Subgradient Method

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Last last time: gradient descent

Consider the problem

$$\min_x f(x)$$

for f convex and differentiable, $\mathrm{dom}(f) = \mathbb{R}^n$.

Gradient descent: choose initial $x^{(0)} \in \mathbb{R}^n$, repeat:

$$x^{(k)} = x^{(k-1)} - t_k \cdot \nabla f(x^{(k-1)}), \quad k = 1, 2, 3, \ldots$$

Step sizes t_k chosen to be fixed and small, or by backtracking line search.

If ∇f Lipschitz, gradient descent has convergence rate $O(1/\varepsilon)$.

Downsides:

- Requires f differentiable — addressed this lecture.
- Can be slow to converge — addressed next lecture.

Subgradient method

Now consider f convex, having $\mathrm{dom}(f) = \mathbb{R}^n$, but not necessarily differentiable.

Subgradient method: like gradient descent, but replacing gradients with subgradients, i.e., initialize $x^{(0)}$, repeat:

$$x^{(k)} = x^{(k-1)} - t_k \cdot g^{(k-1)}, \quad k = 1, 2, 3, \ldots$$

where $g^{(k-1)} \in \partial f(x^{(k-1)})$ any subgradient of f at $x^{(k-1)}$.

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Subgradient method is not necessarily a descent method, so we keep track of best iterate $x^{(k)}_{\mathrm{best}}$ among $x^{(0)}, \ldots x^{(k)}$ so far, i.e.,

$$f(x^{(k)}_{\mathrm{best}}) = \min_{i=0,\ldots,k} f(x^{(i)}).$$

Outline

Today:

- How to choose step sizes
- Convergence analysis
- Intersection of sets
- Projected subgradient method

Step size choices

- Fixed step sizes: $t_k = t$ all $k = 1, 2, 3, \ldots$
- Fixed step length, i.e., $t_k = s/\|g^{(k-1)}\|_2$, and hence $\|t_k g^{(k-1)}\|_2 = s$.
- Diminishing step sizes: choose to meet conditions

$$\sum_{k=1}^{\infty} t_k^2 < \infty, \quad \sum_{k=1}^{\infty} t_k = \infty,$$

  i.e., square summable but not summable. Important here that step sizes go to zero, but not too fast.

There are several other options too, but key difference to gradient descent: step sizes are pre-specified, not adaptively computed.

Convergence analysis

Assume that f convex, $\mathrm{dom}(f) = \mathbb{R}^n$, and also that f is Lipschitz continuous with constant $L > 0$, i.e.,

$$|f(x) - f(y)| \leq L \|x - y\|_2 \quad \text{for all } x, y.$$

Theorem

For a fixed step size t, subgradient method satisfies

$$f(x_{best}^{(k)}) - f^* \leq \frac{\|x^{(0)} - x^*\|_2^2}{2kt} + \frac{L^2 t}{2}.$$

For fixed step length, i.e., $t_k = s/\|g^{(k-1)}\|_2$, we have

$$f(x_{best}^{(k)}) - f^* \leq \frac{L\|x^{(0)} - x^*\|_2^2}{2ks} + \frac{Ls}{2}.$$

For diminishing step sizes, subgradient method satisfies

$$f(x_{best}^{(k)}) - f^* \leq \frac{\|x^{(0)} - x^*\|_2^2 + L^2 \sum_{i=1}^k t_i^2}{2\sum_{i=1}^k t_i},$$

i.e., $\lim_{k\to\infty} f(x_{best}^{(k)}) = f^*$.

Lipschitz continuity

Before the proof let consider the Lipschitz continuity assumption.

Lemma

f is Lipschitz continuous with constant $L > 0$, i.e.,

$$|f(x) - f(y)| \leq L \|x - y\|_2 \quad \text{for all } x, y,$$

is equivalent to

$$\|g\|_2 \leq L \quad \text{for all } x \text{ and } g \in \partial f(x).$$

Chứng minh.

$\Longleftarrow$: Choose subgradients g_x and g_y at x and y. We have

$$g_x^T(x - y) \geq f(x) - f(y) \geq g_y^T(x - y).$$

Apply Cauchy-Schwarz inequality get

$$L\|x - y\|_2 \geq f(x) - f(y) \geq -L\|x - y\|_2.$$

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Chứng minh.

