Discrete Assignment 2

BM Corser

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1. Difference equations

   (a) Here, u_n is an inhomogeneous first order difference equation, of the general form

   $$u_n = f(n)u_{n-1} + g(n) = U \cdot \prod_{i=1}^{n} f(i) + \sum_{i=1}^{n} \left(g(i) \cdot \prod_{j=i+1}^{n} f(j) \right)$$

   where

   $$\begin{aligned} U &= 1 \\ f(n) &= 16^{n^3} \\ g(i) &= 2^{i^2(i+1)^2} \end{aligned}$$

   and

   $$\begin{aligned} \prod_{i=0}^{n} f(i) &= \prod_{i=0}^{n} 16^{i^3} \\ &= 16^{\sum_{i=0}^{n} i^3} \\ &= 16^{\frac{1}{4}n^2(n+1)^2} \\ &= (2^4)^{\frac{1}{4}n^2(n+1)^2} \\ &= 2^{n^2(n+1)^2} \\ &= g(n) \end{aligned}$$

   hence

   $$\begin{aligned} u_n &= g(n) + \sum_{i=1}^{n} \left(g(i) \cdot \prod_{j=i+1}^{n} f(j) \right) \\ &= g(n) + \sum_{i=1}^{n} \left(g(i) \cdot \frac{\prod_{j=1}^{n} f(j)}{\prod_{k=1}^{i} f(k)} \right) \\ &= g(n) + \sum_{i=1}^{n} \left(g(i) \cdot \frac{g(n)}{g(i)} \right) \\ &= (n+1) \cdot 2^{n^2(n+1)^2} \end{aligned}$$

(b) Let $k = n + 2$, then a_{n+2} can be written

$$a_k = 4a_{k-2} + 10 \cdot 3^{k-2}$$

Here, a_k is an inhomogeneous second order difference equation with constant coefficients and can be written in the general form

$$u_n = au_{n-1} + bu_{n-2} + f(n)$$

where $a = 0$, $b = 4$ and $f(n) = 10 \cdot 3^{n-2}$. We know that

$$u_n = G(n) + P(n)$$

where $P(n)$ is a particular solution to the inhomogeneous difference equation and $G(n)$ is a general solution to the homogeneous part of the inhomogeneous difference equation and $G(n) + P(n)$ is a general solution to the inhomogeneous difference equation.

The homogeneous part of the difference equation has characteristic polynomial $\lambda^2 - 4$ with distinct real zeros $w_1 = 2$ and $w_2 = -2$. There must be values c_1 and c_2 such that the initial conditions are satisfied as follows

$$\begin{aligned} c_1 + c_2 &= 9 \\ c_2 &= 9 - c_1 \end{aligned}$$

$$\begin{aligned} c_1w_1 + c_2w_2 &= 4 \\ 2c_1 - 2c_2 &= \\ c_1 - c_2 &= 2 \\ c_1 - (9 - c_1) &= 2 \\ 2c_1 &= 11 \\ c_1 &= \tfrac{11}{2} \end{aligned}$$

$$\begin{aligned} c_2 &= 9 - \tfrac{11}{2} \\ c_2 &= \tfrac{7}{2} \end{aligned}$$

so the general solution to the homogeneous part is

$$G(n) = A \cdot (\tfrac{11}{2})^n + B \cdot (\tfrac{7}{2})^n.$$

$f(n)$ has the form $c\alpha^n$ where $\alpha = 3$ and α is not a zero of the characteristic polynomial, therefore we can try $u_n = M \cdot 3^n$ as a particular solution, hence

$$
\begin{aligned}
M \cdot 3^n &= 4M \cdot 3^{n-2} + 10 \cdot 3^{n-2} \\
M \cdot 3^2 &= 4M + 10 \\
9M &= 4M + 10 \\
5M &= 10 \\
M &= 2
\end{aligned}
$$

and our particular solution is $P(n) = 2 \cdot 3^n$. Now we can write a general solution for the inhomogeneous difference equation u_n as

$$
\begin{aligned}
u_n &= G(n) + P(n) \\
&= A \cdot (\tfrac{11}{2})^n + B \cdot (\tfrac{7}{2})^n + 2 \cdot 3^n.
\end{aligned}
$$

The initial conditions $u_0 = 9$, $u_1 = 4$ imply that

$$
\begin{aligned}
A + B + 2 &= 9 \\
A &= 7 - B
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{11}{2}A + \tfrac{7}{2}B + 6 &= 4 \\
\tfrac{11}{2}(7 - B) + \tfrac{7}{2}B &= -2 \\
\tfrac{77}{2} - \tfrac{11}{2}B + \tfrac{7}{2}B &= -2 \\
\tfrac{5}{2}B &= \tfrac{79}{2} \\
B &= \tfrac{79}{5}
\end{aligned}
$$

$$
\begin{aligned}
A &= 7 - \tfrac{79}{5} \\
&= -\tfrac{44}{5}
\end{aligned}
$$

and therefore our general solution to the inhomogeneous difference equation is $u_n = (\frac{79}{5}) \cdot (\frac{7}{2})^n - (\frac{44}{5}) \cdot (\frac{11}{2})^n + 2 \cdot 3^n$.

