Assignment 2

July 5, 2019

1 Linear Regression

1.1 Linear model

$$y = f(x) = X\beta$$

Given :

- y: dependent variable (a vector with the shape of $Nx1$ where N is the number of samples)
- X: independent variables (a matrix of shape NxD where N is the number of samples and D is the number of features)
- β: model's parameters (a vector of shape $Dx1$)

1.2 Metric for the performance of the model

The performance of the model can be measured by Mean Square Error (MSE) metric, i.e:

$$L = \frac{1}{N}\sum_{i=1}^{N}(y_i - f(x_i))^2 = \frac{1}{N}||y - X\beta||_2^2 = \frac{1}{N}(y - X\beta)^T(y - X\beta)$$

The lower MSE (value of L) the better model

1.3 Construct the optimization problem

$$\beta = argmin_\beta L = argmin_\beta \frac{1}{N}(y - X\beta)^T(y - X\beta) \quad (*)$$

In this assignment, we will use Stochastics Gradient Descent (SGD), Minibatch Gradient Descent and Newton Methods to solve the convex optimization problem (*)

Firstly, we can find out the gradient of L w.r.t β and the Hessian matrix of L w.r.t. β as below:

$$\nabla_\beta L = \nabla_\beta \frac{1}{N}(y - X\beta)^T(y - X\beta) = \frac{-2}{N}X^T(y - X\beta)$$

$$\nabla_\beta^2 L = \frac{d}{d\beta}\frac{-2}{N}X^T(y - X\beta) = \frac{2}{N}X^T X$$

1.4 Stochastic Gradient Descent method

Idea: *In each iteration, we use only 1 record from training datasets to calculate the value of gradient of L w.r.t. β, then update the value of β by using this value. The algorithm will stop at the given number of steps.*

$$\beta^{k+1} = \beta^k - t\nabla L_\beta^k$$

1.5 Minibatch Gradient Descent method

Idea: *Similar to SGD, but in each iteration, we use b (batch size) records from training datasets to calculate the value of gradient of L w.r.t. β instead of 1 record*

$$\beta^{k+1} = \beta^k - t\nabla L_\beta^k$$

1.6 Newton method

Idea: *By using second-order approximation given by Taylor expansion of L and finding the value of beta that minimize the approximation function of L, we get the following formula:*

$$\beta^{(k)} = \beta^{(k-1)} - \left(\nabla^2 L(\beta^{(k-1)})\right)^{-1} \nabla L(\beta^{(k-1)})$$

1.7 Implementation for SGD and Minibatch GD

In [12]:
```
## X: N x D matrix
## y: N x 1 matrix
## beta: D x 1 matrix
import numpy as np
from datetime import datetime
import matplotlib.pyplot as plt

def compute_loss(X, y, beta):
    y_pred = X.dot(beta)
    loss = ((y_pred - y)**2).mean()
    return loss

def compute_gradient(X, y, beta):
    N = X.shape[0]
    grad = 2/N * X.T.dot(X.dot(beta) - y)
    return grad

def compute_loss_v2(y_pred, y):
    loss = ((y_pred - y)**2).mean()
    return loss

def norm_l2(x):
    return np.sqrt((x**2).sum())
```

