PBR

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Abstract

Got tired of forgetting and confusing definitions for various quantities in radiometry and photometry that are needed for physically based rendering, and decided to derive and explain them in a way that I find useful.

1 Poynting Vector S

This section is not necessary for other sections. It just provides ground by showing where the Poynting vector comes from, since it is used to define radiant energy.

Electromagnetic force acting on local charges dq with same velocities $\mathbf{v}$ is the sum of electric and magnetic forces acting on them, and is equal to

$$\mathbf{F} = dq\mathbf{E} + dq\mathbf{v} \times \mathbf{B} = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

To derive the Poynting vector $\mathbf{S}$, we will start with the work-energy theorem that states that the change in kinetic energy is equal to the work done on the system (here charges) by the net force. If we denote the energy by W, assuming that it depends on time and volume, and charge movement vector by $\mathbf{r}$, then the average energy change is

$$\Delta W = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{\Delta r}$$

We get instantaneous energy change for infinitesimal movement $\mathbf{dr}$

$$dW = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{dr}$$

Further, since we know that the velocity is $\mathbf{v}$, then $\mathbf{dr} = \mathbf{v}dt$ and we get

$$dW = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt$$

This equation can be simplified by applying the dot product, since the dot product between $\mathbf{v}$ and $\mathbf{v} \times \mathbf{B}$ is always 0, no matter what $\mathbf{B}$ is (because of orthogonality).

$$dW = dq\mathbf{v} \cdot \mathbf{E}dt + (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt$$
$$dW = dq\mathbf{v} \cdot \mathbf{E}dt$$

Local charges dq that have the same velocity can be expressed via charge density ρ. Since we have velocity $\mathbf{v}$ in equation, we can then combine it with charge density to get current density $\mathbf{J}$. The reason for doing that, is that we can then express $\mathbf{J}$ using Ampere's Law and that way further develop the equation. First, if ρ represents volume charge density, we have

$$dq = \rho dV$$

Secondly, we show that $\mathbf{J} = \rho\mathbf{v}$. Current density $\mathbf{J}$ is defined as the limiting ratio between current flow through area A and the given area A that is orthogonal to the movement direction of charges

$$\mathbf{J} = \lim_{A \to 0} \frac{I}{A}$$

If we now observe some small time period Δt, we know that in that time, all charges that are at the distance $\mathbf{v}\Delta t$ from A will go through A (we observe only orthogonal movement in one direction). The amount of such charge, given the volume charge density ρ, is

$$\Delta Q = \rho \Delta V = \rho A \mathbf{v} \Delta t$$
$$\frac{\Delta Q}{A \Delta t} = \rho \mathbf{v}$$

As the time period and area become infinitesimal, we get

$$\lim_{A \to 0} \lim_{\Delta t \to 0} \frac{\Delta Q}{A \Delta t} = \lim_{A \to 0} \frac{dQ}{A dt} = \lim_{A \to 0} \frac{I}{A} = \mathbf{J}$$

Therefore, we have shown that

$$\mathbf{J} = \rho \mathbf{v}$$

Going back to the equation

$$dW = dq \mathbf{v} \cdot \mathbf{E} dt$$

and substituting $dq = \rho dV$, we get

$$dW = \rho dV \mathbf{v} \cdot \mathbf{E} dt$$

Using $J = \rho \mathbf{v}$, we get

$$dW = \mathbf{J} \cdot \mathbf{E} dV dt$$

Since we now have $\mathbf{J}$ in equation, we can express it using Ampere's Law

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$
$$\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}$$

Substituting this into equation for dW gives us

$$dW = (\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}) \cdot \mathbf{E} dV dt$$

$$dW = (\nabla \times \mathbf{H}) \cdot \mathbf{E} dV dt - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} dV dt$$

Now we can try to somehow transform this formula in order to get $(\nabla \times \mathbf{E})$ term, so that we can use Faraday's Law. For this, we will use the following general identity from vector analysis. If we have two arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$, then this holds

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

In formula for dW, we have $(\nabla \times \mathbf{H}) \cdot \mathbf{E}$, which in the previous general identity is the first term of RHS. In our case, this identity becomes

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}) = (\nabla \times \mathbf{H}) \cdot \mathbf{E} - \mathbf{H} \cdot (\nabla \times \mathbf{E})$$

This gives us

$$(\nabla \times \mathbf{H}) \cdot \mathbf{E} = \nabla \cdot (\mathbf{H} \times \mathbf{E}) + \mathbf{H} \cdot (\nabla \times \mathbf{E})$$

Substituting in formula for dW, we get

$$dW = (\nabla \cdot (\mathbf{H} \times \mathbf{E}) + \mathbf{H} \cdot (\nabla \times \mathbf{E})) dV dt - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} dV dt$$

