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ACM 104 Homework 1

① By the definition of matrix multiplication, we have that $AB = C$, where $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$

We observe that $c_j = \sum_{k=1}^{p} a_k \cdot b^{k_j}$, so $C = [c_1\ c_2 \dots c_n]$

Which can be written as $C = \left[\sum_{k=1}^{p} a_k \cdot b^{k_1} \quad \sum_{k=1}^{p} a_k b^{k_2} \dots \sum_{k=1}^{p} a_k b^{k_n} \right]$

C can also be written as a sum of matrices
$C = \sum_{k=1}^{p} [a_k b^{k_1} \dots a_k b^{k_n}]$

Which is $\sum_{k=1}^{p} a_k b^k$. Thus, $AB = \sum_{k=1}^{p} a_k b^k$

② Let $A \in \mathbb{M}_{m \times n}$ be a strictly upper triangular matrix.
By definition, it must be true that

$$a_{ij} = \begin{cases} 0 & \text{if } i \geq j \\ \mathbb{R} & \text{if } i < j \end{cases}$$

(claim) If A is strictly upper triangular, then for $A^k, k \geq 1$:

$$a_{ij}^k = \begin{cases} 0 & \text{if } i \geq j - k + 1 \\ \mathbb{R} & \text{if } i < j - k + 1 \end{cases}$$

For $k = n$, the condition $i \geq j - k + 1$ will always be true.
Thus, A is nilpotent.

To prove this, we want to show that for A^k, $i \geq j - k + 1 \Rightarrow a_{ij} = 0$

Base case: $k = 1$. This case is trivial. The claim holds by definition if $k = 1$.

Inductive step: Need to show that claim holds for all $k > 1$ as well.
Let p be a positive integer and let $A^p = B$.
Then, $A^{p+1} = A \cdot A^p = A \cdot B$
We let $A \cdot B = C$, where $c_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj}$

We see that $a_{ir} = 0$ for $r \leq i$ (by definition of A) and that $b_{rj} = 0$ for $r \geq j-p+1$.

So for $i \geq j-p$ we have that for any $r \in \{1 \dots n\}$, $c_{ij} = 0$ because it will be true that either $a_{ir} = 0$ for $r \leq i$ or $b_{rj} = 0$ for $r \geq j-p+1$.

Thus, any strictly upper triangular matrix is nilpotent.

(3) We want to use the following definitions for P_n, L_n, and U_n. (Code attached)

$$P_n = I_n, \quad L_n = \begin{bmatrix} 1 & & & & 0 \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & \ddots & \ddots \\ 0 & & & -\left(\frac{n-1}{n}\right) & 1 \end{bmatrix}$$

For all $i = j+1$, $l_{ij} = -\left(\frac{i-1}{i}\right)$
For all $i = j$, $l_{ij} = 1$
All other elements are 0

$$U_n = \begin{bmatrix} \frac{2}{1} & -1 & & & 0 \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & -1 \ddots & -1 \\ & & & \ddots & \\ 0 & & & & \frac{n+1}{n} \end{bmatrix}$$

For all $i = j$, $u_{ij} = \frac{j+1}{j}$
For all $i = j-1$, $u_{ij} = -1$
All other elements are 0

$P_n A_n = L_n U_n \rightarrow$ If $P_n = I_n$ $\rightarrow A_n = L_n U_n$ Let $C_n = L_n U_n$

Then $c_{ij} = \sum_{k=1}^{n} l_{ik} \cdot u_{ij}$

If $i = j = 1$: $c_{11} = \sum_{k=1}^{n} l_{ik} \cdot u_{ij} = 2$

If $i = j-1$: $c_{ij} = -1 \cdot 1 = -1$ (Gives -1 values on super diagonal)

If $i = j \neq 1$: $c_{ij} = -\left(\frac{i-1}{i}\right) \cdot (-1) + (1) \cdot \left(\frac{j+1}{j}\right) = 2$ (Gives 2's on diagonal)

If $i = j+1$: $c_{ij} = -\left(\frac{i-1}{i}\right)\left(\frac{j+1}{j}\right) = -\left(\frac{j}{j+1}\right)\left(\frac{j+1}{j}\right) = -1$ (Gives -1's on sub-diagonal)

If $i > j+1$ or $i < j-1$: No overlap; $c_{ij} = 0$. Thus, the definitions of P_n, L_n, and U_n work.

