Dispersion relations of Drift-Kinetic-MHD

J.W.B., P.J.M. & C.T.

Contents

1 CCS models

These notes compare the following incompressible pressure coupling schemes

$$\rho\partial_t\boldsymbol{U} + \rho\left(\boldsymbol{U}\cdot\nabla\right)\boldsymbol{U} = \Big(\mathbf{J} + \alpha q_h n_h \boldsymbol{U} - \mathbf{J}^{\alpha}_{\mathrm{gc}} - \nabla\times\mathbf{M}^{\beta}_{\mathrm{gc}}\Big)\times\mathbf{B} - \nabla\mathsf{p} \tag{1}$$

$$\nabla\cdot\boldsymbol{U} = 0 \tag{2}$$

$$\partial_t f + \frac{1}{B^*_\parallel}\Big(v_\parallel\mathbf{B}^* - \boldsymbol{b}\times\mathbf{E}^*\Big)\cdot\nabla f + \frac{a_h}{B^*_\parallel}\left(\mathbf{B}^*\cdot\mathbf{E}^*\right)\partial_{v_\parallel} f = 0 \tag{3}$$

$$\partial_t\mathbf{B} = \nabla\times\left(\boldsymbol{U}\times\mathbf{B}\right)\,, \tag{4}$$

where

$$\mathbf{J}^{\alpha}_{\mathrm{gc}} = q_h\int_\mu \Big(v_\parallel\mathbf{B}^* - \boldsymbol{b}\times\mathbf{E}^*_\alpha\Big) f\,\mathrm{d}v_\parallel\,, \qquad \mathbf{E}^*_\alpha = \mathbf{E}^* + (\alpha-1)\mathbf{E} \tag{5}$$

and

$$\mathbf{M}_{\rm gc} = -\int_\mu \Big[\mu B_\parallel^* \boldsymbol{b} - \beta \frac{m_h v_\parallel}{B}\Big(v_\parallel \mathbf{B}_\perp^* - \boldsymbol{b}\times\mathbf{E}^*\Big)\Big]\, f\, \mathrm{d}v_\parallel\,.$$

When

$$\alpha = 1\,, \qquad\qquad \beta = 1$$

then, we have the Hamiltonian model. When

$$\alpha = 0\,, \qquad\qquad \beta = 0$$

then, we have the non-Hamiltonian model (later variants of Todo's with $B_\parallel^* \neq B$).

We wish to linearize the system around the common equilibrium

$$\boldsymbol{U}_0 = 0\,, \qquad \mathbf{B}_0 = B_0\mathbf{e}_z\,, \qquad f_0 = f_0((v_\parallel - V)^2/2 + \mu B_0)\,,$$

In more generality, it might be interesting to consider longitudinal disturbances around the equilibrium

$$\boldsymbol{U}_0 = 0\,, \qquad \mathbf{B}_0 = B_0\mathbf{e}_z\,, \qquad f_0 = f_0\big(\mathbf{x}_\perp, (v_\parallel - V)^2/2 + \mu B_0\big)\,,$$

even in the case when $V_0 = 0$. For example,

$$f_0\big(\mathbf{x}_\perp, (v_\parallel - V)^2/2 + \mu B_0\big) = n(\mathbf{x}_\perp)\exp(-(v_\parallel - V)^2/2 - \mu B_0).$$

so that

$$\mathbf{B}_0^* := \nabla\times\big(\mathbf{A}_0 + a_h^{-1} v_\parallel \boldsymbol{b}_0\big) = \mathbf{B}_0 \tag{6}$$

$$\mathbf{E}_0^* := -\boldsymbol{U}_0\times\mathbf{B}_0 + \frac{v_\parallel}{a_h B_0}[\nabla\times(\boldsymbol{U}_0\times\mathbf{B}_0)]_\perp - q_h^{-1}\mu\nabla B_0 = 0 \tag{7}$$

1.1 Linearized kinetic equation

We wish to linearize the kinetic equation. To this purpose, we shall use the relations

$$\boldsymbol{b} = \mathbf{e}_z + \delta\boldsymbol{b} = \mathbf{e}_z + \frac{1}{B_0}\delta\mathbf{B}_\perp\,, \qquad B = B_0 + \delta B = B_0 + \delta B_z\,, \qquad \nabla\cdot\delta\boldsymbol{U} = 0\,,$$

along with

$$
\begin{aligned}
\mathbf{B}^* &= \mathbf{B}_0 + \delta\mathbf{B}^* = B_0\mathbf{e}_z + \delta\mathbf{B}^* && (8)\\
\mathbf{E}^* &= \delta\mathbf{E}^* = B_0\mathbf{e}_z \times \delta\boldsymbol{U}_\perp + a_h^{-1}v_\parallel\partial_z\delta\boldsymbol{U}_\perp - q_h^{-1}\mu\nabla\delta B_z\,, && (9)
\end{aligned}
$$

where we used $[\nabla\times(\mathbf{e}_z\times\delta\boldsymbol{U})]_\perp = -\partial_z\delta\boldsymbol{U}_\perp$ and

$$
\delta\mathbf{B}^* = \delta\mathbf{B} + \frac{a_h^{-1}v_\parallel}{B_0}\nabla\times\delta\mathbf{B}_\perp\,, \qquad \delta B_\parallel^* = \delta B_z + \frac{a_h^{-1}v_\parallel}{B_0}\mathbf{e}_z\cdot\nabla\times\delta\mathbf{B}_\perp\,.
$$

Then, we compute

$$
\frac{a_h}{B_\parallel^*}\mathbf{B}^*\cdot\mathbf{E}^* = a_h\delta E_z^* = -m_h^{-1}\mu\partial_z\delta B_z
$$

and

$$
\begin{aligned}
\frac{1}{B_\parallel^*}\Big(v_\parallel\mathbf{B}^* - \boldsymbol{b}\times\mathbf{E}^*\Big) &= \frac{v_\parallel}{B_\parallel^*}\mathbf{B}^* - \frac{1}{B_0}\mathbf{e}_z\times\delta\mathbf{E}^*\\
&= \frac{1}{B_0}\left[\Big(B_0 - \delta B_\parallel^*\Big)v_\parallel\mathbf{e}_z + v_\parallel\delta\mathbf{B}^* - \mathbf{e}_z\times\delta\mathbf{E}^*\right]
\end{aligned}
$$

Then, considering that $\nabla_{\mathbf{x}}f_0$, we have

$$
\partial_t\delta f + v_\parallel\partial_z\delta f = m_h^{-1}\mu(\partial_z\delta B_z)\partial_{v_\parallel}f_0
$$

Given that $\nabla\cdot\mathbf{B}=0$, this means that for longitudinal propagation (that is $\delta\mathbf{B}=\delta B_z\mathbf{e}_z$) δf satisfies the Knudsen equation $\partial_t\delta f + v_\parallel\partial_z\delta f = 0$.