The term $(\nabla \times \mathbf{E})$ now appears in the formula and we can use Faraday's Law to substitute another expression in its place. Faraday's Law is given by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Substituting in formula for dW, and rearranging the terms, we get the following

$$dW = (\nabla \cdot (\mathbf{H} \times \mathbf{E}) - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t})dVdt - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E}dVdt$$

$$dW = (-\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t})dVdt - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E}dVdt$$

$$dW = -(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t})dVdt - \nabla \cdot (\mathbf{E} \times \mathbf{H})dVdt$$

We can already see Poynting vector in this expression which is $\mathbf{S} = (\mathbf{E} \times \mathbf{H})$. But, we can transform it further by integrating it across volume and time and applying divergence theorem to the second term i.e. term holding the Poynting vector.

$$W = -\int_T \int_V (\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t})dVdt - \int_T \int_V (\nabla \cdot \mathbf{S})dVdt$$

$$W = -\int_T \int_V (\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t})dVdt - \int_T \int_{\partial V} (\mathbf{S} \cdot \mathbf{n})dAdt$$

In this expression, W represents the energy of the charges in volume V in the time period T. Left RHS term represents the energy that comes from electric and magnetic fields inside the volume V over time period T. Right RHS term represents the energy that enters or exits the given volume i.e. crosses the boundary ∂V over time T. This second term is more important for us because that is what describes the energy that is dissipated by light sources. From the fact that this term is energy, we can conclude that the Poynting vector represents energy per unit area per unit time. Its direction tells us how the energy propagates. We can also characterize it as transferred power per unit area. This notion is called intensity. Intuitively, if we have some energy change per unit time i.e. some power P, we can't conclude anything about the energy distribution. For example, let's imagine a light bulb with power P and a focused flashlight with the same power. Intuitively, we expect that the flashlight will be brighter in specific direction. But, this is not something that we can conclude just based on the value of P. In other words, P only tells us how much the energy itself changed i.e. how much of it passed (entered or exited) through some naturally assigned boundary in unit time and intensity tells us how much energy went through a unit piece of boundary in unit time. Therefore, if the power is equal, but one boundary is twice as large, then every piece of larger boundary will transfer less energy i.e. intensity there will be lower. Finally, we can get power by integrating intensity over some boundary.

2 Radiation direction and spatial angle measurement

Before we move on, one important thing to note about the radiation direction is that it is not modeled just as a unit vector. Rather, it is modeled as an infinitesimal spatial angle. This is because the energy is radiated in many directions, and if we want to measure it, we would choose some spatial angle to capture it. From this, if we are interested in only one direction, we can get to it by observing the limit where the spatial angle (that contains given direction) becomes infinitesimal. Another important thing to mention is how spatial angles are measured. Imagine a light source which is just a point that radiates energy in all directions. Before even talking of spatial angles, we place a unit sphere around that light such that the center is at the light. If we now pick some spatial angle (which doesn't need to be a regular shape, nor it needs to be in only one piece), the problem is how to measure it. The way we do that is by imagining vectors going from the center in all directions that are covered by the chosen spatial angle. These vectors will intersect the unit sphere, which was previously placed. All the points of intersection will form the area (or areas if spatial angle is not in one piece) on the unit sphere. Now, in order to measure the spatial angle, we just need to measure the area/areas formed by intersection points. This is the reason why spatial angles and areas on the unit sphere can be used interchangeably.

3 Radiant Energy Q, Radiant Flux Φ and Radiant Exposure H

If we imagine a closed surface Σ, then the radiant energy is the total energy that passes through that surface during some time period T. This is precisely the second RHS term in the last equation from the previous section (without minus sign, since that only appears because of the relation with W). Here, we just change the symbol for the boundary. Instead of ∂V, which made more sense during Poynting vector derivation, because of the connection to the internal volume V, we now use Σ.

$$Q = \int_T \int_\Sigma (\mathbf{S} \cdot \mathbf{n}) dA dt$$

Since, based on previous section, the Poynting vector $\mathbf{S}$ represents intensity i.e. power per unit area, that means that if we integrate it over oriented surface, we will get power. This power is called radiant flux (any power can be thought of as a flux of intensity through some naturally chosen boundary).

$$\Phi = \int_\Sigma (\mathbf{S} \cdot \mathbf{n}) dA$$

But, this expression is just the time derivative of Q. Therefore, we get the relation between radiant energy and radiant flux.

$$\Phi = \frac{\partial Q}{\partial t}$$

On the other hand, we can also integrate Poynting vector only during some time period. This way, we get the concept of radiant exposure/fluence.

