Smooth IGA basis with polar point

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1 Introduction

We use here the strategy introduced by Toshniwal et al. in [3] to define a smooth (C^1) IGA type map based on B-splines into a domain with a singularity at a unique pole.

2 Spline mapping on a polar domain

We shall consider here a discrete mapping approximating (35)-(36) with uniform B-splines of degree p in each direction with open knots at the endpoints in r and uniform periodic in θ, denoted by $N_i^s(s)$ and $N_j^\theta(\theta)$ respectively. The corresponding logical mesh is defined by $s_i = i/(n_s - 1)$ and $\theta_j = j2\pi/n_\theta$ yielding a uniform mesh of $[0,1] \times [0, 2\pi[$. In the s direction we add to the grid points the duplicated knots $s_{-p}, \ldots s_{-1} = 0$ and $s_{n_s+1}, \ldots s_{n_s+p} = 1$. We shall also need the Greville points in s defined for a B-spline of degree p by $\bar{s}_m = \sum_{i=m+1}^{m+p} s_i/p$, $m = -p, \ldots, n_s - 1$. The dimension of the spline space in s is then $n_s + p - 1$ The polar IGA mapping, with pole at (x_0, y_0) denoted by $\mathbf{F} : [0,1] \times [0, 2\pi[\to \Omega$ is defined by $\mathbf{F}(s, \theta) = (x(s, \theta), y(s, \theta))$ with

$$x(s, \theta) = x_0 N_0^s(s) + \sum_{i=1}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} c_{ij}^x N_i^s(s) N_j^\theta(\theta), \tag{1}$$

$$y(s, \theta) = y_0 N_0^s(s) + \sum_{i=1}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} c_{ij}^y N_i^s(s) N_j^\theta(\theta). \tag{2}$$

This means in particular that all the control points at $i = 0$ are equal: $(c_{0,j}^x, c_{0,j}^y) = (x_0, y_0)$.

Due to the open knot sequence, where the first and last $p+1$ knots defining the spline are duplicated at the two end points of the domain, the spline is interpolating at $s = 0$ and $s = 1$, which means that $N_0^s(0) = 1$ whereas $N_i^s(0) = 0$ for $1 \leq i \leq n_s$ and $N_{n_s}^s(0) = 1$ whereas $N_i^s(n_s) = 0$ for $0 \leq i \leq n_s - 1$. This implies in particular that $x(0, \theta) = x_0$ and $y(0, \theta) = y_0$. The whole $s = 0$ line of the logical space collapses to the polar point $(x_0, 0)$, which introduces a singularity in the mapping. Moreover

$$\frac{\partial x}{\partial \theta}(0, \theta) = \sum_{i=1}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} c_{ij}^x N_i^s(0)(N_j^\theta)'(\theta) = 0, \quad \frac{\partial y}{\partial \theta}(0, \theta) = \sum_{i=1}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} c_{ij}^y N_i^s(0)(N_j^\theta)'(\theta) = 0 \quad \forall \theta. \tag{3}$$

Now, the open knot sequence also implies that $(N_0^s)'(0) = -(N_1^s)'(0) \neq 0$ and $(N_i^s)'(0) = 0$ for $2 \leq i \leq n_s$. Hence

$$\frac{\partial x}{\partial s}(0, \theta) = (N_0^s)'(0) \left(x_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^x N_j^\theta(\theta) \right), \quad \frac{\partial y}{\partial s}(0, \theta) = (N_0^s)'(0) \left(y_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^y N_j^\theta(\theta) \right). \tag{4}$$

3 C^1 approximation of a smooth function based on the spline mapping

Let us know consider a function $g : \Omega \to \mathbb{R}$ defined by

$$g(x, y) = \sum_{i,j} g_{i,j} N_{i,j}(x, y), \tag{5}$$

such that the pullback of $N_{i,j}$ on the logical domain defined by the map $\mathbf{F}$ introduced in the previous section is $N_i^s(s) N_j^\theta(\theta)$, more precisely

$$N_{i,j}(x(s, \theta), y(s, \theta)) = N_i^s(s) N_j^\theta(\theta).$$

The the pullback of g is defined by

$$\hat{g}(s, \theta) = g(x(s, \theta), y(s, \theta)) = \sum_{i,j} g_{i,j} N_i^s(s) N_j^\theta(\theta). \tag{6}$$

