Solutions Chapter 2

Solutions to Exercises 2.1.

1. If f is as given, then $f_y = 6x^3y - 15xy^2 + 12x^2y^3 - 5y^4$ and hence $f_{xy} = 18x^2y - 15y^2 + 24xy^3$.

2. Set $z = 2xy + x^n y^{2n}$. Then

$$\frac{\partial^2 z}{\partial x^2} = n(n-1)x^{n-2}y^{2n}, \qquad \frac{\partial^2 z}{\partial y^2} = 2n(2n-1)x^n y^{2n-2}.$$

Thus

$$2x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + 18z = \big(2n(n-1) - 2n(2n-1) + 18\big)x^n y^{2n} + 36xy.$$

Hence z satisfies

$$2x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + 18z = 36xy$$

if and only if

$$\big(2n(n-1) - 2n(2n-1) + 18\big)x^n y^{2n} = 0.$$

Since this must be true for all x and y, this implies that

$$2n(n-1) - 2n(2n-1) + 18 = 0$$

which implies that $n^2 - 9 = 0$. Thus, $n = \pm 3$.

Solutions to Exercises 2.2.

1. We see that

$$g_x = xy - 2y = y(x-2), \qquad g_y = 1/2x^2 - 2x + 2y^2.$$

We find that $(2,1), (2,-1), (0,0)$ and $(4,0)$ are the stationary points. The second derivatives are $g_{xx} = y, g_{yy} = 4y$ and $g_{xy} = x - 2$ which implies that $\Delta = 4y^2 - (x-2)^2$.

$\Delta(2,1) = 4$ and $g_{xx} > 0$ so $(2,1)$ is a local minimum.

$\Delta(2,-1) = 4$ and $g_{xx} < 0$ so $(2,-1)$ is a local maximum.

$\Delta(0,0) = -4$, so $(0,0)$ is a saddle point.

$\Delta(4,0) = -4$, so $(4,0)$ is a saddle point.

2. For the stationary points $(1,2)$ and $(1,-2)$ we find the Hessian.

$\Delta(1,2) = -16$ implying that $(1,2)$ is a saddle point.

$\Delta(1,-2) = -16$ implying that $(1,-2)$ is a saddle point.

3. We have

$$f_x = 3x^2 - 3 \tag{1}$$
$$f_y = 3y^2 - 3z \tag{2}$$
$$f_z = -3y + 4z. \tag{3}$$

All the above are 0 for stationary points. From (1) we see that $x = \pm 1$; furthermore the values of y and z do not depend on the value of x. Computing $4 \cdot (2) + 3 \cdot (3)$ gives $12y^2 - 9y = 0$, and hence $y = 0$ or $y = \frac{3}{4}$. Using (2) (or (3)) we then compute $z = 0$ for $y = 0$, and $z = \frac{9}{16}$ for $y = \frac{3}{4}$. It follows that that the stationary points of f are

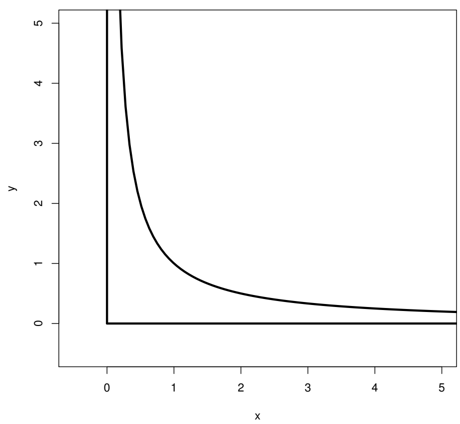
$$(1,0,0), \quad (1, \tfrac{3}{4}, \tfrac{9}{16}), \quad (-1,0,0) \text{ and } (-1, \tfrac{3}{4}, \tfrac{9}{16}).$$

Solutions to Exercises 2.3.