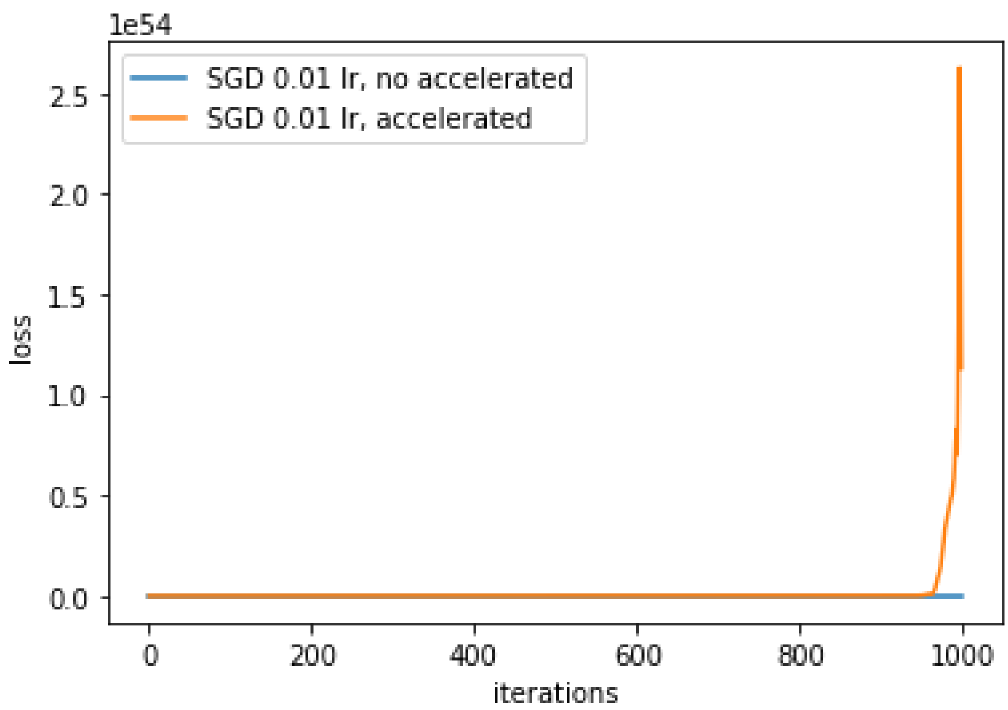
```
In [174]: def gen_batch(X,y, bs=1):
              num_samples = X.shape[0]
              num_batches = num_samples//bs
              rd_id = np.random.permutation(num_samples)
              X,y = X[rd_id],y[rd_id]
              for i in range(num_batches):
                  start_idx = bs * i
                  end_idx = bs *(i+1) if bs *(i+1) <= num_samples else num_samples
                  if start_idx == num_samples:
                      break
                  yield (X[start_idx:end_idx], y[start_idx:end_idx])


          def mini_batch_gd(X, y, lr, beta = None, last_beta = None,bs = 1, accelerated= False
              if beta is None:
                  beta = np.zeros((X_normalized.shape[1],1))
                  last_beta = beta
              g = gen_batch(X, y, bs)
              losses = []
              for X_batch, y_batch in g:
                  step += 1
                  grad = compute_gradient(X_batch, y_batch, beta)
                  loss = compute_loss(X, y, beta)
                  losses.append(loss)
                  if not accelerated:
                      beta = beta - lr * grad
                  else:
                      v = beta + (step-1)/(step+2) *(beta - last_beta)
                      last_beta = beta
                      grad = compute_gradient(X_batch, y_batch, v)
                      beta = v - lr * grad



              return losses, beta, last_beta, step


          def minibatch_training(batch_size, num_iters, lr, accelerated =False):
              ## Minibatch with simple gradient descent
              beta = np.zeros((X_normalized.shape[1],1))
              last_beta = np.zeros((X_normalized.shape[1],1))
              step=0
              losses = []
              num_epoches = num_iters//(num_samples//batch_size)
              for e in range(num_epoches):
                  loss, beta, last_beta, step = mini_batch_gd(X_normalized, y, lr, beta, last_
                  losses += loss
              return losses, beta
```

In [175]:
```python
## Reading data to python
import pandas as pd

import numpy as np
train_data = pd.read_csv("./boston-housing/train.csv")
## Get first 13 columns as X
X = train_data.values[:,1:-1]

## Normalize X
mean = X.mean(axis = 0)
std = X.std(axis = 0)
X_normalized = (X - mean)/std
X_normalized = np.c_[np.ones(X.shape[0]), X_normalized]
## Get last columns as y
y = train_data.values[:,-1].reshape((X.shape[0],1))
```

