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ACM 104 Midterm

① Ⓐ False. The system formed by v and w isn't the same as that made by u and w.

Ⓑ True. Kernels are the same, so the dimensions are too. Thus, the number of free variables and the rank are the same

Ⓒ False. Consider $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Its kernel's basis is $(1,1)$ and so is its image. Thus $\ker A \cap \operatorname{im} A$ isn't $\{0\}$

Ⓓ True. $u^T A u > 0$ and $u^T B u > 0$ for all nonzero vectors u. Thus, $u^T(A+B)u > 0$ is valid as well.

Ⓔ False. $\|v+u\| \leq \|v\| + \|u\|$, which is inconsistent with the claim.

② For every value of x, v_1 and v_4 will be linearly dependent. If x is greater than or less than zero, v_1 can be multiplied by a scalar to become identical. If x is 0, v_4 is the zero vector, making the entire set of vectors linearly dependent.

③ $\{x^2, x, 1\}$ is a basis for the space of all quadratic polynomials (in general). We can have $p(-1)=0$ iff the factor $(x+1)$ divides $p(x)$. Thus we can have the form:

$$p(x) = (x+1)(\alpha x + \beta)$$

which is $\alpha x(x+1) + \beta(x+1) = p(x)$

All polynomials of the above form where α and β are some real numbers will produce $p(-1)=0$.

④ $A = \begin{bmatrix} 1 & -3 \\ 3 & -9 \end{bmatrix} \xrightarrow{R2-3\cdot R1} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\qquad x = 3y$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ Kernel basis and image basis

$A^T = \begin{bmatrix} 1 & 3 \\ -3 & -9 \end{bmatrix} \xrightarrow{R2+3R1} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ $\qquad x = -3y$

$\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ cokernel basis and coimage basis

⑤ $$\cos\theta = \frac{\int_0^{\pi/4} \sin(x)\cos(x)\,dx}{\sqrt{\int_0^{\pi/4} \sin^2(x)\,dx} \cdot \sqrt{\int_0^{\pi/4} \cos^2(x)\,dx}}$$

$$= \frac{0.25}{\sqrt{\frac{\pi-2}{8}} \cdot \sqrt{\frac{2+\pi}{8}}} = 0.8255$$

$$\arccos(0.8255) = \boxed{0.60 \text{ rad}}$$

(6) $f(x,y,z) = z - x - y - x^2 - y^2 - z^2 - xz$

$\frac{\partial f}{\partial x} = -1 - 2x - z$ $\qquad$ $\frac{\partial f}{\partial z} = 1 - 2z - x$

$\frac{\partial f}{\partial y} = -1 - 2y$

$1 - 2z - x = 0$

$\rightarrow x = 1 - 2z$

Plug in

$-1 - 2x - z = 0$

$-1 - 2(1-2z) - z = -1 - 2 + 4z - z = -3 + 3z = 0$

or $z = 1$

$-1 - 2x - 1 = 0 \rightarrow x = -1$

$-1 - 2y = 0 \rightarrow y = -\frac{1}{2}$

Max value:

$f(-1, -\frac{1}{2}, 1) = 1 + 1 + \frac{1}{2} - 1 - \frac{1}{4} - 1 + 1$

$= \boxed{5/4}$

(7) $\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

$$\left(\begin{vmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{vmatrix}^T \begin{vmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{vmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{vmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{vmatrix}^T \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right)$$

$$\begin{pmatrix} 11 & -3 \\ -3 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

$$11x - 3y = -3 \rightarrow x = \frac{-3+3y}{11}$$
$$-3x + 9y = 10$$

$$-3\left(\frac{-3+3y}{11}\right) + 9y = 10 \rightarrow y = \frac{101}{90}$$
$$x = 1/30$$

(8) $\ker(A^TA) = \ker(A)$

For any $x \in \ker(A^TA)$ we have that

$A^TAx = 0$

We then have that: $x^TA^TAx = (Ax)^TAx = 0$

which means Ax must equal 0.

Thus $\ker(A^TA) = \ker(A)$ for any matrix A.