The function g is clearly in $C^1(\Omega) \setminus \{(x_0, y_0)\}$, smooth at all points but the polar point, and, following [3], the condition for the function g to be in $C^1(\Omega)$ is that

$$\lim_{(x,y) \to (x_0, y_0)} g(x, y) = \alpha, \quad \lim_{(x,y) \to (x_0, y_0)} \frac{\partial g}{\partial x}(x, y) = \beta_x, \quad \lim_{(x,y) \to (x_0, y_0)} \frac{\partial g}{\partial y}(x, y) = \beta_y,$$

where α, β_x, β_y are given real numbers. In other words, these limits should yield unique numbers. These constraints restrict the degrees of freedom associated to the basis functions $N_{i,j}$ for $i = 0, 1$.

Let us start with the C^0 constraint. We need

$$\hat{g}(0, \theta) = (x_0, y_0) = \alpha = \sum_j g_{i,j} N_j^\theta(\theta), \quad \forall \theta.$$

This is true only if $g_{i,j}$ is independent of j, which means that a unique value needs to be given to g for all θ at the polar point, which is quite natural.

Using the chain rule we find that

$$\frac{\partial \hat{g}}{\partial s}(0, \theta) = \frac{\partial x}{\partial s} \frac{\partial g}{\partial x}(x_0, y_0) + \frac{\partial y}{\partial s} \frac{\partial g}{\partial y}(x_0, y_0)$$

$$= (N_0^s)'(0) \left(x_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^x N_j^\theta(\theta) \right) \beta_x + (N_0^s)'(0) \left(y_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^y N_j^\theta(\theta) \right) \beta_y, \tag{7}$$

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and because $\frac{\partial x}{\partial \theta}(0, \theta) = \frac{\partial y}{\partial \theta}(0, \theta) = 0$, $\frac{\partial \hat{g}}{\partial \theta}(0, \theta) = 0$ for all θ, which imposes no additional constraint.

Hence, the degrees of freedom $g_{i,j}$ corresponding to $i = 0, 1$ and for all j are constraint by the three values α, β_x, β_y. Hence all the corresponding basis functions need to be replaced by only three basis functions, that can be defined as appropriate linear combinations of the existing ones. In order to maintain the partition of unity property of the splines, Toshniwal et al [3] propose to use barycentric coordinates to construct the degrees of freedom. Taking a triangle enclosing the axis and the first row of control points (c_{1j}^x, c_{1j}^y) for all j, we call $(\lambda_0, \lambda_1, \lambda_2)$ the barycentric coordinates of any point with respect to the vertices of this triangle. And we define the three new basis functions $\tilde{N}_l$, $l = 0, 1, 2$, such that

$$\tilde{N}_l(s, \theta) = \sum_{i=0}^{1} \sum_{j=0}^{N_\theta - 1} e_{i,j}^l N_i^s(s) N_j^\theta(\theta), \tag{8}$$

$$\tilde{N}_l(0, \theta) = \lambda_l(x_0, y_0), \tag{9}$$

$$\frac{\partial \tilde{N}_l}{\partial s}(0, \theta) = (N_0^s)'(0) \left(x_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^x N_j^\theta(\theta) \right) \frac{\partial \lambda_l}{\partial x}(x_0, y_0) \tag{10}$$

$$+ (N_0^s)'(0) \left(y_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^y N_j^\theta(\theta) \right) \frac{\partial \lambda_l}{\partial y}(x_0, y_0). \tag{11}$$

Plugging (9) into (8) yields

$$\sum_{j=0}^{N_\theta - 1} e_{0,j}^l N_j^\theta(\theta) = \lambda_l(x_0, y_0).$$

Due to the partition of unity property $\sum_{j=0}^{N_\theta - 1} N_j^\theta(\theta) = 1$ forall θ, hence a solution is

$$e_{0,j}^l = \lambda_l(x_0, y_0), \quad \forall j. \tag{12}$$

So clearly $\sum_{l=0,2} e_{0,j}^l = 1$ and the $e_{0,j}^l$ are non negative.