$\Longrightarrow$: Assume $\|g\|_2 > L$ for some $g \in \partial f(x)$. Take $y = x + g/\|g\|_2$ we have $\|y - x\|_2 = 1$ and

$$f(y) \geq f(x) + g^T(y - x) = f(x) + \|g\|_2 > f(x) + L,$$

contradiction. □

Convergence analysis - Proof

Can prove both results from same basic inequality. Key steps:

- Using definition of subgradient

$$\|x^{(k)} - x^*\|_2^2 = \|x^{(k-1)} - t_k g^{(k-1)} - x^*\|_2^2$$
$$= \|x^{(k-1)} - x^*\|_2^2 - 2t_k g^{(k-1)}(x^{(k-1)} - x^*) + t_k^2 \|g^{(k-1)}\|_2^2$$
$$\leq \|x^{(k-1)} - x^*\|_2^2 - 2t_k(f(x^{(k-1)}) - f(x^*)) + t_k^2 \|g^{(k-1)}\|_2^2.$$

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- Iterating last inequality

$$\|x^{(k)} - x^*\|_2^2$$
$$\leq \|x^{(0)} - x^*\|_2^2 - 2\sum_{i=1}^{k} t_i(f(x^{(i-1)}) - f(x^*)) + \sum_{i=1}^{k} t_i^2 \|g^{(i-1)}\|_2^2.$$

Convergence analysis - Proof

- Using $\|x^{(k)} - x^*\|_2 \geq 0$ and letting $R = \|x^{(0)} - x^*\|_2$, we have

$$0 \leq R^2 - 2\sum_{i=1}^{k} t_i(f(x^{(i-1)}) - f(x^*)) + \sum_{i=1}^{k} t_i^2 \|g^{(i-1)}\|_2^2.$$

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- Introducing $f(x_{\text{best}}^{(k)}) = \min_{i=0,\dots k} f(x^{(i)})$ and rearranging, we have the basic inequality

$$f(x_{\text{best}}^{(k)}) - f(x^*) \leq \frac{R^2 + \sum_{i=1}^{k} t_i^2 \|g^{(i-1)}\|_2^2}{2\sum_{i=1}^{k} t_i}.$$

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For different step sizes choices, convergence results can be directly obtained from this bound. E.g., theorems for fixed and diminishing step sizes follow.

Convergence analysis - Proof

The basic inequality tells us that after k steps, we have

$$f(x_{\text{best}}^{(k)}) - f(x^*) \leq \frac{R^2 + \sum_{i=1}^k t_i^2 \|g^{(i-1)}\|_2^2}{2\sum_{i=1}^k t_i}.$$

With fixed step size t, this and the Lipschitz continuity assumption give

$$f(x_{\text{best}}^{(k)}) - f^* \leq \frac{R^2}{2kt} + \frac{L^2 t}{2}.$$

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$$f(x_{\text{best}}^{(k)}) - f^* \leq \frac{R^2}{2kt} + \frac{L^2 t}{2}.$$

- Does not guarantee convergence (as $k \to \infty$).
- For large k, $f(x_{\text{best}}^{(k)})$ is approximately $\frac{L^2 t}{2}$-suboptimal.
- To make the gap $\leq \varepsilon$, let's make each term $\leq \varepsilon/2$. So we can choose $t = \varepsilon/L^2$, and $k = R^2/t \cdot 1/\varepsilon = R^2 L^2/\varepsilon^2$.

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I.e., subgradient method guarantees the gap ε in $k = O(1/\varepsilon^2)$ iterations ... compare this to $O(1/\varepsilon)$ rate of gradient descent.

Convergence analysis - Proof

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With fixed step length, i.e., $t_i = s/\|g^{(i-1)}\|_2$, we have

$$f(x_{\text{best}}^{(k)}) - f(x^*) \leq \frac{R^2 + ks^2}{2s\sum_{i=1}^k \|g^{(i-1)}\|_2^{-1}} \leq \frac{R^2 + ks^2}{2s\sum_{i=1}^k L^{-1}} = \frac{LR^2}{2ks} + \frac{Ls}{2}.$$

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- To make the gap $\leq \varepsilon$, let's make each term $\leq \varepsilon/2$. So we can choose $s = \varepsilon/L$, and $k = LR^2/s \cdot 1/\varepsilon = R^2L^2/\varepsilon^2$.

