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The Convexity of Bernstein Polynomials over Triangles

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A necessary and sufficient condition for the convexity of the Bernstein polynomial over the triangle is presented. In particular, it follows that if the nth Bézier net of the function is convex over the triangle, so is the nth Bernstein polynomial.

1. INTRODUCTION

For a function $f(x)$ defined in $[0, 1]$, the nth Bernstein polynomial of f is denoted by $B_n(f; x)$. It is well known that (see [1])

(1) if $f(x)$ is convex in $[0, 1]$, so is $B_n(f; x)$;

(2) if $f(x)$ is convex in $[0, 1]$, then

$$B_n(f; x) \geqslant B_{n+1}(f; x), \qquad n = 1, 2, 3, \ldots,$$

for $x \in [0, 1]$.

We consider the possibility of extending these results to the Bernstein polynomials over triangles. Let us begin with some definitions and notation.

Let T_1, T_2, T_3 be three vertices of a triangle T which is called the base triangle. It is known that every point P of the plane in which the triangle lies can be expressed uniquely by $P = uT_1 + vT_2 + wT_3$ such that

$$u + v + w = 1. \tag{1}$$

(u, v, w) are called the barycentric coordinates of P with respect to the triangle T. We identify the point P with its barycentric coordinates and write $P = (u, v, w)$. It is clear that $T_1 = (1, 0, 0)$, $T_2 = (0, 1, 0)$, and $T_3 = (0, 0, 1)$.

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Barycentric coordinates of points inside or on the boundary of T are characterized by (1) and

$$u \geqslant 0, \qquad v \geqslant 0, \qquad w \geqslant 0. \tag{2}$$

A function $f(P)$ defined on T can be expressed in terms of the barycentric coordinates of P, i.e., $f(P) = f(u, v, w)$. We compute $(n+1)(n+2)/2$ functional values of f:

$$f_{i,j,k} \equiv f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right), \quad i \geqslant 0, j \geqslant 0, k \geqslant 0, i + j + k = n.$$

The nth Bernstein polynomial of f over T is given by

$$B_n(f; P) = \sum_{i+j+k=n} f_{i,j,k} J^n_{i,j,k}(P), \tag{3}$$

where

$$J^n_{i,j,k}(P) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k \tag{4}$$

are called the Bernstein basis polynomials.

Let Ω be a convex set in the plane. A continuous function $f(P)$ is said to be convex in Ω if

$$f\left(\frac{P+Q}{2}\right) \leqslant \frac{1}{2}\,[f(P) + f(Q)]$$

for all points P and Q in Ω.

As we tried to extend (1) and (2) to the Bernstein polynomials over triangles, we found that (2) can be extended while (1) cannot! For example, $f(P)$ is defined by the shaded triangles over T (see Fig. 1), where $f(T_2) = 1$ and $f(T_1) = f(T_3) = f(M) = 0$ and where M is the midpoint of $\overline{T_2 T_3}$. It is clear that $f(P)$ is convex in T. Simple calculation shows that

$$B_2(f; P) = v(u + v),$$

and that

$$B_2(f; T_1) = 0,$$

$$B_2(f; M) = B_2(f; 0, \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{4}.$$

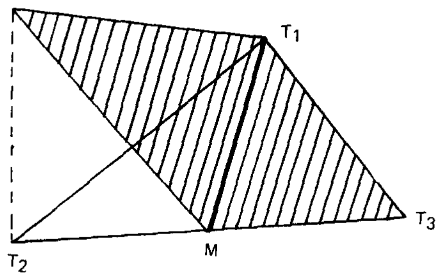

FIGURE 1

Since

$$B_2(f; \tfrac{1}{2}(T_1 + M)) = B_2(f; \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}) = \tfrac{3}{16} > \tfrac{2}{16} = \tfrac{1}{2}(0 + \tfrac{1}{4})$$
$$= \tfrac{1}{2}[B_2(f; T_1) + B_2(f; M)],$$

it follows that $B_2(f; P)$ is not convex!

In this paper, a simple condition which ensures the convexity of $B_n(f; P)$ is given. To formulate our main results, some notation and terminology are needed.

Setting $F_{i,j,k} \equiv (i/n, j/n, k/n; f_{i,j,k})$, this is a point on the surface associated with the function $f(P)$. There are altogether $(n+1)(n+2)/2$ such points in the space. Drawing a triangle with three points

$$F_{i+1,j,k},\ F_{i,j+1,k},\ F_{i,j,k+1}$$

as its vertices, where $i + j + k = n - 1$, a piecewise linear function on T is obtained and is denoted by $\hat{f}_n(P)$. $\hat{f}_n(P)$ is called the nth Bézier net of $f(P)$, in accordance with literature in Computer Aided Geometric Design [2].

The projection of $\hat{f}_n(P)$ onto the triangle T produces a subdivision of T denoted by $S_n(T)$. $S_4(T)$ is illustrated in Fig. 2.

