(12)

Neacșu-Miclea Liviu-Ștefan
Grupa 215

of 1p

Lucrare de control

Subiectul A

rândul 2

-9

3 Problema 1. Justificați cu definiția $\lim_{n\to\infty} \frac{3^n}{2^n+1} = +\infty$.

2 Problema 2. Calculați limita

$$\lim_{n\to\infty} \frac{1\cdot 1! + 2\cdot 2! + \dots + n\cdot n!}{(n+1)!}$$

-3 Problema 3. Calculați derivata de ordinul $n \in \mathbb{N}$ a funcției

$$f(x) = x\sqrt{x-1}, \quad x > 1$$

Problema 1.

Not. $x_n = \dfrac{3^n}{2^n+1}$, $n \in \mathbb{N}$

Dem că

(*) $(\forall)\ \varepsilon > 0\ (\exists)\ n_0 \in \mathbb{N}$ a.î. $x_n > \varepsilon\ (\forall)\ n \geq n_0 \left(\Longleftrightarrow \lim\limits_{n\to\infty} x_n = +\infty\right)$

Fie $\varepsilon > 0$ fixat.

Obs. că $x_n = \dfrac{3^n}{2^n+1} \geq \dfrac{3^n}{2^n+2^n} = \dfrac{3^n}{2\cdot 2^n} = \dfrac{1}{2}\left(\dfrac{3}{2}\right)^n \underset{(\text{impunem})}{>} \varepsilon.$

$\Longleftrightarrow \left(\dfrac{3}{2}\right)^n > 2\varepsilon \Longleftrightarrow n > \log_{\frac{3}{2}} 2\varepsilon.$

Alegem $n_0 = \max\left\{0, \left[\log_{\frac{3}{2}}(2\varepsilon) + 1\right]\right\}$ și obținem că $(\exists)\ n_0 \in \mathbb{N}$ a.î. $x_n > \varepsilon\ (\forall)\ n \geq n_0$. Cum ε a fost ales în mod arbitrar $\Rightarrow$ (*) adevărată $\Rightarrow \lim\limits_{n\to\infty} \dfrac{3^n}{2^n+1} = +\infty.$

Problema 2

$$x_n = 1\cdot 1! + 2\cdot 2! + \dots + n\cdot n! = \sum_{k=1}^{n} k\cdot k! = \sum_{k=1}^{n} (k+1-1)\, k!$$

$$= \sum_{k=1}^{n} \left((k+1)k! - k!\right) = \sum_{k=1}^{n} \left((k+1)! - k!\right) =$$

$$= \cancel{2!} - 1! + \cancel{3!} - \cancel{2!} + \cancel{\dots} + (n+1)! - \cancel{n!} = (n+1)! - 1$$

$$y_n = (n+1)!$$

$$\Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{(n+1)!-1}{(n+1)!} = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)!}\right) = 1 - 0 = 1.$$

Alternativ, $x_n > 0$, $y_n \nearrow$ cu $\lim_{n\to\infty} y_n = +\infty \Rightarrow$

putem aplica relația Stolz–Cesaro:

$$\lim_{n\to\infty} \frac{1\cdot 1! + 2\cdot 2! + \dots + n\cdot n!}{(n+1)!} = \lim_{n\to\infty} \frac{(1\cdot 1! + 2\cdot 2! + \dots + n\cdot n! + (n+1)(n+1)!) - (1\cdot 1! + \dots + n\cdot n!)}{(n+2)! - (n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot(n+1)!}{(n+1)!\,(n+2-1)} = \lim_{n\to\infty} \frac{\cancel{(n+1)}\,\cancel{(n+1)!}}{\cancel{(n+1)}\,\cancel{(n+1)!}} = 1.$$

Problema 3 $f^{(0)}(x) = x\sqrt{x-1}$, Calculăm $f^{(n)}$ pentru $n \geq 1$

Aplicăm formula lui Leibniz $(u \cdot v)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}$

pentru $u(x) = x$, $v(x) = \sqrt{x-1}$, $x \in (1, +\infty)$

$$\Rightarrow f^{(n)}(x) = (x\sqrt{x-1})^{(n)} = \sum_{k=0}^{n} C_n^k \cdot x^{(k)} \cdot (\sqrt{x-1})^{(n-k)}$$

$$= C_n^0\, x \cdot (\sqrt{x-1})^{(n)} + C_n^1\, x' (\sqrt{x-1})^{(n-1)} + C_n^2\, x'' (\sqrt{x-1})^{(n-2)} + \dots + C_n^n x^{(n)} \sqrt{x-1}.$$

$u(x) = x \Rightarrow u'(x) = 1 \Rightarrow u''(x) = 0 \Rightarrow u^{(k)}(x) = 0 \ (\forall)\ k \geq 2$

$$\Rightarrow f^{(n)}(x) = x \cdot (\sqrt{x-1})^{(n)} + n(\sqrt{x-1})^{(n-1)}.$$

Calculăm $v^{(k)}(x)$, $k \in \mathbb{N}^*$

$v(x) = (x-1)^{\frac{1}{2}}$

$v'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}} \overbrace{(x-1)'}^{=1}$

$v''(x) = -\frac{1}{4}(x-1)^{-\frac{3}{2}}$

$v'''(x) = \frac{3}{8}(x-1)^{-\frac{5}{2}}$

Presupunem că $v^{(k)}(x) = (-1)^{k+1} \cdot \frac{(2k-3)!!}{2^k} \cdot (x-1)^{-\frac{2k-1}{2}}$, $k \geq 2$ și demonstrăm inductiv formula pentru $v^{(k+1)}$. Evident relația este adevărată pentru $k=2$.

$$v^{(k+1)}(x) = (v^{(k)}(x))' = (-1)^{k+1} \cdot \frac{(2k-3)!!}{2^k} \cdot \left((x-1)^{-\frac{2k-1}{2}}\right)' =$$

$$= (-1)^{k+1} \cdot \frac{(2k-3)!!}{2^k} \cdot \left(-\frac{2k-1}{2}\right) \cdot (x-1)^{-\frac{2k-1}{2}-1} \Rightarrow$$

$\Rightarrow v^{(k+1)}(x) = (-1)^{k+2} \cdot \frac{(2(k-1))!!}{2^k} \cdot (x-1)^{-\frac{2k+1}{2}}$, relația de demonstrat.

Așadar,

$$f^{(n)}(x) = x \cdot (-1)^{n+1} \cdot \frac{(2n-3)!!}{2^n}(x-1)^{-\frac{2n-1}{2}} + (-1)^n \cdot n \cdot \frac{(2n-5)!!}{2^{n-1}}(x-1)^{-\frac{2n-3}{2}}$$

$$\Rightarrow f^{(n)}(x) = (-1)^n \cdot \frac{(2n-3)!!}{2^{n-1}} \cdot \frac{1}{\sqrt{(x-1)^{2n-3}}}\left(n(2n-5) - \frac{x}{2} \cdot \frac{1}{\sqrt{(x-1)^2}}\right)$$

$$\Rightarrow f^{(n)}(x) = (-1)^n \frac{(2n-3)!!}{2^{n-1}} \cdot \frac{2n^2 - 5n - \cancel{\frac{x}{2(x-1)}}}{\sqrt{(x-1)^{2n-3}}} \quad (\forall)\ n \geq 2,\ x \in (1, +\infty)$$

Pentru $n = 1$, $f'(x) = \frac{x}{2\sqrt{x-1}} + \sqrt{x-1} = \frac{x + 2(x-1)}{2\sqrt{x-1}}$ $(\forall)\ x \in (1, +\infty)$.

$$= \frac{3x-2}{2\sqrt{x-1}}.$$