Convergence analysis - Proof

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$$f(x_{\text{best}}^{(k)}) - f(x^*) \leq \frac{R^2 + \sum_{i=1}^{k} t_i^2 \|g^{(i-1)}\|_2^2}{2\sum_{i=1}^{k} t_i}.$$

From this and the Lipschitz continuity, we have

$$f(x_{\text{best}}^{(k)}) - f(x^*) \leq \frac{R^2 + L^2 \sum_{i=1}^{k} t_i^2}{2\sum_{i=1}^{k} t_i}.$$

With diminishing step size, $\sum_{i=1}^{\infty} t_i = \infty$ and $\sum_{i=1}^{\infty} t_i^2 < \infty$, there holds

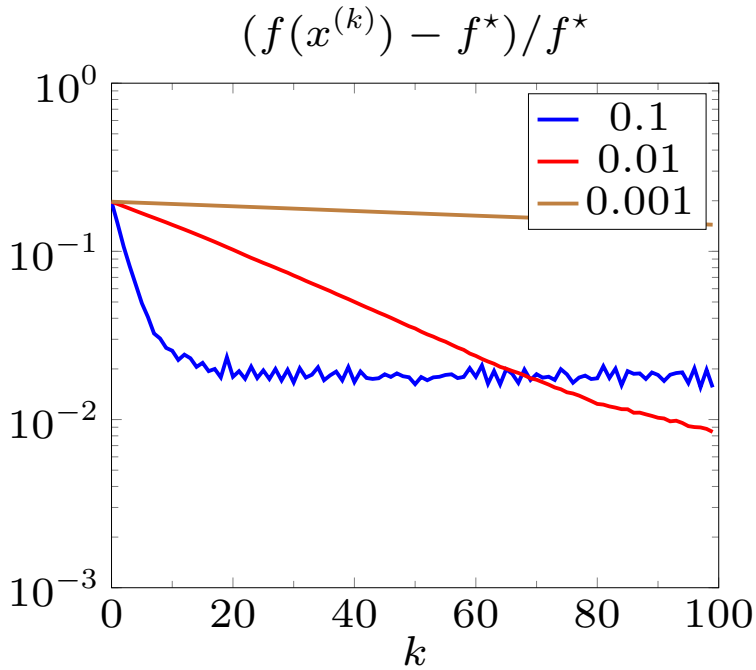
$$\lim_{k\to\infty} f(x_{\text{best}}^{(k)}) = f^*.$$

