Notes on High Dimension Probability

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Abstract

This contains the lecture notes of the course on *High Dimensional Probability* by Roman Vershynin. The course is available for free at online https://www.math.uci.edu/~rvershyn/teaching/hdp/hdp.html.

1 Introduction to High Dimensional Ideas

Big data can come in one of two different ways.

1. # observations is big, this is usually easy, as classical statistical theory tells us how to deal with large number of samples. These are often better.

2. # dimensions is big. This is usually hard.

Empirical observation: it is exponentially harder to deal with larger # of dimensions rather than larger # of observations. To illustrate this, let's consider an example problem.

Example

Example 1. *Let's say we want to numerically compute the integral*

$$\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_d) dx_1 \dots dx_d$$

The usual way is to perform a numerical integration approach, based on Riemann sums. For $d = 1$, we can subdivide the interval $[0, 1]$ into grids of width (or resolution) ϵ, so there are $1/\epsilon$-grids. Then, we have the Riemann sum as

$$\int_{[0,1]} f(x)dx \approx \frac{1}{n} \sum_{i=1}^{n} f(x_i), \ n = (1/\epsilon).$$

Note that,

CHECK FROM HERE

Example Problem: Numerically compute the integral $\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_d) dx_1 \dots dx_d$

$\rightarrow$ Usual numerical integration approach:

Take a grid, $\int_{[0,1]^d} f(x)dx \approx \frac{1}{n} \sum_{i=1}^{n^d} f(x_i)$, resolution $= \epsilon$

In general, for resolution ϵ, we require: $n = (\frac{1}{\epsilon})^d$ points. (exponential in d) (Curse of dimensionality)

Probabilistic method can help, use "Monte Carlo method": Instead of choosing the points on the grid, choose uniformly at random. Pick N points:

$\int_{[0,1]^d} f(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i)$, $x_i \sim \text{Uniform}([0,1]^d)$

Note that, $\mathbb{E}(\frac{1}{N}\sum_{i=1}^{N} f(x_i)) = \mathbb{E}f(x) = \int_{[0,1]^d} f(x)dx$

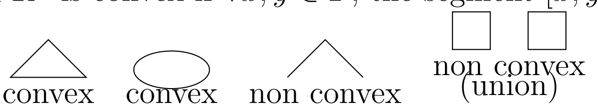
L^2-error: $\mathbb{E}[(\frac{1}{N}\sum_{i=1}^{N} f(x_i) - \int_{[0,1]^d} f(x)dx)^2] = \text{Var}(\frac{1}{N}\sum_{i=1}^{N} f(x_i)) = \frac{\text{Var}(f(x))}{N}$

$\rightarrow$ RMSE $= O(\frac{1}{\sqrt{N}})$, independent of dimension d

Lecture 2

Usually in HD problems, convexity helps a lot.

Def: A set $T \subset \mathbb{R}^n$ is convex if $\forall x, y \in T$, the segment $[x, y] \in T$.

Examples are: 

The convex hull conv(T) of a set $T \subset \mathbb{R}^n$ is the smallest convex set that contains T.

Fact: $\forall z \in \text{conv}(T)$, we have a decomposition, $z = \sum_{i=1}^{m} \lambda_i z_i$, where, $\lambda_i \geq 0$, $\sum_{i=1}^{m} \lambda_i = 1$, and each $z_i \in T$

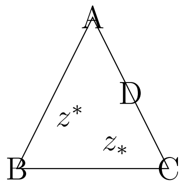
(Note that, the above combination / representation can may not be unique or parsimonious)

Caratheodory Theorem: $\forall z \in \text{conv}(T)$, $\exists$ a representation (convex comb) of $\leq (n+1)$ points in T, where, $T \subset \mathbb{R}^n$.

$\rightarrow$ Note that, the choice of basis in the representation can depend on z.

$\rightarrow$ Usually $(n+1)$ is unreasonable, there is dimension dependence.

$\rightarrow$ But if we allow approximately get z back, we can make use of probability to get far better representation.



