Linear MHD using discrete differential forms

Florian Holderied[1]

[1] *Max Planck Institute for Plasma Physics, Boltzmannstrasse 2, 85748 Garching, Germany*

1 The model

Let us recall the standard form of the ideal magnetohydrodynamic (MHD) equations, which is a system of nonlinear partial differential equations for the mass density ρ, the fluid velocity $\mathbf{U}$, the magnetic induction $\mathbf{B}$ and the pressure p:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U} = \frac{1}{\mu_0}\frac{\nabla \times \mathbf{B}}{\rho} \times \mathbf{B} - \frac{\nabla p}{\rho}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}), \tag{3}$$

$$\frac{\partial p}{\partial t} + \nabla \cdot (p\mathbf{U}) + (\gamma - 1)p\nabla \cdot \mathbf{U} = 0, \tag{4}$$

where $\gamma = 5/3$ is the adiabatic exponent. The model can equivalently be written in terms of coordinate independent differential forms and for physical reasons we assume the mass density to be a 3-form ($\rho \to \rho^3$), the pressure to be a 0-form ($p \to p^0$), the magnetic field to be a 2-form ($\mathbf{B} \to B^2$) and the velocity to be a 1-form ($\mathbf{U} \to U^1$). The model then takes the form

$$\frac{\partial \rho^3}{\partial t} + \mathrm{d}(i_{\#U^1}\rho^3) = 0, \tag{5}$$

$$* \rho^3 \left[\frac{\partial U^1}{\partial t} + \frac{1}{2}\mathrm{d}(i_{\#U^1}U^1) + i_{\#U^1}\mathrm{d}U^1\right] + \mathrm{d}p^0 = \frac{1}{\mu_0} i_{\#*B^2}\mathrm{d} * B^2, \tag{6}$$

$$\frac{\partial B^2}{\partial t} + \mathrm{d}(i_{\#U^1}B^2) = 0, \tag{7}$$

$$\frac{\partial p^0}{\partial t} + *\mathrm{d} * (p^0 U^1) + (\gamma - 1)p^0 * \mathrm{d} * U^1 = 0. \tag{8}$$

We linearize this system about a known equilibrium ($\rho^3 = \rho_0^3 + \rho_1^3$, $p^0 = p_0^0 + p_1^0$, $B^2 = B_0^2 + B_1^2$, $U^1 = U_1^1$) where we assume that there is no equilibrium flow. After relabeling (e.g. $\rho_1^3 \to \rho^3$ and $\rho_0^3 \to \rho_0$) this results in

$$\frac{\partial \rho^3}{\partial t} + \mathrm{d}(i_{\#U^1}\rho_0) = 0, \tag{9}$$

$$(*\rho_0)\frac{\partial U^1}{\partial t} + \mathrm{d}p^0 = \frac{1}{\mu_0} i_{\#*B_0}\mathrm{d} * B^2 + \frac{1}{\mu_0} i_{\#*B^2}\mathrm{d} * B_0, \tag{10}$$

$$\frac{\partial B^2}{\partial t} + \mathrm{d}(i_{\#U^1}B_0) = 0, \tag{11}$$

$$\frac{\partial p^0}{\partial t} + *\mathrm{d} * (p_0 U^1) + (\gamma - 1)p_0 * \mathrm{d} * U^1 = 0. \tag{12}$$

Note that the wedge product with a 0-form is just multiplying with a scalar.

2 Discretization

As a next step, we introduce finite element basis functions satisfying a discrete deRham complex. Thus, we write

$$p^0(\mathbf{q}) \approx p_h^0(\mathbf{q}) = \sum_{\mathbf{i}} p_{\mathbf{i}} \Lambda^0_{\mathbf{i}}(\mathbf{q}), \qquad \mathbf{p}^\top := (p_0, \dots, p_{N-1}) \in \mathbb{R}^N, \tag{13}$$

$$U^1(\mathbf{q}) \approx U_h^1(\mathbf{q}) = \sum_{\mathbf{i}} \sum_{\mu=1}^{3} u_{\mu,\mathbf{i}} \Lambda^1_{\mu,\mathbf{i}}(\mathbf{q}) \mathrm{d}q^\mu, \qquad \mathbf{u}^\top := (\mathbf{u}_1^\top, \mathbf{u}_2^\top, \mathbf{u}_3^\top) \in \mathbb{R}^{3N}, \tag{14}$$

$$B^2(\mathbf{q}) \approx B_h^2(\mathbf{q}) = \sum_{\mathbf{i}} \sum_{\mu=1}^{3} b_{\mu,\mathbf{i}} \Lambda^2_{\mu,\mathbf{i}}(\mathbf{q}) (\mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta)_\mu, \qquad \mathbf{b}^\top := (\mathbf{b}_1^\top, \mathbf{b}_2^\top, \mathbf{b}_3^\top) \in \mathbb{R}^{3N}, \tag{15}$$

$$\rho^3(\mathbf{q}) \approx \rho_h^3(\mathbf{q}) = \sum_{\mathbf{i}} \rho_{123,\mathbf{i}} \Lambda^3_{\mathbf{i}}(\mathbf{q}) \mathrm{d}q^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}q^3, \qquad \boldsymbol{\rho}^\top := (\rho_{123,0}, \dots, \rho_{123,N-1}) \in \mathbb{R}^N, \tag{16}$$

where $\mathbf{i} = (i_1, i_2, i_3)$ is a multi-index and N the total number of basis functions. To simplify the notation, we write for the components of the differential forms

$$p_h^0 \leftrightarrow p_h = (p_0, \dots, p_{N-1}) \begin{pmatrix} \Lambda^0_0 \\ \vdots \\ \Lambda^0_{N-1} \end{pmatrix} = \mathbf{p}^\top \mathbf{\Lambda}^0, \qquad \mathbf{\Lambda}^0 \in \mathbb{R}^N, \tag{17}$$