$$H = \int_T (\mathbf{S} \cdot \mathbf{n}) dA$$

Similarly to radiant flux, we can see that this equation is just a derivative of Q with respect to area.

$$H = \frac{\partial Q}{\partial A}$$

Intuitively, if we imagine a light source, Q represents the total energy that it radiates in all directions (usually modeled by the unit sphere's boundary) during some total period T. In order to measure it, we would place detectors in every direction around the light and keep them there for the total time period T. After this, we would add all of their contributions. On the other hand, flux/power Φ is the energy radiated in all directions, but in unit time. This means that in order to measure it, we would place detectors in all directions around the light and keep them there during some short time t. We would then add their contributions and divide by the small time t to get an approximation for the power. Finally, radiant exposure/fluence H is the energy radiated in only one direction, but during total time T. In order to measure it, we would pick only one direction of interest and place a detector there. We would then keep it for the total time T.

4 Irradiance E and Radiant Exitance/Emittance M

Both of these concepts tell us how much power we are getting per unit area. The difference is in the direction of energy flow. Irradiance describes the case where energy is received by a surface and radiant exitance/emittance the case where it is emitted by a surface. Concept is similar to radiant exposure/fluence H, but should not be confused with it, since H tells us how much of the total energy (over total time period T) we get per unit area. More formally, if we see these quantities as scalars, irradiance and radiant exitance/emittance are defined as

$$M = E = \frac{\partial \Phi}{\partial A}$$

However, if we are in a situation where we are observing some small area that is receiving and emitting the energy, then we would need to take into account the directions of energy flows. In such a case, these quantities are opposite in sign. As an example, given the flux Φ_1 from one side and the flux Φ_2 from the other, and assuming that the irradiance E coincides with the arbitrarily assigned positive direction, the total power per unit area would be

$$E - M = \frac{\partial \Phi_1}{\partial A} - \frac{\partial \Phi_2}{\partial A} = \frac{\partial(\Phi_1 - \Phi_2)}{\partial A}$$

As another example, we can use the concept of irradiance to derive Lambert's cosine law, given the proper assumptions under which it holds. The law simply states that if the perfectly diffuse surface reflects (or radiates) some energy, and we observe it under some angle (with respect to the surface normal), then the observed radiation (more precisely intensity) is scaled by the cosine of the given angle. To derive this, we first assume that the surface is perfectly diffuse (does not exist in practice), meaning that the reflected light will have the same intensity in all directions. What this assumption means is that the energy reflected/radiated in all directions is the same, which means that it can be expressed with a single number, as opposed to a function which would depend on the output direction. This is the reason why it is sometimes useful to approximate certain surfaces as perfectly diffuse. Another thing that we assume is that the radiation is coming from the surface that we observe. This radiation can be the result of reflection or emission (irrelevant in this derivation). If the flux is Φ and the small circular area is A, then the radiant emittance is

$$M = \frac{\Phi}{A}$$

This is what we observe when looking directly at a small circular surface (circular because we capture light within a cone). If we now start looking from the side, at an angle α with respect to the surface normal, we know that we would see the same emittance M if we also rotate the surface by α i.e. move its normal to match our view direction, because it is the same situation as before. This means that viewing it from the side at an angle α is the same as not changing the view direction but rotating the surface by an angle α in opposite direction. Since we made an assumption that the surface is perfectly diffuse, we can take it as granted that the infinitesimal radiation in some angled direction is the same as in any other direction i.e. only area size will impact the amount of radiation in some small direction. Now, our view cone's intersection with the surface is not circular anymore. Instead, it is an ellipse, where the minor axis stays the same as the previous circular area radius, but the major axis, which lies in the plane defined by a surface normal and view direction, is now elongated. We know that the angle between the radius of the circular area r and the major ellipse axis a is α. Since they form a small triangle in the limit where cone rays become parallel, the relation between them is

$$a = \frac{r}{cos(\alpha)}$$

If the prior circular area is A and a new elliptic area is A' then we have

$$A = r^2\pi$$
$$A' = ab\pi$$

Since the minor axis b stays the same and equals r and the major gets elongated as in the previous relation, the elliptic area is then

$$A' = \frac{r}{cos(\alpha)} r\pi = \frac{r^2\pi}{cos(\alpha)} = \frac{A}{cos(\alpha)}$$

Since the Φ for the surface stays the same, we can find the new emittance M' for the angled view

$$M' = \frac{\Phi}{A'} = \frac{\Phi cos(\alpha)}{A} = M cos(\alpha)$$

In the same way, we can derive the identical relation for the irradiance E.