$\mathbb{R}^2$

$z^* \in \text{span}\{A, B, D\}$ $z_* \in \text{span}\{A, B, C\}$

Thm (Approximate Caratheodory Thm): Let $T \subset \mathbb{R}^n$, and diam$(T) \leq 1$ (otherwise rescale) Then, $\forall z \in \text{conv}(T)$, $\forall k \in \mathbb{N}$, $\exists z_1, z_2, \ldots, z_k \in T$ (may be same) s.t.

$$\left\| z - \frac{1}{k}\sum_{i=1}^{k} z_i \right\|_2 \leq \frac{1}{\sqrt{2k}}$$

$\rightarrow$ So if we want error ϵ, the choose, $k = \frac{1}{2\epsilon^2}$ (dimension free)

Proof (empirical method f./ Maurey):

Fix any $z \in \text{conv}(T)$, by Caratheor Fact, we have, $z = \sum_{i=1}^{m} \lambda_i z_i$, $\lambda_i \geq 0$, $\sum_{i=1}^{m} \lambda_i = 1$, $z_i \in T$.

Consider a r.v. Z that takes value z_i, with prob λ_i.

Then, $z = \sum_{i=1}^{m} \lambda_i z_i = \mathbb{E}Z$. Consider iid copies of Z, as, $X_1, X_2, \ldots, X_k \equiv Z$

So, Error $= \mathbb{E}\left(\left\| z - \frac{1}{k}\sum_{i=1}^{k} X_i \right\|_2^2\right) = \mathbb{E}\left\| \frac{1}{k}\sum_{i=1}^{k}(X_i - z) \right\|_2^2 = \frac{1}{k^2}\sum_{i=1}^{k} \mathbb{E}\left\|X_i - \mathbb{E}X_i\right\|_2^2$, since, $z = \mathbb{E}Z = \mathbb{E}X_i$, $\forall i = \frac{1}{k}\mathbb{E}\left\|Z - \mathbb{E}Z\right\|_2^2$ (since $X_i \equiv Z$) $= \frac{1}{2k}\mathbb{E}\left\|Z - Z'\right\|_2^2$ (where, Z' is an indep copy of Z) $\leq \frac{1}{2k}$, as, $\|Z - Z'\|_2^2 \leq \text{diam}(T) \leq 1$.

$\Rightarrow$ So since, the expectation $\leq \frac{1}{2k} \Rightarrow \exists$ a realization of X_i's s.t. error $\leq \frac{1}{2k}$

$\Rightarrow \left\|z - \frac{1}{k}\sum_{i=1}^{k} z_i\right\|_2^2 \leq \frac{1}{2k}$, where, z_i's are the realizations of $X_i \uparrow \in T$.

Lec 3

Applications of ACT:

- Portfolio building - ingredients = stocks

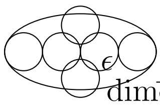
  linear/convex comb of them $\rightarrow$ mutual funds

  Problem: Create a new MF with a given combination of stocks by mixing available MFs. (Fund of funds)

  Solution: ACT provides a fast randomized solution (approximately)

- Covering Numbers:

  Def: The covering # of a set $T \subset \mathbb{R}^n$ at scale $\epsilon > 0$ is the smallest # of Euclidean balls of radius ϵ needed to cover T. Denote by $N(T, \epsilon)$

  example: T 

  $N(T, \epsilon) \leq 6$

  $\overrightarrow{\text{dimension}}$

  Fact 1: $B =$ unit euclidean ball, we have $N(B, \frac{1}{2}) \geq 2^{0.2d}$ (exponentially large)

  Proof: Assume B can be covered by N copies of $(\frac{1}{2}B)$ ball.

  $Vol(B) \leq N \cdot Vol(\frac{1}{2}B)$ (RHS might have some overlap)

  $\Rightarrow Vol(B) \leq N \cdot (\frac{1}{2})^d Vol(B) \Rightarrow N \geq 2^d$.

  Fact 2: Let P be a polytope in $\mathbb{R}^d$ with m vertices, $diam(P) \leq 1$. Then,

  $N(P, \epsilon) \leq m^{\frac{1}{2\epsilon^2}}$ (polynomial in m, nontrivial since, in $\mathbb{R}^3$, $\exists$ a polytope with $m = O(d)$ vertices, we have, RHS = polynomial in d)

  dimension free

  Proof: Consider, P is nonconvex, then $conv(P)$ is a polytop with $\leq m$ vertices.

  So, w.l.o.g. assume P is convex.

  Let $T = \{vertices \text{ of } P\}$, clearly, $P \subset conv(T)$.

  ACT $\Rightarrow \forall z \in P$, is within distance $\frac{1}{\sqrt{2k}}$ from some point in the

  set $\mathcal{N} := \{\frac{1}{k}\sum_{i=1}^{k} z_i : z_i \in T\}$

  $\Rightarrow \forall x \in P$, is covered by a ball of radius $\frac{1}{\sqrt{2k}}$ and center $\in \mathcal{N}$.

  $\Rightarrow N(P, \frac{1}{\sqrt{2k}}) \leq |\mathcal{N}| \leq m^k$ (m vertices, each has k elements)

  $\Rightarrow N(P, \epsilon) \leq m^{\frac{1}{2\epsilon^2}}$

Usually it helps by considering the intuition that, small covering # $\Rightarrow$ small volume $\Rightarrow$ easier to apply union type bounds

Lec 4

Since covering # of polytope is small, we expect volume of polytope is small.

Thm (Carl-Pajor '88): Let B = euclidean ball, $P \subset B$ any polytope with m vertices in $\mathbb{R}^n$

Then, $\frac{Vol(P)}{Vol(B)} \leq \left(4\sqrt{\frac{\log m}{n}}\right)^n$ (unless m is exponential in n, the RHS is exponentially small)

Proof: We consider ϵB-balls and cover P with these.

By def'n of covering #, $Vol(P) \leq N(P,\epsilon) \cdot Vol(\epsilon B) \Rightarrow Vol(P) \leq N(P,\epsilon) \cdot \epsilon^n Vol(B)$
$\Rightarrow \frac{Vol(P)}{Vol(B)} \leq \epsilon^n \cdot m^{\frac{1}{2\epsilon^2}}$ $\quad (m^{\frac{1}{\epsilon^2}} \approx diam(P) \leq diam(B) \leq 2)$
holds for $\forall \epsilon > 0$
$\Rightarrow \frac{Vol(P)}{Vol(B)} \leq \inf_{\epsilon>0} \epsilon^n \cdot m^{\frac{1}{2\epsilon^2}}$

Let $\ell(\epsilon) = \epsilon^n m^{\frac{1}{2\epsilon^2}} \Rightarrow \log \ell(\epsilon) = n \log \epsilon + \frac{1}{2\epsilon^2} \log m \Rightarrow \frac{\partial \log l}{\partial \epsilon} = 0 = \frac{n}{\epsilon} - \frac{1}{\epsilon^3} \log m \Rightarrow \epsilon = \sqrt{\frac{4 \log m}{n}}$

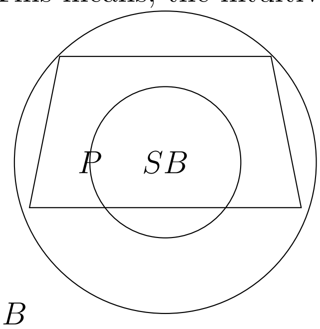
$\Rightarrow \inf_{\epsilon>0} \ell(\epsilon) = \exp\left[n \cdot \log\left(\sqrt{\frac{4\log m}{n}}\right) + \frac{2 \log m}{4 \log m} \cdot n\right] = \exp\left[\frac{n}{2} \log\left(\left(\sqrt{\frac{4\log m}{n}}\right)^2 \cdot e^{\frac{n}{2}}\right)\right] = \left(\sqrt{\frac{4e \log m}{n}}\right)^n \leq \left(4\sqrt{\frac{\log m}{n}}\right)^n$, as req'd

Remarks:

CPT proved a slightly better result, with $\log(m/n) \to$ optimal.