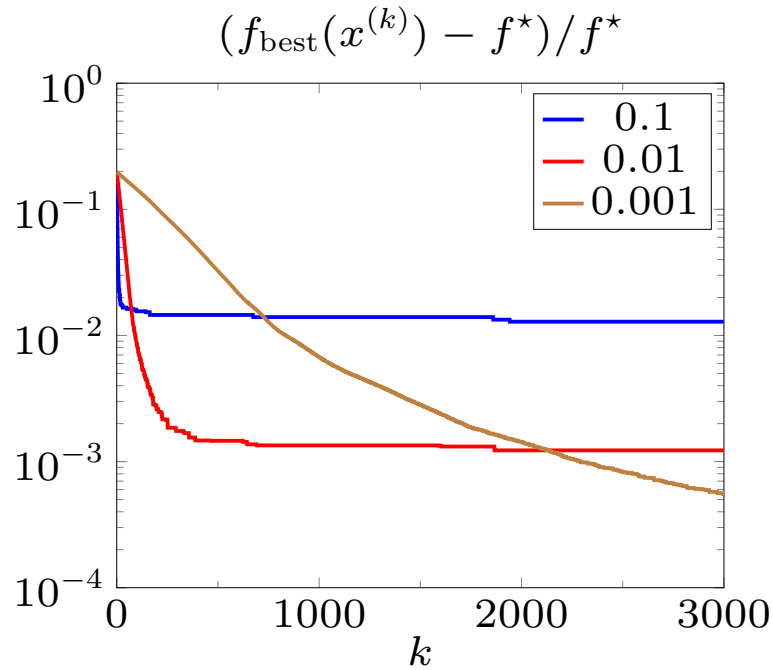
Example: 1-norm minimization

$$\text{minimize } \|Ax - b\|_1$$

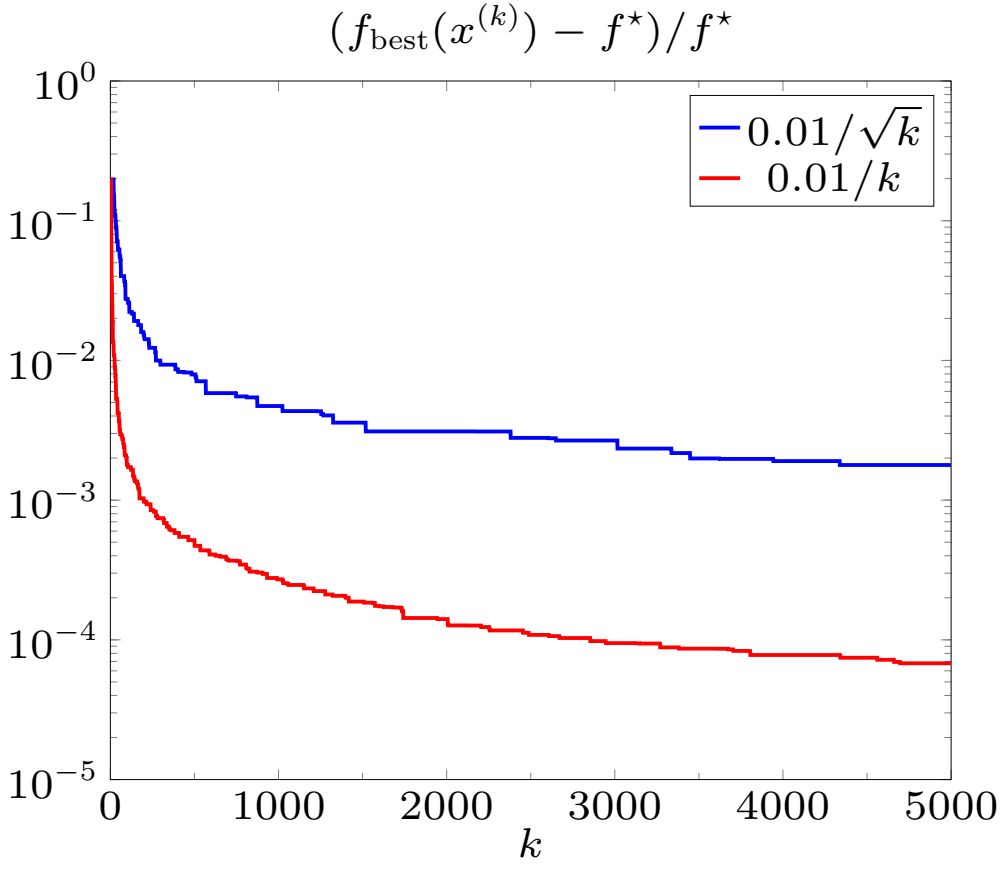
- subgradient is given by $A^T \mathbf{sign}(Ax - b)$
- example with $A \in \mathbf{R}^{500\times 100}$, $b \in \mathbf{R}^{500}$

Fixed steplength $t_k = s/\|g^{(k-1)}\|_2$ for $s = 0.1, 0.01, 0.001$





Diminishing step size: $t_k = 0.01/\sqrt{k}$ and $t_k = 0.01/k$



Example: regularized logistic regression

Given $(x_i, y_i) \in \mathbb{R}^p \times \{0, 1\}$ for $i = 1, \ldots n$, the logistic regression loss is

$$f(\beta) = \sum_{i=1}^{n} \left(-y_i x_i^T \beta + \log(1 + \exp(x_i^T \beta)) \right).$$

This is a smooth and convex with

$$\nabla f(\beta) = \sum_{i=1}^{n} (y_i - p_i(\beta)) x_i,$$

where $p_i(\beta) = \exp(x_i^T \beta)/(1 + \exp(x_i^T \beta))$, $i = 1, \ldots, n$.

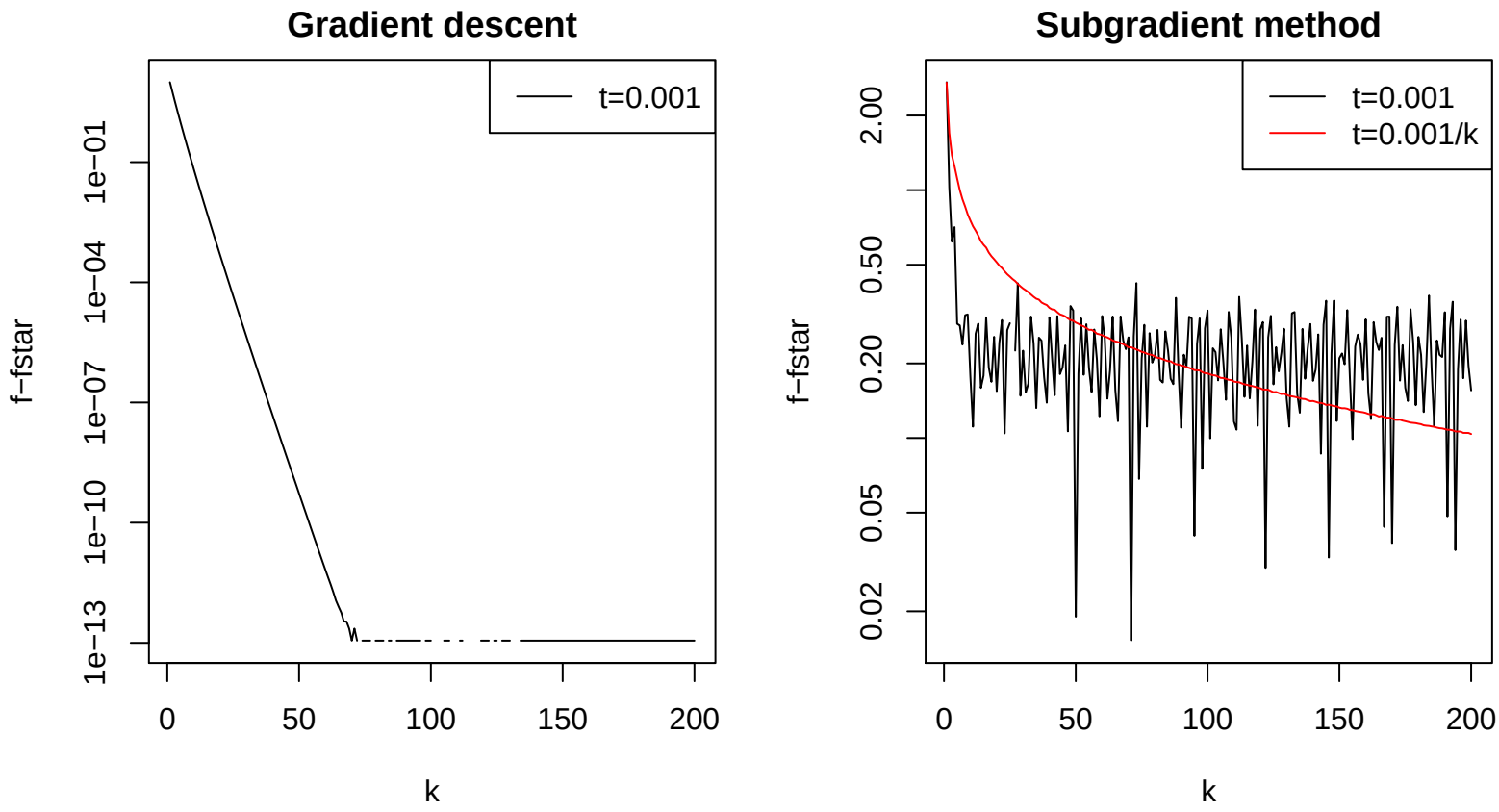
Consider the regularized problem

$$\min_{\beta} f(\beta) + \lambda \cdot P(\beta),$$

where $P(\beta) = \|\beta\|_2^2$ ridge penalty; or $P(\beta) = \|\beta\|_1$ lasso penalty.

Example: regularized logistic regression

Ridge: use gradients; lasso: use subgradients. Example here has $n = 1000, p = 20$.