(c) Here b_n is an inhomogeneous second order difference equation with constant coefficients. We will solve it using the same technique employed above. The homogeneous part of b_n has characteristic polynomial $\lambda^2 + 5 - 6\lambda = (\lambda - 3)^2 - 4$ with zeros $w_1 = 5$ and $w_2 = 1$ and as such there are $c_1, c_2 \in \mathbb{R}$ such that

$$
\begin{aligned}
c_1 + c_2 &= 9 \\
5c_1 + c_2 &= 30
\end{aligned}
$$

so $c_1 = \frac{21}{4}$, $c_2 = \frac{15}{4}$ and the general solution to the homogeneous part is $A \cdot (\frac{21}{4})^n + B \cdot (\frac{15}{4})^n$. The inhomogeneous function is a polynomial of degree 1 in n and 1 is a zero of the characteristic polynomial (of multiplicity 1), so our particular solution will have the form $n(M_0 + nM_1) = n^2 M_1 + nM_0$. So

$$\begin{aligned} b_n &= 6b_{n-1} - 5b_{n-2} + 120n - 33 \\ b_n - 6b_{n-1} + 5b_{n-2} &= 120n - 33 \end{aligned}$$

and

$$\begin{aligned} nM_0 + n^2M_1 - 6(n-1)M_0 - 6(n-1)^2M_1 + 5(n-2)M_0 + 5(n-2)^2M_1 \\ = 120n - 33 \\ nM_0 + n^2M_1 6nM_0 + 6M0 - 6n^2M_1 + 12nM_1 - 6M_1 + 5nM_0 + 10M_0 + 5n^2M_1 - 20nM_1 \\ = 120n - 33 \\ n^2(M_1 - 6M1 + 5M_1) + n(M_0 - 6M_0 + 12M_1 + 5M_1 - 20M_1) - 4M_0 + 14M_1 \\ = 120n - 33 \end{aligned}$$

$$\begin{aligned} -8M_1 &= 120 \\ M_1 &= -15 \\ \\ -4M_0 + 14M_1 &= -33 \\ M_0 &= -\tfrac{177}{4}. \end{aligned}$$

Then $G(n) + P(n) = A \cdot 5^n + B - 15n - \frac{177}{4}n^2$ and (using our values for u_0 and u_1)

$$A = B - 9$$

$$\begin{aligned} 30 &= 5A + B - 15 - \tfrac{177}{4} \\ &= 5(B-9) + B - 15 - \tfrac{177}{4} \\ B &= \tfrac{179}{8} \Leftrightarrow A = \tfrac{501}{4} \end{aligned}$$

and

$$b_n = (\tfrac{501}{4}) \cdot 5^n + \tfrac{179}{8} - 15n - (\tfrac{177}{4})n^2$$

2. The reproduction of flora on planet Zod can be described as a homogeneous second order difference equation with constant coefficients

$$u_n - u_{n-1} - 6u_{n-2} = 0,$$

where $u_0 = u_1 = 1$. Let $g(x) = \sum_{i=0}^{\infty} u_i x^i$ be the generating function for the corresponding sequence $(u_i)_{i=0}^{\infty}$, then

$$\begin{array}{rllll} & N=0 & N=1 & N=2 & \\ \\ g(x) = & u_0 & +u_1 x & +u_2 x^2 & +\cdots \\ -xg(x) = & & -u_0 x & -u_1 x^2 & +\cdots \\ -6x^2 g(x) = & & & -6u_0 x^2 & +\cdots\,. \end{array}$$

The sum of the left hand sides of of the equations above is $(1-x-6x^2)g(x)$. Let the sum of the right hand sides of the equations above be G. We can see that for $N > 1$, the parts of G making up the coefficient of x^N take the same form as the difference equation (repeated up to N), so the coefficient of x^N will be

$$u_N - u_{N-1} - 6u_{N-2} = 0$$

and we can therefore write

$$\begin{aligned} (1 - x - 6x^2)g(x) &= u_0 - x(u_1 - u_0) \\ g(x) &= \frac{1}{1 - x - 6x^2} \\ &= -\frac{1}{(2x+1)(3x-1)} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{(2x+1)(3x-1)} &= \frac{A}{2x+1} + \frac{B}{3x-1} \\ 1 &= A(3x-1) + B(2x+1) \end{aligned}$$

so $A = -\frac{2}{5}$, $B = \frac{3}{5}$ and

$$\begin{aligned} g(x) &= \frac{-\frac{2}{5}}{2x+1} + \frac{\frac{3}{5}}{3x-1} \\ &= \frac{3}{5}(3x-1)^{-1} - \frac{2}{5}(2x+1)^{-1} \\ &= \frac{3}{5}\sum_{i=0}^{\infty}(-3)^i x^i - \frac{2}{5}\sum_{i=0}^{\infty}(-2)^i x^i. \end{aligned}$$

The number of plants that the intergalactic botanist will have after n years (from an initial crop of 3) will be the coefficient of x^n in $3 \cdot g(x) = 3 \cdot \left(\frac{3}{5}(-3)^n - \frac{2}{5}(-2)^n\right)$.

3. (a) When $u_0 = 0$, the sequence is constant $u_n = 0$. Similarly, when $u_0 = 2$, $u_n = 2$.

(b)

(c)