- The optimal bound is attained at a random polytope. (Dafnis et al., 2003, 2009)
- Let $S = 4\sqrt{\frac{\log m}{n}}$, note that, $Vol(SB) = S^n \cdot Vol(B) \Rightarrow \frac{Vol(SB)}{Vol(B)} = S^n \geq \frac{Vol(P)}{Vol(B)}$
  $\Rightarrow Vol(SB) \geq Vol(P)$

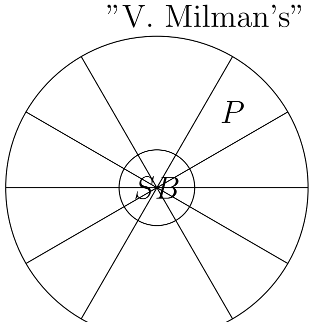
This means, the intuitive low-dimensional picture is wrong.



B

"V. Milman's" hyperbolic correction

B



often called the "core" $\qquad$ P - is convex (not look like so)

Concentration Inequalities:
$X \approx \mathbb{E}X$ with high probability (exponentially close to 1).
Example (Normal dist):

$$X \sim N(\mu, \sigma^2) \quad \text{and} \quad \mathbb{P}(|X - \mu| > t\sigma) \approx 0.9987$$

Prop (Gaussian tails):

$$g \sim N(0,1) \quad \text{and} \quad \mathbb{P}(g > t) \leq \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2}\, dx \approx \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t} \quad \text{(decay exp fast in } t\text{)}$$

Proof:

$$\mathbb{P}(g > t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-(t+y)^2/2}\, dy$$

Using $e^{-(t+y)^2/2} = e^{-t^2/2} \cdot e^{-ty} \cdot e^{-y^2/2}$:

$$\leq \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \int_0^\infty e^{-ty} e^{-y^2/2}\, dy \leq \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

By symmetry, we then have:

$$\mathbb{P}(|X - \mu| > t\sigma) = \mathbb{P}(|g| > t) \leq \frac{2}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}$$

$$X \sim N(\mu, \sigma^2)$$

Turns out that the CLT tells:

$$\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \to Z \sim N(0,1)$$

but the error here is of order $\frac{1}{\sqrt{n}}$ (Berry-Esseen bound).
So, the tail of $\sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right)$ does not go exponential like Gaussian.
Concentration inequalities bridge that gap by controlling the tail bounds.
For general distributions, we have:
- **Markov inequality:** (For any nonnegative X)

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}, \quad \text{where } t > 0$$

- **Chebyshev's inequality:** X r.v. with mean μ, variance σ^2

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}, \quad \text{where } t > 0$$

Example: Toss a fair coin N times. What is $\mathbb{P}$(at least $\frac{3N}{4}$ heads)?
Based on Chebyshev's inequality, $S_N = \#$ of heads $\sim \text{Binomial}(N, \frac{1}{2})$:

$$\mathbb{E}S_N = \frac{N}{2}, \quad \text{var}(S_N) = \frac{N}{4}$$

$$\mathbb{P}\left(S_N \geq \frac{3N}{4}\right) = \frac{1}{2}\mathbb{P}\left(|S_N - \frac{N}{2}| \geq \frac{N}{4}\right) \leq \frac{N/4}{(N/4)^2} = \frac{4}{N} = O\left(\frac{1}{N}\right)$$

Based on CLT:

$$\frac{S_N - \mathbb{E}S_N}{\sqrt{\text{var}(S_N)}} \to Z \sim N(0,1) \quad \text{by CLT}$$

$$\mathbb{P}\left(S_N \geq \frac{3N}{4}\right) = \mathbb{P}\left(\frac{S_N - \frac{N}{2}}{\sqrt{N/4}} \geq \frac{\sqrt{N/4}}{2}\right) \leq e^{-N/8}$$

But by Berry-Esseen bound, this error is $O\left(\frac{1}{\sqrt{N}}\right)$.