$$U_h^1 \leftrightarrow \mathbf{U}_h^\top = (\mathbf{u}_1^\top, \mathbf{u}_2^\top, \mathbf{u}_3^\top) \begin{pmatrix} \mathbf{\Lambda}^1_1 & 0 & 0 \\ 0 & \mathbf{\Lambda}^1_2 & 0 \\ 0 & 0 & \mathbf{\Lambda}^1_3 \end{pmatrix} = \mathbf{u}^\top \mathbf{\Lambda}^1, \qquad \mathbf{\Lambda}^1 \in \mathbb{R}^{3N \times 3}, \tag{18}$$

$$B_h^2 \leftrightarrow \hat{\mathbf{B}}_h^\top = (\mathbf{b}_1^\top, \mathbf{b}_2^\top, \mathbf{b}_3^\top) \begin{pmatrix} \mathbf{\Lambda}^2_1 & 0 & 0 \\ 0 & \mathbf{\Lambda}^2_2 & 0 \\ 0 & 0 & \mathbf{\Lambda}^2_3 \end{pmatrix} = \mathbf{b}^\top \mathbf{\Lambda}^2, \qquad \mathbf{\Lambda}^2 \in \mathbb{R}^{3N \times 3}, \tag{19}$$

$$\rho_h^3 \leftrightarrow \rho_{123,h} = (\rho_{123,0}, \dots, \rho_{123,N-1}) \begin{pmatrix} \Lambda^3_0 \\ \vdots \\ \Lambda^3_{N-1} \end{pmatrix} = \boldsymbol{\rho}^\top \mathbf{\Lambda}^3, \quad \mathbf{\Lambda}^3 \in \mathbb{R}^N, \tag{20}$$

2.1 Continuity equation

We start with the discretization of the mass continuity equation which we shall keep in strong form. In order to stay in the correct polynomial space, we need to project the second term back into the subspace of 3-forms by applying the projector Π_3. We use the same symbol for actions on forms and the components of a form. In the latter case the projector thus returns a vector of coefficients in the new basis.

$$\frac{\partial \rho_{123,h}}{\partial t} + \Pi_3 \left[\nabla \cdot (\rho_0 G^{-1} \mathbf{U}_h) \right] = 0 \tag{21}$$

$$\Leftrightarrow \quad \frac{\partial \boldsymbol{\rho}}{\partial t} + \mathbb{D} \Pi_2 \left[\rho_0 G^{-1} (\mathbf{\Lambda}^1)^\top \right] \mathbf{u} = 0 \tag{22}$$

$$\Leftrightarrow \quad \frac{\partial \boldsymbol{\rho}}{\partial t} + \mathbb{D} \mathbb{P}^1_2 \mathbf{u} = 0 \tag{23}$$

Note that we have used the commuting diagram property for exchanging projectors and differential operators. Furthermore, we have introduced the discrete divergence matrix $\mathbb{D} \in \mathbb{R}^{N \times 3N}$ and the projection matrix $\mathbb{P}^1_2 \in \mathbb{R}^{3N \times 3N}$, where the notation of the latter is supposed to indicate that the projection is performed on the space V_2 and the upper index is just a numbering. Explicitly, we have

$$\mathbb{P}^1_2 := \begin{pmatrix} \Pi_{2,1} \left[\rho_0 G^{11} (\mathbf{\Lambda}^1_1)^\top \right] & \Pi_{2,1} \left[\rho_0 G^{12} (\mathbf{\Lambda}^1_2)^\top \right] & \Pi_{2,1} \left[\rho_0 G^{13} (\mathbf{\Lambda}^1_3)^\top \right] \\ \Pi_{2,2} \left[\rho_0 G^{21} (\mathbf{\Lambda}^1_1)^\top \right] & \Pi_{2,2} \left[\rho_0 G^{22} (\mathbf{\Lambda}^1_2)^\top \right] & \Pi_{2,2} \left[\rho_0 G^{23} (\mathbf{\Lambda}^1_3)^\top \right] \\ \Pi_{2,3} \left[\rho_0 G^{31} (\mathbf{\Lambda}^1_1)^\top \right] & \Pi_{2,3} \left[\rho_0 G^{32} (\mathbf{\Lambda}^1_2)^\top \right] & \Pi_{2,3} \left[\rho_0 G^{33} (\mathbf{\Lambda}^1_3)^\top \right] \end{pmatrix}. \tag{24}$$

As already indicated, each of the projections returns a vector of coefficients in the new basis. This means that the above expression amounts to

$$\mathbb{P}_2^1 = \begin{pmatrix} \mathbf{c}_{11,0} & \mathbf{c}_{11,1} & \cdots & \mathbf{c}_{12,0} & \mathbf{c}_{12,1} & \cdots & \mathbf{c}_{13,0} & \mathbf{c}_{13,1} & \cdots \\ \mathbf{c}_{21,0} & \mathbf{c}_{21,1} & \cdots & \mathbf{c}_{22,0} & \mathbf{c}_{22,1} & \cdots & \mathbf{c}_{23,0} & \mathbf{c}_{23,1} & \cdots \\ \mathbf{c}_{31,0} & \mathbf{c}_{31,1}^1 & \cdots & \mathbf{c}_{32,0} & \mathbf{c}_{32,1} & \cdots & \mathbf{c}_{33,0} & \mathbf{c}_{33,1} & \cdots \end{pmatrix}. \tag{25}$$

Here, e.g. $\mathbf{c}_{23,1}$ are the coefficients resulting from the projection of the basis function with the index 1 in the block 23 in the matrix (24). Unfortunatley, this is a dense matrix, which is problematic from a memory consumption point of view. Therefore, we just save the right-hand sides, which defines a sparse matrix, and perform the final projection in every time step again. Denoting by $(\mathcal{I}_{2,1}, \mathcal{I}_{2,2}, \mathcal{I}_{2,3})$ the mixed interpolation-histopolation matrices and by $(\mathrm{vec}_{2,1}(f), \mathrm{vec}_{2,2}(f), \mathrm{vec}_{2,3}(f))$ the right-hand side vectors for some vector valued function with components (f_1, f_2, f_3), we can write