Now, plugging (11) into (8) and using that $(N_0^s)'(0) = -(N_1^s)'(0) \neq 0$ and $(N_i^s)'(0) = 0$ for $2 \leq i \leq n_s$ we get

$$\sum_{j=0}^{N_\theta - 1} (e_{0,j}^l - e_{1,j}^l) N_j^\theta(\theta) = \left(x_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^x N_j^\theta(\theta) \right) \frac{\partial \lambda_l}{\partial x}(x_0, y_0)$$

$$+ \left(y_0 - \sum_{j=0}^{n_\theta - 1} c_{1j}^y N_j^\theta(\theta) \right) \frac{\partial \lambda_l}{\partial y}(x_0, y_0). \tag{13}$$

Let us denote by $\alpha^l = \lambda_l(x_0, y_0)$, $\beta_x^l = \frac{\partial \lambda_l}{\partial x}(x_0, y_0)$ and $\beta_y^l = \frac{\partial \lambda_l}{\partial y}(x_0, y_0)$. Then using (12), we get

$$\sum_{j=0}^{N_\theta - 1} e_{1,j}^l N_j^\theta(\theta) = \alpha^l - (x_0 \beta_x^l + y_0 \beta_y^l) + \sum_{j=0}^{n_\theta - 1} (c_{1j}^x \beta_x^l + c_{1j}^y \beta_y^l) N_j^\theta(\theta). \tag{14}$$

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A solution is the given by

$$e_{1,j}^l = \alpha^l + (c_{1j}^x - x_0)\beta_x^l + (c_{1j}^y - y_0)\beta_y^l. \tag{15}$$

Taking the derivatives of the relation $\lambda_0 + \lambda_1 + \lambda_2 = 1$ and applying it at the point (x_0, y_0) yields

$$\beta_x^0 + \beta_x^1 + \beta_x^2 = 0, \quad \beta_y^0 + \beta_y^1 + \beta_y^2 = 0, \tag{16}$$

so that $e_{1,j}^0 + e_{1,j}^1 + e_{1,j}^2 = \alpha^0 + \alpha^1 + \alpha^2 = 1$. On the other hand we notice that $e_{1,j}^l$ is the Taylor expansion of λ_l around (x_0, y_0) evaluated at the point $(c_{1,j}^x, c_{1,j}^y)$, and, as the barycentric coordinates λ_l are affine, this Taylor expansion is exact. Hence

$$e_{1,j}^l = \lambda_l(c_{1,j}^x, c_{1,j}^y),$$

and due to the properties of the barycentric coordinates $\lambda_l(c_{1,j}^x, c_{1,j}^y) \geq 0$ for all l if and only if the point $(c_{1,j}^x, c_{1,j}^y)$ lies inside the triangle. This implies that the new basis functions are positive if the control points $(c_{1,j}^x, c_{1,j}^y)$ lie inside the triangle for all j.

More concretely, let us consider like in [3] the equilateral triangle with vertices $T_0 = (x_0+\tau, y_0)$, $T_1 = (x_0 - \frac{\tau}{2}, y_0 + \frac{\sqrt{3}\tau}{2})$, $T_2 = (x_0 - \frac{\tau}{2}, y_0 - \frac{\sqrt{3}\tau}{2})$. The corresponding barycentric coordinates satisfy

$$T_0^x \lambda_0(x, y) + T_1^x \lambda_1(x, y) + T_2^x \lambda_2(x, y) = x, \tag{17}$$

$$T_0^y \lambda_0(x, y) + T_1^y \lambda_1(x, y) + T_2^y \lambda_2(x, y) = y, \tag{18}$$

$$\lambda_0(x, y) + \lambda_1(x, y) + \lambda_2(x, y) = 1, \tag{19}$$

so that

$$\lambda_0(x, y) = \frac{1}{3} + \frac{2(x - x_0)}{3\tau}, \tag{20}$$

$$\lambda_1(x, y) = \frac{1}{3} - \frac{(x - x_0)}{3\tau} + \frac{\sqrt{3}(y - y_0)}{3\tau}, \tag{21}$$

