Calc 2 Final 2012
Solutions

1) a)

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2+3(x+h)-x^2-3x}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2+3x+3h-x^2-3x}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2+3h}{h}$$

$$= \lim_{h\to 0} 2x+h+3$$

$$= 2x+3$$

b)

$$f_x = 2x+3y+3x^2y^2$$
$$f_y = 3x+2x^3y$$

$$f_x(-1,2) = 2(-1)+3\cdot 2+3(-1)^2(2)^2$$
$$= -2+6+12$$
$$= 16$$

$$f_y(-1,2) = 3(-1)+2(-1)^3\cdot 2$$
$$= -3+-4$$
$$= -7$$

$$f(-1,2) = (-1)^2+3(-1)\cdot 2+(-1)^3\cdot 2^2 = 1+-3\cdot 2-4 = -9$$

Then the equation of the tan plane is given by

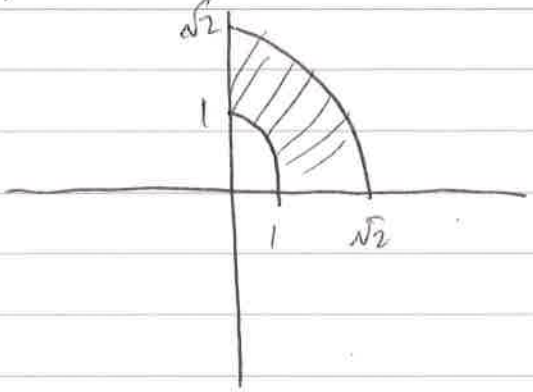
$$\begin{pmatrix} x+1 \\ y-2 \\ z+9 \end{pmatrix} \cdot \begin{pmatrix} 16 \\ -7 \\ -1 \end{pmatrix} = 0$$

$$\Rightarrow 16(x+1)-7(y-2)-(z+9)=0$$

$$\Rightarrow 16x+16-7y+14-z-9=0$$

$$\Rightarrow z = 16x-7y+21$$

2) The region is



Best use polar coords. The $1 \le r^2 \le 2$ and $0 \le \theta \le \pi/2$.
Then

$$\iint_D x^2 \, dx\,dy = \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2 \cos^2\theta \; r \, dr \, d\theta$$

$$= \underbrace{\int_0^{\pi/2} \cos^2\theta \, d\theta}_{①} \cdot \underbrace{\int_1^{\sqrt{2}} r^3 \, dr}_{②}$$

$\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$ ∴ ① becomes

$$\frac{1}{2}\int_0^{\pi/2} \cos 2\theta + 1 \, d\theta = \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta \right|_0^{\pi/2}$$

$$= \frac{1}{2}\left[0 + \pi/2 - 0 \right]$$

$$= \frac{\pi}{4}$$

② evaluates as

$$\int_1^{\sqrt{2}} r^3 \, dr = \left[\frac{r^4}{4} \Big|_1^{\sqrt{2}} \right]$$

$$= \frac{4}{4} - \frac{1}{4} = 3/4$$

∴ ① * ② $= \dfrac{\pi}{4} \cdot \dfrac{3}{4} = \dfrac{3\pi}{16}$

3) The Lagrangian is

$$L = x+y - \lambda(x^2+y^2-1)$$

$\therefore$ $L_x = 1 - 2x\lambda$ ①

1 $L_y = 1 - 2y\lambda.$ ②

$L_\lambda = x^2+y^2-1.$ ③

Setting all the above to 0 we get

$$\lambda(2x-2y) = 0 \quad (\text{from } ①=0 \text{ and } ②=0)$$

1 $\therefore$ $x=y$ and setting $x=y$ in ③ we get

$$2x^2 = 1$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ We have two ~~solution~~ extrema $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$. The extreme values are

2

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

and $$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}}.$$

1 These are global extrema on our constraint because our constraint is a circle (closed & bounded.

Note: −0.5 for not finding values.