$$\begin{aligned}\mathbb{P}_2^1 &= \begin{pmatrix} \mathcal{I}_{2,1}^{-1} & 0 & 0 \\ 0 & \mathcal{I}_{2,2}^{-1} & 0 \\ 0 & 0 & \mathcal{I}_{2,3}^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{vec}_{2,1}\left[\rho_0 G^{11}(\mathbf{\Lambda}_1^1)^\top\right] & \mathrm{vec}_{2,1}\left[\rho_0 G^{12}(\mathbf{\Lambda}_2^1)^\top\right] & \mathrm{vec}_{2,1}\left[\rho_0 G^{13}(\mathbf{\Lambda}_3^1)^\top\right] \\ \mathrm{vec}_{2,2}\left[\rho_0 G^{21}(\mathbf{\Lambda}_1^1)^\top\right] & \mathrm{vec}_{2,2}\left[\rho_0 G^{22}(\mathbf{\Lambda}_2^1)^\top\right] & \mathrm{vec}_{2,2}\left[\rho_0 G^{23}(\mathbf{\Lambda}_3^1)^\top\right] \\ \mathrm{vec}_{2,3}\left[\rho_0 G^{31}(\mathbf{\Lambda}_1^1)^\top\right] & \mathrm{vec}_{2,3}\left[\rho_0 G^{32}(\mathbf{\Lambda}_2^1)^\top\right] & \mathrm{vec}_{2,3}\left[\rho_0 G^{33}(\mathbf{\Lambda}_3^1)^\top\right] \end{pmatrix} &(26)\\ &=: \mathcal{I}_2^{-1}\tilde{\mathbb{P}}_2^1. &(27)\end{aligned}$$

Thus, we only precompute the sparse matrix $\mathbb{R}_2^1$ with entries defined by the mixed interpolation-histopolation problem

$$\mathrm{vec}_{2,1}(f) = \int_{\xi_{i_2}}^{\xi_{i_2+1}} \int_{\xi_{i_3}}^{\xi_{i_3+1}} f_1(\xi_{i_2}, q_2, q_3)\mathrm{d}q^2\mathrm{d}q^3, \tag{28}$$

$$\mathrm{vec}_{2,2}(f) = \int_{\xi_{i_1}}^{\xi_{i_1+1}} \int_{\xi_{i_3}}^{\xi_{i_3+1}} f_2(q_1, \xi_{i_2}, q_3)\mathrm{d}q^1\mathrm{d}q^3, \tag{29}$$

$$\mathrm{vec}_{2,3}(f) = \int_{\xi_{i_1}}^{\xi_{i_1+1}} \int_{\xi_{i_2}}^{\xi_{i_2+1}} f_3(q_1, q_2, \xi_{i_3})\mathrm{d}q^1\mathrm{d}q^2. \tag{30}$$

The sparsity of R_2^1 follows immediately from the local support of the basis functions.

2.2 Induction equation

Like the continuity equation we keep the induction equation in strong form. This time we have to use the projector Π_2 which commutes with the curl operator.

$$\begin{aligned}&\frac{\partial \hat{\mathbf{B}}_h}{\partial t} + \Pi_2\left[\nabla \times (\hat{\mathbf{B}}_0 \times G^{-1}\mathbf{U}_h)\right] = 0 &(31)\\ \Leftrightarrow\quad &\frac{\partial \mathbf{b}}{\partial t} + \mathbb{C}\Pi_1\left[\mathbb{B}_0 G^{-1}(\mathbf{\Lambda}^1)^\top\right]\mathbf{u} = 0 &(32)\\ \Leftrightarrow\quad &\frac{\partial \mathbf{b}}{\partial t} + \mathbb{C}\mathbb{P}_1^1\mathbf{u} = 0. &(33)\end{aligned}$$

Here, we have introduced the discrete curl matrix $\mathbb{C} \in \mathbb{R}^{3N\times 3N}$ and we have written the vector product of the background magnetic field with the velocity field in terms of a matrix-vector product by using the matrix

$$\mathbb{B}_0 := \begin{pmatrix} 0 & -B_{0,12} & B_{0,31} \\ B_{0,12} & 0 & -B_{0,23} \\ -B_{0,31} & B_{0,23} & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}. \tag{34}$$

Following the same steps as for the continuity equation, we obtain for the projection matrix

$$\mathbb{P}_1^1 =: \boldsymbol{\mathcal{I}}_1^{-1}\mathbb{R}_1^1 \in \mathbb{R}^{3N\times 3N}, \tag{35}$$

where

$$\mathbb{R}_1^1 := \begin{pmatrix} \mathrm{vec}_{1,1}\left[(B_{0,31}G^{31} - B_{0,12}G^{21})(\mathbf{\Lambda}_1^1)^\top, (B_{0,31}G^{32} - B_{0,12}G^{22})(\mathbf{\Lambda}_2^1)^\top, (B_{0,31}G^{33} - B_{0,12}G^{23})(\mathbf{\Lambda}_3^1)^\top\right] \\ \mathrm{vec}_{1,2}\left[(B_{0,12}G^{11} - B_{0,23}G^{31})(\mathbf{\Lambda}_1^1)^\top, (B_{0,12}G^{12} - B_{0,23}G^{32})(\mathbf{\Lambda}_2^1)^\top, (B_{0,12}G^{13} - B_{0,23}G^{33})(\mathbf{\Lambda}_3^1)^\top\right] \\ \mathrm{vec}_{1,3}\left[(B_{0,23}G^{21} - B_{0,31}G^{11})(\mathbf{\Lambda}_1^1)^\top, (B_{0,23}G^{22} - B_{0,31}G^{12})(\mathbf{\Lambda}_2^1)^\top, (B_{0,23}G^{23} - B_{0,31}G^{13})(\mathbf{\Lambda}_3^1)^\top\right] \end{pmatrix} \tag{36}$$

and the the right-hand sides of the mixed interpolation-histopolation problem are defined by

$$\mathrm{vec}_{1,1}(f) = \int_{\xi_{i_1}}^{\xi_{i_1+1}} f_1(q_1, \xi_{i_2}, \xi_{i_3})\mathrm{d}q^1, \tag{37}$$

$$\mathrm{vec}_{1,2}(f) = \int_{\xi_{i_2}}^{\xi_{i_2+1}} f_2(\xi_{i_1}, q_2, \xi_{i_3})\mathrm{d}q^2, \tag{38}$$

$$\mathrm{vec}_{1,3}(f) = \int_{\xi_{i_3}}^{\xi_{i_3+1}} f_3(\xi_{i_1}, \xi_{i_2}, q_3)\mathrm{d}q^3. \tag{39}$$

