De Rham sequence of conforming finite element spaces for polar mappings

Florian Holderied[2] and Stefan Possanner[1,2]

[1]Technical University of Munich, Department of Mathematics, Boltzmannstraße 3, 85748 Garching, Germany
[2]Max Planck Institute for Plasma Physics, Boltzmannstraße 2, 85748 Garching, Germany

June 14, 2020

Abstract

We construct the de Rham sequence of conforming tensor product B-spline spaces for mappings with a polar singularity. The sequence is exact ($\mathrm{grad} V_0 = \ker V_1$, $\mathrm{curl} V_1 = \ker V_2$) and can be fit into the commuting diagram of finite element exterior calculus (FEEC). In the mapped domain, basis functions are C^2 at the pole in the first space V_0 and are C^0 at the pole in the third space V_3.

Contents

1 Introduction **1**

2 Problem statement **2**
2.1 Polar mapping . . . 2
2.2 Hilbert spaces of differential forms . . . 3
2.3 Construction of polar basis functions . . . 5

1 Introduction

2 Problem statement

2.1 Polar mapping

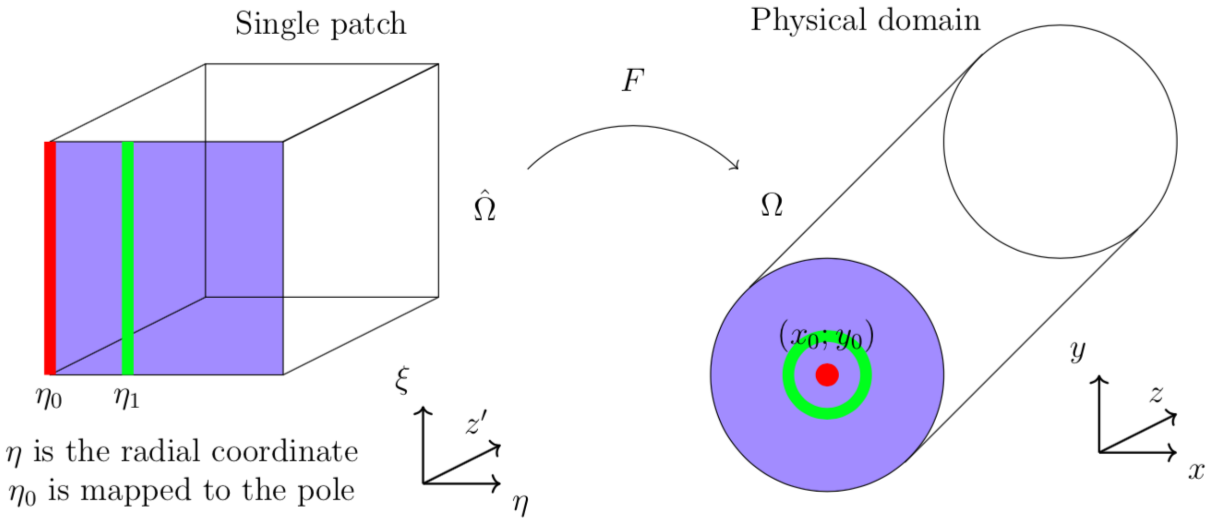

Figure 1: Cylindrical coordinates.

Let us denote the "physical domain" by $\Omega \subset \mathbb{R}^3$ and its Cartesian coordinates by $\mathbf{x} = (x, y, z) \in \Omega$. The "logical domain" $\hat{\Omega} \subset \mathbb{R}^3$ is assumed to be box-shaped, suitable for tensor product construction, and with logical (or patch) coordinates $\boldsymbol{\eta} = (\eta, \xi, z') \in \hat{\Omega}$. The two domains are related by the mapping

$$F : \hat{\Omega} \to \Omega\,, \qquad (\eta, \xi, z') \mapsto (x, y, z)\,, \qquad F^{-1} \in C^p(\Omega \setminus (x_0, y_0))\,. \tag{1}$$

The mapping F is C^p, $p \geq 1$ (later the spline degree), and invertible everywhere except at the pole (x_0, y_0). As a generic example, let us consider cylindrical coordinates defined on $\hat{\Omega} = [0, 1] \times [0, 2\pi) \times [0, L]$ via

$$F : \boldsymbol{\eta} \mapsto \mathbf{x} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(\eta) \cos \xi \\ f(\eta) \sin \xi \\ z' \end{pmatrix}\,, \tag{2}$$

where we assume f to be some function with the properties

$$f : [0, 1] \to \mathbb{R}\,, \quad f(0) = 0\,, \quad 0 < f' < \infty\,. \tag{3}$$

Hence, in the following η denotes the "radial coordinate" while ξ plays the role of the angular coordinate. The pole is attained for $\eta \to 0$. The Jacobian DF of F and its inverse are given by

$$DF = \begin{pmatrix} f' \cos \xi & -f \sin \xi & 0 \\ f' \sin \xi & f \cos \xi & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad DF^{-1} = \begin{pmatrix} 1/f' \cos \xi & 1/f' \sin \xi & 0 \\ -1/f \sin \xi & 1/f \cos \xi & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{4}$$

from which follow the metric tensor G and its inverse,

$$G = DF^\top DF = \begin{pmatrix} (f')^2 & 0 & 0 \\ 0 & f^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} 1/(f')^2 & 0 & 0 \\ 0 & 1/f^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

with the determinant $g = \det G = (ff')^2$. The cylindrical mapping is illustrated in Figure 1.

2.2 Hilbert spaces of differential forms

$$\begin{tikzcd}
H^1(\hat\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi_0"'] & H(\mathrm{curl}, \hat\Omega) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi_1"'] & H(\mathrm{div}, \hat\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_2"'] & L^2(\hat\Omega) \arrow[d, "\Pi_3"'] \\
V_0 \arrow[r, "\mathrm{grad}"] & V_1 \arrow[r, "\mathrm{curl}"] & V_2 \arrow[r, "\mathrm{div}"] & V_3
\end{tikzcd}$$

Figure 2: Commuting diagram for the logical domain $\hat\Omega$.

Conforming FE methods in three dimensions can be built upon the commuting diagram depicted in Figure 2. All spaces in this diagram refer to functions on the logical domain $\hat\Omega$. The upper line contains the continuous spaces well-known in FE analysis. In the framework of FEEC, these spaces refer to the components of differentiable n-forms, with $0 \le n \le 3$. We use the symbol

$$H^1(\hat\Omega) = \left\{ a : \hat\Omega \to \mathbb{R} \ s.t. \ |a|_0 + |\mathrm{grad}\, a|_1 < \infty \right\} \qquad \text{(0-forms)}, \tag{6}$$

$$H(\mathrm{curl}, \hat\Omega) = \left\{ \mathbf{a} : \hat\Omega \to \mathbb{R}^3 \ s.t. \ |\mathbf{a}|_1 + |\mathrm{curl}\, \mathbf{a}|_2 < \infty \right\} \qquad \text{(1-forms)}, \tag{7}$$

$$H(\mathrm{div}, \hat\Omega) = \left\{ \mathbf{a} : \hat\Omega \to \mathbb{R}^3 \ s.t. \ |\mathbf{a}|_2 + |\mathrm{div}\, \mathbf{a}|_3 < \infty \right\} \qquad \text{(2-forms)}, \tag{8}$$

$$L^2(\hat\Omega) = \left\{ a : \hat\Omega \to \mathbb{R} \ s.t. \ |a|_3 < \infty \right\} \qquad \text{(3-forms)}, \tag{9}$$

where the seminorms $|\cdot|_{0\leq n\leq 3}$ are given by

$$|a|_0^2 := \int_{\hat{\Omega}} a^2\,\sqrt{g}\,\mathrm{d}\boldsymbol{\eta}\,, \tag{10}$$

$$|\mathbf{a}|_1^2 := \int_{\hat{\Omega}} \mathbf{a}\,G^{-1}\mathbf{a}\,\sqrt{g}\,\mathrm{d}\boldsymbol{\eta}\,, \tag{11}$$

$$|\mathbf{a}|_2^2 := \int_{\hat{\Omega}} \mathbf{a}\,G\,\mathbf{a}\,\frac{1}{\sqrt{g}}\,\mathrm{d}\boldsymbol{\eta}\,, \tag{12}$$

$$|a|_3^2 := \int_{\hat{\Omega}} a^2\,\frac{1}{\sqrt{g}}\,\mathrm{d}\boldsymbol{\eta}\,. \tag{13}$$

Denoting $\hat{\nabla} = (\partial_\eta, \partial_\xi, \partial_{z'})$ in logical coordinates, the differential operators can be written as

$$\mathrm{grad} = \hat{\nabla}\,, \qquad \mathrm{curl} = (\hat{\nabla}\times)\,, \qquad \mathrm{div} = (\hat{\nabla}\cdot)\,. \tag{14}$$

In cylindrical coordinates, the above Hilbert spaces are defined as follows:

$$a \in H^1(\hat{\Omega}):\ \int_{\hat{\Omega}} a^2\,ff'\,\mathrm{d}\boldsymbol{\eta} + \int_{\hat{\Omega}} \left[(\partial_\eta a)^2\frac{f}{f'} + (\partial_\xi a)^2\frac{f'}{f} + (\partial_{z'}a)^2\,ff'\right]\mathrm{d}\boldsymbol{\eta} < \infty\,, \tag{15}$$

$$\begin{aligned}\mathbf{a} \in H(\mathrm{curl},\hat{\Omega}):\ &\int_{\hat{\Omega}} \left[a_\eta^2\frac{f}{f'} + a_\xi^2\frac{f'}{f} + a_{z'}^2\,ff'\right]\mathrm{d}\boldsymbol{\eta} \\ &+ \int_{\hat{\Omega}} \left[(\partial_\xi a_{z'} - \partial_{z'}a_\xi)^2\frac{f'}{f} + (\partial_{z'}a_\eta - \partial_\eta a_{z'})^2\frac{f}{f'} + (\partial_\eta a_\xi - \partial_\xi a_\eta)^2\,\frac{1}{ff'}\right]\mathrm{d}\boldsymbol{\eta} < \infty\,,\end{aligned} \tag{16}$$

$$\begin{aligned}\mathbf{a} \in H(\mathrm{div},\hat{\Omega}):\ &\int_{\hat{\Omega}} \left[a_\eta^2\frac{f'}{f} + a_\xi^2\frac{f}{f'} + a_{z'}^2\,\frac{1}{ff'}\right]\mathrm{d}\boldsymbol{\eta} \\ &+ \int_{\hat{\Omega}} \left[(\partial_\eta a_\eta)^2 + (\partial_\xi a_\xi)^2 + (\partial_{z'}a_{z'})^2\right]\frac{1}{ff'}\,\mathrm{d}\boldsymbol{\eta} < \infty\,,\end{aligned} \tag{17}$$

$$a \in L^2(\hat{\Omega}):\ \int_{\hat{\Omega}} a^2\,\frac{1}{ff'}\,\mathrm{d}\boldsymbol{\eta} < \infty\,. \tag{18}$$

For $f(\eta) = \eta^q$ with $q > 0$ we have $f/f' = \eta/q$ and $ff' = q\eta^{2q-1}$. Then $q = 1/2$ yields $f/f' = 2\eta$ and $ff' = 1/2$ such that integrals featuring the factor f'/f must be handled with care on the discrete level. The Hilbert spaces form an exact sequence, meaning that

$$\mathrm{grad}\,H^1 = \ker(\mathrm{curl}\,H(\mathrm{curl}))\,, \qquad \mathrm{curl}\,H(\mathrm{curl}) = \ker(\mathrm{div}\,H(\mathrm{div}))\,. \tag{19}$$

The operators Π_j, $0 \leq j \leq 3$ project onto the finite-dimensional subspaces V_j, $0 \leq j \leq 3$, which will be spanned by tensor product basis functions, constructed from univariate B-splines of degree p, denoted by $\hat{N}_i^p(\eta)$, $0 \leq i \leq \hat{n}_N - 1$. The sequence of $\hat{n}_N$ splines $(\hat{N}_i^p)_i$ is constructed from the knot vector $\mathcal{T}_p = \{\eta_i\}_{0\leq i\leq n+2p}$, composed of $n + 2p + 1$ non-decreasing points η_i in a logical interval $\hat{I} \subset \mathbb{R}$. Here, n is the number of cells partitioning the interval $\hat{I}$ to define the 1D space grid. Each spline $\hat{N}_i^p$ is defined by $p+2$ neighbouring knots, such that we can fit $n+p$ spline functions into the knot vector $\mathcal{T}_p$. The ensuing spline basis $(\hat{N}_i^p)_i$ can be either periodic or "clamped". In the periodic case we relate the first p and the last p splines to obtain $\hat{n}_N = n$ basis functions. In the clamped case we have $\hat{n}_N = n + p$ basis functions. Moreover, for clamped splines $\hat{N}_0^p(\eta_0) = \hat{N}_{\hat{n}_N-1}^p(\eta_{n+2p}) = 1$, where η_0 is the left and η_{n+2p} is the right boundary of $\hat{I}$. Because of partition of unity we have

$$\text{clamped:} \qquad \hat{N}_i^p(\eta_0) = \hat{N}_i^p(\eta_{n+2p}) = 0\,, \qquad 1 \le i \le \hat{n}_N - 2\,. \tag{20}$$

The derivative of $\hat{N}_i^p(\eta)$ can be written as

$$(\hat{N}_i^p)'(\eta) = \hat{D}_{i-1}^{p-1}(\eta) - \hat{D}_i^{p-1}(\eta)\,, \tag{21}$$

where we introduced the "D-splines" of degree $p-1$ as

$$\hat{D}_i^{p-1}(\eta) = \frac{p}{\eta_{i+p+1} - \eta_{i+1}} \hat{N}_{i+1}^{p-1}(\eta)\,, \qquad -1 \le i \le \hat{n}_N - 1\,, \tag{22}$$

It is convenient to view D-splines as usual B-splines of degree $p-1$ created from the same knot vector $\mathcal{T}_p$ as the $\hat{N}_i^p$, and multiplied by the factor $p/(\eta_{i+p+1} - \eta_{i+1})$. We can fit $n+p+1$ basis splines of degree $p-1$ into the knot vector $\mathcal{T}_p$. In the periodic case we relate the first $p+1$ D-splines with the last $p+1$ D-splines. In the clamped case we have $\hat{D}_{-1}^{p-1}(\eta) = \hat{D}_{\hat{n}_N-1}^{p-1}(\eta) = 0$. Thus, we finally end up with the D-spline sequence $(\hat{D}_i^{p-1})_i$, $0 \le i \le \hat{n}_D - 1$, where $\hat{n}_D = \hat{n}_N$ for periodic and $\hat{n}_D = \hat{n}_N - 1$ for clamped splines.

2.3 Construction of polar basis functions

We start from the tensor product space V_0 defined by

$$V_0 = \text{span}(\hat{\Lambda}_i^0)\,, \qquad \hat{\Lambda}_i^0 = \hat{N}_{i_1}^{p_1}(\eta)\,\hat{N}_{i_2}^{p_2}(\xi)\,\hat{N}_{i_3}^{p_3}(z')\,, \quad i = i_1(\hat{n}_N^2\hat{n}_N^3) + i_2\,\hat{n}_N^3 + i_3\,, \tag{23}$$

for $0 \le i_j \le \hat{n}_N^j - 1$, $j = 1,2,3$. We assume $\hat{N}_{i_1}^{p_1}(\eta)$ to be clamped splines, whereas the other two directions are periodic. In order to maintain the exact sequence property (19) we construct the other spaces $V_{1\le j\le 3}$ as follows:

$$V_1 := \text{span}\left(\begin{pmatrix} \partial_\eta \hat{\Lambda}_i^0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_\xi \hat{\Lambda}_i^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \partial_{z'} \hat{\Lambda}_i^0 \end{pmatrix} \right), \tag{24}$$

$$V_2 := \text{span}\left(\begin{pmatrix} \partial_\xi \partial_{z'} \hat{\Lambda}_i^0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_\eta \partial_{z'} \hat{\Lambda}_i^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \partial_\eta \partial_\xi \hat{\Lambda}_i^0 \end{pmatrix} \right), \tag{25}$$

$$V_3 := \text{span}(\partial_\eta \partial_\xi \partial_{z'} \hat{\Lambda}_i^0)\,. \tag{26}$$

We shall hold on to this construction even when the basis $\hat{\Lambda}^0$ is not a tensor product basis anymore. One problem of the tensor product basis in the case of cylindrical coordinates is immediately obvious, namely that $\hat{\Lambda}_i^0$ is not single-valued as $\eta \to 0$, hence at the pole. This means:

- Tensor product V_0-basis functions $\Lambda_i^0(\mathbf{x})$ are not C^0 at the pole in the physical domain.
- If we construct $\Lambda_i^0(\mathbf{x})$ to be C^0 somehow, V_1-basis functions are not single-valued at the pole.
- If we construct $\Lambda_i^0(\mathbf{x})$ to be C^1 somehow, the third V_2-basis functions and the V_3-basis functions (mixed derivatives $\partial_\eta \partial_\xi$) are not single-valued at the pole.
- Our goal is thus as follows: $\Lambda_i^0(\mathbf{x})$ **must be** C^1 **at the pole and** $\partial_\eta \partial_\xi \hat{\Lambda}_i^0$ **must be single-valued at the pole. We also want an IGA-compatible basis.**