Our main results are the following:

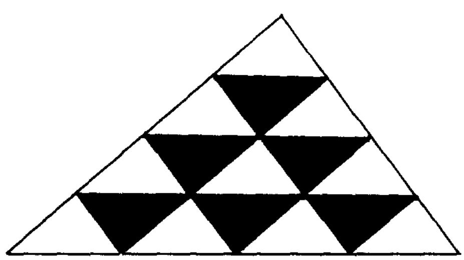

FIGURE 2

(1) if the nth Bézier net $\hat{f}_n(P)$ is convex, so is the nth Bernstein polynomial $B_n(f; P)$;

(2) if $f(P)$ is convex in T, then we have

$$B_n(f; P) \geqslant B_{n+1}(f; P), \qquad n = 1, 2, 3, \ldots,$$

for $P \in T$.

2. Preliminaries

It is clear that

$$\begin{aligned} B_n(f; 1, 0, 0) &= f(1, 0, 0), \\ B_n(f; 0, 1, 0) &= f(0, 1, 0), \\ B_n(f; 0, 0, 1) &= f(0, 0, 1), \end{aligned}$$

i.e., $B_n(f; P)$ interpolates to function f at the vertices of the base triangle T. Since

$$J^n_{i,j,k}(P) \geqslant 0 \qquad \text{for} \quad P \in T,$$

and

$$\sum_{i+j+k=n} J^n_{i,j,k}(P) = (u + v + w)^n = 1,$$

it follows that $B_n(f; P)$ is a convex linear combination of $\{f_{i,j,k}\}$. This means that the surface over T represented by $B_n(f; P)$ is contained in the convex hull of the set of points $\{F_{i,j,k}\}$.

There is a recursive algorithm for the evaluation of $B_n(f; P)$ (see [2]). Define

$$f^0_{i,j,k}(P) = f_{i,j,k} \qquad (i + j + k = n), \tag{5}$$

and

$$f^l_{i,j,k}(P) = u f^{l-1}_{i+1,j,k}(P) + v f^{l-1}_{i,j+1,k}(P) + w f^{l-1}_{i,j,k+1}(P), \tag{6}$$

where $l = 1, 2, \ldots, n$; $i + j + k + l = n$. Introducing three formal "partial shift" operators E_1, E_2, E_3 by

$$\begin{aligned} E_1 f_{i,j,k} &= f_{i+1,j,k}, \\ E_2 f_{i,j,k} &= f_{i,j+1,k}, \\ E_3 f_{i,j,k} &= f_{i,j,k+1}, \end{aligned}$$

then (6) can be rewritten as

$$f^{l}_{i,j,k}(P) = (uE_1 + vE_2 + wE_3)\, f^{l-1}_{i,j,k}(P). \tag{7}$$

Using (7) repeatedly, we have

$$f^{l}_{i,j,k}(P) = (uE_1 + vE_2 + wE_3)^l f_{i,j,k}. \tag{8}$$

Since E_1, E_2, E_3 commute, we can expand $(uE_1 + vE_2 + wE_3)^l$ in (8) by the trinomial formula and get

$$f^{l}_{i,j,k}(P) = \sum_{r+s+t=l} \frac{l!}{r!\,s!\,t!} u^r v^s w^t E_1^r E_2^s E_3^t f_{i,j,k},$$

i.e.,

$$f^{l}_{i,j,k}(P) = \sum_{r+s+t=l} J^{l}_{r,s,t}(P) f_{i+r,j+s,k+t}, \tag{9}$$

where $i+j+k+l=n$. Putting $l=n$ in (9) we obtain

$$f^{n}_{0,0,0}(P) = B_n(f, P), \tag{10}$$

(10) implies that (6) together with (5) provides a stable recursive algorithm for evaluating the nth Bernstein polynomial over triangles.

Replacing $f_{i,j,k}$ by $F_{i,j,k}$ in both (5) and (6), we will have a recursive algorithm for determining the point on the Bernstein triangular surface, i.e.,

$$F^{n}_{0,0,0}(P) = [P; B_n(f; P)]. \tag{11}$$

We shall prove in the next section that the following three points

$$F^{n-1}_{1,0,0}(P) = \left[\frac{1+(n-1)u}{n}, \frac{(n-1)v}{n}, \frac{(n-1)w}{n}; f^{n-1}_{1,0,0}(P)\right],$$

$$F^{n-1}_{0,1,0}(P) = \left[\frac{(n-1)u}{n}, \frac{1+(n-1)v}{n}, \frac{(n-1)w}{n}; f^{n-1}_{0,1,0}(P)\right],$$

$$F^{n-1}_{0,0,1}(P) = \left[\frac{(n-1)u}{n}, \frac{(n-1)v}{n}, \frac{1+(n-1)w}{n}; f^{n-1}_{0,0,1}(P)\right],$$

determine a plane which is tangential to the Bernstein surface at the point $F^{n}_{0,0,0}(P)$.

Figure 3 shows the construction for $B_n(f; \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

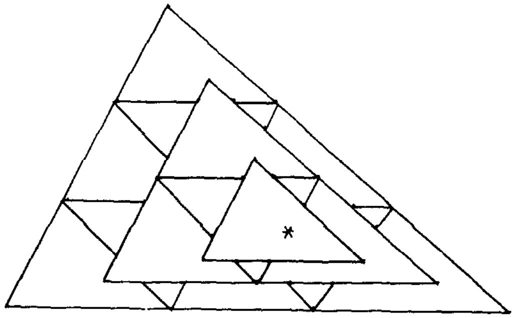

FIGURE 3

3. THE TAYLOR EXPANSION

We mention that not only $f^{n}_{0,0,0}(P)$, but also all the other $f^{l}_{i,j,k}(P)$, can be directly related to the Bernstein polynomial $B_n(f; P)$. We have the following

LEMMA. *For $l = 0, 1, 2, \ldots, n$ and $i + j + k = l$ we have*

$$f^{n-l}_{i,j,k}(P) = \frac{(n-l)!}{n!} \frac{\partial^l}{\partial u^i \partial v^j \partial w^k} B_n(f; P), \tag{12}$$

where u, v, w are treated as independent variables.