2.3 Momentum equation

Unlike the previous equations we choose a weak formulation for the momentum equation and consequently take the inner product with a test function $V^1 \in H\Lambda^1(\Omega)$ to obtain

$$\left(*\rho_0\frac{\partial U^1}{\partial t}, V^1\right) + \left(\mathrm{d}p^0, V^1\right) = \frac{1}{\mu_0}\left(i_{\#*B_0}\mathrm{d}*B^2, V^1\right) + \frac{1}{\mu_0}\left(i_{\#*B^2}\mathrm{d}*B_0, V^1\right) \quad \forall V^1 \in H\Lambda^1(\Omega). \tag{40}$$

We use the Galerkin approximation and compute each term separately:

$$\left(*\rho_0\frac{\partial U^1}{\partial t}, V^1\right) = \int_{\hat{\Omega}} *\rho_0\dot{\mathbf{U}}^\top G^{-1}\mathbf{V}\sqrt{g}\mathrm{d}^3q \approx \int_{\hat{\Omega}} \Pi_1\left(*\rho_0\dot{\mathbf{U}}_h^\top\right) G^{-1}\mathbf{V}_h\sqrt{g}\mathrm{d}^3q \tag{41}$$

$$= \dot{\mathbf{u}}^\top(\mathbb{P}_1^2)^\top \int_{\hat{\Omega}} \mathbf{\Lambda}^1 G^{-1}(\mathbf{\Lambda}^1)^\top\sqrt{g}\mathrm{d}^3q\mathbf{v} = \dot{\mathbf{u}}^\top(\mathbb{P}_1^2)^\top\mathbb{M}^1\mathbf{v} \quad \forall\mathbf{v} \in \mathbb{R}^{3N}, \tag{42}$$

where $\mathbb{M}^1 \in \mathbb{R}^{3N\times3N}$ is the mass matrix in the space V_1. The projection matrix is

$$\mathbb{P}_1^2 = \boldsymbol{\mathcal{I}}_1^{-1}\mathbb{R}_1^2 \in \mathbb{R}^{3N\times3N}, \tag{43}$$

where

$$\mathbb{R}_1^2 := \begin{pmatrix} \mathrm{vec}_{1,1}\left[\rho_0/\sqrt{g}(\mathbf{\Lambda}_1^1)^\top\right] & 0 & 0 \\ 0 & \mathrm{vec}_{1,2}\left[\rho_0/\sqrt{g}(\mathbf{\Lambda}_2^1)^\top\right] & 0 \\ 0 & 0 & \mathrm{vec}_{1,3}\left[\rho_0/\sqrt{g}(\mathbf{\Lambda}_3^1)^\top\right] \end{pmatrix}. \tag{44}$$

For the second term including the pressure, we get

$$\left(\mathrm{d}p^0, V^1\right) = \int_{\hat{\Omega}} (\nabla p)^\top G^{-1}\mathbf{V}\sqrt{g}\mathrm{d}^3q \approx (\mathbb{G}\mathbf{p})^\top \int_{\hat{\Omega}} \mathbf{\Lambda}^1 G^{-1}(\mathbf{\Lambda}^1)^\top\sqrt{g}\mathrm{d}^3q\mathbf{v} = \mathbf{p}^\top\mathbb{G}^\top\mathbb{M}^1\mathbf{v} \quad \forall\mathbf{v} \in \mathbb{R}^{3N}, \tag{45}$$

with $\mathbb{G} \in \mathbb{R}^{3N\times N}$ being the discrete gradient matrix. Using the identites $\langle i_{\#\gamma^1}\alpha^2, \beta^1\rangle = \langle\alpha^2, \gamma^1 \wedge \beta^1\rangle$ and $*(*B_0 \wedge V^1) = i_{\#V^1}B_0$ the third term yields (omitting the $1/\mu_0$)

$$\left(i_{\#*B_0}\mathrm{d}*B^2, V^1\right) = \left(d*B^2, *B_0 \wedge V^1\right) = \left(*\mathrm{d}*B^2, *(*B_0 \wedge V^1)\right) = \left(d^*B^2, i_{\#V^1}B_0\right), \tag{46}$$

where we have introduced the co-differential operator $\mathrm{d}^*\alpha^p = (-1)^p * \mathrm{d} * \alpha^p$. Applying the Green formula for differential forms and assuming that the boundary term vanishes yields

$$\left(i_{\#*B_0}\mathrm{d}*B^2, V^1\right) = \left(B^2, \mathrm{d}i_{\#V^1}B_0\right) = \int_{\hat{\Omega}} \frac{1}{g}\hat{\mathbf{B}}^\top G\left(\nabla \times (B_0 \times G^{-1}\mathbf{V})\right)\sqrt{g}\mathrm{d}^3q \tag{47}$$

$$\approx \mathbf{b}^\top \int_{\hat{\Omega}} \frac{1}{\sqrt{g}}\mathbf{\Lambda}^2 G(\mathbf{\Lambda}^2)^\top\mathrm{d}^3q\mathbb{C}\Pi_1\left(\mathbb{B}_0 G^{-1}(\mathbf{\Lambda}^1)^\top\right)\mathbf{v} = \mathbf{b}^\top\mathbb{M}^2\mathbb{C}\mathbb{P}_1^1\mathbf{v} \quad \forall\mathbf{v} \in \mathbb{R}^{3N}. \tag{48}$$

We recognize that we end up with the same projection matrix as in the induction equation. Furthermore, we have introduced the mass matrix $\mathbb{M}^2 \in \mathrm{R}^{3N\times3N}$ in the space V_2.