Theorem: (Berry-Esseen)

Let X_i be i.i.d. r.v.s with mean 0, variance 1. Then

$$\left|\mathbb{P}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \geq t\right) - \mathbb{P}(Z \geq t)\right| \leq \frac{\mathbb{E}|X_1|^3}{\sqrt{n}} = O\left(\frac{1}{\sqrt{n}}\right)$$

This is an optimal order $\frac{1}{\sqrt{n}} + e^{-Nt^2/2}$. This is worse than Chebyshev's.

So, the CLT method yields a bound of $O\left(\frac{1}{\sqrt{n}} + e^{-Nt^2/2}\right)$. So the idea is to sidestep CLT and directly aim at controlling the tails.

Theorem: (Hoeffding's Inequality)

Let $X_1, X_2, \ldots, X_n$ be symmetric Bernoulli r.v.: $\mathbb{P}(X_i = \pm 1) = \frac{1}{2}$. Then

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \geq t\right) \leq e^{-t^2/2}, \quad \forall t \geq 0 \quad \text{(Gaussian tail)}$$

Proof (MGF method):

Let $\lambda > 0$ be a parameter.

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \geq t\right) = \mathbb{P}\left(e^{\lambda\sum_{i=1}^{n} X_i} \geq e^{\lambda t\sqrt{n}}\right) \leq e^{-\lambda t\sqrt{n}}\mathbb{E}\left(e^{\lambda\sum_{i=1}^{n} X_i}\right)$$

(By Markov)

$$= e^{-\lambda t\sqrt{n}}\prod_{i=1}^{n}\mathbb{E}\left(e^{\lambda X_i}\right) \quad \text{(since } X_i \text{ are i.i.d)}$$

$$\leq e^{-\lambda t\sqrt{n}}\left(\frac{e^{\lambda} + e^{-\lambda}}{2}\right)^n \quad [\text{Note, } \cosh(\lambda) = \frac{e^{\lambda} + e^{-\lambda}}{2}]$$

$$\leq e^{-\lambda t\sqrt{n}}e^{n\lambda^2/2} = \exp\left(-\lambda t\sqrt{n} + \frac{n\lambda^2}{2}\right)$$

Minimize over $\lambda > 0$.

Application (Mean Estimation):

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$.

Classical estimator:

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{n} X_i, \quad \mathbb{E}\hat{\mu} = \mu \quad \text{(unbiased)}$$

$$\mathbb{E}\left(\hat{\mu}-\mu\right)^2 = \mathrm{Var}(\hat{\mu}) = \frac{\sigma^2}{N} \Rightarrow \mathrm{RMSE} = \frac{\sigma}{\sqrt{N}}$$

Confidence interval:

$$\mathbb{P}\left(|\hat{\mu}-\mu| \geq t\frac{\sigma}{\sqrt{N}}\right) \leq \frac{\sigma^2/N}{(t\sigma/\sqrt{N})^2} = \frac{1}{t^2} \quad = ¿ \text{ not very sharp bound.}$$

Can we get sharper exponentially close to 1 confidence for general distributions? Surprisingly YES! (Note that we only assume $\mathbb{E}|X|^2 < \infty$, not higher order moments).

"Median of means" estimator:

Partition the sample into K blocks of size M:

$$X_1, \ldots, X_M \quad X_{M+1}, \ldots, X_{2M} \quad \ldots \quad X_{(K-1)M+1}, \ldots, X_{KM}$$

(Assume $N = MK$)

Let $\hat{\mu}_j = \frac{1}{M}\sum_{i\in B_j} X_i$ and $\hat{\mu} = \mathrm{Med}(\hat{\mu}_1, \ldots, \hat{\mu}_K)$.

$$\text{Error for each } \hat{\mu}_j, \text{ we have } \mathbb{P}\left(\hat{\mu}_j \geq \mu + \frac{t\sigma}{\sqrt{N}}\right) \leq \frac{\sigma^2/M}{(t\sigma/\sqrt{N})^2} = \frac{N/t^2M}{Kt^2/M} = \frac{K}{t^2}$$

Let us choose $K = \frac{t^2}{4}$, so:

$$\mathbb{P}\left(\hat{\mu}_j \geq \mu + \frac{t\sigma}{\sqrt{N}}\right) \leq \frac{1}{4}$$