4) a) $x = e^t$ then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$= \frac{dy}{dx} x$$

1

Since $\frac{dx}{dt} = x$. Also,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{dx}{dt}\frac{d}{dx}\left(x\frac{dy}{dx}\right)$$

2

$$= x \cdot \left(1 \cdot \frac{dy}{dx} + x\frac{d^2y}{dx^2}\right)$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

b) From parta) we have.

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = x^2\frac{d^2y}{dx^2}$$

∴

$$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 12y = 0$$

2

$$\Rightarrow \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 5\frac{dy}{dt} + 12y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 12y = 0$$

→ const coefficients.

5) This is a linear first order DE. We can make it exact by finding integrating factor. The integrating factor is.

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

$$\therefore \quad x\frac{dy}{dx} + y - 5x^2 = 0$$

is exact.

Let $f(x,y) = C$ be the solution. Then $f_y = x \Rightarrow f = xy + g(x)$. Then $f_x = y - 5x^2$ which implies

$$y + g'(x) = y - 5x^2$$

$$\Rightarrow \quad g'(x) = -5x^2$$

$$\Rightarrow \quad g(x) = -\frac{5}{3}x^3.$$

$$\therefore \quad f = xy - \frac{5}{3}x^3 = c.$$

Since $y(1) = 1$, we have

$$1 \cdot 1 - \frac{5}{3} 1 = c$$

$$\Rightarrow \quad -\frac{2}{3} = c$$

$\therefore$ our solution is

$$xy - \frac{5}{3}x^3 = -\frac{2}{3}$$

6a) We use a Taylor series about 1 ie

$$y(x) = y(1) + y'(1)(x-1) + y''(1)\frac{(x-1)^2}{2!} + \dots$$ 1/2

$y' = x+y \Rightarrow y'' = 1+y'$

$\Rightarrow y''' = y''$

$\Rightarrow y^{(iv)} = y'''$

etc.

We are given $y(1) = 1$, so

$$y'(1) = 1 + y(1)$$
$$= 2.$$

Further,

$$y''(1) = 1 + y'(1)$$
$$= 1+2$$
$$= 3$$

~~and $y^{(n)}(1) = y''(1) = 3$~~

and $y^{(n)}(1) = y''(1) = 3$. 1

∴ Our solution is

$$y(x) = 1 + 2(x-1) + \frac{3(x-1)^2}{2!} + \frac{3(x-1)^3}{3!} + \dots$$ 1/2

b) The higher derivative Euler method with 3 terms gives

$$y_{i+1} \approx y_i + h y_i' + \frac{h^2}{2} y_i''$$

$y(1) = 1$, so $y_0 = 1$ and $x_0 = 1$, $h = 0.1$

Now, $y' = x+y$ and $y'' = 1+y' = 1+x+y$.

$$\therefore \quad y_{i+1} = y_i + h(x_i+y_i) + \frac{h^2}{2!}(1+x_i+y_i)$$

$$\therefore \quad y_1 = 1 + 0.1(1+1) + \frac{0.1^2}{2}(1+1+1)$$

$$= 1 + 0.2 + 0.015$$

$$= 1.215$$

$\therefore \quad x_1 = 1.1$ and $y_1 = 1.215$

$$y_2 = 1.215 + 0.1(1.1+1.215) + \frac{0.1^2}{2}(1+1.1+1.215)$$

$$= 1.215 + 0.1(2.315) + 0.005(3.315)$$

$$= 1.215 + 0.2315 + .0166$$

$$= 1.4631$$

7) a) By Newton's law, $ma = \Sigma$ Forces, $\therefore$ $ma = +F_{drag} + F_{spring}$.

$$m\ddot{x} = -4\dot{x} - 4x$$

$$\Rightarrow m\ddot{x} + 4\dot{x} + 4x = 0 \quad \text{with } x(0) = 1,\ x'(0) = 0.$$

$$\Rightarrow \ddot{x} + 4\dot{x} + 4x = 0$$

b) The aux equation is $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$. $\therefore$ the general solution is