1. We first find the stationary points of f in the open interval $(0,5)$. We have $f'(x) = 1 + \cos x$, hence $f'(x) = 0$ if and only if $\cos x = -1$, which is the case for $x = \pm\pi, \pm 3\pi, \ldots$. Only π is in the interval $(0,5)$, and $f(\pi) = \pi \approx 3.14$. Checking the endpoints of the closed interval $[0,5]$ we find $f(0) = 0$ and $f(5) = 5 + \sin 5 \approx 4.04$.
(a) For the closed interval $[0,5]$ the global maximum value is $5+\sin 5$ occurring at the endpoint 5 and the global minimum value is 0 occurring at the endpoint 0.
(b) For the interval $(0,5]$ the global maximum value is $5 + \sin 5$ occurring at the endpoint 5. However f does not have a global minimum on the interval, that is, there is no point $a \in (0,5]$ such that $f(a) \le f(x)$ for all $x \in (0,5]$.

2. The boundary is the set consisting of the branch of the hyperbola $xy = 1$ lying in the first quadrant, the non-negative x-axis (i.e. the positive x-axis together with the origin $(0,0)$) and the positive y-axis. In set notation this can be written as

$$\{(x,y) : x > 0 \text{ and } xy = 1\} \cup \{(x,0) : x \ge 0\} \cup \{(0,y) : y > 0\}.$$



The set U is not closed because it does not contain all of its boundary. The set U is not bounded because it contains points arbitrarily far from the origin.

Solutions to Exercises 2.4.

1. Set $L = xy - \lambda(18x^2 + 2y^2 - 25)$. The derivatives are

$$L_x = y - 36x\lambda \tag{4}$$
$$L_y = x - 4y\lambda \tag{5}$$
$$L_\lambda = -(18x^2 + 2y^2 - 25). \tag{6}$$

Set all the above $= 0$ to find possible maxima and minima. We see that $y \cdot (4) - 9x \cdot (5)$ implies that $y = \pm 3x$. Substituting $y = \pm 3x$ in (6) gives $-(18x^2 + 2 \cdot 9x^2 - 25) = 0$ and hence $x = \pm\frac{5}{6}$. Thus the stationary points of L (with the last coordinate omitted) are $(\pm\frac{5}{6}, \pm\frac{5}{2})$. Finding the values of f on all of these gives $f(\frac{5}{6}, \frac{5}{2}) = f(-\frac{5}{6}, -\frac{5}{2}) = \frac{25}{12}$ and $f(\frac{5}{6}, -\frac{5}{2}) = f(-\frac{5}{6}, \frac{5}{2}) = -\frac{25}{12}$. Thus the maximum of f is $\frac{25}{12}$ and the minimum is $-\frac{25}{12}$.

2. Set $L = 2x + y - \lambda \cdot g$, where $g(x, y) = x^2 + xy + 4y^2 + 2x + 16y + 7$ is the given constraint. The partial derivatives are

$$L_x = 2 - \lambda(2x + y + 2) \tag{7}$$
$$L_y = 1 - \lambda(x + 8y + 16) \tag{8}$$
$$L_\lambda = -g. \tag{9}$$

Setting $L_x = L_y = L_\lambda = 0$ and solving for λ in (7) and (8) we get

$$\frac{2}{2x + y + 2} = \frac{1}{x + 8y + 16}$$

which implies that $y = -2$. Using this and (9) we get that $x = \pm 3$.

So, possible points for extreme values are $(\pm 3, -2)$. We find $f(-3, -2) = -8$ and $f(3, -2) = 4$. Thus -8 is the minimum value and 4 is the maximum value.

3. Set $f = xy^2$ and $g = x^2 + y^2 - 1$. With $L = xy^2 - \lambda g$ we get

$$L_x = y^2 - 2x\lambda \tag{10}$$
$$L_y = 2xy - 2y\lambda \tag{11}$$
$$L_\lambda = -(x^2 + y^2 - 1). \tag{12}$$

Equate all the above to 0. We note that $y \cdot (10) - x \cdot (11) = y(y^2 - 2x^2)$, implying that $y = 0$ or $y = \pm\sqrt{2}x$.