$$\lambda_2(x, y) = \frac{1}{3} - \frac{(x - x_0)}{3\tau} - \frac{\sqrt{3}(y - y_0)}{3\tau}. \tag{22}$$

$$\tag{23}$$

So, the control points of the mapping being known, we need to choose τ such that

$$\tau \geq -2(c_{1,j}^x - x_0), \quad \tau \geq (c_{1,j}^x - x_0) - \sqrt{3}(c_{1,j}^y - y_0), \quad \tau \geq (c_{1,j}^x - x_0) - \sqrt{3}(c_{1,j}^y - y_0), \quad \forall j, \tag{24}$$

which is obtained by taking

$$\tau = \max\left(\max_j(-2(c_{1,j}^x - x_0)), \max_j((c_{1,j}^x - x_0) - \sqrt{3}(c_{1,j}^y - y_0)), \max_j((c_{1,j}^x - x_0) + \sqrt{3}(c_{1,j}^y - y_0))\right). \tag{25}$$

We are now in position to summarise our construction. Given a polar mapping of the form (1)-(2), we construct the triangle with vertices (T_0, T_1, T_2), τ being defined by (25), and the associated barycentric coordinates (20)-(22). We can then define the three new C^1 basis functions

$$\tilde{N}_l(s, \theta) = \lambda_l(x_0, y_0)N_0^s(s) + \left(\sum_{j=0}^{n_\theta-1} \lambda_l(c_{1,j}^x, c_{1,j}^y)N_j^\theta(\theta)\right)N_1^s(s) \quad l = 0, 1, 2. \tag{26}$$

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Remark 1 *As the barycentric coordinates sum up to one, we easily see that*

$$\sum_{l=0}^{2} \tilde{N}_l(s,\theta) = N_0^s(s) + \left(\sum_{j=0}^{n_\theta-1} N_j^\theta(\theta) \right) N_1^s(s) = N_0^s(s) + N_1^s(s),$$

is independent of θ. It is the spline whose coefficients corresponding to the first two rows (in s) of control points are one and all the others zero. On the other hand the unique C^0 basis functions at the center is obtained by setting only the first row of control points at the center to one.

So if we want only C^1 functions we need to replace the initial representation of $\hat{g}$ in (6) by

$$\hat{g}(s,\theta) = \sum_{l=0}^{2} \tilde{g}_l \tilde{N}_l(s,\theta) + \sum_{i=2}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} g_{i,j} N_i^s(s) N_j^\theta(\theta). \tag{27}$$

Plugging in the expression for $\tilde{N}_l(s,\theta)$ given by (26), we get the original expression

$$\hat{g}(s,\theta) = \sum_{i=0}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} g_{i,j} N_i^s(s) N_j^\theta(\theta), \tag{28}$$

but with the constrained values of $g_{0,j}$ and $g_{1,j}$:

$$g_{0,j} = \sum_{l=0}^{2} \tilde{g}_l \lambda_l(x_0, y_0), \quad g_{1,j} = \sum_{l=0}^{2} \tilde{g}_l \lambda_l(c_{1,j}^x, c_{1,j}^y), \tag{29}$$

depending only on the three coefficients $\tilde{g}_0, \tilde{g}_1, \tilde{g}_2$.

4 Interpolation on the C^1 basis

In this section we shall modify the classical tensor product spline information to take into account that the first two rows are constrained and that our interpolation spline is now given by (27). First we notice applying this formula for $s = 0$ that for all θ

$$\hat{g}(0,\theta) = \sum_{l=0}^{2} \tilde{g}_l \lambda_l(x_0, y_0),$$

so that $\hat{g}(0,\theta)$ has a single value for all θ. This means that the interpolation values corresponding to $s = 0$ need to all be taken identical, *e.g.* to $\hat{g}(0,0)$. Then assume g_{1,j_1} and g_{1,j_2} for two given different integers j_1 and j_2 and that the points (x_0, y_0), $(c_{1,j_1}^x, c_{1,j_1}^y)$, $(c_{1,j_2}^x, c_{1,j_2}^y)$ are not aligned, which is equivalent to

