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Hamiltonian approach to hybrid plasma models

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Abstract

The Hamiltonian structures of several hybrid kinetic-fluid models are identified explicitly, upon considering collisionless Vlasov dynamics for the hot particles interacting with a bulk fluid. After presenting different pressure-coupling schemes for an ordinary fluid interacting with a hot gas, the paper extends the treatment to account for a fluid plasma interacting with an energetic ion species. Both current-coupling and pressure-coupling MHD schemes are treated extensively. In particular, pressure-coupling schemes are shown to require a transport-like term in the Vlasov kinetic equation, in order for the Hamiltonian structure to be preserved. The last part of the paper is devoted to studying the more general case of an energetic ion species interacting with a neutralizing electron background (hybrid Hall-MHD). Circulation laws and Casimir functionals are presented explicitly in each case.

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1. Introduction

1.1. Hybrid plasma models

While fluid models are widely successful in plasma physics, kinetic effects have been shown to be relevant in many situations involving coexistence of cold plasmas and energetic hot particles. The latter play an important role in different contexts, ranging from fusion research [36] to astrophysical plasmas [45]. The need for multiscale models that accommodate the statistical kinetic effects of hot particles is a subject of current research, particularly involving computer simulations [35]. These models are usually realized by following a hybrid philosophy that couples ordinary fluid models to appropriate kinetic equations governing the phase-space distribution of the energetic particle species.

The two main research directions involve either the coexistence of a fluid MHD plasma with an energetic ion component [36] or the coexistence of an electron neutralizing background with an energetic ion component [45, 47] (hybrid Hall-MHD). The first direction splits into two possible approaches: the current-coupling scheme [1, 36, 43] and the pressure-coupling scheme [7, 11, 36, 42], depending on how the fluid equation is coupled to the kinetic equation for the hot particles.

In particular, the pressure-coupling scheme possesses some variants that will be discussed in this paper. In all the variants appearing in the literature, the kinetic-fluid coupling occurs solely as an extra pressure term in the fluid momentum equation, while the hot particles are affected by the cold background plasma only through the electromagnetic field terms appearing in the kinetic equation. The fluid plasma density is conventionally transported by the background fluid and this transport generates the usual (barotropic) pressure effects in the evolution of the fluid momentum. On the other hand, the kinetic pressure corresponding to the hot particles has never been considered as arising from fluid transport terms appearing in the accompanying kinetic equation. The absence of these fluid transport terms destroys certain relevant properties of ordinary fluid models, such as the celebrated Kelvin circulation theorem that is of paramount importance, for example, in geophysical fluid dynamics. Also, while the momentum equation usually conserves the prescribed energy, the absence of transport terms in the kinetic equation breaks the corresponding Hamiltonian structure, which is of interest in various plasma models [3, 27, 28, 31]. However, even when considering hypothetical transport terms, their inclusion in the kinetic equation cannot be achieved by ad hoc arguments and their insertion must follow from general principles. An interesting work in this direction was carried out by Flå [9], who considered a linearized approach. Flå's results follow from the geometric properties underlying kinetic Vlasov-type equations and in particular their Lie-group symmetries [25, 44].

The pressure-coupling scheme has been adopted in many simulation codes, probably because of its simplicity (compared to the current-coupling scheme). On the other hand, this scheme assumes that the averaged kinetic momentum of the hot particles is negligible at all times. This assumption is not required by the current-coupling scheme, which then possesses a wider range of validity. The current-coupling scheme couples the fluid and kinetic equations through the Lorentz force that is exerted by the hot particles on the background fluid plasma. Although this scheme has been used in [1, 43], it remains less popular in the plasma physics community. Nevertheless, this paper stands in favor of this model, because of the strong assumption on the hot particle momentum that is required by the pressure-coupling scheme.

All existing hybrid codes usually replace the phase-space kinetic equation for the energetic component by particle dynamics. The distribution function is then reconstructed in order to calculate the macroscopic variables, such as pressure and momentum. In particular, the hot particles are often advanced by appropriate gyrokinetic equations [2, 13, 23]. Nevertheless, this paper will describe the energetic ions in terms of the more general Vlasov kinetic equation, whose Hamiltonian formulation is easier than its gyrokinetic counterpart. The particle picture can always be recovered by the δ-like particle solution, which is well known to preserve the Hamiltonian structure [19].

1.2. Hamiltonian methods in plasma physics

The use of Hamiltonian methods in plasma physics goes back to the early 1980s, when many well-known plasma models were found to possess a Hamiltonian structure. The latter is composed of a Poisson bracket and a Hamiltonian function(al), which unambiguously identifies the total energy of the system. For example, celebrated Hamiltonian structures are those for multi-fluid plasmas [40, 41], for magnetohydrodynamics [17, 34] and for the Maxwell–Vlasov equations [25, 33, 44]. Another important discovery was the Hamiltonian formulation of guiding-center motion, which led to the modern theory of gyrokinetic equations

[2, 13, 23]. The development of Hamiltonian techniques is still an active area of research, as shown by the recent works on magnetic reconnection and gyrofluid models (see [31] and references therein).

The Hamiltonian formulation is particularly advantageous when it is accompanied by a variational principle on the Lagrangian side, which is the case for most plasma models [31]. When this happens, this is due to the fact that the Poisson bracket arises naturally from the relabeling symmetry of the plasma [27]. The advantage of such a Hamiltonian structure resides in several aspects, whose most celebrated example is the possibility of carrying an exact nonlinear stability analysis [18]. This analysis is made possible by the existence of invariant functions (Casimirs) that arise from the Poisson bracket. On the Lagrangian side, Noether's theorem produces conserved quantities, thereby establishing the existence of explicit circulation laws that are a fundamental feature of fluid models. Moreover, the variational principle associated with the Hamiltonian structure can be approached to apply asymptotic methods or Lagrangian averaging (or Lie-transform) techniques. Some recent reviews on these topics, with special emphasis on plasma physics, are available in [3, 28, 31].

In certain cases, the Hamiltonian structure does not possess an immediate variational principle formulation in terms of purely Lagrangian variables. For example, although the Maxwell–Vlasov system [25, 33, 44] possesses several variational formulations [24, 37–39, 46], these are based either on Eulerian variables or on a mixture of Eulerian and Lagrangian variables. On the other hand, purely Lagrangian variables were used in [5], upon suitably modifying Low's Lagrangian [24]. The method in [5] is general enough to be applicable to any Vlasov-type system. This paper relies on this approach to produce the Lagrangian formulation of the Hamiltonian plasma models that will be presented.

1.3. Goal of the paper

This paper aims to present the explicit Hamiltonian structure of several hybrid kinetic-fluid models, for either ordinary barotropic fluids, MHD and Hall-MHD. In particular, this paper derives a whole class of hybrid kinetic-fluid models by making use of well-established Hamiltonian Poisson bracket methods. Upon taking the direct sum of ordinary Poisson bracket structures, the hybrid models are derived by simply transforming the bracket appropriately, so that the Hamiltonian structure is always carried along in a natural fashion. The current-coupling MHD scheme is found to possess an intrinsic Poisson bracket, which is presented explicitly. On the other hand, pressure-coupling schemes are derived by neglecting the momentum contribution of the hot particles directly in the form of the Hamiltonian, rather than approximating the equations of motion in their final form. This key step of approximating the Hamiltonian instead of the equations leads to the preservation of all the usual properties of fluid models, such as circulation theorems and even helicity preservation. All the considerations in this paper restrict to consider non-collisional Vlasov dynamics and barotropic fluid flows. The more general case of adiabatic fluid flows accounting for specific entropy transport is a straightforward extension. After presenting the most basic (pressure-coupling) hybrid models for ordinary neutral fluids, the paper formulates hybrid models for electromagnetic fluids. Hybrid versions of magnetohydrodynamics are presented, together with their associated circulation theorems. The last part of the paper is devoted to presenting the Hamiltonian structure of the hybrid Hall-MHD scheme [45, 47].

Remark 1 (The role of Lie symmetries)**.** The whole paper makes use of well-established Lie-symmetry techniques to produce the various Hamiltonian structures. Indeed, the use of Lie-symmetry concepts (e.g. momentum maps and group actions) is of central importance to the treatment, although the discussion ignores all technical points. Consequently, all the resulting Poisson brackets possess an intrinsic geometric nature, which goes back to the invariance properties underlying MHD [26] and the Maxwell–Vlasov system [25, 27]. Thus, the emergence of Lie–Poisson systems is an immediate consequence of the treatment. Restricting to consider these symmetric Lie–Poisson structures is a deliberate choice of the author and the possibility of other forms of Poisson brackets is not taken under consideration.

2. Elementary hybrid models

This section presents the most basic ideas underlying hybrid models. We consider two simple situations: the hybrid description of a single fluid and the hybrid model for an ideal fluid interacting with an ensemble of hot particles.

2.1. Hybrid formulation of ordinary fluid dynamics

The most fundamental hybrid model already arises in the well-known derivation of fluid dynamics from kinetic theory [22]. Upon restricting to the non-collisional case, one considers a Vlasov equation of the form

$$\frac{\partial f}{\partial t}+\mathbf{p}\cdot\frac{\partial f}{\partial \mathbf{x}}+\mathbf{F}\cdot\frac{\partial f}{\partial \mathbf{p}}=0, \tag{1}$$

where $\mathbf{p}$ is the momentum coordinate (the single-particle mass is taken to be unitary for simplicity) and $f(\mathbf{x},\mathbf{p},t)$ is the Vlasov distribution on phase space. The collective force field $\mathbf{F}=-\nabla\phi(f)$ arises from a potential ϕ, which is usually a linear functional of f. Equation (1) conserves the total energy

$$H(f)=\frac{1}{2}\int f\,(|\mathbf{p}|^2+\phi(f))\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \tag{2}$$

and possesses the Poisson structure [27–32]

$$\{F,G\}=\int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \tag{3}$$

where $\{\cdot,\cdot\}$ denotes the canonical Poisson bracket on phase space. Thus, equation (1) becomes

$$\frac{\partial f}{\partial t}+\left\{f,\frac{\delta H}{\delta f}\right\}=0. \tag{4}$$

The equations of fluid dynamics follow by closing the equations of the moments [12]

$$A_s(\mathbf{x},t)=\int \mathbf{p}^{\otimes s}\,f(\mathbf{x},\mathbf{p},t)\,\mathrm{d}^3\mathbf{p} \tag{5}$$

where $\mathbf{p}^{\otimes s}$ is the sth tensor power on the momentum coordinate. The moment hierarchy is conventionally closed to consider the first two moments (A_0,A_1) as the basic dynamical variables, while the second-order moment dynamics is usually assumed to depend on (A_0,A_1). Evidently, $n=A_0$ is the particle density, while $\mathbf{K}=A_1$ is the averaged kinetic momentum, that is the fluid momentum. We note that the Hamiltonian (2) can be rewritten in terms of the first two moments as

$$\bar{H}(n,\mathbf{K},f)=\frac{1}{2}\int f\left|\mathbf{p}-\frac{\mathbf{K}}{n}\right|^2\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}+\frac{1}{2}\int\frac{|\mathbf{K}|^2}{n}\,\mathrm{d}^3\mathbf{x}+\Phi(n), \tag{6}$$

where Φ denotes a potential energy functional depending exclusively on the particle density n.

Remark 2 (Mean and fluctuation terms in the Vlasov Hamiltonian)**.** Note that the above form (6) of the Vlasov Hamiltonian (2) has the advantage of splitting explicitly the mean and fluctuation parts of the kinetic energy, corresponding to the second and first term respectively. When the fluctuation part is absent, the system reduces to a 'cold plasma' system, which usually arises from the simple moment closure $f = n\delta(\mathbf{p} - \mathbf{K}/n)$. On the other hand, in certain situations involving small amounts of energetic particles, the energy contribution of the averaged momentum $\mathbf{K}$ is often neglected. Then, the second term in (6) is small and all the kinetic energy is concentrated in the fluctuation part, which is the first term. As we shall see, this point is of central interest in hybrid models for plasma physics.

In order to express the Vlasov equation (1) in terms of its macroscopic fluid quantities, one inserts the chain rule formula

$$\frac{\delta F}{\delta f} = \frac{\delta \bar{F}}{\delta n} + \mathbf{p}\cdot\frac{\delta \bar{F}}{\delta \mathbf{K}} + \frac{\delta \bar{F}}{\delta f} \tag{7}$$

in the Poisson bracket (3), thereby obtaining the Poisson structure

$$\begin{aligned}\{\bar{F}, \bar{G}\}(n, \mathbf{K}, f) = &\int \mathbf{K}\cdot\left[\frac{\delta \bar{F}}{\delta \mathbf{K}}, \frac{\delta \bar{G}}{\delta \mathbf{K}}\right] \mathrm{d}^3\mathbf{x} - \int n\left(\frac{\delta \bar{F}}{\delta \mathbf{K}}\cdot\nabla\frac{\delta \bar{G}}{\delta n} - \frac{\delta \bar{G}}{\delta \mathbf{K}}\cdot\nabla\frac{\delta \bar{F}}{\delta n}\right)\mathrm{d}^3\mathbf{x}\\ &+\int f\left(\left\{\frac{\delta \bar{F}}{\delta f}, \frac{\delta \bar{G}}{\delta n} + \mathbf{p}\cdot\frac{\delta \bar{G}}{\delta \mathbf{K}}\right\} - \left\{\frac{\delta \bar{G}}{\delta f}, \frac{\delta \bar{F}}{\delta n} + \mathbf{p}\cdot\frac{\delta \bar{F}}{\delta \mathbf{K}}\right\}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}\\ &+\int f\left\{\frac{\delta \bar{F}}{\delta f}, \frac{\delta \bar{G}}{\delta f}\right\}\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}.\end{aligned} \tag{8}$$

Here, $[\boldsymbol{X}, \boldsymbol{Y}] = -(\boldsymbol{X}\cdot\nabla)\boldsymbol{Y} + (\boldsymbol{Y}\cdot\nabla)\boldsymbol{X}$ is minus the commutator on vector fields. Upon dropping the bar notation for convenience, Hamilton's equations are written as

$$\frac{\partial n}{\partial t} + \operatorname{div}\left(n\frac{\delta H}{\delta \mathbf{K}}\right) = -\int\left\{f, \frac{\delta H}{\delta f}\right\}\mathrm{d}^3\mathbf{p} \tag{9}$$

$$\frac{\partial \mathbf{K}}{\partial t} + \operatorname{div}\left(\frac{\delta H}{\delta \mathbf{K}}\mathbf{K}\right) + \left(\nabla\frac{\delta H}{\delta \mathbf{K}}\right)\cdot\mathbf{K} = -n\nabla\frac{\delta H}{\delta n} - \int \mathbf{p}\left\{f, \frac{\delta H}{\delta f}\right\}\mathrm{d}^3\mathbf{p} \tag{10}$$

$$\frac{\partial f}{\partial t} + \left\{f, \frac{\delta H}{\delta n} + \mathbf{p}\cdot\frac{\delta H}{\delta \mathbf{K}}\right\} = -\left\{f, \frac{\delta H}{\delta f}\right\}. \tag{11}$$

These must be accompanied by the constraints

$$\mathbf{K} - \int \mathbf{p} f\,\mathrm{d}\mathbf{p} = 0, \qquad n - \int f\,\mathrm{d}\mathbf{p} = 0,$$

which enforce the moments $(\mathbf{K}, n)$ to depend on the distribution function f at all times. Indeed, although the hydrodynamical quantities $(\mathbf{K}, n)$ are mutually independent, they both depend on the statistical distribution f. (In more geometric terms, the above constraints identify the zero-level set of a momentum map, known as plasma-to-fluid map [27].)

At this point, the Hamiltonian (6) yields

$$\frac{\delta \bar{H}}{\delta f} = \frac{1}{2}\left|\mathbf{p} - \frac{\mathbf{K}}{n}\right|^2, \qquad \frac{\delta \bar{H}}{\delta \mathbf{K}} = \frac{\mathbf{K}}{n}, \qquad \frac{\delta \bar{H}}{\delta n} = -\frac{1}{2}\left|\frac{\mathbf{K}}{n}\right|^2 + \frac{\delta \Phi}{\delta n},$$

thereby producing the pressure term in equation (10) as follows:

Proposition 1. *With the notation above, the following relation holds:*

$$\frac{1}{2}\int \mathbf{p}\{f, |\mathbf{p} - \boldsymbol{V}|^2\}\,\mathrm{d}^3\mathbf{p} = \operatorname{div}\int(\mathbf{p} - \boldsymbol{V})^{\otimes 2} f\,\mathrm{d}^3\mathbf{p},$$

where we have defined the velocity $\boldsymbol{V} = \mathbf{K}/n$.

Proof. The proof is given by direct verification as follows:

$$
\begin{aligned}
\frac{1}{2}\int \mathbf{p}\{f,\,|\mathbf{p}-\boldsymbol{V}|^2\}\,\mathrm{d}^3\mathbf{p} &= \int \mathbf{p}\left\{f,\,\left(\frac{1}{2}|\mathbf{p}|^2-\mathbf{p}\cdot\boldsymbol{V}+\frac{1}{2}|\boldsymbol{V}|^2\right)\right\}\mathrm{d}^3\mathbf{p}\\
&= \operatorname{div}\int \mathbf{p}\mathbf{p}f\,\mathrm{d}^3\mathbf{p}-(\boldsymbol{V}\cdot\nabla)\mathbf{K}-\nabla\boldsymbol{V}\cdot\mathbf{K}-(\operatorname{div}\boldsymbol{V})\mathbf{K}+\frac{1}{2}n\nabla|\boldsymbol{V}|^2\\
&= \operatorname{div}\int \mathbf{p}\mathbf{p}f\,\mathrm{d}^3\mathbf{p}-(\boldsymbol{V}\cdot\nabla)\mathbf{K}-(\operatorname{div}\boldsymbol{V})\mathbf{K}\\
&= \operatorname{div}\left(n\boldsymbol{V}\boldsymbol{V}+\int(\mathbf{p}-\boldsymbol{V})(\mathbf{p}-\boldsymbol{V})f\,\mathrm{d}^3\mathbf{p}\right)-(\boldsymbol{V}\cdot\nabla)\mathbf{K}-(\operatorname{div}\boldsymbol{V})\mathbf{K}\\
&= \operatorname{div}\int(\mathbf{p}-\boldsymbol{V})(\mathbf{p}-\boldsymbol{V})f\,\mathrm{d}^3\mathbf{p}.
\end{aligned}
$$

□

Therefore, the hybrid equations of motion read

$$\frac{\partial n}{\partial t}+\operatorname{div}(n\boldsymbol{V})=0 \tag{12}$$

$$\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}=-\nabla\phi-\frac{1}{n}\operatorname{div}\int(\mathbf{p}-\boldsymbol{V})^{\otimes 2}f\,\mathrm{d}^3\mathbf{p} \tag{13}$$

$$\frac{\partial f}{\partial t}=-\mathbf{p}\cdot\frac{\partial f}{\partial \mathbf{x}}+\frac{\partial \phi}{\partial \mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{p}}, \tag{14}$$

where we have defined the potential energy density $\phi=\delta\Phi/\delta n$ and $\boldsymbol{V}=\mathbf{K}/n$.

At this point, one usually proceeds by invoking thermodynamic principles leading to an equation of state. This simplifies the pressure tensor on the right-hand side of the velocity equation and allows us to discard the kinetic effects contained in the Vlasov equation for f. This process leads to ordinary fluid equations for inviscid barotropic fluids. However, when kinetic effects cannot be neglected, the Vlasov kinetic equation must be retained in the above system, which then constitutes the hybrid formulation of ordinary fluid dynamics.

We note that the terms on the left-hand side of the kinetic equation in (11) play a key role in the Hamiltonian structure of the system by coupling fluid and kinetic terms in the Poisson bracket. Even more importantly, the term $\mathbf{p}\cdot\delta H/\delta\mathbf{K}$ automatically generates the pressure term in the velocity equation: this presence of the pressure tensor in the velocity equation is not allowed when the term $\mathbf{p}\cdot\delta H/\delta\mathbf{K}$ is absent in equation (11). Moreover, the same term ensures the validity of the Kelvin circulation theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t(\boldsymbol{V})}\boldsymbol{V}\cdot\mathrm{d}\mathbf{x}=-\oint_{\gamma_t(\boldsymbol{V})}\left(\frac{1}{\rho}\operatorname{div}\int(\mathbf{p}-\boldsymbol{V})^{\otimes 2}f\,\mathrm{d}^3\mathbf{p}\right)\cdot\mathrm{d}\mathbf{x}, \tag{15}$$

where $\gamma_t(\boldsymbol{V})$ is an arbitrary loop moving with velocity $\boldsymbol{V}$ and the right-hand side evidently arises from the kinetic energy associated with the fluctuation velocity $\mathbf{p}-\boldsymbol{V}$. When this fluctuation velocity is absent in the Hamiltonian (6) the right-hand side above vanishes, thereby returning the usual Kelvin's theorem $\oint_{\gamma_t}\boldsymbol{V}=\text{const}$.

2.2. *Hamiltonian hybrid kinetic-fluid systems*

This section considers the interaction of an ideal compressible fluid interacting with an ensemble of energetic particles. This basic example is of fundamental importance because it embodies all the main properties that will be used later to derive hybrid pressure-coupling plasma models.

In order to derive the hybrid system, we begin by considering the following equations for the fluid density ρ and velocity $\boldsymbol{u}$ and for the Vlasov distribution f:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0 \tag{16}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla \mathsf{p} - \nabla \frac{\delta \Phi}{\delta \rho} \tag{17}$$

$$\frac{\partial f}{\partial t} + \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{x}} - \nabla \frac{\delta \Phi}{\delta f} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0. \tag{18}$$

Here p denotes the scalar fluid pressure while the total potential energy $\Phi(\rho, f)$ is a functional of both the fluid mass density and the Vlasov distribution. For instance, the case of electrostatic interactions involves a potential energy functional of the form [27, 30]

$$\Phi = -\frac{1}{2} \iint \left(q_c \rho(\mathbf{x}) + q_h \int f(\mathbf{x}, \mathbf{p})\, \mathrm{d}^3\mathbf{p} \right) \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} \left(q_c \rho(\mathbf{x}') + q_h \int f(\mathbf{x}', \mathbf{p}')\, \mathrm{d}^3\mathbf{p}' \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}',$$

where the charge labels c and h denote the fluid (cold) component and the energetic (hot) component respectively (again, we consider unitary particle masses for simplicity). Then, the potential energy represents the only coupling term of the above equations, which otherwise are completely decoupled.

Since the Hamiltonian structures of ideal compressible fluids and kinetic Vlasov equations are well known [27], it is easy to see that the above system possesses a Hamiltonian formulation with the direct sum Poisson bracket

$$\{F, G\} = \int \mathbf{m} \cdot \left[\frac{\delta F}{\delta \mathbf{m}}, \frac{\delta G}{\delta \mathbf{m}} \right] \mathrm{d}^3\mathbf{x} - \int \rho \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \rho} \right) \mathrm{d}^3\mathbf{x} + \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} \tag{19}$$

and the Hamiltonian

$$H(\mathbf{m}, \rho, f) = \frac{1}{2} \int \frac{|\mathbf{m}|^2}{\rho}\, \mathrm{d}^3\mathbf{x} + \frac{1}{2} \int f |\mathbf{p}|^2\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} + \Phi(\rho, f) + \int \rho \mathcal{U}(\rho)\, \mathrm{d}^3\mathbf{x}, \tag{20}$$

which are both written in terms of the fluid momentum $\mathbf{m} = \rho\, \boldsymbol{u}$. Note that we have introduced the fluid internal energy $\mathcal{U}$ such that $\mathsf{p} = \rho^2\, \mathrm{d}\mathcal{U}/\mathrm{d}\rho$.

At this point, in the search for a hybrid kinetic-fluid model, one defines the total momentum

$$\mathbf{M} = \mathbf{m} + \int \mathbf{p} f\, \mathrm{d}^3\mathbf{p}$$

and expresses the equations of motion in terms of the new variable. This can be done by replacing the chain-rule formulas

$$\frac{\delta F}{\delta \mathbf{m}} = \frac{\delta \bar{F}}{\delta \mathbf{M}}, \qquad \frac{\delta F}{\delta \rho} = \frac{\delta \bar{F}}{\delta \rho}, \qquad \frac{\delta F}{\delta f} = \mathbf{p} \cdot \frac{\delta \bar{F}}{\delta \mathbf{M}} + \frac{\delta \bar{F}}{\delta f},$$

in the Poisson bracket (19), thereby obtaining the structure

$$\{\bar{F}, \bar{G}\}(\mathbf{M}, \rho, f) = \int \mathbf{M} \cdot \left[\frac{\delta \bar{F}}{\delta \mathbf{M}}, \frac{\delta \bar{G}}{\delta \mathbf{M}} \right] \mathrm{d}^3\mathbf{x} - \int \rho \left(\frac{\delta \bar{F}}{\delta \mathbf{M}} \cdot \nabla \frac{\delta \bar{G}}{\delta \rho} - \frac{\delta \bar{G}}{\delta \mathbf{M}} \cdot \nabla \frac{\delta \bar{F}}{\delta \rho} \right) \mathrm{d}^3\mathbf{x} + \int f \left\{ \frac{\delta \bar{F}}{\delta f}, \frac{\delta \bar{G}}{\delta f} \right\} \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} + \int f \left(\left\{ \frac{\delta \bar{F}}{\delta f}, \mathbf{p} \cdot \frac{\delta \bar{G}}{\delta \mathbf{M}} \right\} - \left\{ \frac{\delta \bar{G}}{\delta f}, \mathbf{p} \cdot \frac{\delta \bar{F}}{\delta \mathbf{M}} \right\} \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p}. \tag{21}$$

Then, upon denoting $\mathbf{K} = \int \mathbf{p}\, f\, d^3\mathbf{p}$, the total energy expression

$$\bar{H}(\mathbf{M}, \rho, f) = \frac{1}{2}\int \frac{|\mathbf{M} - \mathbf{K}|^2}{\rho}\, d^3\mathbf{x} + \frac{1}{2}\int f\,|\mathbf{p}|^2\, d^3\mathbf{x}\, d^3\mathbf{p} + \Phi(\rho, f) + \int \rho\,\mathcal{U}(\rho)\, d^3\mathbf{x} \tag{22}$$

yields the equations of motion

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0 \tag{23}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\operatorname{div}\int \mathbf{p}\mathbf{p} f\, d^3\mathbf{p} - \frac{1}{\rho}\nabla \mathsf{p} - \nabla \frac{\delta \Phi}{\delta \rho} \tag{24}$$

$$\frac{\partial f}{\partial t} + \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}}\frac{\delta \Phi}{\delta f} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \tag{25}$$

where $\boldsymbol{U} = \mathbf{M}/\rho$ and $\boldsymbol{u} = \mathbf{m}/\rho = (\mathbf{M} - \mathbf{K})/\rho$. These last equations of motion are totally equivalent to the system (16)–(18), that is they yet involve no approximations.

2.2.1. First pressure-coupling scheme. As we remarked in the preceding section, one normally considers small amounts of energetic particles, so that the total energy contribution of the averaged momentum is often neglected. In the plasma physics literature, this assumption is made by simply replacing the convective term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ by $\boldsymbol{U} \cdot \nabla \boldsymbol{U}$ in the velocity equation. On the other hand, this simple step inexorably destroys the Hamiltonian structure of the equations of motion, which then lack an energy balance equation and therefore need substantial modifications. A simple possibility of preserving the Hamiltonian structure is provided by the assumption $\mathbf{M} = \mathbf{m} + n\,\boldsymbol{V} \simeq \mathbf{m}$ in the Hamiltonian (22). Then, the total energy

$$\bar{H}(\mathbf{M}, \rho, f) = \frac{1}{2}\int \frac{|\mathbf{M}|^2}{\rho}\, d^3\mathbf{x} + \frac{1}{2}\int f\,|\mathbf{p}|^2\, d^3\mathbf{x}\, d^3\mathbf{p} + \Phi(\rho, f) + \int \rho\,\mathcal{U}(\rho)\, d^3\mathbf{x} \tag{26}$$

yields the equations of motion

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{U}) = 0 \tag{27}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\operatorname{div}\int \mathbf{p}\mathbf{p} f\, d^3\mathbf{p} - \frac{1}{\rho}\nabla \mathsf{p} - \nabla \frac{\delta \Phi}{\delta \rho} \tag{28}$$

$$\frac{\partial f}{\partial t} + \{f, \mathbf{p} \cdot \boldsymbol{U}\} + \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}}\frac{\delta \Phi}{\delta f} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0. \tag{29}$$

While the velocity equation is exactly the same as that of some hybrid schemes frequently used in plasma physics [7, 11, 36] (up to Lorentz force terms that will be included in the following sections), the Vlasov equation carries the extra term

$$\{f, \mathbf{p} \cdot \boldsymbol{U}\} = \boldsymbol{U} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial \boldsymbol{U}}{\partial \mathbf{x}} \cdot \mathbf{p},$$

which evidently arises from the velocity shift $\boldsymbol{U} = \boldsymbol{u} + \mathbf{K}/\rho$.

Remark 3 (Transport of the Vlasov distribution)**.** We emphasize that the term $\{f, \mathbf{p} \cdot \boldsymbol{U}\}$ is required by the Hamiltonian structure of the original equations and cannot arise from direct approximations on the equations of motion (16)–(18). The same term was found by Flå [9], who used more sophisticated techniques. We note that the term $\boldsymbol{U} \cdot \nabla f$ leads to the interpretation of a Vlasov equation that is 'transported' along the macroscopic velocity $\boldsymbol{U}$.

This transport property generates the circulation force term $\partial_{\mathbf{p}} f \cdot \nabla \boldsymbol{U} \cdot \mathbf{p}$ that is related to the change of reference $\boldsymbol{u} \to \boldsymbol{U}$. The kinetic-fluid interaction term $\{f, \mathbf{p} \cdot \boldsymbol{U}\}$ is the main novelty of this model.

The Vlasov equation may be rewritten as $\partial_t f + \{f, 1/2(|\mathbf{p} + \boldsymbol{U}|^2 - |\boldsymbol{U}|^2) + \delta_f \Phi\} = 0$. Note that the equation $\partial_t n + \operatorname{div}(n\,\boldsymbol{U}) = -\operatorname{div} \mathbf{K}$ for the energetic particle density $n = \int f \, \mathrm{d}^3\mathbf{p}$ involves the continuous source term $\operatorname{div} \mathbf{K}$.

2.2.2. Second pressure-coupling scheme. So far, we only made the simple assumption $\mathbf{M} \simeq \mathbf{m}$ in the expression of the total energy. This was justified by the fact that the energetic particles exist only in small amounts, so that the averaged momentum $\mathbf{K}$ is assumed not to contribute to the total energy. However, as we have remarked in section 2.1 (see remark 2), under this assumption the kinetic energy of the energetic component may be modified to neglect its average value $|\mathbf{K}|^2/(2n)$. Then, the total energy becomes

$$\bar{H}(\mathbf{M}, \rho, f) = \frac{1}{2}\int \frac{|\mathbf{M}|^2}{\rho}\,\mathrm{d}^3\mathbf{x} + \frac{1}{2}\int f \left|\mathbf{p} - \frac{\mathbf{K}}{n}\right|^2 \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \Phi(\rho, f) + \int \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x}, \qquad (30)$$

which in turn produces the equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{U}) = 0 \qquad (31)$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\operatorname{div}\int (\mathbf{p} - \boldsymbol{V})(\mathbf{p} - \boldsymbol{V}) f \,\mathrm{d}^3\mathbf{p} - \frac{1}{\rho}\nabla \mathsf{p} - \nabla \frac{\delta \Phi}{\delta \rho} \qquad (32)$$

$$\frac{\partial f}{\partial t} + \left\{ f, \mathbf{p} \cdot \left(\boldsymbol{U} - \boldsymbol{V}\right) + \frac{1}{2}|\boldsymbol{V}|^2 \right\} + \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} \frac{\delta \Phi}{\delta f} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0. \qquad (33)$$

The same form of the above velocity equation also appears in certain hybrid schemes [42], although the above Vlasov equation differs substantially from those models because the present system accounts for energy balance, contrarily to the pressure-coupling schemes appearing in the physics literature. We note that the above Vlasov equation can be written in the form $\partial_t f + \{f, 1/2(|\mathbf{p} - \boldsymbol{V}|^2 + |\mathbf{p} + \boldsymbol{U}|^2 - |\mathbf{p}|^2 - |\boldsymbol{U}|^2) + \delta_f \Phi\} = 0$, which emphasizes the various kinetic energy contributions. A natural consequence of this kinetic equation is that its zeroth moment n satisfies the simple advection relation $\partial_t n + \operatorname{div}(n\,\boldsymbol{U}) = 0$, so that the total density $D = \rho + n$ satisfies the equation $\partial_t D + \operatorname{div}(D\,\boldsymbol{U}) = 0$, as it appears in [21].

2.2.3. Third pressure-coupling scheme. The advection equation for the total density D arising in the previous case, suggests looking at the equations of motion in terms of this density variable. To this purpose, one again replaces the chain-rule formulas

$$\frac{\delta F}{\delta \rho} = \frac{\delta \bar{F}}{\delta D}, \qquad \frac{\delta F}{\delta \mathbf{m}} = \frac{\delta \bar{F}}{\delta \mathbf{M}}, \qquad \frac{\delta F}{\delta f} = \frac{\delta \bar{F}}{\delta D} + \mathbf{p} \cdot \frac{\delta \bar{F}}{\delta \mathbf{M}} + \frac{\delta \bar{F}}{\delta f},$$

in the Poisson bracket (19), thereby obtaining the following Poisson bracket:

$$\begin{aligned}\{\bar{F}, \bar{G}\}(\mathbf{M}, D, f) = &\int \mathbf{M} \cdot \left[\frac{\delta \bar{F}}{\delta \mathbf{M}}, \frac{\delta \bar{G}}{\delta \mathbf{M}}\right] \mathrm{d}^3\mathbf{x} - \int D \left(\frac{\delta \bar{F}}{\delta \mathbf{M}} \cdot \nabla \frac{\delta \bar{G}}{\delta D} - \frac{\delta \bar{G}}{\delta \mathbf{M}} \cdot \nabla \frac{\delta \bar{F}}{\delta D} \right) \mathrm{d}^3\mathbf{x} \\ &+ \int f \left(\left\{ \frac{\delta \bar{F}}{\delta f}, \frac{\delta \bar{G}}{\delta D} + \mathbf{p} \cdot \frac{\delta \bar{G}}{\delta \mathbf{M}} \right\} - \left\{ \frac{\delta \bar{G}}{\delta f}, \frac{\delta \bar{F}}{\delta D} + \mathbf{p} \cdot \frac{\delta \bar{F}}{\delta \mathbf{M}} \right\} \right) \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \\ &+ \int f \left\{ \frac{\delta \bar{F}}{\delta f}, \frac{\delta \bar{G}}{\delta f} \right\} \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p},\end{aligned}$$

which is identical to (8). Under the assumption of a rarefied energetic gas, one has $D \simeq \rho$ and each of the Hamiltonians used above yields a slightly different system. For example, upon replacing ρ by D, the energy in (30) produces the equations

$$\frac{\partial D}{\partial t} + \operatorname{div}(D\boldsymbol{U}) = 0 \tag{34}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{U} = -\frac{1}{D}\operatorname{div}\int(\mathbf{p}-\boldsymbol{V})(\mathbf{p}-\boldsymbol{V})f\,\mathrm{d}^3\mathbf{p} - \frac{1}{\rho}\nabla\mathsf{p} - \nabla\frac{\delta\Phi}{\delta D} \tag{35}$$

$$\frac{\partial f}{\partial t} + \left\{f, \mathbf{p}\cdot(\boldsymbol{U}-\boldsymbol{V}) + \frac{1}{2}(|\boldsymbol{V}|^2 - |\boldsymbol{U}|^2)\right\} + \mathbf{p}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}}\frac{\delta\Phi}{\delta f}\cdot\frac{\partial f}{\partial \mathbf{p}} = 0, \tag{36}$$

which involve no substantial modification with respect to the previous hybrid schemes.

2.3. *Conservation laws*

The Hamiltonian structure of fluid models has the main advantage of preserving most of the geometric features possessed by ordinary ideal fluids. One of the most significant features is the Kelvin circulation theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t(\boldsymbol{u})} \boldsymbol{u}\cdot\mathrm{d}\mathbf{x} = 0 \tag{37}$$

that holds for the velocity equation (17). Here the curve $\gamma_t(\boldsymbol{u})$ is a loop moving with velocity $\boldsymbol{u}$. The hybrid schemes available in the physics literature often lack this fundamental feature, which is instead available for the hybrid models presented above. Indeed, the momentum shift $\mathbf{M} = \mathbf{m} + \int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p}$ together with its resulting Poisson bracket (21) guarantees the Kelvin circulation theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t(\boldsymbol{U})}\left(\boldsymbol{U} - \frac{1}{\rho}\int \mathbf{p}\, f\,\mathrm{d}^3\mathbf{p}\right)\cdot\mathrm{d}\mathbf{x} = 0. \tag{38}$$

It is important to observe that the circulation law (38) is *different* from (37), since the loop γ_t moves with velocity $\boldsymbol{u}$ in (37) and with velocity $\boldsymbol{U} = \boldsymbol{u} + \rho^{-1}\int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p}$ in (38). This is due to the fact that modifying the fluid kinetic energy in the Hamiltonian (22) yields a different fluid velocity.

Remark 4 (Relation to turbulence models)**.** Note that exactly the same phenomenon occurs in certain regularization models for fluid turbulence [6], such as the Navier–Stokes-α model [10]. These models have also been called 'Kelvin-filtered' turbulence models. Another property of the preceding pressure-coupling schemes that is shared with Kelvin-filtered models is that the fluid vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ is transported along the modified velocity $\boldsymbol{U}$, that is $\partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{U}\times\boldsymbol{\omega}) = 0$. This equation evidently possesses vortex filament solutions that project to point vortices in two dimensions. The appearance of this close analogy between pressure-coupling hybrid schemes and fluid turbulence models looks promising for further investigation, especially in relation to the energy splitting appearing in the Hamiltonian (6).

The circulation conservation (38) is valid in any of the pressure schemes presented previously. In the third pressure-coupling scheme, the Kelvin circulation theorem becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t(\boldsymbol{U})}\left(\boldsymbol{U} - \left(D - \int f\,\mathrm{d}^3\mathbf{p}\right)^{-1}\int \mathbf{p} f\,\mathrm{d}^3\mathbf{p}\right)\cdot\mathrm{d}\mathbf{x} = 0.$$

Moreover, the previous hybrid schemes possess the helicity conservation relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\boldsymbol{U}-\frac{1}{\rho}\int\mathbf{p}f\,\mathrm{d}^3\mathbf{p}\right)\cdot\nabla\times\left(\boldsymbol{U}-\frac{1}{\rho}\int\mathbf{p}f\,\mathrm{d}^3\mathbf{p}\right)\mathrm{d}^3\mathbf{x}=0.$$

Note the above conserved quantity is a Casimir of the Poisson bracket (21) and in two dimensions, this Casimir can be used to approach the problem of nonlinear stability, following [18]. Another Casimir of the Poisson bracket (21) is given by $C=\int\Phi(f)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$, for an arbitrary function Φ. This is the Casimir typically associated with Vlasov dynamics and it includes the total mass $\int f\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$. The search for other Casimir functionals is left for future investigation.

3. Hamiltonian hybrid MHD models

The purpose of this section is to formulate a kinetic-multifluid model that leads to the formulation of hybrid MHD schemes. As we shall see, the pressure-coupling schemes can be obtained by repeating systematically all the steps outlined in the previous sections, upon including electromagnetic fields appropriately. On the other hand there is another relevant hybrid MHD scheme, which involves a current coupling. This section derives both current and pressure coupling schemes and provides their Hamiltonian structures. At the end of this section, the two schemes are compared in terms of their energy-balance properties.

3.1. Kinetic-multifluid system

The equations of motion for a multifluid plasma in the presence of an energetic component read

$$\rho_s\frac{\partial\boldsymbol{u}_s}{\partial t}+\rho_s\left(\boldsymbol{u}_s\cdot\nabla\right)\boldsymbol{u}_s=a_s\rho_s\left(\mathbf{E}+\boldsymbol{u}_s\times\mathbf{B}\right)-\nabla\mathsf{p}_s \tag{39}$$

$$\frac{\partial\rho_s}{\partial t}+\mathrm{div}\left(\rho_s\boldsymbol{u}_s\right)=0 \tag{40}$$

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{m_h}\cdot\frac{\partial f}{\partial\mathbf{x}}+q_h\left(\mathbf{E}+\frac{\mathbf{p}}{m_h}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial\mathbf{p}}=0 \tag{41}$$

$$\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}=\nabla\times\mathbf{B}-\mu_0\sum_s a_s\rho_s\boldsymbol{u}_s-\mu_0a_h\int\mathbf{p}f\,\mathrm{d}^3\mathbf{p} \tag{42}$$

$$\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E} \tag{43}$$

$$\epsilon_0\,\mathrm{div}\,\mathbf{E}=\sum_s a_s\rho_s+q_h\int f\,\mathrm{d}^3\mathbf{p},\qquad \mathrm{div}\,\mathbf{B}=0 \tag{44}$$

where $a_s=q_s/m_s$ is the charge-to-mass ratio of the fluid species s, while ρ_s and $\boldsymbol{u}_s$ are its mass density and velocity. Note that in the above system we have restored the hot particle mass m_h. When the energetic component is absent (i.e. $f=0$), the Hamiltonian structure of this system was discovered by Spencer [40, 41] and analyzed further in [27]. Combining this Hamiltonian structure with that of the Maxwell–Vlasov system [25, 33, 44] yields the following Poisson bracket for the kinetic-multifluid system:

$$
\begin{aligned}
\{F,G\} = {} & \sum_s \int \mathbf{m}_s \cdot \left[\frac{\delta F}{\delta \mathbf{m}_s}, \frac{\delta G}{\delta \mathbf{m}_s}\right] \mathrm{d}^3\mathbf{x} + \int f \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \\
& - \sum_s \int \rho_s \left(\frac{\delta F}{\delta \mathbf{m}_s} \cdot \nabla \frac{\delta G}{\delta \rho_s} - \frac{\delta G}{\delta \mathbf{m}_s} \cdot \nabla \frac{\delta F}{\delta \rho_s}\right) \mathrm{d}^3\mathbf{x} \\
& + \sum_s \int a_s \rho_s \left(\frac{\delta F}{\delta \mathbf{m}_s} \cdot \frac{\delta G}{\delta \mathbf{D}} - \frac{\delta G}{\delta \mathbf{m}_s} \cdot \frac{\delta F}{\delta \mathbf{D}} + \mathbf{B} \cdot \frac{\delta F}{\delta \mathbf{m}_s} \times \frac{\delta G}{\delta \mathbf{m}_s}\right) \mathrm{d}^3\mathbf{x} \\
& + q_h \int f \left(\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \cdot \frac{\delta G}{\delta \mathbf{D}} - \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f} \cdot \frac{\delta F}{\delta \mathbf{D}} + \mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f}\right)\right) \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \\
& + \int \left(\frac{\delta F}{\delta \mathbf{D}} \cdot \operatorname{curl} \frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{D}} \cdot \operatorname{curl} \frac{\delta F}{\delta \mathbf{B}}\right) \mathrm{d}^3\mathbf{x},
\end{aligned}
\tag{45}
$$

where evidently $\mathbf{m}_s = \rho_s \boldsymbol{u}_s$. To complete the Hamiltonian structure of the kinetic-multifluid model, one writes the Hamiltonian

$$
\begin{aligned}
H(\mathbf{m}_s, \rho_s, f, \mathbf{D}, \mathbf{B}) = {} & \frac{1}{2} \sum_s \int \frac{|\mathbf{m}_s|^2}{\rho_s} \mathrm{d}^3\mathbf{x} + \frac{1}{2m_h} \int f |\mathbf{p}|^2 \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \sum_s \int \rho_s \mathcal{U}(\rho_s)\, \mathrm{d}^3\mathbf{x} \\
& + \frac{1}{2} \left(\frac{1}{\epsilon_0} \int |\mathbf{D}|^2\, \mathrm{d}^3\mathbf{x} + \frac{1}{\mu_0} \int |\mathbf{B}|^2\, \mathrm{d}^3\mathbf{x}\right),
\end{aligned}
\tag{46}
$$

where the electric induction $\mathbf{D} = \epsilon_0 \mathbf{E}$ is (minus) the conjugate variable to the vector potential $\mathbf{A}$, as explained in [14, 15].

Remark 5 (Extending to several energetic components)**.** The above kinetic-multifluid system can be easily extended to the case of several energetic components, each described by its own Vlasov equation. Then, each of these Vlasov equations can be rewritten in terms of the first two moments $(n, \mathbf{K})$ in order to isolate kinetic pressure terms. A similar approach was followed in [8].

3.2. Current-coupling hybrid MHD scheme

In usual situations, one is interested in one-fluid models. Thus, it is customary to specialize to the two-fluid system and to neglect the inertia of one species (electrons). This last approximation is equivalent to taking the limit $m_2 \to 0$ for the second species in the total fluid momentum equation. Under this assumption, the sum of the equations (39) for $s = 1, 2$ produces the equation

$$
\rho_1 \frac{\partial \boldsymbol{u}_1}{\partial t} + \rho_1 (\boldsymbol{u}_1 \cdot \nabla) \boldsymbol{u}_1 = (a_1\rho_1 + a_2\rho_2)\, \mathbf{E} + (a_1\rho_1\boldsymbol{u}_1 + a_2\rho_2\boldsymbol{u}_2) \times \mathbf{B} - \nabla \mathsf{p}_1. \tag{47}
$$

Also, upon assuming neutrality by letting $\epsilon_0 \to 0$, the electromagnetic fields satisfy the equations

$$
\sum_s a_s \rho_s \boldsymbol{u}_s = \frac{1}{\mu_0} \nabla \times \mathbf{B} - a_h \int \mathbf{p} f\, \mathrm{d}^3\mathbf{p} \tag{48}
$$

$$
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{49}
$$

$$
\sum_s a_s \rho_s = -q_h \int f\, \mathrm{d}^3\mathbf{p}, \qquad \operatorname{div} \mathbf{B} = 0. \tag{50}
$$

Then, equation (47) becomes

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u}=-\left(q_h\int f\,\mathrm{d}^3\mathbf{p}\right)\mathbf{E}+\left(\frac{1}{\mu_0}\nabla\times\mathbf{B}-a_h\int\mathbf{p}\,f\,\mathrm{d}^3\mathbf{p}\right)\times\mathbf{B}-\rho\nabla\mathsf{p},\qquad(51)$$

where we have dropped labels for convenience. Finally, inserting Ohm's ideal law $\mathbf{E}+\boldsymbol{u}\times\mathbf{B}=0$, the kinetic two-fluid system becomes

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u}=\left(q_h\boldsymbol{u}\int f\,\mathrm{d}^3\mathbf{p}-a_h\int\mathbf{p}f\,\mathrm{d}^3\mathbf{p}+\frac{1}{\mu_0}\nabla\times\mathbf{B}\right)\times\mathbf{B}-\rho\nabla\mathsf{p}\qquad(52)$$

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\left(\rho\boldsymbol{u}\right)=0\qquad(53)$$

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{m_h}\cdot\frac{\partial f}{\partial\mathbf{x}}+q_h\left(\frac{\mathbf{p}}{m_h}-\boldsymbol{u}\right)\times\mathbf{B}\cdot\frac{\partial f}{\partial\mathbf{p}}=0\qquad(54)$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times\left(\boldsymbol{u}\times\mathbf{B}\right),\qquad(55)$$

which is identical to the current-coupling hybrid scheme presented in [1, 11, 36], except the fact that particle dynamics is governed by the Vlasov equation rather than its gyrokinetic counterpart. Note that the above system does not make any assumption on the form of the Vlasov distribution for the energetic particles. Therefore, this system applies to a whole variety of possible situations.

We now turn our attention to the energy balance, that is we ask whether the above current-coupling system possesses a Hamiltonian structure. Remarkably, a positive answer is provided by the following Poisson bracket:

$$\begin{aligned}\{F,G\}=&\int\mathbf{m}\cdot\left[\frac{\delta F}{\delta\mathbf{m}},\frac{\delta G}{\delta\mathbf{m}}\right]\mathrm{d}^3\mathbf{x}-\int\rho\left(\frac{\delta F}{\delta\mathbf{m}}\cdot\nabla\frac{\delta G}{\delta\rho}-\frac{\delta G}{\delta\mathbf{m}}\cdot\nabla\frac{\delta F}{\delta\rho}\right)\mathrm{d}^3\mathbf{x}\\&+q_h\int f\mathbf{B}\cdot\left(\frac{\delta F}{\delta\mathbf{m}}\times\frac{\delta G}{\delta\mathbf{m}}-\frac{\delta F}{\delta\mathbf{m}}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f}+\frac{\delta G}{\delta\mathbf{m}}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}\\&+\int f\left(\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}+q_h\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}\\&-\int\mathbf{B}\cdot\left(\frac{\delta F}{\delta\mathbf{m}}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}-\frac{\delta G}{\delta\mathbf{m}}\times\nabla\times\frac{\delta F}{\delta\mathbf{B}}\right)\mathrm{d}^3\mathbf{x},\end{aligned}\qquad(56)$$

together with the Hamiltonian

$$H(\mathbf{m},\rho,f,\mathbf{B})=\frac{1}{2}\int\frac{|\mathbf{m}|^2}{\rho}\mathrm{d}^3\mathbf{x}+\frac{1}{2m_h}\int f|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}+\int\rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x}+\frac{1}{2\mu_0}\int|\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x},\qquad(57)$$

where the fluid velocity is replaced by the fluid momentum $\mathbf{m}=\rho\boldsymbol{u}$. The nature of the Hamiltonian structure (56) will be discussed later in relation to the MHD bracket (see theorem 2).

3.3. Pressure-coupling hybrid MHD schemes

In this section we show how the Hamiltonian structure of the previous current-coupling scheme represents a helpful tool for the formulation of a pressure-coupling scheme. This scheme establishes an equation for the total momentum

$$\mathbf{M}=\mathbf{m}+\int\mathbf{p}f\,\mathrm{d}^3\mathbf{p},$$

under the assumption that the averaged kinetic momentum $\mathbf{K} = \int \mathbf{p}\, f \,\mathrm{d}^3\mathbf{p}$ does not contribute to the total energy of the system. The definition of the momentum coordinate $\mathbf{M}$ yields the simple functional relations

$$\frac{\delta F}{\delta \mathbf{m}} = \frac{\delta \bar{F}}{\delta \mathbf{M}}, \qquad \frac{\delta F}{\delta \rho} = \frac{\delta \bar{F}}{\delta \rho}, \qquad \frac{\delta F}{\delta \mathbf{B}} = \frac{\delta \bar{F}}{\delta \mathbf{B}}, \qquad \frac{\delta F}{\delta f} = \frac{\delta \bar{F}}{\delta f} + \mathbf{p}\cdot\frac{\delta \bar{F}}{\delta \mathbf{M}}$$

which can be easily substituted in bracket (56) to produce the new Hamiltonian structure

$$\begin{aligned}\{F, G\} = &\int \mathbf{M}\cdot\left[\frac{\delta F}{\delta \mathbf{M}}, \frac{\delta G}{\delta \mathbf{M}}\right]\mathrm{d}^3\mathbf{x} - \int \rho\left(\frac{\delta F}{\delta \mathbf{M}}\cdot\nabla\frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{M}}\cdot\nabla\frac{\delta F}{\delta \rho}\right)\mathrm{d}^3\mathbf{x}\\ &+\int f\left(\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} + q_h\mathbf{B}\cdot\frac{\partial}{\partial \mathbf{p}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial \mathbf{p}}\frac{\delta G}{\delta f}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}\\ &+\int f\left(\left\{\frac{\delta F}{\delta f}, \mathbf{p}\cdot\frac{\delta G}{\delta \mathbf{M}}\right\} - \left\{\frac{\delta G}{\delta f}, \mathbf{p}\cdot\frac{\delta F}{\delta \mathbf{M}}\right\}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}\\ &+\int \mathbf{B}\cdot\left(\frac{\delta F}{\delta \mathbf{M}}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{M}}\times\nabla\times\frac{\delta F}{\delta \mathbf{B}}\right)\mathrm{d}^3\mathbf{x}.\end{aligned} \tag{58}$$

Let us now express the Hamiltonian (57) in terms of the total momentum $\mathbf{M}$: we obtain

$$H = \frac{1}{2}\int\frac{|\mathbf{M}-\mathbf{K}|^2}{\rho}\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mathsf{m}_h}\int f\,|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \int \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int|\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x}. \tag{59}$$

Then, the Poisson bracket (58) yields the velocity equation

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla\mathsf{p} - \frac{1}{\mathsf{m}_h\rho}\,\mathrm{div}\int\mathbf{p}\mathbf{p} f\,\mathrm{d}^3\mathbf{p} - \frac{1}{\mu_0\rho}\mathbf{B}\times\nabla\times\mathbf{B},$$

while the other equations remain identical to those in (53)–(55). Before proceeding further, we remark that neglecting all terms involving the averaged kinetic momentum $\int\mathbf{p}\,f\,\mathrm{d}^3\mathbf{p}$ (so that $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \simeq \boldsymbol{U}\cdot\nabla\boldsymbol{U}$) produces the hybrid MHD model in [7] (although the general Vlasov equation is adopted here, rather than a gyrokinetic equation). However, this crucial step breaks the energy-conserving nature of the system and, when an energy balance equation is required, the model needs substantial modifications.

3.3.1. First pressure-coupling MHD scheme. A structure-preserving approximation can be easily obtained by neglecting the averaged kinetic momentum directly in the expression of the hybrid Hamiltonian (59) (so that $\mathbf{M}\simeq\mathbf{m}$), which then becomes

$$H(\mathbf{M},\rho,f,\mathbf{B}) = \frac{1}{2}\int\frac{|\mathbf{M}|^2}{\rho}\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mathsf{m}_h}\int f\,|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \int\rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int|\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x}. \tag{60}$$

Then, the Poisson bracket (58) yields the final equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = -\frac{1}{\rho}\nabla\mathsf{p} - \frac{1}{\mathsf{m}_h\rho}\,\mathrm{div}\int\mathbf{p}\mathbf{p} f\,\mathrm{d}^3\mathbf{p} - \frac{1}{\mu_0\rho}\mathbf{B}\times\nabla\times\mathbf{B} \tag{61}$$

$$\frac{\partial f}{\partial t} + \left(\boldsymbol{U} + \frac{\mathbf{p}}{m_h}\right)\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{p}}\cdot\nabla\boldsymbol{U}\cdot\mathbf{p} + a_h\mathbf{p}\times\mathbf{B}\cdot\frac{\partial f}{\partial \mathbf{p}} = 0 \tag{62}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\boldsymbol{U}) = 0, \qquad \frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\boldsymbol{U}\times\mathbf{B}), \tag{63}$$

where the term $\{f, \mathbf{p}\cdot\boldsymbol{U}\}$ again appears to balance the pressure term in the energy conservation. Except for the Vlasov equation (62), the above system is identical to the scheme presented in [11, 36].

3.3.2. Second pressure-coupling MHD scheme. Note that the following expression of the Hamiltonian is also possible (see section 2.2.2, remark 2):

$$H = \frac{1}{2}\int \frac{|\mathbf{M}|^2}{\rho}\,\mathrm{d}^3\mathbf{x} + \frac{1}{2m_h}\int f\left|\mathbf{p} - \frac{\mathbf{K}}{n}\right|^2 \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \int \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int |\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x}, \tag{64}$$

which in turn yields the equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = -\frac{1}{\rho}\nabla\mathsf{p} - \frac{1}{\mathsf{m}_h\rho}\,\mathrm{div}\int(\mathbf{p}-\boldsymbol{V})^{\otimes 2} f\,\mathrm{d}^3\mathbf{p} + \frac{1}{\mu_0\rho}\mathrm{curl}\,\mathbf{B}\times\mathbf{B} \tag{65}$$

$$\frac{\partial f}{\partial t} + \left(\frac{\mathbf{p}-\boldsymbol{V}}{m_h} + \boldsymbol{U}\right)\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{p}}\cdot\left(\frac{\partial \boldsymbol{U}}{\partial \mathbf{x}}\cdot\mathbf{p} + \frac{\partial \boldsymbol{V}}{\partial \mathbf{x}}\cdot\frac{\mathbf{p}-\boldsymbol{V}}{m_h}\right) + a_h\,(\mathbf{p}-\boldsymbol{V})\times\mathbf{B}\cdot\frac{\partial f}{\partial \mathbf{p}} = 0 \tag{66}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\boldsymbol{U}) = 0, \qquad \frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\boldsymbol{U}\times\mathbf{B}), \tag{67}$$

where $\boldsymbol{V} = \mathbf{K}/n$. Note that although complicated, the above kinetic equation yields the simple advection equation $\partial_t n + \mathrm{div}(n\,\boldsymbol{U}) = 0$ for the zeroth moment $n = \int f\,\mathrm{d}^3\mathbf{p}$. Then, the total density $D = \rho + m_h\,n$ also satisfies $\partial_t D + \mathrm{div}(D\,\boldsymbol{U}) = 0$, which is the total density equation appearing in Cheng's hybrid model [7]. This transport equation for D and the velocity equation above both appear exactly the same in the hybrid scheme presented in [21, 42]. As a conclusion to this section, we remark that in the present Hamiltonian setting a purely advection equation for D is necessarily accompanied by the presence of the relative pressure tensor $\int(\mathbf{p}-\boldsymbol{V})^{\otimes 2} f\,\mathrm{d}^3\mathbf{p}$ in the velocity equation, instead of the absolute pressure $\int \mathbf{p}^{\otimes 2} f\,\mathrm{d}^3\mathbf{p}$. This fact is suggestive that the scheme in [21, 42] is preferable to the one presented in [7].

3.4. The Hamiltonian structure and its consequences

In the previous sections, the current-coupling hybrid kinetic scheme (52)–(55) was derived by inserting neutrality and Ohm's ideal law in the general Hamilton's equations arising from the Poisson bracket (45). Then, upon assuming negligible density and momentum for the energetic component, a simple momentum shift produced the pressure-coupling hybrid schemes. This section explains the origin of the hybrid Hamiltonian structures, starting from a direct sum Poisson bracket. In particular, the reason why the hybrid Poisson brackets (56) and (58) both satisfy the Jacobi identity is not at all evident and it needs particular care.

We shall proceed by first deriving the bracket (56) from the direct sum bracket

$$\begin{aligned}\{F, G\} = &\int \mathbf{N}\cdot\left[\frac{\delta F}{\delta \mathbf{N}}, \frac{\delta G}{\delta \mathbf{N}}\right]\mathrm{d}^3\mathbf{x} - \int \rho\left(\frac{\delta F}{\delta \mathbf{N}}\cdot\nabla\frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{N}}\cdot\nabla\frac{\delta F}{\delta \rho}\right)\mathrm{d}^3\mathbf{x}\\ &- \int \mathbf{A}\cdot\left(\frac{\delta F}{\delta \mathbf{N}}\,\mathrm{div}\frac{\delta G}{\delta \mathbf{A}} + \frac{\delta G}{\delta \mathbf{N}}\,\mathrm{div}\frac{\delta F}{\delta \mathbf{A}} - \nabla\times\left(\frac{\delta F}{\delta \mathbf{N}}\times\frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{N}}\times\frac{\delta F}{\delta \mathbf{A}}\right)\right)\mathrm{d}^3\mathbf{x}\\ &+ \int \hat{f}\left\{\frac{\delta F}{\delta \hat{f}}, \frac{\delta G}{\delta \hat{f}}\right\}\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p},\end{aligned} \tag{68}$$

where the first two lines coincide with the bracket of ideal compressible MHD [17, 31, 34], while the last term determines the Poisson bracket of the Vlasov equation. The momentum variable $\mathbf{N}$ is a new variable that will be given physical meaning in theorem 2. Note that in the above bracket, the Vlasov distribution $\hat{f}(\mathbf{x}, \boldsymbol{p})$ is expressed as a function of the mechanical momentum variable $\boldsymbol{p} = m_h\mathbf{v} + q_a\mathbf{A}$ and *not* as a function of the variable $\mathbf{p} = m_h\mathbf{v}$. Thus one has

$$\hat{f}(\mathbf{x}, \boldsymbol{p}) = \hat{f}(\mathbf{x}, \mathbf{p} + q_h\mathbf{A}) = f(\mathbf{x}, \mathbf{p}).$$

(Note the different notation: the mechanical momentum is denoted by the italic bold font $\boldsymbol{p}$, while the roman bold notation $\mathbf{p}$ corresponds to $m_h\mathbf{v}$.)

Remark 6 (Lie algebra of the current-coupling scheme)**.** The Poisson bracket (68) is a Lie–Poisson bracket on the direct-sum Lie algebra

$$C^\infty(\mathbb{R}^6) \oplus (\mathfrak{X}(\mathbb{R}^3)\circledS(C^\infty(\mathbb{R}^3) \oplus \Omega^2(\mathbb{R}^3)))$$

where $C^\infty(\mathbb{R}^6)$ is the algebra of Hamiltonian functions (dual to Vlasov distributions), while the variables $(\mathbf{N}, \rho, \mathbf{A})$ belong to the dual space of the semidirect-product Lie algebra

$$\mathfrak{X}(\mathbb{R}^3)\circledS(C^\infty(\mathbb{R}^3) \oplus \Omega^2(\mathbb{R}^3)).$$

Here $\Omega^2(\mathbb{R}^3)$ is the space of differential two-forms containing the electric induction $\mathbf{D} \cdot \mathrm{d}\mathbf{S}$ (where $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$ denotes the product of an infinitesimal surface element $\mathrm{d}S$ with its normal unit vector $\hat{\mathbf{n}}$).

The Poisson bracket (56) is equivalent to (68) in the following sense:

Theorem 2. *The Poisson bracket (68) transforms to bracket (56) under the momentum shift*

$$\mathbf{m} = \mathbf{N} + q_h\mathbf{A}\int \hat{f}(\mathbf{x}, \boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p} = \mathbf{N} + q_h\mathbf{A}\int f(\mathbf{x}, \mathbf{p})\,\mathrm{d}^3\mathbf{p}.$$

This result is proved in the appendix. Perhaps the most direct physical consequence resides in the following circulation theorem:

Corollary 3 (Kelvin's theorem for the current-coupling scheme)**.** *For every loop $\gamma_t(\boldsymbol{u})$ moving with velocity $\boldsymbol{u} = \mathbf{m}/\rho$, the current-coupling scheme (52)–(55) possesses the following circulation theorem:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t(\boldsymbol{u})}\left(\boldsymbol{u} - q_h\frac{n}{\rho}\mathbf{A}\right)\cdot\mathrm{d}\mathbf{x} = -\oint_{\gamma_t(\boldsymbol{u})}\frac{1}{\rho}\mathbf{B}\times\left(\mu_0^{-1}\nabla\times\mathbf{B} - a_h\left(\mathbf{K} - m_h n\boldsymbol{u}\right)\right)\cdot\mathrm{d}\mathbf{x}$$
$$+ a_h\oint_{\gamma_t(\boldsymbol{u})}\frac{1}{\rho}(\nabla\cdot(\mathbf{K} - m_h n\boldsymbol{u}))\mathbf{A}\cdot\mathrm{d}\mathbf{x},$$

where $\mathbf{K} = \int \mathbf{p}\, f\,\mathrm{d}^3\mathbf{p}$.

Proof. From theorem 2, one concludes that the current-coupling scheme (52)–(55) possesses the Poisson bracket (68) and the Hamiltonian (57) in the form

$$H = \frac{1}{2}\int\frac{1}{\rho}\left|\mathbf{N} + q_h\mathbf{A}\int\hat{f}\,\mathrm{d}^3\boldsymbol{p}\right|^2\mathrm{d}^3\mathbf{x} + \frac{1}{2m_h}\int\hat{f}\,|\boldsymbol{p} - q_h\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\boldsymbol{p}$$
$$+ \int\rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int|\nabla\times\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}. \tag{69}$$

Then, using the Hamiltonian (69) in the bracket (68) yields the momentum equation

$$\left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{u}}\right)\mathbf{N} = -\rho\,\nabla\frac{\delta H}{\delta\rho} + \frac{\delta H}{\delta\mathbf{A}}\times(\nabla\times\mathbf{A}) - \mathbf{A}\,\nabla\cdot\frac{\delta H}{\delta\mathbf{A}} \tag{70}$$

where $\pounds_{\boldsymbol{u}}$ denotes the Lie derivative along the velocity vector field $\boldsymbol{u}$. After dividing by ρ and calculating

$$\frac{\delta H}{\delta \mathbf{A}} = q_h \frac{n}{\rho} (\mathbf{N} + q_h n \mathbf{A}) + \mu_0^{-1} \nabla \times \nabla \times \mathbf{A} - a_h \int \hat{f}\, (\boldsymbol{p} - q_h \mathbf{A})\, \mathrm{d}^3\boldsymbol{p}$$
$$= \mu_0^{-1} \nabla \times \mathbf{B} - a_h (\mathbf{K} - m_h n \boldsymbol{u}),$$

the proof follows easily by taking the circulation integral on both sides of (70). □

Another relevant consequence of theorem 2 is the existence of Casimir functionals

Corollary 4 (Casimir functionals)**.** *The Poisson bracket (56) possesses the following Casimir functional:*

$$C(f) = \int \Phi(f)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p}.$$

Proof. Although it can be proven by direct verification, this is a direct consequence of the Casimir functional $C(\hat{f}) = \int \Lambda(\hat{f})\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p}$ for the Poisson bracket (68), which is inherited from the general properties of the Hamiltonian structure of the Vlasov equation. □

Note that the magnetic helicity $\mathcal{H} = \int \mathbf{A} \cdot \nabla \times \mathbf{A}\, \mathrm{d}^3\mathbf{x}$ is also a Casimir for the bracket (68), although it is not strictly a Casimir for bracket (56), which is rather expressed in terms of the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$.

Analogous properties as those above also hold for the pressure-coupling scheme. This can be seen by the following relation between the Poisson brackets (58) and (68):

Theorem 5. *The Poisson bracket (68) transforms to bracket (58) under the change of variable*

$$\mathbf{M} = \mathbf{N} + \int \boldsymbol{p} \hat{f}(\mathbf{x}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p} = \mathbf{N} + \int \mathbf{p} f(\mathbf{x}, \mathbf{p})\, \mathrm{d}^3\mathbf{p} + q_h \mathbf{A} \int f(\mathbf{x}, \mathbf{p})\, \mathrm{d}^3\mathbf{p}.$$

The proof of the above theorem is presented in the appendix. Again, the above statement yields

Corollary 6 (Kelvin's theorems for pressure-coupling schemes)**.** *The Kelvin circulation laws for the pressure-coupling schemes (61)–(63) and (65)–(67) read, respectively,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t(\boldsymbol{U})} \left(\boldsymbol{U} - \frac{1}{\rho} \int \boldsymbol{p} \hat{f}(\mathbf{x}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p} \right) \cdot \mathrm{d}\mathbf{x} = - \oint_{\gamma_t(\boldsymbol{U})} \frac{1}{\rho} \mathbf{B} \times \left(\mu_0^{-1} \nabla \times \mathbf{B} - a_h \mathbf{K} \right) \cdot \mathrm{d}\mathbf{x}$$
$$+ a_h \oint_{\gamma_t(\boldsymbol{U})} \frac{1}{\rho} (\nabla \cdot \mathbf{K}) \mathbf{A} \cdot \mathrm{d}\mathbf{x},$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t(\boldsymbol{U})} \left(\boldsymbol{U} - \frac{1}{\rho} \int \boldsymbol{p} \hat{f}(\mathbf{x}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p} \right) \cdot \mathrm{d}\mathbf{x} = -\mu_0^{-1} \oint_{\gamma_t(\boldsymbol{U})} \rho^{-1} \mathbf{B} \times \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{x},$$

where $\mathbf{K} = \int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p}$ *and the loop* $\gamma_t(\boldsymbol{U})$ *moves with the velocity* $\boldsymbol{U} = \mathbf{M}/\rho$.

Proof. Because of theorem 5, the pressure-coupling scheme (61)–(63) can be written in terms of the momentum variable $\mathbf{N} = \mathbf{M} - \int \boldsymbol{p} \hat{f}(\mathbf{x}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p}$. Indeed, the equation for $\mathbf{N}$ is given immediately by the Poisson bracket (68) together with the Hamiltonian (60) in the form

$$H = \frac{1}{2} \int \frac{1}{\rho} \left| \mathbf{N} + \int \boldsymbol{p} \hat{f}(\mathbf{x}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p} \right|^2 \mathrm{d}^3\mathbf{x} + \frac{1}{2\mathrm{m}_h} \int \hat{f} |\boldsymbol{p} - q_h \mathbf{A}|^2\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p}$$
$$+ \int \rho\, \mathcal{U}(\rho)\, \mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0} \int |\nabla \times \mathbf{A}|^2\, \mathrm{d}^3\mathbf{x}. \tag{71}$$

At this point, the Poisson bracket (68) produces the momentum equation (70), while the Hamiltonian (71) yields

$$\frac{\delta H}{\delta \mathbf{A}} = {\mu_0}^{-1}\nabla \times \mathbf{B} - a_h\mathbf{K}.$$

Then, after replacing the above variational derivative in equation (70), the remaining steps coincide with those already followed in the proof of corollary 3.

The same arguments also hold for the second pressure-coupling scheme (65)–(67). In this case, the Hamiltonian (64) is written explicitly in terms of $\mathbf{N}$ as

$$\begin{aligned} H = \frac{1}{2}\int \frac{1}{\rho}\left|\mathbf{N} + \int \boldsymbol{p}\hat{f}(\mathbf{x},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}\right|^2 \mathrm{d}^3\mathbf{x} + \frac{1}{2\mathrm{m}_h}\int \hat{f}|\boldsymbol{p} - q_h\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\boldsymbol{p} \\ - \frac{1}{2m_h}\int \frac{1}{\int \hat{f}(\mathbf{x},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}}\left|\int \boldsymbol{p}\hat{f}(\mathbf{x},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p} - q_h\mathbf{A}\int \hat{f}(\mathbf{x},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}\right|^2 \mathrm{d}^3\mathbf{x} \\ + \int \rho\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int |\nabla\times\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}, \end{aligned} \tag{72}$$

which in turn yields the following variational derivative:

$$\frac{\delta H}{\delta \mathbf{A}} = {\mu_0}^{-1}\nabla \times \mathbf{B}.$$

Then, inserting the following relation in equation (70) arising from the Poisson bracket (68) and following the same steps as in the previous cases completes the derivation of the circulation law for the second pressure-coupling scheme (65)–(67). □

It is easy to verify that functionals of the type $\int \Phi(f)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$ are also Casimirs of the Poisson bracket (58). The identification of Casimir functionals is a fundamental feature of the Hamiltonian approach, which allows for a nonlinear stability analysis as extensively described in [18]. The study of the stability properties of the hybrid MHD schemes is a considerable part of future work plans.

Thus, we have the following consequence:

Corollary 7. *The brackets (56) and (58) are Poisson brackets, whose Jacobi identity is inherited from the direct sum Poisson bracket (68).*

Remark 7 (Lie algebra of pressure-coupling schemes)**.** From an algebraic point of view, the Poisson bracket (58) is a Lie–Poisson bracket [26] on the semidirect-product Lie algebra

$$\mathfrak{X}(\mathbb{R}^3)\circledS(C^\infty(\mathbb{R}^3)\times\Omega^1(\mathbb{R}^3)\times C^\infty(\mathbb{R}^6))$$

where we have denoted by $\mathfrak{X}(\mathbb{R}^3)$ the space of velocity vector fields, while $\Omega^1(\mathbb{R}^3)$ denotes the space of differential one-forms (i.e. the space of magnetic vector potentials). Note that the space $C^\infty(\mathbb{R}^6)$ is a Poisson algebra, which is dual to the Poisson manifold of Vlasov distributions. This construction containing the semidirect-product Lie algebra $\mathfrak{X}(\mathbb{R}^3)\circledS C^\infty(\mathbb{R}^6)$ appears in a continuum model for the first time.

3.5. Arguments in favor of the current-coupling scheme

The current-coupling scheme for hybrid MHD is often considered to be equivalent to ordinary pressure-coupling schemes [11, 35, 36]. However, the assumption of a rarefied energetic ion plasma poses nontrivial problems in the mathematical formulation. Indeed, we recognize that the assumption of the negligible particle density n may lead to some ambiguities. For example, we saw that in the second pressure-coupling scheme (65)–(67), the condition $n = 0$ is preserved by the dynamics, although this reflects in a possible divergence of the Hamiltonian, which indeed contains the term $n^{-1}|\mathbf{K}|^2$. Then, the only possibility of avoiding this divergence is to set $\mathbf{K} = 0$. For example, the whole velocity $\mathbf{K}/n$ is sometimes neglected completely [42]. However, the well-known pressure term $\operatorname{div}\int(\mathbf{p} - \boldsymbol{V})^{\otimes 2} f \, \mathrm{d}^3\mathbf{p}$ appearing in the equation for $\mathbf{K}$ forbids this possibility unless one accepts that this pressure term is also negligible, which contradicts the starting hypothesis of non-negligible pressure effects in the total momentum dynamics. This non-negligible pressure term is a continuous source of hot particle averaged momentum, whose initial low levels are not maintained by the dynamics. Note that this phenomenon is not peculiar of the pressure-coupling schemes presented in this paper; rather, this problem is intrinsically unavoidable and it is present in all pressure-coupling schemes.

The current-coupling scheme provides an exceptional way of avoiding this sort of problems, since it makes no assumption on the moments of the Vlasov distribution. Some advantages of the current-coupling scheme were also emphasized in [1]. Moreover, the existence of the newly discovered Hamiltonian structure for this model, together with its consequent circulation theorem and Casimir functions, enriches this model significantly. In conclusion, the pressure-coupling scheme is very different from the current-coupling scheme and the latter solves all the significant problems emerging from pressure-coupling schemes.

4. Hybrid kinetic-fluid formulation of Hall-MHD

In the preceding sections we have investigated the hybrid kinetic-fluid formulation of MHD models taking into account the dynamics of energetic ions. On the other hand, the existence of energetic ions is also a peculiar feature of Hall effects in magnetized plasmas. Indeed, one of the main consequences of Hall effects is the progressive acceleration of ions that become particularly fast. In Hall-MHD models, this scenario leads to the simple idea of treating the whole ion dynamics by a kinetic approach, while the electrons are still considered as a neutralizing background. This model is usually referred to as 'kinetic-particle-ion/fluid electron' hybrid scheme [4, 35, 45, 47]. Its equations are easily derived from the kinetic-multifluid system of section 3.1, upon letting the only label $s = 2$ identify the electron species. Then, the plasma equations of motion reduce to the Vlasov kinetic equation (41) and the electron fluid equation (39) (with $s = 2$). Neglecting electron inertia (that is letting $m_e \to 0$) and assuming neutrality (that is letting $\epsilon_0 \to 0$) yields the well-known equations [4, 35, 45]

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m_i} \cdot \frac{\partial f}{\partial \mathbf{x}} + a_i \mathbf{p} \times \mathbf{B} \cdot \frac{\partial f}{\partial \mathbf{p}} + \frac{q_i}{a_e \rho_e}\left[a_e \nabla \mathsf{p}_e - \left(a_i \int \mathbf{p} f \, \mathrm{d}^3\mathbf{p} - \mu_0^{-1} \nabla \times \mathbf{B}\right) \times \mathbf{B}\right] \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \tag{73}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\frac{1}{a_e \rho_e} \mathbf{B} \times \left(a_i \int \mathbf{p} f \, \mathrm{d}^3\mathbf{p} - \mu_0^{-1} \nabla \times \mathbf{B}\right)\right]. \tag{74}$$

In the search for a Hamiltonian structure of the above equations, it is helpful to recall the Hamiltonian structure of Hall-MHD, which was presented in [16]. Proceeding by analogy to the Hall-MHD bracket in [16], one introduces the Poisson bracket

$$\{F, G\} = \int \left(\mathsf{Q}_e^{-1} \nabla \times \mathbf{A}\right) \cdot \frac{\delta F}{\delta \mathbf{A}} \times \frac{\delta G}{\delta \mathbf{A}} \, \mathrm{d}^3\mathbf{x} - \int \left(\frac{\delta F}{\delta \mathbf{A}} \cdot \nabla \frac{\delta G}{\delta \mathsf{Q}_e} - \frac{\delta G}{\delta \mathbf{A}} \cdot \nabla \frac{\delta F}{\delta \mathsf{Q}_e}\right) \mathrm{d}^3\mathbf{x}$$
$$+ \int \hat{f} \left\{\frac{\delta F}{\delta \hat{f}}, \frac{\delta G}{\delta \hat{f}}\right\} \mathrm{d}^3\mathbf{x} \, \mathrm{d}^3\boldsymbol{p}, \tag{75}$$

where $\mathsf{Q}_e = a_e \rho_e$ is the electron charge density and the kinetic Vlasov term in the last line replaces the fluid bracket terms appearing in the Hamiltonian structure of Hall-MHD [16].

Note that the first line of (75) is equivalent to a fluid bracket in the density-momentum variables $(\mathsf{Q}_e, \mathsf{Q}_e\mathbf{A})$ [16]. At this point, the Hamiltonian structure of equations (73)–(74) arises by simply expressing the above bracket in terms of the Vlasov distribution $f(\mathbf{x},\mathbf{p}) = f(\mathbf{x}, \boldsymbol{p} - q_i\mathbf{A}) = \hat{f}(\mathbf{x}, \boldsymbol{p})$. Then, the final result is

$$
\begin{aligned}
\{F, G\} = & \int \left(\mathsf{Q}_e^{-1}\nabla\times\mathbf{A}\right)\cdot\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\mathrm{d}^3\mathbf{x} - \int\left(\frac{\delta F}{\delta\mathbf{A}}\cdot\nabla\frac{\delta G}{\delta\mathsf{Q}_e} - \frac{\delta G}{\delta\mathbf{A}}\cdot\nabla\frac{\delta F}{\delta\mathsf{Q}_e}\right)\mathrm{d}^3\mathbf{x} \\
& + \int f\left(\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\} + q_i\nabla\times\mathbf{A}\cdot\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \\
& - q_i\int f\left(\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta\mathsf{Q}_e}\right\} - \left\{\frac{\delta G}{\delta f},\frac{\delta F}{\delta\mathsf{Q}_e}\right\}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \\
& + \int \mathsf{Q}_e^{-1} f\nabla\times\mathbf{A}\cdot\left(\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f}\times\frac{\delta F}{\delta\mathbf{A}} - \frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\times\frac{\delta G}{\delta\mathbf{A}}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} \\
& + q_i^2\int\mathsf{Q}_e^{-1}\nabla\times\mathbf{A}\cdot\left(\int f\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\mathrm{d}^3\mathbf{p}\right)\times\left(\int f\frac{\partial}{\partial\mathbf{p}'}\frac{\delta G}{\delta f}\mathrm{d}^3\mathbf{p}'\right)\mathrm{d}^3\mathbf{x},
\end{aligned}
\tag{76}
$$

which is indeed the Poisson bracket of equations (73)–(74) and it is accompanied by the Hamiltonian

$$
H(f, \mathbf{A}, \mathsf{Q}_e) = \frac{1}{2\mathsf{m}_i}\int f|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \int\phi(\mathsf{Q}_e)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int|\nabla\times\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}, \tag{77}
$$

where $\phi(\mathsf{Q}_e)$ appropriately denotes the internal energy expressed in terms of the charge Q_e. Note that the charge neutrality $\mathsf{Q}_e + q_i\,n = 0$ is preserved by the dynamics, as it is shown by the equation

$$
\frac{\partial\mathsf{Q}_e}{\partial t} = a_i\mathrm{div}\int\mathbf{p}f\,\mathrm{d}^3\mathbf{p} = -q_i\frac{\partial}{\partial t}\int\mathbf{p}f\,\mathrm{d}^3\mathbf{p},
$$

which arises from the Hamiltonian structure (76).

It is also interesting to note that the class of Casimir functionals $C(\hat{f}) = \int\Lambda(\hat{f})\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\boldsymbol{p}$ for the Vlasov equation again produces the Casimir functionals $C(f) = \int\Phi(f)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$ for the Hamiltonian structure (75) under consideration. Here, the magnetic helicity

$$
\mathcal{H} = \int\mathbf{A}\cdot\nabla\times\mathbf{A}\,\mathrm{d}^3\mathbf{x}
$$

is a Casimir of both Poisson brackets (75) and (76). These Casimir functionals provide the opportunity for a nonlinear stability analysis, which will be the subject of future work.

Remark 8 (Lie algebraic structure)**.** The Hamiltonian structure (76) of the hybrid Hall-MHD model is inherited from the Lie–Poisson bracket (75) on the direct-sum Lie algebra

$$
(\mathfrak{X}(\mathbb{R}^3)\circledS C^\infty(\mathbb{R}^3))\oplus C^\infty(\mathbb{R}^6),
$$

where $C^\infty(\mathbb{R}^6)$ is the Poisson algebra of Hamiltonian functions, while the variables $(\mathsf{Q}_e\mathbf{A}, \mathsf{Q}_e)$ belong to the dual space of the semidirect-product Lie algebra $\mathfrak{X}(\mathbb{R}^3)\circledS C^\infty(\mathbb{R}^3)$ [16]. The Hamiltonian corresponding to the Poisson bracket (75) is given by

$$
H(\hat{f}, \mathbf{A}, \mathsf{Q}_e) = \frac{1}{2\mathsf{m}_i}\int\hat{f}\,|\boldsymbol{p} - q_i\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\boldsymbol{p} + \int\phi(\mathsf{Q}_e)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int|\nabla\times\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}. \tag{78}
$$

The absence of a fluid velocity equation in the hybrid formulation of Hall-MHD makes this model particularly difficult to implement numerically. However, one can introduce a velocity equation by simply coupling the ion kinetic equation to the equation for its first two moments $(\rho_i\boldsymbol{V}_i, \rho_i) = (\int\mathbf{p}\,f\,\mathrm{d}^3\mathbf{p},\ \mathsf{m}_i\int f\,\mathrm{d}^3\mathbf{p})$. Evidently, this operation does not produce any changes to the physics, as long as the form of the Hamiltonian is not modified. On the other hand, the ion momentum equation might be useful to isolate kinetic pressure terms explicitly [35]. When these kinetic pressure terms are omitted, the ion momentum equation consistently returns the ordinary equations of Hall-MHD.

5. Conclusions and future work plans

This paper has presented the Hamiltonian structure of several hybrid kinetic-fluid models. Upon making assumptions on the form of the Hamiltonian, new pressure-coupling schemes were derived for either ordinary fluids and MHD, whereas the existing current-coupling MHD scheme [35, 36] was shown to possess an intrinsic Hamiltonian structure. All Poisson brackets were derived from first principles, so that the Jacobi identity is always guaranteed. In turn, the Hamiltonian structure produces a Casimir functional and a circulation law for each of these models. The same approach was used to present the Hamiltonian structure of the existing hybrid Hall-MHD scheme [45, 47].

The next steps will focus on the nonlinear stability analysis, which is now made possible by the newly discovered Hamiltonian structures. Another relevant open question concerns the extension of these Hamiltonian structures to the case of gyrokinetic equations, which are frequently used in simulations. The Hamiltonian structure of gyrokinetic equations is well known [2, 13, 23] and it is possible that the same methods used in this paper can be fruitful also in that context.

While the Hamiltonian approach was used in this paper, its Lagrangian counterpart may also deserve further investigation. This direction can be easily approached by the methods developed in [5] and is part of current work [20].

As a conclusive remark, it is worth mentioning that while pressure coupling schemes were shown to require transport-like terms in the accompanying kinetic equation, one cannot exclude *a priori* that removing these terms still allows for other exotic Poisson brackets. Indeed, although this is unlikely, it cannot be excluded. However, even if this is the case, the alternative Poisson brackets would not be of Lie–Poisson form and thus they would not possess well-defined symmetry properties, which are the main focus of this paper.

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Appendix

A.1. Proof of theorem 2

Proof. First, one computes the variational derivatives

$$\frac{\delta \bar{F}}{\delta \mathbf{m}} = \frac{\delta F}{\delta \mathbf{N}}, \qquad \frac{\delta F}{\delta \mathbf{A}} = \frac{\delta \bar{F}}{\delta \mathbf{A}} + q_h \frac{\delta \bar{F}}{\delta \mathbf{m}} \int \hat{f}(\mathbf{x}, \boldsymbol{p})\, \mathrm{d}^3 \boldsymbol{p}, \qquad \frac{\delta F}{\delta \hat{f}} = \frac{\delta \bar{F}}{\delta \hat{f}} + q_h \mathbf{A} \cdot \frac{\delta \bar{F}}{\delta \mathbf{m}}.$$

Then, the Poisson bracket (68) becomes

$$
\begin{aligned}
\{F, G\} = & \int \mathbf{N} \cdot \left[\frac{\delta F}{\delta \mathbf{m}}, \frac{\delta G}{\delta \mathbf{m}}\right] \mathrm{d}^3\mathbf{x} - \int \rho \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \rho}\right) \mathrm{d}^3\mathbf{x} \\
& - \int \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \operatorname{div} \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{m}} \operatorname{div} \frac{\delta F}{\delta \mathbf{A}} - \nabla \times \left(\frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{m}} \times \frac{\delta F}{\delta \mathbf{A}}\right)\right) \mathrm{d}^3\mathbf{x} \\
& - q_h \int \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \operatorname{div}\left(n \frac{\delta G}{\delta \mathbf{m}}\right) - \frac{\delta G}{\delta \mathbf{m}} \operatorname{div}\left(n \frac{\delta F}{\delta \mathbf{m}}\right) - 2\nabla \times \left(n \frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{m}}\right)\right) \mathrm{d}^3\mathbf{x} \\
& + \int \hat{f} \left\{\frac{\delta F}{\delta \hat{f}}, \frac{\delta G}{\delta \hat{f}}\right\} \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p} + q_h \int \hat{f} \left(\left\{\frac{\delta F}{\delta \hat{f}}, \mathbf{A} \cdot \frac{\delta G}{\delta \mathbf{m}}\right\} - \left\{\frac{\delta G}{\delta \hat{f}}, \mathbf{A} \cdot \frac{\delta F}{\delta \mathbf{m}}\right\}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p}
\end{aligned}
$$

where we have dropped the bar notation for convenience and we have denoted $n = \int \hat{f}\, \mathrm{d}^3\boldsymbol{p}$. Upon expanding the third line above, one has

$$
\begin{aligned}
\{F, G\} = & \int \mathbf{N} \cdot \left[\frac{\delta F}{\delta \mathbf{m}}, \frac{\delta G}{\delta \mathbf{m}}\right] \mathrm{d}^3\mathbf{x} - \int \rho \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \rho}\right) \mathrm{d}^3\mathbf{x} \\
& - \int \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \operatorname{div} \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{m}} \operatorname{div} \frac{\delta F}{\delta \mathbf{A}} - \nabla \times \left(\frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{m}} \times \frac{\delta F}{\delta \mathbf{A}}\right)\right) \mathrm{d}^3\mathbf{x} \\
& - q_h \int \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \left(\frac{\delta G}{\delta \mathbf{m}} \cdot \nabla n\right) - \frac{\delta G}{\delta \mathbf{m}} \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla n\right)\right) \\
& - q_h \int n\mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \operatorname{div}\left(\frac{\delta G}{\delta \mathbf{m}}\right) - \frac{\delta G}{\delta \mathbf{m}} \operatorname{div}\left(\frac{\delta F}{\delta \mathbf{m}}\right)\right) \mathrm{d}^3\mathbf{x} \\
& + 2q_h \int \mathbf{A} \cdot \nabla \times \left(n \frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{m}}\right) \mathrm{d}^3\mathbf{x} + \int \hat{f} \left\{\frac{\delta F}{\delta \hat{f}}, \frac{\delta G}{\delta \hat{f}}\right\} \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p} \\
& + q_h \int \hat{f} \left(\left\{\frac{\delta F}{\delta \hat{f}}, \mathbf{A} \cdot \frac{\delta G}{\delta \mathbf{m}}\right\} - \left\{\frac{\delta G}{\delta \hat{f}}, \mathbf{A} \cdot \frac{\delta F}{\delta \mathbf{m}}\right\}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p}.
\end{aligned}
$$

Then, the standard vector identities

$$
\begin{aligned}
& \mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} \\
& \nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\, (\nabla \cdot \mathbf{b}) - \mathbf{b}\, (\nabla \cdot \mathbf{a}) + [\mathbf{a}, \mathbf{b}]
\end{aligned}
$$

yield

$$
\begin{aligned}
\{F, G\} = & \int \mathbf{N} \cdot \left[\frac{\delta F}{\delta \mathbf{m}}, \frac{\delta G}{\delta \mathbf{m}}\right] \mathrm{d}^3\mathbf{x} - \int \rho \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \rho}\right) \mathrm{d}^3\mathbf{x} \\
& - \int \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \operatorname{div} \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{m}} \operatorname{div} \frac{\delta F}{\delta \mathbf{A}} - \nabla \times \left(\frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{m}} \times \frac{\delta F}{\delta \mathbf{A}}\right)\right) \mathrm{d}^3\mathbf{x} \\
& - q_h \int \mathbf{A} \cdot \left(\nabla n \times \frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{m}} + n \nabla \times \left(\frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{m}}\right) - n \left[\frac{\delta F}{\delta \mathbf{m}}, \frac{\delta G}{\delta \mathbf{m}}\right]\right) \mathrm{d}^3\mathbf{x} \\
& + 2q_h \int \mathbf{A} \cdot \nabla \times \left(n \frac{\delta F}{\delta \mathbf{m}} \times \frac{\delta G}{\delta \mathbf{m}}\right) \mathrm{d}^3\mathbf{x} + \int \hat{f} \left\{\frac{\delta F}{\delta \hat{f}}, \frac{\delta G}{\delta \hat{f}}\right\} \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p} \\
& + q_h \int \hat{f} \left(\left\{\frac{\delta F}{\delta \hat{f}}, \mathbf{A} \cdot \frac{\delta G}{\delta \mathbf{m}}\right\} - \left\{\frac{\delta G}{\delta \hat{f}}, \mathbf{A} \cdot \frac{\delta F}{\delta \mathbf{m}}\right\}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\boldsymbol{p}.
\end{aligned}
$$

Upon collecting all vector field commutators and simplifying according to the general formula

$$
\nabla \times (k\mathbf{a}) = k \nabla \times \mathbf{a} + \nabla k \times \mathbf{a},
$$

we obtain

$$\{F,G\} = \int \mathbf{m}\cdot\left[\frac{\delta F}{\delta\mathbf{m}},\frac{\delta G}{\delta\mathbf{m}}\right]\mathrm{d}^3\mathbf{x} - \int \rho\left(\frac{\delta F}{\delta\mathbf{m}}\cdot\nabla\frac{\delta G}{\delta\rho} - \frac{\delta G}{\delta\mathbf{m}}\cdot\nabla\frac{\delta F}{\delta\rho}\right)\mathrm{d}^3\mathbf{x}$$
$$- \int \mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{m}}\,\mathrm{div}\frac{\delta G}{\delta\mathbf{A}} - \frac{\delta G}{\delta\mathbf{m}}\,\mathrm{div}\frac{\delta F}{\delta\mathbf{A}} - \nabla\times\left(\frac{\delta F}{\delta\mathbf{m}}\times\frac{\delta G}{\delta\mathbf{A}} - \frac{\delta G}{\delta\mathbf{m}}\times\frac{\delta F}{\delta\mathbf{A}}\right)\right)\mathrm{d}^3\mathbf{x}$$
$$+ q_h\int n\nabla\times\mathbf{A}\cdot\frac{\delta F}{\delta\mathbf{m}}\times\frac{\delta G}{\delta\mathbf{m}}\mathrm{d}^3\mathbf{x} + \int \hat{f}\left\{\frac{\delta F}{\delta\hat{f}},\frac{\delta G}{\delta\hat{f}}\right\}\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\boldsymbol{p}$$
$$+ q_h\int \hat{f}\left(\left\{\frac{\delta F}{\delta\hat{f}},\mathbf{A}\cdot\frac{\delta G}{\delta\mathbf{m}}\right\} - \left\{\frac{\delta G}{\delta\hat{f}},\mathbf{A}\cdot\frac{\delta F}{\delta\mathbf{m}}\right\}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\boldsymbol{p}. \tag{A.1}$$

At this point, one expresses the above Poisson bracket in terms of the distribution $f(\mathbf{x},\mathbf{p}) = \hat{f}(\mathbf{x},\mathbf{p}+q_h\mathbf{A})$. Then, the relations [25, 27]

$$\{\hat{h},\hat{k}\} = \{h,k\} + q_h\nabla\times\mathbf{A}\cdot\frac{\partial h}{\partial\mathbf{p}}\times\frac{\partial k}{\partial\mathbf{p}},\qquad \frac{\delta F}{\delta\mathbf{A}} = \frac{\delta\bar{F}}{\delta\mathbf{A}} - q_h f\frac{\partial}{\partial\mathbf{p}}\frac{\delta\bar{F}}{\delta f}$$

and

$$\left\{\frac{\delta F}{\delta f},\mathbf{A}\cdot\frac{\delta G}{\delta\mathbf{m}}\right\} = -\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\cdot\nabla\left(\frac{\delta G}{\delta\mathbf{m}}\cdot\mathbf{A}\right) \tag{A.2}$$

transform the bracket to

$$\{F,G\} = \int \mathbf{m}\cdot\left[\frac{\delta F}{\delta\mathbf{m}},\frac{\delta G}{\delta\mathbf{m}}\right]\mathrm{d}^3\mathbf{x} - \int \rho\left(\frac{\delta F}{\delta\mathbf{m}}\cdot\nabla\frac{\delta G}{\delta\rho} - \frac{\delta G}{\delta\mathbf{m}}\cdot\nabla\frac{\delta F}{\delta\rho}\right)\mathrm{d}^3\mathbf{x}$$
$$- \int \mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{m}}\,\mathrm{div}\frac{\delta G}{\delta\mathbf{A}} - \frac{\delta G}{\delta\mathbf{m}}\,\mathrm{div}\frac{\delta F}{\delta\mathbf{A}} - \nabla\times\left(\frac{\delta F}{\delta\mathbf{m}}\times\frac{\delta G}{\delta\mathbf{A}} - \frac{\delta G}{\delta\mathbf{m}}\times\frac{\delta F}{\delta\mathbf{A}}\right)\right)\mathrm{d}^3\mathbf{x}$$
$$- q_h\int f\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{m}}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f} - \frac{\delta G}{\delta\mathbf{m}}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f} - \frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f}\right)\mathrm{d}^3\mathbf{x}$$
$$+ q_h\int n\nabla\times\mathbf{A}\cdot\frac{\delta F}{\delta\mathbf{m}}\times\frac{\delta G}{\delta\mathbf{m}}\mathrm{d}^3\mathbf{x} + \int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}. \tag{A.3}$$

Introducing the magnetic induction $\mathbf{B} = \nabla\times\mathbf{A}$ by the formula

$$\frac{\delta F}{\delta\mathbf{A}} = \nabla\times\frac{\delta F}{\delta\mathbf{B}}$$

completes the proof, thereby yielding bracket (56). $\square$

A.2. Proof of theorem 5

Proof. The momentum shift $\mathbf{M} = \mathbf{N} + \int \boldsymbol{p}\, f\,\mathrm{d}^3\boldsymbol{p}$ takes the Poisson bracket (68) to the form

$$\{F,G\} = \int \mathbf{M}\cdot\left[\frac{\delta F}{\delta\mathbf{M}},\frac{\delta G}{\delta\mathbf{M}}\right]\mathrm{d}^3\mathbf{x} - \int \rho\left(\frac{\delta F}{\delta\mathbf{M}}\cdot\nabla\frac{\delta G}{\delta\rho} - \frac{\delta G}{\delta\mathbf{M}}\cdot\nabla\frac{\delta F}{\delta\rho}\right)\mathrm{d}^3\mathbf{x}$$
$$- \int \mathbf{A}\cdot\left(\pounds_{\delta F/\delta\mathbf{M}}\frac{\delta G}{\delta\mathbf{A}} - \pounds_{\delta G/\delta\mathbf{M}}\frac{\delta F}{\delta\mathbf{A}}\right)\mathrm{d}^3\mathbf{x}$$
$$+ \int \hat{f}\left(\left\{\frac{\delta F}{\delta\hat{f}},\frac{\delta G}{\delta\hat{f}}\right\} + \left\{\frac{\delta F}{\delta\hat{f}},\boldsymbol{p}\cdot\frac{\delta G}{\delta\mathbf{M}}\right\} - \left\{\frac{\delta G}{\delta\hat{f}},\boldsymbol{p}\cdot\frac{\delta F}{\delta\mathbf{M}}\right\}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\boldsymbol{p} \tag{A.4}$$

where $\pounds_{\boldsymbol{u}}$ denotes Lie derivative with respect to the velocity $\boldsymbol{u}$. Then, substitution of the formulas

$$\{\hat{h},\hat{k}\} = \{h,k\} + q_h\nabla\times\mathbf{A}\cdot\left(\frac{\partial h}{\partial\mathbf{p}}\times\frac{\partial k}{\partial\mathbf{p}}\right),\qquad \frac{\delta F}{\delta\mathbf{A}} = \frac{\delta\bar{F}}{\delta\mathbf{A}} - q_h\int f\frac{\partial}{\partial\mathbf{p}}\frac{\delta\bar{F}}{\delta f}\mathrm{d}^3\mathbf{p}$$

and some integration by parts yield the bracket

$$
\begin{aligned}
\{F, G\} = & \int \mathbf{M} \cdot \left[\frac{\delta F}{\delta \mathbf{M}}, \frac{\delta G}{\delta \mathbf{M}}\right] \mathrm{d}^3\mathbf{x} - \int \rho \left(\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \frac{\delta F}{\delta \rho}\right) \mathrm{d}^3\mathbf{x} \\
& - \int \mathbf{A} \cdot \left(\pounds_{\delta F/\delta \mathbf{M}} \frac{\delta G}{\delta \mathbf{A}} - \pounds_{\delta G/\delta \mathbf{M}} \frac{\delta F}{\delta \mathbf{A}}\right) \mathrm{d}^3\mathbf{x} \\
& + q_h \int f \left(\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \cdot \pounds_{\delta G/\delta \mathbf{M}} \mathbf{A} - \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f} \cdot \pounds_{\delta F/\delta \mathbf{M}} \mathbf{A}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} \\
& + \int f \left(\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} + q_h \nabla \times \mathbf{A} \cdot \left(\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f}\right)\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} \qquad (A.5) \\
& + \int f \left(\left\{\frac{\delta F}{\delta \hat{f}}, (\mathbf{p} + q_h \mathbf{A}) \cdot \frac{\delta G}{\delta \mathbf{M}}\right\} - \left\{\frac{\delta G}{\delta \hat{f}}, (\mathbf{p} + q_h \mathbf{A}) \cdot \frac{\delta F}{\delta \mathbf{M}}\right\}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} \qquad (A.6) \\
& + q_h \nabla \times \mathbf{A} \cdot \int f \left(\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \times \frac{\delta G}{\delta \mathbf{M}} - \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f} \times \frac{\delta F}{\delta \mathbf{M}}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p}. \qquad (A.7)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \cdot \pounds_{\delta G/\delta \mathbf{M}} \mathbf{A} &= \frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \cdot \left(\nabla \left(\frac{\delta G}{\delta \mathbf{M}} \cdot \mathbf{A}\right) - \frac{\delta G}{\delta \mathbf{M}} \times \nabla \times \mathbf{A}\right) \qquad (A.8) \\
&= -\left\{\frac{\delta F}{\delta f}, \mathbf{A} \cdot \frac{\delta G}{\delta \mathbf{M}}\right\} - \nabla \times \mathbf{A} \cdot \frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \times \frac{\delta G}{\delta \mathbf{M}} \qquad (A.9)
\end{aligned}
$$

takes to the Poisson bracket

$$
\begin{aligned}
\{F, G\} = & \int \mathbf{M} \cdot \left[\frac{\delta F}{\delta \mathbf{M}}, \frac{\delta G}{\delta \mathbf{M}}\right] \mathrm{d}^3\mathbf{x} - \int \rho \left(\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \frac{\delta F}{\delta \rho}\right) \mathrm{d}^3\mathbf{x} \\
& - \int \mathbf{A} \cdot \left(\pounds_{\delta F/\delta \mathbf{M}} \frac{\delta G}{\delta \mathbf{A}} - \pounds_{\delta G/\delta \mathbf{M}} \frac{\delta F}{\delta \mathbf{A}}\right) \mathrm{d}^3\mathbf{x} \\
& + \int f \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} + q_h \nabla \times \mathbf{A} \cdot \int f \left(\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} \\
& + \int f \left(\left\{\frac{\delta F}{\delta f}, \mathbf{p} \cdot \frac{\delta G}{\delta \mathbf{M}}\right\} - \left\{\frac{\delta G}{\delta f}, \mathbf{p} \cdot \frac{\delta F}{\delta \mathbf{M}}\right\}\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p}. \qquad (A.10)
\end{aligned}
$$

Then, using the formula

$$
\frac{\delta F}{\delta \mathbf{A}} = \nabla \times \frac{\delta F}{\delta \mathbf{B}}
$$

yields the Poisson structure (58). □

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