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ACM 104 Final

① (A) True. If we have $Av = \lambda_1 v$ and $Bv = \lambda_2 v$ then $(A+B)v = (\lambda_1 + \lambda_2)v$ implies this is true

(B) False. As before we have $Av = \lambda v$ and $Bv = \lambda v$ so $(A+B)v = (\lambda + \lambda)v = 2\lambda v$

(C) True. The two pairings of eigenvalues and eigenvectors can create a symmetric matrix.

(D) True. The singular values are the square root of the eigenvalues of $A^H A$. $A^H A = \begin{bmatrix} 0 & 0 \\ 0 & (2019)^2 \end{bmatrix} \rightsquigarrow \lambda = 2019^2 \rightsquigarrow \sqrt{2019^2} = 2019$

(E) False. If A is non-singular, then $A^+ = A^{-1}$. Then the statement would be true. However, A is not necessarily non-singular.

② (A) The polynomials are linear, so we can use the basis $1, x$. Then we can have $p_0(x) = 1$

$$p_1(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1$$

$$= x - \frac{\int_0^1 x \cdot x \cdot 1\, dx}{\int_0^1 x\, dx} = x - \frac{1/3}{1/2} = x - \frac{2}{3}$$

So we have $p_0(x) = 1$ and $p_1(x) = x - \frac{2}{3}$

(B)

(3) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rightsquigarrow \det(A-\lambda I) = \det \begin{bmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{bmatrix}$

$= \lambda^2 + 1 = 0$

Eigenvalues $\lambda = i, -i$

$V = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$

$V^{-1} = \frac{1}{-i-i} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} = \begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix}$

So we have

$A = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix}$

(4) $AA^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\det(AA^T - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 9-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = (1-\lambda)\cdot(4-\lambda)(9-\lambda)\cdot -\lambda$

$= 0 \rightarrow \lambda = 0, 1, 4, 9$

$\lambda = 0 \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda = 1 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 4 \rightsquigarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \lambda = 9 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\Sigma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Eigenvalues are the same.

$$\lambda = 0 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda = 1 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 4 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda = 9 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Q^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) $$A^TA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \qquad A^Tb = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2019 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^TAx = A^Tb$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array}\right]$$

⑥ For any $m \times n$ matrix A, $\ker(A^T A) = \ker(A)$. Additionally, $\dim(\operatorname{img}(A)) + \dim(\ker(A)) = n$

⑦ $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(8) To find the eigenvalues of a matrix we solve $\det(A-\lambda I)$. The same goes for finding the eigenvalues of A^T: $\det(A^T-\lambda I)$. However, the determinant of the matrix A is the same as that of A^T, determinant doesn't change from transpose. Thus $\det(A-\lambda I)=0$ is equivalent to $\det(A^T-\lambda I)=0$. This mean the singular values of A and A^T coincide.