Physica 7D (1983) 330–333
North-Holland Publishing Company

NONCANONICAL HAMILTONIAN FORMULATION OF IDEAL MAGNETOHYDRODYNAMICS

Darryl D. HOLM and Boris A. KUPERSHMIDT
Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

A noncanonical Poisson structure for ideal magnetohydrodynamics is presented and identified with a differential Lie algebra.

1. Introduction

Nonlinear hydrodynamics is still an active field of research in mathematical physics even though it has 250 years of history since the Bernoulli brothers and Euler. The most active areas in nonlinear hydrodynamics today are dynamical systems and Hamiltonian structures. In dynamical systems, much of the activity centers upon chaotic behavior of truncated modal expansions in convective flow. Such chaos is the main subject of this conference. In Hamiltonian structure studies, the main activity is to identify order – in the form of Lie algebras and related objects which underly the Lie–Poisson brackets responsible for fluid flows.

Noncanonical Poisson brackets for fluid systems first appear in Gardner's 1971 paper [1] on the Korteweg–de Vries equation. Since then, these noncanonical structures have proliferated; now they are known for a great many fluid dynamical theories.

In this paper we present a Poisson structure for ideal magnetohydrodynamics (MHD) in which the physical variables appear explicitly. Thus, the Galilean symmetries of MHD, for example, are realized as canonical transformations whose generators are physical quantities. In addition, the physical variables appear *linearly* in the Poisson bracket; which means that this bracket is intimately connected with a certain differential Lie algebra [see the next paper for details].

The Lie algebra given here and all of the other Lie algebras connected with Poisson brackets for hydrodynamically-related systems turn out to be semi-direct products of varying complexity (see, e.g. [2, 3]). From the calculational point of view, the presence of semi-direct products is an "experimental" observation, which is open to mathematical interpretations (see, e.g. [4]). The brackets themselves, though, can often be obtained by intuitive, physical reasoning [5] or even, sometimes, by trial and error [6].

In the ideal MHD model, electrically neutral plasma convects like an adiabatic fluid which carries an embedded magnetic field. The MHD fluid has mass density, ρ, and specific entropy, s. It moves through Euclidean space $\mathbb{R}^n$ with positions x_i and velocities v_j and carries an embedded magnetic field $B_{ij}(x, t)$. The magnetic field components B_{ij} are skew-symmetric and are derived from a vector potential, A_j, according to $B_{ij} = A_{i,j} - A_{j,i}$ with subscript notation also for partial derivatives. During convection, induced electrical currents flow: $J_i = -B_{ij,j}$ according to Ampere's Law. These induced currents oppose any change of magnetic flux through each co-moving surface. The resultant magnetic stresses alter the convective motion of the plasma by opposing bending of magnetic field lines.

In terms of momentum density $M_j = \rho v_j$, the MHD equations are:

$$\partial M_i/\partial t = -[M_i M_j/\rho + \delta_{ij}(p - 1/4\,\mathrm{Tr}\,B^2) - B_{ik}B_{kj}]_{,j}, \tag{1}$$

0167-2789/83/0000–0000$03.00

$$\partial\rho/\partial t = -M_{j,j}, \tag{2}$$

$$\partial s/\partial t = -\rho^{-1}M_j s_{,j}, \tag{3}$$

$$\partial B_{ij}/\partial t = -(B_{ik}M_k/\rho)_{,j} + (B_{jk}M_k/\rho)_{,i}. \tag{4}$$

Throughout, we sum on repeated indices. Eq. (1) is the hydrodynamic motion equation expressed in conservative form as the divergence of the stress tensor for MHD. In the stress tensor the fluid pressure p is determined as a function of ρ and s from a prescribed relation for the specific internal energy, $U(\rho, s)$, combined with the first law of thermodynamics,

$$\mathrm{d}U = U_\rho\,\mathrm{d}\rho + U_s\,\mathrm{d}s = \rho^{-2}p\,\mathrm{d}\rho + T\,\mathrm{d}s, \tag{5}$$

where T is temperature. Eq. (2) expresses local mass conservation. Eq. (3) expresses the adiabatic flow condition, that no heat is exchanged between fluid elements as they convect. Eq. (4) is the Maxwell induction equation with electric field eliminated by Ohm's law, which is written for perfect electrical conductivity as $E_i = B_{ij}M_j/\rho$.