$$x = Ae^{-2t} + Bte^{-2t}$$

$$x(0) = 1 \Rightarrow A \cdot 1 + B \cdot 0 = 1$$

$$\Rightarrow A = 1$$

$$x'(0) = 0 \Rightarrow -2Ae^{-2t} + Be^{-2t} + -2Bte^{-2t}\Big|_{t=0} = 0$$

$$\Rightarrow -2 + B = 0$$

$$\Rightarrow B = 2.$$

$$\therefore x = e^{-2t} + 2te^{-2t},$$

c) The mass passes through equilibrium if there is a positive time solution to $x(t) = 0$

$$0 = e^{-2t} + 2te^{-2t}$$

$$\Rightarrow 0 = 1 + 2t$$

$$\Rightarrow t = -1/2$$

Since this is the only time solution for $x(t) = 0$ and it is less ~~that~~ than 0, the mass does not pass the equilibrium pt.

8) a)

$$\frac{d}{dx}\sinh 2x = 2\cosh 2x.$$
$$= 2\left(\cosh^2 x+\sinh^2 x\right)$$
$$= 2\cosh^2 x+2\sinh^2 x.$$

or

$$\frac{d}{dx}\sinh 2x = \frac{d}{dx}2\sinh x\cosh x$$
$$= 2\left(\cosh^2 x + \sinh^2 x\right) \quad \text{(by product rule)}$$
$$= 2\cosh^2 x+2\sinh^2 x$$

b) Set $\frac{dv}{dt} = e^{-t}$ and $u = t^{x-1}$

Integration by parts says

$$\int u\frac{dv}{dt}\,dt = uv - \int v\frac{du}{dt}\,dt$$

$$\therefore \quad \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$
$$= -t^{x-1}e^{-t}\Big|_0^\infty - \int_0^\infty (x-1)\,t^{x-2}\left(-e^{-t}\right)dt$$
$$= -t^{x-1}e^{-t}\Big|_0^\infty + \int_0^\infty (x-1)\,t^{x-2}e^{-t}\,dt.$$

The second summand is clearly $(x-1)\Gamma(x-1)$. The first is

$$-t^{x-1}e^{-t}\Big|_0^\infty = \lim_{r\to\infty} -r^{(x-1)}e^{-r} + 0\cdot 1$$
$$= 0+0$$
$$= 0$$

$$\therefore \quad \Gamma(x) = (x-1)\,\Gamma(x-1)$$

9a) $z = x^3 + y^3 - 3xy^2 + 3/_2\, y^2 + 5$.

i) $z_x = 3x^2 - 3y^2$

$z_y = 3y^2 - 6xy + 3y$

ii) $z_x = z_y = 0$ for stat pts.

Therefore

$z_x = 0 \Rightarrow 3x^2 - 3y^2 = 0 \Rightarrow x = \pm y$.

$z_y = 0 \Rightarrow 3y^2 - 6xy + 3y = 0$

$\Rightarrow 3y(y - 2x + 1) = 0$

$\Rightarrow y = 0$ or $y = 2x - 1$

If $y = 0$ and $x = \pm y$, we have $(0,0)$ is a stationary pt.

If $y = 2x - 1$ and $x = \pm y$ we have

$(x = y)$: $y = 2y - 1 \Rightarrow y = 1 \Rightarrow x = \cancel{\pm}1$

$x = -y$: $y = -2y - 1 \Rightarrow y = \frac{-1}{3} \Rightarrow x = \frac{1}{3}$

∴ Two additional pts $(1,1)$ and $\left(\frac{1}{3}, -1/_3\right)$

iii) $z_{xx} = 6x$ $\quad z_{yy} = 6y - 6x + 3$

$z_{xy} = -6y$.

iv) $\Delta = z_{xx} z_{yy} - z_{xy}^2$

$= 6x(6y - 6x + 3) - 36y^2$

$= 36xy - 36x^2 + 18x - 36y^2$.

$\Delta(0,0) = 0$ no information from Hessian

$\Delta(1,1) = 36 - 36 + 18 - 36 < 0$

$\therefore$ $(1,1)$ is a saddle pt.

$\Delta(\frac{1}{3}, \frac{-1}{3}) = \frac{-36}{9} - \frac{36}{9} + \frac{18}{3} - \frac{36}{9} < 0$

$(\frac{1}{3}, -\frac{1}{3})$ ~~also~~ is also a saddle pt.

b) $g(x,y) = f(u)$ and $u = 2x^3 + 3y^2$.

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} 6x^2.$$

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial f}{\partial u} 6y.$$

$$\therefore \quad y \cdot \frac{\partial g}{\partial x} = x^2 \frac{\partial g}{\partial y}$$

c) $f(1+h, 1+k) = f(1,1) + h f_x(1,1) + k f_y'(1,1)$

1

$$+\frac{1}{2}\left(h^2 f_{xx}(1,1)\right) + 2hk\, f_{xy}(1,1) + k^2 f_y(1,1)\Big)$$

$f(1,1) = e.$

$f_x(1,1) = y^2 e^{xy} \Big|_{(1,1)} = e$

$f_y(1,1) = e^{xy} + xye^{xy} \Big|_{(1,1)} = 2e$

3

$f_{xx}(1,1) = y^3 e^{xy} \Big|_{(1,1)} = e$

$f_{xy}(1,1) = 2ye^{xy} + y^2 x e^{xy} \Big|_{(1,1)} = 2e + e = 3e$

$f_{yy}(1,1) = xe^{xy} + xe^{xy} + x^2 y e^{xy} \Big|_{(1,1)} = 3e.$

$\therefore f(1+h, 1+k) = e + he + 2ek + \frac{1}{2}\left(h^2 e + 6ehk + 3ek^2\right)$

$f(1.1, 0,9) \approx e + 0.1e + 2e(-0,1) + \frac{1}{2}\left(0.1^2 e + 6e(0.1)(-0.1) + 3e(0.1)^2\right)$

$\cancel{\approx}\ e\left(1 + 0,1 - 0,2 + \frac{1}{2}\left(0,1^2 - 6(0.1)^2 + 3(0.1)^2\right)\right)$

1

$= e\left(0.9 + \frac{1}{2}(0.1)^2(1 - 6 + 3)\right)$

$= e\left(0,9 + .005(-2)\right)$

$= e\left(0,9 - 0,01\right)$

$= e \cdot 0.89$

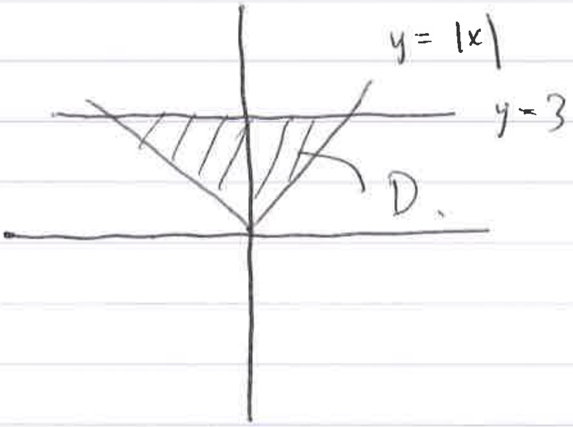
10) a) Since $\lim_{h\to0} e^h - \cos h = 0$ and $\lim_{h\to0} h = 0$, we can apply L'Hôpital's rule. ∴

$$\lim_{h\to0} \frac{e^h - \cos h}{h} = \lim_{h\to0} \frac{e^h + \sin h}{1}$$

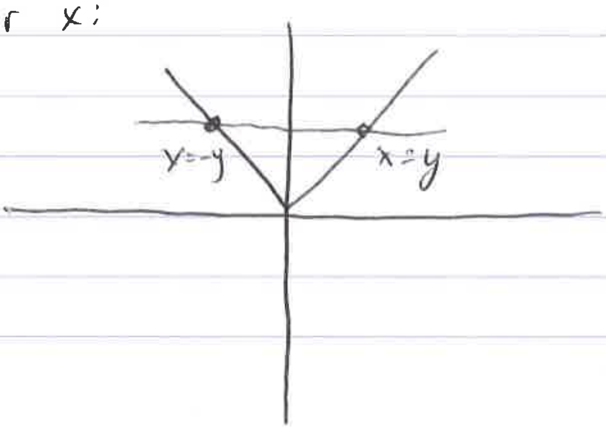
$$= \frac{\lim_{h\to0} e^h + \lim_{h\to0} \sin h}{\lim_{h\to0} 1}$$

$$= \frac{1+0}{1} = 1.$$

b) We sketch D.



We can evaluate the integral by noting limits for x:



$x: -y \to y$

limits for y: 0→3.

2

$$\iint_D 5x^2y+2\,dx\,dy = \int_0^3\int_{-y}^{y} 5x^2y+2\,dx\,dy$$

$$= \int_0^3 \left(\frac{5x^3}{3}y+2x \Big|_{-y}^{y}\right) dy$$

$$= \int_0^3 \frac{5}{3}\cdot y^3y + 2y - \left(\frac{5}{3}(-y)^3y + 2(-y)\right) dy$$

$$= 2\int_0^3 5/3\, y^4 + 2y\ dy$$

3

$$= 2\left(\frac{y^5}{3} + y^2 \Big|_0^3\right)$$

$$= 2\left(\frac{3^5}{3} + 3^2 - 0\right)$$

$$= 2\left(3^4 + 3^2\right)$$

$$= 2\cdot(81+9)$$

$$= 180$$

c) We make the substitution $u = xy$ and $v = \frac{y}{x^2}$ ①

Recall that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2} \end{vmatrix} = \frac{y}{x^2} + \frac{2y}{x^2} = \frac{3y}{x^2}$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{x^2}{3y}$$ ②

limits for u, v: $1 \le u \le 3$ and $1 \le v \le 2$. ①

$$\therefore \quad \iint_R y^3 \, dx \, dy = \int_1^3 \int_1^2 y^3 \cdot \frac{\partial(x,y)}{\partial(u,v)} \cdot dv \, du$$

$$= \int_1^3 \int_1^2 y^3 \cdot \frac{x^2}{3y} \, dv \, du$$

$$= \frac{1}{3} \int_1^3 \int_1^2 x^2 y^2 \, dv \, du$$

③ $$= \frac{1}{3} \int_1^3 \int_1^2 u^2 \, dv \, du$$

$$= \frac{1}{3} \int_1^3 u^2 \, du \int_1^2 dv$$

$$= \frac{1}{3} \left(\frac{u^3}{3} \Big|_1^3 \right) \cdot \left(v \Big|_1^2 \right)$$

$$= \frac{1}{3} \left(9 - \frac{1}{3} \right) \cdot (2-1)$$

$$= \frac{1}{3} \cdot \frac{26}{3} = \frac{26}{9}$$

d) $\underbrace{(x+y)}_{N}\frac{dy}{dx} + \underbrace{y+e^{x}}_{M} = 0$

1 $\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial M}{\partial y} = 1 \qquad \therefore$ Exact.

Let $f(x,y)=c$ be the solution.

$f_x = M \Rightarrow f = yx+e^{x}+g(y)$

$f_y = N \Rightarrow x+g'(y) = x+y$

2 $\therefore g'(y) = y$

$\Rightarrow g(y) = \frac{y^2}{2}$

$\therefore f(x,y) = yx+e^{x}+y^2/2 = c$

$y(0)=1 \Rightarrow 1\cdot 0+1+1/2 = C$

$C=3/2.$

1 $\therefore$ The solution is

$$yx+e^{x}+\frac{y^2}{2} = 3/2.$$

1) a) i) Using Newton's laws, since gravity is the only force on the object we have

$$ma = -\frac{mgR}{(R+y)^2}$$

$$\Rightarrow \quad m\frac{dv}{dt} = \frac{-mgR^2}{(R+y)^2}$$

ii) $$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

∴ We have

$$mv\frac{dv}{dy} = \frac{-mgR^2}{(R+y)^2}$$

$$\Rightarrow \quad v\frac{dv}{dy} = -\frac{gR^2}{(R+y)^2}$$

iii) The equation is separable.

$$v\,dv = \frac{-gR^2}{(R+y)^2}\,dy$$

$$\Rightarrow \quad \frac{v^2}{2} = +\frac{gR^2}{(R+y)} + c.$$

b) i) $4x+y=2$ and $x+y=-1$. Subtracting equations we have $3x=3$ $\Rightarrow x=1 \Rightarrow y=-2$. $\therefore$ $(1,-2)$ is the intersection. 1

ii) According to the notes, the good substitution $X=x-1$ and $Y=y+2$. 1

Then

$$\frac{4x+y-2}{x+y+1} = \frac{4(X+1)+(Y-2)-2}{X+1+Y-2+1}$$

$$= \frac{4X+4+Y-2-2}{X+Y}$$

$$= \frac{4X+Y}{X+Y}$$ 1

Since LHS is $\frac{dy}{dx} = \frac{dY}{dX}$, we have

1 $$\frac{dY}{dX} = \frac{4X+Y}{X+Y} \Rightarrow \frac{dY}{dX}(X+Y)-4X-Y=0 \quad ①$$

iii) Let $M(X,Y)=X+Y$ and $N(X,Y)=-4X-Y$. Then

2

$$M(\lambda X,\lambda Y) = \lambda X+\lambda Y = \lambda(X+Y) = \lambda M(X,Y)$$

and $$N(\lambda X,\lambda Y) = -4(\lambda X)-\lambda Y = \lambda(-4X-Y) = \lambda N(X,Y)$$

$\therefore$ both are homogeneous of degree 1

iv) The good substitution is $Y=VX$. Then $\frac{dY}{dX} = V+X\frac{dV}{dX}$.
$\therefore$ ① becomes 1

$$(X+VX)\left(V+X\frac{dV}{dX}\right) - 4X-XV=0$$

$$\Rightarrow (1+V)\left(V+X\frac{dV}{dX}\right)-4-V=0$$

$$\Rightarrow \quad V+V^2 + (1+V)X\frac{dV}{dX} - 4 - V = 0$$

$$\Rightarrow \quad (1+V)X\frac{dV}{dX} = 4 - V^2$$

$$\Rightarrow \quad \frac{1+V}{4-V^2}\,\frac{dV}{\cancel{dX}} = \frac{dX}{X} \qquad ②$$

$$\frac{1+V}{4-V^2} = \frac{A}{2-V} + \frac{B}{2+V} = \frac{A(2+V)+B(2-V)}{4-V^2}$$

$$= \frac{2A+2B+V(A-B)}{4-V^2}$$

$\therefore$ $2A+2B=1$

$A-B=1 \Rightarrow 2A-2B=2$

$\Rightarrow A=3/4 \quad B=-1/4.$

$\therefore$ ② becomes

$$\left(\frac{3}{4}\,\frac{1}{2-V} - \frac{1}{4}\,\frac{1}{2+V}\right)dV = \frac{dX}{X}$$

$$\Rightarrow \quad \frac{3\,dV}{2-V} - \frac{dV}{2+V} = \frac{4\,dX}{X}$$

$$\Rightarrow \quad -3\ln(2-V) - \ln(2+V) = 4\ln X + C$$

$$\Rightarrow \quad \ln(2-V)^{+3} + \ln(2+V) = \ln X^{-4} + C$$

v) $$\Rightarrow \quad (2-V)^3\cdot(2+V) = AX^{-4}$$

$$\Rightarrow \quad \left(2-\frac{Y}{X}\right)^3\left(2+\frac{Y}{X}\right) = AX^{-4} \quad (A \text{ a constant})$$

$$\Rightarrow \quad (2X-Y)^3\cdot(2X+Y) = A$$

$$\Rightarrow \left(2(x-1)-(y+2)\right)^3 \cdot \left(2(x-1)+(y+2)\right) = A$$

$$\Rightarrow (2x-y-4)^3 \cdot (2x+y) = A.$$

12a) i) $$\int_{\sqrt{3}}^{3} \frac{1}{\sqrt{3+x^2}}\,dx = \frac{1}{\sqrt{3}} \int_{\sqrt{3}}^{3} \frac{1}{\sqrt{1+\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx$$

$$= \frac{\sqrt{3}}{\sqrt{3}} \operatorname{arc\,sinh}\left(\frac{x}{\sqrt{3}}\right)\Big|_{\sqrt{3}}^{3}$$

$$= \operatorname{arc\,sinh}(\sqrt{3}) - \operatorname{arc\,sinh}(1) \quad ①$$

Using $\operatorname{arcsinh} x = \ln(x+\sqrt{x^2+1})$ we have ① becomes

$$= \ln(\sqrt{3}+\sqrt{3+1}) - \ln(1+\sqrt{1+1})$$

$$= \ln(\sqrt{3}+2) - \ln(1+\sqrt{2})$$

$$= \ln\left(\frac{\sqrt{3}+2}{1+\sqrt{2}}\right)$$

2

ii) Let $z = x+iy$. Then ~~cos z~~

$$\cos z = \cos(x+iy)$$
$$= \cos x \cos iy - \sin x \sinh iy$$
$$= \cos x \cosh y - i \sin x \sinh y$$

If $\cos z = 2$, $\sin x = 0$ or $\sinh y = 0$

$\Rightarrow x = n\pi,\ n \in \mathbb{Z}$ or $y = 0$.

If $y = 0$, then $\cosh y = 1 \Rightarrow \cos x = 2$. No solution.

If $x = n\pi$, then $\cos x = (-1)^n$. Then $(-1)^n \cosh y = 2$

2. If n is odd there is no solution. If n is even

$y = \pm \operatorname{arc\,cosh}(2) = \pm \ln(2+\sqrt{2^2-1}) = \pm \ln(2+\sqrt{3})$

$\therefore$ The solutions are

$$z = 2n\pi \pm i \ln(2+\sqrt{3}),\quad n \in \mathbb{Z}$$

b) i) Set $t = \sin^2\theta$ 1. Then $t=0 \Rightarrow \theta = 0$ and 1 $t = 1 \Rightarrow \theta = \pi/2$.
Also, $\frac{dt}{d\theta} = 2\sin\theta\cos\theta$. 1 Therefore,

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^{\pi/2} \left(\sin^2\theta\right)^{x-1}\left(\cos^2\theta\right)^{y-1}\frac{dt}{d\theta}\,d\theta$$

2 $$= \int_0^{\pi/2} \sin^{2x-2}\theta\,\cos^{2y-2}\theta\;2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,d\theta$$

ii) $$\int_0^{\pi} \cos^6\theta\,d\theta = \underset{\textcircled{1}}{\underset{\shortparallel}{\int_0^{\pi/2} \cos^6\theta\,d\theta}} + \underset{\boxed{2}}{\underset{\shortparallel}{\int_{\pi/2}^{\pi} \cos^6\theta\,d\theta}}$$ 1

$$\textcircled{1} = \frac{1}{2}\,2\int_0^{\pi/2} \cos^6\theta\,d\theta$$

$$= \frac{1}{2}\,B\left(\tfrac{1}{2}, \tfrac{7}{2}\right) \qquad \text{(by part b)i))}$$

$$= \frac{1}{2}\,\frac{\Gamma(1/2)\,\Gamma(7/2)}{\Gamma(4)}$$

$$= \frac{1}{2}\,\frac{\sqrt{\pi}\cdot 5/2 \cdot 3/2 \cdot 1/2\,\pi^{1/2}}{3!}$$ ✓

$$= \frac{15}{2^4 \cdot 3!}\,\pi = \frac{5\pi}{32}$$

② becomes by a change of vars $u = \theta - \pi/2$:
when $\theta = \pi/2$, $u = 0$ and when $\theta = \pi$, $u = \pi/2$.
Also $\frac{du}{d\theta} = 1$. Also,

$$\cos\theta = \cos\left(u + \pi/2\right) = \cos u \cos\pi/2 - \sin u \sin\pi/2$$
$$= -\sin u.$$

∴ ② becomes

$$\int_0^{\pi/2} (-\sin u)^6 \, du = \int_0^{\pi/2} \sin^6 u \, du$$

$$= \frac{1}{2} B(7/2, 1/2)$$

$$= \frac{5\pi}{32}$$

Since $B(x,y) = B(y,x)$, ∴ ① + ② $= \frac{5\pi}{16}$ 2

Note: By making the subs $t \to \pi - t$, you can show ② = ①