If $y = 0$ then $x = \pm 1$ from (12).

If $y = \pm\sqrt{2}x$ then $x = \pm\frac{1}{\sqrt{3}}$ which implies that $y = \pm\frac{\sqrt{2}}{\sqrt{3}}$.

Thus, the points of interest are $(\pm 1, 0)$ and $(\pm\frac{1}{\sqrt{3}}, \pm\frac{\sqrt{2}}{\sqrt{3}})$. Evaluating f at these points gives

$$f(\pm 1, 0) = 0, \quad f(\tfrac{1}{\sqrt{3}}, \pm\tfrac{\sqrt{2}}{\sqrt{3}}) = \tfrac{2}{3\sqrt{3}}, \quad f(-\tfrac{1}{\sqrt{3}}, \pm\tfrac{\sqrt{2}}{\sqrt{3}}) = -\tfrac{2}{3\sqrt{3}}.$$

Hence $-\frac{2}{3\sqrt{3}}$ is the minimum and $\frac{2}{3\sqrt{3}}$ is the maximum.

Solutions to Exercises 2.5.

1. The distance function from the origin is given by $\sqrt{x^2+y^2+z^2}$. We will minimize the function $f(x,y,z) = x^2+y^2+z^2$; this is the distance function squared, therefore the minimum of this function will be the square of the minimum distance. So, we want to minimize $x^2+y^2+z^2$ subject to $g = 2y+4z-15 = 0$ and $h = z^2-4x^2-4y^2 = 0$. Let

$$L = x^2+y^2+z^2 - \lambda(2y+4z-15) - \mu(z^2-4x^2-4y^2).$$

Then

$$\begin{aligned} L_x &= 2x + 8x\mu && (13)\\ L_y &= 2y - 2\lambda + 8y\mu && (14)\\ L_z &= 2z - 4\lambda - 2\mu z && (15)\\ L_\lambda &= -(2y+4z-15) && (16)\\ L_\mu &= -(z^2-4x^2-4y^2). && (17) \end{aligned}$$

Set $L_x = L_y = L_z = L_\lambda = L_\mu = 0$. From (13), we see that either $x = 0$ or $\mu = -1/4$.

If $x = 0$ then (17) implies $z = \pm 2y$. From this along with (16) we get $y = 3/2$ and $y = -5/2$, respectively. Thus, points of interest are $(0, 3/2, 3)$ and $(0, -5/2, 5)$.

If $\mu = -1/4$ then $\lambda = 0$ by (14). But then (15) implies that $z = 0$, and from $z = 0$ we can deduce that $x = 0$ and $y = 0$ by (17). But $z = 0$ and $y = 0$ contradicts (16), thus we don't get any more points of interest.

Evaluating f at the two points we found, we obtain $f(0, 3/2, 3) = 45/4$ and $f(0, -5/2, 5) = 125/4$. The smaller of these two values is the global minimum of f subject to the constraints. Hence $\sqrt{45/4} = \frac{3}{2}\sqrt{5}$ is the minimum distance.

Solutions to Exercises 2.6.

1.

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}t} =& \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\\ =&(2x+2y)\cdot 3 + (2x-2y)\cdot 2t\\ =&(2(3t-1)+2t^2)\cdot 3 + (2(3t-1)-2t^2)\cdot 2t\\ =&-4t^3+18t^2+14t-6 \end{aligned}$$

2. (a) Let $z = f(u)$ with $u = xy$. Then,

$$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du}y$$
$$\frac{\partial z}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = \frac{df}{du}x.$$

Hence $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{df}{du}xy - \frac{df}{du}xy = 0$ as required.

(b) For $f(t) = 2t^3 - t^2 + 3t$ we see that $z = f(xy) = 2x^3y^3 - x^2y^2 + 3xy$. So,

$$z_x = 6x^2y^3 - 2xy^2 + 3y$$
$$z_y = 6x^3y^2 - 2x^2y + 3x.$$

Hence $xz_x - yz_y = 0$.