2.4 Energy equation

The appearance of the co-differential operator all terms of the energy equation already indicates that it is convenient to solve this equation once more weakly. Taking the inner product with a test function $r^0 \in H\Lambda^0(\Omega)$ yields

$$\left(\frac{\partial p^0}{\partial t}, r^0\right) - \left(\mathrm{d}^*(p_0 U^1), r^0\right) - (\gamma - 1)\left(\mathrm{d}^* U^1, p_0 r^0\right) = 0 \quad \forall r^0 \in H\Lambda^0(\Omega) \tag{49}$$

$$\Leftrightarrow \quad \left(\frac{\partial p^0}{\partial t}, r^0\right) - \left(p_0 U^1, \mathrm{d}r^0\right) - (\gamma - 1)\left(U^1, \mathrm{d}(p_0 r^0)\right) = 0 \quad \forall r^0 \in H\Lambda^0(\Omega), \tag{50}$$

if we again assume all boundary terms to vanish. For the first term we simply get

$$\left(\frac{\partial p^0}{\partial t}, r^0\right) = \int_{\hat{\Omega}} \dot{p} r \sqrt{g} \mathrm{d}^3 q \approx \dot{\mathbf{p}}^\top \int_{\hat{\Omega}} \mathbf{\Lambda}^0 (\mathbf{\Lambda}^0)^\top \sqrt{g} \mathrm{d}^3 q \mathbf{r} = \dot{\mathbf{p}}^\top \mathbb{M}^0 \mathbf{r}, \tag{51}$$

where $\mathbb{M}^0$ is the mass matrix in the space V_0. The second term amounts to

$$\left(p_0 U^1, \mathrm{d}r^0\right) = \int_{\hat{\Omega}} p_0 \mathbf{U}^\top G^{-1} \nabla r \sqrt{g} \mathrm{d}^3 q \approx \int_{\hat{\Omega}} \Pi_1(p_0 \mathbf{U}_h^\top) G^{-1} \nabla r_h \sqrt{g} \mathrm{d}^3 q \tag{52}$$

$$= \mathbf{u}^\top (\mathbb{P}_1^3)^\top \int_{\hat{\Omega}} \mathbf{\Lambda}^1 G^{-1} (\mathbf{\Lambda}^1)^\top \sqrt{g} \mathrm{d}^3 q \mathbb{G} \mathbf{r} = \mathbf{u}^\top (\mathbb{P}_1^3)^\top \mathbb{M}^1 \mathbb{G} \mathbf{r}, \tag{53}$$

with $\mathbb{P}_1^3 \in \mathbb{R}^{3N \times 3N}$ being the projection matrix

$$\mathbb{P}_1^3 := \mathcal{I}_1^{-1} \mathbb{R}_1^3 = \mathcal{I}_1^{-1} \begin{pmatrix} \mathrm{vec}_{1,1}\left[p_0 (\mathbf{\Lambda}_1^1)^\top\right] & 0 & 0 \\ 0 & \mathrm{vec}_{1,2}\left[p_0 (\mathbf{\Lambda}_2^1)^\top\right] & 0 \\ 0 & 0 & \mathrm{vec}_{1,3}\left[p_0 (\mathbf{\Lambda}_3^1)^\top\right] \end{pmatrix}. \tag{54}$$

Finally, the last term is given by

$$\left(U^1, \mathrm{d}(p_0 r^0)\right) = \int_{\hat{\Omega}} \mathbf{U}^\top G^{-1} \nabla (p_0 r) \sqrt{g} \mathrm{d}^3 q \approx \mathbf{u}^\top \int_{\hat{\Omega}} \mathbf{\Lambda}^1 G^{-1} (\mathbf{\Lambda}^1)^\top \mathrm{d}^3 q \mathbb{G} \Pi_0 \left(p_0 (\mathbf{\Lambda}^0)^\top\right) \mathbf{r} \tag{55}$$

$$= \mathbf{u}^\top \mathbb{M}^1 \mathbb{G} \mathbb{P}_0^1 \mathbf{r}, \tag{56}$$

where $\mathbb{P}_0^1 \in \mathbb{R}^{N \times N}$ is the projection matrix

$$\mathbb{P}_0^1 = \mathcal{I}_0^{-1} \mathbb{R}_0^1 = \mathcal{I}_0^{-1} \mathrm{vec}_0 \left[p_0 (\mathbf{\Lambda}^0)^\top\right], \tag{57}$$

whose right-hand side follow from a pure interpolation problem defined by

$$\mathrm{vec}_0(f) = f(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}). \tag{58}$$

In summary, we obtain

$$\mathbf{r}^\top \mathbb{M}^0 \dot{\mathbf{p}} - \mathbf{r}^\top \mathbb{G}^\top \mathbb{M}^1 \mathbb{P}_1^3 \mathbf{u} - (\gamma - 1) \mathbf{r}^\top (\mathbb{P}_0^1)^\top \mathbb{G}^\top \mathbb{M}^1 \mathbf{u} = 0 \quad \forall \mathbf{r} \in \mathbb{R}^N \tag{59}$$

$$\Leftrightarrow \quad \mathbb{M}^0 \dot{\mathbf{p}} - \mathbb{G}^\top \mathbb{M}^1 \mathbb{P}_1^3 \mathbf{u} - (\gamma - 1)(\mathbb{P}_0^1)^\top \mathbb{G}^\top \mathbb{M}^1 \mathbf{u} = 0. \tag{60}$$

3 Implementation

For simplicity we shall restrict ourselves on the moment on periodic boundary conditions in all directions as well as a smooth analytical mapping $F : \hat{\Omega} \to \Omega$.