$$\det \begin{pmatrix} \lambda_0(x_0, y_0) & \lambda_1(x_0, y_0) & \lambda_2(x_0, y_0) \\ \lambda_0(c_{1,j_1}^x, c_{1,j_1}^y) & \lambda_1(c_{1,j_1}^x, c_{1,j_1}^y) & \lambda_2(c_{1,j_1}^x, c_{1,j_1}^y) \\ \lambda_0(c_{1,j_2}^x, c_{1,j_2}^y) & \lambda_1(c_{1,j_2}^x, c_{1,j_2}^y) & \lambda_2(c_{1,j_2}^x, c_{1,j_2}^y) \end{pmatrix} \neq 0.$$

In order for this determinant not to be numerically close to 0, we propose to choose $j_1 = 0$ and $j_2 = n_\theta/4$ or a close integer. Then the $\tilde{g}_l$ are the unique solutions of

$$\lambda_0(x_0, y_0)\tilde{g}_0 + \lambda_1(x_0, y_0)\tilde{g}_1 + \lambda_2(x_0, y_0)\tilde{g}_2 = g_{0,0} = \hat{g}(0, 0) \qquad (30)$$

$$\lambda_0(c^x_{1,j_1}, c^y_{1,j_1})\tilde{g}_0 + \lambda_1(c^x_{1,j_1}, c^y_{1,j_1})\tilde{g}_1 + \lambda_2(c^x_{1,j_1}, c^y_{1,j_1})\tilde{g}_2 = g_{1,j_1} \qquad (31)$$

$$\lambda_0(c^x_{1,j_2}, c^y_{1,j_2})\tilde{g}_0 + \lambda_1(c^x_{1,j_2}, c^y_{1,j_2})\tilde{g}_1 + \lambda_2(c^x_{1,j_2}, c^y_{1,j_2})\tilde{g}_2 = g_{1,j_2} \qquad (32)$$

and can be obtained using for example Cramer's rule.

We can now proceed with our interpolation algorithm. The classical interpolation points would be $(\bar{s}_m, \theta_n)$, defined by the Greville points $\bar{s}_m$ in s and the logical grid points θ_n in θ. However in our methods the first to rows of degrees of freedom corresponding to $i = 0, 1$ have been replaced by only three degrees of freedom. For this reason we consider only one point on the first row of Greville points, namely $(\bar{s}_{-p} = 0, \theta_0 = 0)$. And on the second row of Greville points, we consider only the two points $(\bar{s}_{-p+1} = 1/(p(n_s-1)), \theta_0 = 0)$ and $(\bar{s}_{-p+1} = 1/(p(n_s - 1))), \theta_{n_\theta/4} = \frac{\pi}{2})$.

Denote by g the smooth function that we want to interpolate at the interpolation points $(\bar{s}_m, \theta_n)$, with $-p + 2 \leq m \leq n_s - 1$ and $0 \leq n \leq n_\theta - 1$ and the additional three points $(0, 0)$, $(1/(p(n_s - 1)), 0)$ and $(1/(p(n_s - 1)), \pi/2)$.

Then the steps of our algorithm are:

1. $g_{0,j} = g(0, 0)$, for $j = 0, \ldots, n_\theta - 1$.

2. For $m = -p + 1, \ldots, n_s - 1$ solve the periodic 1D interpolation problem

$$g(\bar{s}_m, \theta_n) = \sum_{j=0}^{n_\theta-1} h_{j,m} N^\theta_j(\theta_n),$$

   to obtain $h_{j,m}$. Note that we also use the second row of Greville points here, to simplify the algorithm.

