(0) = -3$ we deduce that $A = -1$ and $B = 6$. Hence the final solution is
$$y = -e^{-5x} + 6e^{-2x} + 4xe^{-2x}.$$

Solutions to Exercises 4.6.

1. (a) We use the first way to solve this Euler equation. So we start by looking for a solution of the form $y = x^k$ for the homogeneous part $x^2y'' - 2y = 0$. We find $y = x^2$ (and also $y = x^{-1}$).

   Now to solve the nonhomogeneous differential equation $x^2y'' - 2y = x$, we put $y = vx^2$. With this change of variable, we obtain the differential equation $x^4v'' + 4x^3v' = x$ for v. Dividing by x^4 and putting $w = v'$ gives the first order linear equation $w' + 4x^{-1}w = x^{-3}$ for w. This has the general solution $w = \frac{1}{2}x^{-2} + cx^{-4}$, where c is a constant. Then $v = \int w \, \mathrm{d}x = -\frac{1}{2}x^{-1} - \frac{c}{3}x^{-3} + a$, where a is a constant. Hence the final solution is
$$\begin{aligned} y = vx^2 &= -\frac{1}{2}x - \frac{c}{3}x^{-1} + ax^2 \\ &= -\frac{1}{2}x + bx^{-1} + ax^2, \end{aligned}$$
   where a and $b = -\frac{c}{3}$ are constants.

   (b) We use the second way to solve this Euler euqation, i.e. we use the change of variable $x = e^t$. Then the differential equation becomes
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = t.$$

This is a second order linear equation with constant coefficients. Its general solution is

$$y = e^t\big(A\cos(t) + B\sin(t)\big) + \frac{1}{2}t + \frac{1}{2},$$

where A and B are constants. Now substituting $e^t = x$ and $t = \ln x$ gives the final solution

$$y = x\big(A\cos(\ln x) + B\sin(\ln x)\big) + \frac{1}{2}\ln x + \frac{1}{2},$$

where A and B are constants.

(c) We use the first way to solve this Euler equation. So we start by looking for a solution of the form $y = x^k$ for the homogeneous part $x^2y'' + 5xy' + 4y = 0$. We find $y = x^{-2}$.
Now to solve the nonhomogeneous differential equation $x^2y'' + 5xy' + 4y = x^2 + 16(\ln x)^2$, we put $y = x^{-2}v$. With this change of variable, we obtain the differential equation $v'' + x^{-1}v' = x^2 + 16(\ln x)^2$ for v. Putting $w = v'$ gives the first order linear equation $w' + x^{-1}w = x^2 + 16(\ln x)^2$ for w. This has the general solution

$$w = 8x(\ln x)^2 - 8x(\ln x) + \frac{x^3}{4} + 4x + \frac{c}{x},$$

where c is a constant. Then

$$v = \int w\,\mathrm{d}x = 4x^2(\ln x)^2 - 8x^2(\ln x) + \frac{x^4}{16} + 6x^2 + c(\ln x) + d,$$

where d is a constant. Hence the final solution is

$$y = x^{-2}v = 4(\ln x)^2 - 8(\ln x) + \frac{x^2}{16} + 6 + \frac{c\ln x}{x^2} + \frac{d}{x^2},$$

where c and d are constants.

2. Dividing the differential equation by x^3 and setting $w = y'$ gives

$$w' + \frac{3}{x}w = \frac{1+x}{x^3}.$$

This is a first order linear equation in w. Its general solution is $w = \frac{1}{2x} + \frac{1}{x^2} + \frac{c}{x^3}$ where c is a constant. Hence the final solution is

$$\begin{aligned} y &= \int w\,\mathrm{d}x \\ &= \frac{1}{2}\ln x - \frac{1}{x} - \frac{c}{2x^2} + d, \end{aligned}$$

where c and d are constants.