Derivation of MHD/drift-kinetic hybrid model

Contents

1 Drift kinetic equation one species

1.1 Equation

We start from the drift kinetic setup that can be found in [1]. Meaning we take a constant external magnetic field in the toroidal direction

$$\mathbf{B}_0 = B_0\mathbf{b},\ \mathbf{b} = \mathbf{e}_\phi,$$

Remark 1.1 *In what follows we will consider a cylinder (we neglect the $1/R$ term in Jorek framework).*

We can express the vector potential $\mathbf{A}$ (see [1]) as

$$\mathbf{A}(x,t) = \mathbf{A}_0 + A_{\|}(x,t)\mathbf{b}, \tag{1.1}$$

where $\nabla \times \mathbf{A}_0 = B_0\mathbf{b}$, $A_{\|}$ is the vector potential in the parallel direction. Meaning

$$\mathbf{B}(x,t) = \nabla \times \mathbf{A} = B_0\mathbf{b} + \nabla A_{\|} \times \mathbf{b}. \tag{1.2}$$

We then start from the drift-kinetic equation obtained from this framework, and found in [1]

$$\begin{cases} \partial_t f + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla f + v_{\|}\mathbf{b}^{\star}\cdot\nabla f - \frac{q}{m}\left(\mathbf{b}^{\star}\cdot\nabla\Phi + \frac{1}{c}\partial_t A_{\|}\right)\partial_{v_{\|}} f = 0 \\ -\Delta A_{\|} = J_{\|} \\ -\Delta\Phi = \rho. \end{cases} \tag{1.3}$$

where

$$\mathbf{b}^{\star} = \mathbf{b} + \frac{\nabla A_{\|}\times\mathbf{b}}{B_0} = \frac{\mathbf{B}}{B_0} \tag{1.4}$$

Remark 1.2 *For simplicity of notation and because the only direction of interest while considering drift-kinetic model is the parallel direction we will denote $v_{\|}$ by v, and later on, $u_{\|}$ by u.*

1.2 Energy conservation

Lemma 1

Assuming that all the quantities are equal to zero at the boundary, the total energy satisfies

$$\partial_t \int_{\Omega}\left(\frac{1}{2}m_h \mid v_{\|}\mid^2 f_h + \frac{\mid\nabla A_{\|}\mid^2}{2c} + \frac{\mid\nabla\Phi\mid^2}{2}\right) = 0$$

Proof. We multiply by q and we integrate on the velocity space to obtain the continuity equation

$$\partial_t\rho + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho + \cdot\mathbf{b}^{\star}\cdot\nabla J_{\|} = 0$$

with $\int_{\mathbb{R}} f_h = \rho$ and $q_h\int_{\mathbb{R}} v_{\|} f_h = J_{\|}$.

We multiply the kinetic equation by $\frac{1}{2}m_h \mid v_{\|}\mid^2$ and we integrate with respect to space and velocity

$$\partial_t\int_{\Omega}\int_{\mathbb{R}}\frac{1}{2}m_h\mid v_{\|}\mid^2 f_h + \int_{\Omega}\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\left(\int_{\mathbb{R}}\frac{1}{2}m_h\mid v_{\|}\mid^2 f_h\right) + \int_{\Omega} v_{\|}\mathbf{b}^{\star}\cdot\nabla\left(\int_{\mathbb{R}}\frac{1}{2}m_h\mid v_{\|}\mid^2 f_h\right)$$
$$-\int_{\Omega}\int_{\mathbb{R}}\frac{1}{2}q_h\mid v_{\|}\mid^2\left(\mathbf{b}^{\star}\cdot\nabla\Phi + \frac{1}{c}\partial_t A_{\|}\right)\partial_{v_{\|}} f_h = 0$$

since $\nabla\cdot\left(\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\right) = 0$ and $\nabla\cdot(v_{\|}\mathbf{b}^{\star}) = 0$ we obtain that the second and third terms can be written in the divergence form. Using the flux theorem and the null boundary condition we obtain

$$
\begin{aligned}
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h - \int_\Omega \int_{\mathbb{R}} \frac{1}{2} q_h \mid v_\parallel \mid^2 \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\parallel \right) \partial_{v_\parallel} f_h = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\parallel \right) \int_{\mathbb{R}} f_h \partial_{v_\parallel} (\frac{1}{2} q_h \mid v_\parallel \mid^2) = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\parallel \right) \int_{\mathbb{R}} f_h q_h v_\parallel = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\parallel \right) J_\parallel = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \int_\Omega \mathbf{b}^\star \cdot \nabla \Phi J_\parallel + \int_\Omega \frac{1}{c} \partial_t A_\parallel J_\parallel = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h - \int_\Omega \frac{1}{c} \partial_t A_\parallel \Delta A_\parallel - \int_\Omega \Phi \mathbf{b}^\star \cdot \nabla J_\parallel = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \partial_t \int_\Omega \frac{\mid \nabla A_\parallel \mid^2}{2c} + \int_\Omega \Phi \partial_t \rho + \int_\Omega \frac{c}{B_0} (\mathbf{b} \times \nabla \Phi) \cdot \nabla \rho = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \partial_t \int_\Omega \frac{\mid \nabla A_\parallel \mid^2}{2c} - \int_\Omega \Phi \partial_t \Delta \Phi + \int_\Omega \nabla \cdot \left(\frac{c}{B_0} (\mathbf{b} \times \nabla \Phi) \rho \right) = 0 \\
&\partial_t \int_\Omega \left(\frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \partial_t \int_\Omega \frac{\mid \nabla A_\parallel \mid^2}{2c} + \partial_t \int_\Omega \frac{\mid \nabla \Phi \mid^2}{2} \right) = 0.
\end{aligned}
$$

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1.3 Fluid limite model

In this section we write the fluid model associated with the kinetic equation.

Notations

- $n = \int f(x,v,t) \mathrm{d}v$
- $nu = \int v f(x,v,t) \mathrm{d}v$
- $\rho = mn$

1.3.1 Motion equation

We integrate the first equation of (1.3) with respect to v and multiply it by m, we get

$$
\boxed{\partial_t \rho + \frac{c}{B_0} (\mathbf{b} \times \nabla \Phi) \cdot \nabla \rho + \mathbf{b}^\star \cdot \nabla (\rho u) = 0} \tag{1.5}
$$

1.3.2 Momentum equation

We integrate (1.3) $\times v$ with respect to v and multiply it by m

Step 1 We take $k = v - u$

$$m\mathbf{b}^{\star} \cdot \int \nabla v^2 f \mathrm{d}v = m\mathbf{b}^{\star} \cdot \nabla \int (k+u)^2 f \mathrm{d}v$$
$$= m\mathbf{b}^{\star} \cdot \nabla \int k^2 f \mathrm{d}v + \int 2(ku) f \mathrm{d}v + \int u^2 f \mathrm{d}v$$

But since u does not depend on v

$$\int (ku) f \mathrm{d}v = \int (v-u) u f \mathrm{d}v = u \int v f \mathrm{d}v - \rho u^2 = \rho u^2 - \rho u^2 = 0.$$

We define

- Parallel pressure $P = \int m k^2 f \mathrm{d}v$

Hence, we obtain,

$$m\mathbf{b}^{\star} \cdot \int \nabla v^2 f \mathrm{d}v = m\mathbf{b}^{\star} \cdot \big(\nabla(\rho u^2) + \nabla P\big) \tag{1.6}$$

Step 2

$$\int mv \frac{q}{m}(\mathbf{b}^{\star} \cdot \nabla\Phi + \frac{1}{c}\partial_t A_{\|})\partial_v f \mathrm{d}v = \int v \cdot \partial_v \left(q(\mathbf{b}^{\star} \cdot \nabla\Phi + \frac{1}{c}\partial_t A_{\|}) f \right) \mathrm{d}v$$
$$= -\int \partial_v(v) q(\mathbf{b}^{\star} \cdot \nabla\Phi + \frac{1}{c}\partial_t A_{\|}) f \mathrm{d}v = -\frac{q}{m}\left(\mathbf{b}^{\star} \cdot \nabla\Phi + \frac{1}{c}\partial_t A_{\|}\right)\rho.$$

which completes our computation and establishes the equation of motion

$$\boxed{\partial_t(\rho u) + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla(\rho u) + \mathbf{b}^{\star} \cdot \nabla(\rho u^2) + \mathbf{b}^{\star} \cdot \nabla P + a\left(\mathbf{b}^{\star} \cdot \nabla\Phi + \frac{1}{c}\partial_t A_{\|}\right)\rho = 0 \quad (1.7)}$$

1.3.3 Energy equation

We integrate (1.3) $\times |v|^2/2$ with respect to v and multiply it by m

Step 1 The first term leads to

$$\int m \frac{|v|^2}{2} \partial_t f = \partial_t \int m \frac{|v|^2}{2} f \mathrm{d}v$$
$$= \partial_t \int m \left(\frac{|k|^2}{2} f + kuf + \frac{|u|^2}{2} f \right) \mathrm{d}v$$
$$= \partial_t P + \frac{1}{2}\rho|u|^2$$
$$= \partial_t(\rho\epsilon).$$

The same computations lead to $\int m\frac{|v|^2}{2}\frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla f = \frac{c}{B_0}(\mathbf{b}\nabla\Phi)(\rho\epsilon)$.

Step 2

$$\int m\frac{|v|^2}{2}v\mathbf{b}^\star\cdot\nabla f = \mathbf{b}^\star\cdot\nabla\int m\frac{|v|^2}{2}vf\mathrm{d}v$$
$$= \mathbf{b}^\star\cdot\left(\nabla\int m\frac{|v|^2}{2}kf\mathrm{d}v + \nabla u\int m\frac{|v|^2}{2}f\mathrm{d}v\right),$$

But

$$\nabla\int m\frac{|v|^2}{2}kf\mathrm{d}v = \nabla\left(\int m\frac{|k|^2}{2}kf\mathrm{d}v + \int m(ku)kf\mathrm{d}v + \int m\frac{|u|^2}{2}kf\right)$$
$$= \nabla Q + \nabla(Pu),$$

because $\int u^2/2(v-u)f\mathrm{d}v = u^3\rho - u^3\rho f = 0$, and with $Q = \int k\frac{|k|^2}{2}f\mathrm{d}v$ parallel component of scalar heat flux ?. Which gives

$$\int m\frac{|v|^2}{2}v\mathbf{b}^\star\cdot\nabla f = \mathbf{b}^\star\cdot\nabla(\rho u\epsilon) + \mathbf{b}^\star\cdot\nabla Q + \mathbf{b}^\star\cdot\nabla(Pu)$$

Step 3

$$\int q\left(\frac{|v|^2}{2}\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\partial_v f\right)\mathrm{d}v = -\int q\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)f\cdot v\mathrm{d}v$$
$$= -q\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\int vf\mathrm{d}v$$

Which finally gives the equation of momentum

$$\boxed{\begin{aligned}&\partial_t(\rho\epsilon) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla(\rho\epsilon) + \mathbf{b}^\star\cdot\nabla(\rho u\epsilon) + \mathbf{b}^\star\cdot\nabla Q + \mathbf{b}^\star\cdot\nabla(Pu) +\\ &\qquad + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho u = 0\end{aligned}}$$

Now we assume that $Q = 0$.

$$\partial_t(\rho u) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla(\rho u) + \mathbf{b}^\star\cdot\nabla(\rho u^2) + \mathbf{b}^\star\cdot\nabla P + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho = 0$$

$$\rho\partial_t u + \rho\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla u + \rho u\mathbf{b}^\star\cdot\nabla u + u\left(\partial_t\rho + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho + \mathbf{b}^\star\cdot\nabla(\rho u)\right)$$
$$+ \mathbf{b}^\star\cdot\nabla P + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho = 0$$

$$\rho\partial_t u + \rho\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla u + \rho u\mathbf{b}^\star\cdot\nabla u + \mathbf{b}^\star\cdot\nabla P + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho = 0$$