5 Radiant Intensity I

We already encountered the general notion of intensity when describing the Poynting vector. In general, it describes the power density i.e. how much power we get per unit area and therefore has the form

$$\frac{\partial P}{\partial A}$$

or expressed directly with energy E

$$\frac{\partial^2 E}{\partial A \partial t}$$

In the context of radiometry, power P is flux Φ, so the general intensity is just irradiance E. This is NOT the same as Radiant Intensity I. Radiant Intensity I also represents power density, but in the context of the spatial angle and not the area. If we denote spatial angle by ω, then I is formally defined as

$$I = \frac{\partial \Phi}{\partial \omega}$$

But, previously it was mentioned (in section 2) that there is no real difference between the spatial angle and the area when we use the unit sphere. In that situation, both are measured in the same way by simply measuring the area on the unit sphere. In that context, the radiant intensity I and the general concept of intensity i.e. irradiance E are the same

$$I = \frac{\partial \Phi}{\partial \omega} = \frac{\partial \Phi}{\partial A} = E$$

However, if we do not work in the context of the unit sphere, for example because the distance between the light source and some surface is more than a unit radius, then these concepts are different. In that case, the radiant intensity I would remain the same regardless of the distance between the light and the surface (since the angle stays the same), but the irradiance E would need to be scaled using the inverse square law (since the same angle encompasses a larger area as the distance grows).

6 Radiance L

Before defining this concept, let's first go back to the concept of irradiance E. We imagine a surface and we focus on an infinitesimal patch of this surface for which we assume that every point on it receives the same amount of energy in unit time (same amount of power). Irradiance describes this power. If there are two different light sources emitting light from different directions, then the irradiance E can only answer the questions about the total received power from both lights, by a point on the infinitesimal patch of surface. It can't tell us how much power did a point on infinitesimal patch receive from some direction i.e. from some spatial angle. This is what radiance L tries to describe. Since both direction and area are at play in the definition for the radiance, we will need to take into account scaling that is the result of specific direction, just like we did in the example for Lambert's cosine law (section 4). The reason for this is that under some angle, the same amount of energy falls on the larger area, meaning that one point of that area receives less energy. First, if we only focus on the case where the infinitesimal spatial angle $\partial\omega$ is orthogonal to the infinitesimal surface ∂A, a natural definition, given what radiance tries to describe, is

$$L = \frac{\partial^2 \Phi}{\partial A \partial \omega}$$

However, if the flux is coming from some direction that is at the angle θ with respect to the surface normal, then the infinitesimal area ∂A becomes elongated (just like in the example in section 4). If we use $\partial A^{\perp}$ to denote the infinitesimal area as it would be if the angle θ was 0 i.e. the infinitesimal area that, when protected orthogonally to the surface, gives infinitesimal area ∂A, then we get the relation

$$\partial A = \frac{\partial A^{\perp}}{cos(\theta)}$$

Combining the two formulas, we get

$$L = \frac{\partial^2 \Phi}{\partial A^{\perp} \partial \omega} cos(\theta)$$

Now, important thing to note is that this is a valid definition for radiance, but it is not the standard definition. In fact, the standard definition just drops the cosine term

$$L = \frac{\partial^2 \Phi}{\partial A^{\perp} \partial \omega}$$

The reason for this is the following. Often, when things are defined, we want to express them concisely, in a sense that we want to include only the necessary information. This means that if there are additional terms like the scaling terms or some other transformation terms, we realise that they are not fundamental i.e. intrinsic to the notion being defined and instead depend on the context in which we use the given notion. Therefore, we don't include them in the definition and instead worry about adding them when the context is known. For example, when defining a parabola, we don't usually express it in a general way, where we would also cover the cases of rotated parabolas in the plane. Instead, if we have the rotated parabola in some context, we simply define the rotation transformation R and apply it to the "ordinary" parabola. In the same way, if we are looking for the radiance under some angle θ, we will use the scaling transformation $cos(\theta)$ on the original definition to take the context into account.

7 Frequencies

In all previous derivations, different frequencies were not explicitly taken into account. The reason for this is that the way they are handled is the same for all defined quantities. If the given radiometric quantity is X, then, in addition to other variables, it also depends on the frequency λ. Therefore, as an example, if radiation carries more power in some frequency range (λ_1, λ_2), we would expect that irradiance E for that range has a higher value. With this in mind, we can define spectral versions of all previous radiometric quantities. Given a quantity X, its spectral version X_λ, for some frequency λ is

$$X_\lambda = \frac{\partial X}{\partial \lambda}$$

This then allows us to talk about the given quantity X over some frequency range (λ_1, λ_2) by integrating the spectral versions in that range

$$X = \int_{\lambda_1}^{\lambda_2} X_\lambda d\lambda$$