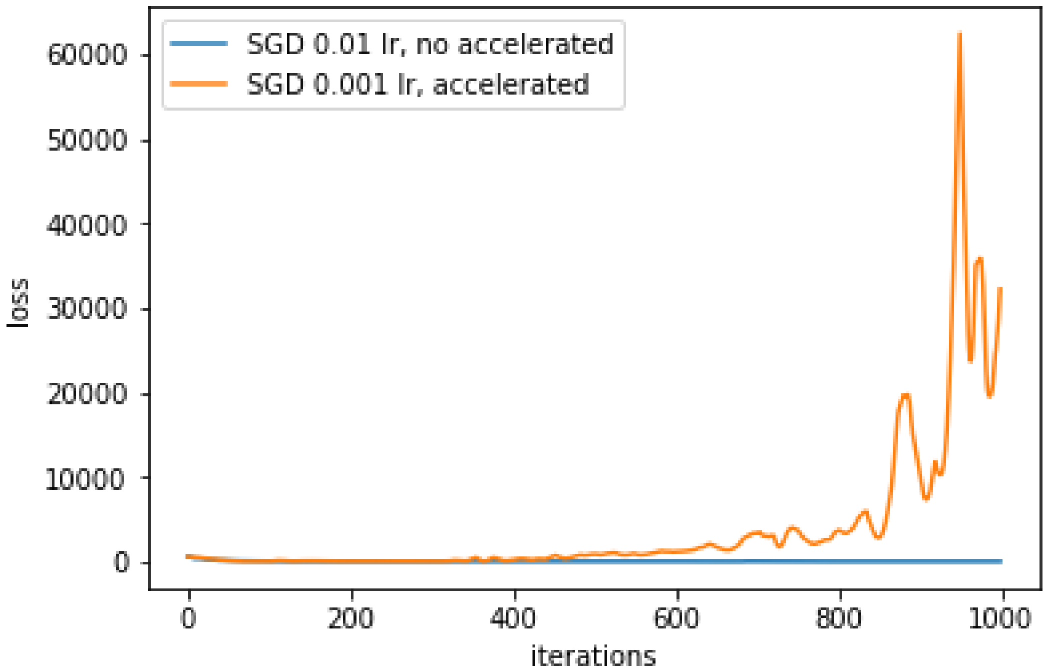
In [192]:

In [193]:
```python
sgd_losses, sgd_beta = minibatch_training(1, 1000, 0.01, False)
```

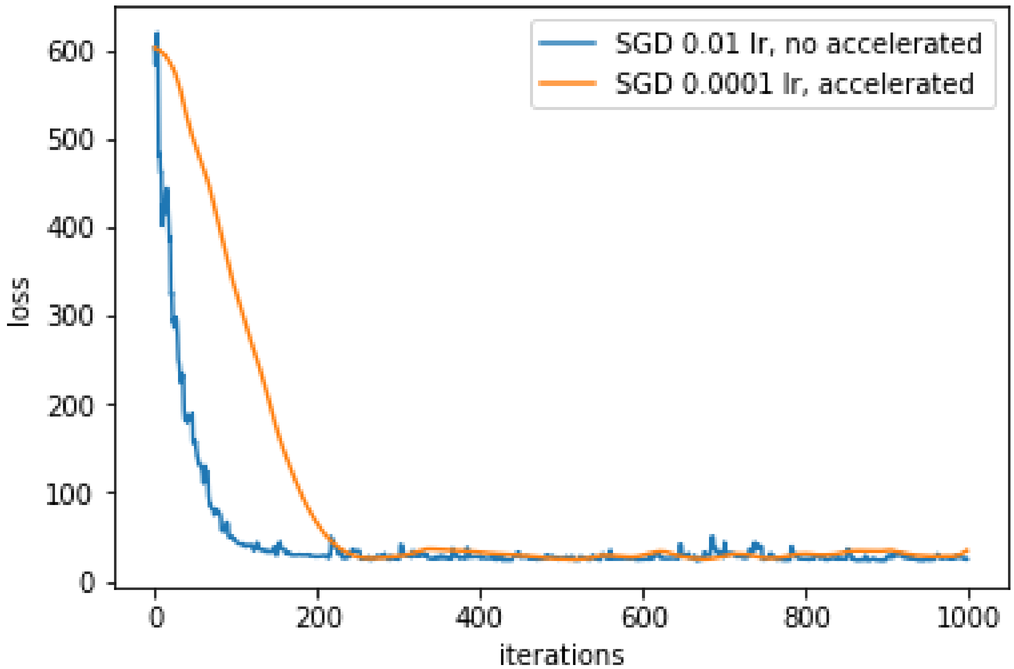
In [178]:
```python
plt.plot(sgd_losses, label = 'SGD 0.01 lr, no accelerated')
acc_sgd_losses, acc_sgd_beta = minibatch_training(1, 1000, 0.01, True)
plt.plot(acc_sgd_losses, label = 'SGD 0.01 lr, accelerated')
plt.legend()
plt.xlabel("iterations")
plt.ylabel("loss")
plt.show()
```