3.1 Discrete differential operators

We use uniform tensor product B-splines of degree $p = (p_1, p_2, p_3)$ as a basis for the space V_0 which are created from knot vectors $\hat{T}^{p_\mu} = \{-p_\mu\Delta q_\mu, -(p_\mu - 1)\Delta q_\mu, \dots, 0, \Delta q_\mu, 2\Delta q_\mu, \dots, 1, 1 + \Delta q_\mu, \dots, 1 + p_\mu\Delta q_\mu\}$, one for each of the coordinates on the logical domain $\hat{\Omega}$, i.e. $\mu = \{1, 2, 3\}$. Δq_μ is just the element size of the discretized logical domain in μ-direction. A family of B-splines is then recursively defined by

$$\hat{N}^{p_\mu}_{i_\mu}(q_\mu) = \frac{q_\mu - \hat{T}^\mu_{i_\mu}}{\hat{T}^\mu_{i_\mu+p_\mu} - \hat{T}^\mu_{i_\mu}} \hat{N}^{p_\mu-1}_{i_\mu}(q_\mu) + \frac{\hat{T}^\mu_{i_\mu+p_\mu+1} - q_\mu}{\hat{T}^\mu_{i_\mu+p_\mu+1} - \hat{T}^\mu_{i_\mu+1}} \hat{N}^{p_\mu-1}_{i_\mu+1}(q_\mu), \tag{61}$$

$$\hat{N}^0_{i_\mu}(q_\mu) = \begin{cases} 1 & q_\mu \in [\hat{T}^{p_\mu}_{i_\mu}, \hat{T}^{p_\mu}_{i_\mu+1}], \\ 0 & \text{else} \end{cases} . \tag{62}$$

This defines a spline space with $N_\mu = \text{len}(T^\mu) - 2p_\mu - 1$ distinct B-splines. We will also need a compatible spline space of one degree less which is created from a reduced knot vector $\hat{t}^{p_\mu-1} = \hat{T}^{p_\mu}(1 : -1)$, i.e. from deleting the first and last entry of the original knot vector. We denote the resulting spline family weighted with the element size Δq_μ by $\hat{D}^{p_\mu-1}_{i_\mu} = \hat{N}^{p_\mu-1}_{i_\mu}/\Delta q_\mu$. Note that the reduced space has the same number of basis functions which is specific to periodic boundary conditions. With this choice for the reduced space, the derivative of a spline in the original space can simply be written as

$$(N^{p_\mu}_{i_\mu})'(q_\mu) = D^{p_\mu-1}_{i_\mu-1} - D^{p_\mu-1}_{i_\mu}. \tag{63}$$

This has the consequence that the derivative of a finite element field of the form

$$f_\mu(q_\mu) = \sum_{i_\mu} f_{i_\mu} N^{p_\mu}_{i_\mu}(q_\mu) \tag{64}$$

is just an operation on the vector of coefficients $\mathbf{f}_\mu$ with a matrix that contains 1,-1 and 0 only, i.e.

$$(f_\mu)'(q_\mu) = \sum_{i_\mu} f_{i_\mu}(D^{p_\mu-1}_{i_\mu-1} - D^{p_\mu-1}_{i_\mu}) = \sum_{i_\mu}(f_{i_\mu+1} - f_{i_\mu}) D^{p_\mu-1}_{i_\mu} := \sum_{i_\mu} \hat{f}_{i_\mu} D^{p_\mu-1}_{i_\mu} \tag{65}$$

$$\Rightarrow \quad \hat{\mathbf{f}}_\mu = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{pmatrix} \mathbf{f}_\mu = \mathbb{G}_\mu \mathbf{f}_\mu, \quad \mathbb{G}_\mu \in \mathbb{R}^{N_\mu \times N_\mu} \tag{66}$$

If we use a tensor-product basis for functions in the space V_0, i.e. we write

$$f^0(q_1, q_2, q_3) = \sum_{i_1, i_2, i_3} f_{i_1 i_2 i_3} N^{p_1}_{i_1}(q_1) N^{p_2}_{i_2}(q_2) N^{p_3}_{i_3}(q_3), \tag{67}$$

we can construct the sequence:

$$N^{p_1}_{i_1} N^{p_2}_{i_2} N^{p_3}_{i_3} \xrightarrow{\mathbb{G}\,(\nabla)} \begin{pmatrix} D^{p_1-1}_{i_1} N^{p_2}_{i_2} N^{p_3}_{i_3} \\ N^{p_1}_{i_1} D^{p_2-1}_{i_2} N^{p_3}_{i_3} \\ N^{p_1}_{i_1} N^{p_2}_{i_2} D^{p_3-1}_{i_3} \end{pmatrix} \xrightarrow{\mathbb{C}\,(\nabla\times)} \begin{pmatrix} N^{p_1}_{i_1} D^{p_2-1}_{i_2} D^{p_3-1}_{i_3} \\ D^{p_1-1}_{i_1} N^{p_2}_{i_2} D^{p_3-1}_{i_3} \\ D^{p_1-1}_{i_1} D^{p_2-1}_{i_2} N^{p_3}_{i_3} \end{pmatrix} \xrightarrow{\mathbb{D}\,(\nabla\cdot)} D^{p_1-1}_{i_1} D^{p_2-1}_{i_2} D^{p_3-1}_{i_3}, \tag{68}$$

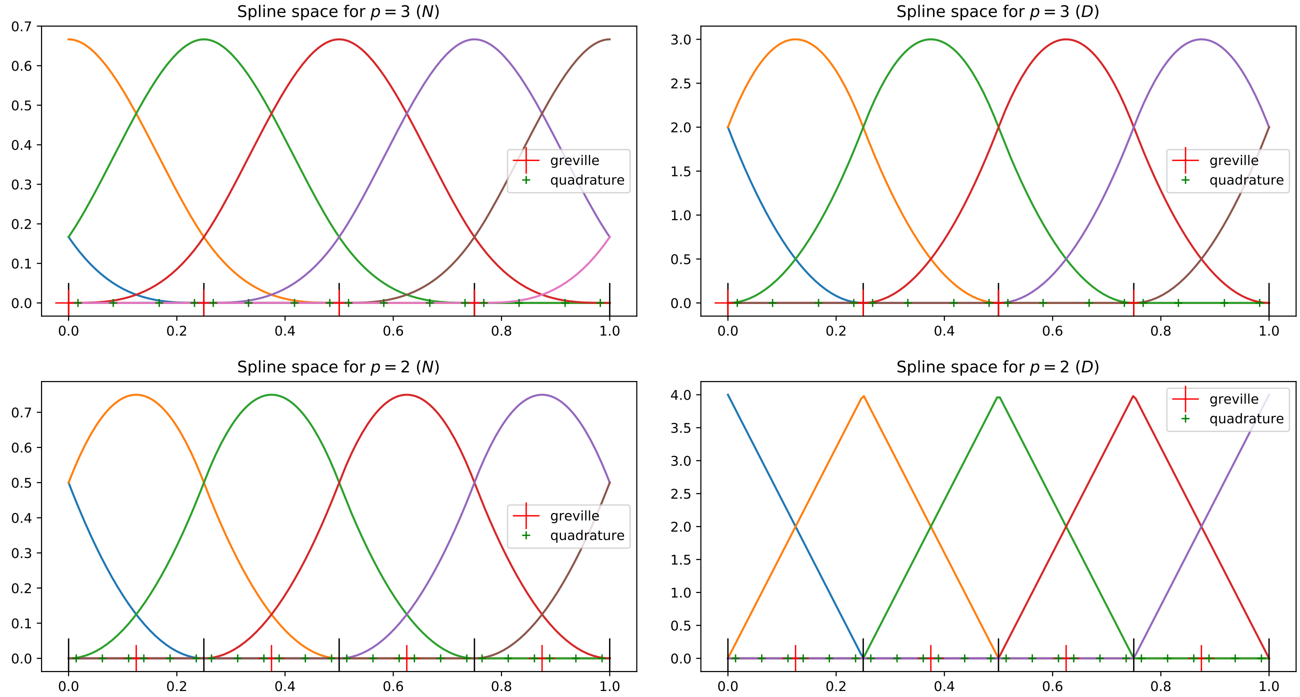
where the discrete derivatives are given by

$$\mathbb{G} = \begin{pmatrix} \mathbb{G}_1 \otimes \mathbb{I}_{N_2} \otimes \mathbb{I}_{N_3} \\ \mathbb{I}_{N_1} \otimes \mathbb{G}_2 \otimes \mathbb{I}_{N_3} \\ \mathbb{I}_{N_1} \otimes \mathbb{I}_{N_2} \otimes \mathbb{G}_3 \end{pmatrix} \in \mathbb{R}^{3N \times N}, \tag{69}$$

$$\mathbb{C} = \begin{pmatrix} 0 & -\mathbb{I}_{N_1} \otimes \mathbb{I}_{N_2} \otimes \mathbb{G}_3 & \mathbb{I}_{N_1} \otimes \mathbb{G}_2 \otimes \mathbb{I}_{N_3} \\ \mathbb{I}_{N_1} \otimes \mathbb{I}_{N_2} \otimes \mathbb{G}_3 & 0 & -\mathbb{G}_1 \otimes \mathbb{I}_{N_2} \otimes \mathbb{I}_{N_3} \\ -\mathbb{I}_{N_1} \otimes \mathbb{G}_2 \otimes \mathbb{I}_{N_3} & \mathbb{G}_1 \otimes \mathbb{I}_{N_2} \otimes \mathbb{I}_{N_3} & 0 \end{pmatrix} \in \mathbb{R}^{3N \times 3N}, \tag{70}$$

$$\mathbb{D} = \begin{pmatrix} \mathbb{G}_1 \otimes \mathbb{I}_{N_2} \otimes \mathbb{I}_{N_3} & \mathbb{I}_{N_1} \otimes \mathbb{G}_2 \otimes \mathbb{I}_{N_3} & \mathbb{I}_{N_1} \otimes \mathbb{I}_{N_2} \otimes \mathbb{G}_3 \end{pmatrix} \in \mathbb{R}^{N \times 3N}. \tag{71}$$

Here $N = N_1 N_2 N_3$ is the total number of basis functions in the space V_0. Note that we have $\mathbb{C}\mathbb{G} = 0$ and $\mathbb{D}\mathbb{C} = 0$ which are just the discrete counterparts of the well-known identities $\nabla \times (\nabla) = 0$ and $\nabla \cdot (\nabla\times) = 0$.



3.2 Projections

In order to perform projections on the right finite element spaces, we have to deal with interpolation and histopolation problems. For the latter, we need to perform integration between the so-called Greville points, one associated to every basis function. In the case of periodic boundary conditions, these points are just the element vertices for odd polynomial degrees and the element centers for even degrees. We use a Gauss-Legendre quadrature rule, which means that we first define a suitable set of quadrature points. If not specified differently, we use a quadrature rule of degree $\mathrm{nq} = \{p_1 + 1, p_2 + 1, p_3 + 1\}$. We denote the set of global quadrature points and weights by $\mathrm{pts} = \{\mathrm{pts}_1, \mathrm{pts}_2, \mathrm{pts}_3\}$ and $\mathrm{wts} = \{\mathrm{wts}_1, \mathrm{wts}_2, \mathrm{wts}_3\}$, respectively.

4 Test case 1: Shear Alfvén waves

The most simple first test case is the simulation of shear Alfvén waves in a homogeneous equilibrium and slab geometry (i.e. $(q_1, q_2, q_3) = (x, y, z)$ and $\Omega = \hat{\Omega}$). Due to the fact that these waves are pure transverse waves with perturbations in the magnetic field and the velocity, only the momentum and induction equation are needed. The model then takes the reduced form

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{u}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & (\mathbb{M}^1)^{-1}(\mathbb{P}^1_1)^\top\mathbb{C}^\top\mathbb{M}^2/(\rho_0\mu_0)\\ -\mathbb{C}\mathbb{P}^1_1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{b}\end{pmatrix} = \mathbb{A}\mathbf{S}, \tag{72}$$

which the state vector $\mathbf{S} = (\mathbf{u}, \mathbf{b})$. If we additionally introduce the total energy

$$H = \frac{1}{2}\int_\Omega \rho_0 U^2 \mathrm{d}^3x + \frac{1}{2\mu_0}\int_\Omega B^2\mathrm{d}^3x \approx \frac{1}{2}\rho_0\mathbf{u}^\top\mathbb{M}^1\mathbf{u} + \frac{1}{2\mu_0}\mathbf{b}^\top\mathbb{M}^2\mathbf{b}, \tag{73}$$

we can write the system in the canonical Hamiltonian form

$$\frac{\partial\mathbf{S}}{\partial t} = \mathbb{J}\nabla_\mathbf{S} H \quad\Leftrightarrow\quad \frac{\partial}{\partial t}\begin{pmatrix}\mathbf{u}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & (\mathbb{M}^1)^{-1}(\mathbb{P}^1_1)^\top\mathbb{C}^\top/\rho_0\\ -\mathbb{C}\mathbb{P}^1_1(\mathbb{M}^1)^{-1}/\rho_0 & 0\end{pmatrix}\begin{pmatrix}\rho_0\mathbb{M}^1\mathbf{u}\\ \mathbb{M}^2\mathbf{b}/\mu_0\end{pmatrix}. \tag{74}$$

We observe that the resulting Poisson matrix is skew-symmetric $\mathbb{J}^\top = -\mathbb{J}$ which means that the Hamiltonian is conserved by the dynamics:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \nabla_\mathbf{S} H^\top\frac{\partial\mathbf{S}}{\partial t} = \nabla_\mathbf{S} H^\top\mathbb{J}\nabla_\mathbf{S} H = -\nabla_\mathbf{S} H^\top\mathbb{J}\nabla_\mathbf{S} H = 0. \tag{75}$$

4.1 Time discretization

Since we are dealing with a Hamiltonian system, there are some naturals choices for integrating the dynamical system in time: First, we shall look at the Hamiltonian splitting which consists of splitting the Hamiltonian in $H = H_U + H_B$ while keeping the full Poisson matrix. Second, we shall use a discrete gradient method which should yield exact energy conservation.

4.1.1 Hamiltonian splitting

Splitting the Hamiltonian in the aforementioned way leads to the two sub-systems

$$H_U: \quad \mathbb{J}\nabla_{\mathbf{S}} H_U = \begin{pmatrix} 0 \\ -\mathbb{C}\mathbb{P}_1^1 \mathbf{u} \end{pmatrix} \quad \Rightarrow \quad \mathbf{b}(t) = \mathbf{b}(t_0) - t\mathbb{C}\mathbb{P}_1^1 \mathbf{u}, \tag{76}$$

$$H_B: \quad \mathbb{J}\nabla_{\mathbf{S}} H_U = \begin{pmatrix} (\mathbb{M}^1)^{-1}(\mathbb{P}_1^1)^\top \mathbb{C}^\top \mathbb{M}^2/(\rho_0\mu_0) \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{u}(t) = \mathbf{u}(t_0) + t(\mathbb{M}^1)^{-1}(\mathbb{P}_1^1)^\top \mathbb{C}^\top \mathbb{M}^2/(\rho_0\mu_0)\mathbf{b}, \tag{77}$$

which can easily be solved analytically.

4.1.2 Discrete gradient method

Denoting the n- th time step by $t_n = n\Delta t$ and using an implicit midpoint rule gives the energy conserving method

$$\frac{\mathbf{S}^{n+1} - \mathbf{S}^n}{\Delta t} = \frac{1}{2}\mathbb{A}(\mathbf{S}^{n+1} + \mathbf{S}^n) \quad \Leftrightarrow \quad (\mathbb{I} - \frac{\Delta t}{2}\mathbb{A})\mathbf{S}^{n+1} = (\mathbb{I} + \frac{\Delta t}{2}\mathbb{A})\mathbf{S}^n, \tag{78}$$

which allows larger time steps but has the drawback of solving a linear system in each time step.

5 Test case 2: Full system

As a next step, we can simulate the full system, however still in slab geometry with periodic boundary conditions and a homogeneous equilibrium. Without loss of generality we can align the z-axis of our coordinate system to the direction of wave propagation ($\mathbf{k} = k\mathbf{e}_z$). In this case one can derive the dispersion relations

$$\omega_{\mathrm{M}}^{\pm}(k)^2 = \frac{1}{2}k^2(c_{\mathrm{S}}^2 + v_{\mathrm{A}}^2)(1 \pm \sqrt{1-\delta}), \quad \delta = \frac{4B_{0z}^2 c_{\mathrm{S}}^2 v_{\mathrm{A}}^2}{(c_{\mathrm{S}}^2 + v_{\mathrm{A}}^2)^2 B_0^2}, \quad c_{\mathrm{S}}^2 = \frac{\gamma p_0}{2\rho_0}, \quad v_{\mathrm{A}}^2 = \frac{B_0^2}{\mu_0\rho_0}, \tag{79}$$

$$\omega_{\mathrm{S}}^2(k) = v_{\mathrm{A}}^2 k^2 \cos^2\vartheta = v_{\mathrm{A}}^2 k^2 \frac{B_{0z}}{B_0}, \tag{80}$$

where we have defined the two characteristic veclocities in the system, that is the sound speed c_{S} and the Alfvén velocity v_{A}, respectively. Hence there can be three types of waves depending on the orientation of the background magnetic field $\mathbf{B}_0$. The two waves corresponding to $\omega_{\mathrm{M}}^{\pm}$ are referred to as the fast (+) and slow (-) magnetosonic wave, whereas the wave corresponding to ω_{S} is the already known shear Alfvén wave. In the special case of $\mathbf{B}_0 = B_0\mathbf{e}_z$, the slow magnetosonic wave is just a "normal" sound wave with perturbations parallel to the direction of propagation (longitudinal wave) and the fast magnetosonic wave is identical to the shear Alfvén wave which is a transverse wave. If the background magnetic field is perpendicular to the direction of propagation, i.e. lies in the xy-plane, there is only the fast magnetosonic wave (or compressional Alfvén wave.

The semi-discrete system which shouls be able to describe these waves reads

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{\rho} \\ \mathbf{u} \\ \mathbf{b} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{D}\mathbb{P}_2^1 & 0 & 0 \\ 0 & 0 & (\mathbb{M}^1)^{-1}(\mathbb{P}_1^1)^\top \mathbb{C}^\top \mathbb{M}^2/(\rho_0\mu_0) & -\mathbb{G}/\rho_0 \\ 0 & -\mathbb{C}\mathbb{P}_1^1 & 0 & 0 \\ 0 & p_0\gamma(\mathbb{M}^0)^{-1}\mathbb{G}^\top \mathbb{M}^1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\rho} \\ \mathbf{u} \\ \mathbf{b} \\ \mathbf{p} \end{pmatrix}. \tag{81}$$

We shall test different time integration schemes for this system.