3. Solve the two 1D interpolation problems in r corresponding to $j = 0$ and $j = n_\theta/4$

$$h_{0,m} - g_{0,0} = \sum_{i=1}^{n_s+p-2} g_{i,0} N^s_i(\bar{s}_m), \qquad m = -p + 1, \ldots, n_s - 1.$$

$$h_{n_\theta/4,m} - g_{0,n_\theta/4} = \sum_{i=1}^{n_s+p-2} g_{i,n_\theta/4} N^s_i(\bar{s}_m), \qquad m = -p + 1, \ldots, n_s - 1.$$

4. Use the values of $g_{1,0}$ and $g_{1,n_\theta/4}$ as well as $g(0, 0)$, obtained at the previous step in roder to compute $\tilde{g}_0, \tilde{g}_1, \tilde{g}_2$ as a solution of (30)-(32).

5. Compute

$$g_{1,j} = \sum_{l=0}^{2} \tilde{g}_l \lambda_l(c^x_{1,j}, c^y_{1,j}).$$

6. For $j \neq 0$ and $j \neq n_\theta/4$ solve the 1D interpolation problems in r

$$h_{j,m} - g_{0,j} - g_{1,j} N^s_1(\bar{s}_m) = \sum_{i=2}^{n_s+p-2} g_{i,0} N^s_i(\bar{s}_m), \qquad m = -p + 2, \ldots, n_s - 1.$$

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5 Finite Element Poisson solver on the C^1 basis

Let us consider the following Poisson equation on our polar domain with vanishing Dirichlet boundary conditions at the outer boundary

$$- \operatorname{div}(n_0(x, y)\nabla\phi) = \rho. \tag{33}$$

The weak formulation pulled back on the logical mesh reads: Find $\hat{\phi} \in H^1(\hat{\omega})$ such that

$$\int \hat{n}_0(s, \theta)\hat{\nabla}\hat{\varphi} \cdot G^{-1}\hat{\nabla}\hat{\phi}J\,\mathrm{d}s\mathrm{d}\theta = \int \hat{\rho}\hat{\varphi}J\,\mathrm{d}s\mathrm{d}\theta \tag{34}$$

where $J(s, \theta)$ is the Jacobian of the mapping $\mathbf{F}$ and G the metric matrix defined from the Jacobian matrix $D\mathbf{F}$ by $G = (D\mathbf{F})^\top D\mathbf{F}$. We also denote by $\hat{\nabla} = (\partial_s, \partial_\theta)^\top$ and the hatted functions are defined on the logical domain as in (6).

6 Analytical equilibria

We consider the geometry proposed in [1] defined by

$$x(s, \theta) = x_0 - d_0 r^2 + (1 - e_0)r\cos\theta \tag{35}$$
$$y(s, \theta) = (1 + e_0)r\sin\theta \tag{36}$$

where the parameters x_0, d_0, e_0 define the polar point, the Shafranov shift and the elongation respectively. Taking them all equal to zero, we recover the standard polar geometry. The values proposed in [1] are $x_0 = 0.08$, $d_0 = 0.2$, $e_0 = 0.3$.

Special solutions of the Grad-Shafranov equations are the Soloviev equilibria used in [2] defined for vanishing triangularity by

$$\psi(x, y) = \left(x - \frac{1}{2}e(1 - x^2)\right)^2 + \left(\left(1 - \frac{1}{4}e^2\right)(1 + ex)^2\right)\frac{y^2}{b^2}. \tag{37}$$

The isolines of ψ can be parametrized by (s, θ) such that

$$X = x - \frac{1}{2}e(1 - x^2) = r\cos\theta \tag{38}$$

$$Y = \left(\sqrt{1 - \frac{e^2}{4}}\right)(1 + ex)y = br\sin\theta \tag{39}$$

Solving for x with respect to X yields, taking the root with the negative sign

$$ex = 1 - \sqrt{1 + e(e + 2 * X)}. \tag{40}$$

We shall consider here a discrete mapping approximating (35)-(36) with uniform B-splines with open knots at the endpoints in r and uniform periodic in θ, denoted by $N_i^s(s)$

and $N_j^\theta(\theta)$ respectively. The IGA mapping denoted by $\mathbf{F} : [0,1] \times [0,2\pi[\to \Omega$ is defined by $\mathbf{F}_h(s,\theta) = (x(s,\theta), y(s,\theta))$ with