We have the non conservative equation on the velocity. Now we consider the energy equation. For this we multiply by u and the density energy by $u^2/2$, we have

$$u\rho\partial_t u + u\rho\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla u + u\rho u\mathbf{b}^\star\cdot\nabla u + u\mathbf{b}^\star\cdot\nabla P + ua\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho$$
$$+\frac{u^2}{2}\partial_t\rho + \frac{u^2}{2}\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho + \frac{u^2}{2}\mathbf{b}^\star\cdot\nabla(\rho u) = 0$$

We obtain

$$\partial_t(\rho\frac{u^2}{2}) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla(\rho\frac{u^2}{2}) + \mathbf{b}^\star\cdot\nabla(\rho\frac{u^2}{2}u) + u\mathbf{b}^\star\cdot\nabla P + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho u$$

we subtract this equation to the energy equation we obtain

$$\partial_t(\frac{P}{\gamma-1}) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\frac{P}{\gamma-1} + \mathbf{b}^\star\cdot\nabla(\frac{P}{\gamma-1}u) + \mathbf{b}^\star\cdot\nabla(Pu) - u\mathbf{b}^\star\cdot\nabla P = 0$$
$$\partial_t(\frac{P}{\gamma-1}) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\frac{P}{\gamma-1} + \mathbf{b}^\star\cdot\nabla(\frac{\gamma P}{\gamma-1}u) - u\mathbf{b}^\star\cdot\nabla P = 0$$
$$\partial_t(\frac{P}{\gamma-1}) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\frac{P}{\gamma-1} + u\mathbf{b}^\star\cdot\nabla(\frac{P}{\gamma-1}) + \frac{\gamma}{\gamma-1}P\mathbf{b}^\star\cdot\nabla u = 0$$

1.3.4 Model

We then have the total conservative model

$$\begin{cases}\partial_t\rho + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho + \mathbf{b}^\star\cdot\nabla(\rho u) = 0\\ \partial_t(\rho u) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla(\rho u) + \mathbf{b}^\star\cdot\nabla(\rho u^2) + \mathbf{b}^\star\cdot\nabla P + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho = 0\\ \partial_t(\rho\epsilon) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla(\rho\epsilon) + \mathbf{b}^\star\cdot\nabla(\rho u\epsilon) + \mathbf{b}^\star\cdot\nabla(Pu) + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho u = 0\\ -\Delta A_\parallel = J_\parallel\\ -\Delta\Phi = \rho.\end{cases} \tag{1.8}$$

and the non-conservative form is given by

$$\begin{cases}\partial_t\rho + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho + \mathbf{b}^\star\cdot\nabla(\rho u) = 0\\ \rho\partial_t u + \rho\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla u + \rho u\mathbf{b}^\star\cdot\nabla u + \mathbf{b}^\star\cdot\nabla P + a\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho = 0\\ \partial_t(\frac{P}{\gamma-1}) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\frac{P}{\gamma-1} + u\mathbf{b}^\star\cdot\nabla(\frac{P}{\gamma-1}) + \frac{\gamma}{\gamma-1}P\mathbf{b}^\star\cdot\nabla u = 0\\ -\Delta A_\parallel = J_\parallel\\ -\Delta\Phi = \rho\end{cases} \tag{1.9}$$

2 Reduced MHD

2.1 Two species bulk plasma

Let us consider for the bulk plasma two species, denoted by 1 for the bulk ions and 2 for the bulk electrons, we got

$$\partial_t \rho_1 + \frac{c}{B_0}(\mathbf{b} \times \nabla \Phi) \cdot \nabla \rho_1 + \mathbf{b}^\star \cdot \nabla(\rho_1 u_1) = 0 \tag{2.10}$$

$$\partial_t \rho_2 + \frac{c}{B_0}(\mathbf{b} \times \nabla \Phi) \cdot \nabla \rho_2 + \mathbf{b}^\star \cdot \nabla(\rho_2 u_2) = 0 \tag{2.11}$$

$$\rho_1 \partial_t u_1 + \rho_1 \frac{c}{B_0}(\mathbf{b} \times \nabla \Phi) \cdot \nabla u_1 + \rho_1 u_1 \mathbf{b}^\star \cdot \nabla u_1 + \mathbf{b}^\star \cdot \nabla P_1 + a_1 \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c}\partial_t A_\parallel \right) \rho_1 = 0 \tag{2.12}$$

$$\rho_2 \partial_t u_2 + \rho_2 \frac{c}{B_0}(\mathbf{b} \times \nabla \Phi) \cdot \nabla u_2 + \rho_2 u_2 \mathbf{b}^\star \cdot \nabla u_2 + \mathbf{b}^\star \cdot \nabla P_2 + a_2 \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c}\partial_t A_\parallel \right) \rho_2 = 0, \tag{2.13}$$

$$\partial_t (\frac{P_1}{\gamma - 1}) + \frac{c}{B_0}(\mathbf{b} \times \nabla \Phi) \cdot \nabla \frac{P_1}{\gamma - 1} + u_1 \mathbf{b}^\star \cdot \nabla (\frac{P_1}{\gamma - 1}) + \frac{\gamma}{\gamma - 1} P_1 \mathbf{b}^\star \cdot \nabla u_1 = 0, \tag{2.14}$$

$$\partial_t (\frac{P_2}{\gamma - 1}) + \frac{c}{B_0}(\mathbf{b} \times \nabla \Phi) \cdot \nabla \frac{P_2}{\gamma - 1} + u_2 \mathbf{b}^\star \cdot \nabla (\frac{P_2}{\gamma - 1}) + \frac{\gamma}{\gamma - 1} P_2 \mathbf{b}^\star \cdot \nabla u_2 = 0, \tag{2.15}$$

$$\tag{2.16}$$

with $a_s = q_s / m_s$.

Lemma 2

Assuming that all the quantities are equal to zero at the boundary, the total energy satisfies

$$\partial_t \int_\Omega \left(\frac{1}{2}\rho_e u_e^2 + \frac{P_e}{\gamma - 1} + \frac{1}{2}\rho_i u_i^2 + \frac{P_i}{\gamma - 1} + \frac{|\nabla A_\parallel|^2}{2c} \right) = 0$$

2.2 Derivation of MHD + Kinetic model

2.2.1 Elliptic equations

We take the equation of Maxwell which gives

$$\nabla \times \mathbf{B} = \mathbf{J}$$

by definition of the magnetic field we obtain that

Parallel current

$$-\Delta A_\parallel = J_\parallel = \mathbf{J} \cdot \mathbf{e}_z = \sum_i a_i \rho_i u_i, \tag{2.17}$$

Then, we take the equation on the electric field and the charge density

$$\nabla \cdot \mathbf{E} = \sum_i a_i \rho_i, \quad \partial_t (\sum_i a_i \rho_i) + \frac{c}{B_0}(\mathbf{b} \times \nabla \Phi) \cdot \nabla (\sum_i a_i \rho_i) + \mathbf{b}^\star \cdot \nabla (J_\parallel) = 0$$

Now we assume the quasi-neutrality $n_2 = n_1$. Consequently $\sum_i a_i \rho_i = |q| (n_1 - n_2) = 0$. We obtain

Continuity charge equation

$$\mathbf{b}^\star \cdot \nabla(J_\parallel) = 0$$

2.2.2 Mass and momentum equation

We neglect the electron inertia by taking the limit $m_2 \to 0$. That leads us to use the notations

$$u = \frac{\rho_1 u_1 + \rho_2 u_2}{\rho} \approx u_1, \ \rho = \rho_1 + \rho_2 \approx \rho_1. \tag{2.18}$$

We sum equations (3.36) and (3.37) we obtain the equation on the full density. Using the quasi-neutrality we obtain

Mass equation

$$\partial_t \rho + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla\rho + \mathbf{b}^\star \cdot \nabla(\rho u) = 0$$

Momentum equation

$$\rho\partial_t u + \rho_1 \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla u + \rho u \mathbf{b}^\star \cdot \nabla u + \mathbf{b}^\star \cdot \nabla P = 0$$

2.2.3 Equation on the poloidal Flux

Now we propose to compute the equation on the magnetic flux. For this we take the momentum equation on the electron and multiply by q_2

$$m_2\left(\partial_t J_2 + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla(J_2) + u_2 \mathbf{b}^\star \cdot \nabla(J_2)\right) + q_2 \mathbf{b}^\star \cdot \nabla P_2 + q_2 a_2 \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho_2 = 0,$$

$$\frac{m_2}{a_2 q_2 \rho_2}\left(\partial_t J_2 + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla(J_2) + u_2 \mathbf{b}^\star \cdot \nabla(J_2)\right) + \frac{1}{a_2\rho_2}\mathbf{b}^\star \cdot \nabla P_2 + \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right) = 0,$$

we neglect the the term linked to m_e, we use $n_2 \approx \frac{\rho}{m_1}$ and we take $P_2 = \frac{1}{1+T_1/T_2}$ to obtain

Magnetic flux equation

$$\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel = -\frac{\tau}{\rho}\mathbf{b}^\star \cdot \nabla P$$

with $\tau = \frac{m_1}{q_2(1+T_1/T_2)}$.

2.2.4 Pressure equation

To compute the energy equation we sum the two pressure equations. Using the definition of the current, the quasi-neutrality $n_2 = \frac{\rho}{m_1}$ we obtain that

$$u_2 = u_1 + \frac{m_1}{q_2\rho}J_\parallel.$$

Additionally to the classical pressure equation we have

$$\frac{\tau}{\rho}J_{\|}\mathbf{b}^{\star}\cdot\nabla(\frac{P}{\gamma-1})+\tau\frac{\gamma}{\gamma-1}P\mathbf{b}^{\star}\cdot\nabla\left(\frac{J_{\|}}{\rho}\right)=0 \tag{2.19}$$

$$\frac{\tau}{\rho}J_{\|}\mathbf{b}^{\star}\cdot\nabla(\frac{P}{\gamma-1})-\tau\frac{\gamma}{\gamma-1}\frac{J_{\|}P}{\rho}\mathbf{b}^{\star}\cdot\left(\frac{\nabla\rho}{\rho}\right)=0 \tag{2.20}$$

$$\frac{1}{\gamma-1}\frac{\tau J_{\|}}{\rho}\mathbf{b}^{\star}\cdot\left(\nabla P-\gamma\frac{\nabla\rho}{\rho}\right)=0 \tag{2.21}$$

$$\tag{2.22}$$

MHD model

$$\partial_t\rho+\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho+\mathbf{b}^{\star}\cdot\nabla(\rho u)=0 \tag{2.23}$$

$$\rho\partial_t(u)+\rho\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla u+\rho u\mathbf{b}^{\star}\cdot\nabla u+\mathbf{b}^{\star}\cdot\nabla P=0 \tag{2.24}$$

$$\partial_t(\frac{P}{\gamma-1})+\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\frac{P}{\gamma-1}+u\mathbf{b}^{\star}\cdot\nabla(\frac{P}{\gamma-1})+\frac{\gamma}{\gamma-1}P\mathbf{b}^{\star}\cdot\nabla u$$
$$+\frac{1}{\gamma-1}\frac{\tau J_{\|}}{\rho}\mathbf{b}^{\star}\cdot\left(\nabla P-\gamma\frac{\nabla\rho}{\rho}\right)=0 \tag{2.25}$$

$$-\Delta A_{\|}=J_{\|}=\sum_i a_i\rho_i u_i,\sum_i a_i\rho_i=0 \tag{2.26}$$

$$\frac{1}{c}\partial_t A_{\|}+\mathbf{b}^{\star}\cdot\nabla\Phi+\frac{\tau}{\rho}\mathbf{b}^{\star}\cdot\nabla P=0. \tag{2.27}$$

Lemma 3

Assuming that all the quantities are equal to zero at the boundary, the total energy satisfy

$$\partial_t\int_\Omega\left(\frac{1}{2}\rho u^2+\frac{P}{\gamma-1}+\frac{|\nabla A_{\|}|^2}{2c}\right)=0$$

Proof. We multiply the first equation by $\frac{u^2}{2}$ and the second one by u. We obtain

$$\partial_t(\rho\frac{u^2}{2})+\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla(\rho\frac{u^2}{2})+\mathbf{b}^{\star}\cdot\nabla(\frac{u^2}{2}u)+u\mathbf{b}^{\star}\cdot\nabla P=0$$

we sum with the pressure equation to obtain

$$\partial_t\left(\rho\frac{u^2}{2}+\frac{P}{\gamma-1}\right)+\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\left(\rho\frac{u^2}{2}+\frac{P}{\gamma-1}\right)+\mathbf{b}^{\star}\cdot\nabla\left((\frac{u^2}{2}+\frac{P}{\gamma-1})u\right) \tag{2.28}$$

$$+\frac{1}{\gamma-1}\frac{\tau J_{\|}}{\rho}\mathbf{b}^{\star}\cdot\left(\nabla P-\gamma\frac{\nabla\rho}{\rho}\right)=0 \tag{2.29}$$

we then multiply 2.27 by $\Delta A_\|$

$$J_\| \left(\frac{1}{c}\partial_t A_\| + \mathbf{b}\cdot\nabla\Phi + \frac{\tau}{\rho}\mathbf{b}^\star\cdot\nabla P\right) = 0$$
$$-\Delta A_\| \left(\frac{1}{c}\partial_t A_\| + \mathbf{b}\cdot\nabla\Phi + \frac{\tau}{\rho}\mathbf{b}^\star\cdot\nabla P\right) = 0$$

and integrate this equation, using an integration by parts we obtain

$$\int \left(\partial_t \frac{(\nabla A_\|)^2}{2c} + J_\|\mathbf{b}\cdot\nabla\Phi + \frac{\tau}{\rho}J_\|\mathbf{b}^\star\cdot\nabla P\right) = 0 \tag{2.30}$$

we integrate (2.31) and substract (2.30)

$$\int \partial_t \left(\rho\frac{u^2}{2} + \frac{P}{\gamma-1} + \frac{\mid\nabla A_\|\mid^2}{2c} + \frac{\mid\nabla\Phi\mid^2}{2}\right) + \int \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\left(\rho\frac{u^2}{2} + \frac{P}{\gamma-1}\right) + \int \mathbf{b}^\star\cdot\nabla\left((\rho\frac{u^2}{2} + \frac{P}{\gamma-1})u\right)$$
$$+\frac{1}{\gamma-1}\int\frac{\tau J_\|}{\rho}\mathbf{b}^\star\cdot\left(\nabla P - \gamma\frac{\nabla\rho}{\rho}\right) + \int J_\|\mathbf{b}^\star\cdot\nabla\Phi + \int\frac{\tau}{\rho}J_\|\mathbf{b}^\star\cdot\nabla P \tag{2.31}$$

In the classical case, where $b^\star$ is a divergence free vector field, we do

$$\int \mathbf{b}^\star\cdot\nabla\left((\rho\frac{u^2}{2} + \frac{P}{\gamma-1})u\right) = \int\nabla\cdot\left(\mathbf{b}^\star(\rho\frac{u^2}{2} + \frac{P}{\gamma-1})u\right) = 0$$

using the flux-divergence theorem and the boundary conditions. As $\nabla\cdot(b\times\nabla\phi)$ is also a divergence free vector fields, $\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\left(\rho\frac{u^2}{2} + \frac{P}{\gamma-1}\right)$ vanishes as well. As

$$\begin{aligned}
&\frac{1}{\gamma-1}\int\frac{\tau J_\|}{\rho}\mathbf{b}^\star\cdot\left(\nabla P - \gamma\frac{\nabla\rho}{\rho}\right) + \int\frac{\tau}{\rho}J_\|\mathbf{b}^\star\cdot\nabla P \\
&= \frac{1}{\gamma-1}\int\frac{\tau J_\|}{\rho}\mathbf{b}^\star\cdot\left(\nabla P - \gamma\frac{\nabla\rho}{\rho}\right) + \frac{\gamma-1}{\gamma-1}\int\frac{\tau}{\rho}J_\|\mathbf{b}^\star\cdot\nabla P \\
&= \frac{\gamma}{\gamma-1}\int\frac{\tau J_\|}{\rho}\mathbf{b}^\star\cdot\nabla P - \frac{\gamma}{\gamma-1}\int\tau J_\| P\mathbf{b}^\star\cdot\frac{\nabla\rho}{\rho^2} \\
&= \frac{\gamma}{\gamma-1}\int\frac{\tau J_\|}{\rho}\mathbf{b}^\star\cdot\nabla P + \frac{\gamma}{\gamma-1}\int\tau J_\| P\mathbf{b}^\star\cdot\nabla(\frac{1}{\rho}) \\
&= \frac{\gamma}{\gamma-1}\int\tau J_\|\mathbf{b}^\star\cdot\nabla(\frac{P}{\rho}) \\
&= \frac{\gamma}{\gamma-1}\int\tau\mathbf{b}^\star\cdot\nabla(J_\|\frac{P}{\rho}) = 0
\end{aligned}$$

using the flux-divergence theorem and the boundary conditions. Now we have

$$\int \partial_t\left(\rho\frac{u^2}{2} + \frac{P}{\gamma-1} + \frac{(\nabla A_\|)^2}{2}\right) + \int J_\|\mathbf{b}^\star\cdot\nabla\Phi = 0 \tag{2.32}$$

$$\int \partial_t\left(\rho\frac{u^2}{2} + \frac{P}{\gamma-1} + \frac{(\nabla A_\|)^2}{2}\right) - \int \Phi\mathbf{b}^\star\cdot\nabla J_\| = 0 \tag{2.33}$$

$$\int \partial_t\left(\rho\frac{u^2}{2} + \frac{P}{\gamma-1} + \frac{(\nabla A_\|)^2}{2}\right) = 0 \tag{2.34}$$

(2.35)

This result is obtained using the continuity equation. ■

3 Coupling Three species model

3.1 Two species bulk plasma+ kinetic equation

Let us consider for the bulk plasma two species, denoted by 1 for the bulk ions and 2 for the bulk electrons, we got

$$\partial_t \rho_1 + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho_1 + \mathbf{b}^\star\cdot\nabla(\rho_1 u_1) = 0 \tag{3.36}$$

$$\partial_t \rho_2 + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\rho_2 + \mathbf{b}^\star\cdot\nabla(\rho_2 u_2) = 0 \tag{3.37}$$

$$\rho_1\partial_t u_1 + \rho_1\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla u_1 + \rho_1 u_1\mathbf{b}^\star\cdot\nabla u_1 + \mathbf{b}^\star\cdot\nabla P_1 + a_1\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\|\right)\rho_1 = 0 \tag{3.38}$$

$$\rho_2\partial_t u_2 + \rho_2\frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla u_2 + \rho_2 u_2\mathbf{b}^\star\cdot\nabla u_2 + \mathbf{b}^\star\cdot\nabla P_2 + a_2\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\|\right)\rho_2 = 0, \tag{3.39}$$

$$\partial_t(\frac{P_1}{\gamma-1}) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\frac{P_1}{\gamma-1} + u_1\mathbf{b}^\star\cdot\nabla(\frac{P_1}{\gamma-1}) + \frac{\gamma}{\gamma-1}P_1\mathbf{b}^\star\cdot\nabla u_1 = 0, \tag{3.40}$$

$$\partial_t(\frac{P_2}{\gamma-1}) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla\frac{P_2}{\gamma-1} + u_2\mathbf{b}^\star\cdot\nabla(\frac{P_2}{\gamma-1}) + \frac{\gamma}{\gamma-1}P_2\mathbf{b}^\star\cdot\nabla u_2 = 0, \tag{3.41}$$

$$\partial_t f_h + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla f_h + v_\|\mathbf{b}^\star\cdot\nabla f_h - \frac{q}{m}\left(\mathbf{b}^\star\cdot\nabla\Phi + \frac{1}{c}\partial_t A_\|\right)\partial_{v_\|} f = 0 \tag{3.42}$$

$$-\Delta A_\| = J_\| = \sum_i q_i\int v_\| f_i q_h\int v_\| f_h \tag{3.43}$$

$$-\Delta\Phi = \rho = \sum_i q_i\int f_i + q_i\int f_h \tag{3.44}$$

$$\tag{3.45}$$

with $a_s = q_s/m_s$.

We write the equation on the charge density $\sigma = \sum_i a_i\rho_i + a_h\rho_h$

$$\partial_t(\sum_i a_i\rho_i + a_h\rho_h) + \frac{c}{B_0}(\mathbf{b}\times\nabla\Phi)\cdot\nabla(\sum_i a_i\rho_i + a_h\rho_h) + \mathbf{b}^\star\cdot\nabla(J_\|) = 0$$

Now we assume the quasi-neutrality therefore we have $q_2n_2 + q_1n_1 + q_hn_h = 0$. We obtain

Continuity charge equation

$$\mathbf{b}^\star\cdot\nabla(J_\|) = 0$$

Lemma 4

Assuming that all the quantities are equal to zero at the boundary, the total energy satisfies

$$\partial_t\int_\Omega\left(\frac{1}{2}\rho_e u_e^2 + \frac{P_e}{\gamma-1} + \frac{1}{2}\rho_i u_i^2 + \frac{P_i}{\gamma-1} + \frac{1}{2}m_h\mid v_\|\mid^2 f_h + \frac{\mid\nabla A_\|\mid^2}{2c}\right) = 0$$

Proof. Using the same computation as in the first part

$$
\begin{aligned}
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) \int_{\mathbb{R}} f_h q_h v_\parallel = 0 \\
&\partial_t \int_\Omega (\rho_2\varepsilon_2) + \int_\Omega \left(\frac{c}{B_0} (\mathbf{b} \times \nabla\Phi) \cdot \nabla(\rho_2\varepsilon_2) + \mathbf{b}^\star \cdot \nabla(\rho_2\varepsilon_2 u_2) + \mathbf{b}^\star \cdot \nabla(P_2 u_2) + a_2 \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) \rho_2 u_2 \right) = 0 \\
&\partial_t \int_\Omega (\rho_1\varepsilon_1) + \int_\Omega \left(\frac{c}{B_0} (\mathbf{b} \times \nabla\Phi) \cdot \nabla(\rho_1\varepsilon_1) + \mathbf{b}^\star \cdot \nabla(\rho_1\varepsilon_1 u_1) + \mathbf{b}^\star \cdot \nabla(P_1 u_1) + a_1 \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) \rho_1 u_1 \right) = 0
\end{aligned}
$$

since the magnetic field is divergence free and also $(\mathbf{b} \times \nabla\Phi)$ we obtain

$$
\begin{aligned}
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) \int_{\mathbb{R}} f_h q_h v_\parallel = 0 \\
&\partial_t \int_\Omega (\rho_2\varepsilon_2) + \int_\Omega \left(a_2 \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) \rho_2 u_2 \right) = 0 \\
&\partial_t \int_\Omega (\rho_1\varepsilon_1) + \int_\Omega \left(a_1 \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) \rho_1 u_1 \right) = 0
\end{aligned}
$$

We sum the three equation and we obtain

$$
\begin{aligned}
&\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + (\rho_2\varepsilon_2) + (\rho_1\varepsilon_1) \right) + \int_\Omega (a_1\rho_1 u_1 + a_2\rho_2 u_2 + a_h\rho_h u_h) \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) = 0 \\
&\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + (\rho_2\varepsilon_2) + (\rho_1\varepsilon_1) \right) + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel \right) J_\parallel = 0 \\
&\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + (\rho_2\varepsilon_2) + (\rho_1\varepsilon_1) \right) + \int_\Omega \mathbf{b}^\star \cdot \nabla\Phi J_\parallel + \int_\Omega \frac{1}{c}\partial_t A_\parallel J_\parallel = 0 \\
&\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\parallel \mid^2 f_h + (\rho_2\varepsilon_2) + (\rho_1\varepsilon_1) \right) - \int_\Omega \frac{1}{c}\partial_t A_\parallel \Delta A_\parallel - \int_\Omega \Phi \mathbf{b}^\star \cdot \nabla J_\parallel = 0
\end{aligned}
$$

Integrate by part the second term and using the continuity equation we conclude. ■

3.2 MHD + kinetic model

In this section we propose to replace the 2 fluid model ion-electron by the MHD model associated.

3.2.1 Mass and momentum equation

We neglect the electron inertia by taking the limit $m_2 \to 0$. That leads us to use the notations

$$
u = \frac{\rho_1 u_1 + \rho_2 u_2}{\rho} \approx u_1, \;\; \rho = \rho_1 + \rho_2 \approx \rho_1. \tag{3.46}
$$

We sum equations (3.36) and (3.37) we obtain the equation on the full density. Using the quasi-neutrality we obtain

Mass equation

$$\partial_t \rho + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla\rho + \mathbf{b}^\star \cdot \nabla(\rho u) = 0$$

Momentum equation

$$\rho\partial_t u + \rho_1 \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla u + \rho u \mathbf{b}^\star \cdot \nabla u + \mathbf{b}^\star \cdot \nabla P - a_h \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\| \right) \rho_h = 0$$

3.2.2 Equation on the poloidal Flux

Now we propose to compute the equation on the magnetic flux. For this we take the momentum equation on the electron and multiply by q_2

$$m_2 \left(\partial_t a_2 J_2 + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla(a_2 J_2) + u_2 \mathbf{b}^\star \cdot \nabla(a_2 J_2) \right) + q_2 \mathbf{b}^\star \cdot \nabla P_2 + q_2 a_2 \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\| \right) \rho_2 = 0,$$

$$\frac{m_2}{a_2 q_2 \rho_2} \left(\partial_t a_2 J_2 + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla(a_2 J_2) + u_2 \mathbf{b}^\star \cdot \nabla(a_2 J_2) \right) + \frac{1}{a_2 \rho_2}\mathbf{b}^\star \cdot \nabla P_2 + \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\| \right) = 0,$$

we neglect the the term linked with m_e, we use $n_2 \approx \frac{\rho}{m_1}$ and we take $P_2 = \frac{1}{1+T_1/T_2}P$ to obtain

Magnetic flux equation

$$\frac{1}{c}\partial_t A_\| + \mathbf{b}^\star \cdot \nabla\Phi = -\frac{\tau}{\rho}\mathbf{b}^\star \cdot \nabla P$$

with $\tau = \dfrac{m_1}{q_2(1 + T_1/T_2)}$.

3.2.3 Pressure equation

To compute the energy equation we sum the two pressure equations. We assume that $n_h << n_1 \approx n_2$ consequently using the quasi-neutrality $q_1 n_1 + q_2 n_2 + q_h n_h = 0$ and $m_2 << m_1$ $n_2 \approx \frac{\rho}{m_1}$ we obtain that

$$\begin{aligned}
u_2 &= -\frac{a_1 \rho_1}{a_2 \rho_2} u_1 + \frac{1}{a_2 \rho_2} J_\| - \frac{a_h \rho_h}{a_2 \rho_2} u_h \\
u_2 &= (\frac{a_2 \rho_2}{a_2 \rho_2} + \frac{a_h \rho_h}{a_2 \rho_2}) u_1 + \frac{1}{a_2 \rho_2} J_\| - \frac{a_h \rho_h}{a_2 \rho_2} u_h \\
u_2 &= u_1 + \frac{1}{a_2 \rho_2} J_\| - \frac{a_h \rho_h}{a_2 \rho_2}(u_h - u_1) \\
u_2 &\approx u_1 + \frac{\tau}{\rho} J_\| - \frac{\tau}{\rho} a_h \rho_h (u_h - u_1) \\
u_2 &\approx u + \frac{\tau}{\rho} J_\| - \frac{\tau}{\rho} a_h \rho_h (u_h - u)
\end{aligned}$$

At the end we obtain that

Coupling MHD Kinetic model

$$\partial_t \rho + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla\rho + \mathbf{b}^\star \cdot \nabla(\rho u) = 0 \tag{3.47}$$

$$\rho\partial_t(u) + \rho\frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla u + \rho u\mathbf{b}^\star \cdot \nabla u + \mathbf{b}^\star \cdot \nabla P - a_h \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho_h = 0 \tag{3.48}$$

$$\partial_t(\frac{P}{\gamma - 1}) + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla\frac{P}{\gamma - 1} + u\mathbf{b}^\star \cdot \nabla(\frac{P}{\gamma - 1}) + \frac{\gamma}{\gamma - 1}P\mathbf{b}^\star \cdot \nabla u$$
$$+ \frac{1}{\gamma - 1}\frac{\tau J_\parallel}{\rho}\mathbf{b}^\star \cdot \left(\nabla P - \gamma\frac{\nabla\rho}{\rho}\right) \tag{3.49}$$
$$- \frac{\tau a_h\rho_h(u_h - u)}{\rho}\mathbf{b}^\star \cdot \nabla(\frac{P}{\gamma - 1}) - \tau\frac{\gamma}{\gamma - 1}P\mathbf{b}^\star \cdot \nabla\left(\frac{a_h\rho_h(u_h - u)}{\rho}\right) = 0 \tag{3.50}$$

$$\partial_t f_h + \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla f_h + v_\parallel\mathbf{b}^\star \cdot \nabla f_h - \frac{q_h}{m_h}\left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\partial_{v_\parallel} f_h = 0 \tag{3.51}$$

$$-\Delta A_\parallel = J_\parallel = \sum_i a_i\rho_i u_i + a_h K, \quad \sum_i a_i\rho_i = 0 \tag{3.52}$$

$$\frac{1}{c}\partial_t A_\parallel + \mathbf{b}^\star \cdot \nabla\Phi + \frac{\tau}{\rho}\mathbf{b}^\star \cdot \nabla P = 0 \tag{3.53}$$

Lemma 5

Assuming that all the quantities are equal to zero at the boundary, the total energy satisfies

$$\partial_t \int_\Omega \left(\frac{1}{2}m_h \mid v_\parallel \mid^2 f_h + \frac{1}{2}\rho u^2 + \frac{P}{\gamma - 1} + \frac{\mid \nabla A_\parallel \mid^2}{2c}\right) = 0$$

Proof. Using the same proof as for the non coupled case we obtain

$$\partial_t \int_\Omega \left(\frac{1}{2}\rho u^2 + \frac{P}{\gamma - 1} + \frac{\mid \nabla A_\parallel \mid^2}{2c}\right) - \int_\Omega a_h \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\rho_h u = 0$$

We multiply the kinetic equation by $\frac{1}{2}m_h \mid v_\parallel \mid^2$ and we integrate with respect to space and velocity

$$\partial_t \int_\Omega \int_\mathbb{R} \frac{1}{2}m_h \mid v_\parallel \mid^2 f_h + \int_\Omega \frac{c}{B_0}(\mathbf{b} \times \nabla\Phi) \cdot \nabla\left(\int_\mathbb{R} \frac{1}{2}m_h \mid v_\parallel \mid^2 f_h\right) + \int_\Omega v_\parallel\mathbf{b}^\star \cdot \nabla\left(\int_\mathbb{R} \frac{1}{2}m_h \mid v_\parallel \mid^2 f_h\right)$$
$$- \int_\Omega \int_\mathbb{R} \frac{1}{2}q_h \mid v_\parallel \mid^2 \left(\mathbf{b}^\star \cdot \nabla\Phi + \frac{1}{c}\partial_t A_\parallel\right)\partial_{v_\parallel} f_h = 0$$

since $\nabla \cdot (\frac{c}{B_0}(\mathbf{b} \times \nabla\Phi)) = 0$ and $\nabla \cdot (v_\parallel\mathbf{b}^\star) = 0$ we obtain that the second and third terms can be written in the divergence form. Using the flux theorem and the null boundary conditions we obtain

$$
\begin{aligned}
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h - \int_\Omega \int_{\mathbb{R}} \frac{1}{2} q_h \mid v_\| \mid^2 \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\| \right) \partial_{v_\|} f_h = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\| \right) \int_{\mathbb{R}} f_h \partial_{v_\|} (\frac{1}{2} q_h \mid v_\| \mid^2) = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\| \right) \int_{\mathbb{R}} f_h q_h v_\| = 0 \\
&\partial_t \int_\Omega \int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\| \right) a_h \rho_h u_h = 0
\end{aligned}
$$

Now we sum this last term with the energy balance of the MHD

$$
\begin{aligned}
&\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h + \frac{1}{2} \rho u^2 + \frac{P}{\gamma - 1} + \frac{\mid \nabla A_\| \mid^2}{2c} \right) + \int_\Omega \left(\mathbf{b}^\star \cdot \nabla \Phi + \frac{1}{c} \partial_t A_\| \right) a_h \rho_h (u_h - u) = 0 \\
&- \int \frac{\tau a_h \rho_h (u_h - u)}{\rho} \mathbf{b}^\star \cdot \nabla (\frac{P}{\gamma - 1}) - \int \tau \frac{\gamma}{\gamma - 1} P \mathbf{b}^\star \cdot \nabla \left(\frac{a_h \rho_h (u_h - u)}{\rho} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h + \frac{1}{2} \rho u^2 + \frac{P}{\gamma - 1} + \frac{\mid \nabla A_\| \mid^2}{2c} \right) - \int_\Omega \frac{\tau}{\rho} \mathbf{b}^\star \cdot \nabla P a_h \rho_h (u_h - u) = 0 \\
&- \int \frac{\tau a_h \rho_h (u_h - u)}{\rho} \mathbf{b}^\star \cdot \nabla (\frac{P}{\gamma - 1}) - \int \tau \frac{\gamma}{\gamma - 1} P \mathbf{b}^\star \cdot \nabla \left(\frac{a_h \rho_h (u_h - u)}{\rho} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h + \frac{1}{2} \rho u^2 + \frac{P}{\gamma - 1} + \frac{\mid \nabla A_\| \mid^2}{2c} \right) \\
&\int - \frac{\gamma}{\gamma - 1} \frac{\tau a_h \rho_h (u_h - u)}{\rho} \mathbf{b}^\star \cdot \nabla P - \int \tau \frac{\gamma}{\gamma - 1} P \mathbf{b}^\star \cdot \nabla \left(\frac{a_h \rho_h (u_h - u)}{\rho} \right)
\end{aligned}
$$

$$
\partial_t \int_\Omega \left(\int_{\mathbb{R}} \frac{1}{2} m_h \mid v_\| \mid^2 f_h + \frac{1}{2} \rho u^2 + \frac{P}{\gamma - 1} + \frac{\mid \nabla A_\| \mid^2}{2c} \right) \quad - \frac{\gamma}{\gamma - 1} \int \mathbf{b}^\star \cdot \nabla \left(\frac{\tau a_h \rho_h (u_h - u)}{\rho} P \right)
$$

Using the flux divergence theorem we conclude. ■

References

[1] Shinji Tokuda, Hiroshi Naitou, and William Wei-li Lee. A particle-fluid hybrid simulation model based on nonlinear gyrokinetics. *submitted to J. Plasma and Fusion Research*, 1998.