Step sizes hand-tuned to be favorable for each method (of course comparison is imperfect, but it reveals the convergence behaviors).

Polyak step sizes

Polyak step sizes: when the optimal value f^* is known, take

$$t_k = \frac{f(x^{(k-1)}) - f^*}{\|g^{(k-1)}\|_2^2}, \quad k = 1, 2, 3, \ldots$$

Can be motivated from first step in subgradient proof:

$$\|x^{(k)} - x^*\|_2^2 \leq \|x^{(k-1)} - x^*\|_2^2 - 2t_k(f(x^{(k-1)}) - f(x^*)) + t_k^2 \|g^{(k-1)}\|_2^2.$$

Polyak step size minimizes the right-hand side.

With Polyak step sizes, can show subgradient method converges to optimal value. Convergence rate is still $O(1/\varepsilon^2)$

$$f(x_{\text{best}}^{(k)}) - f(x^*) \leq \frac{LR}{\sqrt{k}}.$$

(Proof: see slide 11, http://www.seas.ucla.edu/~vandenbe/236C/lectures/sgmethod.pdf).

Example: intersection of sets

Suppose we want to find $x^* \in C_1 \cap \cdots \cap C_m$, i.e., find a point in intersection of closed, convex sets $C_1, \ldots, C_m$.

First define

$$f_i(x) = \text{dist}(x, C_i), \quad i = 1, \ldots, m$$
$$f(x) = \max_{i=1,\ldots,m} f_i(x)$$

and now solve

$$\min_x f(x).$$

Check: is this convex?

Note that $f^* = 0 \iff x^* \in C_1 \cap \cdots \cap C_m$.

Example: intersection of sets

Recall the distance function $\text{dist}(x, C) = \min_{y \in C} \|y - x\|_2$. Last time we computed its gradient

$$\nabla \text{dist}(x, C) = \frac{x - P_C(x)}{\|x - P_C(x)\|_2}$$

where $P_C(x)$ is the projection of x onto C.

Also recall subgradient rule: if $f(x) = \max_{i=1,\ldots m} f_i(x)$, then

$$\partial f(x) = \text{conv}\left(\bigcup_{i: f_i(x) = f(x)} \partial f_i(x) \right).$$

So if $f_i(x) = f(x)$ and $g_i \in \partial f_i(x)$, then $g_i \in \partial f(x)$.

Example: intersection of sets

Put these two facts together for intersection of sets problem, with $f_i(x) = \text{dist}(x, C_i)$: if C_i is farthest set from x (so $f_i(x) = f(x)$), and

$$g_i = \nabla f_i(x) = \frac{x - P_{C_i}(x)}{\|x - P_{C_i}(x)\|_2}$$

then $g_i \in \partial f(x)$.

Now apply subgradient method, with Polyak size $t_k = f(x^{(k-1)})$. At iteration k, with C_i farthest from $x^{(k-1)}$, we perform update

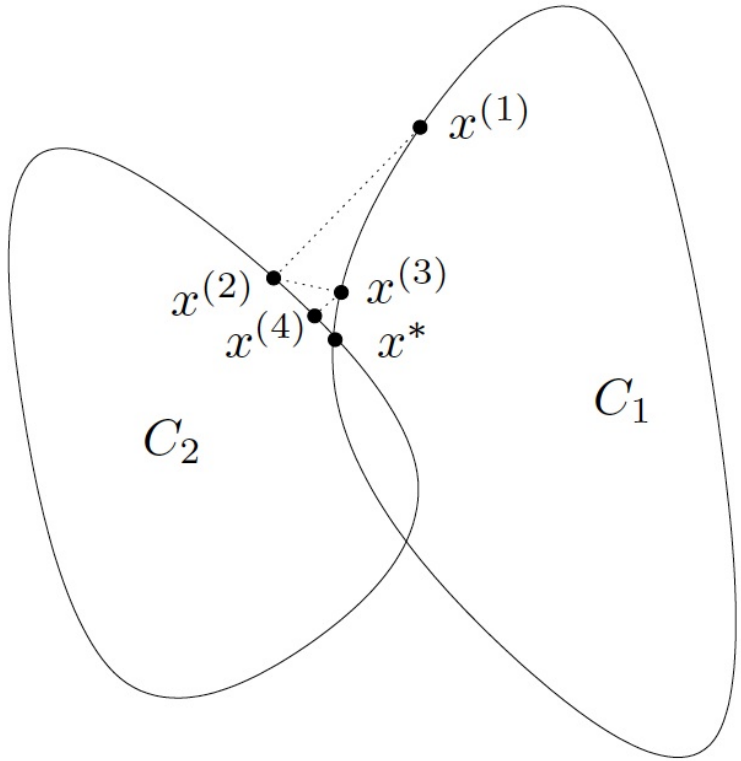
$$\begin{aligned} x^{(k)} &= x^{(k-1)} - f(x^{(k-1)}) \frac{x^{(k-1)} - P_{C_i}(x^{(k-1)})}{\|x^{(k-1)} - P_{C_i}(x^{(k-1)})\|} \\ &= P_{C_i}(x^{(k-1)}), \end{aligned}$$

since

$$f(x^{(k-1)}) = \text{dist}(x^{(k-1)}, C_i) = \|x^{(k-1)} - P_{C_i}(x^{(k-1)})\|.$$

Example: intersection of sets

For two sets, this is the famous alternating projections[1] algorithm, i.e., just keep projecting back and forth.



[1] von Neumann (1950), "Functional operators, volume II: The geometry of orthogonal spaces"

Projected subgradient method

To optimize a convex function f over a convex set C,

$$\min_x f(x) \quad \text{subject to} \quad x \in C$$

we can use the projected subgradient method.

Just like the usual subgradient method, except we project onto C at each iteration:

$$x^{(k)} = P_C(x^{(k-1)} - t_k \cdot g^{(k-1)}), \quad k = 1, 2, 3, \ldots$$

Assuming we can do this projection, we get the same convergence guarantees as the usual subgradient method, with the same step size choices.

Projected subgradient method

What sets C are easy to project onto? Lots, e.g.,

- Affine images: $\{Ax + b : x \in \mathbb{R}^n\}$
- Solution set of linear system: $\{x : Ax = b\}$
- Nonnegative orthant: $\mathbb{R}^n_+ = \{x : x \geq 0\}$
- Some norm balls: $\{x : \|x\|_p \leq 1\}$ for $p = 1, 2, \infty$
- Some simple polyhedra and simple cones.

Warning: it is easy to write down seemingly simple set C, and P_C can turn out to be very hard! E.g., generally hard to project onto arbitrary polyhedron $C = \{x : Ax \leq b\}$.

Note: projected gradient descent works too, more next time ...

Can we do better?

Upside of the subgradient method: broad applicability.

Downside: $O(1/\varepsilon^2)$ convergence rate over problem class of convex, Lipschitz functions is really slow.

Nonsmooth first-order methods: iterative methods updating $x^{(k)}$ in

$$x^{(0)} + \text{span}\{g^{(0)}, g^{(1)}, \ldots, g^{(k-1)}\}$$

where subgradients $g^{(0)}, g^{(1)}, \ldots, g^{(k-1)}$ come from weak oracle.

Theorem (Nesterov)

For any $k \leq n-1$ and starting point $x^{(0)}$, there is a function in the problem class such that any nonsmooth first-order method satisfies

$$f(x^{(k)}) - f^* \geq \frac{RG}{2(1+\sqrt{k+1})}.$$

Improving on the subgradient method

In words, we cannot do better than the $O(1/\varepsilon^2)$ rate of subgradient method (unless we go beyond nonsmooth first-order methods).

So instead of trying to improve across the board, we will focus on minimizing composite functions of the form

$$f(x) = g(x) + h(x)$$

where g is convex and differentiable, h is convex and nonsmooth but "simple".

For a lot of problems (i.e., functions h), we can recover the $O(1/\varepsilon)$ rate of gradient descent with a simple algorithm, having important practical consequences.

References and further reading

- S. Boyd, *Lecture notes for EE 264B*, Stanford University, Spring 2010-2011
- Y. Nesterov (1998), *Introductory lectures on convex optimization: a basic course*, Chapter 3
- B. Polyak (1987), *Introduction to optimization*, Chapter 5
- L. Vandenberghe, *Lecture notes for EE 236C*, UCLA, Spring 2011-2012