$$x(s,\theta) = \sum_{i=0}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} [(x_0 - d_0 r_i^2 + (1-e_0)r_i \cos\theta_j)]N_i^s(s)N_j^\theta(\theta) \tag{41}$$

$$y(s,\theta) = \sum_{i=0}^{n_s+p-2} \sum_{j=0}^{n_\theta-1} [(1+e_0)r_i \sin\theta_j]N_i^s(s)N_j^\theta(\theta) \tag{42}$$

where $r_i = i/n_s$ and $\theta_j = j2\pi/n_\theta$ define a uniform mesh of $[0,1] \times [0,2\pi[$.

In order to define the IGA mapping we need from these analytical equilibria. These are interpolated to our splines at the Greville points in s and the periodic knots in θ. They are displayed in Figure 1.

References

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[2] Olivier Czarny and Guido Huysmans. Bézier surfaces and finite elements for mhd simulations. *Journal of computational physics*, 227(16):7423–7445, 2008.

[3] Deepesh Toshniwal, Hendrik Speleers, René R Hiemstra, and Thomas JR Hughes. Multi-degree smooth polar splines: A framework for geometric modeling and isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering*, 316:1005–1061, 2017.
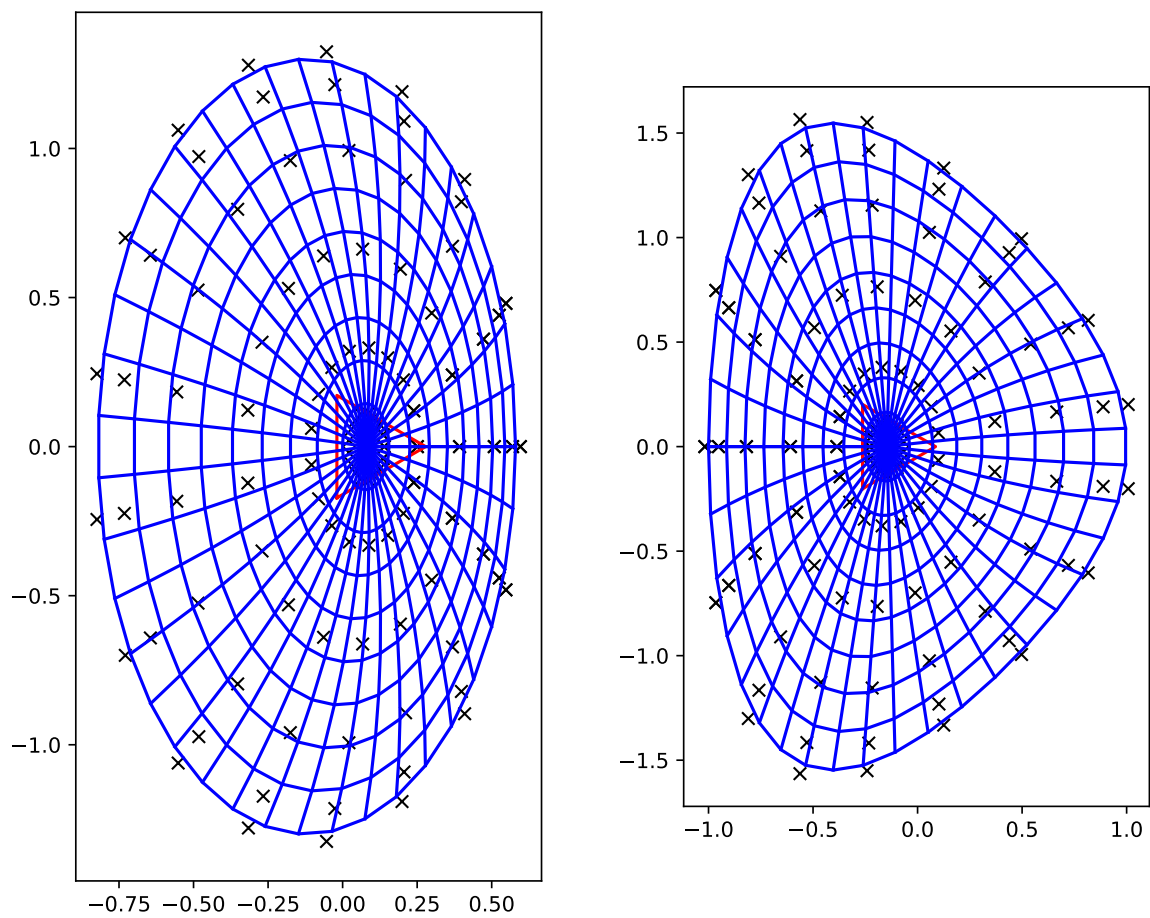
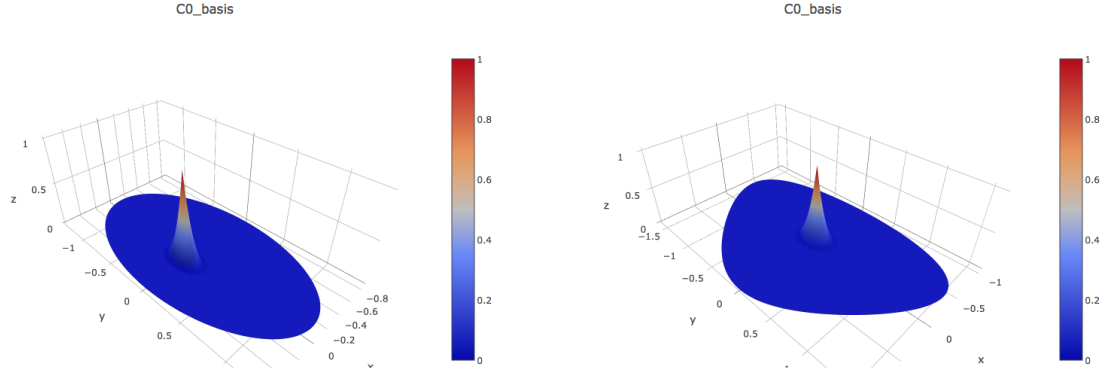Figure 1: Target grid (left). Soloviev grid (right). Crosses are control points.

