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ACM 104 Homework 2

① (A) W is not a subspace of V.
Consider 2 matrices A and B, $\in V$ (determinants are 0)

$$A=\begin{bmatrix}1&1\\0&0\end{bmatrix} \quad B=\begin{bmatrix}0&0\\1&2\end{bmatrix}$$

The sum $A+B=\begin{bmatrix}1&1\\1&2\end{bmatrix}$ has a determinant of 1.

A+B is not in V; W is not a subspace of V

(B) W is a subspace of V. Zero vector is the zero matrix.
For any $A, B \in W$ (trace is 0) we have

$$A+B\in W: \ tr(A+B) = tr(A)+tr(B) = 0+0=0$$

$$\alpha\cdot A\in W: \ tr(\alpha\cdot A) = \alpha\cdot\sum_{i=1}^{n} a_{ii} = \alpha\cdot 0 = 0$$

$$\alpha A+\beta B\in W: \ tr(\alpha A+\beta B) = \alpha\cdot\sum_{i=1}^{n} a_{ii} + \beta\sum_{i=1}^{n} b_{ii} = \alpha\cdot 0+\beta\cdot 0 = 0$$

(C) W is not a subspace of V. Let $g(x)=2-1.5x$, so $g(x)\in W$;
$g(0)g(1) = 2\cdot\frac{1}{2} = 1$
Also let $h(x)=\alpha g(x)$ where $\alpha\neq 1$.
Now, $h(0)h(1) = \alpha g(0)\alpha g(1) = \alpha\cdot 2\cdot\alpha\cdot\frac{1}{2} = \alpha^2$ but $\alpha^2\neq 1$

(D) W is a subspace of V. Let $f(x)$ and $g(x)\in W$.

$$f(x)+g(x)\in W: \ \int_0^1 (f(t)+g(t))\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = f(0.5)+g(0.5)$$

$$\alpha\cdot f(x)\in W: \ \int_0^1 \alpha\cdot f(t)\,dt = \alpha\int_0^1 f(t)\,dt = \alpha\cdot f(0.5)$$

$$\alpha f(x)+\beta g(x)\in W: \ \int_0^1 \alpha\cdot f(t)\,dt + \int_0^1 \beta\cdot g(t)\,dt = \alpha f(0.5)+\beta g(0.5)$$

Zero vector is $f(x)=0$; $f(\tfrac{1}{2})=0$, $\int_0^1 f(t)\,dt = 0$

(E) W is a subspace of V. Zero vector is $v(x,y) = [0\ 0]^T \in W$.
Consider $a(x,y), b(x,y) \in W$

$$a(x,y) + b(x,y) = \begin{bmatrix} a_1(x,y) + b_1(x,y) \\ a_2(x,y) + b_2(x,y) \end{bmatrix}$$

$$a+b \in W: \nabla \cdot (a+b) = \frac{\partial(a_1+b_1)}{\partial x} + \frac{\partial(a_2+b_2)}{\partial y} = \frac{\partial a_1}{\partial x} + \frac{b_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial b_2}{\partial y} = 0+0=0$$

$$\alpha \cdot a \in W: \nabla \cdot (\alpha \cdot a) = \frac{\partial(\alpha \cdot a_1)}{\partial x} + \frac{\partial(\alpha \cdot a_2)}{\partial y} = \alpha \cdot 0 = 0$$

(2) (A) The three polynomials
$$p_1(x) = x^2 + 0x - 3$$
$$p_2(x) = 0x^2 - x + 2$$
$$p_3(x) = x^2 + 2x + 1$$
can be written as a single matrix
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$
Which has the row echelon form of:

$$\xrightarrow{REF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

The matrix's rank = the number of columns, so p_1, p_2, and p_3 are linearly independent

(B) $P^{(2)}$ has a dimension of 3; it depends on 3 variables. In this case, the variables are polynomial coefficients. p_1, p_2, and p_3 are linearly independent, so they span $P^{(2)}$

(C) Because p_1, p_2, and p_3 are linearly independent and span $P^{(2)}$, they form a basis of $P^{(2)}$. The coordinates of $q(x) = 1$ in this basis are $\left(-\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right)$

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ACM 104 Homework 2 Part 2

③ (A) For any f, x_1 and x_2 are free. Thus, it just needs to be shown that $x_n = x_{n-1} + x_{n-2}$ holds for addition and scalar multiplication.