1.2 Linearized current forces

We have

$$
\delta\mathbf{E}_\alpha^* = \alpha B_0\mathbf{e}_z\times\delta\boldsymbol{U}_\perp + a_h^{-1}v_\parallel\partial_z\delta\boldsymbol{U}_\perp - q_h^{-1}\mu\nabla\delta B_z
$$

so that the current is

$$
\begin{aligned}
\mathbf{J}^{\alpha}_{\rm gc} &= q_h \int_\mu \Big(v_\parallel f_0 \delta\mathbf{B}^* + v_\parallel B_0 \mathbf{e}_z \delta f - f_0 \mathbf{e}_z \times \delta\mathbf{E}^*_\alpha + v_\parallel B_0 f_0 \mathbf{e}_z \Big)\, {\rm d}v_\parallel \\
&= \int_\mu \bigg[q_h v_\parallel f_0 \bigg(\delta\mathbf{B} + \frac{m_h v_\parallel}{B_0} \nabla \times \delta\mathbf{B}_\perp \bigg) + q_h v_\parallel B_0 \mathbf{e}_z \delta f \\
&\qquad - q_h f_0 \mathbf{e}_z \times \big(\alpha B_0 \mathbf{e}_z \times \delta\boldsymbol{U}_\perp + a_h^{-1} v_\parallel \partial_z \delta\boldsymbol{U}_\perp - q_h^{-1} \mu \nabla \delta B_z \big) \bigg]\, {\rm d}v_\parallel \\
&= q_h \frac{K_0}{B_0} \Big(\delta\mathbf{B} + B_0 \mathbf{e}_z \Big) + \frac{p_\parallel}{B_0^2} \nabla \times \delta\mathbf{B}_\perp + q_h \delta K \mathbf{e}_z + q_h n_0 \alpha \delta\boldsymbol{U}_\perp \\
&\qquad - m_h \frac{K_0}{B_0} \mathbf{e}_z \times \partial_z \delta\boldsymbol{U}_\perp - \frac{M_0}{B_0} \mathbf{e}_z \times \nabla \delta B_z \,,
\end{aligned}
$$

where (see C.Z. Cheng's hybrid paper for the definition of the pressure used here)

$$
p_\parallel = m_h B_0 \int_\mu v_\parallel^2 f_0 \, {\rm d}v_\parallel \,, \qquad n_0 = B_0 \int_\mu f_0 \, {\rm d}v_\parallel \,, \qquad K_0 = B_0 \int_\mu f_0 \, v_\parallel \, {\rm d}v_\parallel
$$

$$
\delta K = B_0 \int_\mu v_\parallel \delta f \, {\rm d}v_\parallel \,, \qquad M_0 = -B_0 \int_\mu \mu f_0 \, {\rm d}v_\parallel = -\frac{p_\perp}{B_0} \,.
$$

Thus,

$$
\begin{aligned}
\mathbf{J}^{\alpha}_{\rm gc} \times \mathbf{B} = -q_h K_0 \mathbf{e}_z \times \delta\mathbf{B}_\perp - \frac{p_\parallel}{B_0} \mathbf{e}_z \times \nabla \times \delta\mathbf{B}_\perp& \\
- q_h n_0 \alpha B_0 \mathbf{e}_z \times \delta\boldsymbol{U}_\perp - m_h K_0 \partial_z \delta\boldsymbol{U}_\perp - M_0 \nabla_\perp \delta B_z \,.&
\end{aligned}
$$

Since

$$
n_h \boldsymbol{U} \times \mathbf{B} = \boldsymbol{U} \times \mathbf{B} \int_\mu B_\parallel^* f \, {\rm d}v_\parallel = B_0 \delta\boldsymbol{U} \times \mathbf{e}_z \int_\mu B_0 f_0 \, {\rm d}v_\parallel = n_0 B_0 \delta\boldsymbol{U}_\perp \times \mathbf{e}_z \,,
$$

we notice that the $E \times B-$drift contribution does indeed cancel with the electric field term in the Lorentz force. (After all, we know that those terms don't contribute to the energy balance).

1.3 Linearized magnetization forces

The first term in the magnetization is

$$
\begin{aligned}
\mathbf{M}_{\rm gc}^{(1)} &= -\int_\mu \mu B_\parallel^* \boldsymbol{b} f \, \mathrm{d}v_\parallel \\
&= M_0 \mathbf{e}_z + \delta M \mathbf{e}_z - \int_\mu \left[\mathbf{e}_z \mu \delta B_\parallel^* + \mu \delta \mathbf{B}_\perp \right] f_0 \, \mathrm{d}v_\parallel \\
&= M_0 \mathbf{e}_z + \delta M \mathbf{e}_z - \int_\mu \left[\mu (\delta B_z + \frac{a_h^{-1} v_\parallel}{B_0} \mathbf{e}_z \cdot \nabla \times \delta \mathbf{B}_\perp) \mathbf{e}_z + \mu \delta \mathbf{B}_\perp \right] f_0 \, \mathrm{d}v_\parallel \\
&= M_0 \mathbf{e}_z + \delta M \mathbf{e}_z + \frac{M_0}{B_0} (\delta B_z \mathbf{e}_z + \delta \mathbf{B}_\perp) + \frac{a_h^{-1} Q_0}{B_0^2} (\mathbf{e}_z \cdot \nabla \times \delta \mathbf{B}_\perp) \mathbf{e}_z \\
&= M_0 \mathbf{e}_z + \delta M \mathbf{e}_z + \frac{M_0}{B_0} \delta \mathbf{B} + \frac{a_h^{-1} Q_0}{B_0^2} (\mathbf{e}_z \cdot \nabla \times \delta \mathbf{B}_\perp) \mathbf{e}_z \,,
\end{aligned}
$$

where we have defined

$$
\delta M = -B_0 \int_\mu \mu \, \delta f \, \mathrm{d}v_\parallel \,, \qquad Q_0 = -B_0 \int_\mu \mu \, v_\parallel \, \delta f \, \mathrm{d}v_\parallel
$$

Then,

$$
\nabla \times \mathbf{M}_{\rm gc}^{(1)} = \nabla \times \left(\frac{M_0}{B_0} \delta \mathbf{B} + \frac{a_h^{-1} Q_0}{B_0^2} (\mathbf{e}_z \cdot \nabla \times \delta \mathbf{B}_\perp) \mathbf{e}_z \right) - \mathbf{e}_z \times \nabla \delta M \,.
$$

The second term is

$$
\begin{aligned}
\mathbf{M}_{\rm gc}^{(2)} &= -\boldsymbol{b} \times \int_\mu \frac{m_h v_\parallel}{B} \Big(v_\parallel \boldsymbol{b} \times \mathbf{B}^* + \mathbf{E}^* \Big) f \, \mathrm{d}v_\parallel \\
&= -\mathbf{e}_z \times \int_\mu \frac{m_h v_\parallel}{B_0} \Big(v_\parallel \frac{\delta \mathbf{B}_\perp}{B_0} \times \mathbf{B}_0 + v_\parallel \mathbf{e}_z \times \delta \mathbf{B}^* + \delta \mathbf{E}^* \Big) f_0 \, \mathrm{d}v_\parallel \\
&= -\mathbf{e}_z \times \int_\mu \frac{m_h v_\parallel^2}{B_0} \Big(\delta \mathbf{B}_\perp \times \mathbf{e}_z + \mathbf{e}_z \times \delta \mathbf{B} \Big) f_0 \, \mathrm{d}v_\parallel \\
&\quad - \mathbf{e}_z \times \int_\mu \frac{m_h v_\parallel}{B_0} \Big(B_0 \mathbf{e}_z \times \delta \boldsymbol{U}_\perp + a_h^{-1} v_\parallel \partial_z \delta \boldsymbol{U}_\perp - q_h^{-1} \mu \nabla \delta B_z \Big) f_0 \, \mathrm{d}v_\parallel \\
&= -\frac{a_h^{-1} p_\parallel}{B_0^2} \mathbf{e}_z \times \partial_z \delta \boldsymbol{U}_\perp + \frac{m_h K_0}{B_0} \delta \boldsymbol{U}_\perp - \frac{a_h^{-1} Q_0}{B_0^2} \mathbf{e}_z \times \nabla \delta B_z \,,
\end{aligned}
$$

so that

$$\begin{aligned}
\nabla\times\mathbf{M}^{(2)}_{\rm gc} =& -\frac{a_h^{-1}p_\|}{B_0^2}\nabla\times(\mathbf{e}_z\times\partial_z\delta\boldsymbol{U}_\perp)+\nabla\times\left(\frac{m_hK_0}{B_0}\delta\boldsymbol{U}_\perp-\frac{a_h^{-1}Q_0}{B_0^2}\mathbf{e}_z\times\nabla\delta B_z\right)\\
=&\frac{a_h^{-1}p_\|}{B_0^2}\partial_z^2\delta\boldsymbol{U}+\frac{m_hK_0}{B_0}\mathbf{e}_z\times\partial_z\delta\boldsymbol{U}_\perp-\frac{m_hK_0}{B_0}\mathbf{e}_z\,{\rm div}(\mathbf{e}_z\times\delta\boldsymbol{U}_\perp)\\
&-\frac{a_h^{-1}Q_0}{B_0^2}\left(\mathbf{e}_z\Delta\delta B_z+\nabla\partial_z\delta B_z\right)\,,
\end{aligned}$$

where we have used $\nabla\times\delta\boldsymbol{U}_\perp=-\mathbf{e}_z\,{\rm div}(\mathbf{e}_z\times\delta\boldsymbol{U}_\perp)+\mathbf{e}_z\times\partial_z\delta\boldsymbol{U}_\perp$. Therefore

$$\begin{aligned}
\mathbf{B}\times\nabla\times\mathbf{M}^{\beta}_{\rm gc}=M_0\mathbf{e}_z\times\nabla\times\delta\mathbf{B}+B_0\nabla_\perp\delta M+\beta\frac{a_h^{-1}p_\|}{B_0}\,\mathbf{e}_z\times\partial_z^2\delta\boldsymbol{U}\\
+\frac{a_h^{-1}Q_0}{B_0^2}\mathbf{e}_z\times\nabla\times((\mathbf{e}_z\cdot\nabla\times\delta\mathbf{B}_\perp)\mathbf{e}_z)-\beta m_hK_0\partial_z\delta\boldsymbol{U}_\perp-\beta\frac{a_h^{-1}Q_0}{B_0}\mathbf{e}_z\times\nabla\partial_z\delta B_z\,.
\end{aligned}$$

1.4 Linearized current-coupling forces

Here, we deal only with longitudinal propagation so that the Q_0-terms do not contribute. Eventually, the linearized current-coupling forces are

$$\begin{aligned}
\left(\alpha q_hn_h\boldsymbol{U}-\mathbf{J}^{\alpha}_{\rm gc}-\nabla\times\mathbf{M}^{\beta}_{\rm gc}\right)\times\mathbf{B}=\left(\frac{p_\|}{B_0}+M_0\right)\mathbf{e}_z\times\nabla\times\delta\mathbf{B}_\perp+M_0\nabla_\perp\delta B_z\\
+B_0\nabla_\perp\delta M+q_hK_0\mathbf{e}_z\times\delta\mathbf{B}_\perp+\beta\frac{a_h^{-1}p_\|}{B_0}\,\mathbf{e}_z\times\partial_z^2\delta\boldsymbol{U}+(1-\beta)m_hK_0\partial_z\delta\boldsymbol{U}_\perp
\end{aligned}\tag{10}$$

or, upon Fourier transforming and setting $\mathbf{k}_\perp=0$,

$$\begin{aligned}
q_hK_0\mathbf{e}_z\times\widetilde{\mathbf{B}}_\perp-\beta\frac{p_\|}{\omega_c}k_z^2\mathbf{e}_z\times\widetilde{\boldsymbol{U}}_\perp+(1-\beta)im_hK_0k_z\widetilde{\boldsymbol{U}}_\perp\\
=-q_hK_0\mathbb{J}\widetilde{\mathbf{B}}_\perp+\beta\frac{p_\|}{\omega_c}k_z^2\mathbb{J}\widetilde{\boldsymbol{U}}_\perp+(1-\beta)im_hK_0k_z\widetilde{\boldsymbol{U}}_\perp\,,
\end{aligned}\tag{11}$$

where $\mathbb{J}$ is the canonical symplectic form. The linearized equation of motion reads

$$(\omega^2-k^2v_A^2)\mathbf{1}=i\left(q_h\frac{K_0B_0}{\rho_0}+\beta\frac{p_\|}{\rho_0}\frac{\omega}{\omega_c}k\right)k\mathbb{J}-(1-\beta)q_h\frac{K_0B_0}{\rho_0}\frac{\omega}{\omega_c}k\mathbf{1}$$

or, equivalently,

$$\left(\omega^2 + (1-\beta)\bar{n}kV_0\omega - k^2v_A^2\right)\mathbf{1} = i\left(\omega_c\bar{n}V_0 + \beta\bar{n}T_\parallel\frac{\omega}{\omega_c}k\right)k\mathbb{J}\,,$$

where $\bar{n} = m_h n_0/\rho_0$, $K_0 = n_0V_0$, and $T_\parallel = p_\parallel/n_0$. Then, the dispersion relation reads

$$\left(\omega^2 + (1-\beta)k\bar{V}\omega - k^2v_A^2\right)^2 = k^2\left[\beta\bar{T}_\parallel\frac{\omega}{\omega_c}k + \omega_c\bar{V}\right]^2\,,$$

where $\bar{V} = \bar{n}V_0$ and $\bar{T}_\parallel = \bar{n}T_\parallel$. We consider the roots

$$\omega^2 + (1-\beta)k\bar{V}\omega - k^2v_A^2 = \pm k\left(\beta\bar{T}_\parallel\frac{\omega}{\omega_c}k + \bar{V}\omega_c\right)$$

so that the four roots are

$$\omega_{1,\pm} = \frac{1}{2}(\beta-1)k\bar{V} \pm \beta\frac{k^2\bar{T}_\parallel}{\omega_c} - \frac{1}{2}\sqrt{\left[(1-\beta)k\bar{V} \mp \beta\frac{k^2\bar{T}_\parallel}{\omega_c}\right]^2 + 4\left(k^2v_A^2 \pm k\bar{V}\omega_c\right)}$$

$$\omega_{2,\pm} = \frac{1}{2}(\beta-1)k\bar{V} \pm \beta\frac{k^2\bar{T}_\parallel}{\omega_c} + \frac{1}{2}\sqrt{\left[(1-\beta)k\bar{V} \mp \beta\frac{k^2\bar{T}_\parallel}{\omega_c}\right]^2 + 4\left(k^2v_A^2 \pm k\bar{V}\omega_c\right)}$$

Let us look closer at the last two roots. In the Hamiltonian case, we have

$$\omega_{2,\pm}^{\mathrm{H}} = \pm\frac{k^2\bar{T}_\parallel}{\omega_c} + \frac{1}{2}\sqrt{\frac{k^4\bar{T}_\parallel^2}{\omega_c^2} + 4k^2v_A^2 \pm 4k\bar{V}\omega_c}$$

thereby leading to an instability for sufficient values of $\bar{V}$. In the non-Hamiltonian case, we have

$$\omega_{2,\pm}^{\mathrm{nH}} = -\frac{1}{2}k\bar{V} + \frac{1}{2}\sqrt{k^2\bar{V}^2 + 4k^2v_A^2 \pm 4k\bar{V}\omega_c}$$

and we notice that no instability occurs for sufficiently high values of $\bar{V}$.

CT: this frame is Josh's comment on the case when $\bar{V} = 0$

Hi Cesare and Phil,

I've gone and carefully normalized the dispersion relation for the CCS case. I normalize length to L, which is the length scale of the waves. I normalize time to $T = L/v_a$, which corresponds to the Alfven timescale at length scale L. When you use these normalizations, the dispersion relation Cesare derived looks like this:

$$(w^2 - k^2)^2 = \alpha\beta_h(\rho_h/L)^2(wk^2)^2$$

Here β_h and ρ_h are the hot species beta and gyroradius, respectively. In the regime the guiding center theory is designed to work in, the product of constants on the RHS is always small, so it makes sense to introduce the small parameter

$$\epsilon^2 = \alpha\beta_h(\rho_h/L)^2$$

The solution for w^2 is then

$$w^2 = k^2(1 + .5\epsilon^2k^2)(+-)\epsilon k^3 sqrt(1 + .25\epsilon^2k^2).$$

For k= O(1), the Taylor expansion about eps = 0 is

$$w^2 = k^2 \pm \epsilon k^3 + \epsilon^2.5k^4 + ...$$

In this regime we recover Cesare's statement: there is no instability. Actually, Cesare's statement is more generally true: there is never a negative root for w^2, meaning no instability ever. This can be shown by carefully looking at the exact dispersion relation.

I also looked into the group velocity to get a sense for what's going on with the dispersion. Specifically I looked at $w(k)$ for large k at fixed eps.

As $k- > \infty$, you can show that one root goes like $w = \pm\epsilon k^2$, which means the magnitude of the group velocity increases with k. These waves are therefore highly dispersive. These are surely the whistler waves. You would expect the high-k content in these waves to quickly disperse away, leaving the low-k content behind – just like wave function spreading for Schrodinger. Physically, the group velocity of the whistler wave should *level off to zero, but that requires introducing physics at the electron cyclotron timescale, which this model clearly does not include.*

The other root has $w- > cost./\epsilon$ *as* $k- > \infty$*. Thus, for this root, the new Hamiltonian terms actually cause the group velocity to level off to zero as* $k- > \infty$*, which is clearly a strong modification to the alfven dispersion relation. Do these waves correspond to anything physical? Kinetic Alfven waves maybe? (I admit I don't know much about those.)*

Best wishes,

Josh

2 PCS models

These notes compare the following incompressible pressure coupling schemes

$$\rho\frac{\partial \boldsymbol{U}}{\partial t} + \rho(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = -\nabla \mathsf{p} - \nabla\cdot\mathbb{P}_{\mathrm{gc}}^{\alpha} + \mathbf{J}\times\mathbf{B}\,, \tag{12}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{U}) = 0\,, \qquad \frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\boldsymbol{U}\times\mathbf{B})\,, \tag{13}$$

$$\frac{\partial f}{\partial t} + \Big((v_\parallel + \alpha U_\parallel)\frac{\mathbf{B}^*}{B_\parallel^*} - \frac{\boldsymbol{b}}{B_\parallel^*}\times(\mathbf{E}^* - \alpha a_h^{-1} v_\parallel \nabla U_\parallel)\Big)\cdot\nabla f + \frac{\mathbf{B}^*}{B_\parallel^*}\cdot\big(a_h\mathbf{E}^* - \alpha v_\parallel\nabla U_\parallel\big)\,\partial_{v_\parallel} f = 0\,, \tag{14}$$

where

$$\mathbb{P}_{\mathrm{gc}}^{\alpha} = \int_\mu B_\parallel^*\left(m_h v_\parallel^2 \boldsymbol{bb} + \mu B(\mathbf{1} - \boldsymbol{bb}) + \alpha m_h v_\parallel \mathbf{w}_\perp\boldsymbol{b} + \alpha m_h v_\parallel \boldsymbol{b}\mathbf{w}_\perp\right) f\,\mathrm{d}v_\parallel\,, \tag{15}$$

and

$$\mathbf{w} = \big[(v_\parallel + U_\parallel)\big]\,\frac{\mathbf{B}^*}{B_\parallel^*} - \frac{\boldsymbol{b}}{B_\parallel^*}\times\big[\mathbf{E}^* - a_h^{-1} v_\parallel\nabla U_\parallel\big] - \boldsymbol{U}\,, \tag{16}$$

When

$$\alpha = 1$$

then, we have the Hamiltonian model. When

$$\alpha = 0$$

then, we have the non-Hamiltonian model (see Park et al.).

We wish to linearize the compressible system around the common equilibrium

$$\boldsymbol{U}_0 = 0\,, \qquad \mathbf{B}_0 = B_0\mathbf{e}_z\,, \qquad f_0 = f_0(v_\parallel^2/2 + \mu B_0)\,,$$

In more generality, it might be interesting to consider longitudinal disturbances around the equilibrium

$$\boldsymbol{U}_0 = 0\,, \qquad \mathbf{B}_0 = B_0\mathbf{e}_z\,, \qquad f_0 = f_0\big(\mathbf{x}_\perp, (v_\parallel - V)^2/2 + \mu B_0\big)\,,$$

even in the case when $V_0 = 0$. For example,

$$f_0\big(\mathbf{x}_\perp, (v_\parallel - V)^2/2 + \mu B_0\big) = n(\mathbf{x}_\perp)\exp(-(v_\parallel - V)^2/2 - \mu B_0).$$

so that

$$
\begin{aligned}
\mathbf{B}_0^* &:= \nabla \times \left(\mathbf{A}_0 + a_h^{-1} v_\parallel \boldsymbol{b}_0\right) = \mathbf{B}_0 && (17)\\
\mathbf{E}_0^* &:= -\boldsymbol{U}_0 \times \mathbf{B}_0 + \frac{v_\parallel}{a_h B_0}[\nabla \times (\boldsymbol{U}_0 \times \mathbf{B}_0)]_\perp - \mu \nabla B_0 = 0 && (18)
\end{aligned}
$$

2.1 Linearized kinetic equation

We wish to linearize the kinetic equation. To this purpose, we shall use the relations

$$
\boldsymbol{b} = \mathbf{e}_z + \delta \boldsymbol{b} = \mathbf{e}_z + \frac{1}{B_0} \delta \mathbf{B}_\perp \,, \qquad B = B_0 + \delta B = B_0 + \delta B_z \,,
$$

along with

$$
\begin{aligned}
\mathbf{B}^* &= \mathbf{B}_0 + \delta \mathbf{B}^* = B_0 \mathbf{e}_z + \delta \mathbf{B}^* && (19)\\
\mathbf{E}^* &= \delta \mathbf{E}^* = B_0 \mathbf{e}_z \times \delta \boldsymbol{U}_\perp + a_h^{-1} v_\parallel \partial_z \delta \boldsymbol{U}_\perp - \mu \nabla \delta B_z \,, && (20)
\end{aligned}
$$

and

$$
\delta \mathbf{B}^* = \delta \mathbf{B} + \frac{a_h^{-1} v_\parallel}{B_0} \nabla \times \delta \mathbf{B}_\perp \,, \qquad \delta B_\parallel^* = \delta B_z + \frac{a_h^{-1} v_\parallel}{B_0} \mathbf{e}_z \cdot \nabla \times \delta \mathbf{B}_\perp \,.
$$

Then, we compute

$$
\frac{\mathbf{B}^*}{B_\parallel^*} \cdot \left(a_h \mathbf{E}^* - \alpha v_\parallel \nabla U_\parallel\right) = -a_h \mu \partial_z \delta B_z - \alpha v_\parallel \partial_z \delta U_z
$$

and

$$
\begin{aligned}
(v_\parallel + \alpha U_\parallel) \frac{\mathbf{B}^*}{B_\parallel^*} - \frac{\boldsymbol{b}}{B_\parallel^*} \times (\mathbf{E}^* - \alpha a_h^{-1} v_\parallel \nabla U_\parallel) &= \frac{v_\parallel}{B_\parallel^*} \mathbf{B}^* + \alpha \delta U_z \mathbf{e}_z \\
&\quad - \frac{1}{B_0} \mathbf{e}_z \times \left(\delta \mathbf{E}^* - \alpha a_h^{-1} v_\parallel \nabla \delta U_z\right) \\
&= \left(1 - \frac{\delta B_\parallel^*}{B_0}\right) v_\parallel \mathbf{e}_z + \frac{v_\parallel}{B_0} \delta \mathbf{B}^* + \alpha \delta U_z \mathbf{e}_z \\
&\quad - \frac{1}{B_0} \mathbf{e}_z \times \left(\delta \mathbf{E}^* - \alpha a_h^{-1} v_\parallel \nabla \delta U_z\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbf{w}_\perp = & -\mathbf{e}_z \times \delta\boldsymbol{b} \times \mathbf{w}^{(0)} + \delta\mathbf{w}_\perp \\
= & \; \delta\mathbf{w}_\perp - v_\parallel \delta\boldsymbol{b}_\perp \\
= & \; \frac{a_h^{-1} v_\parallel^2}{B_0^2} (\nabla \times \delta\mathbf{B}_\perp)_\perp - \frac{1}{B_0}\mathbf{e}_z \times \big(\delta\mathbf{E}^* - a_h^{-1} v_\parallel \nabla \delta U_z\big) - \delta\boldsymbol{U}_\perp
\end{aligned}
$$