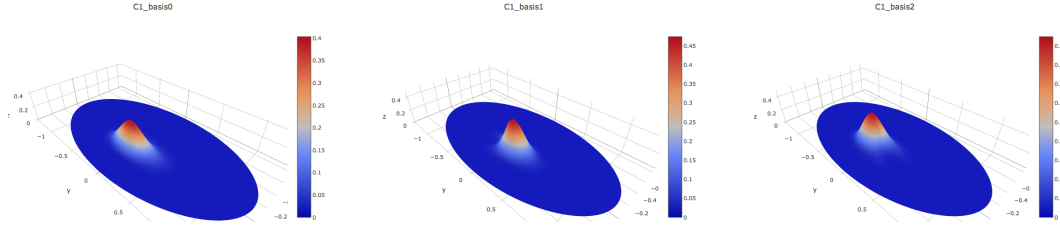Figure 2: Target grid (left). Soloviev grid (right).The C^0 basis function
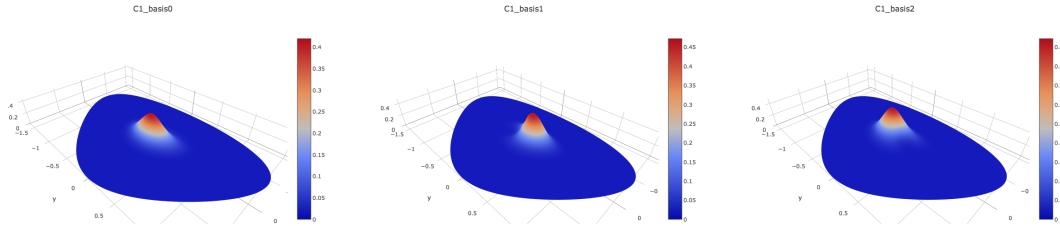

Figure 3: Target grid. The three C^1 basis functions

Figure 4: Soloviev grid. The three C^1 basis functions