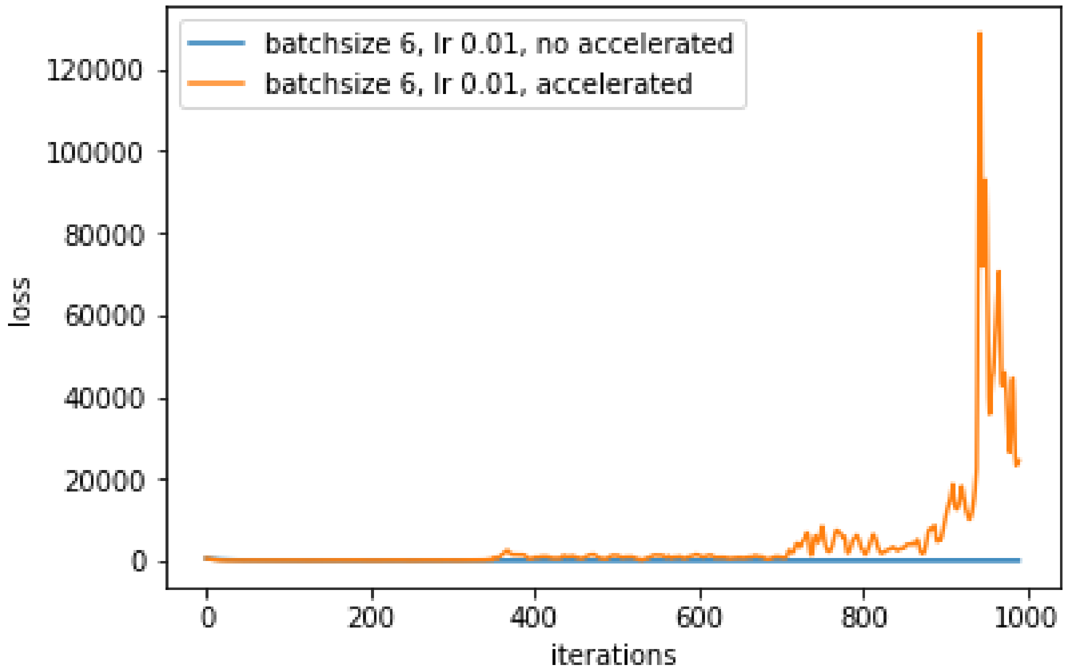
```
In [184]: plt.plot(sgd_losses, label = 'SGD 0.01 lr, no accelerated')
          acc_sgd_losses, acc_sgd_beta = minibatch_training(1, 1000, 0.001, True)
          plt.plot(acc_sgd_losses, label = 'SGD 0.001 lr, accelerated')
          plt.xlabel("iterations")
          plt.ylabel("loss")
          plt.legend()
          plt.show()
```