Then, considering that $\nabla_{\mathbf{x}} f_0$, we have

$$
\partial_t \delta f + v_\parallel \partial_z \delta f = \big[\partial_z\big(a_h \mu \delta B_z + \alpha v_\parallel \delta U_z\big)\big]\partial_{v_\parallel} f_0
$$

Given that $\nabla\cdot\mathbf{B} = 0$, this means that the case of longitudinal propagation yields

$$
\widetilde{f} = -\alpha \widetilde{U}_z \frac{k_z v_\parallel}{\omega - k_z v_\parallel}\frac{\partial f_0}{\partial v_\parallel}
$$

The only kinetic fluid coupling term is the Hamiltonian term.

2.2 Linearized pressure tensor

We have

$$
\mathbb{P}^\alpha_{\rm gc} = \int_\mu B^*_\parallel \left(m_h v_\parallel^2 \boldsymbol{b}\boldsymbol{b} + \mu B(\mathbf{1} - \boldsymbol{b}\boldsymbol{b}) + \alpha m_h v_\parallel \boldsymbol{b}\mathbf{w}_\perp + \alpha m_h v_\parallel \mathbf{w}_\perp \boldsymbol{b}\right) f \,\mathrm{d}v_\parallel \,, \tag{21}
$$

and, from the previous section,

$$
\mathbf{w} = \left(1 - \frac{\delta B^*_\parallel}{B_0}\right) v_\parallel \mathbf{e}_z + \frac{v_\parallel}{B_0}\delta\mathbf{B}^* - \delta\boldsymbol{U}_\perp - \frac{1}{B_0}\mathbf{e}_z \times \left(\delta\mathbf{E}^* - a_h^{-1} v_\parallel \nabla \delta U_z\right) \tag{22}
$$

so that $\mathbf{w}_\perp^{(0)} = 0$.

We compute

$$
\begin{aligned}
\mathbb{P}_{\mathrm{gc}}^{(1)} &= \int_\mu B_\parallel^* m_h v_\parallel^2 \boldsymbol{b}\boldsymbol{b} f \,\mathrm{d}v_\parallel \\
&= \int_\mu (B_0 + \delta B_\parallel^*) m_h v_\parallel^2 (\mathbf{e}_z\mathbf{e}_z + \mathbf{e}_z\delta\boldsymbol{b} + \delta\boldsymbol{b}\mathbf{e}_z)(f_0 + \delta f)\,\mathrm{d}v_\parallel \\
&= \int_\mu B_0 m_h v_\parallel^2 (\mathbf{e}_z\mathbf{e}_z + \mathbf{e}_z\delta\boldsymbol{b} + \delta\boldsymbol{b}\mathbf{e}_z) f_0 \,\mathrm{d}v_\parallel + \mathbf{e}_z\mathbf{e}_z \int_\mu \delta B_z m_h v_\parallel^2 f_0 \,\mathrm{d}v_\parallel \\
&\quad + \int_\mu B_0 m_h v_\parallel^2 \mathbf{e}_z\mathbf{e}_z \delta f \,\mathrm{d}v_\parallel \\
&= p_\parallel (\mathbf{e}_z\mathbf{e}_z + \mathbf{e}_z\delta\boldsymbol{b} + \delta\boldsymbol{b}\mathbf{e}_z) + \frac{p_\parallel}{B_0}\delta B_z \mathbf{e}_z\mathbf{e}_z + \delta p_\parallel \mathbf{e}_z\mathbf{e}_z
\end{aligned}
$$

so that, for longitudinal propagation

$$
\nabla\cdot\mathbb{P}_{\mathrm{gc}}^{(1)} = \frac{p_\parallel}{B_0}\partial_z\delta\mathbf{B}_\perp + (\partial_z\delta p_\parallel)\mathbf{e}_z\,,
$$

where we have used $\nabla\cdot\delta\mathbf{B} = \partial_z\delta B_z = 0$.

Also, we compute (for longitudinal propagation, so that $\delta B_\parallel^* = \delta B_z$)

$$
\begin{aligned}
\mathbb{P}_{\mathrm{gc}}^{(2)} &= \int_\mu B_\parallel^* B (\mathbf{1} - \boldsymbol{b}\boldsymbol{b})\mu f \,\mathrm{d}v_\parallel \\
&= \int_\mu (B_0 + \delta B_\parallel^*)(B_0 + \delta B_z)(\mathbf{1} - \mathbf{e}_z\mathbf{e}_z - \mathbf{e}_z\delta\boldsymbol{b} - \delta\boldsymbol{b}\mathbf{e}_z)\mu(f_0 + \delta f)\,\mathrm{d}v_\parallel \\
&= (B_0 + 2\delta B_z)(\mathbf{1} - \mathbf{e}_z\mathbf{e}_z - \mathbf{e}_z\delta\boldsymbol{b} - \delta\boldsymbol{b}\mathbf{e}_z)\int_\mu \mu B_0 f_0 \,\mathrm{d}v_\parallel \\
&\quad + \int_\mu B_0^2 (\mathbf{1} - \mathbf{e}_z\mathbf{e}_z)\mu\delta f \,\mathrm{d}v_\parallel \\
&= -M_0(B_0 + 2\delta B_z)(\mathbf{1} - \mathbf{e}_z\mathbf{e}_z) + M_0 B_0(\mathbf{e}_z\delta\boldsymbol{b} + \delta\boldsymbol{b}\mathbf{e}_z) - B_0\delta M(\mathbf{1} - \mathbf{e}_z\mathbf{e}_z)
\end{aligned}
$$

so that, for longitudinal propagation,

$$
\nabla\cdot\mathbb{P}_{\mathrm{gc}}^{(2)} = M_0\partial_z\delta\mathbf{B}_\perp
$$

In addition, we also compute

$$
\begin{aligned}
\mathbb{P}^{(3)}_{\rm gc} &= \int_\mu B^*_\parallel m_h v_\parallel \boldsymbol{b}\mathbf{w}_\perp f\,{\rm d}v_\parallel \\
&= \int_\mu B_0 m_h v_\parallel \mathbf{e}_z \left[\frac{a_h^{-1}v_\parallel^2}{B_0^2}(\nabla\times\delta\mathbf{B}_\perp)_\perp - \frac{1}{B_0}\mathbf{e}_z\times\big(\delta\mathbf{E}^* - a_h^{-1}v_\parallel\nabla\delta U_z\big) - \delta\boldsymbol{U}_\perp\right] f_0\,{\rm d}v_\parallel \\
&= -\,\frac{a_h^{-1}p_\parallel}{B_0}\mathbf{e}_z\big[\mathbf{e}_z\times(\partial_z\delta\boldsymbol{U}_\perp - \nabla\delta U_z)\big]
\end{aligned}
$$

so that, for longitudinal propagation (i.e. $\nabla_\perp = 0$),

$$
\nabla\cdot\mathbb{P}^{(3)}_{\rm gc} = -\frac{a_h^{-1}p_\parallel}{B_0}\,\mathbf{e}_z\times\partial_z^2\delta\boldsymbol{U}_\perp\,.
$$

One obtains exactly the same result in the presence of a drift velocity.

2.3 Linearized pressure-coupling forces

For longitudinal propagation, one has

$$
\begin{aligned}
-\nabla\cdot\mathbb{P}^{\alpha}_{\rm gc} &= -\nabla\cdot\mathbb{P}^{(1)}_{\rm gc} - \nabla\cdot\mathbb{P}^{(2)}_{\rm gc} - \alpha\nabla\cdot\mathbb{P}^{(3)}_{\rm gc} \\
&= -\left(M_0 + \frac{p_\parallel}{B_0}\right)\partial_z\delta\mathbf{B}_\perp - (\partial_z\delta p_\parallel)\mathbf{e}_z + \alpha\frac{a_h^{-1}p_\parallel}{B_0}\,\mathbf{e}_z\times\partial_z^2\delta\boldsymbol{U}_\perp
\end{aligned}
$$

It is easy to recognize that the first and the last terms already appeared in the CCS.

Upon recalling that $M_0 = -p_\perp/B_0$, the linearized momentum equation becomes (cf. Fitzpatrick)

$$
-i\rho_0\omega\widetilde{\boldsymbol{U}} + ik_z\mathbf{e}_z\widetilde{\mathsf{p}} = i\left(\mu_0^{-1}B_0 + \frac{p_\perp - p_\parallel}{B_0}\right)k_z\widetilde{\mathbf{B}}_\perp - ik_z\tilde{p}_\parallel\mathbf{e}_z - \alpha k_z^2\frac{a_h^{-1}p_\parallel}{B_0}\,\mathbf{e}_z\times\widetilde{\boldsymbol{U}}_\perp\,.
$$

In turn, this is complemented by

$$
\tilde{\rho} = \rho_0\frac{k_z\widetilde{U}_z}{\omega}\,,\qquad \tilde{\mathsf{p}} = \Gamma\mathsf{p}_0\frac{k_z\widetilde{U}_z}{\omega}\,,\qquad \widetilde{\mathbf{B}}_\perp = -\frac{k_zB_0}{\omega}\widetilde{\boldsymbol{U}}_\perp\,.
$$

Then, assuming $p_\perp = p_\parallel$ yields

$$\rho_0 \omega \widetilde{\boldsymbol{U}} = -\mu_0^{-1} k_z B_0 \widetilde{\mathbf{B}}_\perp + k_z(\tilde{p}_\parallel + \widetilde{\mathsf{p}})\mathbf{e}_z - i\alpha k_z^2 \frac{a_h^{-1} p_\parallel}{B_0}\, \mathbf{e}_z \times \widetilde{\boldsymbol{U}}_\perp$$

or, more explicitly,

$$\omega^2 \widetilde{\boldsymbol{U}} = k_z^2 v_A^2 \widetilde{\boldsymbol{U}}_\perp + \left(k_z \omega \frac{\tilde{p}_\parallel}{\rho_0} + k_z^2 v_S^2 \widetilde{U}_z \right) \mathbf{e}_z - i\alpha \frac{p_\parallel k_z^2}{\rho_0} \frac{\omega}{\omega_c}\, \mathbf{e}_z \times \widetilde{\boldsymbol{U}}_\perp \,,$$

where $v_S^2 = \Gamma \mathsf{p}_0/\rho_0$ and

$$\frac{\tilde{p}_\parallel}{\rho_0} = m_h \frac{B_0}{\rho_0} \int v_\parallel^2 \widetilde{f}\, \mathrm{d}v_\parallel = -\alpha \frac{m_h \mu_0}{B_0} v_A^2 \widetilde{U}_z \int_\mu v_\parallel^2 \,\frac{k_z v_\parallel}{\omega - k_z v_\parallel} \frac{\partial f_0}{\partial v_\parallel}\, \mathrm{d}v_\parallel$$

Eventually,

$$\omega^2 \widetilde{\boldsymbol{U}} - k_z^2 v_A^2 \widetilde{\boldsymbol{U}}_\perp = \left(k_z^2 v_S^2 - \alpha \omega \frac{m_h \mu_0}{B_0} v_A^2 \int_\mu v_\parallel \,\frac{k_z^2 v_\parallel^2}{\omega - k_z v_\parallel} \frac{\partial f_0}{\partial v_\parallel}\, \mathrm{d}v_\parallel \right) \widetilde{U}_z \mathbf{e}_z - i\alpha \frac{p_\parallel k_z^2}{\rho_0} \frac{\omega}{\omega_c}\, \mathbf{e}_z \times \widetilde{\boldsymbol{U}}_\perp \,.$$

We compute

$$\begin{aligned} \int_\mu v_\parallel \,\frac{k_z^2 v_\parallel^2}{\omega - k_z v_\parallel} \frac{\partial f_0}{\partial v_\parallel}\, \mathrm{d}v_\parallel &= \frac{n_0}{B_0}\omega + \frac{\omega^2}{k_z} \int_\mu \frac{k_z v_\parallel}{\omega - k_z v_\parallel} \frac{\partial f_0}{\partial v_\parallel}\, \mathrm{d}v_\parallel \\ &= \frac{n_0}{B_0}\omega + \frac{\omega^3}{k_z} \int_\mu \frac{1}{\omega - k_z v_\parallel} \frac{\partial f_0}{\partial v_\parallel}\, \mathrm{d}v_\parallel \end{aligned}$$

and thus the dispersion relation for sound waves becomes

$$\omega^2 = k_z^2 v_S^2 - \alpha \omega^2 \frac{m_h n_0}{\rho_0} \left(1 - \omega^2 \int_\mu \frac{B_0 \bar{f}_0}{(\omega - k_z v_\parallel)^2}\, \mathrm{d}v_\parallel \right),$$

where $\bar{f}_0 = f_0/n_0$. By integrating by parts, one has

$$\boxed{\omega^2 = k_z^2 v_S^2 - \alpha \omega^2 \frac{m_h n_0}{\rho_0} \left(1 - \frac{\omega^2 B_0}{k_z} \int_\mu \frac{\partial \bar{f}_0/\partial v_\parallel}{k_z v_\parallel - \omega}\, \mathrm{d}v_\parallel \right)}$$

2.4 Weak damping

Application of the Sokhotski–Plemelj theorem yields

$$\int_\mu \frac{\partial \bar{f}_0/\partial v_\parallel}{k_z v_\parallel - \omega}\,\mathrm{d}v_\parallel = \int \mathrm{d}\mu \int \frac{\partial \bar{f}_0/\partial v_\parallel}{k_z v_\parallel - \omega}\,\mathrm{d}v_\parallel \simeq \frac{1}{k_z}\int \mathrm{d}\mu \left[\fint \frac{\partial \bar{f}_0/\partial v_\parallel}{v_\parallel - \omega_r/k_z}\,\mathrm{d}v_\parallel + \frac{i}{\mathrm{sgn}(k_z)} \frac{\partial \bar{f}_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega_r/k_z} \right]$$

and we compute

$$\begin{aligned}
\frac{1}{k_z}\int \mathrm{d}\mu \fint \frac{\partial \bar{f}_0/\partial v_\parallel}{v_\parallel - \omega_r/k_z}\,\mathrm{d}v_\parallel &\approx \frac{1}{k_z}\int \mathrm{d}\mu \int \frac{\partial \bar{f}_0}{\partial v_\parallel}\left[\frac{1}{\omega_r/k_z} + \frac{v_\parallel}{(\omega_r/k_z)^2} + \frac{v_\parallel^2}{(\omega_r/k_z)^3} + \frac{v_\parallel^3}{(\omega_r/k_z)^4}\right]\mathrm{d}v_\parallel \\
&= \frac{1}{k_z}\frac{1}{(\omega_r/k_z)^2}\int \mathrm{d}\mu \int \bar{f}_0 \left[1 + 3\frac{v_\parallel^2}{(\omega_r/k_z)^2}\right]\mathrm{d}v_\parallel \\
&= \frac{k_z}{\omega_r^2 B_0}\left[1 + \frac{3T_\parallel}{m_h}\frac{k_z^2}{\omega_r^2}\right]
\end{aligned}$$

Therefore, the dispersion relation reads $D(k,\omega) = 0$ with

$$D(k,\omega) = 1 - \frac{k_z^2 v_S^2}{\omega^2} + \alpha \frac{m_h n_0}{\rho_0}\left\{1 - \frac{\omega^2 B_0}{k_z}\left[\frac{k_z}{\omega_r^2 B_0}\left(1 + \frac{3T_\parallel}{m_h}\frac{k_z^2}{\omega_r^2}\right) + \frac{i}{|k_z|}\int \mathrm{d}\mu\, \frac{\partial \bar{f}_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega_r/k_z}\right]\right\}$$

so that

$$D(k,\omega_r) = 1 - \frac{k_z^2 v_S^2}{\omega_r^2} - \alpha\frac{m_h n_0}{\rho_0}\left[\frac{3T_\parallel}{m_h}\frac{k_z^2}{\omega_r^2} + i\frac{\omega_r^2 B_0}{|k_z| k_z}\int \mathrm{d}\mu\, \frac{\partial \bar{f}_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega_r/k_z}\right]$$

and

$$D_r(k,\omega_r) = 1 - \frac{k_z^2}{\omega_r^2}\left(v_S^2 + \alpha\frac{3n_0}{\rho_0}T_\parallel\right), \qquad D_i(k,\omega_r) = -\alpha\frac{m_h n_0}{\rho_0}\frac{\omega_r^2 B_0}{|k_z| k_z}\int \mathrm{d}\mu\, \frac{\partial \bar{f}_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega_r/k_z}.$$

The real frequency is given by

$$\boxed{\omega_r = k_z\sqrt{v_S^2 + \alpha\frac{3n_0}{\rho_0}T_\parallel}}\,.$$

We now apply the weak-damping formula

$$\boxed{\omega_i \approx -\frac{D_i(k,\omega_r)}{\partial D_r(k,\omega_r)/\partial\omega_r} = \alpha\frac{m_h n_0}{\rho_0}\left(v_S^2 + \alpha\frac{3n_0}{\rho_0}T_\parallel\right)\omega_r \int \mathrm{d}\mu\, B_0 \frac{\partial \bar{f}_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega_r/k_z}.}$$

For $\omega_r, k_z > 0$ and in the case

$$\bar{f}_0 \propto e^{-\frac{1}{2}m_h v_\parallel^2 - \mu B_0}\,,$$

we have $\partial_{v_\parallel} \bar{f}_0|_{v_\parallel=\omega_r/k_z} < 0$ so that

$$\omega_i < 0.$$

However, in the presence of a drift velocity V_0, one obtains an acoustic instability as long as $k_z V_0/\omega_r > 1$.

References

[1] Podesta, J.J. *Plasma dispersion function for the kappa distribution.* NASA Report NASA/CR–2004-212770