Proof. Since

$$\frac{\partial^l}{\partial u^i \partial v^j \partial w^k} (uE_1 + vE_2 + wE_3)^n$$

$$= \frac{n!}{(n-l)!} (uE_1 + vE_2 + wE_3)^{n-l} E_1^i E_2^j E_3^k,$$

we have by (8),

$$\frac{\partial^l}{\partial u^i \partial v^j \partial w^k} B_n(f; P) = \frac{\partial^l}{\partial u^i \partial v^j \partial w^k} (uE_1 + vE_2 + wE_3)^n f_{0,0,0}$$

$$= \frac{n!}{(n-l)!} (uE_1 + vE_2 + wE_3)^{n-l} f_{i,j,k}$$

$$= \frac{n!}{(n-l)!} f^{n-l}_{i,j,k}(P).$$

Equation (10) is a special case of (12) in which $l = 0$. ∎

We are now in a position to present the Taylor expansion for the Bernstein polynomial $B_n(f; P)$. Put $P = (u, v, w), P' = (u', v', w')$ and $P' - P = (u' - u, v' - v, w' - w)$, then we have

THEOREM 1. *For any P and P', we have the identity*

$$B_n(f; P') = \sum_{l=0}^{n} \binom{n}{l} \sum_{i+j+k=l} f_{i,j,k}^{n-l}(P)\, J_{i,j,k}^{l}(P' - P). \tag{13}$$

Proof. Using the Taylor expansion for polynomials of degree n in three variables, we get

$$\begin{aligned}
B_n(f; P') &= \sum_{l=0}^{n} \frac{1}{l!} \left[(u' - u)\frac{\partial}{\partial u} + (v' - v)\frac{\partial}{\partial v} + (w' - w)\frac{\partial}{\partial w}\right]^l B_n(f; P) \\
&= \sum_{l=0}^{n} \frac{1}{l!} \sum_{i+j+k=l} \frac{l!}{i!\, j!\, k!} (u' - u)^i (v' - v)^j (w' - w)^k \frac{\partial^l B_n(f; P)}{\partial u^i \partial v^j \partial w^k} \\
&= \sum_{l=0}^{n} \frac{1}{l!} \sum_{i+j+k=l} J_{i,j,k}^{l}(P' - P) \frac{\partial^l B_n(f; P)}{\partial u^i \partial v^j \partial w^k},
\end{aligned}$$

and by (12) we have

$$B_n(f; P') = \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \sum_{i+j+k=l} f_{i,j,k}^{n-l}(P)\, J_{i,j,k}^{l}(P' - P).$$

This completes the proof of Theorem 1. ∎

The Taylor expansion of functions (13) provides a powerful tool for the investigation of the local analytical behavior of the Bernstein polynomial in the neighborhood of P. Let us write the first three terms of the right-hand side of (13) in more detail:

$$\begin{aligned}
B_n(f; P') = {} & f_{0,0,0}^{n}(P) + n[f_{1,0,0}^{n-1}(P)(u' - u) \\
& + f_{0,1,0}^{n-1}(P)(v' - v) + f_{0,0,1}^{n-1}(P)(w' - w)] \\
& + \frac{n(n-1)}{2} [u' - u, v' - v, w' - w] \\
& \times \begin{bmatrix} f_{2,0,0}^{n-2} & f_{1,1,0}^{n-2} & f_{1,0,1}^{n-2} \\ f_{1,1,0}^{n-2} & f_{0,2,0}^{n-2} & f_{0,1,1}^{n-2} \\ f_{1,0,1}^{n-2} & f_{0,1,1}^{n-2} & f_{0,0,2}^{n-2} \end{bmatrix} \begin{bmatrix} u' - u \\ v' - v \\ w' - w \end{bmatrix} \\
& + \cdots .
\end{aligned} \tag{14}$$

Note that the elements in the 3×3 matrix should be evaluated at the point P.

The first four terms in the right-hand side of (14) form a linear function in u', v', w', which has the contact of at least second degree with the surface at the point $F^n_{0,0,0}(P)$. Hence

$$
\begin{aligned}
z = f^n_{0,0,0}(P) + n[& f^{n-1}_{1,0,0}(P)(u' - u) \\
& + f^{n-1}_{0,1,0}(P)(v' - v) + f^{n-1}_{0,0,1}(P)(w' - w)]
\end{aligned}
\tag{15}
$$

is the tangent plane of the Bernstein surface at the point $F^n_{0,0,0}(P)$. It is easy to show that the plane determined by three points $F^{n-1}_{1,0,0}(P)$, $F^{n-1}_{0,1,0}(P), F^{n-1}_{0,0,1}(P)$ also has Eq. (15). Thus the conclusion in the end of the previous section is justified.