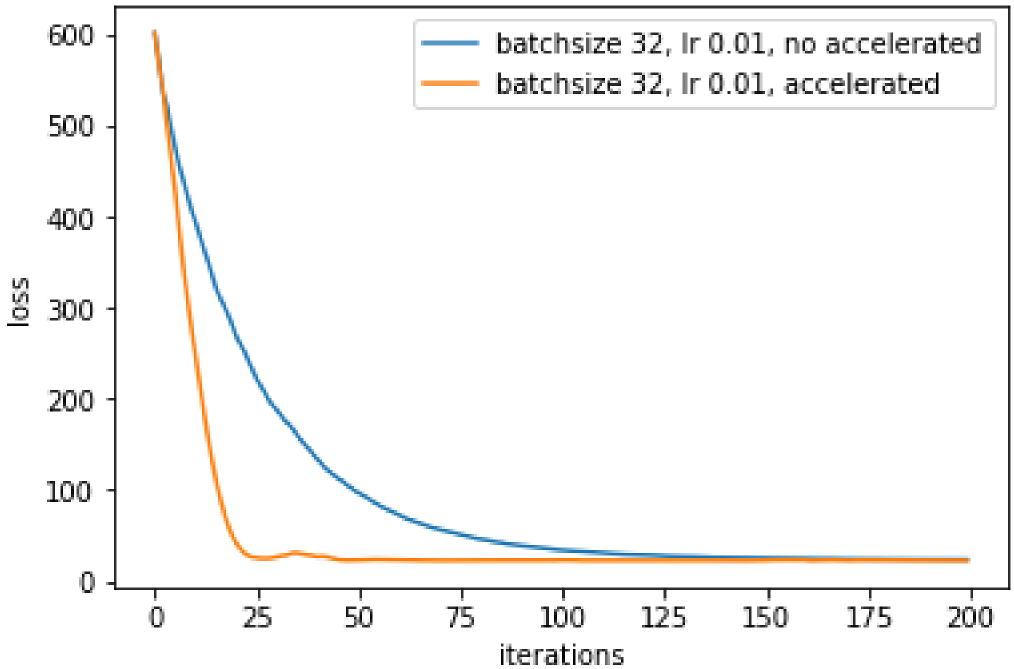

```
In [185]: plt.plot(sgd_losses, label = 'SGD 0.01 lr, no accelerated')
          acc_sgd_losses, acc_sgd_beta = minibatch_training(1, 1000, 0.0001, True)
          plt.plot(acc_sgd_losses, label = 'SGD 0.0001 lr, accelerated')
          plt.xlabel("iterations")
          plt.ylabel("loss")
          plt.legend()
          plt.show()
```



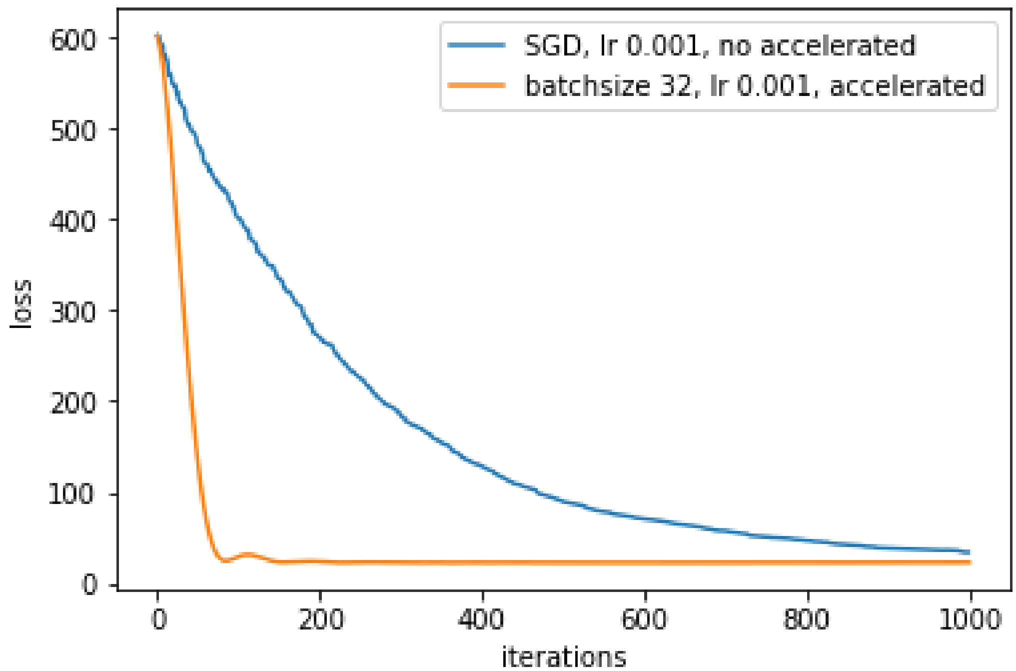

```
In [186]: minibatch_losses, minibatch_beta = minibatch_training(6, 1000, 0.01, False)
          plt.plot(minibatch_losses, label = 'batchsize 6, lr 0.01, no accelerated')
          minibatch_losses, minibatch_beta = minibatch_training(6, 1000, 0.01, True)
          plt.plot(minibatch_losses, label = 'batchsize 6, lr 0.01, accelerated')
          plt.xlabel("iterations")
          plt.ylabel("loss")
          plt.legend()
          plt.show()
```




```
In [187]: minibatch_losses, minibatch_beta = minibatch_training(32, 200, 0.01, False)
          plt.plot(minibatch_losses, label = 'batchsize 32, lr 0.01, no accelerated')
          minibatch_losses, minibatch_beta = minibatch_training(32, 200, 0.01, True)
          plt.plot(minibatch_losses, label = 'batchsize 32, lr 0.01, accelerated')
          plt.xlabel("iterations")
          plt.ylabel("loss")
          plt.legend()
          plt.show()
```




```
In [191]: sgd_losses

Out[191]: []

In [223]: sgd_losses, sgd_beta = minibatch_training(1, 1000, 0.001, False)
          plt.plot(sgd_losses, label = 'SGD, lr 0.001, no accelerated')
          minibatch_losses, minibatch_beta = minibatch_training(32, 1000, 0.001, True)
          plt.plot(minibatch_losses, label = 'batchsize 32, lr 0.001, accelerated')
          plt.xlabel("iterations")
          plt.ylabel("loss")
          plt.legend()
          plt.show()
```