2. Poisson bracket relations

The Hamiltonian for the MHD system (1)–(4) is

$$H = \int \mathrm{d}^n x[M^2/(2\rho) + \rho U(s,\rho) - 1/4\,\mathrm{Tr}\,B^2], \tag{6}$$

where $\mathrm{Tr}\,B^2 = B_{ij}B_{ji}$. The Hamiltonian functional H is the sum of the kinetic energy, thermal energy, and magnetic energy of the fluid.

We introduce the following Poisson bracket defined over functionals $F[\rho, \sigma, M_i, A_j]$, where the variables $\sigma = \rho s$ and A_j are entropy density and magnetic vector potential, respectively,

$$\begin{aligned}\{F,G\} = \int \mathrm{d}^n x\Bigg[&\rho\left(\frac{\delta F}{\delta M_j}\partial_j\frac{\delta G}{\delta\rho} - \frac{\delta G}{\delta M_j}\partial_j\frac{\delta F}{\delta\rho}\right)\\ &+ \sigma\left(\frac{\delta F}{\delta M_j}\partial_j\frac{\delta G}{\delta\sigma} - \frac{\delta G}{\delta M_j}\partial_j\frac{\delta F}{\delta\sigma}\right)\\ &+ M_k\left(\frac{\delta F}{\delta M_j}\partial_j\frac{\delta G}{\delta M_k} - \frac{\delta G}{\delta M_j}\partial_j\frac{\delta F}{\delta M_k}\right)\\ &- \left(\frac{\delta F}{\delta M_i}\frac{\delta G}{\delta A_k} - \frac{\delta G}{\delta M_i}\frac{\delta F}{\delta A_k}\right)(A_{k,i} - A_{i,k})\\ &+ A_k\left(\frac{\delta F}{\delta M_k}\partial_i\frac{\delta G}{\delta A_i} - \frac{\delta G}{\delta M_k}\partial_i\frac{\delta F}{\delta A_i}\right)\Bigg].\end{aligned} \tag{7}$$

The MHD eqs. (1)–(4) are then equivalent to the following bracket relations:

$$\dot F = \{H, F\}, \quad F \in \{\rho, \sigma, M_i, A_j\} \tag{8}$$

for Hamiltonian H given by eq. (6). The proof of (8) follows by comparison of the righthand side of eqs. (1)–(4) with terms in the identity below,

$$\begin{aligned}\{H,F\} = -\int \mathrm{d}^n x\Bigg\{&\frac{\delta F}{\delta\rho}(\partial_j M_j) + \frac{\delta F}{\delta\sigma}\partial_j(\sigma M_j/\rho)\\ &+ \frac{\delta F}{\delta M_k}\partial_j[M_j M_k/\rho\\ &+ \delta_{jk}(p - 1/4\,\mathrm{Tr}\,B^2) - B_{ji}B_{ik}]\\ &+ \frac{\delta F}{\delta A_k}[B_{ki}M_i/\rho + \partial_k(M_i A_i/\rho)]\Bigg\}.\end{aligned} \tag{9}$$

Thus the Hamiltonian H in (6) generates time evolution for MHD as a canonical transformation.

It is possible to transfer bracket (7) from **A**-space to **B**-space (see formula (55) of [7]). The resulting cumbersome bracket simplifies greatly for $n \leqslant 3$, when **B** can be treated not as a 2-form but as a vector (for $n = 3$) or scalar (for $n = 2$). This bracket in **B** space, for $n = 3$, was first found by Greene and Morrison [6].