3. Let $F = F(u, v)$. We find

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{\partial F}{\partial u}2x + \frac{\partial F}{\partial v}y, \\ \frac{\partial F}{\partial y} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \\ &= \frac{\partial F}{\partial u}2y + \frac{\partial F}{\partial v}x.\end{aligned}$$

Then, we see that

$$x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}(2x^2 - 2y^2),$$

so

$$\frac{\partial F}{\partial u} = \frac{1}{2x^2 - 2y^2}\left(x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y}\right).$$

Similarly

$$y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} = \frac{\partial F}{\partial v}(y^2 - x^2),$$

so

$$\frac{\partial F}{\partial v} = \frac{1}{y^2 - x^2}\left(y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y}\right).$$

4. (a) We have

$$\begin{aligned}g(\lambda x, \lambda y) &= \frac{3(\lambda x)(\lambda y)^3 - 2(\lambda x)^2(\lambda y)^2}{4(\lambda x) + 5(\lambda y)} \\ &= \frac{\lambda^4(3xy^3 - 2x^2y^2)}{\lambda(4x + 5y)} \\ &= \lambda^3 g(x, y).\end{aligned}$$

(b) We have

$$\begin{aligned}
x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} =& x\frac{(3y^3 - 4xy^2)(4x+5y) - (3xy^3 - 2x^2y^2)\cdot 4}{(4x+5y)^2} \\
&+ y\frac{(9xy^2 - 4x^2y)(4x+5y) - (3xy^3 - 2x^2y^2)\cdot 5}{(4x+5y)^2} \\
=& \frac{(12xy^3 - 8x^2y^2)(4x+5y) - (3xy^3 - 2x^2y^2)(4x+5y)}{(4x+5y)^2} \\
=& \frac{9xy^3 - 6x^2y^2}{4x+5y} \\
=& 3g(x,y)
\end{aligned}$$

5. Let $u : \mathbb{R} \to \mathbb{R}^n$ be the function $u(t) = \big(u_1(t), u_2(t), \dots, u_n(t)\big)$. Then $F(t) = f(u(t))$. The chain rule gives the equation of matrices $\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t}$, i.e.

$$\left(\tfrac{\mathrm{d}F}{\mathrm{d}t}\right) = \begin{pmatrix} \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} & \cdots & \frac{\partial f}{\partial u_n} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}t} \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}u_n}{\mathrm{d}t} \end{pmatrix},$$

where $\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)$ is the derivative $\frac{\mathrm{d}F}{\mathrm{d}t}$ considered as a 1×1-matrix. Hence

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial f}{\partial u_1}\frac{\mathrm{d}u_1}{\mathrm{d}t} + \frac{\partial f}{\partial u_2}\frac{\mathrm{d}u_2}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial u_n}\frac{\mathrm{d}u_n}{\mathrm{d}t}.$$

Solutions to Exercises 2.7.

1. For the Taylor polynomial of degree 2 centred at $(0,0)$ we need

$$\begin{aligned}
g(0+h, 0+k) \approx& g(0,0) + hg_x(0,0) + kg_y(0,0) \\
&+ \frac{1}{2}(h^2 g_{xx}(0,0) + 2hkg_{xy}(0,0) + k^2 g_{yy}(0,0)).
\end{aligned}$$

Computing all the relevant derivatives and evaluating them at $(0,0)$ gives

$$\begin{aligned}
g(0,0) &= \frac{1}{2}, \quad g_x(0,0) = -\frac{1}{4}, \quad g_y(0,0) = 0, \\
g_{xx}(0,0) &= \frac{1}{4}, \quad g_{xy}(0,0) = \frac{1}{2}, \quad g_{yy}(0,0) = 0.
\end{aligned}$$

Hence the Taylor polynomial is

$$\frac{1}{2} - \frac{1}{4}h + \frac{1}{8}h^2 + \frac{1}{2}hk.$$

Substituting $h = 0.1$ and $k = 0.2$, we estimate $g(0.1, 0.2) \approx 0.48625$. Note that $g(0.1, 0.2) = 0.48571\ldots$.