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ACM 104 Homework 1 Part 2

(4)(A) A permutation matrix P represents a set of elementary row operations. The operations perform row swaps of the matrix that P is being applied to, so its operations can be undone by reapplying P. Because of this, $P = P^{-1}$

An example of a permutation matrix could be:

$$P = \begin{array}{cc} \left[\begin{matrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{matrix}\right] & \begin{matrix} \\ \text{row } i \\ \\ \text{row } j \\ \\ \end{matrix} \\ \begin{matrix} \quad & \text{col } i & \quad & \text{col } j & \quad \end{matrix} & \end{array}$$

P swaps rows i and j of the matrix it is applied to. $P = P^T$, and $P = P^{-1} = P^T$. Thus, any permutation matrix P is orthogonal.

(B) It is not true that an orthogonal matrix is necessarily a permutation matrix. A counterexample is: Let $M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. $M = M^T$

$$M^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

In this case, $M = M^{-1} = M^T$, but M is not a permutation matrix; it has values other than 1 or 0.

(5) Let $A \in \mathbb{M}_{n\times n}$ where A is some matrix. Let constant S be defined as: $s_{ij} = \begin{cases} a_{ij} & \text{if } i=j \\ \frac{a_{ij}+a_{ji}}{2} & \text{if } i \neq j \end{cases}$ (symmetric)

And let J be defined as: $j_{ij} = \begin{cases} 0 & \text{if } i=j \\ \frac{a_{ij}-a_{ji}}{2} & \text{if } i\neq j \end{cases}$ (skew symmetrix)

Let $C = S + J$, then we have

$$c_i = \begin{cases} a_{ij} & \text{if } i=j \\ \frac{a_{ij}+a_{ji}}{2} + \frac{a_{ij}-a_{ji}}{2} = \frac{2a_{ij}}{2} = a_{ij} & \text{if } i\neq j \end{cases}$$

Thus every matrix A can be written as $A = S + J$, $S^T = S$, $J^T = -J$

(6) Attached

(7)(A) The difference between each adjacent column is $[1\ 1\ 1 \cdots 1]$ (column of size n)

Because of this, any 2 columns from A can be used to represent any of the other columns. Thus, the rank of A is 2.

(B) Code attached. $x = [0\ 0\ 0 \cdots 0\ 0.01]$ (size n)

$X_{n-1} = 0$ and $X_n = 0.01$

$X_n = 0.01$ is the only non-zero component of x.

```
% code for problem 3
n = 10;
A = zeros(n);

for i = 1:n
    for j = 1:n
        if i == j
            A(i, j) = 2;
        end
        if i == j + 1 || i == j - 1
            A(i, j) = -1;
        end
    end
end

% get L, U, and P
[L, U, P] = lu(A)
```

```
L =

  Columns 1 through 7

    1.0000         0         0         0         0         0         0
   -0.5000    1.0000         0         0         0         0         0
         0   -0.6667    1.0000         0         0         0         0
         0         0   -0.7500    1.0000         0         0         0
         0         0         0   -0.8000    1.0000         0         0
         0         0         0         0   -0.8333    1.0000         0
         0         0         0         0         0   -0.8571    1.0000
         0         0         0         0         0         0   -0.8750
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0

  Columns 8 through 10

         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
    1.0000         0         0
   -0.8889    1.0000         0
         0   -0.9000    1.0000


U =

  Columns 1 through 7
```

```
    2.0000   -1.0000         0         0         0         0         0
         0    1.5000   -1.0000         0         0         0         0
         0         0    1.3333   -1.0000         0         0         0
         0         0         0    1.2500   -1.0000         0         0
         0         0         0         0    1.2000   -1.0000         0
         0         0         0         0         0    1.1667   -1.0000
         0         0         0         0         0         0    1.1429
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0

  Columns 8 through 10

         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
   -1.0000         0         0
    1.1250   -1.0000         0
         0    1.1111   -1.0000
         0         0    1.1000


P =

     1     0     0     0     0     0     0     0     0     0
     0     1     0     0     0     0     0     0     0     0
     0     0     1     0     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0     0
     0     0     0     0     1     0     0     0     0     0
     0     0     0     0     0     1     0     0     0     0
     0     0     0     0     0     0     1     0     0     0
     0     0     0     0     0     0     0     1     0     0
     0     0     0     0     0     0     0     0     1     0
     0     0     0     0     0     0     0     0     0     1
```