If $f^{n-1}_{1,0,0}(P)$, $f^{n-1}_{0,1,0}(P), f^{n-1}_{0,0,1}(P)$ are not all equal, then without loss of generality we can assume that $f^{n-1}_{1,0,0}(P) \neq f^{n-1}_{0,1,0}(P)$. In this case we put $w' = w$ and since $u' - u = -(v' - v)$,

$$
\begin{aligned}
& f^{n-1}_{1,0,0}(P)(u' - u) + f^{n-1}_{0,1,0}(P)(v' - v) + f^{n-1}_{0,0,1}(P)(w' - w) \\
& \quad = [f^{n-1}_{1,0,0}(P) - f^{n-1}_{0,1,0}(P)](u' - u),
\end{aligned}
$$

and this will assume both positive and negative values no matter how small $|u' - u|$ is. Thus we have

THEOREM 2. *For $f^n_{0,0,0}(P)$ to be a local extreme value it is necessary that*

$$
f^{n-1}_{1,0,0}(P) = f^{n-1}_{0,1,0}(P) = f^{n-1}_{0,0,1}(P).
$$

This means geometrically that the tangent plane of the Bernstein surface at the point $F^n_{0,0,0}(P)$ must be parallel to the plane determined by the base triangle. To determine whether $f^n_{0,0,0}(P)$ is a local extreme value or not, we need further information coming from the third term in the right-hand side of (14), i.e., from the following quadratic form

$$
[\xi, \eta, \zeta] \begin{bmatrix} f^{n-2}_{2,0,0} & f^{n-2}_{1,1,0} & f^{n-2}_{1,0,1} \\ f^{n-2}_{1,1,0} & f^{n-2}_{0,2,0} & f^{n-2}_{0,1,1} \\ f^{n-2}_{1,0,1} & f^{n-2}_{0,1,1} & f^{n-2}_{0,0,2} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{15}
$$

where $\xi + \eta + \zeta = (u' + v' + w') - (u + v + w) = 1 - 1 = 0$. This quadratic form will be studied carefully in the next section.

4. CONVEXITY

On p. 80, Sect. 99 of the book [3], the investigation of convexity of a function Φ with the rectangular Cartesian coordinates as its variables, is shifted to that of nonnegativity of the quadratic form with the second order partial derivatives of Φ as its coefficients. A necessary and sufficient condition for the convexity of Φ is presented there. With obvious modifications we can state

THEOREM 3. *Ω is a convex set in the plane. A necessary and sufficient condition that $B_n(f; P)$ should be convex in Ω is that the quadratic form* (15) *should be nonnegative for all P in Ω and all (ξ, η, ζ) such that $\xi + \eta + \zeta = 0$.*

Setting for simplicity

$$A = f^{n-2}_{2,0,0}(P), \qquad B = f^{n-2}_{0,2,0}(P), \qquad C = f^{n-2}_{0,0,2}(P),$$
$$a = f^{n-2}_{0,1,1}(P), \qquad b = f^{n-2}_{1,0,1}(P), \qquad c = f^{n-2}_{1,1,0}(P),$$

Eq. (15) becomes

$$[\xi, \eta, \zeta] \begin{bmatrix} A & c & b \\ c & B & a \\ b & a & C \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{16}$$

where $\xi + \eta + \zeta = 0$. Insertion of $\zeta = -\xi - \eta$ in (16) gives

$$[\xi, \eta] \begin{bmatrix} A + C - 2b & C + c - a - b \\ C + c - a - b & B + C - 2a \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{17}$$

where now there are no longer any restrictions on ξ and η. Note that the quadratic form (17) is nonnegative if and only if

$$\begin{gathered} A + C \geqslant 2b, \qquad B + C \geqslant 2a, \\ (A + C - 2b)(B + C - 2a) \geqslant (C + c - a - b)^2. \end{gathered} \tag{18}$$

The second inequality is equivalent to

$$\begin{aligned} &BC + CA + AB + 2(bc + ca + ab) \\ &\qquad \geqslant a^2 + b^2 + c^2 + 2(Aa + Bb + Cc). \end{aligned} \tag{19}$$

Hence Theorem 3 can be reformulated by

THEOREM 3′. *A necessary and sufficient condition that $B_n(f; P)$ should be convex in Ω is that* (18) *and* (19) *hold for all P in Ω.*

The following theorem provides a sufficient condition for the convexity of $B_n(f; P)$ in Ω. This condition is easier to check.

THEOREM 4. *If for all P in Ω we have that*

$$A + a \geqslant b + c, \tag{20}$$

$$B + b \geqslant c + a, \tag{21}$$

$$C + c \geqslant a + b, \tag{22}$$

then $B_n(f; P)$ is convex in Ω.

Proof. It is clear that (20), (21), (22) imply

$$A \geqslant b + c - a, \tag{23}$$

$$B \geqslant c + a - b, \tag{24}$$

$$C \geqslant a + b - c, \tag{25}$$

and that

$$\tfrac{1}{2}(B + C) - a, \tag{26}$$

$$\tfrac{1}{2}(C + A) - b, \tag{27}$$

$$\tfrac{1}{2}(A + B) - c \tag{28}$$

are nonnegative numbers. Multiplying both sides of (23), (24), (25) by the numbers in (26), (27), (28), respectively, adding, and simplifying, we get (19). The nonnegativity of numbers in (26) and (27) implies (18). Hence Theorem 4 comes from Theorem 3$'$. ∎

It will be desirable if we can find some conditions for convexity of $B_n(f; P)$ in terms of $f_{i,j,k}$, the values of the primitive function.

The set $\{f_{i,j,k}; i + j + k = n\}$ is said to be convex in the u-direction if inequalities

$$f_{i+1,j,k} + f_{i-1,j+1,k+1} \geqslant f_{i,j+1,k} + f_{i,j,k+1} \tag{29}$$

hold for all i, j, k such that $i > 0$ and $i + j + k = n - 1$. Let us say a few words about inequality (29). In the subdivision $S_n(T)$ there are altogether $n(n-1)/2$ parallelograms each of which has the diagonal parallel to the side $u = 0$ of the base triangle. A typical parallelogram with its vertices $((i+1)/n, j/n, k/n)$, $(i/n, (j+1)/n, k/n)$, $((i-1)/n, (j+1)/n, (k+1)/n)$, $(i/n, j/n, (k+1)/n)$ and the valuations of the function f at these vertices are shown in Fig. 4. Inequality (29) has the following interpretations: in each of these parallelograms the sum of values of f at two vertices connected by the

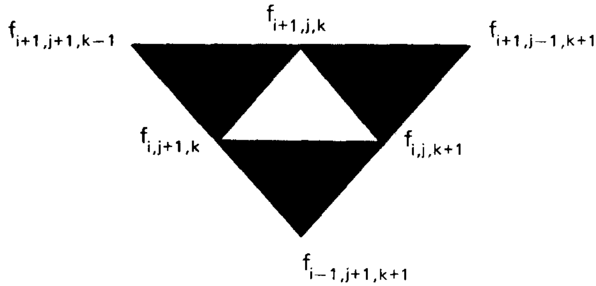

FIGURE 4

explicit diagonal is less than or equal to that of values of f at other two vertices. Similar definitions may be applied to the v- direction and the w direction by

$$f_{i,j+1,k}+f_{i+1,j-1,k+1}\geqslant f_{i+1,j,k}+f_{i,j,k+1} \qquad (j>0 \text{ and } i+j+k=n-1), \tag{30}$$

$$f_{i,j,k+1}+f_{i+1,j+1,k-1}\geqslant f_{i+1,j,k}+f_{i,j+1,k} \qquad (k>0 \text{ and } i+j+k=n-1), \tag{31}$$

respectively (see Fig. 4). By the recursive algorithm (6) we have

$$f^1_{i+2,j-1,k-1}(P)=uf_{i+3,j-1,k-1}+vf_{i+2,j,k-1}+wf_{i+2,j-1,k},$$
$$f^1_{i,j,k}(P)=uf_{i+1,j,k}+vf_{i,j+1,k}+wf_{i,j,k+1},$$

and

$$f^1_{i+1,j,k-1}(P)=uf_{i+2,j,k-1}+vf_{i+1,j+1,k-1}+wf_{i+1,j,k},$$
$$f^1_{i+1,j-1,k}(P)=uf_{i+2,j-1,k}+vf_{i+1,j,k}+wf_{i+1,j-1,k+1}.$$

Inequalities (29) imply that for $u\geqslant 0$, $v\geqslant 0$, $w\geqslant 0$,

$$f^1_{i+2,j-1,k-1}(P)+f^1_{i,j,k}(P)\geqslant f^1_{i+1,j,k-1}(P)+f^1_{i+1,j-1,k}(P).$$

In other words, the convexity of the set $\{f^1_{i,j,k}\}$ in the u- direction implies the convexity of the set $\{f^1_{i,j,k}(P)\}$ in the u- direction for P inside the base triangle T. Repeating this argument we conclude that the convexity of the set $\{f_{i,j,k}\}$ in the u- direction implies the convexity of $\{f^{n-2}_{i,j,k}(P)\}$ in the u-direction for $P\in T$; or equivalently implies the inequality (20). Similar reasoning can be applied for the convexity in the v- direction and in the w-direction. Hence inequalities in (29), (30), (31) imply inequalities (20), (21), (22) for $P\in T$. Thus we have

THEOREM 5. *If* $\{f_{i,j,k}\}$ *satisfy inequalities* (29), (30), (31), *then* $B_n(f; P)$ *is convex over the base triangle* T.

COROLLARY. *If the nth* Bézier net $\hat{f}_n(P)$ is convex in T, so is $B_n(f; P)$.

Proof. In this case we have

$$f_{i,j+1,k} + f_{i,j,k+1} = f\left(\frac{i}{n}, \frac{j+1}{n}, \frac{k}{n}\right) + f\left(\frac{i}{n}, \frac{j}{n}, \frac{k+1}{n}\right)$$
$$= \hat{f}_n\left(\frac{i}{n}, \frac{j+1}{n}, \frac{k}{n}\right) + \hat{f}_n\left(\frac{i}{n}, \frac{j}{n}, \frac{k+1}{n}\right).$$

By the definition of $\hat{f}_n$, $\hat{f}_n$ is linear along the line segment between points $(i/n, (j+1)/n, k/n)$ and $(i/n, j/n, (k+1)/n)$, hence the value of $\hat{f}_n$ at the midpoint of the segment is half of the sum of the values of $\hat{f}_n$ at two endpoints. Thus we have

$$f_{i,j+1,k} + f_{i,j,k+1} = 2\hat{f}_n\left(\frac{i}{n}, \frac{j+(1/2)}{n}, \frac{k+(1/2)}{n}\right),$$

and by the convexity of $\hat{f}_n$,

$$2\hat{f}_n\left(\frac{i}{n}, \frac{j+(1/2)}{n}, \frac{k+(1/2)}{n}\right)$$
$$\leqslant \hat{f}_n\left(\frac{i+1}{n}, \frac{j}{n}, \frac{k}{n}\right) + \hat{f}_n\left(\frac{i-1}{n}, \frac{j+1}{n}, \frac{k+1}{n}\right)$$
$$= f\left(\frac{i+1}{n}, \frac{j}{n}, \frac{k}{n}\right) + f\left(\frac{i-1}{n}, \frac{j+1}{n}, \frac{k+1}{n}\right)$$
$$= f_{i+1,j,k} + f_{i-1,j+1,k+1},$$

as the point $(i/n, (j+\frac{1}{2})/n, (k+\frac{1}{2})/n)$ is the midpoint of the line segment between points $((i+1)/n, j/n, k/n)$ and $((i-1)/n, (j+1)/n, (k+1)/n)$ too. Hence we get

$$f_{i,j+1,k} + f_{i,j,k+1} \leqslant f_{i+1,j,k} + f_{i-1,j+1,k+1},$$

for all i, j, k such that $i > 0$ and $i + j + k = n - 1$. This inequality is just (29). Hence the convexity of $\hat{f}_n(P)$ in T implies (29), (30) and (31). By Theorem 5 we conclude that $B_n(f; P)$ is convex in the base triangle T.

5. Condition for $B_n(f; P) = B_{n+1}(f; P)$

If $f(P)$ is continuous in Ω, then the convexity of f in Ω can be defined equivalently by (see [3])

$$f\left(\sum_{k=1}^{m} \lambda_k P_k\right) \leqslant \sum_{k=1}^{m} \lambda_k f(P_k) \tag{32}$$

for any $P_1, P_2, ..., P_m$ in Ω and for any nonnegative numbers $\lambda_1, \lambda_2, ..., \lambda_m$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1. \tag{33}$$

Lemma. *We have the identity*

$$\begin{aligned} J^n_{i,j,k}(P) = \frac{1}{n+1} [(i+1)\, J^{n+1}_{i+1,j,k}(P) + (j+1)\, J^{n+1}_{i,j+1,k}(P) \\ + (k+1)\, J^{n+1}_{i,j,k+1}(P)], \end{aligned} \tag{34}$$

where $i + j + k = n$.

This lemma can be verified by simple calculations. Equation (34) enables us to write the nth Bernstein polynomial $B_n(f; P)$ in terms of the $(n+1)$th Bernstein basis polynomials:

$$\begin{aligned} B_n(f; P) = &\sum_{i+j+k=n} \frac{i+1}{n+1} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) J^{n+1}_{i+1,j,k}(P) \\ &+ \sum_{i+j+k=n} \frac{j+1}{n+1} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) J^{n+1}_{i,j+1,k}(P) \\ &+ \sum_{i+j+k=n} \frac{k+1}{n+1} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) J^{n+1}_{i,j,k+1}(P). \end{aligned} \tag{35}$$

Replacing $(i+1)$ by i, the first term of the right-hand side of (35) becomes

$$\sum_{i+j+k=n+1} \frac{i}{n+1} f\left(\frac{i-1}{n}, \frac{j}{n}, \frac{k}{n}\right) J^{n+1}_{i,j,k}(P).$$

Even though $f((i-1)/n, j/n, k/n)$ makes no sense for $i = 0$, the coefficient $i/(n+1)$ standing before f will annihilate the corresponding term. Applying similar manipulations to the second and the third term in the right-hand side of (35), we obtain

$$B_n(f;P)=\sum_{i+j+k=n+1}\frac{1}{n+1}\left[if\left(\frac{i-1}{n},\frac{j}{n},\frac{k}{n}\right)\right.$$
$$\left.+jf\left(\frac{i}{n},\frac{j-1}{n},\frac{k}{n}\right)+kf\left(\frac{i}{n},\frac{j}{n},\frac{k-1}{n}\right)\right]J^{n+1}_{i,j,k}(P). \tag{36}$$

If $f(P)$ is convex and continuous in T, since $i+j+k=n+1$ and

$$\frac{i}{n+1}\left(\frac{i-1}{n},\frac{j}{n},\frac{k}{n}\right)+\frac{j}{n+1}\left(\frac{i}{n},\frac{j-1}{n},\frac{k}{n}\right)+\frac{k}{n+1}\left(\frac{i}{n},\frac{j}{n},\frac{k-1}{n}\right)$$
$$=\left(\frac{i}{n+1},\frac{j}{n+1},\frac{k}{n+1}\right), \tag{37}$$

then by (32) we get

$$\frac{1}{n+1}\left[if\left(\frac{i-1}{n},\frac{j}{n},\frac{k}{n}\right)+jf\left(\frac{i}{n},\frac{j-1}{n},\frac{k}{n}\right)+kf\left(\frac{i}{n},\frac{j}{n},\frac{k-1}{n}\right)\right]$$
$$\geqslant f\left(\frac{i}{n+1},\frac{j}{n+1},\frac{k}{n+1}\right). \tag{38}$$

By (36) and (38) we see that if the continuous function $f(P)$ is convex in T, then

$$B_n(f;P)\geqslant B_{n+1}(f;P) \tag{39}$$

for all $P\in T$ and $n=1, 2, 3,\ldots$. We propose the following problem: Under what conditions does the equality in (39) hold? From (36) and the linear independence of $J^{n+1}_{i,j,k}(P)$ we see that for any function $f(P)$ (not necessarily convex) $B_n(f;P)\equiv B_{n+1}(f;P)$ if and only if

$$\frac{1}{n+1}\left[if\left(\frac{i-1}{n},\frac{j}{n},\frac{k}{n}\right)+jf\left(\frac{i}{n},\frac{j-1}{n},\frac{k}{n}\right)+kf\left(\frac{i}{n},\frac{j}{n},\frac{k-1}{n}\right)\right]$$
$$=f\left(\frac{i}{n+1},\frac{j}{n+1},\frac{k}{n+1}\right), \tag{40}$$

where $i+j+k=n+1$. If we call each point $F_{i,j,k}$ the vertex of the nth Bézier net $\hat{f}_n(P)$, we can state (40) geometrically as the following

THEOREM 6. *Let f be any function defined in T. Then $B_n(f;P)\equiv B_{n+1}(f;P)$ if and only if all vertices of $\hat{f}_{n+1}(P)$ lie on $\hat{f}_n(P)$.*

For a convex function f, we can reformulate Theorem 6 in a different way which has a stronger geometric implication. We have mentioned in Section 1 that the projection of $\hat{f}_n(P)$ onto the triangle T produces the subdivision of T denoted by $S_n(T)$. Each point $(i/n, j/n, k/n)$ with $i+j+k=n$ is called a node of $S_n(T)$. $S_n(T)$ has $(n+1)(n+2)/2$ nodes altogether. Denote the boundary of T by ∂T and $T^\circ \equiv T \backslash \partial T$. The nodes in T° are called interior nodes while the others are called boundary nodes. Clearly $S_n(T)$ has $3n$ boundary nodes and $(n-2)(n-1)/2$ interior nodes. There are n^2 subtriangles in $S_n(T)$. Each subtriangle with vertices

$$\left(\frac{i-1}{n}, \frac{j}{n}, \frac{k}{n}\right), \left(\frac{i}{n}, \frac{j-1}{n}, \frac{k}{n}\right), \left(\frac{i}{n}, \frac{j}{n}, \frac{k-1}{n}\right), \tag{41}$$

where $i, j, k \geqslant 1$ and $i+j+k=n+1$, is called a downward subtriangle. $S_n(T)$ has $(n-1)\,n/2$ downward subtriangles. All downward subtriangles of $S_4(T)$ are colored by black in Fig. 2.

Let us observe the relationship between $S_n(T)$ and $S_{n+1}(T)$. All nodes of $S_{n+1}(T)$ can be put into three categories:

(1) Interior nodes are characterized by $(i/(n+1), j/(n+1), k/(n+1))$ with $i, j, k \geqslant 1$ and $i+j+k=n+1$. From (37) and (41) we see that each interior node of $S_{n+1}(T)$ lies inside one and only one downward triangle of $S_n(T)$. See Fig. 5.

(2) Nodes on just one side of T are characterized by $(i/(n+1), j/(n+1), k/(n+1))$ with only one of i, j, k equal to zero. From (37) we can say that each of these nodes of $S_{n+1}(T)$ lies inside one and only one boundary segment of $S_n(T)$.

(3) Three vertices of T.

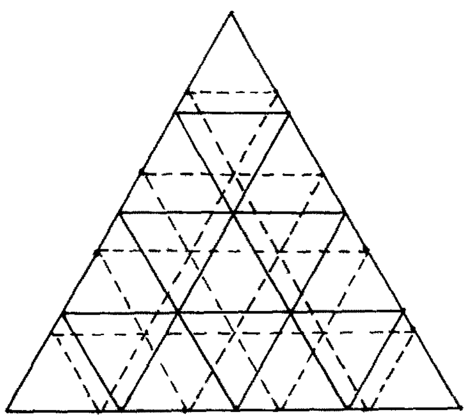

FIGURE 5

If $f(P)$ is convex in T, naturally $f(P)$ is convex in each subtriangle with vertices shown in (41). Hence (40) will imply $f(P)$ is linear over this subtriangle. Thus we have

THEOREM 7. *Let function $f(P)$ be convex and continuous in T. Let D_n be the union of all downward subtriangles of $S_n(T)$. Then $B_{n+1}(f;P)=B_n(f;P)$ if and only if*

$$f(P)=\hat{f}_n(P) \qquad \text{for} \quad P\in D_n\cup\partial T, \tag{42}$$

otherwise we have $B_n(f;P)>B_{n+1}(f;P)$ for $P\in T^{\circ}$.

6. CONVEXITY OVER A CHANGED TRIANGLE

Even if $f(P)$ is defined on the base triangle T only, the Bernstein polynomial (3) is well defined in the whole plane. In some practical applications, we relax the restrictions (2) for more flexibility. Let $T^*=\Delta T_1^* T_2^* T_3^*$ be any triangle in the same plane of the triangle T. We are interested in the convexity of the Bernstein polynomial $B_n(f;P)$ restricted to T^*. Assume T_i^* has barycentric coordinates (u_i, v_i, w_i) with respect to T, $i=1,2,3$. We define the following $(n+1)(n+2)/2$ numbers

$$\begin{aligned} f^*_{i,j,k}=&\sum_{r+s+t=i}\ \sum_{\alpha+\beta+\gamma=j}\ \sum_{\lambda+\mu+\nu=k} J^i_{r,s,t}(u_1,v_1,w_1)\\ &\times J^j_{\alpha,\beta,\gamma}(u_2,v_2,w_2)\,J^k_{\lambda,\mu,\nu}(u_3,v_3,w_3)\\ &\times f_{r+\alpha+\lambda,s+\beta+\mu,t+\gamma+\nu}, \end{aligned} \tag{43}$$

or briefly,

$$\begin{aligned} f^*_{i,j,k}=&(u_1E_1+v_1E_2+w_1E_3)^i(u_2E_1+v_2E_2+w_2E_3)^j\\ &\times(u_3E_1+v_3E_2+w_3E_3)^k f_{0,0}, \end{aligned}$$

where $i+j+k=n$. We have

THEOREM 8. *Let (u, v, w) be the barycentric coordinates of P with respect to the triangle T^*, then the expression*

$$\sum_{i+j+k=n} f^*_{i,j,k}J^n_{i,j,k}(u,v,w) \tag{44}$$

represents the Bernstein polynomial $B_n(f;P)$ restricted to T^*.

Proof. Setting $l=r+\alpha+\lambda$, $m=s+\beta+\mu$, $p=t+\gamma+\nu$, we have $l+m+p=i+j+k=n$ and

$$\begin{aligned}
&J^n_{i,j,k}(u,v,w)\,J^i_{r,s,t}(u_1,v_1,w_1)\,J^j_{\alpha,\beta,\gamma}(u_2,v_2,w_2)\,J^k_{\lambda,\mu,\nu}(u_3,v_3,w_3)\\
&\quad=\frac{n!}{i!\,j!\,k!}\cdot\frac{i!}{r!\,s!\,t!}\cdot\frac{j!}{\alpha!\beta!\gamma!}\cdot\frac{k!}{\lambda!\,\mu!\,\nu!}\,u^iv^jw^ku_1^rv_1^sw_1^tu_2^\alpha v_2^\beta w_2^\gamma u_3^\lambda v_3^\mu w_3^\nu\\
&\quad=\frac{n!}{l!\,m!\,p!}\cdot\frac{l!}{r!\,\alpha!\,\lambda!}\cdot\frac{m!}{s!\,\beta!\,\mu!}\cdot\frac{p!}{t!\,\gamma!\,\nu!}\\
&\qquad\times(uu_1)^r(vu_2)^\alpha(wu_3)^\lambda\cdot(uv_1)^s(vv_2)^\beta(wv_3)^\mu(uw_1)^t(vw_2)^\gamma(ww_3)^\nu\\
&\quad=\frac{n!}{l!\,m!\,p!}\,J^l_{r,\alpha,\lambda}(uu_1,vu_2,wu_3)\\
&\qquad\times J^m_{s,\beta,\mu}(uv_1,vv_2,wv_3)\cdot J^p_{t,\gamma,\nu}(uw_1,vw_2,ww_3).
\end{aligned}$$

Insertion of (43) into (44) gives

$$\begin{aligned}
&\sum_{i+j+k=n} f^*_{i,j,k}J^n_{i,j,k}(u,v,w)\\
&\quad=\sum_{l+m+p=n}\frac{n!}{l!\,m!\,p!}\left[\sum_{r+\alpha+\lambda=l}J^l_{r,\alpha,\lambda}(uu_1,vu_2,wu_3)\right]\\
&\qquad\qquad\left[\sum_{s+\beta+\mu=m}J^m_{s,\beta,\mu}(uv_1,vv_2,wv_3)\right]\\
&\qquad\qquad\left[\sum_{t+\gamma+\nu=p}J^p_{t,\gamma,\nu}(uw_1,vw_2,ww_3)\right]f_{l,m,p}\\
&\quad=\sum_{l+m+p=n}\frac{n!}{l!\,m!\,p!}(uu_1+vu_2+wu_3)^l\cdot\\
&\qquad(uv_1+vv_2+wv_3)^m(uw_1+vw_2+ww_3)^pf_{l,m,p}\\
&\quad=B_n(f;uu_1+vu_2+wu_3,uv_1+vv_2+wv_3,uw_1+vw_2+ww_3).
\end{aligned}$$

Note that

$$[u,v,w]\begin{bmatrix}u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\end{bmatrix}$$

are the barycentric coordinates of a point inside T^* with respect to the triangle T. ∎

Now we can use the methods presented in Section 4 on $f^*_{i,j,k}$ to check the convexity of $B_n(f; P)$ restricted to the triangle T^*.

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