By def. of median,

$$\mathbb{P}\left(\hat{\mu} > \mu + \frac{t\sigma}{\sqrt{N}}\right) \leq \mathbb{P}\left(\text{at least } \frac{K}{2} \text{ of } \hat{\mu}_j \text{ are } \geq \frac{t\sigma}{\sqrt{N}}\right) = \mathbb{P}\left(\text{Binomial}\left(K, \frac{1}{2}\right)\right) \leq e^{-Ct^2}$$

Let $\hat{\mu}_j = \frac{1}{M}\sum_{i\in B_j} X_i$ in Bernoulli(p), with $p \leq \frac{1}{4}$ (as shown before),
and $S_k = \frac{1}{K}\sum_{j=1}^{k} \hat{\mu}_j \sim \mathrm{Bin}(K, p)$, then

$$\mathbb{P}\left(S_k > \frac{1}{2}\right) \leq \mathbb{P}\left(S_k - \mathbb{E}S_k \geq \frac{1}{2} - p\right) \leq e^{-\lambda\left(\frac{1}{2}-p\right)}\mathbb{E}\left(e^{\lambda(S_k-\mathbb{E}S_k)}\right)$$

(By Markov)

$$\mathbb{P}\left(\hat{\mu}_j \geq \mu + \frac{t\sigma}{\sqrt{N}}\right) \leq e^{-Ct^2} \quad \text{By Hoeffding's}$$

Hence,

$$\mathbb{P}\left(\hat{\mu} > \mu + \frac{t\sigma}{\sqrt{N}}\right) \leq e^{-C\cdot t^2}$$

■ (QED)

Hoeffding's Inequality (General):

Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v. such that $X_i \in [a_i, b_i]$.
Then, $S_n = \sum_{i=1}^n X_i$ satisfies

$$\mathbb{P}(S_n - \mathbb{E}S_n \geq t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

$\rightarrow$ Problem with Hoeffding's inequality: it does not help if we know variance concentration.

Maybe, X_i in Bern(p), p is very small. So, we expect more rapid decay. Note: Hoeffding only uses the fact that $X_i \in [0,1]$.

(Empirical approximation):

Let $X_1, X_2, \ldots, X_n \sim \text{Poi}(P)$, with $P \to 0$, $nP \to \mu$.

Then,

$$\mathbb{P}\left(S_n = \sum_{i=1}^n X_i \geq t\right) \to \text{Poisson}(\mu)$$

Consider Poisson tails,

$$\mathbb{P}\left(\sum_{i=1}^n X_i \geq t\right) = e^{-\mu} \sum_{k \geq t} \frac{\mu^k}{k!} \quad \text{(Stirling's bounds: } k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k)$$

$$\leq e^{-\mu}\mu^t \left(\frac{e}{t}\right)^t \quad \text{(only dominating term is } t)$$

$$= e^{-\mu}\left(\frac{\mu e}{t}\right)^t \quad \text{This is the tail we expect, not a Gaussian tail like } \exp\left(-\frac{t^2}{2}\right)$$

Chernoff's Inequality:

Let $X_i \sim \text{Bernoulli}(p_i)$, $S_n = \sum_{i=1}^n X_i$, has mean $\mathbb{E}S_n = \sum_{i=1}^n p_i = \mu$, and satisfies

$$\mathbb{P}(S_n \geq t) \leq \exp\left(-\mu\left(\frac{t}{\mu}\right)^t\right), \quad \forall t \geq \mu$$

Proof: Using the MGF method,

$$\mathbb{P}(S_n \geq t) \leq e^{-\lambda t} \prod_{i=1}^n \mathbb{E}\left(e^{\lambda X_i}\right)$$

- Now, $\mathbb{E}\left(e^{\lambda X_i}\right) = e^{\lambda p_i} + (1 - p_i) \leq 1 + \left(e^\lambda - 1\right)p_i \leq \exp\left(\left(e^\lambda - 1\right)p_i\right) \quad (\text{as } 1 + x \leq e^x)$