The entire set of Galilean symmetries of the MHD system (1)–(4) may be realized as a subgroup of the canonical transformations defined in terms of the Poisson bracket (7). The generators of the Galilean transformations are expressible as physical quantities,

$$H = \int \mathrm{d}^n x[M^2/(2\rho) + \rho U(s,\rho) - 1/4\,\mathrm{Tr}\,B^2],$$

$$P_i = \int d^n x M_i ,$$

$$L_{ij} = \int d^n x (x_i M_j - x_j M_i), \tag{10}$$

$$G_i = -tP_i .$$

The functionals H, P_i, and L_{ij}, are respectively the total energy, kinetic momentum, and kinetic angular momentum of the MHD fluid.

As a result of the Poisson bracket (7) the generators (10) form a realization of the Lie Algebra of the Galilean group, viz.

$$\begin{aligned}
&\{H, G_i\} = P_i ,\\
&\{P_k, L_{ij}\} = P_j\delta_{ik} - P_i\delta_{jk} ,\\
&\{G_k, L_{ij}\} = G_j\delta_{ik} - G_i\delta_{jk} ,\\
&\{L_{ij}, L_{kl}\} = \delta_{jk}L_{il} - \delta_{jl}L_{ik} + \delta_{il}L_{jk} - \delta_{ik}L_{jl} ,\\
&0 = \langle H, P_k\} = \{H, L_{ij}\} = \{P_i, P_k\}\\
&\quad = \{P_i, G_k\} = \{G_i, G_k\}.
\end{aligned} \tag{11}$$

In addition, the following quantities Poisson-commute with all functionals defined over $\{\rho, \sigma, M_i, A_j\}$. That is, with

$$\mathcal{M} = d^n x \rho(x), \quad S = d^n x \sigma(x), \tag{12}$$

one finds that $\{\mathcal{M}, F\} = 0 = \{S, F\}$ for arbitrary $F[\rho, \sigma, M_i, A_j]$. So the total mass and entropy of the fluid each generate the identity transformation.

We have already seen in (9) that the total energy H generates time translations. We notice also that the kinetic momentum and angular momentum of the fluid generate spatial translations and rotations, respectively. Those results follow from the identities

$$\{P_k, F\} = -\int d^n x \left[\frac{\delta F}{\delta\rho}\rho_{,k} + \frac{\delta F}{\delta\sigma}\sigma_{,k} + \frac{\delta F}{\delta M_l}M_{l,k} + \frac{\delta F}{\delta A_l}A_{l,k}\right], \tag{13}$$

$$\{L_{ij}, F\} = -\int d^n x (x_i\delta_{jk} - x_j\delta_{ik}) \times \left[\frac{\delta F}{\delta\rho}\rho_{,k} + \frac{\delta F}{\delta\sigma}\sigma_{,k} + \frac{\delta F}{\delta M_l}M_{l,k} + \frac{\delta F}{\delta A_l}A_{l,k}\right]. \tag{14}$$

Note that for MHD the magnetic field plays no role in the canonical momentum and angular momentum. Finally we mention that the functional $G_i = -tP_i$ is the generator of Galilean boosts in the ith direction, cf. eq. (13). Thus all of the transformations in terms of the Galilean group are realized as canonical transformations in terms of physical variables with the Poisson bracket (7).

3. Lie-algebraic interpretation of the Poisson structure

We comment briefly on the mathematical origin of formula (7) for the Poisson bracket.

Recall that if $\mathfrak{G}$ is a finite-dimensional Lie algebra, with a basis $\{e_1, \dots, e_n\}$ and structure constants $c_{ij}^k: e_i \Delta e_j = c_{ij}^k e_k$, then (smooth) functions on the dual space $\mathfrak{G}^*$ form a Lie algebra with the Poisson bracket

$$\{f, g\} = \sum_{ijk} \frac{\partial g}{\partial u_i} c_{ij}^k u_k \frac{\partial f}{\partial u_j}, \tag{15}$$

where u_i's are coordinates on $\mathfrak{G}^*$ in the basis dual to $\{e_i\}$. A formula analogous to (15) exists when one has a "differential" Lie algebra, that is, when c_{ij}^k are linear differential operators (examples of this sort first appeared in [8, 9]; more details can be found in [10]).