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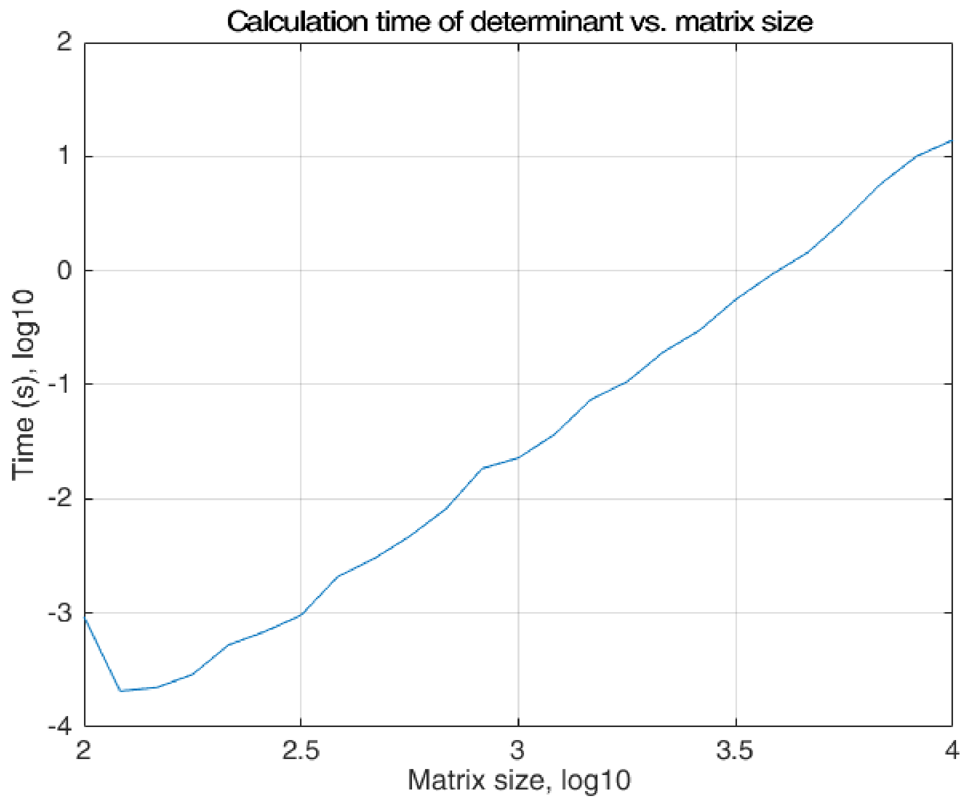
```
% code for problem 6

numvals = 25;
ns = logspace(2, 4, numvals); % part a
nLogs = zeros(1, numvals);

% part b
arrays = cell(1, numvals);
for i = 1 : numvals
    nInt = round(ns(i));
    arrays{i} = randn(nInt);
    nLogs(i) = log10(nInt);
end

%parts c and d
timeTaken = zeros(1, numvals);

for i = 1 : numvals
    tic
    det(arrays{i});
    timeTaken(i) = log10(toc);
end

% set up the plot
plot(nLogs, timeTaken)
title('Calculation time of determinant vs. matrix size')
xlabel('Matrix size, log10')
ylabel('Time (s), log10')
grid on
```



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```
% problem 7 code
n = 100;

% Creating A
A = zeros(n);
for i = 1:n
    for j = 1:n
        A(i,j) = i * n - (n - j);
    end
end
b = (1:n)';
% We want the first (n-rank(A)) = 2 components to be zero
% thus we want to find the values of the last 2 columns
B = A(:, n-1:n);

x0 = pinv(B)*b; % last 2 values of x
x = zeros(n, 1);
x(n-1) = x0(1);
x(n) = x0(2);
x
```

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x =

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