Proof of the relation between GCD and LCM

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Abstract

This is a proof of relation $a \cdot b = GCD(a,b) \cdot LCM(a,b)$ which directly relies only on divisibility of natural numbers and doesn't use any existing result that is not, in itself, immediately obvious.

1 Proof

We need to prove that $a \cdot b = GCD(a,b) \cdot LCM(a,b)$. Given two numbers $a, b \in \mathbb{N}$, the following relations always hold

$$\exists!\, k_1 \in \mathbb{N}\ ,\ a = k_1 \cdot GCD(a,b)$$
$$\exists!\, k_2 \in \mathbb{N}\ ,\ b = k_2 \cdot GCD(a,b)$$

We will first handle trivial cases where at least one of the numbers a or b is equal to $GCD(a,b)$.

$$a = GCD(a,b) \iff b = k_2 \cdot a \iff b = LCM(a,b) \iff a \cdot b = GCD(a,b) \cdot LCM(a,b)$$
$$b = GCD(a,b) \iff a = k_1 \cdot b \iff a = LCM(a,b) \iff a \cdot b = GCD(a,b) \cdot LCM(a,b)$$
$$a = b = GCD(a,b) \iff a = b = LCM(a,b) \iff a \cdot b = GCD(a,b) \cdot LCM(a,b)$$

Now we will prove that the original relation also holds in non-trivial case ie. when neither a nor b is equal to $GCD(a,b)$. We will use shorthand $GCD(a,b) = k$. From expressions for a and b, we get

$$a \cdot b = k_1 \cdot k_2 \cdot k \cdot k = k_1 \cdot k_2 \cdot k \cdot GCD(a,b)$$

In order to prove the original relation, we need to show that $k_1 \cdot k_2 \cdot k$ is the minimal number divisible by both a and b. Divisibility is easy to show

$$a \mid k_1 \cdot k_2 \cdot k = k_2 \cdot a$$
$$b \mid k_1 \cdot k2 \cdot k = k_1 \cdot b$$

We must show that there does not exist some natural number L that is divisible by both a and b, and smaller than $k_1 \cdot k_2 \cdot k$. We will assume that a number with such properties exists

$$\exists\, L \in \mathbb{N}$$
$$a \mid L \iff \exists!\, l_1 \in \mathbb{N},\ L = l_1 \cdot a = l1 \cdot k_1 \cdot k$$
$$b \mid L \iff \exists!\, l_2 \in \mathbb{N},\ L = l_2 \cdot b = l_2 \cdot k_2 \cdot k$$
$$L < k_1 \cdot k_2 \cdot k$$

Firstly, using previous relations, we can find the order between k_1 and l_2, and also between k_2 and l_1.

$$l_1 \cdot k_1 \cdot k < k_1 \cdot k_2 \cdot k \iff l_1 < k_2$$
$$l_2 \cdot k_2 \cdot k < k_1 \cdot k_2 \cdot k \iff l_2 < k_1$$

Secondly, we can find a relation between all k_1, k_2, l_1 and l_2.

$$l_1 \cdot k_1 \cdot k = l_2 \cdot k_2 \cdot k \iff l_1 \cdot k_1 = l_2 \cdot k_2 \iff \frac{l_1}{l_2} \cdot k_1 = k_2$$

Since k_2 is a natural number, the left side of the previous relation must also be a natural number. Now we show that $\frac{l_1}{l_2}$ can't be a natural number. If we assume that it is, then

$$\exists!\, p \in \mathbb{N},\ \frac{l_1}{l_2} = p \iff l_1 = p \cdot l_2$$

If we now use this result instead of l_1 in the relation for L, we get

$$L = l_1 \cdot a = p \cdot l_2 \cdot a$$

Combining this with relation for L that involves b, we get

$$p \cdot l_2 \cdot a = l_2 \cdot b \iff p \cdot a = b$$

If this was true, then we would have $GCD(a, b) = GCD(a, p \cdot a) = a$, instead of $GCD(a, b) = k$. Therefore, our assumption was wrong and we know that $\frac{l_1}{l_2}$ can't be a natural number. Since the left side must be a natural number, this implies that k_1 must be a multiple of l_2

$$\exists!\, g \in \mathbb{N},\ k_1 = g \cdot l_2$$

If we use this instead of k_1 in equation for L, and combine it with another equation for L, we get

$$l_1 \cdot g \cdot l_2 \cdot k = l_2 \cdot k_2 \cdot k \iff k_2 = g \cdot l_1$$

We know that $a = k_1 \cdot k$ and $b = k_2 \cdot k$, and by using new expressions for k_1 and k_2, we get

$$a = g \cdot l_2 \cdot k$$
$$b = g \cdot l_1 \cdot k$$

Since $GCD(a, b) = k$, this is only possible if $g = 1$, because we would otherwise have $GCD(a, b) = g \cdot k$. This gives us $k_1 = l_2$ and $k_2 = l_1$. This is in contradiction with the order constraints between them that are the consequence of assumption regarding existence of L, namely

$$l_1 < k_2$$
$$l_2 < k_1$$

This shows that the assumption was wrong and that the number L does not exist, and therefore, we know that $k1 \cdot k_2 \cdot k$ is the minimal number that is divisible by both a and b. This means that it is equal to LCM(a, b). From this, it follows that

$$a \cdot b = GCD(a, b) \cdot LCM(a, b)$$

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