Let N be a (smooth) manifold, $C^\infty(N)$ be a ring of smooth functions, $\mathscr{D}(N)$ be a Lie algebra of vector fields on N (i.e., derivations of $C^\infty(N)$), $\wedge^i(N)$ be a $C^\infty(N)$ module of differential i-forms on N.

Consider $\mathbb{R}^p$ with coordinates $y_1, \dots, y_p$. Denote by $\mathscr{D}^p(N)$ a Lie subalgebra of $\mathscr{D}(N \times R^p)$ consisting of vector fields X of the form

$$\left\{X = X' + \sum_{s=1}^{p} f_s \partial/\partial y_s, \quad X' \in \mathscr{D}(N), f_s \in C^\infty(N)\right\}.$$

Finally, denote by $\wedge^{i,P}(N)$ i-forms on N lifted to $N \times \mathbb{R}^P$ by pullback of the projection $N \times \mathbb{R}^P \to N$.

It is easy to see that $\mathfrak{G} = \mathscr{D}^P(N) \ominus \wedge^{i,P}(N)$ (direct sum of $C^\infty(N)$-modules) is a *Lie algebra* with respect to multiplication Δ given by

$$(X; \omega)\Delta(Y; \nu) = ([X, Y]; X(\nu) - Y(\omega)), \tag{16}$$

where, e.g., $X(\nu)$ is the Lie derivative of the i-form, ν, with respect to the vector field, X.

Formula (7) now can be gotten from (16) for our $\mathfrak{G}$ by direct computation, when one takes $p = 2$, $i = n - 1$, $N = R^n = \{(x_1, \ldots, x_n)\}$ and denotes coordinates on $\mathfrak{G}^*$ in the following manner: ρ and σ are dual to $\partial/\partial y_1$ and $\partial/\partial y_2$, M_i's are dual to $\partial/\partial x_i$'s, $i = 1, \ldots, n$ and A_i's are dual to $\partial/\partial x_i \lrcorner (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$, $i = 1, \ldots, n$.

The resulting bracket (7) necessarily satisfies the Jacobi identity and the other defining properties of a Poisson bracket (linearity and antisymmetry) because (7) has been constructed from a (differential) Lie algebra.

Thus, MHD fits into an algebraic Hamiltonian setting, directly in terms of the physical variables.

Acknowledgments

It is a pleasure to thank John Greene, for discussions of the Morrison–Greene bracket before its publication and J. Marsden, for his stimulating comments about the mathematical significance of the Morrison–Greene bracket.

This work was performed at the Center for Nonlinear Studies of the Los Alamos National Laboratory, sponsored by the United States Department of Energy.

References

[1] C.S. Gardner, J. Math. Phys. 12 (1971) 1548.

[2] J. Gibbons, D.D. Holm and B. Kupershmidt, Phys. Lett. 90A (1982) 281.

[3] D.D. Holm and B. Kupershmidt, Poisson Structures of Superfluids, Phys. Lett. 91A (1982) 425.

[4] J.E. Marsden, T. Ratiu and A. Weinstein, Semidirect Products and Reduction in Mechanics (preprint).

[5] J.E. Dzyaloshinskii and G.E. Volovick, Ann. Phys. 125 (1980) 67.

[6] P.J. Morrison and J.M. Greene, Phys. Rev. Lett. 45 (1980) 790; errata, ibid. 48 (1982) 569.

[7] D.D. Holm and B. Kupershmidt, "Poisson Brackets and Clebsch Representations for Magneto-hydrodynamics, Multifluid Plasmas, and Elasticity", Physica 6D (1983) 347.

[8] D.R. Lebedev and Yu.I. Manin, "Hamiltonian Operator of Gel'fand-Dikii and Coadjoint Representation of Volterra Group", Funk. Anal. Appl. 13:4 (1979) 40–46.

[9] D.R. Lebedev, "Benney's Long Waves Equations: Hamiltonian Formalism", Lett. Math. Phys. 3 (1979) 481–488.

[10] B.A. Kupershmidt, "On Dual Spaces of Differential Lie Algebras", these proceedings.