1.8 Implementation for Newton method

```
In [197]: ## Hessian matrix:
          N = X_normalized.shape[0]
          hess_matrix = 2/N * X_normalized.T.dot(X_normalized)
          hess_inversed = np.linalg.pinv(hess_matrix)

In [198]: newton_beta =  np.zeros((X_normalized.shape[1],1))
          losses = []
          grad = compute_gradient(X_normalized, y, newton_beta)
          grad_norm = np.sqrt(grad.T.dot(grad))
          num_iters = 0
          while grad_norm > 1e-6 and num_iters < 1e3:
              loss = compute_loss(X_normalized, y, newton_beta)
              grad = compute_gradient(X_normalized, y, newton_beta)
              grad_norm = np.sqrt(grad.T.dot(grad))
              newton_beta = newton_beta - hess_inversed.dot(grad)

              losses.append(loss)
              num_iters += 1

In [199]: losses

Out[199]: [602.3166366366366, 22.389549423647892]
```

In [200]: newton_beta

Out[200]:
```
array([[22.76876877],
       [-0.38533599],
       [ 1.07418454],
       [ 0.37631929],
       [ 0.89927653],
       [-1.80662577],
       [ 2.64908845],
       [-0.12996621],
       [-3.06379849],
       [ 2.87156628],
       [-2.19483476],
       [-1.84128775],
       [ 1.00856953],
       [-4.23652255]])
```

1.9 Compare result with sklearn library

In [204]:
```
from sklearn.linear_model import LinearRegression
reg = LinearRegression()
start_time = datetime.now()
reg.fit(X_normalized, y)
print('running_time:', datetime.now() - start_time)
```

```
running_time: 0:00:00.010165
```

In [203]:
```
beta = reg.coef_[0]
beta[0] = reg.intercept_
beta
```

Out[203]:
```
array([22.76876877, -0.38533599,  1.07418454,  0.37631929,  0.89927653,
       -1.80662577,  2.64908845, -0.12996621, -3.06379849,  2.87156628,
       -2.19483476, -1.84128775,  1.00856953, -4.23652255])
```

In [227]:
```
print("Difference between Newton method's result and sklearn library: ",norm_l2(beta-
```

```
Difference between Newton method's result and sklearn library:  6.295378177329821e-15
```

In [225]:
```
print("Difference between Minibatch GD method's result and sklearn library: ",norm_l2
```

```
Difference between Minibatch GD method's result and sklearn library:  0.4248816868514875
```

In [ ]: