BIRKBECK

(University of London)

BSc EXAMINATION
SCHOOL OF BUSINESS, ECONOMICS AND INFORMATICS

Calculus 2: Multivariate Differential Equations BUEM001S5

30 credits

Friday 5th June, 2015
10:00 a.m. - 13:00p.m.

This examination contains two sections: Section A (8 questions) and Section B (4 questions). Questions in Section A are worth 5 *marks each and questions in Section B are worth* 20 *marks each.*
Candidates should attempt **all** *of the questions in Section A and* **two** *questions out of the four in Section B.*
Candidates can use their own calculator, provided the model is on the circulated list of authorised calculators or has been approved by the chair of the Mathematics and Statistics Examination Sub-board.

Please turn over

Section A

1. (a) Explain why

$$\lim_{x\to 0}\frac{e^x+e^{-x}}{x^2}=\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{2}=1$$

is *not* correct.
Evaluate

$$\lim_{x\to 0}\frac{e^x+e^{-x}}{x^2}.$$

[2]

(b) Evaluate

$$\lim_{x\to\infty}\frac{\sqrt{9x^2+2}}{3-4x}.$$

[3]

2. (a) Let $U\subseteq\mathbb{R}$ and let $f:U\to\mathbb{R}$ be a function.

(i) Define what it means for f to be continuous at a point $a\in U$. [1]

(ii) State the definition of the derivative of f at $x\in U$. [1]

(b) Let $f:\{x\in\mathbb{R}:x\geq -1\}\to\mathbb{R}$ be the function $f(x)=\sqrt{x+1}$. Use the formal definition of the derivative to find $f'(x)$ for $x>-1$. If any limits appear as part of your computation, you must evaluate them without using L'Hôpital's rule. [3]

3. Evaluate the integral

$$\iint_D (x^3+4y)\,dxdy,$$

where D is the region in the $xy-$ plane bounded by the graphs of the equations $y=x^2$ and $y=2x$. [5]

4. Consider the differential equation

$$y'=1+x^2+y.$$

Assume that the initial condition is $y(1)=0$. Use the method of *Taylor series* about the point $x=1$ to find the first five terms of the Taylor series of y about that point. [5]

5. Solve the following differential equation using any appropriate method.

$$\frac{1}{x}\frac{dy}{dx}-y=e^{x^2/2}.$$

[5]

Please turn over

6. Show that the differential equation

$$4x^3 - y^2 = 2xy\,\frac{dy}{dx}$$

is an exact differential equation. Solve it using the appropriate method subject to the condition that $y = 3$ when $x = 2$. Express y explicitly as a function of x. [5]

7. Radium decays exponentially and has a half-life of approximately 1600 years; that is, given any quantity, one-half of it will disintegrate in 1600 years.

   (a) Find a formula for the quantity $q(t)$ of radium (in mg) at time t (in years). It is known that at time $t = 0$ years the quantity of pure radium is $50\,mg$. [3]

   (b) When will there be $20\,mg$ of radium left? [2]

8. Recall that the hyperbolic function $\tanh x$ is defined, for all $x \in \mathbb{R}$, by

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

   (a) State the domain and codomain for the *inverse* hyperbolic function $\operatorname{arctanh} x$. [1]

   (b) Deduce the explicit formula for the *inverse* hyperbolic function $\operatorname{arctanh} x$. [4]

Please turn over

Section B

9. (a) Let $U \subseteq \mathbb{R}^2$ and consider $f : U \to \mathbb{R}$ be a function.
Define the following terms, for point $(a, b) \in U$:

(i) stationary point of f,

(ii) local minimum of f, and

(iii) global minimum of f. [3]

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^2 + 6xy + 6y^2 + 7.$$

(i) Compute the partial derivatives f_x, f_y, f_{xx}, f_{yy} and f_{xy}. [3]

(ii) Find and classify the stationary points of f. [5]

(iii) Investigate if this function has any global extrema and, if yes, identify them. [2]

(c) Consider now the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^2 \cos y + y^2.$$

(i) Find the *quadratic* Taylor approximation to f at the point $(0, 0)$. Note that $(0, 0)$ is a stationary point of the function f. [5]

(ii) Using the (c)(i) classify the point $(0, 0)$ as a local minimum, local maximum or saddle point. [2]

Please turn over

10. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = 5y^2 - x^2$.

(a) (i) Find the gradient vector of f and evaluate it at the point $(x, y) = (1, 1)$. [2]

(ii) Find the directional derivative of the function f at $(1, 1)$ in the direction of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. [3]

(iii) Write down the equation of the tangent plane to the surface $z = f(x, y) = 5y^2 - x^2$ at the point $(2, 0, f(2, 0))$. [3]

(b) Consider again the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 5y^2 - x^2$.

(i) Show that f is a homogeneous function of degree 2. [2]

(ii) Verify *Euler's Theorem* on homogeneous functions by evaluating $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$. [2]

(c) Consider the differential equation

$$5xy\frac{dy}{dx} = x^2 - 5y^2.$$

Our aim is to find the general solution of the differential equation.

(i) By setting $y = vx$, change the above differential equation into

$$5xv\frac{dv}{dx} = 1 - 10v^2.$$

Explain with a sentence why one would set $y = vx$ in the first instance. [3]

(ii) By solving the equation in (i), find the general solution to the original differential equation. [5]

Please turn over

11. (a) If D is the region bounded by $y = x^2,\ x = 3$ and $y = 0$, evaluate

$$\iint_D e^{x^3}\, dxdy.$$

[4]

(b) Consider the integral

$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2 + y^2)^{3/2}\, dydx.$$

(i) Assume the change of variables $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that the *Jacobean* of this change of variables is r. [2]

(ii) Use polar coordinates to evaluate the integral. [4]

(c) (i) State the definition of the Gamma function. State clearly the domain of the function. [1]

(ii) Show that the Gamma function $\Gamma(x)$ satisfies

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

for $x > 1$. [3]

(d) (i) Evaluate

$$\int_0^{\infty} x^4 e^{-3x} dx.$$

[3]

(ii) Prove that

$$2\int_0^2 x\sqrt[3]{1 - \left(\frac{x}{2}\right)^3}\, dx = \frac{8}{3}B\left(\frac{2}{3}, \frac{4}{3}\right).$$

[3]

Please turn over

12. (a) Suppose that a population of bacteria grows according to the logistic growth model. That is

$$\frac{dP}{dt} = rP(1 - \frac{P}{k}),$$

where $P := P(t)$ is the bacteria population as a function of time t, r is the growth proportionality constant and k is the carrying capacity.

(i) Suppose that $r = 20$ and $k = 1000$. Find the general solution to the differential equation. You may keep your answer in the form

$$\frac{P}{a - P} = Ae^{bt},$$

where a, b are numbers and A is the constant of integration. [4]

(ii) Find A if the initial population of bacteria is 500. [1]

(iii) How long will it take the population to reach 80% of its maximum size? Assume that the time is measured in days. [3]

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = e^{3x} + e^{2x}.$$

[7]

(c) Consider the differential equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 8y = 0.$$

(i) By substituting $x = e^t$ and using the chain rule, show that

$$\frac{dy}{dt} = x\frac{dy}{dx} \quad \text{and} \quad \frac{d^2y}{dt^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}.$$

[3]

(ii) Show that the above substitution (in (i)) turns the original differential equation into one with constant coefficients of the form

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 8y = 0.$$

You do not have to solve the equation. [2]