$f_1 + f_2 \in f$: $f_1, f_2 \in f$ $\quad f_1 + f_2 = (x_1 + y_1, x_2 + y_2, \ldots) = (z_1, z_2, \ldots)$

$$z_n = z_{n-1} + z_{n-2}$$
$$= (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2}) = x_n + y_n$$

Thus $f_1 + f_2 \in f$

$\alpha \cdot f \in f$: $\quad z_n = z_{n-1} + z_{n-2} = \alpha \cdot x_{n-1} + \alpha \cdot x_{n-2} = \alpha \cdot x_n$

Thus $\alpha \cdot f \in f$

$\alpha \cdot f_1 + \beta \cdot f_2 \in f$: $\quad z_n = z_{n-1} + z_{n-2} = (\alpha \cdot x_{n-1} + \beta \cdot y_{n-1}) + (\alpha \cdot x_{n-2} + \beta \cdot y_{n-2})$

$$= \alpha (x_{n-1} + x_{n-2}) + \beta \cdot (y_{n-1} + y_{n-2}) = \alpha \cdot x_n + \beta \cdot y_n$$

Thus $\alpha \cdot f_1 + \beta + f_2 \in f$

$x_n = x_{n-1} + x_{n-2}$ holds for addition and scalar multiplication, thus f is a vector space.

(B) Each $f \in f$ has 2 free parameters that other parameters are dependent on: x_1 and x_2. Thus, f has a dimension of 2. The vectors $f_1 = (1, 0, 1, 1, 2, \ldots)$ and $f_2 = (0, 1, 1, 2, 3, \ldots)$ form a basis of f.

(C) The coordinates of the original sequence f^* in the basis from part (B) are $(1, 1)$.

(4) The difference between any two adjacent columns is the vector $(\{1\}^n)^T$.

Any column c_i of A can be written as

$$c_i = c_1 + k \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{where } 0 \leq k \leq n-1$$

$c_2 - c_1 = (\{1\}^n)^T$ so we can have $c_i = c_1 + k \cdot (c_2 - c_1)$ or $(1-k) \cdot c_1 + k \cdot c_2$. c_1 and c_2 are linearly independent and $\text{span}(c_1, c_2) = \text{span}(c_1, c_2, \cdots c_n)$. Thus, $\text{im} A = \text{span}(c_1, c_2)$ so c_1 and c_2 form a basis of the image of A. (c_1 and c_2 can be be any 2 adjacent columns of A).

For the basis of A's coimage, the adjacent columns of A^T have a common difference of $(\{n\}^n)^T$ and can be expressed as $r_i^T = (1-k) \cdot r_1^T + k \cdot r_2^T$. Using the same line of reasoning, r_1^T and r_2^T span the column space of A^T; they are linearly independent. Thus, r_1^T, r_2^T form a basis for the coimage of A.

For the basis of the Kernel of A, the row echelon form must first be found. By starting at the bottom row and subtracting the above row continuously, then simplifying, we get

$$\text{REF}(A) = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & -n & -2n & \cdots & n-n^2 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix} \quad \text{and thus } \text{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & \cdots & 2-n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix}$$

The RREF has 2 basic variables (x_1 and x_2) and $n-2$ free variables ($x_3, x_4, \dots x_n$).

Then our solution for $Ax = 0$ is

$$x = \begin{bmatrix} \sum_{i=3}^{n} (2-i) \cdot x_i \\ \sum_{i=3}^{n} (i-1) \cdot x_i \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad x_3, \dots, x_n \in \mathbb{R}$$

$n-2$ vectors generated from x above with $x_k = 1$ and

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ACM 104 Homework 2 Part 3

with all other free variables 0 will be linearly independent and span the kernel of A. These vectors form a basis of the kernel of A.

For the basis of the cokernel of A, the same process is done, but for A^T. For REF of A^T we get

$$\begin{bmatrix} 1 & n+1 & 2n+1 & \cdots & n^2-n+1 \\ 0 & -n & -2n & \cdots & n-n^2 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 0 \end{bmatrix}$$

And the RREF turns out to be the same:

$$\begin{bmatrix} 1 & 0 & -1 & \cdots & 2-n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix}$$

Thus the basis for the cokernel of A is the same as that of the kernel.

(5) Attached

```
function [K, I, cK, cI] = ps2_5_TyLimpasuvan(A)

    [m, n] = size(A);
    r = rank(A);

    if r == 0
        I = zeros(m, 1);
        cI = zeros(n, 1);
        K = eye(n);
        cK = eye(m);
        return;
    end

    if r == n
        K = zeros(n, 1);
        cK = zeros(m, 1);
        I = eye(m);
        cI = eye(n);
        return;
    end

    K = null(A);
    cK = null(A');
    rowEch = rref(A)';
    I = zeros(n, r);

    for i = 1:r
        I(:, i) = rowEch(:, i);
    end

    rowEchTrans = rref(A')';
    cI = zeros(m, r);
    for i = 1:r
        cI(:, i) = rowEchTrans(:, i);
    end

end
```

```
Not enough input arguments.

Error in ps2_5_TyLimpasuvan (line 3)
    [m, n] = size(A);
```

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